Simplicial Sets

Exercice 1 (*Modules over cdgas*)**.** Let A be a cdga (commutative differential graded algebra) over $\mathbb{Q}$. Let $\mathsf{Mod}(A)$ denote the category of dg modules over A. An object in $\mathsf{Mod}(A)$ is thus a cochain complex M together with a morphism of complexes $A \otimes_{\mathbb{Q}} M \to M$ satisfying the module axioms (i.e. $(a \cdot b) \cdot m = a \cdot (b \cdot m)$, $1 \cdot m = m$).

1. Show that the forgetful functor $U : \mathsf{Mod}(A) \to \mathsf{Ch}(\mathbb{Q})$ is a right adjoint and describe its left adjoint F.
2. Show that there is a model structure on $\mathsf{Mod}(A)$ where
   - weak equivalences are the morphisms f such that $U(f)$ is a quasi-isomorphism,
   - fibrations are the morphisms f such that $U(f)$ is surjective.
3. Show that the functor $- \underset{A}{\otimes} -: \mathsf{Mod}(A) \times \mathsf{Mod}(A) \longrightarrow \mathsf{Mod}(A)$ admits a total left derived functor $$- \overset{\mathbb{L}}{\underset{A}{\otimes}} - : \mathrm{Ho}(\mathsf{Mod}(A) \times \mathsf{Mod}(A)) \cong \mathrm{Ho}(\mathsf{Mod}(A)) \times \mathrm{Ho}(\mathsf{Mod}(A)) \longrightarrow \mathrm{Ho}(\mathsf{Mod}(A)).$$
4. Let $f : A \to B$ be a morphism of cdgas. Show that the functor $f_* : \mathsf{Mod}(B) \to \mathsf{Mod}(A)$, given by $A \otimes_{\mathbb{Q}} M \overset{f \otimes id}{\to} B \otimes_{\mathbb{Q}} M \to M$, is a right Quillen functor.
5. Assume $f : A \to B$ is a quasi-isomorphism of cdgas. Show that f_* is a Quillen equivalence.

Exercice 2 (Playing with simplicial sets)**.** We recall that Δ is the category whose objects are finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$ and morphisms are order-preserving maps. Denote by $\Delta[n] = \Delta^n_\bullet \in \mathsf{sSet}$ the Yoneda embedding: $\Delta[n] := \mathrm{Hom}_\Delta(-, [n])$. We recall that if X is a simplicial set, the data of a n-simplex of X, corresponds to the data of a simplicial set morphism $\Delta[n] \to X$.

1. Write d_i and ϵ_j the face and degeneracies. Check that any map $f : [m] \to [n]$ in Δ can be factored in a unique way as $f =$ $$[m] \xrightarrow{\epsilon_{j_1}} [m-1] \xrightarrow{\epsilon_{j_2}} \dots \xrightarrow{\epsilon_{j_t}} [m-t] \xrightarrow{\partial_{i_1}} [m-t+1] \xrightarrow{\partial_{i_2}} \dots \xrightarrow{\partial_{i_k}} [m-t+k] = [n]$$ where $j_t < j_{t-1} < \cdots < j_1$ are the elements of $[m]$ with $f(j) = f(j+1)$ and $i_1 < i_2 < \cdots < i_k$ are the values in $[n]$ that are not in the image of f. Conclude that Δ is the free category generated by the objects $[n]$ and morphisms ∂_i and ϵ_j submitted to the simplicial relations.
2. Check that a morphism $f : [m] \to [n]$ is an epimorphism if and only if it is a non-decreasing surjection and that the simplicial relations imply that every epimorphism is split.
3. (Eilenberg-Zilber Lemma) Let X be a simplicial set. Show that for each m-simplex $\sigma : \Delta[m] \to X$ there is an epimorphism $s : \Delta[m] \to \Delta[n]$ and a non-degenerate n-simplex $x : \Delta[n] \to X$ such that $y \circ s = \sigma$. Show that the pair (y, s) is unique.
4. (Skeletons) We denote by $\mathrm{sk}_n(X)$ the subsimplicial set of $X \in \mathsf{sSet}$ given by the non-degenerate simplices of X of dimension less than n. Thus its p-simplices are the p-simplices σ of X such that there exists an epimorphism $s : \Delta[p] \to \Delta[q]$ with $q \leq n$ and a q-simplex $x : \Delta[q] \to X$ such that $x \circ s = \sigma$. In other words, for $q \leq n$ the q-cells of $\mathrm{sk}_n(X)$ coincide precisely with the q-cells of X. For $m > n$, the m-cells of $\mathrm{sk}_n(X)$ are given by the m-cells of X which are degenerate.

   The construction $X \mapsto \mathrm{sk}_n(X)$ can be seen as a right adjoint: let $\Delta_{\leq n}$ denote the full subcategory of Δ spanned by those objects $[k]$ with $k \leq n$. Write $i_n : \Delta_{\leq n} \hookrightarrow \Delta$ for the inclusion functor.

(a) Let $T \in \mathrm{Fun}(\Delta^{op}_{\leq n}, \mathsf{Set})$. Prove that the formula[1] $(i_n)_!(T)_* := \operatorname*{colim}_{* \to k \leq n} T(k)$ defines a functor $(i_n)_! : \mathrm{Fun}(\Delta^{op}_{\leq n}, \mathsf{Set}) \to \mathsf{sSet}$ and that the functor $(i_n)_!$ admits a right adjoint $(i_n)^*$.

(b) Show that for any $X \in \mathrm{Fun}(\Delta^{op}_{\leq n}, \mathsf{Set})$, the unit of the adjunction $X \to (i_n)^*(i_n)_! X$ is an isomorphism. Conclude that $(i_n)^*$ is fully faithful.

(c) Show that for any simplicial set X, the co-unit of the adjunction $(i_n)_!(i_n)^*(X) \to X$ is injective and show that its image in X coincides with the sub-simplicial set $\mathrm{sk}_n(X)$;

(d) Show that the canonical map $\operatorname*{colim}_{n \geq 0} \mathrm{sk}_n(X) \to X$ is an isomorphism.

5. (Boundaries) We give an alternative presentation of $\partial\Delta[n]$ as the result of gluings all the $n-1$-simplices of $\Delta[n]$ along the $n-2$-simplices. Consider the diagram

$$\coprod_{0 \leq i < j \leq n} \Delta[n-2] \underset{v}{\overset{u}{\rightrightarrows}} \coprod_{0 \leq i \leq n} \Delta[n-1] \xrightarrow{p} \Delta[n]$$

where the map p is induced by the inclusions of the faces of $\Delta[n]$. Each copy of $\Delta[n-2]$ on the l.h.s corresponds to a copy of $[n]$ where both i and j are missing. Similarly, each copy of $\Delta[n-1]$ on the r.h.s corresponds to a copy of $[n]$ where a single element i is missing. The map u is induced by the boundary maps $\partial^{j-1}_{n-1} : \Delta[n-2] \to \Delta[n-1]$ and the maps v are induced by the boundary maps $\partial^{i}_{n-1} : \Delta[n-2] \to \Delta[n-1]$. Check that the image of p is the set of simplices in $\Delta[n]$ belonging to $\partial\Delta[n]$ and conclude that $\partial\Delta[n]$ is isomorphic to the co-equalizer of (u, v).

6. Let X be a simplicial set. Show that for each $n \geq 0$ the squares

$$\begin{tikzcd}
\coprod_{\sigma \in X_n,\ \sigma \text{ non-deg}} \partial\Delta[n]_\sigma \arrow[r] \arrow[d, "\text{inclusion}"'] & Sk_{n-1}(X) \arrow[d] \\
\coprod_{\sigma \in X_n,\ \sigma \text{ non-deg}} \Delta[n]_\sigma \arrow[r] & Sk_n(X)
\end{tikzcd}$$

are cocartesian. This allows us to construct X by induction on n.

7. (Horns) Recall the notion of j-horn Λ^j_n the sub simplicial set of $\Delta[n]$ in which the jth face and the interior have been removed.

(a) Prove that the m-simplices of Λ^j_n are the order preserving maps $p : [m] \to [n]$ whose image does not contain the set $[n] - \{j\}$.

(b) Describe the horn using boundaries and skeletons.

(c) Deduce that $\mathrm{Hom}_{\mathsf{sSet}}(\Lambda^r_n, X)$ is in bijection with the set of n-tuples of $(n-1)$-simplices $(x_0, \ldots, \widehat{x_r}, \ldots x_n)$ of X such that for all $i, j \neq r$ and $i < j$, one has $d_i x_j = d_{j-1} x_i$.

(d) Prove that a simplicial set is fibrant if and only if, for any $k \leq n$ and n-tuple of $(n-1)$-simplices $(x_0, \ldots, \widehat{x_r}, \ldots x_n)$ of X satisfying that, for all $i, j \neq r$ and $i < j$, $d_i x_j = d_{j-1} x_i$, then there exists a n-simplex $x \in X$ such that $d_i(x) = x_i$ for all $i \neq k$.

8. Deduce that the simplicial set $\Delta[n]$ is not fibrant for $n \geq 1$.

[1]this is nothing more than the left Kan extension along the inclusion i_n

Exercice 3 (Detailled construction of the Nerve of a category)**.** In this exercise and the following, we establish a link between the theory of categories and the theory of simplicial sets. More precisely, we check that we can translate the information provided by a category $\mathcal{C}$ into a simplicial set, called the nerve of $\mathcal{C}$ and denoted by $N(\mathcal{C})$. We will see that this translation does not lose any information and that in fact the theory of categories can be seen as a sub-theory of that of simplicial sets.

1. The category of simplexes Δ can be canonically identified with a full subcategory of Cat, spanned by the categories of the form $[n] := [0 \to 1 \to \cdots \to n]$. Use this inclusion and the previous exercise to produce an adjunction $\begin{tikzcd} \mathsf{sSet} \arrow[r, bend left, "\tau"] & \mathsf{Cat} \arrow[l, bend left, "N"] \end{tikzcd}$ sending $\tau(\Delta[n]) = [n]$.

2. Let $\mathcal{C}$ be a small category. Check that the functor N is characterized as follows: $N(\mathcal{C})_n$ consists of composable strings of morphims in $\mathcal{C}$ of lenght n:
$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} X_n.$$
In particular, the 0-simplexes of $N(\mathcal{C})$ are the objects of $\mathcal{C}$ and the 1-cells are morphisms in $\mathcal{C}$. Describe the face and degeneracy maps in terms of compositions and identity morphims.

3. Show that the canonical morphism induced by the inclusion $\tau(\partial\Delta[n]) \to \tau(\Delta[n]) = [n]$ is an isomorphism of categories for $n \geq 3$. Describe both $\tau(\partial\Delta[1])$ and $\tau(\partial\Delta[2])$.
(*Hint:* use the construction of $\partial\Delta[n]$ as a cokernel).

4. Deduce that the canonical map $\tau(\mathrm{sk}_2(X)) \to \tau(X)$ is an isomorphism of categories for every simplicial set X. In other words, the category $\tau(X)$ only depends on the 2-skeleton of X.

5. Let X be a simplicial set. Check that the category $\tau(\mathrm{sk}_2(X))$ is isomorphic to the quotient of the free category with X_0 as objects and X_1 as morphisms under the following relation on morphisms:
   - for every 2-simplex $\sigma : \Delta[2] \to X$, we identify $\partial_1(\sigma)$ with the composition $\partial_0(\sigma) \circ \partial_2(\sigma)$.
   - for every $x \in X_0$, identify $\epsilon_0(x)$ with Id_x

6. Let $\mathcal{C}$ be a category and describe the category $\tau(\mathrm{sk}_2(N(\mathcal{C}))$. Conclude that the adjunction map $\tau(N(\mathcal{C})) \to \mathcal{C}$ is an isomorphism of categories and that N is fully faithful.

7. Let I_n denote the sub-simplicial set (subfunctor) of $\Delta[n]$ given by $\bigcup_i^n \mathrm{im}\, \alpha_i \subseteq \Delta[n]$ where $\alpha_i : \Delta[1] \to \Delta[n]$ is the map sending $0 \to i$ and $1 \mapsto i+1$. Show that I_n is the colimit of the diagram
$$\begin{tikzcd} \Delta[1] & & \Delta[1] & & \dots & & \Delta[1] \\ & \Delta[0] \arrow[ul, "\partial_1"] \arrow[ur, "\partial_0"'] & & \Delta[0] \arrow[ul, "\partial_1"] \arrow[ur, "\partial_0"'] & \dots & \Delta[0] \arrow[ul, "\partial_1"] \arrow[ur, "\partial_0"'] & \end{tikzcd}$$
where $\Delta[1]$ appears n times.

8. Let $\mathcal{C}$ be a category and let $N(\mathcal{C})$ denote its nerve. Show that the composition with the inclusion $I_n \subseteq \Delta[n]$ produces a bijection
$$\mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], N(\mathcal{C})) \cong \mathrm{Hom}_{\mathsf{sSet}}(I_n, N(\mathcal{C}))$$
for all $n \geq 2$. Conclude that the canonical map $\tau(I_n) \to \tau(\Delta[n]) = [n]$ is an isomorphism of categories for $n \geq 2$.

Exercice 4 (Segal conditions and categories)**.** Let $N : \mathsf{Cat} \to \mathsf{sSet}$ be the nerve functor.

1. (Grothendieck-Segal condition) Show that a simplicial set X belongs to the essential image of the functor N (ie, X encodes the information of a category) if and only if the composition map

$$\mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], X) \longrightarrow \mathrm{Hom}_{\mathsf{sSet}}(I_n, X)$$

is a bijection for all $n \geq 2$.

2. (Grothendieck-Segal condition via horns) Let X be a simplicial set. Show that X is in the essential image of N if and only if for all $n \geq 2$, all $0 < i < n$ and for any map of simplicial sets $u : \Lambda^i_n \to X$ there exists a unique factorization of u along the canonical inclusion

$$\begin{tikzcd} \Lambda^i_n \arrow[r, "u"] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

In other words, the composition map gives a bijection

$$\mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], X) \cong \mathrm{Hom}_{\mathsf{sSet}}(\Lambda^i_n, X).$$

3. Following the previous question, show that a simplicial set X is the nerve of a groupoid if and only if for all $n \geq 1$ and $0 \leq i \leq n$ and for any map of simplicial sets $u : \Lambda^i_n \to X$ there exists a unique factorization of u along the canonical inclusion (as in the above diagram).
In particular, if $\mathcal{C}$ is a groupoid, then $N(\mathcal{C})$ is a Kan complex.

Exercice 5 (*Classifying space)***.** Let G be a group and let $\mathcal{G}$ be the category with one object, whose endomorphisms are given by G.

1. Verify that $\mathcal{G}$ is a category and describe the n-simplices of its nerve $N\mathcal{G}$.
2. Show that $N\mathcal{G}$ is a Kan complex.
3. Let $\mathcal{E}_G$ be the category whose objects are the elements of G and with a unique morphism between every two objects. Show that $\mathcal{E}_G$ is well-defined and describe the n-simplices of its nerve.
4. Show that there exists a functor $\phi : \mathcal{E}_G \to \mathcal{G}$ sending a morphism $g \to g'$ to $g' \cdot g^{-1} \in \mathrm{End}_{\mathcal{G}}(*)$.
5. Prove that the induced morphism of simplicial sets $N\phi : N\mathcal{E}_G \to N\mathcal{G}$ is a Kan fibration.
6. Show that $N\mathcal{E}_G$ is contractible. Deduce the homotopy type of $N\mathcal{G}$.

Exercice 6 (Universal Property of Presheaves of Sets)**.** Let $\mathcal{C}$ be a small category. We denote by $\mathcal{P}(\mathcal{C})$ the category of functors $\mathcal{C}^{op} \to \mathsf{Set}$ with natural transformations as morphisms. Objects in this category are called *presheaves* (of sets) over $\mathcal{C}$. We have a canonical way to go from $\mathcal{C}$ to $\mathcal{P}(\mathcal{C})$, namely, to each object $X \in \mathcal{C}$ we assign the functor $h(X) := \mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{op} \to \mathsf{Set}$. We let h denote this functor.

1. Let $F : \mathcal{C}^{op} \to \mathsf{Set}$ be a presheaf over $\mathcal{C}$ and let $X \in \mathcal{C}$. Given a natural transformation $u : h(X) \to F$ we can produce an element in $F(X)$ as follows: evaluating both $h(X)$ and F at the object X, u induces a map $u_X : h(X)(X) := \mathrm{Hom}_{\mathcal{C}}(X, X) \to F(X)$. We consider the element $u_X(\mathrm{Id}_X) \in F(X)$. Show that the assignment $\mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(h(X), F) \to F(X)$ given by $u \mapsto u_X(\mathrm{Id}_X)$ is an isomorphism of sets. Use this to deduce that h is fully faithful. [2]

[2] h is also called the Yoneda functor.

2. Show that $\mathcal{P}(\mathcal{C})$ admits all small colimits. (Hint: Construct the colimits objectwise.)

3. Let $F \in \mathcal{P}(\mathcal{C})$ and denote by $\mathcal{C}/F$ the full subcategory of objects over F in $\mathcal{P}(\mathcal{C})$ whose source is of the form $h(X)$ for some $X \in C$. Consider the diagram $\mathcal{C}/F \to \mathcal{P}(\mathcal{C})$ sending $(h(X) \to F)$ to $h(X)$. Show that the canonical arrow

$$\operatorname*{colim}_{u:h(X)\to F} h(X) \longrightarrow F$$

is an isomorphism in $\mathcal{P}(\mathcal{C})$ [3]. In other words, every presheaf F is the colimit of all representable presheaves defined over F.

4. Show that h has the following universal property: for any functor $\Phi : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is a category which admits all small colimits, there exists a unique functor (up to canonical equivalence of categories) $\widetilde{\Phi} : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ which commutes with small colimits and makes the following diagram commute up to natural isomorphism

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\Phi"] \arrow[d, "h"] & \mathcal{D} \\ \mathcal{P}(\mathcal{C}) \arrow[ur, dashed, "\widetilde{\Phi}"'] & \end{tikzcd}$$

In other words, $\mathcal{P}(\mathcal{C})$ is the universal way to complete $\mathcal{C}$ with all small colimits.

5. Check that the previous universal property can be reformulated as follows: if $\mathcal{D}$ is a category having all small colimits, then composition with h induces an equivalence of categories

$$\mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

where $\mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D})$ is by definition the full subcategory of $\mathrm{Fun}(\mathcal{P}(\mathcal{C}), \mathcal{D})$ spanned by all functors which commute with colimits.

6. Show that in the context of the previous question, the functor $\widetilde{\Phi}$ always admits a right adjoint.

7. Let F be a presheaf over $\mathcal{C}$. Show that the category of presheaves over the comma category of representables over F is equivalent to the category of all presheaves over F, that is,

$$\mathcal{P}(\mathcal{C}/F) \simeq \mathcal{P}(\mathcal{C})/F.$$

8. Use the conclusion of this exercise to show that in order to produce an adjunction

$$F : \mathsf{sSet} \rightleftarrows \mathcal{D} : G$$

where $\mathcal{D}$ is a category having all small colimits, it is enough to give a functor $\Delta \to \mathcal{D}$ (also called a *cosimplicial object* in $\mathcal{D}$). Observe that the following examples arise in this way:

- $(|-| \dashv \mathrm{Sing})$: the geometric realization of simplicial sets and the singular chain functors,
- $(\tau \dashv N)$: the truncation (or categorical realization) and the nerve of categories,
- $(\mathfrak{C} \dashv \mathrm{N})$: the rigidification functor and the homotopy-coherent nerve of simplicial categories.

[3] This is a small colimit because the indexing category is small as by hypothesis $\mathcal{C}$ is small.