Derived Functors and Homotopy Colimits

Exercice 1 (Composition of Derived Functors)**.**

1. Let $F_1 : \mathcal{C}_1 \to \mathcal{C}_2$ and $F_2 : \mathcal{C}_2 \to \mathcal{C}_3$ be functors and let $\mathcal{W}_i$ be a class of morphisms in $\mathcal{C}_i$. Assuming all the relevant total left derived functors exist, use their universal properties to construct a natural transformation $\mathbb{L}F_2 \circ \mathbb{L}F_1 \to \mathbb{L}(F_2 \circ F_1)$.
2. Suppose now that $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ are model categories and that F_1 and F_2 are left Quillen functors. Show that all derived functors exist and the natural transformation of the previous exercise is a natural isomorphism.

Exercice 2 (Homotopy colimits)**.** In this exercise, we first deal with generalities on homotopy pushouts and then specialize to chain complexes with the projective model structure. Let $\mathcal{C}$ be a model category and let I be the category given by the diagram-shape

$$\begin{tikzcd} b \arrow[r] \arrow[d] & c \\ a & \end{tikzcd}$$

1. Let $f : X \to Y$ be a natural transformation of diagrams $X, Y \in \mathrm{Fun}(I, \mathcal{C})$. Show that f has the left lifting property with respect to all projective acyclic fibrations if and only if the the natural maps
$$X(a) \coprod_{X(b)} Y(b) \to Y(a), \quad X(b) \to Y(b), \quad X(c) \coprod_{X(b)} Y(b) \to Y(c)$$
are cofibrations in $\mathcal{C}$. (Here we mean the usual pushouts in $\mathcal{C}$.)
Deduce that a diagram $Y : I \to \mathcal{C}$ is cofibrant if and only if $Y(b)$ is cofibrant in $\mathcal{C}$ and the maps $Y(a) \to Y(b)$ and $Y(a) \to Y(c)$ are cofibrations. Moreover, show that $X \to Y$ has the left lifting property with respect to projective fibrations if and only the above three maps are acyclic cofibrations.
2. Show that the category of diagrams $\mathrm{Fun}(I, \mathcal{C})$ admits the projective model structure (without using the result seen in class that such a structure exists since I is very small).
3. Show that the colimit functor $\mathrm{colim}\colon \mathrm{Fun}(I, \mathcal{C}) \to \mathcal{C}$ is a left Quillen functor.
4. Assume that $\mathcal{C}$ is left proper (i.e. weak equivalences are stable under pushouts along cofibrations). Show that any pushout diagram
$$\begin{tikzcd} B \arrow[r, tail, "f"] \arrow[d] & C \arrow[d] \\ A \arrow[r] & A \coprod_B C \end{tikzcd}$$
where $f\colon B \longrightarrow C$ a cofibration, is also a homotopy pushout diagram.
5. **Case of Topological spaces.** Assume now that $\mathcal{C} = \mathbf{Top}$.
   (a) Using that **Top** is proper (as seen in exercise 3. from the sheet on Quillen model structure), show that there is a canonical isomorphism
$$\mathbb{L}\,\mathrm{colim}(X \leftarrow A \to Y) \cong X \coprod_A^{\mathrm{h}} Y = X \coprod_{A \times \{0\}} \mathrm{Cyl}(A \to Y)$$

in **Ho**(**Top**) between the homotopy pushout computed by the projective model structure and the formula given by the mapping cylinder.

(b) Give a formula for computing the homotopy colimit of a tower $(X_0 \to X_1 \to X_2 \to \dots)$ as well as the homotopy limit of a tower $(\dots \to Y_2 \to Y_1 \to Y_0)$.

6. **Case of chain complexes.** Assume now that $\mathcal{C}$ is the model category of chain complexes over a ring R.

(a) Show that $\mathcal{C}$ is left proper.

(b) Let $g : A \to B$ be a map of chain complexes. Recall that the *mapping cone* of g, denoted $C(g)$, is the chain complex given in level n by $B_n \oplus A_{n-1}$ and whose differential $B_{n+1} \oplus A_n \to B_n \oplus A_{n-1}$ is given $(b, a) \mapsto (\partial_B(b) + g(a), -\partial_A(a))$. Let I denote the chain complex given by $R \oplus R$ in degree 0 and R in degree 1 with differential given by $\partial_R : R \to R \oplus R$ given by $r \mapsto (-r, r)$. We define the *mapping cylinder* of g, denoted $\mathrm{Cyl}(g)$, as the pushout in chain complexes of

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i_0"] & B \arrow[d] \\ I \otimes A \arrow[r] & \mathrm{Cyl}(g) \end{tikzcd}$$

where the vertical arrow $A \to I \otimes A$ is induced by the inclusion $i_0 : R \to I$ corresponding to the inclusion of the second factor $R \hookrightarrow R \oplus R$ in degree 0. The differential on $I \otimes A$ is given by $r \otimes a \mapsto \partial_R(x) \otimes a + (-1)^{\deg(r)} r \otimes \partial_A(a)$. Show that the mapping cone of g is the pushout of

$$\begin{tikzcd} I \otimes A \arrow[r] \arrow[d] & \mathrm{Cyl}(g) \arrow[d] \\ C(\mathrm{Id}_A) \arrow[r] & C(g). \end{tikzcd}$$

(c) Let Δ^1 be the category with two objects and one non trivial morphism in between them. Show that the construction of the mapping cone defines a functor $C : \mathrm{Fun}(\Delta^1, \mathrm{Ch}(R)) \to \mathrm{Ch}(R)$ sending natural transformations objectwise given by quasi-isomorphisms to quasi-isomorphisms.

(d) Let $Y := (0 \longleftarrow A \xrightarrow{g} B)$ be a diagram in $\mathcal{C}$. Show that there exists a diagram of the form $Y' := (0 \longleftarrow A' \xrightarrow{g'} B')$ with g' a cofibration and A' and B' cofibrant, together with a natural transformation $u : Y' \to Y$ which is objectwise a weak-equivalence. Notice that by the previous question the induced map $C(g') \to C(g)$ is a weak-equivalence.

(e) Let $Y := (0 \longleftarrow A \xrightarrow{g} B)$ be a diagram in $\mathcal{C}$ with A and B cofibrant and g a cofibration. Show that $A \to A \otimes \Delta^1$ is a weak-equivalence and show that we can construct a zigzag of diagrams $Y \leftarrow Y' \to Y''$ of the form

$$\begin{tikzcd} 0 & A \arrow[l] \arrow[r, "g"] & B \\ C(A) \arrow[u] \arrow[d] & A \arrow[l] \arrow[r, "g"] \arrow[u] \arrow[d] & B \arrow[u] \arrow[d] \\ C(A) & I \otimes A \arrow[l] \arrow[r, "g"] & \mathrm{Cyl}(g) \end{tikzcd}$$

where each vertical arrow is a weak-equivalence and the map $I \otimes A \to \mathrm{Cyl}(g)$ is a cofibration.

(f) Let $Y := (0 \longleftarrow A \xrightarrow{g} B)$ be any diagram. Conclude that the mapping cone $C(g)$ is a model for the homotopy colimit of the diagram Y.

Exercice 3 (**Bad behavior of Gabriel-Zisman Localization**). Let A be a ring and let $D(A) := \mathbf{Ho}(\mathrm{Ch}(A))$ denote the derived category of A; it is the Gabriel-Zisman localization of the category $\mathrm{Ch}(A)$ of chain complexes in A along quasi-isomorphisms of complexes. We have seen in class that $D(A)$ is the homotopy category of a model structure in $\mathrm{Ch}(A)$ with weak-equivalences given by quasi-isomorphisms and fibrations given by levelwise surjections.

1. Show that if E and H are two A-modules seen as complexes concentrated in degree zero, then

$$\mathrm{Hom}_{D(A)}(E, H[n]) \simeq \mathrm{Ext}^n_A(E, H).$$

2. Show that if A is a field, then $D(A)$ is an abelian category[1], equivalent to the category $A^{\mathbb{Z}}$ of $\mathbb{Z}$-graded A-vector spaces.
3. Show that $D(A[X])$ does not admit colimits in general. (*Hint*: take a non-trivial element $f : A \to A[1]$ and show that if it has a kernel to get a contradiction.)
4. Let A be a field and let I be the category with one object and $\mathbb{N}$ as endomorphisms. Show that $\mathrm{Fun}(I, D(A))$ is not equivalent to $D(\mathrm{Fun}(I, \mathrm{Ch}(A))$.
   The conclusion is that the theory of diagrams does not interact well with derived categories.

Exercice 4. Let $\mathbf{Top}_*$ be the category of *pointed* topological spaces and $U : \mathbf{Top}_* \to \mathbf{Top}$ be the functor forgetting the base point.

1. Prove that U is a right adjoint and compute its left adjoint.
2. We endow $\mathbf{Top}$ with Quillen model structure. Find a model structure on $\mathbf{Top}_*$ such that U is right Quillen.
3. Generalize the previous construction to any model category $\mathcal{C}$.

[1] see links to homological algebra exercises on the web page, if you are not familiar with this.