Introduction to Riemann surfaces

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1 Introduction: Elliptic curves

Definition 1.1 *A complex manifold of dimension n is a differential manifold (which we suppose Hausdorff and second countable) equipped with a cover $M = \cup U_\alpha$ by open sets U_α and homeomorphisms (charts) $z_\alpha : U_\alpha \to \mathbb{C}^n$ such that the maps $z_\beta \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U_\beta) \to z_\beta(U_\alpha \cap U_\beta)$ are biholomorphisms.*

Once a chart cover is defined, one can consider a maximal family of charts compatible with the given cover.

Definition 1.2 *A Riemann surface is a one dimensional complex manifold.*

Example 1.3 *The Riemann sphere $\mathbb{C}P^1$ is a Riemann surface whose underlying topological manifold is the two dimensional sphere S^2. We write $S^2 = \mathbb{C} \cup \{\infty\}$. There are two natural charts:*

1. $z_1 : \mathbb{C} \cup \{\infty\} \setminus \{0\} = U_1 \to \mathbb{C}$ *defined by* $z_1(z) = 1/z$ *if* $z \neq 0$ *and* $z_1(\infty) = 0$
2. $z_2 : \mathbb{C} = U_2 \to \mathbb{C}$ *defined by* $z_1(z) = z$

In the intersection $U_1 \cap U_2 = \mathbb{C} \setminus \{0\} = z_1(\mathbb{C} \setminus \{0\}) = z_2(\mathbb{C} \setminus \{0\})$ *we obtain*

$$z_2 \circ z_1^{-1} : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$$

given by $z_2 \circ z_1^{-1}(z) = 1/z$ *which is a biholomorphism.*

This is the most symmetric example. It has the largest group of automorphisms (the group of diffeomorphisms which are holomorphic) namely, the group of Möbius transformations. It contains the complex plane whose automorphism group (the similarity group) is a subgroup of the Möbius group.

Holomorphic (meromorphic) functions on a Riemann surface are those continuous functions which viewed through the charts are holomorphic (meromorphic). We denote the field of meromorphic functions defined on a Riemann surface X by $\mathcal{M}(X)$. Meromorphic functions can be seen as holomorphic functions with values in $\mathbb{C}P^1$.

We will show later that meromorphic functions on $\mathbb{C}P^1$ are very simple to describe. They are all rational:

Definition 1.4 *A rational function $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ is a function of the form*

$$f(z) = \frac{p(z)}{q(z)}$$

where $p(z)$ and $q(z)$ are polynomials.

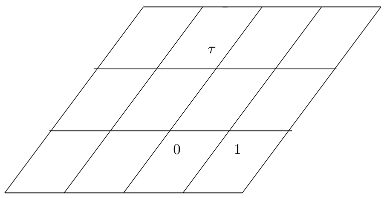
Writing a meromorphic function in the neighborhood of a point a as $f(z) = (z-a)^k g(z)$ where $g(a) \neq 0$, we define the order of $f(z)$ at a to be k. The order of the function defined on $\mathbb{C}P^1$ at ∞ is computed using the chart $w = 1/z$. So that if $p(z)$ has degree n and $q(z)$ degree m, then ∞ will have order $-(n-m)$. Therefore, for a rational function, the sum of the orders of the zeros and poles is zero. Conversely, an easy construction gives

Proposition 1.5 *Let a_i and b_j be two finite disjoint families of points in $\mathbb{C}P^1$ with the same number of elements. Then there exists a rational function vanishing precisely at a_i and having poles precisely at b_j. The order of the function at each point being the number of times the point appears in the families.*

This is not true for other Riemann surfaces. First of all, it is much more difficult to prove that there exists a meromorphic function and secondly one cannot fix arbitrarily the structure of zeros and poles of a meromorphic function.

The next examples of compact Riemann surfaces, after $\mathbb{C}P^1$, consists of complex structures on a torus. We will show later that any such structure arises as a quotient of $\mathbb{C}$ by a translation group generated by two independent directions one of each we may suppose (by a conjugation by a similarity transformation $z \to az + b$) to be $z \to z + 1$ and the other one $z \to z + \tau$ with $\tau \in \mathbb{C}$. More precisely, we will show that any compact Riemann surface whose underlying manifold is a torus is biholomorphic to an elliptic curve:

Definition 1.6 *Let $\tau \in \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$ and $\Gamma_\tau = \mathbb{Z} + \mathbb{Z}\tau$ be the additive group generated by $1, \tau \in \mathbb{C}$. We say that $E_\tau = \mathbb{C}/\Gamma_\tau$ is the complex torus associated to Γ_τ.*



The set of points inside the parallelogram defined by 1 and τ is called a fundamental region. Its closure, with some identifications on the boundary, is homeomorphic to a torus. Observe that any translation of that parallelogram also is a fundamental domain in the sense that any two points in its interior are contained in different orbits and each orbit has a point in the domain or its closure.

Meromorphic functions $\mathcal{M}(E_\tau)$ defined on E_τ are identified with meromorphic functions defined on $\mathbb{C}$ which are invariant under Γ_τ (called elliptic functions) but holomorphic functions which are invariant reduce to constants due to the maximum principle. It is not obvious that a non-constant function exists but several of its properties, assuming existence, are simple to state. The following is a basic property.

Proposition 1.7 *Let $f \in \mathcal{M}(E_\tau)$ be a meromorphic function without poles on the boundary of a fundamental region. Then, the sum of its residues in the fundamental region is zero.*

Proof. The sum of residues in the interior is given by $\frac{1}{2\pi i}\int_{\partial P} f(z)dz$ where P is a parallelogram which is a fundamental domain. By translation invariance the integrals on opposite sides cancel. □

This shows that, in order to construct a meromorphic function on E_τ with only one pole, its order has to be at least two. A related proposition counts the number of zeros.

Proposition 1.8 *Suppose there are no poles or zeros in the boundary of a fundamental domain. Then the number of zeros is the same as the number of poles counting multiplicities.*

Proof. The proof is simply a corollary to the previous proposition applied to the the function f'/f. In fact the sum of the residues is equal to the number of zeros minus the number of poles counting multiplicity by the following

Exercise 1.9 *If f has no poles nor zeros in ∂P, prove that*

$$\frac{1}{2\pi i}\int_{\partial P}\frac{f'(z)}{f(z)}dz = \textit{number of zeros in } P - \textit{number of poles in } P$$

where P is a domain with boundary ∂P.

□

Another necessary condition on the zeros and poles of a meromorphic function is given in the following proposition. It turns out that these necessary conditions are also sufficient (Abel's theorem).

Proposition 1.10 *Suppose there are no poles or zeros in the boundary of a fundamental domain P. Let a_i and b_j be two finite disjoint families of points inside P and f an elliptic function vanishing precisely at a_i and having poles precisely at b_j (we repeat the points according to the multiplicity of the zero or pole). Then*

$$\sum a_i - \sum b_j \in \Gamma_\tau.$$

Proof. The same as above using the following

Exercise 1.11

$$\frac{1}{2\pi i}\int_{\partial P}\frac{zf'(z)}{f(z)}dz = \sum a_i - \sum b_i.$$

Indeed, taking into account the invariance of $f(z)$ under translations and supposing that P is the parallelogram with corners $0, 1, 1+\tau, \tau$, we get

$$\frac{1}{2\pi i}\int_{\partial P}\frac{zf'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_0^1 \frac{(-\tau)f'(z)}{f(z)}dz + \frac{1}{2\pi i}\int_0^\tau \frac{f'(z)}{f(z)}dz$$

$$\frac{1}{2\pi i}\left(-\tau\log(f(z))|_0^1 - \log(f(z))|_0^\tau\right) = n_1\tau + n_2.$$

□

The next goal is to construct at least one meromorphic function on E_τ. In the following discussion we fix a translation τ and let Γ_τ be the lattice generated by 1 and τ. Several objects will depend on τ although we will not make it explicit. A direct construction of elliptic functions is obtained by means of the series

$$F_n(z) = \sum_{\gamma \in \Gamma_\tau} \frac{1}{(z-\gamma)^n}.$$

One can prove that the series converges absolutely and uniformly on compact sets for $n \geq 3$ so that $F_n(z)$ is meromorphic. To see that, we start with the following

Lemma 1.12 *The series*

$$\sum_{\gamma \in \Gamma_\tau - \{0\}} \frac{1}{|\gamma|^s}$$

is convergent for $s > 2$.

Proof. Consider the description of the lattice by the layers $n_1 + n_2\tau \in \Gamma_\tau$ with $max(|n_1|, |n_2|) = n$. There are $8n$ elements of Γ_τ in that layer. If we let r be the radius of an inscribed circle inside the first layer (that is the parallelogram defined by $\pm(1+\tau), \pm(\tau - 1)$), then $|n_1 + n_2\tau| \geq r\, max(|n_1|, |n_2|)$. Therefore

$$\sum_{\gamma \in \Gamma_\tau - \{0\}} \frac{1}{|\gamma|^s} \leq \sum_{n \geq 1} \frac{8n}{r^s n^s} = \sum_{n \geq 1} \frac{8}{r^s n^{s-1}}$$

which is convergent for $s > 2$. □

Lemma 1.13 *The series*

$$\sum_{\gamma \in \Gamma_\tau} \frac{1}{(z-\gamma)^s}$$

is uniformly convergent on compact sets of $\mathbb{C} - \Gamma_\tau$ for any integer $s > 2$.

Proof. If $K \subset \mathbb{C}$ is a compact subset we can assume that, except for finitely many γ, $|\gamma| \geq 2|z|$ for $z \in K$. In that case $|z - \gamma| \geq |\gamma| - |z| \geq |\gamma| - \frac{|\gamma|}{2} = \frac{|\gamma|}{2}$. Therefore for all $z \in K$ and γ on the complement of a finite subset in Γ_τ,

$$\sum_{\gamma} \frac{1}{|z-\gamma|^s} \leq \sum_{\gamma} \frac{2^s}{|\gamma|^s}$$

which is convergent for $s > 2$. Together with the previous lemma, this implies the series is uniformly convergent by Weierstrass M-test. □

Having proved convergence, for each $\omega \in \Gamma_\tau$ we obtain

$$F_n(z+\omega) = \sum_{\gamma \in \Gamma_\tau} \frac{1}{(z+\omega-\gamma)^n} = \sum_{\gamma \in \Gamma_\tau} \frac{1}{(z-\gamma)^n}$$

so that $F_n(z)$ is elliptic. In particular the function $F_3(z)$ is elliptic. It has a pole of order 3 at 0. To obtain a meromorphic function with a pole of order 2 we solve the equation

$$\mathcal{P}'(z) = -2F_3(z).$$

A solution is given by the Weierstrass function

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{\gamma \in \Gamma_\tau - \{0\}} \left(\frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2} \right).$$

The naive idea would be to start with a function with a pole of order two, namely $\frac{1}{z^2}$, and add the term $\sum_{\gamma \in \Gamma_\tau - \{0\}} \frac{1}{(z-\gamma)^2}$ which would make it invariant under Γ_τ but unfortunately this sum is not convergent.

Lemma 1.14 *The series*

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{\gamma \in \Gamma_\tau - \{0\}} \left(\frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2} \right).$$

defines an elliptic function with only one pole of order two modulo the lattice.

Proof. To show convergence, the argument is the same as in the previous lemma. The general term of the series $\mathcal{P}$ satisfies, as in the previous lemma, for $|\gamma| \geq 2|z|$

$$\left| \frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2} \right| = \left| \frac{z(z-2\gamma)}{\gamma^2 (z-\gamma)^2} \right| \leq \frac{4|z|(5/2|\gamma|)}{|\gamma|^2 |\gamma|^2} \leq \frac{10|z|}{|\gamma|^3}.$$

Therefore, as before, we conclude that the series converges absolutely and uniformly on compact sets.

The periodicity is not clear from the formula. But we can use the periodicity of its derivative to conclude that $\mathcal{P}(z) - \mathcal{P}(z+1)$ and $\mathcal{P}(z) - \mathcal{P}(z+\tau)$ are constants. The value of the constants are seen to be zero. In fact, $\mathcal{P}(-1/2) - \mathcal{P}(-1/2+1) = 0$ and $\mathcal{P}(-\tau/2) - \mathcal{P}(-\tau/2+\tau) = 0$ because $\mathcal{P}(z)$ is clearly even. □

The following existence theorem of meromorphic functions on an elliptic curve should be contrasted to the corresponding existence theorem of rational functions on the Riemann sphere. The only if part was proven in a previous proposition.

Theorem 1.15 *(Abel's theorem) Let E_τ be a complex torus with corresponding group Γ_τ. Let a_i and b_j be two finite disjoint families of points in a fundamental domain P with the same number of elements (greater or equal than 2). Then there exists an elliptic function vanishing (inside P) precisely at a_i and having poles (inside P) precisely at b_j if and only if*

$$\sum a_i - \sum b_j \in \Gamma_\tau.$$

Proof. (sketch) A constructive proof of this theorem can be given by considering the Weierstrass sigma functions (they are examples of theta functions)

$$\sigma(z) = z \Pi_{\gamma \in \Gamma'} (1 - \frac{z}{\gamma}) e^{\frac{z}{\gamma} + \frac{z^2}{2\gamma^2}},$$

which have only simple zeros at points of Γ. They are not functions defined on the quotient but their behavior with respect to the lattice is quite simple. In fact

$$\sigma(z+\gamma) = (-1)^{n_\gamma} \sigma(z) e^{\alpha_\gamma (z + \frac{1}{2}\gamma)}$$

where α_γ and n_γ depend only on $\gamma \in \Gamma$. We define the meromorphic function in the theorem as

$$f(z) = \frac{\sigma(z-a_1)\cdots\sigma(z-a_n)}{\sigma(z-b_1)\cdots\sigma(z-b_n)}.$$

It is easy to verify that $f(z)$ is indeed defined on the quotient. □

Another description of the set of meromorphic functions is given through its subsets of meromorphic functions with less singularities than a fixed arbitrary data. More precisely we fix a divisor, that is, a formal linear combination

$$D = \sum_z n_z[z]$$

where $n_z \in \mathbb{Z}$ are different from zero only for a finite number of $z \in E$. We think of a divisor as giving the order n_z of a possible function at z, except that a function with precisely these orders might not exist. The degree of a divisor will be the total order $deg(D) = \sum_z n_z$. In particular we call divisor of f the divisor (called a principal divisor)

$$div(f) = \sum_{z\in E} ord_z(f)[z],$$

and we will show that it has zero degree for any meromorphic function defined on any compact Riemann surface. Define the vector space

$$L(D) = \{f \in \mathcal{M}(E) \mid f = 0 \ \text{ or } div(f) \geq -D\ \}$$

where $div(f) \geq -D$ means that, for each $z \in E$, the order of f at z is greater than or equal to $-n_z$. We will prove later the following:

Theorem 1.16 *(Riemann-Roch) Let D be a divisor on E with $d = deg(D) \geq 1$. Then $dim(L(D)) = d$.*

The field of meromorphic functions is described in the following

Theorem 1.17 *$\mathcal{M}(E_\tau) = \mathbb{C}(P, P')$, that is, the field of meromorphic functions is generated by $\mathbb{C}$, the Weierstrass function and its derivative.*

A meromorphic function on the elliptic curve can be interpreted as a function $E_\tau \to \mathbb{C}P^1$. In general, the meromorphic function is locally a bijection but it has ramification points when its derivatives vanishes. It is important then to determine the zeros of $\mathcal{P}'$:

Lemma 1.18 *The zeros of $\mathcal{P}'$ in a fundamental parallellogram containing the points 0, $\frac{1}{2}$, $\frac{\tau}{2}$ and $\frac{1+\tau}{2}$ are*

$$\frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$$

Proof. As $\mathcal{P}'$ has order 3, it has only three zeros in the fundamental domain. We have $\mathcal{P}'(z) = -\mathcal{P}'(-z)$ because $\mathcal{P}'$ is odd. On the other hand, because $\mathcal{P}'$ is periodic, $\mathcal{P}'(z) = \mathcal{P}'(z-\gamma)$. Therefore, for $z = \gamma/2$, $\mathcal{P}'$ vanishes. □

One can prove that the Weierstrass function defined on E_τ assumes each value on the Riemann sphere exactly twice except for 4 points; three corresponding to the vanishing of its derivative $\mathcal{P}(\frac{1}{2})$, $\mathcal{P}(\frac{\tau}{2})$, $\mathcal{P}(\frac{1+\tau}{2})$ and the last one corresponding to the unique pole of order 2, ∞. That gives an interpretation of the Weierstrass function as a branched covering of the Riemann sphere by the torus.

Another way to obtain this interpretation is via the study of a differential equation satisfied by $\mathcal{P}(z)$ which, in fact, establishes an algebraic relation between $\mathcal{P}(z)$ and $\mathcal{P}'(z)$.

Proposition 1.19 *The Weierstrass function satisfies the equation*

$$\mathcal{P}'(z)^2 = 4\mathcal{P}^3(z) - g_2(\tau)\mathcal{P}(z) - g_3(\tau)$$

where

$$g_2(\tau) = 60 \sum_{\gamma\in\Gamma_\tau-\{0\}} \frac{1}{\gamma^4}$$

and

$$g_3(\tau) = 140 \sum_{\gamma\in\Gamma_\tau-\{0\}} \frac{1}{\gamma^6}.$$

Proof. A simple proof can be given by computing the Laurent series at the origin of $\mathcal{P}(z)$ and $\mathcal{P}'(z)$. One must show that the two sides of the equality have equal Laurent series up to the constant term. In that case their difference would be a a bounded holomorphic function vanishing at the origin and therefore, by Liouville, vanishing everywhere.

In order to obtain the Laurent series of $\mathcal{P}(z)$ it is useful to consider the series bellow satisfying $\zeta'(z) = -\mathcal{P}(z)$.

$$\zeta(z) = \frac{1}{z} + \sum_{\gamma\in\Gamma_\tau-\{0\}} \left(\frac{1}{(z-\gamma)} + \frac{1}{\gamma} + \frac{z}{\gamma^2}\right)$$

Exercise 1.20 *The series is uniformly convergent and its Laurent series is*

$$\zeta(z) = \frac{1}{z} - G_4 z^3 - G_6 z^5 + \cdots$$

where

$$G_n = \sum_{\gamma\in\Gamma_\tau-\{0\}} \frac{1}{\gamma^n}.$$

We obtain the following developments

$$\mathcal{P}(z) = -\zeta'(z) = \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + \cdots$$

$$4\mathcal{P}(z)^3 = \frac{4}{z^6} - \frac{36G_4}{z^2} - 60G_6 + \cdots$$

$$\mathcal{P}'(z) = -\frac{2}{z^3} + 6G_4 z + 20G_6 z^3 + \cdots$$

$$\mathcal{P}'(z)^2 = \frac{4}{z^6} - \frac{24G_4}{z^2} - 80G_6 + \cdots$$

and then a simple computation shows that the Laurent series of each side of the equation is equal up to zero order. $\square$

Writing $t = \mathcal{P}(z)$ and the differential equation as $\left(\frac{dt}{dz}\right)^2 = 4t^3 - g_2t - g_3$ we see that the inverse function of $\mathcal{P}(z)$, $\mathcal{P}^{-1}(t)$, is formally given by

$$\int \frac{1}{\sqrt{4t^3 - g_2t - g_3}} dt.$$

But those integrals are not well defined in general. The problem is that the function $\sqrt{4t^3 - g_2t - g_3}$ is not well defined in $\mathbb{C}$. For each path of integration (which does not meet the roots) one can define the integral by analytically extending the function along the path, but different paths will give, sometimes, different integrals.

In fact, the study of integrals of the form

$$\int \frac{1}{\sqrt{p(t)}} dt$$

were the motivation for the whole theory. In particular one can think of the elliptic functions as generalizations of the circular functions. For instance

$$\int \frac{1}{\sqrt{1-t^2}} dt$$

is $\arcsin(t)$ and the inverse function of that integral is a periodic function. The elliptic functions are inverse functions of the integrals as above with $p(t)$ of degree three and they have the remarkable property of being doubly periodic.

The map $E_\tau - [0] \to \mathbb{C}^2$ given by $z \to (\mathcal{P}(z), \mathcal{P}'(z))$ defined on the complement of the pole ([0] is the projection of the lattice on the quotient space) is a holomorphic embedding whose image is the curve

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau).$$

But one can extend that embedding to complex projective space.

Theorem 1.21 *The map $z \to (\mathcal{P}(z), \mathcal{P}'(z), 1)$ for $z \in \mathbb{C} - \Gamma_\tau$ and $z \to (0, 1, 0)$ for $z \in \Gamma_\tau$ defines a holomorphic embedding $E_\tau \to \mathbb{C}P^2$ whose image is the algebraic curve*

$$y^2z = 4x^3 - g_2(\tau)xz^2 - g_3(\tau)z^3.$$

Several results about elliptic curves are generalized for any compact Riemann surface. In particular, we will

1. Describe any Riemann surface as a quotient of $\mathbb{C}$, D, the unit disc, or the Riemann sphere by a discrete group Γ.

2. Prove that there exist meromorphic functions on any compact surface and, more generally, give a generalization of Abel's theorem, Riemann-Roch theorem and describe the structure of its field of meromorphic functions.

3. Prove that there exists an embedding of a compact Riemann surface as a submanifold of a complex projective space.

1.1 Apendix: Projective space

Complex projective space $\mathbb{C}P^n$ is the quotient of $\mathbb{C}^{n+1}-0$ by the $\mathbb{C}^*$-action $\lambda(z_1,\cdots,z_{n+1}) = (\lambda z_1,\cdots,\lambda z_{n+1})$. The orbit containing the point $(z_1,\cdots,z_{n+1})$ is denoted $[z_1,\cdots,z_{n+1}]$ (the homogeneous coordinates).

Natural charts are given by defining the open sets $U_i = \{\ [z_1,\cdots,z_{n+1}] \mid z_i \neq 0\ \}$ and $\varphi_i : U_i \to \mathbb{C}^n$ as

$$\varphi_i([z_1,\cdots,z_{n+1}]) = (\frac{z_1}{z_i},\cdots,1,\cdots,\frac{z_{n+1}}{z_i})$$

where the coordinate 1 corresponding to z_i/z_i should be deleted in the identification with $\mathbb{C}^n$. The transition functions are given by

$$\varphi_j \circ \varphi_i^{-1}(w_1,\cdots,w_{n+1}) = (\frac{w_1}{w_j},\cdots,\frac{w_{n+1}}{w_j})$$

where we think $(w_1,\cdots,w_{n+1})$ as having the i-coordinate equal to 1 and $(\frac{w_1}{w_j},\cdots,\frac{w_{n+1}}{w_j})$ having the j-coordinate equal to 1.

We denote $\Pi : \mathbb{C}^{n+1}\setminus\{0\} \to \mathbb{C}P^n$ the projection. $\mathbb{C}P^n$ is a compact manifold as the projection Π is continuous and its restriction to the sphere $S^{2n-1} \subset \mathbb{C}^{n+1}$ is surjective.

The group $GL(n+1,\mathbb{C})$ of invertible $(n+1)\times(n+1)$ matrices acts on $\mathbb{C}P^n$: just use the action on C^{n+1} and observe that it passes to the quotient. The subgroup $\mathbb{C}^* \subset GL(n+1,\mathbb{C})$ of multiples of the identity acts trivialy on the quotient. In fact one can prove the following.

Proposition 1.22 *The group of biholomorphism of* $\mathbb{C}P^n$ *is*

$$PGL(n+1,\mathbb{C}) = GL(n+1,\mathbb{C})/\mathbb{C}^*.$$

Important submanifolds of complex projective space are $k-planes$. These are the image under Π of linear subspaces (with the origin deleted) of dimension $k+1$ in $\mathbb{C}^{n+1}$.(restricted to the fundamental domain) They are determined by $k+1$ points such that their lifts to $\mathbb{C}^{n+1}$ are linearly independent. 1-planes are called complex lines and $n-1$-planes are called hyperplanes. In fact, given n points in $\mathbb{C}P^n$ not contained in a hyperplane, we can find a projective transformation such that, after composition, the points are given by

$$[1,0,\cdots,0],\cdots[0,\cdots,0,1].$$

That follows by choosing a basis of $\mathbb{C}^{n+1}$ using lifts of those points and defining the projective transformation by sending a standard basis to that basis.

Given a point $p \in \mathbb{C}P^n$ and a hyperplane L not containing it we may define the projection from p to L, $\pi : \mathbb{C}P^n\setminus\{p\} \to L$; choose lifts $\tilde{L}$ and $\tilde{p}$ and, given $z \in \mathbb{C}P^n\setminus\{p\}$, $\tilde{z}$ of the hyperplane and the point in $\mathbb{C}^{n+1}$, then $\pi(z) = \Pi(span(\tilde{z},\tilde{p})\cap\tilde{L})$. Here $span(\tilde{z},\tilde{p})$ is the vector space generated by $\tilde{z},\tilde{p}$ and the intersection is not empty as $dim(\tilde{L}) = n$ and $dim(\tilde{z},\tilde{p}) = 2$. More generally, we can define projections from a k-plane M to a $n-k-1$-plane L by the analogous formula $\pi(z) = \Pi(span(\tilde{z},\tilde{M})\cap\tilde{L})$.

2 Topology of surfaces

A two dimensional topological manifold is called a surface. That is a connected Hausdorff topological space M having a cover by open sets U_α and a collection of homeomorphisms $\varphi_\alpha : U_\alpha \to \mathbb{R}^2$ which are compatible in the sense that the transition functions

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

are homeomorphisms.

It is convenient to have a combinatorial description of surfaces by means of a triangulation. This allows a direct computation of some topological invariants of the surface as the Euler characteristic and the fundamental group.

To be more precise define first the standard 2-simplex Δ given by the convex envelope of the points (vertices) $(0,0),(1,0),(0,1)$ in $\mathbb{R}^2$. Each boundary segment is called an edge. If $\varphi : \Delta \to \varphi(\Delta) \subset M$ is a homeomorphism, we call $\varphi(\Delta)$ a triangle and the images of the vertices and edges of the standard simplex are also called vertices and edges of the triangle.

Definition 2.1 *A triangulation of a compact surface M is a finite set of homeomorphisms $\varphi_i : \Delta \to \varphi_i(\Delta) \subset M$ such that the intersection of two triangles is either*

- *empty,*
- *a vertex or*
- *an edge of each of the triangles.*

In particular the interior of the triangles are disjoint. We can now state the theorem whose first rigorous proof was given by Radó in 1924.

Theorem 2.2 *Any compact surface has a triangulation.*

Remark 2.3

1. *In fact, Radó proved that any surface which has a countable basis of open sets can be triangulated. For non-compact surfaces, as the number of triangles is not finite, we need to impose that each point has a neighborhood intersecting only a finite number of triangles.*
2. *The existence of a triangulation for a compact manifold dimension 3 was established by Moise in 1952, but in dimensions higher than three a topological manifold might not have a triangulation.*
3. *One can define orientability for triangulated surfaces by saying that there exists a compatible orientation on all triangles (they induce opposite orientations on common edges).*
4. *Any triangulation of a compact surface may be obtained from another one by a finite sequence of the following elementary moves:*
   - *the creation of a vertex inside a triangle and thereby introducing three new triangles in the place of the original one and the corresponding inverse operation,*
   - *replacing the common side of two adjacent triangles of the triangulation by the other diagonal of the quadrilateral formed by these two triangles (this is called a flip).*

A reference for the classification of compact surfaces is the first chapter of [Massey] and we will only summarize the main results bellow without proofs. Riemann surfaces are orientable surfaces because the transition functions, being holomorphic, preserve orientation so we state the theorem of classification only for orientable surfaces. A basic surgery construction is that of connected sum. We start with two surfaces and remove one disc from each and glue the two surfaces along the boundary of the discs. In fact we can obtain any surface, apart the sphere, by this surgery procedure applied to tori.

Theorem 2.4 *A compact orientable surface is homeomorphic to a sphere or to a connected sum of tori.*

Proof. Sketch: Once we know the surface is triangulated, one can prove the theorem of classification of compact surfaces by spreading the triangulation of the surface in the plane to form a polygon with boundary identifications. More precisely, given a triangulated surface we enumerate its triangles $T_1, T_2, \cdots, T_n$ in a way that each T_i has an edge in common with one of the previous triangles in the sequence. If T_i has two edges in common, we choose one of them to identify to one of the edges on the plane but leave the other one as a boundary of the polygon thus obtained. The union of the first two triangles along the common edge gives a parallelogram with possible boundary identifications. Adding each triangle makes the number of sides of this polygon jump by two. At the end we obtain a polygon with a number of sides identifications.

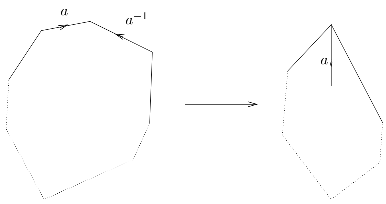
The idea now is to find a normal form for this polygon describing the surface. A usual normal form is the one which describes the surface as a connected sum of tori. A torus corresponds to a sequence $aba^{-1}b^{-1}$ and a handle to sequence $aba^{-1}b^{-1}c$. Clearly $a_1b_1{a_1}^{-1}{b_1}^{-1}a_2b_2{a_2}^{-1}{b_2}^{-1}$ corresponds to a connected sum of two tori. In other words, adding a handle to a torus. The normal form we look for a surface with g handles is therefore

$$a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}$$

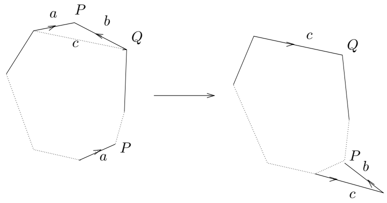
as in Figure 1) with $g \geq 1$ or aa^{-1} which is a sphere.

This is done using a sequence of operations which simplify the structure of the identifications on the boundary. As a first observation, note that a sequence of edges of the form $a \cdots a \cdots$ cannot occur because this implies the existence of a Möbius band inside the surface which would not be orientable. We divide the proof in several steps which are better explained by the figures:

1. Elimination of adjacent edges of the form aa^{-1} (in the case there are more than two sides).



2. Obtain a polygon whose vertices are all identified. In the figure we show how to get rid of a vertex P at the expense of introducing more vertices identified to Q. Each time we perform this operation we return to step one to simplify the configaration. If there is only one point P in the boundary, then we obtain a configuration as in step one so either we obtain a sphere or we can simplify, therefore suppressing the point P.



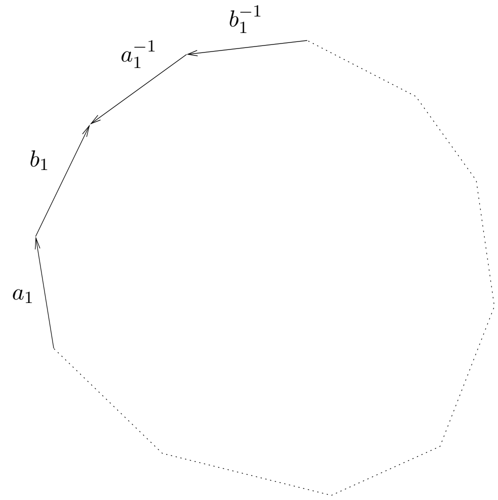


Figure 1: A surface obtained by boundary identifications on a disc.

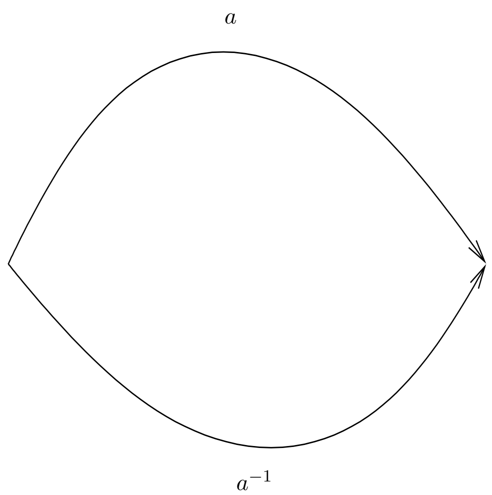


Figure 2: A sphere obtained by boundary identifications on a disc.

3. Change to a sequence of handles: exercise.

□

The Euler characteristic can be defined by the formula $\chi = T - E + V$, where T is the number of triangles, E is the number of edges and V is the number of vertices of a triangulation. It is given by

$$\chi = 2 - 2g.$$

3 Covering spaces and the fundamental group

A curve in a topological space X is a continuous map $c : [0,1] \to X$. Two curves c_1 and c_2 with $c_1(0) = c_2(0)$ and $c_1(1) = c_2(1)$ are homotopic (with fixed end points) if there exists a continuous map $F : [0,1] \times [0,1] \to X$ such that

1. $F_{|\{0\}\times[0,1]} = c_1(0)$ and $F_{|\{1\}\times[0,1]} = c_1(1)$
2. $F_{|[0,1]\times\{0\}} = c_1$ and $F_{|[0,1]\times\{1\}} = c_2$.

A loop in X is a curve c with $c(0) = c(1)$. We can define the product of two loops c_1 and c_2 such that $c_1(0) = c_2(0) = x_0$ (we say the loops are based at x_0) as the loop $c_2c_1 : [0,1] \to X$ given by $c_2c_1(t) = c_1(2t)$ for $0 \leq t \leq 1/2$ and $c_2c_1(t) = c_2(2(t - 1/2))$ for $1/2 \leq t \leq 1$. The constant loop is defined to be $c(t) = x_0$ for all t, and the inverse of a loop c is the loop c^{-1} defined by $c^{-1}(t) = c(1 - t)$. Two loops c_1 and c_2 based at x_0 are homotopic if there exists a continuous map $F : [0,1] \times [0,1] \to X$ such that

1. $F_{|\{0\}\times[0,1]} = F_{|\{1\}\times[0,1]} = x_0$
2. $F_{|[0,1]\times\{0\}} = c_1$ and $F_{|[0,1]\times\{1\}} = c_2$

We say that two loops are freely homotopic if the second condition is verified but not necessarily the first one.

Let X be a manifold and $x_0 \in X$ a base point. We denote by $\pi_1(X, x_0)$, *the fundamental group*, the space of homotopy classes of loops based at x_0. It has a group structure induced by the multiplication on loops. Usually we denote by $[\gamma]$ the class containing the loop γ.

If x_0' is another base point, $\pi_1(X, x_0')$ is isomorphic to $\pi_1(X, x_0)$. In fact, let c be a curve with $c(0) = x_0$ and $c(1) = x_0'$. Then, one can define an isomorphism of groups $\pi_1(X, x_0) \to \pi_1(X, x_0')$ by $\gamma \to c\gamma c^{-1}$.

Example 3.1 *The fundamental group of S^1 is isomorphic to $\mathbb{Z}$.*

A continuous function $f : X \to Y$ between topological spaces such that $f(x_0) = y_0$ induces a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$. A homeomorphism induces an isomorphism but an isomorphism between fundamental groups does not imply that the corresponding topological spaces are homeomorphic. A typical situation of isomorphic fundamental groups arises in the case of deformation retracts. They are very useful for computations.

Definition 3.2 *A subset $K \subset X$ of a topological space is a deformation retract of X if there exists a homotopy $F : X \times [0,1] \to X$ such that*

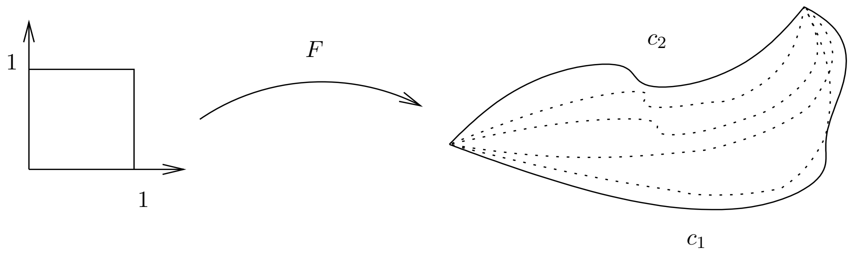
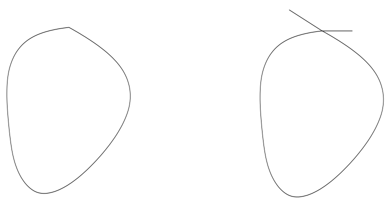


Figure 3: A homotopy between two curves c_1 and c_2.

- *For all* $x \in X$, $F(x, 0) = x$.
- *For all* $x \in K$, $F(x, .) = x$.
- $F(., 1)(X) \subset K$.



3.1 Group presentations and computations of the fundamental group.

A presentation of a group Γ is given by

$$\Gamma = \langle \gamma_1, \cdots \ \ | r_1, \cdots \rangle.$$

The γ_i are the generators and the r_i reduced words on the generators (words constructed with γ_i or γ_i^{-1} which don't contain the sequence $\gamma_i \gamma_i^{-1}$). By definition, Γ is the quotient of the free group on the generators γ_i by the normal subgroup generated by the relators. We say that Γ is finitely presented if there exists a presentation with a finitely number of generators and relators.

Example 3.3

$$\mathbb{Z} \oplus \mathbb{Z} = \langle \ \gamma_1, \gamma_2 \ \ | \ [\gamma_1, \gamma_2] \ \rangle.$$

To give the fundamental group by a presentation is very useful for computations. An application of that description is the following theorem which we quote without proof.

Theorem 3.4 (Seifert-Van Kampen Theorem) *Let $M = M_1 \cup M_2$ be the union of two path-connected open sets with $I = M_1 \cap M_2$ path-connected. Suppose the fundamental groups of M_1 and M_2 at a base point $x_0 \in I$ are $\Gamma_1 = \langle \gamma_1, \cdots \ | r_1, \cdots \rangle$. and $\Gamma_2 = \langle \delta_1, \cdots \ | s_1, \cdots \rangle$. Suppose $\pi_1(I, x_0)$ is generated by the elements η_i. Write each η_i as φ_{i1} and φ_{i2} using the generators of Γ_1 and Γ_2 respectively. Then*

$$\pi_1(M, x_0) = \langle \gamma_1, \cdots \delta_1 \cdots \ | r_1, \cdots, s_1, \cdots, \varphi_{i1}\varphi_{i2}^{-1} \rangle.$$

As a first application of the theorem we compute

Exercise 3.5 *The fundamental group of the infinity symbol ∞ is the free group with two generators. More generally, the fundamental group of a bouquet of g circles is the free group with g generators.*

We use the theorem of Seifert-Van Kampen to provide presentations for surface groups.

Exercise 3.6 *The fundamental group of a compact Riemann surface of genus g with a point deleted is the free group with $2g$ generators.*

We say a surface is of finite type if it is homeomorphic to a compact surface with a finite number of points (or disjoint discs) deleted.

Theorem 3.7 *The fundamental group of an orientable surface of finite type has a presentation of the form*

$$\left\langle \ a_1, b_1, \cdots, a_g, b_g, h_1, \cdots h_t \mid \Pi_{j=1}^{g}[a_j, b_j]h_1 \cdots h_t = 1 \ \right\rangle.$$

The elements h_i correspond to loops around the boundaries. In particular, from the presentation, we see that if $t \neq 0$ the fundamental group is free of rank $2g + t - 1$.

3.2 Covering spaces

In the following we suppose that the topological spaces are all arc connected and locally arc connected. We denote by $\varphi : (Y, y_0) \to (X, x_0)$ a continuous map $\varphi : Y \to X$ such that $\varphi(y_0) = x_0$. Recall that it induces the homomorphism $\varphi_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ defined by $[\gamma] \to [\varphi \circ \gamma]$.

Definition 3.8 *A map $p : \tilde{X} \to X$ between topological spaces is a covering if each point $x \in X$ has a neighborhood U_x such that $p^{-1}(U_x)$ is a disjoint union of open sets homeomorphic to U_x under p.*

We say that two coverings $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are equivalent if there exists a homeomorphism $p : \tilde{X}_1 \to \tilde{X}_2$ such that $p_2 \circ p = p_1$. Coverings have the fundamental path lifting property:

Proposition 3.9 *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space. A path $\varphi : ([0, 1], 0) \to (X, x_0)$ can be lifted to a unique path $\tilde{\varphi} : ([0, 1], 0) \to (\tilde{X}, \tilde{x}_0)$ satisfying $p \circ \tilde{\varphi} = \varphi$.*

Proof. Let $L = \{ t \in [0,1] \mid \varphi_{|[0,t]}$ can be lifted $\}$. We will show that this set is open and closed. It is clearly non-empty as $0 \in L$. If $t_0 \in L$ then $\tilde{\varphi}(t_0)$ is contained in a unique component U of $p^{-1}(V)$ homeomorphic to V, a sufficiently small neighborhood of $\varphi(t_0)$. There exists therefore a lift of the curve in a neighborhood of t_0 by taking $(p_{|U})^{-1} \circ \varphi$. Similarly if t_0 is a limit of points t_n in L we observe that there exists a sufficiently small neighborhood of $\varphi(t_0)$ such that $\tilde{\varphi}(t_n)$ are contained in a component U of $p^{-1}(V)$. As U is a homeomorphism we can define $\tilde{\varphi}(t_0)$. Uniqueness follows by a similar argument. □

Using a similar proof we may lift homotopies on X to homotopies on a covering $\tilde{X}$:

Proposition 3.10 *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space. A homotopy $F : [0,1] \times [0,1] \to X$ between two paths $\varphi_1 : ([0,1], 0) \to (X, x_0)$ and $\varphi_2 : ([0,1], 0) \to (X, x_0)$ has a lift to a unique homotopy $\tilde{F} : [0,1] \times [0,1] \to \tilde{X}$ between $\tilde{\varphi}_1 : ([0,1], 0) \to (\tilde{X}, \tilde{x}_0)$ and $\tilde{\varphi}_2 : ([0,1], 0) \to (\tilde{X}, \tilde{x}_0)$. In particular, $\tilde{\varphi}_1(1) = \tilde{\varphi}_2(1)$.*

Remark 3.11

1. *The proposition above shows that $p_* \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective.*
2. *If $\tilde{x}_0'$ is another base point for $\tilde{X}$ over x_0 then $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$ and $p_*(\pi_1(\widetilde{X}, \tilde{x}_0'))$ are conjugate.*

Definition 3.12 *The subgroup $p_* \pi_1(\tilde{X}, \tilde{x}_0) \subset \pi_1(X, x_0)$ is called the defining subgroup of the covering.*

Definition 3.13 *The universal covering of a topological space (arc connected and locally arc connected) is the covering having trivial defining group.*

The basic result about covering spaces is the following:

Theorem 3.14 *There exists a bijection between conjugacy classes of subgroups of $\pi_1(X, x_0)$ and equivalence classes of coverings.*

The construction of the covering space associated to a given subgroup $\Gamma \subset \pi_1(X, x_0)$ can be accomplished by considering the set of equivalence classes of paths $c : [0,1] \to X$ with $c(0) = x_0$. Equivalence between paths c_1 and c_2 meaning that $c_1(1) = c_2(1)$ and that $[{c_2}^{-1} c_1] \in \Gamma$. The map $p : \tilde{X} \to X$ is given by $p([c]) = c(1)$. For details see [Massey].

Remark 3.15 *If X is simply connected any covering is homeomorphic to X.*

The covering transformations (or deck transformations) of a covering $p : \widetilde{X} \to X$ are those homeomorphisms $\varphi : \widetilde{X} \to \widetilde{X}$ satisfying $\pi \circ \varphi = \pi$. The description of the covering group is given in the following theorem.

Theorem 3.16 *The group of covering transformations is isomorphic to*

$$N(p_* \pi_1(\widetilde{X}, \tilde{x}_0)) / p_* \pi_1(\widetilde{X}, \tilde{x}_0)$$

where N denotes the normalizer of the group in $\pi_1(X, x_0)$.

A covering whose defining subgroup is normal is called a *regular* or *normal* covering. In particular the universal covering is regular and $\pi_1(X, x_0)$ is the group of covering transformations.

3.3 Exercises

1. Recall that a map $\varphi : X \to Y$ is proper if for any compact $K \subset Y$, $\varphi^{-1}(K)$ is compact. Show that a local homeomorphism between manifolds is a finite covering if and only if φ is proper.

2. The punctured unit disc D^* has the upper half-plane as a universal covering. An explicit map is given by $e^{2\pi it}$. The fundamental group is $\mathbb{Z}$ acting on the half-plane by integer translations. The regular covering corresponding to the subgroup generated by $e^{2\pi im}$ also is the disc with covering group isomorphic to $\mathbb{Z}/m\mathbb{Z}$. The finite coverings of the punctured unit disc are equivalent to the maps $\varphi_m : D^* \to D^*$ given by $z \to z^m$.

3. The torus $S^1 \times S^1$ is covered by the plane. Find its regular coverings.

4. Let the annulus $A = \{ r < |w| < 1 \}$. The map $z \to \exp(2\pi i \log z / \log \lambda)$, where $r = \exp(-2\pi^2/\log \lambda)$ defines a covering $D \to A$ of A by the unit disc D. The covering group is generated by $z \to \lambda z$.

5. Give an example of a surjective map which is a local homeomorphism but which is not a covering.

6. Let X be a simply connected Riemann surface and $f : X \to \mathbb{C}^*$ a holomorphic function. Prove that there exists a function $\tilde{f} : X \to \mathbb{C}$ such that $exp \circ \tilde{f} = f$.

7. Let M_1 and M_2 be two manifolds which have the same universal covering $\tilde{M}$ with projections $p_1 : \tilde{M} \to M_1$ and $p_2 : \tilde{M} \to M_2$ and covering transformations group G_1 and G_2 respectively. If $\varphi : M_1 \to M_2$ is a homeomorphism, then we can lift it to a homeomorphism $\tilde{\varphi} : \tilde{M} \to \tilde{M}$. Prove that $G_2 = \tilde{\varphi} \circ G_1 \circ \tilde{\varphi}^{-1}$.

3.4 Group actions

Let G be a group and X a topological manifold.

Definition 3.17 *G acts by homeomorphisms on X if there exists a map $G \times X \to X$ such that*

1. *for fixed $g \in G$, the induced map $g : X \to X$ is a homeomorphism.*
2. *$(gh)x = g(hx)$ for all $x \in X$ and $g, h \in G$*
3. *$1x = x$ for all $x \in X$*

If $G \times X \to X$ is an action we call the set $G_x = \{g \in G \mid gx = x\}$ the *stabilizer* or *isotropy of the action at* x. The *orbit* of $x \in X$ is the set Gx. The action is said to be *transitive* if the orbit of every point coincides with the whole space. The set of all orbits is denoted X/G and we define a topology on it by imposing that $U \subset X/G$ is open if and only if $\pi^{-1}(U) \subset X$ is open, where $\pi : X \to X/G$ is the canonical projection. A very special action is related to covering spaces. We need the following definitions:

Definition 3.18 *Let $G \times X \to X$ be an action.*

1. *The action of G is free if no point of X is fixed by an element of G different from the identity (that is, the isotropy of each element of X is trivial).*

2. *The action is properly discontinuous if for any compact $K \subset X$ the set of all $\gamma \in G$ such that $\gamma K \cap K \neq \emptyset$ is finite.*

Proposition 3.19 *Let $G \times X \to X$ be an action on a manifold X. The quotient X/G is a manifold with projection $X \to X/G$ a covering if the action is free and properly discontinuous.*

Proof. Suppose $x \in X$ and U_x is a relatively compact neighborhood. As the action is properly discontinuous there exists only a finite number of elements in G such that $g\bar{U}_x \cap \bar{U}_x \neq \emptyset$. As the action is free, for each one of those elements, $gx \neq x$. As the space is Hausdorff, we can choose a neighborhood $V_x \subset U_x$ such that for all $g \in G$, $g\bar{V}_x \cap \bar{V}_x = \emptyset$. This proves that the projection $X \to X/G$ is a covering.

The quotient is Hausdorff: suppose $x, y \in X$ are two points in distinct orbits. As X is a manifold, there exists two relatively compact neighborhoods U_x and U_y with $\bar{U}_x \cap \bar{U}_y = \emptyset$. As before, because the action is properly discontinuous and free, we may suppose $g\bar{U}_x \cap \bar{U}_x = \emptyset$ and $g\bar{U}_y \cap \bar{U}_y = \emptyset$. Consider $K = \bar{U}_x \cup \bar{U}_y$. As the action is properly discontinuous, the set of elements $g \in G$ such that $gK \cap K = (g\bar{U}_x \cap \bar{U}_y) \cup (\bar{U}_x \cap g\bar{U}_y) \neq \emptyset$ is finite, and by the same argument as before (using the fact that the action is free), we can choose U_x and U_y smaller such that $gK \cap K = \emptyset$ for all g. □

In fact the fundamental group of a manifold X acts freely and properly discontinuously in the universal cover $\tilde{X}$ such that the quotient map $\tilde{X} \to \tilde{X}/\pi_1(X, x_0)$ is equivalent to the covering $\tilde{X} \to X$.

Exercise 3.20 *A discrete subgroup Γ of a topological group G acts freely properly discontinuously on G by the natural action $\Gamma \times G \to G$ given by $(\gamma, g) \to \gamma g$.*

Example 3.21 *A subgroup of $\mathbb{R}^n$ is discrete if and only if it is generated by a set of linearly independent vectors.*

Proof. Suppose that the group is generated by a set of linearly independent vectors. By a linear transformation we can transform the set into a subset of the canonical base vectors. It is clear that the group is discrete as 0 is an isolated point of the group.

Conversely, suppose that the subgroup $\Gamma \subset \mathbb{R}^n$ is discrete and use induction on the dimension. For $n = 1$, let v be the smallest positive vector. Without loss of generality, suppose $\gamma \in \Gamma$ is positive and let k be the largest integer such that $kv \leq \gamma$. Then $\gamma - kv \in \Gamma$ and is smaller then v. A contradiction unless $\gamma = kv$. We conclude that Γ is generated by v.

Suppose now that any discrete subgroup in $\mathbb{R}^{n-1}$ is generated by a set of linearly independent vectors. Let $\Gamma \subset \mathbb{R}^n$ be discrete and v a vector with minimum norm. Because of the first step of the induction $\Gamma \cap \mathbb{R}v = \mathbb{Z}v$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{R}v$ be the quotient map. We claim that $\pi(\Gamma)$ is discrete. Suppose v_i is a sequence in Γ such that $\pi(v_i) \to 0$, that is, $v_i - r_i v \to 0$ (where we can suppose that $r_i \leq 1/2$). Then for large i, $v_i < v$. This implies that $v_i = 0$ for large i so that $\pi(\Gamma)$ is discrete. By the induction hypothesis we can find linearly independent vectors $\{\pi(w_1), \cdots, \pi(w_{m-1})\}$ generating $\pi(\Gamma)$. $\{v, w_1, \cdots, w_{m-1}\}$ are linearly independent and generate Γ.

□

4 Branched coverings and holomorphic functions

Holomorphic and meromorphic functions $\varphi : X \to \mathbb{C}$ on a Riemann surface are defined to be continuous functions for which the composition $\varphi \circ p^{-1}$, where p^{-1} is any chart, is holomorphic or meromorphic.

The local behavior of a holomorphic map between Riemann surfaces is described in the following lemma.

Lemma 4.1 *Let $\varphi : Y \to X$ be a non-constant holomorphic map between Riemann surfaces with $\varphi(y_0) = x_0$. There exist local coordinates p_Y and p_X around y_0 and x_0 respectively such that $p_Y(y_0) = p_X(x_0) = 0$ and $p_X \circ \varphi \circ {p_Y}^{-1}(z) = z^n$ for some $n \geq 1$.*

Proof. Clearly we can assume that there exists local coordinates p'_Y (we will change that coordinate next) and p_X around y_0 and x_0, respectively, such that $p_Y(y_0) = p_X(x_0) = 0$. Now, if $p_X \circ \varphi \circ {p'_Y}^{-1}$ is non-constant we may suppose that there exists a holomorphic function $f(w)$ such that $p_X \circ \varphi \circ {p'_Y}^{-1}(w) = w^n f(w)$ with $n \geq 1$ and $f(0) \neq 0$. Therefore, on some neighborhood of the origin, there exists a holomorphic function $h(w)$ such that $h^n(w) = f(w)$. Observe that the map $p : w \to wh(w)$ is a biholomorphism in a neighborhood of the origin so that $p_Y = p \circ p'_Y$ is a new chart around y_0. For $z = wh(w)$ we obtain $p_X \circ \varphi \circ {p_Y}^{-1}(z) = p_X \circ \varphi \circ {p'_Y}^{-1}(w) = w^n f(w) = (wh(w))^n = z^n$. □

Observe that in the case $n = 1$ the map φ is a local biholomorphism at $y_0 \in Y$.

Definition 4.2 *A point $y_0 \in Y$ with $n \geq 2$ in the above theorem is called a ramification point and the point $x_0 \in X$ as above is a branching point of order n of the map φ.*

Definition 4.3 *A map $\varphi : Y \to X$ between surfaces is a branched covering if*

1. *The restriction $\varphi|_{\varphi^{-1}(X-S)}$, where S is a discrete subset of X, is a covering.*
2. *For each point in $y_0 \in \varphi^{-1}(S)$ there are coordinates p_Y around y_0 and p_X around $x_0 = \varphi(y_0)$ such that $p_X \circ \varphi \circ {p_Y}^{-1}(z) = z^n$. The integer n is called the ramification order of the ramification point y_0.*

Definition 4.4 *Let $\varphi : Y \to X$ be a branched covering. The ramification divisor is the formal sum*

$$R_\varphi = (\sum n_i - 1) y_i$$

where y_i are the ramification points and n_i their ramification order.

Topological coverings of Riemann surfaces inherit a unique complex structure such that the covering map is holomorphic. The equivalence between two coverings with their induced complex structure is a biholomorphism. This implies that the classification of coverings up to equivalence by conjugacy classes of subgroups of the fundamental group is in fact a classification of holomorphic coverings up to holomorphic equivalence.

A finite covering of a Riemann surface with a number of points deleted can always be extended to a branched covering. This follows from the following:

Exercise: The finite coverings, up to equivalence, of the punctured disc $D \setminus \{0\}$ are given by $\varphi_n : D \setminus \{0\} \to D \setminus \{0\}$ where $\varphi_n(z) = z^n$.

More precisely:

Proposition 4.5 *If X is a Riemann surface and $S \subset X$ is a closed discrete subset, then any finite covering $\varphi' : Y' \to X' = X \setminus S$ can be extended to a proper holomorphic map $\varphi : Y \to X$, where Y is a Riemann surface containing Y' such that $Y \setminus Y'$ is a closed discrete subset.*

Proof. At a point $s \in S$ there exists a neighborhood U_s with $U_s \cap S = \{s\}$ and a coordinate chart $\varphi_s : U_s \to D$ where D is the unit disc centered at the origin. As φ' is a finite covering, there exists a finite number of components $\varphi'^{-1}(U_s \setminus \{s\})$. Let V' be one of the components. As $\varphi'_{|V'}$ is a finite covering of the unit punctured disc, there exists a map $\psi' : V' \to D \setminus \{0\}$ so that $\varphi_s \circ \varphi \circ \psi'^{-1} : D \setminus \{0\} \to D \setminus \{0\}$ such that $\varphi_s \circ \varphi \circ \psi'^{-1}(z) = z^k$ and therefore we can add the point 0 to $D \setminus \{0\}$ and obtain a holomorphic map from D to D. Let V be the set obtained by adding an abstract point to V' so that $\psi : V \to D$ is a homeomorphism and defines a holomorphic chart and $\varphi_{|V}$ becomes a branched holomorphic covering. Repeating the procedure for each component above every $U_s \setminus \{s\}$ for $s \in S$ we obtain the Riemann surface Y. □

4.0.1 Exercises

1. Prove Liouville's theorem: every bounded holomorphic function defined on $\mathbb{C}$ is constant.
2. Let $\varphi : Y \to X$ be a non-constant holomorphic map between Riemann surfaces. Show that φ is an open map.
3. Let $\varphi : X \to \mathbb{C}$ be a non-constant holomorphic map. Show that $|\varphi|$ does not attain its maximum. Conclude that every holomorphic function on a compact Riemann surface is constant.
4. Let $\varphi : X \to \mathbb{C}$ be a non-constant holomorphic map. Show that $\mathrm{Re}\,\varphi$ does not attain its maximum.
5. Show that the meromorphic functions on $\mathbb{C}P^1$ are quotients of two polynomials.
6. Let $\varphi : Y \to X$ be a non-constant holomorphic map between compact Riemann surfaces. Show that φ is surjective. Prove the fundamental theorem of algebra by considering a polynomial as a holomorphic map between $\mathbb{C}P^1$.

4.1 Algebraic functions

Definition 4.6 *The field of meromorphic functions defined on a Riemann surface X is denoted by $\mathcal{M}(X)$.*

Let $\varphi : Y \to X$ be a branched holomorphic covering of degree n between Riemann surfaces. The map $\varphi^* : \mathcal{M}(X) \to \mathcal{M}(Y)$ defined by $g \to g \circ \varphi$ is clearly a monomorphism. Considering the field extension $\varphi^*(\mathcal{M}(X)) \subset \mathcal{M}(Y)$ we show the following

Theorem 4.7 *Let $\varphi : Y \to X$ be a branched holomorphic covering of degree n between Riemann surfaces. Then $\varphi^*(\mathcal{M}(X)) \subset \mathcal{M}(Y)$ is an algebraic field extension of degree less than n.*

Proof. Let $f \in \mathcal{M}(Y)$. Let $S \subset X$ be a closed discrete subset such that $\varphi : Y \setminus \varphi^{-1}(S) \to X \setminus S$ is a covering. Consider the restriction of f to the meromorphic function $f \in \mathcal{M}(Y \setminus \varphi^{-1}(S))$. We can define meromorphic functions on $X \setminus S$ by taking the elementary symmetric functions $s_1, \cdots s_n$ of the n functions $f \circ \varphi_i^{-1} : U \to \mathbb{C}$ where $\varphi_i = \varphi_{|U_i} : U_i \to Y$ and U_i is a component of $\varphi^{-1}(U)$ (supposing that each component of $\varphi^{-1}(U)$ is homeomorphic to U). Observe that, by construction, f is a solution of the equation

$$\Pi_{i=1}^n (w - \varphi^*(f \circ \varphi_i^{-1})) = w^n - \varphi^* s_1 w^{n-1} + \cdots + (-1)^n \varphi^* s_n = 0.$$

To conclude that the extension is algebraic we need to show that the coefficients s_i extend to meromorphic functions on X. We divide the proof in two steps:

1. If f is holomorphic then s_i are bounded holomorphic functions on a neighborhood of a point $s \in S$. By Riemann's removable singularity theorem we can extend s_i to a holomorphic function.

2. If f is meromorphic at all points in $\varphi^{-1}(s)$, consider a coordinate chart $z : U \to D$ such that $z(s) = 0$. Then $(\varphi^* z)^m f$ is holomorphic if m is large and therefore the elementary symmetric functions of $(\varphi^* z)^m f$ can be extended to holomorphic functions of the form $z^{m_i} s_i$ and therefore the s_i can be extended to meromorphic functions.

Suppose $f_0 \in \mathcal{M}(Y)$ is an element such that the minimal polynomial is of maximal degree n_0. We show now that $\mathcal{M}(X)(f_0) = \mathcal{M}(Y)$, thereby proving that the degree of the extension is less than n. In fact if $f \in \mathcal{M}(Y)$ is another element we have, by the existence of a primitive element ($\mathcal{M}(X)$ is of characteristic 0), $\mathcal{M}(X)(f_0, f) = \mathcal{M}(X)(g)$ and then

$$n_0 = dim_{\mathcal{M}(X)} \mathcal{M}(X)(f_0) \leq dim_{\mathcal{M}(X)} \mathcal{M}(X)(f_0, f) = dim_{\mathcal{M}(X)} \mathcal{M}(X)(g) \leq n_0$$

so that $\mathcal{M}(X)(f_0) = \mathcal{M}(X)(f_0, f)$. □

In fact the degree of the extension in the theorem above is precisely n but, to prove that, we need a result (see corollary 13.5) that guarantees the existence of a meromorphic function which assumes pairwise different values at points of a generic fiber (that is, whose points are not ramification points).

In the following we will prove a converse to that theorem. One of origins of Riemann surface theory concerns the study of algebraic equations of the form

$$w^n + a_1(z) w^{n-1} + \cdots + a_n(z) = 0,$$

where the coefficients $a_i(z)$ are meromorphic functions on $\mathbb{C}$. The idea is that the solution to that equation is, in fact, defined on a Riemann surface Y which is a branched covering $Y \to \mathbb{C}P^1$. We state the theorem in a more general form substituting $\mathbb{C}P^1$ for a general Riemann surface X.

Theorem 4.8 *Let X be a Riemann surface and*

$$P(w) = w^n + a_1 w^{n-1} + \cdots + a_n$$

an irreducible polynomial in $\mathcal{M}(X)[w]$ of degree n. Then there exists a Riemann surface Y, a branched holomorphic covering $p : Y \to X$ of degree n and a meromorphic function $F \in \mathcal{M}(Y)$ such that

$$P(F) = F^n + p^* a_1 F^{n-1} + \cdots + p^* a_n = 0.$$

Definition 4.9 *We say that Y is the Riemann surface associated to the irreducible polynomial P.*

Remark 4.10 *1. As $\mathcal{M}(X)$ is a field of characteristic 0, we know that the irreducible polynomial $P(w) \in \mathcal{M}(X)[w]$ is separable. That is, its roots in the algebraic closure of $\mathcal{M}(X)$ are all distinct.*

2. Recall that the elementary symmetric polynomial $s_i(t_1, \cdots, t_n)$ $(1 \leq i \leq n)$ of the variables t_i generate the algebra of symmetric polynomials of those variables. Observe that the functions $a_i \in \mathcal{M}(X)$ are the elementary symmetric functions of the roots of the polynomial $P(w)$. That is

$$\Pi_{1 \leq i \leq n}(w - t_i) = w^n - s_1 w^{n-1} + \cdots + (-1)^n s_n.$$

Therefore, the polynomial $\Delta = \Pi_{i<j}(t_i - t_j)^2$ which is clearly symmetric belongs to $\mathcal{M}(X)$. It is called the discriminant of $P(w)$. In particular, by the previous remark, the discriminant vanishes identically only if $P(w)$ is reducible.

Proof. The discriminant Δ of $P(w)$ vanishes at points of X where there are multiple roots. Therefore, because $P(w)$ is irreducible, Δ vanishes only on a closed discrete set of points S which we also suppose contains the poles of a_i. Let $X' = X \setminus S$ and define Y' to be the set of all points in $(z, w) \in (X \setminus S) \times \mathbb{C}$ satisfying the equation $P(w) = 0$. By the implicit function theorem (Proposition 4.11 bellow) and its corollary, $\varphi' : Y' \to X'$ is a covering map. We extend then this covering to a branched covering $\varphi : Y \to X$. The meromorphic function is defined first as a holomorphic function on Y' as $(z, w) \to w$ and then by extension (with a similar argument as in the previous theorem) to the whole of Y. To show that Y is connected, suppose that $Y = Y_1 \cup \cdots \cup Y_k$ is a decomposition in connected components with $\varphi_i : Y_i \to X$ branched coverings. Then, for each φ_i the meromorphic function F restricted to Y_i defines a polynomial $P_i(w) \in \mathcal{M}(X)$ such that $P(w) = P_1(w) \cdots P_k(w)$ contradicting the irreducibility of $P(w)$. □

Proposition 4.11 *Let f be a holomorphic function in two complex variables defined on $\{ (z, w) \mid |z| < \epsilon_1,\ |w| < \epsilon_2 \}$. Suppose that $f(0,0) = 0$ and $\frac{\partial f}{\partial w}(0,0) \neq 0$. Then, there exists $\delta_1, \delta_2 > 0$ and a unique function φ defined on $|z| < \delta_1$ with $|\varphi| < \delta_2$ such that $f(z, \varphi(z)) = 0$. Moreover, φ is holomorphic.*

Proof. As $\frac{\partial f}{\partial w}(0,0) \neq 0$, there exists $\delta_2 > 0$ such that $f(0, w) \neq 0$ for $|w| = \delta_2$. There exists therefore, by compactness, $\delta_1 > 0$ such that $f(z, w) \neq 0$ for $|z| < \delta_1, |w| = \delta_2$. Writing $f_w(z, w) = \frac{\partial f(z,w)}{\partial w}$, for each z, the number of zeros of $f(z, w)$ in $|w| < \delta_2$ is given by the holomorphic function

$$N(z) = \frac{1}{2\pi i} \int_{|w|=\delta_2} \frac{f_w(z, w)}{f(z, w)} dw$$

which is therefore constant equal to one. The explicit solution is given by the residue theorem (writing $f(z, w) = (w - \varphi(z))h(z, w)$ for a non-vanishing function $h(z, w)$):

$$\varphi(z) = \frac{1}{2\pi i} \int_{|w|=\delta_2} w \frac{f_w(z, w)}{f(z, w)} dw$$

which is holomorphic in z. □

Corollary 4.12 *Suppose that $P(w) = w^n + a_1 w^{n-1} + \cdots + a_n$ (with a_i holomorphic functions defined on a neighborhood of z) has n distinct solutions $w_1, \cdots, w_n$ at z. Then there exists unique holomorphic functions $f_1, \cdots, f_n$ (defined on perhaps smaller neighborhood of z) with $f_i(z) = w_i$ satisfying $P(f_i) = 0$ so that $P(w) = \Pi_1^n (w - f_i)$.*

4.1.1 Riemann-Hurwitz formula

Any compact Riemann surface can be described as a branched covering of $\mathbb{C}P^1$ once we admit the existence of at least one non-constant meromorphic function. From that description we can easily compute the genus of the surface. We state a more general version of that computation valid for a covering between compact surfaces.

Theorem 4.13 *Let $Y \to X$ be a branched covering of degree d between compact surfaces. For each ramification point $y \in Y$, let $o(y)$ be its ramification order. Then*

$$\chi(Y) = d\chi(X) - \sum (o(y) - 1).$$

Proof. The proof of the theorem follows from the existence of a triangulation with vertices containing the branching locus, that is, the image of all ramification points by the covering map. We will assume the existence of that triangulation of X. If the simplices of this triangulation are sufficiently small, the inverse image of the triangulation is a triangulation of Y. The number of its simplices is d times the number of original simplices, except for the vertices. Each ramification point diminishes by $(o(y) - 1)$ the maximum number of d times the number of vertices of the original triangulation. □

4.1.2 Hyperelliptic Riemann surfaces

Let $f(z) = (z - a_1) \cdots (z - a_k) \in \mathcal{M}(\mathbb{C}P^1)$ with distinct roots $a_i \in \mathbb{C}$. The algebraic function defined by $P(z, w) = w^2 - f$ is a Riemann surface together with a branched covering of degree two which is branched on $a_1, \cdots a_k$ if k is even and on $a_1, \cdots a_k, \infty$ if k is odd. These Riemann surfaces are called hyperelliptic.

Observe that in that case the algebraic curve $\{ (z, w) \in \mathbb{C}^2 \mid P(z, w) = 0 \}$ is a Riemann surface by the implicit function theorem as at each solution (z, w) we have $P_z \neq 0$ or $P_w \neq 0$.

To understand the topology of hyperelliptic Riemann surfaces, consider the Riemann-Hurwitz formula to compute their genera. Let X_f be the Riemann surface as defined by $P(z, w) = w^2 - f$. If k is even we obtain

$$\chi(X_f) = 2\chi(\mathbb{C}P^1) - k = 4 - k$$

and as the Euler characteristic is given by $\chi = 2 - 2g$, we obtain $g = \frac{-2+k}{2} = k/2 - 1$. In the case k is odd we obtain

$$\chi(X_f) = 2\chi(\mathbb{C}P^1) - (k + 1) = 3 - k$$

so that $g = \frac{-1+k}{2} = (k - 1)/2$. In particular, for $k = 3$ we obtain an elliptic curve.

4.1.3 Exercises

1. Determine the Riemann surface defined by $P(z,w) = z^2 - w^3$ over $\mathbb{C}P^1$.

2. Determine the genus of the Riemann surface defined by $P(z,w) = z^n + w^n - 1$ over $\mathbb{C}P^1$.

3. The field $\mathcal{M}(\mathbb{C}P^1)$ is $\mathbb{C}(z)$, a purely transcendental extension of $\mathbb{C}$.

4. The field of meromorphic functions of a compact Riemann surface X is finitely generated over $\mathbb{C}$ and of transcendence degree one. (Use the fact that there exists a meromorphic function defined on X).

5. Any finitely generated field of transcendence degree one over $\mathbb{C}$ is isomorphic to the field of meromorphic functions of a compact Riemann surface.

4.1.4 Belyi's theorem

As an application of the construction of a Riemann surface of an algebraic function we will describe a relation between the number of branching points of the covering and the field of definition of an algebraic function.

We say that the Riemann surface X is defined over $\bar{\mathbb{Q}}$ if it is constructed as above starting with an irreducible polynomial in $\bar{\mathbb{Q}}[z,w]$, where $\bar{\mathbb{Q}}$ is the field of algebraic numbers.

Theorem 4.14 (Belyi) *A compact Riemann surface X is defined over $\bar{\mathbb{Q}}$ if and only if there exists a holomorphic covering $\pi : X \to \mathbb{C}P^1$ branched on three points.*

Proof. We will prove the "only if" part. The other implication being outside our scope because it needs basic algebraic geometry. We start with a polynomial $P \in \bar{\mathbb{Q}}[z,w]$. By theorem 4.7 there exists $\varphi : X \to \mathbb{C}P^1$ which is branched over a finite set S of algebraic points. We divide the proof in two steps:

1. We first modify this branched covering to a covering which is branched over rational points. Take $s \in S$ and let $h \in \mathbb{Q}[X]$ be its minimal polynomial. The map $h \circ \varphi : X \to \mathbb{C}P^1$ is a branched covering with branching points contained in $h(S) \cup \{h(z) \mid h'(z) = 0\ \}$. Observe that $h(s) = 0$ so we made one of the branching points in S rational at the cost of introducing new branching points. But the minimal polynomial of a point $z_0 \in \{z|\ h'(z) = 0\ \}$ is of degree strictly smaller than the degree of h and therefore the minimal polynomial of $h(z_0) \in \{h(z)|\ h'(z) = 0\ \}$ has strictly smaller degree too (being in the same field extension as $\mathbb{Q}(z_0)$). We repeat this procedure with each element in S and obtain, by composing with each minimal polynomial, a branched covering where the new branching points have minimal polynomials of strictly smaller degrees. Eventually the degree is one and we obtain only rational branching points.

2. By the previous step, we may suppose that $\varphi : X \to \mathbb{C}P^1$ is branched on rational points. Now we reduce the number of branching points to at most three. Supposing it is greater than three, we can always assume that $\{0, 1, \infty\}$ are among those points by composing with an automorphism of $\mathbb{C}P^1$. For $m, n \in \mathbb{Z}^*$ such that $m + n \neq 0$, consider the map $f_{mn} : \mathbb{C}P^1 \to \mathbb{C}P^1$ defined by

$$f_{mn}(z) = \frac{(m+n)^{m+n}}{m^n n^n} z^m (1-z)n.$$

The critical values are computed solving $f'_{mn}(z) = 0$ and we obtain that they are contained in $\{0, 1, \infty, \frac{m}{m+n}\}$. But the branching points are contained in $\{0, 1, \infty\}$. We conclude that for each rational branching point of φ outside $\{0, 1, \infty\}$ we can find a map f_{mn} so that $f_{mn} \circ \varphi$ transforms this branching point to one of $\{0, 1, \infty\}$. This concludes the proof.

□

5 Riemann surfaces as quotients.

One of the problems concerning Riemann surfaces is their classification. A natural classification is up to equivalence under biholomorphisms.

We first define maps between complex manifolds.

Definition 5.1 *Let M, N be two complex manifolds. A continuous map $F : M \to N$ is said to be a holomorphic map if for all charts $z_\alpha : U_\alpha \to \mathbb{C}^n$ and $w_\beta : W_\beta \to \mathbb{C}^n$ of M and N respectively,*

$$w_\beta \circ F \circ z_\alpha^{-1} : z_\alpha(U_\alpha) \to w_\beta(W_\beta)$$

is holomorphic.

A homeomorphism between complex manifolds which is holomorphic and whose inverse is also holomorphic is called a biholomorphism. The group of biholomorphisms of a complex manifold M is also called the automorphism group of M.

Example 5.2 *The map $z \to (\mathcal{P}(z), \mathcal{P}'(z), 1)$ for $z \in \mathbb{C} - \Gamma_\tau$ and $z \to (0, 1, 0)$ for $z \in \Gamma_\tau$ defines a holomorphic embedding $E_\tau \to \mathbb{C}P^2$ whose image is the algebraic curve*

$$y^2 z = 4x^3 - g_2(\tau)xz^2 - g_3(\tau)z^3.$$

Example 5.3 *The disc and the half plane are biholomorphic complex manifolds of dimension 1. In fact, the map $z \to \frac{z-i}{z+i}$ is a biholomorphism from the half plane to the disc.*

Example 5.4 *The Möbius transformations preserving the disc are of the form*

$$z \to e^{i\theta} \frac{z + a}{\bar{a}z + 1}$$

.

The most fundamental theorem concerning complex manifolds of dimension one is Riemann's mapping theorem. We will state it without proof.

Theorem 5.5 (Riemann mapping theorem) *A simply connected one dimensional manifold is biholomorphic to either*

1. *$\mathbb{C}P^1$ (the Riemann sphere)*
2. *$\mathbb{C}$*
3. *$H^1_{\mathbb{C}} = \{z \in \mathbb{C}, |z| < 1\}$.*

5.1 Automorphism groups

It will be important to determine for each manifold M its group of biholomorphisms Aut(M) In the following theorem we need to recall Schwarz lemma:

Lemma 5.6 *If $f : H^1_{\mathbb{C}} \to H^1_{\mathbb{C}}$ be a holomorphic map and $f(0) = 0$ then $|f(z)| \leq |z|$ for all $z \in H^1_{\mathbb{C}}$ and $|f'(0)| \leq 1$. If $|f'(0)| = 1$ or if $f(z) = z$ for some $z \neq 0$ then $f(z) = e^{i\theta}z$.*

Theorem 5.7 *The automorphism groups of the simply connected Riemann surfaces are*

1. *$Aut(\mathbb{C}P^1) = PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\{\pm I\}$, all Möbius transformations*
2. *$Aut(\mathbb{C}) = \{az + b \mid a \neq 0 \ , \ b \in \mathbb{C}\}$*
3. *$Aut(H^1_{\mathbb{C}}) = PSU(1, 1) = SU(1, 1)/\{\pm I\}$, Möbius transformations preserving the disc.*

Proof. We first describe $f \in Aut(\mathbb{C})$. We have $f(z) = a_0 + a_1 z + \cdots$. As f is an automorphism, the image of a neighborhood of infinity is a neighborhood of infinity. Therefore it can be extended to a holomorphic function at infinity. Therefore $f(z)$ is a polynomial and by the fundamental theorem of algebra, it must be linear.

To show 1. observe that we can write in homogeneous coordinates $\mathbb{C}P^1 = \{[z_0, z_1]\}$, where z_0, z_1 are not both null. Any transformation of the form $[z_0, z_1] \to [az_0+bz_1, cz_0+dz_1]$, with $ad-bc \neq 0$ is an automorphism. So we have an action $PSL(2, \mathbb{C})\times\mathbb{C}P^1 \to \mathbb{C}P^1$. Given an element $\gamma \in Aut(\mathbb{C}P^1)$ we can find an element $\gamma_1 \in PSL(2, \mathbb{C})$ such that $\gamma\circ\gamma_1(\infty) = \infty$. So $\gamma \circ \gamma_1 \in Aut(\mathbb{C})$ and we conclude using the description of $Aut(\mathbb{C})$.

To show 3. we observe first that $PSU(1, 1) \subset PSL(2, \mathbb{C})$. That is, $SU(1, 1) = \{A \in SL(2, \mathbb{C}) \mid h(Az, Az) = h(z, z)\}$, where $h(z, w) = z_0\overline{w}_0 - z_1\overline{w}_1$ is a hermitian form. So $PSU(1, 1)$ preserves the disc $H^1_{\mathbb{C}} = \{z \in \mathbb{C}P^1 \mid h(z, z) < 0\}$. If $\gamma \in Aut(H^1_{\mathbb{C}})$, there exists an element $\gamma_1 \in PSU(1, 1)$ such that $\gamma \circ \gamma_1(0) = 0$. By Schwarz's lemma we obtain $|f'(0)| \leq 1$ and, as f is a biholomorphism, the same inequality for the inverse function gives $|f'(0)| = 1$. By Schwarz's lemma we conclude that $\gamma \circ \gamma_1(z) = e^{i\theta}z$ and that concludes the proof. □

Corollary 5.8 *A Riemann surface covered by $\mathbb{C}P^1$ is biholomorphic to $\mathbb{C}P^1$.*

Proof. This follows from the fact that any Möbius transformation has a fixed point. It implies that there is no subgroup of the Möbius group acting freely on $\mathbb{C}P^1$. □

On the other hand observe that the involution $\iota : z \to -\frac{1}{\bar{z}}$ defined on $\mathbb{C}P^1$ does not have fixed points. The quotient space $\mathbb{C}P^1/\langle\iota\rangle$ is the real projective plane which is not a Riemann surface.

Exercise 5.9 *A meromorphic function on $\mathbb{C}P^1$ is a holomorphic map of $\mathbb{C}P^1$ on itself. They are all rational functions, that is $f(z) = \frac{p(z)}{q(z)}$ where $p(z)$ and $q(z)$ are polynomials.*

Exercise 5.10 *If K is a field $PSL(n, K) = PGL(n, K)$ if and only if every element of K has an n-th root. For instance $PSL(2, \mathbb{R}) \neq PGL(2, \mathbb{R})$.*

Exercise 5.11 *$PU(1, 1)$ acts doubly transitively on the boundary. That is given $x_1, y_1, x_2, y_2 \in \partial H^1\mathbb{C}$ with $x_i \neq y_i$, there exists an element $\gamma \in PU(1, 1)$ such that $\gamma x_1 = x_2$ and $\gamma y_1 = y_2$.*

5.1.1 Conjugacy classes

It is very important to understand the conjugacy classes of elements in the automorphism groups. Elements in the same conjugacy class act in an "equivalent" way.

Lemma 5.12 *An element of $PSL(2,\mathbb{C})$ has one or two fixed points. We have*

1. *If it has only one fixed point then it is conjugate to $z \to z+1$.*
2. *If it has two fixed points it is conjugate to $z \to \lambda z$, $\lambda \neq 1, 0$.*

Proof. Given any Möbius transformation we solve the equation

$$\frac{az+b}{cz+d} = z.$$

It has one or two solutions. If it has only one solution by conjugating with an element of $PSL(2,\mathbb{C})$ we can suppose that ∞ is that fixed point. In that case the element must be of the form $z \to az+b$. We immediately see that $a = 1$ otherwise there would be a second fixed point. Moreover, by conjugating with $z \to \frac{1}{b}z$ we obtain $z \to z+1$. To show the second part we observe that we can conjugate an element with two fixed points to one fixing 0 and ∞. That gives clearly the form $z \to \lambda z$.

□

We can further refine that lemma to obtain the orbit space by the conjugation action of $PSL(2,\mathbb{C})$. The proof of the following theorem is a simple consequence of the lemma.

Theorem 5.13 *The conjugacy classes of $PSL(2,\mathbb{C})$ are uniquely represented by the following elements*

1. *$z \to z+1$ called parabolic.*
2. *$z \to e^{i\theta}z$, $0 \le \theta \le \pi$, called elliptic.*
3. *$z \to \lambda z$, $\lambda \in \mathbb{C}$ $|\lambda| > 1$, called loxodromic. In the case $\lambda \in \mathbb{R}$ we call it a hyperbolic transformation.*

Proof. The first part is contained in the previous lemma. For the second and third part we observe that if $\gamma(z) = \lambda z$, in order to preserve the fixed points, we are allowed to conjugate by elements of the form $z \to az$, which commute with γ (so irrelevant), or $z \to a/z$. In that case γ is transformed to $g\gamma g^{-1}(z) = \frac{1}{\lambda}z$. This shows the result.

□

Considering only elements in $PSU(1,1)$ we describe conjugacy classes in the following definition.

Definition 5.14 $\gamma \in PSU(1,1)$ *is called*

1. *Elliptic if it has a fixed point in $H^1_{\mathbb{C}}$.*
2. *Parabolic if it has a unique fixed point in $\partial H^1_{\mathbb{C}}$.*
3. *Hyperbolic if it has two fixed points in $\partial H^1_{\mathbb{C}}$.*

Theorem 5.15 *Let $\gamma \in PSU(1,1)$ and consider a lift $\tilde{\gamma} \in SU(1,1)$. Then γ is*

1. *elliptic if and only if $tr^2\tilde{\gamma} < 4$,*
2. *parabolic if and only if $tr^2\tilde{\gamma} = 4$ and γ is not the identity,*
3. *hyperbolic if and only if $tr^2\tilde{\gamma} > 4$.*

Observe, however, that conjugation in $PSU(1,1)$ splits certain conjugacy classes in $PSL(2,\mathbb{C})$ (of course, some disappear). For instance, the parabolic class is split in two: $z \to z+1$ and $z \to z-1$. Analogously, the elliptic class $z \to e^{i\theta}$, $0 \le \theta \le \pi$ splits in two, so that $0 \le \theta < 2\pi$ is the parameterization of the classes. On the other hand, the only loxodromic classes which appear in $PSL(2,\mathbb{C})$ are those with $\lambda > 1$ and they don't split.

Remark 5.16 *Let $\widehat{P}SU(1,1) = \langle PSU(1,1), z \to \bar{z} \rangle$. Using conjugation on that group we can collapse again the splitting. In particular $z \to z+1$ and $z \to z-1$ are conjugate in the corresponding group $\widehat{P}SL(2,\mathbb{R})$.*

5.2 The complex plane $\mathbb{C}$ and its quotients

Theorem 5.17 *A Riemann surface is covered by $\mathbb{C}$ if and only if it is biholomorphic to $\mathbb{C}$, $\mathbb{C} - 0$ or a torus.*

Proof. We prove the only if part. The other implication is a consequence of the next proposition. Let $\Gamma \subset Aut(\mathbb{C})$ be the covering group, where $Aut(\mathbb{C})$ is the group of biholomorphisms of $\mathbb{C}$. If $\gamma(z) = az + b$ is a non trivial element of Γ then $a = 1$, otherwise γ would have a fixed point. So Γ is generated by translations. We saw in theorem 3.21 that a discrete subgroup of $Aut(\mathbb{C})$ generated by translations is one of the following:

1. $\{id\}$
2. $< \gamma >= \mathbb{Z}$, a group generated by one translation $\gamma(z) = z + \omega$
3. $< \gamma_1, \gamma_2 >= \mathbb{Z} \oplus \mathbb{Z}$, a group generated by two translation $\gamma_1(z) = z + \omega_1$ and $\gamma_2(z) = z + \omega_2$ with ω_1 and ω_2 linearly independent over $\mathbb{R}$.

The first case corresponds to $\mathbb{C}$. For the second case the function $z \to e^{2\pi z/\omega}$ establishes a biholomorphism between $\mathbb{C}/ < \gamma >$ and $\mathbb{C} - 0$. In the third case the quotient manifold is diffeomorphic to a torus.

□

To complete the theorem we need to show that any torus is covered by $\mathbb{C}$. That is, the complex disc cannot cover a torus. This follows from the following proposition.

Proposition 5.18 *Let $\Gamma \subset Aut(H^1_{\mathbb{C}})$ be a discrete group without fixed points. If Γ is abelian, then it is cyclic.*

Proof. We separate in two cases. If $\gamma \in \Gamma$ is parabolic we can, without loss of generality, suppose that $\gamma(z) = z + a$. A computation then shows that any commuting element is parabolic . In fact

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

implies

$$\begin{pmatrix} a+xc & b+xd \\ c & d \end{pmatrix} \begin{pmatrix} a & ax+b \\ c & cx+d \end{pmatrix}$$

So $xc = 0$ and $x(a-d) = 0$ which implies $c = 0$ and $a = d$. That is, the commuting element is parabolic. By discreteness we obtain that it is cyclic.
Analogously, if γ is hyperbolic, without loss of generality, suppose that $\gamma(z) = \lambda z$. We easily conclude (by the lemma bellow) that an element commuting with it is of the same form and using discreteness we conclude that the subgroup is cyclic.

□

Lemma 5.19 *Two hyperbolic elements commute if and only if they have the same fixed points.*

Proof. We write one element as $z \to \lambda z$ and the other by a general Möbius transformation. Then, by commutativity

$$\begin{pmatrix} \lambda^{-1/2} & 0 \\ 0 & \lambda^{1/2} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

A computation shows that $b = c = 0$.

□

A proof without computations runs as follows. Observe that if G (any group) acts on M (any space) and g_1 commutes with g_2 the fixed points of g_1 are preserved by g_2 and the fixed points of g_2 are preserved by g_1, in fact

$$g_1(x) = x \to g_2 g_1(x) = g_2(x) \to g_1(g_2(x)) = g_2(x).$$

The proposition is a refinement of the observation: If γ has only one fixed point any commuting element will have precisely the same fixed point (so if γ is parabolic the commuting element is also parabolic). If γ has two fixed points the commuting element either has the same fixed points or it interchanges the two fixed points. But in the last case, if $\gamma \in Aut(H^1_{\mathbb{C}})$ then the commuting element must have only one fixed point (in fact it is elliptic) and therefore γ has only one fixed point, a contradiction.

5.3 Fuchsian groups

Definition 5.20 *A Fuchsian group is a discrete subgroup of $PSU(1,1)$.*

In order to define a quotient of the disc by a discrete group as a Riemann surface we need to verify that the action is free and properly discontinuous. The action is free if there are no elliptic elements, also called torsion elements. On the other hand, the action is always properly discontinuous as is shown by the next theorem.

Theorem 5.21 *A subgroup $\Gamma \subset Aut(H^1_{\mathbb{C}})$ is Fuchsian if and only if it acts properly discontinuously.*

Proof. Clearly if Γ acts properly discontinuously then it is discrete. Now suppose it is discrete and it does not act properly discontinuously.
Recall the normal family theorem:

Theorem 5.22 (Normal family theorem) *Suppose $f_n : \Omega \to \mathbb{C}$ is a family of holomorphic functions defined on a region of $\mathbb{C}$. If f_n is uniformly bounded on each compact subset of Ω (a normal family) then there exists a subsequence which converges uniformly on compact subsets (the limit function will then be holomorphic)*

We need the following lemma

Lemma 5.23 *If a sequence $\gamma_n \in Aut(H^1_{\mathbb{C}})$ converges uniformly on compact subsets to γ then*

1. *$\gamma \in Aut(H^1_{\mathbb{C}})$ or*
2. *γ is a constant function with value some $e^{i\theta}$.*

Proof. If $\gamma_n(x_0) \to b$ with $|b| = 1$ then by the maximum modulus principle $\gamma(x_0) = b = \gamma(z)$, for all $z \in H^1_{\mathbb{C}}$. Otherwise we have $\gamma : H^1_{\mathbb{C}} \to H^1_{\mathbb{C}}$ and taking a subsequence γ_n^{-1} converges uniformly on compact subsets to γ^{-1} so $\gamma \in Aut(H^1_{\mathbb{C}})$.
□

Back to the proof: if the action is not properly discontinuous there exists a compact $K \subset H^1_{\mathbb{C}}$ and a sequence of distinct elements $\gamma_n \in \Gamma$ such that $\gamma_n(K) \cap K \neq \emptyset$. Clearly the sequence γ_n is a normal family. Therefore, taking perhaps a subsequence, it converges uniformly on compact subsets to a holomorphic function. Taking a subsequence if necessary we have $\gamma_n(x_n) = y_n$ with $\lim x_n = x$ and $\lim y_n = y$, therefore $\lim \gamma_n(x) = y$. We conclude, using the lemma, that γ_n converges to an element of $Aut(H^1_{\mathbb{C}})$, therefore the group is not discrete.
□

The following lemma is an important technical component of the next theorem.

Lemma 5.24 (Shimizu) *If $z \to z + 1$ belongs to a Fuchsian group in $PSL(2, \mathbb{R})$, then every other element γ of the form*

$$\frac{az + b}{cz + d}$$

satisfies $|c| \geq 1$, provided $c \neq 0$.

Proof. We set

$$A_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} A_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and define by induction for $n \geq 1$,

$$A_{n+1} = A_n A_0 A_n^{-1}.$$

We compute the coeficients of A_{n+1} obtaining

$$\begin{aligned} a_{n+1} &= 1 - c_n a_n \\ b_{n+1} &= a_n^2 \\ c_{n+1} &= -c_n^2 \\ d_{n+1} &= 1 + a_n c_n \end{aligned}$$

If $c < 1$ then c_n converges, in fact $|c_n| = |c|^{2^{n-1}}$. We claim that $\lim a_n = 1$. Observe that $|a_{n+1}| \leq 1 + |a_n c_n| \leq 1 + |a_n|$. By induction then $|a_{n+1}| \leq n + |a|$. We obtain then $|a_{n+1}| \leq 1 + |a_n c_n| \leq 1 + |c_n|(n + |a|) \leq 1 + |c|^{2^{n-1}}(n + |a|)$ and the result follows. $\square$

A Fuchsian group $\Gamma \subset PSL(2, \mathbb{R})$ is said to be co-compact if the quotient $H^1_{\mathbb{C}}/\Gamma$ compact. From Shimizu lemma we conclude the following theorem which says that if a Riemann surface is compact and not the sphere or a quotient of the complex plane then its fundamental group does not have parabolics.

Theorem 5.25 *If $\Gamma \subset PSL(2, \mathbb{R})$ is co-compact without torsion then any non-trivial element is hyperbolic.*

Proof. If there were a parabolic element, by conjugation we may suppose it $z \to z + 1$ and generator of the parabolic group Γ_∞ fixing ∞. As

$$Im(\gamma(z)) = \frac{Im(z)}{|cz + d|^2}$$

for any $\gamma(z) = \frac{az+b}{cz+d}$ in Γ we estimate using Shimizu's lemma that if $Im(z) > 1$ then

$$Im(\gamma(z)) \leq \frac{1}{|c|^2 Im(z)} < 1$$

for γ not in Γ_∞. Therefore the set $\{\ z \mid -\frac{1}{2} < \operatorname{Re} z < \frac{1}{2}\ ,\ Im(z) > 1\ \}$ passes to the quotient, but it is not compact, a contradiction.

$\square$

5.4 Fundamental domains

Definition 5.26 *A fundamental domain of a properly discontinous action on a topological manifold, $\Gamma \times X \to X$ is an open set $F \subset X$ such that*

1. *$\bigcup_{\gamma \in \Gamma} \gamma \overline{F} = X$, where $\overline{F}$ is the closure of F*
2. *If $x, y \in F$ they are not in the same orbit.*

We do not suppose that the action is free but observe that a fixed point of an element in Γ is never contained in F. It might be contained in the closure of F.

Example 5.27 *A fundamental domain for the action of the additive group generated by the translations $z \to z + 1$ and $z \to z + \tau$ is the parallelogram defined by the sides $1, \tau$.*

5.4.1 $PSL(2, \mathbb{Z})$

Theorem 5.28 *$D = \{\ z \in H^1_{\mathbb{C}} \mid |z| > 1, -1/2 < Re(z) < 1/2\ \}$ is a fundamental domain for $PSL(2, \mathbb{Z})$.*

Proof. Again we use

$$Im(\gamma(z)) = \frac{Im(z)}{|cz + d|^2}$$

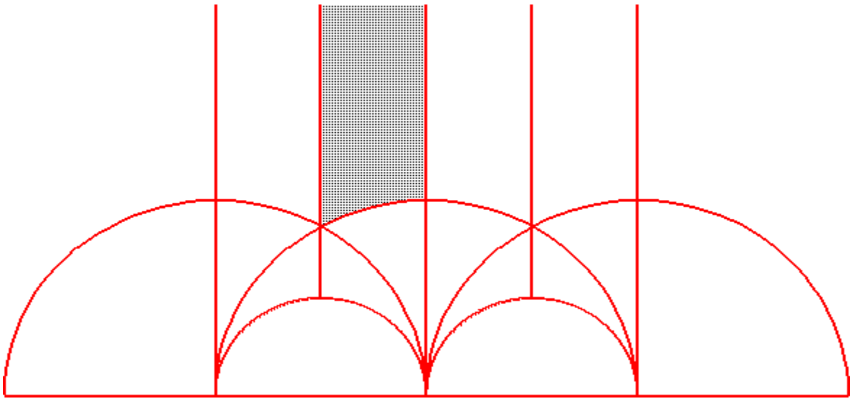


Figure 4: A fundamental domain for a triangle group containing $PSL(2,\mathbb{Z})$ as an index two subgroup. The fundamental domain for $PSL(2,\mathbb{Z})$ is the symmetric double of the grey region.

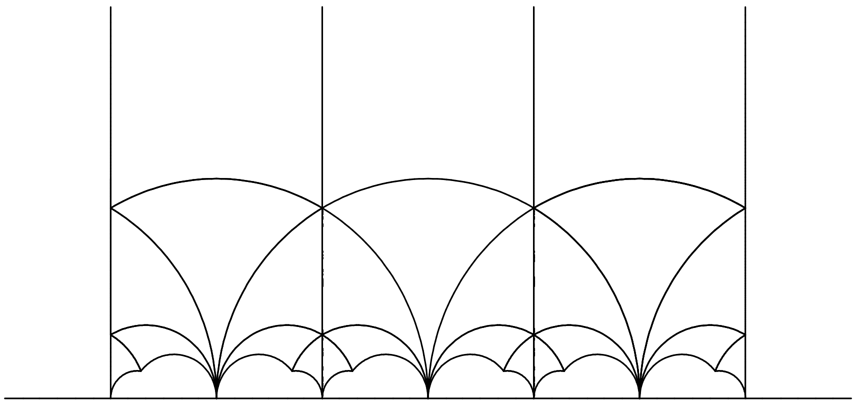


Figure 5: A fundamental domain for $PSL(2,\mathbb{Z})$ and some of its translates.

to observe that fixing $\tau \in \mathbb{C}$, there is only a finite number of elements $\gamma \in PSL(2,\mathbb{Z})$ with $|c\tau + d|^2 < M$ for a fixed bound M. This follows because $\mathbb{Z}\tau + \mathbb{Z}$ is a discrete group. Take γ such that $Im(\gamma(\tau))$ is maximum. Using the translation we can suppose without loss of generality that $-1/2 \leq Re(\tau) \leq 1/2$. We claim that $|\gamma(\tau)| \geq 1$, otherwise using the inversion $s(z) = -1/z$ we would get $Im(s\gamma(\tau)) = \frac{Im(\gamma(\tau))}{|\gamma(\tau)|^2} > Im(\gamma(\tau))$. A contradiction.

Suppose now that τ and $\gamma(\tau)$ belong to $\bar{D}$. Without loss of generality we may assume that $Im(\gamma(\tau)) \geq Im(\tau)$. Therefore

$$|c\tau + d| \leq 1.$$

Just looking at the imaginary part, that is, $\mathrm{Im}\,(c\tau + d) = c\mathrm{Im}\,\tau \geq c\frac{\sqrt{3}}{2}$, we obtain that the only possibilities are $c = 0, 1, -1$. If $c = 0$ it follows easily that γ is either the translation or the identity. If $c = 1$, we must have $|z + d| \leq 1$. We claim that that is only possible if $z = \omega$ or $z = -\bar{\omega}$ or $z = i$. That can be seen easily in the picture. Analogously we obtain those two points if $c = -1$.

□

5.5 $\Gamma(2)$

Let $\pi_N : SL(2,\mathbb{Z}) \to SL(2,\mathbb{Z}_N)$ be the homomorphism obtained by reducing modulo N. It passes to the quotients

$$\varphi_N : SL(2,\mathbb{Z})/\{I,-I\} \to SL(2,\mathbb{Z}_N)/\{I,-I\}.$$

The kernel of this homomorphism is called the principal congruence group of level N, $\Gamma(N) \subset PSL(2,\mathbb{Z})$.

The simplest case, $\Gamma(2)$, acts freely on the complex disc so that $H^1_{\mathbb{C}}/\Gamma(2)$ is a sphere with three points deleted.

To understand the action, observe first that the homomorphism φ_N is clearly surjective and, as $SL(2,\mathbb{Z}_2) = PSL(2,\mathbb{Z}_2)$ has 6 elements which can easily be enumerated:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

we have, therefore, that $\Gamma(2) \subset PSL(2,\mathbb{Z})$ is of index 6.

The fundamental domain of subgroups of finite index can be computed using the following lemma.

Lemma 5.29 *Suppose D is a fundamental domain for a group G acting on a space M. Let $H \subset G$ be a subgroup of index k and $Hg_1, \cdots, Hg_k$ be its left cosets. Then $D_H = g_1D \cup \cdots \cup g_kD$ is a fundamental domain for H.*

Proof. If $x, y \in D_H$ and there exists $h \in H$ such that $y = hx$ then, as $x \in g_iD$ and $y \in g_jD$, we might suppose that $g_j\bar{y} = hg_i\bar{x}$ for $\bar{x}, \bar{y} \in D$. That is, $\bar{y} = g_j^{-1}hg_i\bar{x}$ which contradicts the fact that D is a fundamental domain for G. On the other hand, $H\overline{D_H} = M$ follows because $G = \bigcup Hg_i$. □

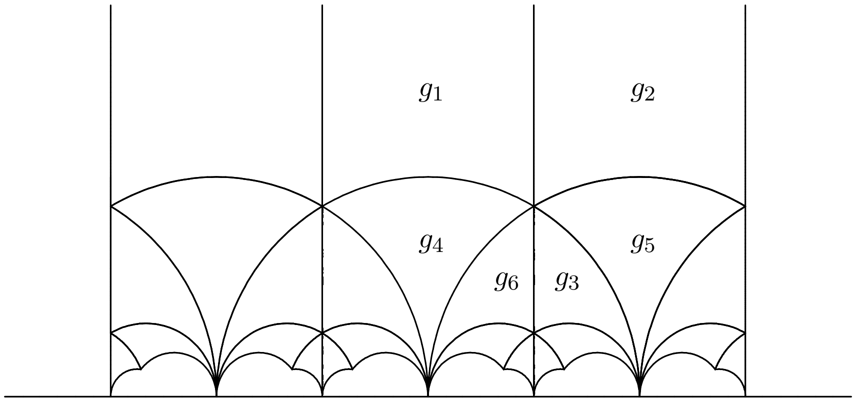


Figure 6: A fundamental domain for $\Gamma(2)$ showing the six translates of the fundamental region of $PSL(2,\mathbb{Z})$ corresponding to each coset.

Left coset representatives of $\Gamma(2)$ are obtained by chosing an inverse image for each element of $SL(2,\mathbb{Z}_2)$:

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} g_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} g_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$g_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} g_5 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} g_6 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

The boundary of the fundamental domain consists of 2 vertical half lines paired by the parabolic element

$$\gamma_1 = z \to z + 2$$

and two pairs of arcs paired by parabolic elements in the group:

$$\gamma_2 = g_4\gamma_1 g_4^{-1} = z \to \frac{z}{2z+1}$$

for the sides of the region $g_4D \cup g_6D$ (where D is the fundamental domain for $PSL(2,\mathbb{Z})$ found before),

$$z \to \frac{3z-2}{2z-1}$$

for the sides of the region $g_3D \cup g_5D$. One should observe that the three points of $H^1_{\mathbb{C}}$ in the boundary of the region are identified by those pairings and, around that point, the regions match together to form a complex disc. The quotient is the sphere where 3 points are deleted.

6 Algebraic curves

6.1 Affine plane curves

Let

$$F(x,y) = \sum_{r,s} c_{r,s} x^r y^s$$

be a polynomial in two variables with complex coefficients. That is, $F \in \mathbb{C}[x,y]$.

Definition 6.1 *The affine complex plane curve defined by a non-constant polynomial F is the set*

$$C_F = \{(x,y) \in \mathbb{C}^2 \mid F(x,y) = 0 \}$$

Examples:

1. A complex line is given by the equation $ax + by + c = 0$.
2. A conic is given by the equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$.
3. (Exercise) A homogeneous polynomial in two variables can be factored as a product of linear polynomials.

The definition has some obvious problems. Namely, two different polynomials might define the same curve (think of $F(x,y)$ and $F(x,y)^2$) and the set C_F might not be connected ($F(x,y) = x(x+1)$). Another problem is that the set C_F might not be a smooth subvariety of $\mathbb{C}^2$.

The important notion to address the first problem is that of irreducible polynomial. F (non-constant polynomial) is irreducible if it cannot be written as $F = Q.R$ where Q and R are non-constant polynomials. Any polynomial can be written in a unique way (up to multiplicative constants and permutation of factors) as a product of irreducible factors. The following theorem shows that C_F is determined by the irreducible factors of F. One can also show that if F is irreducible C_F is connected. We say that a curve C_F is irreducible if F is irreducible.

Theorem 6.2 (Hilbert Nullstellensatz) *If F and Q are two polynomials, then $C_F = C_Q$ if and only if they have the same irreducible factors.*

We will say that the curves defined by the irreducible factors of F are the irreducible components of C_F.

Definition 6.3 *The degree of a curve C_F defined by F is the degree of F, that is*

$$d = max\{ \ r + s \mid c_{r,s} \neq 0 \ \}.$$

Definition 6.4 *A point $(x_0, y_0) \in C_F$ is singular if*

$$\frac{\partial F}{\partial x}(x_0, y_0) = \frac{\partial F}{\partial y}(x_0, y_0) = 0.$$

Otherwise, it is called a non-singular point.

By the implicit function theorem, the curve $C_F - \{ \ \textit{singular points} \ \}$ is a complex submanifold. At a singular point (x_0, y_0), we can further analyse the curve by computing the Taylor polynomial

$$F(x,y) = \sum_{m \geq 1} \sum_{i+j=m} \frac{1}{i!j!} \frac{\partial^m F}{\partial x^i \partial y^j}(x_0, y_0)(x - x_0)^i (y - y_0)^j.$$

The smallest m with $\frac{\partial^m F}{\partial x^i \partial y^j}(x_0, y_0) \neq 0$ is the order of the singular point. Then, the homogeneous polynomial

$$\sum_{i+j=m} \frac{1}{i!j!} \frac{\partial^m F}{\partial x^i \partial y^j}(x_0, y_0)(x - x_0)^i (y - y_0)^j$$

has linear irreducible components. Each irreducible component defines a line which is tangent to the curve at the singular point. We say that the singular point is ordinary if the number of lines equals the order of the singular point.

6.2 Projective plane curves

Affine curves are never compact as fixing any arbitrarily large x we can always solve for y in $F(x, y) = 0$. In order to consider compact surfaces we define projective curves in $\mathbb{C}P^2$. We start with a homogeneous polynomial $F(x, y, z)$ defined on $\mathbb{C}^3$.

Definition 6.5 *The projective complex curve defined by F is the set*

$$C_F = \{[x, y, z] \in \mathbb{C}P^2 \mid F(x, y, z) = 0 \}.$$

We define, as for affine curves, the irreducible components of C_F to be the projective curves defined by the irreducible factors of F.

Definition 6.6 *The degree of a curve C_F defined by F is the degree of F.*

Definition 6.7 *A point $[x_0, y_0, z_0] \in C_F$ is singular if*

$$\frac{\partial F}{\partial x}(x_0, y_0, z_0) = \frac{\partial F}{\partial y}(x_0, y_0, z_0) = \frac{\partial F}{\partial z}(x_0, y_0, z_0) = 0.$$

Otherwise, it is called a non-singular point.

Example: A projective line in $\mathbb{C}P^2$ is defined by the equation $ax + by + cz = 0$.

The relation between affine curves and projective curves is made explicit by writing $\mathbb{C}P^2 = \mathbb{C}^2 \cup \mathbb{C}P^1 = \{ [x, y, z] \mid z \neq 0 \} \cup \{ [x, y, 0] \}$. A homogeneous polynomial of degree d, $F(x, y, z)$, which does not have z as a factor, defines a polynomial $F(x, y, 1)$ on $\mathbb{C}^2$ of degree d. And reciprocaly, if $F(x, y) = \sum_{r,s} c_{r,s} x^r y^s$ is a polynomial of degree d on $\mathbb{C}^2$ we define a degree d homogeneous polynomial on three variables

$$\tilde{F}(x, y, z) = \sum_{r,s} c_{r,s} x^r y^s z^{d-r-s}.$$

One can interpret the projective curve $C_{\tilde{F}}$ as the compactification of the affine curve C_F. The points at infinity are

$$\{ [x, y, 0] \mid \sum_{0 \leq r \leq d} c_{r,d-r} x^r y^{d-r} = 0 \}.$$

To each infinity point (a_i, b_i) corresponds an asymptote line in $\mathbb{C}^2$ given by

$$a_i x - b_i y = 0.$$

The tangent line at a non-singular point is the projective line defined by the equation

$$\frac{\partial F}{\partial x}(x_0, y_0, z_0)x + \frac{\partial F}{\partial y}(x_0, y_0, z_0)y + \frac{\partial F}{\partial z}(x_0, y_0, z_0)z = 0.$$

Exercise : Prove Euler's relation: If F is homogeneous of degree d then

$$\frac{\partial F}{\partial x}(x_0, y_0, z_0)x_0 + \frac{\partial F}{\partial y}(x_0, y_0, z_0)y_0 + \frac{\partial F}{\partial z}(x_0, y_0, z_0)z_0 = dF(x_0, y_0, z_0).$$

The following Lemma relates non-singular points of a projective curve and its affine curve. It follows imediately from Euler's relation.

Lemma 6.8 $[x_0, y_0, z_0]$, *with* $z_0 \neq 0$ *is a non-singular point of a projective curve defined by* $F(x, y, z)$ *if and only if* $(x_0/z_0, y_0/z_0)$ *is a non-singular point of the affine curve defined by* $F(x, y, 1)$*. The tangent line of* $C_{F(x,y,z)}$ *at* $[x_0, y_0, z_0]$ *(restricted to* $\mathbb{C}^2 \subset \mathbb{C}P^2$*) coincides with the tangent line of* $C_{F(x,y,1)}$ *at* $(x_0/z_0, y_0/z_0)$*.*

Using the previous lemma for each affine coordinate chart of $\mathbb{C}P^2$ we conclude that a projective curve whose points are non-singular is a Riemann surface. They are called smooth projective plane curves.

Exercise: Any projective line is biholomorphic to $\mathbb{C}P^1$.

Exercise: A conic in $\mathbb{C}P^2$ is defined by a degree two homogeneous polynomial

$$F(x, y, z) = ax^2 + dy^2 + fz^2 + 2bxy + 2cxz + 2eyz$$

which can be written as $X^T A_F X$ where

$$A_F = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

and

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

1. Prove that C_F is non-singular if and only if $det A_F \neq 0$.
2. Prove that any smooth projective conic is isomorphic to $\mathbb{C}P^1$.

6.3 Algebraic sets and algebraic curves

In order to give some perspective we give in this section a very short introduction to algebraic geometry. Indeed, algebraic sets in $\mathbb{C}^n$ of any dimension are defined as follows.

Consider $A = \mathbb{C}[x_1, \cdots, x_n]$ the polynomial ring in n-variables over $\mathbb{C}$.

Definition 6.9 *An affine algebraic set defined by a subset* $T \subset A$ *is*

$$Z(T) = \{x \in \mathbb{C}^n \mid F(x) = 0 \text{ for all } F \in T \}.$$

So the empty set, any finite subset of $\mathbb{C}^n$, the whole $\mathbb{C}^n$ and affine algebraic curves are examples of algebraic sets. An hypersurface, is an algebraic set defined by one polynomial. In particular, if the polynomial is linear, the algebraic set is called an hyperplane. Again, the fact that $Z(T)$ might have different defining sets is an obvious problem. One can show that any algebraic set is a finite union of *irreducible* algebraic sets which are themselves related to prime ideals of A.

Definition 6.10 *An irreducible affine algebraic set (or algebraic variety) X is an algebraic set whose ideal*

$$I(X) = \{F \in A \mid F(x) = 0 \text{ for all } x \in X \quad \}$$

is prime.

Recall that a prime ideal $I \subset A$ is a proper ideal such that if $ab \in I$, either $a \in I$ or $b \in I$. As an example, if $F \in \mathbb{C}[x, y]$ is an irreducible polynomial, then the ideal generated bt F is prime and the complex algebraic curve is therefore an algebraic variety. We define projective algebraic varieties analogously by considering homogeneous polynomials. In principle, in $\mathbb{C}P^n$ we need $n-1$ equations but one sometimes need more equations. The best possible situation is given in the following Definition.

Definition 6.11 *A smooth complete intersection curve is the set*

$$C = \{[x] \in \mathbb{C}P^n \mid F_1(x) = \cdots = F_{n-1}(x) = 0 \ \}.$$

where F_i are homogeneous polynomials in $\mathbb{C}^{n+1}$ such that the $(n-1) \times (n+1)$ matrix

$$\left(\frac{\partial F_i}{\partial x_j}\right)$$

has maximal rank at each point in C.

As for plane curves we can prove, using the implicit function theorem, that a complete intersection is a complex submanifold. It defines therefore a compact Riemann surface.

Not all projective curves are complete intersections. But one can show that every embedding of a Riemann surface in projective space $\mathbb{C}P^n$ is a local complete intersection, meaning that it is a projective curve defined by a finite number of homogeneous polynomials which is locally defined by only $(n-1)$ polynomials satisfying the rank condition above.

6.4 All projective curves can be embedded in $\mathbb{C}P^3$

Proposition 6.12 *Any smooth projective curve can be embedded in $\mathbb{C}P^3$.*

The proof is obtained by projecting a curve embedded in $\mathbb{C}P^n$ from a linear space into a convenient $\mathbb{C}P^3$. If we want that the projection be an embedding we need to be careful. The linear space from where we should project should avoid secants and tangents.

Definition 6.13 *A complex line passing through two points of a projective curve is called a secant.*

Suppose that L is a k-plane and X a projective curve disjoint from L. The projection from L is injective along X if and only if L does not intersect any of the secants to X. Indeed, if there is an intersection, the space spanned by the lifts of the two points is the same and therefore the intersection with a complementary space is the same.

Lemma 6.14 *Let $p \in X$ be a point in a smooth projective curve and L a disjoint linear space in $\mathbb{C}P^n$ not intersecting the secants of X. The projection from L restricted to X is an embedding at p if and only if L is disjoint from the tangent line to X at p.*

Proof. We may suppose that $p = [1, 0, \cdots, 0]$ and $L = \{[0, \cdots, 0, x_{n-k}, \cdots, x_n]\}$. The projection from L is given by $[x_0, \cdots, x_n] \to [x_0, \cdots, x_{n-k-1}, 0 \cdots, 0]$. On a neighborhood of p, the smooth projective curve is given by $[1, g_1(z), \cdots, g_n(z)]$ with $g_i'(z)$ for some $1 \leq i \leq n-k-1$ if we impose that the tangent line does not intersect L. That completes the proof. □

To prove the theorem, we start with a projective curve. Define the complex manifold defined by triples of points (x, y, z) such that $x \neq y$ are points in X and z a point in the secant between x and y. It is of dimension 3 and therefore, its image by the projection $(x, y, z) \to z$ is of maximal dimension 3. We conclude that there are points in $\mathbb{C}P^n$ which are not contained in any secant. Analogously, we may conclude that the set of points contained in a tangent line are of dimension at most 2. If the projective curve is embedded into a projective space of dimension greater than or equal to 4 we obtain a point not contained in any secant or tangent line and the projection from that point embeds X in a projective space of one dimension smaller.

6.5 Intersections of projective curves: Bézout's theorem

In this section we prove a formula which counts the intersection number of two projective curves. The formula involves a definition of multiplicity and is best described using the notion of a divisor. Meromorphic functions on projective curves are obtained by taking quotients of homogeneous polynomials of the same degree.

Consider a smooth projective curve X and a non-zero homogeneous polynomial F of degree d.

Definition 6.15 *The intersection divisor of F on X, $div(F) = \sum n_p p$, is the formal sum of points $p \in X$ where $F(p) = 0$ with n_p being the order of the meromorphic function obtained from F by dividing it by a homogeneous polynomial G of the same degree which is non-vanishing at p.*

Observe that the order of the meromorphic function does not depend on the choice of the non-vanishing homogeneous polynomial G because $G(p) \neq 0$. If F is linear, we call $div(F)$ a hyperplane divisor.

In general, the degree of a divisor $D = \sum n_p p$ is $deg(D) = \sum n_p$. If F_1 and F_2 are homogeneous polynomials of the same degree than $div(F_1) - div(F_2) = div(F_1/F_2)$ which is the divisor of a meromorphic function. But the degree of a principal divisor is 0 so $deg(div(F_1)) = deg(div(F_2))$. In particular all hyperplane divisors have the same degree.

Definition 6.16 *The degree of a smooth projective curve, $deg(X)$ is the degree of a hyperplane divisor.*

Exercise: The degree of a smooth plane projective curve coincides with the degree of the irreducible polynomial defining it. Bézout's theorem computes the degree of an

intersection divisor:

Theorem 6.17 (Bézout's theorem) *Let X be a smooth curve and F a non-zero homogeneous polynomial . Then*

$$deg(div(F)) = deg(X)deg(F).$$

Proof. Let H a homogeneous polynomial of degree 1. Then $deg(div(H^{degF})) = deg(div(F))$. Now $deg(div(H^{degF})) = deg(F)deg(div(H)) = deg(F)deg(X)$. □

6.6 Algebraic curves and ramified covers

Given a smooth projective plane curve $X \subset \mathbb{C}P^2$, not containing the point $[0, 1, 0]$, defined by a homogeneous polynomial F we can define a ramified cover $\pi : X \to \mathbb{C}P^1$ by taking the projection from the point $[0, 1, 0]$, that is $\pi : [x, y, z] \to [x, z]$. We obtain the Riemann surface which as a ramified cover of $\mathbb{C}P^1$:

Proposition 6.18 *Let X be a smooth algebraic curve defined by the homogeneous polynomial F in $\mathbb{C}P^2$ missing the point $[0, 1, 0]$ and $\pi : X \to \mathbb{C}P^1$ the projection as above. Then, the ramification divisor $R_\pi \subset X$ is equal to $div(\frac{\partial F}{\partial y})$.*

7 Hyperbolic geometry

An important development was the discovery by Poincaré was that Möbius transformations preserving the disc were, in fact, isometries of the disc equipped with a metric of constant negative curvature.

7.1 Riemannian manifolds

A Riemannian manifold is a manifold equipped with a positive definite scalar product $\langle \, , \, \rangle$ defined on the tangent space at each point. Using the Riemannian metric one defines the length of curves and a metric on the manifold so that the distance between two points is the infimum of all lengths of curves joining them:

$$d(p, q) = \inf_{\gamma(0)=p, \gamma(1)=q} L(\gamma)$$

where

$$L(\gamma) = \int_0^1 \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle} \, dt$$

The group of isometries, that is, distance preserving diffeomorphisms of a metric space M, will be denoted by $Isom(M)$. Isometries are determined by their derivative at one point:

Theorem 7.1 *Let M be a connected Riemannian manifold, $\varphi : M \to M$ be an isometry with $\varphi(p) = p$. Then $\varphi_* : T_pM \to T_pM$ determines φ.*

Exercise 7.2 *Let E^n be the n-dimensional Euclidean space. Show that $Isom(E^n)$ is the group $\{x \to Ax + B\}$ where A is orthogonal.*

Exercise 7.3 *Prove the following exact sequence*

$$0 \to \mathbb{R}^n \to Isom(E^n) \to O(n) \to 1$$

Exercise 7.4 *The finite subgroups of O(2) are the cyclic group generated by a rotation and the dihedral group generated by two reflections.*

The discrete subgroups of $Isom(E^2)$ were classified in the 19th century. The classification starts writing the discrete group Γ inside the exact sequence

$$0 \to T \to \Gamma \to H \to 1$$

where T is the subgroup of translations of Γ and H is a subgroup of $O(2)$. As Γ is discrete, T is also discrete. Therefore it is either trivial or $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$. If T is trivial Γ is either finite cyclic or dihedral. If T has one generator it is one of the seven strip patterns. If T is a lattice it is one of the 17 crystallographic groups.

Example 7.5 *The triangle groups are those groups generated by reflections in three lines. If the angles are π/p, π/q and π/r for positive integers p, q, r we should have $\pi/p+\pi/q+\pi/r = \pi$ and in this case the group is discrete. That gives three possibilities for (p, q, r), that is, $(3, 3, 3)$, $(2, 3, 6)$ and $(2, 4, 4)$. The region inside the triangle is a fundamental domain for the triangle group. (reflections on the sides of the triangle of angles $2\pi/3$, $\pi/6$ and $\pi/6$ also defines a discrete group, this is the only non-obtuse triangle leading to a discrete group)*

Example 7.6 *The index two subgroup of orientation preserving isometries of a triangle group has two generators. If we denote r_1, r_2 and r_3 the reflections on the sides of the triangles, the subgroup of orientation preserving isometries is generated by $r_1 \circ r_2$ and $r_1 \circ r_3$. A fundamental domain consists of any two adjacent triangles.*

Exercise 7.7

1. *Consider the Riemannian manifold obtained by identifying the two vertical lines $\{\mathrm{Re}\, z = 1\}$ and $\{\mathrm{Re}\, z = 2\}$ on the upper half-plane via the isometry $z \to z + 1$. Prove that this manifold is complete. Hint: show that any geodesic is defined on $\mathbb{R}$ by glueing copies of the vertical band to form the complete Poincaré half-plane.*

2. *Consider the Riemannian manifold obtained by identifying the vertical lines $\{\mathrm{Re}\, z = 1\}$ and $\{\mathrm{Re}\, z = 2\}$ on the upper half-plane via the isometry $z \to 2z$. That manifold is not complete. Prove that the sequence $(1, 2^i)$ is a Cauchy sequence but it is not convergent.*

Local isometries between Riemannian spaces are very special:

Theorem 7.8 *Let $d : M \to N$ be a surjective local isometry between Riemannian manifolds. If M is complete and connected then d is a covering.*

7.2 Hyperbolic surfaces

We will start with the half-plane model and define the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{|dz|^2}{Im(z)^2}$$

Given a metric g on a Riemannian manifold we can define a volume form dv by imposing $dv(X_1, \cdots, X_n) = 1$ for an orthonormal basis. In local coordinates we have $dv = \sqrt{det(g_{ij})}dx^1 \cdots dx^n$. For hyperbolic geometry we get

$$dv = \frac{1}{y^2}dxdy.$$

Theorem 7.9 *$PSL(2,\mathbb{R}) \subset Isom(H^1_{\mathbb{C}})$.*

Proof. We need to show that

$$\frac{|d\gamma(z)|^2}{(Im\gamma(z))^2} = \frac{|dz|^2}{(Imz)^2}.$$

This follows from a simple computation. □

Theorem 7.10 *The geodesics of $H^1_{\mathbb{C}}$ are vertical lines or circles perpendicular to the $\mathbb{R}$-axis.*

Proof. We first observe that given two points with the same x-coordinate, $p = (x, y_1)$ and $q = (x, y_2)$ (without loss of generality we suppose $y_2 > y_1$), then

$$d(p,q) = \inf \int \frac{\sqrt{dx^2 + dy^2}}{y}.$$

But $\int \frac{\sqrt{dy^2}}{y} \leq \int \frac{\sqrt{dx^2+dy^2}}{y}$. As $\int \frac{\sqrt{dy^2}}{y} \geq \ln(y_2/y_1)$ we conclude that

$$d(p,q) = \ln(y_2/y_1).$$

We use now the fact that geodesics are preserved by isometries and that vertical lines are transformed to circles orthogonal to the real axis or to vertical lines by $PSL(2,\mathbb{R})$. □

In the following we will call a hyperbolic triangle a simplex in $H^1_{\mathbb{C}}$ whose boundary is formed by three geodesic segments.

Theorem 7.11 *Let Δ be an hyperbolic triangle with angles α, β and γ. Then*

$$Area(\Delta) = \pi - \alpha - \beta - \gamma$$

Proof. Suppose first that the triangle has an ideal point, that is, one of the angles is null, or, equivalently, one of its vertices is in the boundary of $H^1_{\mathbb{C}}$. Without loss of generality we might suppose that the vertex is ∞ and one of the geodesics is the half circle of radius one centered at the origin. The other two are vertical lines which form angle α and β with the circle. then

$$Area(\Delta) = \int\int \frac{dxdy}{y^2} = \int_a^b dx \int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2} = \int_a^b \frac{dx}{\sqrt{1-x^2}}.$$

By a change of coordinate $x = \cos\theta$ we get

$$\int_{\pi-\alpha}^{\beta} \frac{-\sin\theta}{\sin\theta} d\theta = \pi - \alpha - \beta$$

.

If the triangle Δ_1 is compact we choose one of the vertices (say the one with angle γ) and prolong one of the sides containing it up to the boundary of $H^1_{\mathbb{C}}$. We have three triangles one (containing an ideal point) being the union of the other two. Comparing their areas:

$$\Delta_1 = \Delta_1 + \Delta_2 - \Delta_2$$

$$A(\Delta_1) = \pi - \alpha - (\beta - \theta) - (\pi - (\pi - \gamma) - \theta) = \pi - \alpha - \beta - \gamma.$$

□

Decomposing a polygon in triangles we obtain the following

Corollary 7.12 *For a geodesic polygon with n sides denote by α the sum of the internal angles. Then*

$$A = n\pi - 2\pi - \alpha.$$

Using a geodesic triangulation one can prove Gauss-Bonnet theorem:

Theorem 7.13 *If S_g is a hyperbolic surface, then*

$$A = -2\pi\chi.$$

Proof. We have $A = \sum(\pi - \alpha_i - \beta_i - \gamma_i)$ summing over all triangles, say F of them. The angles sum to 2π times the number of vertices, say V. Therefore $A = \pi(F - 2V)$. On the other hand the number of edges is precisely $E = 3F/2$. We conclude that $\chi = F - E + V = F - 3F/2 + V = -F/2 + V = A/(-2\pi)$ □

The Poincaré metric on the disc is given by

$$ds^2 = \frac{4|dz|^2}{(1 - |z|^2)^2}.$$

The geodesics of the hyperbolic disc are described in the following

Proposition 7.14 *The geodesics of the hyperbolic disc are sub-arcs of circles orthogonal to the boundary of the disc.*

Proposition 7.15 *Let S be a Riemann surface with a Fuchsian model $H^1_{\mathbb{C}}/\Gamma$. If γ is a hyperbolic element and L_γ the geodesic obtained by projection of its axis then*

$$|tr(\gamma)| = 2\cosh\left(\frac{l(L_\gamma))}{2}\right)$$

where $l(L_\gamma)$ is the length of the geodesic.

Proof. We may assume that $\gamma(z) = \lambda^2 z$. Then $l(L_\gamma) = \int_1^{\lambda^2} \frac{dy}{y} = \ln \lambda^2$. □

7.3 Poincaré's theorem

Poincaré's theorem is an efficient method to prove that a given set of transformations of $H^1_{\mathbb{C}}$ generates a discrete group and to determine the topology of the quotient.

Consider a domain $P \subset H^1_{\mathbb{C}}$ whose boundary is a finite union of geodesics segments c_i (called sides). Suppose that the sides are paired. That is, for each c_i there exists another side c_i' and an isometry (called a side-pairing) γ_i such that $c_i' = \gamma_i c_i$ and $c_i = \gamma_i^{-1} c_i'$ (the side might be paired to itself). We will supoose the side pairings reverse the orientation of the segments. For simplicity we may orient the boundary in the direct sense and define for each vertex of v_0 the image $v_1 = \gamma v_0$ where γ is the side-pairing associated to the side starting at v_0. The vertices of the polygonal boundary are then partioned into cycles. We define the dyhedral angle θ_v at a vertex v to be the positive internal angle betwwen the sides meeting at v.

Theorem 7.16 *Suppose P is a domain with geodesic sides $\{c_i\}$ and side pairings γ_i. Suppose that for each cycle $\mathcal{C}$,*

$$\sum_{v_i \in \mathcal{C}} \theta_{v_i} = 2\pi$$

where the sum is over all vertices of the cycle. Then, the group Γ generated by the side-pairings is discrete and the quotient $H^1_{\mathbb{C}}/\Gamma$ is a Riemann surface. For each cycle $\mathcal{C}_k$, $1 \leq k \leq N$, let $\gamma_1^k, \cdots, \gamma_{n_k}^k$ be the sequence of side pairings such that $\gamma_1^k v_0^k = v_1^k, \cdots, v_0^k = \gamma_{n_k}^k v_n^k$. A presentation of Γ is then given by

$$\langle \gamma_i^k \ (1 \leq k \leq N) \mid \gamma_1^k \cdots \gamma_{n_k}^k = 1 \ \ \textit{for } 1 \leq k \leq N \ \rangle.$$

Example: Consider the n-roots of unity in S^1. Take the geodesics (circle segments perpendicular to the boundary) centred at each of these roots with the same radii. If the radius is near 0 we get n disjoint circle segments. On the opposite case, if the radius approaches 1, then ,near the origin, we obtain a region which is nearly a regular euclidean polygon. The angle at a vertex, therefore, varies from $\pi - 2\pi/n$ (the almost euclidean regular polygon) to 0 (the ideal regular polygon). Clearly, the angle is a continuous function and there exists a radius such that the angle between the circles will be

$$\theta = \frac{2\pi}{n}.$$

In that case we can apply Poincaré's theorem to side pairings as in the canonical polygon defining a surface of genus $g \geq 2$. We obtain a Riemann surface of genus g as a quotient of the disc by the discrete subgroup generated by the side-pairings.

Remarks:

1. To obtain non-compact Riemann surfaces with finite volume we may allow certain vertices in the boundary. The angles at these vertices are 0. We need a further hypothesis: the cycle map $\gamma_1 \cdots \gamma_n$ defined as before starting with an ideal vertex should be parabolic. One can prove that this is equivalent to suppose that the space $P/\equiv$, obtained by identifying the sides with the pairings, is a complete space. In

the previous example, if the geodesic segments touch at infinity we obtain a Riemann surfce of genus g with one puncture.

2. To obtain subgroups with torsion elements we impose that the cycle satisfies

$$\sum_{v_i \in \mathcal{C}} \theta_{v_i} = 2\pi/r$$

Then, the group Γ generated by the side-pairings (in that case we allow a side to be paired to itself) is discrete with a presentation given by

$$\langle \gamma_i^k \ (1 \leq k \leq N) \mid (\gamma_1^k \cdots \gamma_{n_k}^k)^{r_k} = 1 \ \text{ for } 1 \leq k \leq N \ \rangle.$$

We might also suppose that the side-pairings are not holomorphic (isometries which don't preserve the orientation). In that case the quotient is not a Riemann surface but there will exist a subgroup of finite index which does not have torsion elements whose quotient is a Riemann surface. The simplest examples of discrete groups obtained using that version of Poincaré's theorem are the triangle groups. We consider a geodesic triangle with angles $\pi/p, \pi/q, \pi/r$, with positive integers p, q, r, at the three vertices. The necessary and sufficient condition for the existence of the triangle is that $\pi/p + \pi/q + \pi/r < \pi$. Granted that condition, the subgroup generated by reflections on each side is discrete and has a presentation of the form

$$\langle r_1, r_2, r_3 \mid r_i^2 = (r_1 \circ r_2)^p = (r_2 \circ r_3)^q = (r_3 \circ r_1)^r = 1 \ \rangle.$$

3. To obtain surfaces which are not of finite volume we allow the polygon to have sides on the boundary. There is no side-pairing between them. A vertex which is in a boundary side is paired to another vertex of the same type by a loxodromic element (it is a side pairing of the corresponding sides in the interior of hyperbolic space). The simplest case is that of Schottky groups. The interior sides of the polygon are given by an even number of non-intersecting geodesics.

8 Teichmüller space and Moduli space

From the theory of covering spaces we obtain the following

Proposition 8.1 *Let $f : M \to M'$ be a continuous function. Then there exists a continuous function $\tilde{f} : \tilde{M} \to \tilde{M}'$, lift of f to the universal covers. It is unique up to a choice of an origin on a fiber of $\tilde{M}$ and a corresponding point on the fiber of $\tilde{M}'$.*

Clearly, if f is a homeomorphism the lift will be a homeomorphism. Also, if the manifolds are Riemannian and f is an isometry, the lift will be an isometry. Suppose now that $M = X/\Gamma$ and $M' = X/\Gamma'$ are Riemannian. Then a lift $\tilde{f}$ is an isometry $g : X \to X$ which is "equivariant" with respect to the action of Γ and Γ'. That is, by unicity of the lift and transitivity of the group of covering transformations, for any $\gamma \in \Gamma$ there exists $\gamma' \in \Gamma'$ such that

$$g\gamma x = \gamma' g x$$

valid for every $x \in X$, we conclude that

$$\Gamma' = g\Gamma g^{-1}.$$

We proved the following

Proposition 8.2 *If $M = X/\Gamma$ and $M' = X/\Gamma'$ are isometric then $\Gamma' = g\Gamma g^{-1}$.*

Of course, if M and M' are homeomorphic we conclude that $\Gamma' = f \circ \Gamma \circ f^{-1}$, where $f : X \to X$ is a homeomorphism. We will suppose $X = G/K$ is a homogeneous space with compact isotropy K. That is always the case if X is Riemannian with a transitive group of isometries.

8.1 The representation space $Hom(\Gamma, G)$

Assume Γ is finitely generated and G a Lie group. Two actions on $Hom(\Gamma, G)$ are essential

$$Aut(G) \times Hom(\Gamma, G) \to Hom(\Gamma, G)$$

given by

$$(\varphi, \rho) \to \varphi \circ \rho$$

and

$$Aut(\Gamma) \times Hom(\Gamma, G) \to Hom(\Gamma, G)$$

given by

$$(\varphi, \rho) \to \rho \circ \varphi^{-1}.$$

Definition 8.3 *The Teichmüller space $\mathbf{T}(\Gamma)$ in G is the set $Hom(\Gamma, G)/Aut(G)$.*

Remark: Sometimes one defines $R(\Gamma, G) = Hom(\Gamma, G)/G^0$, where G^0 is the connected component of G. If G is an algebraic group, a choice of presentation of Γ leads to a description of $Hom(\Gamma, G)$ as an algebraic variety. In order to obtain a structure of algebraic variety we consider the Mumford quotient which is written $\mathbb{R}(\Gamma, G) = Hom(\Gamma, G)//G^0$. This is called the *character variety*. The idea is to consider the subring of the ring defining $Hom(\Gamma, G)$ which consist of invariant functions.

We consider the action

$$Aut(\Gamma) \times \mathbf{T}(\Gamma) \to \mathbf{T}(\Gamma)$$

and observe that it is trivial when restricted to the inner automorphisms of Γ. We have therefore the action

$$Out(\Gamma) \times \mathbf{T}(\Gamma) \to \mathbf{T}(\Gamma)$$

where $Out(\Gamma) = Aut(\Gamma)/Inn(\Gamma)$.

Definition 8.4 *The Moduli space $\mathbf{M}(\Gamma)$ in G is the set $\mathbf{T}(\Gamma)/Out(\Gamma)$.*

We will also define $R(\Gamma, G) \subset Hom(\Gamma, G)$ as those representations which are faithful and discrete.

An equivalent description of the moduli space relating to the uniformization problem runs as follows

Definition 8.5 *The moduli space $\mathbf{M}$ of Γ in G is the set $Hom(\Gamma, G)/\equiv$, where we identify two representations if the image groups are conjugate.*

As observed above, a practical description is obtained if we mark generators for Γ. Suppose $\langle \gamma_i \rangle$ is a family of such generators. A representation is determined by the values $\pi(\gamma_i)$.

Definition 8.6 *The Teichmüller space* $\mathbf{T}$ *of the marked* Γ *in* G *is the set* $Hom(\Gamma, G)/\equiv$, *where we identify two representations if* $\pi(\gamma_i) = g\pi'(\gamma_i)g^{-1}$.

Clearly there is a canonical projection $\mathbf{T} \to \mathbf{M}$. The problem is that even if $\pi'(\Gamma) = g\pi(\Gamma)g^{-1}$ globally, it might not be true that $\pi'(\gamma_i) = g\pi(\gamma_i)g^{-1}$. It could happen that $g\pi(\gamma_i)g^{-1} = \pi'(\beta_i) \neq \pi'(\gamma_i)$. But if that is the case β_i would be new generators of Γ satisfying the same relations. This is possible if an element $\varphi \in Aut(\Gamma)$ satisfies $\varphi(\gamma_i) = \beta_i$.

8.2 Tori

8.2.1 Teichmüller space

A complex torus is covered by the complex plane $\mathbb{C}$. The Teichmüller space of representations of $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$ in $Aut(\mathbb{C})$ is obtained by identifying generators (ω_1, ω_2) and (ω_1', ω_2') when they differ by a conjugation with an element of $\widehat{Aut}(\mathbb{C})$ (those are the automorphisms of the group of automorphisms of $\mathbb{C}$). That is $\omega_i' = \lambda\omega_i$ or $\omega_i' = \bar{\omega}_i$. We can therefore find an element of the form

$$(1, \tau)$$

with $\mathrm{Im}\,(\tau) > 0$. Therefore

Theorem 8.7 $\mathbf{T}(\Gamma, Aut(\mathbb{C}))$ *is identified with the upper half plane.*

8.2.2 Moduli space

We need to determine $Aut(\Gamma) = Aut(\mathbb{Z} \oplus \mathbb{Z})$. Fixing generators γ_1 and γ_2 for Γ, new generators are determined clearly by a matrix in $SL(2, \mathbb{Z})$. To find the action $SL(2, \mathbb{Z}) \times \mathbf{T}(\Gamma) \to \mathbf{T}(\Gamma)$ we act

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}),$$

where a, b, c, d are integers with $ad - bc = 1$, on an element $(1, \tau)$ and obtain then

Theorem 8.8 *Two tori determined by* τ *and* τ' *are biholomorphic if and only if*

$$\tau' = \frac{a\tau + b}{c\tau + d}.$$

So obtain,

Corollary 8.9 $\mathbf{M}(\Gamma, Aut(\mathbb{C})) = H/PSL(2, \mathbb{Z})$.

9 Vector bundles

Let X be a topological space and $\pi : E \to X$ a (complex) vector bundle over X. By this we mean

1. a locally trivial bundle in the sense that for each $x \in X$ there exists an open neighborhood $U_x \subset X$ and a homeomorphism $h_U : \pi^{-1}(U) \to U \times \mathbb{C}^n$ such that $p_U \circ h_U(e) = \pi(e)$, where $p_U : U \times \mathbb{C}^n \to U$ is the projection in the first factor. We call h_U a trivialization of E over U.

2. For each $x \in X$ the fiber $\pi^{-1}(x)$ is a vector space and for any trivialization h_U : $\pi^{-1}(U) \to U \times \mathbb{C}^n$, the map $p'_U \circ h_U|_{\pi^{-1}(x)} \to \mathbb{C}^n$, where $p'_U : U \times \mathbb{C}^n \to \mathbb{C}^n$ (the projection on the second factor), is an isomorphism.

We say that the vector bundle is C^∞ (holomorphic) if all the manifolds and maps are C^∞ (holomorphic). The dimension of the fibers is called the rank of the vector bundle and in the case the dimension is one the vector bundle is said to be a line bundle. Morphisms between vector bundles E and F are maps $\varphi : E \to F$ which map linearly fibers into fibers. An isomorphism is a morphism which is a diffeomorphism whose restriction to each fiber is an isomorphism between vector spaces. A trivial vector bundle over X is a vector bundle isomorphic to $X \times \mathbb{C}^n$. The definition of real vector bundles is the same with $\mathbb{R}$ substituted for $\mathbb{C}$.

We usually work with vector bundles over a fixed base space X. We restrict then the category of vector bundles so that a morphism $\varphi : E \to F$ satisfies $\pi_F \circ \varphi = \pi_E$, where π_E and π_F are the projections.

9.1 Transition Cocycles

Given a vector bundle $\pi : E \to X$ over X and trivialisations $h_{U_i} : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n$ ($\mathbb{R}^n$ in the case of real vector bundles) defined over a covering $X = \bigcup U_i$ one can define the maps

$$h_{U_i} \circ h_{U_j}^{-1} : (U_i \cap U_j) \times \mathbb{C}^n \to (U_i \cap U_j) \times \mathbb{C}^n$$

which have the form

$$h_{U_i} \circ h_{U_j}^{-1}(x, v) = (x, g_{ij}(x)v)$$

where $g_{ij} : U_i \cap U_j \to GL(n, \mathbb{C})$, $GL(n, \mathbb{R})$ in the case of real vector bundles, are called transition functions of the vector bundle associated to the covering $\bigcup U_i$. If the vector bundle is C^∞ (holomorphic) then the transition functions are C^∞ (holomorphic).

The transition functions satisfy a cocycle condition, namely, on $U_i \cap U_j \cap U_k$ we have

$$g_{ij} g_{jk} = g_{ik}.$$

Conversely, given a family of transition functions satisfying the cocycle condition one can construct a vector bundle. To see that, we construct the disjoint union $\tilde{E} = \bigcup_i U_i \times \mathbb{C}^n$ with projection $\tilde{\pi}(x, v)_i = x$ where $(x, v)_i \in U_i \times \mathbb{C}^n$. To obtain the vector bundle we quotient this space by the equivalence relation $(x, v)_i \equiv (y, w)_j$ if and only if $x = y$ and $v = g_{ij} w$.

Let $\varphi : E \to F$ be an isomorphism whose projection on the base is the identity. Let $\bigcup U_i$ be a covering and chose vector bundle trivializations h_U and h'_U on the source and on the target, respectively. We can write then

$$h'_{U_i} \circ \varphi \circ h_{U_i}^{-1}(x, v) = (x, \varphi_i(x)v)$$

where $\varphi_i(x) \in GL(n, \mathbb{C})$. On $U_i \cap U_j$ we obtain on one hand

$$h'_{U_j} \circ h'^{-1}_{U_i}(x, v) = (x, g'_{ji} v)$$

and, on the other hand,

$$h'_{U_j} \circ h'^{-1}_{U_i}(x, v) = h'_{U_j} \circ \varphi \circ \varphi^{-1} \circ h'^{-1}_{U_i}(x, v) = h'_{U_j} \circ \varphi \circ h^{-1}_{U_j} \circ h_{U_j} \circ h^{-1}_{U_i} \circ h_{U_i} \circ \varphi^{-1} \circ h'^{-1}_{U_i}(x, v)$$

That is

$$g'_{ji} = \varphi_j g_{ji} \varphi_i^{-1}.$$

In particular, if φ is the identity we obtain the description of all possible transition functions over a fixed cover.

Example 9.1 *The tangent space of a surface:*

Let (U_i, φ_i) be an atlas of a surface X. We define transition cocycles as the Jacobian matrices $g_{ij} : U_i \cap U_j \to GL(2, \mathbb{R})$ defined by

$$g_{ij}(x) = D(\varphi_i \circ \varphi_j^{-1})(x).$$

The corresponding vector bundle associated to these transition functions is called the vector bundle of X and is denoted by TX.

Example 9.2 *The holomorphic tangent space of a surface:*

If the surface has a complex structure we can use a holomorphic atlas (U_i, z_i) to define the rank one complex vector space, called holomorphic tangent bundle, with transition functions

$$g_{ij} = \frac{\partial z_i}{\partial z_j}.$$

Example 9.3 *The cotangent space of a surface:*

Let (U_i, φ_i) be an atlas of a surface X. We define transition cocycles by taking the transpose of the Jacobian matrix

$$g^*_{ij}(x) = D(\varphi_i \circ \varphi_j^{-1})^*(x).$$

The corresponding vector bundle is denoted by T^*X.

Example 9.4 *The canonical line bundle K over a surface or holomorphic cotangent space:*

From the holomorphic atlas (U_i, z_i) we define the rank one complex vector space, called holomorphic cotangent bundle, with transition functions

$$g_{ij} = \frac{\partial z_j}{\partial z_i}.$$

Example 9.5 *Line bundles over $\mathbb{C}P^1$*

The line bundles over $\mathbb{C}P^1$ can be described by the transition cocycles defined over the covering $U_0 = \mathbb{C}$ and $U_1 = \mathbb{C}^* \cup \{\infty\}$. We let $g_{01} = z^n$ and denote by $\mathcal{O}(n)$ the corresponding holomorphic line bundle. Observe that from

$$dz_1 = \frac{-1}{z_0^2} dz_0,$$

the canonical line bundle over $\mathbb{C}P^1$ is identified to $\mathcal{O}(-2)$.

Remark : All vector bundles over $\mathbb{C}P^1$ can be obtained using those building blocks. In fact, a theorem of Birkhoff and Gothendieck shows that any holomorphic vector bundle of rank n over $\mathbb{C}P^1$ is isomorphic to $\mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_k)$, for some $n_i \in \mathbb{Z}$.

Suppose X is a fixed Riemann surface and consider the space of line bundles over X with isomorphism between bundles defined as isomorphism of vector bundles whose induced map on the base is the identity. Given two line bundles L_1 and L_2 one can form their product $L \otimes L'$ by defining the transition functions $g_{ij} g'_{ij}$. The inverse of a line bundle L with transition functions g_{ij} is denoted by L^* it has transition functions g_{ij}^{-1}. This product is clearly commutative and defines a group structure on the space of line bundles modulo isomorphisms.

Definition 9.6 *The Picard group $Pic(X)$ associated to a Riemann surface is the abelian group of all holomorphic line bundles modulo isomorphisms which induce the identity on the surface.*

9.2 Sections of vector bundles

Definition 9.7 *A section of a vector bundle $\pi : E \to X$ is a map $s : X \to E$ such that $\pi \circ s(x) = x$. If the vector bundle is C^∞ (holomorphic) then we can consider C^∞ (holomorphic) sections.*

In local trivialisations over a covering $\bigcup U_i$, a section is given by functions $f_i : U_i \to \mathbb{C}^n$ satisfying the compatibility condition

$$f_i(x) = g_{ij} f_j(x)$$

on $U_i \cap U_j$ where g_{ij} are the transition functions.

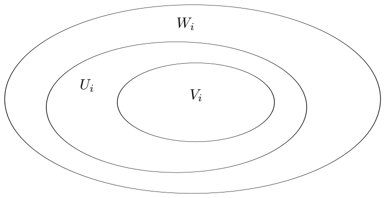
The space of sections of a vector bundle is a vector space.

Definition 9.8 *The space of holomorphic sections of a holomorphic vector bundle E over a complex manifold M is denoted by $H^0(M, E)$. The space of C^∞ sections will be denoted by $C^\infty(M, E)$.*

A simple but very important theorem is that the space of holomorphic sections of a vector bundle over a compact manifold is finite dimensional. We prove the theorem in the case of line bundles over a Riemann surface. The general proof is similar.

Theorem 9.9 *For any holomorphic line bundle $\pi : L \to X$ over a compact Riemann surface X, $H^0(X, L)$ is finite dimensional.*

Proof. Choose a finite covering $\mathcal{U}$ of X by open sets U_i satisfying the following conditions:



1. There exists charts $\varphi_i : U_i \to \Delta(1)$.
2. $\mathcal{V}$ defined by $V_i = {\varphi_i}^{-1}(\Delta(1/2))$ is a covering.
3. For each i, $U_i \subset W_i$, an open set such that $p_i : \pi^{-1}(W_i) \to W_i \times \mathbb{C}$ is a trivialization with transition functions g_{ij}.

Given a $s \in H^0(X, L)$, that is a section of L we define its norm with respect to a covering $\mathcal{U}$ subordinated to the trivialization defined by $\mathcal{W}$ as

$$||s||^{\mathcal{U}} = \max_i \ sup_{z \in U_i} |s_i(z)|$$

where $s_i(z)$ is defined by $p_i(s(z)) = (z, s_i(z))$, that is the expression of $s(z)$ in the coordinates defined by the trivialization. Analogously

$$||s||^{\mathcal{V}} = max_i \ sup_{z \in V_i} |s_i(z)|.$$

Let $z_i^0 \in V_i$ be such that $\varphi_i(z_i^0) = 0$. The idea of the proof is that if a holomorphic section vanishes with sufficiently high order at z_i^0 for all i than it vanishes everywhere. This clearly proves the theorem because over each V_i the holomorphic sections modulo the ones vanishing with order k at z_i^0 is a finite dimensional vector space (given by the coefficients of the series expansion to order $k-1$ at z_i^0).

There are two relevant inequalities:

- The first one is local. For each $V_i \subset U_i$ and s a section over W_i vanishing to order k we have for $w \in V_i$

$$|s_i(w)| \leq sup_{z \in V_i} |z^k \frac{s_i(z)}{z^k}| \leq \frac{1}{2^k} sup_{z \in U_i} |\frac{s_i(z)}{z^k}|$$
$$= \frac{1}{2^k} sup_{z \in \partial U_i} |\frac{s_i(z)}{z^k}| = \frac{1}{2^k} sup_{z \in \partial U_i} |s_i(z)| = \frac{1}{2^k} sup_{z \in U_i} |s_i(z)|$$

  where we use the maximum principle in the last inequalities. So

$$sup_{z \in V_i} |s_i(z)| \leq \frac{1}{2^k} sup_{z \in U_i} |s_i(z)|$$

- The other inequality is where compactness comes into play. In fact, although $\mathcal{U}$ is a bigger covering there exists a constant $C > 0$ such that

$$||s||^{\mathcal{U}} \leq C||s||^{\mathcal{V}}.$$

  To prove this, take a point $z_0 \in U_i$ realizing $||s||^{\mathcal{U}}$. Then $z_0 \in V_j$ for some j. and then $s_i(z_0) = g_{ij} s_j(z_0)$ so writing $C = max_{i,j} \ max_{z \in U_i \cap U_j} |g_{ij}(z)|$ we obtain

$$||s||^{\mathcal{U}} = |s_i(z_0)| \leq C|s_j(z_0)| \leq C||s||^{\mathcal{V}}$$

  as we wished.

From the two inequalities we obtain

$$||s||^{\mathcal{U}} \leq C||s||^{\mathcal{V}} \leq \frac{C}{2^k}||s||^{\mathcal{U}}$$

which implies that s vanishes for k large enough. □

9.2.1 Meromorphic sections

Let X be a Riemann surface and $\pi : E \to X$ a holomorphic vector bundle.

Definition 9.10 *A meromorphic section of E is a holomorphic section $s : X \setminus D \to E$ where $D \subset X$ is a discrete set such that for each $p \in D$ there exists a chart $z : U \to \mathbb{C}$ with $U \cap D = \{p\}$ satisfying*

- *$z(p) = 0$,*
- *there exists $k \geq 0$ such that $z^k(x)s(x)$ is the restriction of a holomorphic section over U to $U \setminus \{p\}$.*

We set $-ord_p(s)$ to be the minimum k as above and call it the order of the meromorphic section at p.

In local coordinates this means that the n-tuple of functions f_i defined over each U_i are meromorphic. Meromorphic functions are simply meromorphic sections of the trivial holomorphic bundle $X \times \mathbb{C}$.

If a meromorphic section $s : X \to E$ vanishes at $p \in X$, consider a chart $z : U \to \mathbb{C}$ vanishing at p as above. De define $ord_p(s) = k$ to be the positive integer such that $s = z^k g$ where g is an n-tuple of holomorphic functions over U with $g(p) \neq 0$. Those definitions are clearly independent on the chosen charts.

9.2.2 Operators

Let $E_1 \to X_1$ and $E_2 \to X_2$ be two vector bundles.

Definition 9.11 *A linear map $P : C^\infty(X_1, E_1) \to C^\infty(X_2, E_2)$ is called an operator.*

The most important example is the following. We fix a Riemann surface X. If $E \to X$ is a holomorphic vector space we define the Cauchy-Riemann map

$$\bar{\partial}_E : C^\infty(X, E) \to C^\infty(X, E \otimes K)$$

by fixing a trivialization e_i over a neighborhood U and writing any section over U as $s(z) = \sum f_i(z)e_i$. Define

$$\bar{\partial}_E(s) = \bar{\partial}_E(\sum f_i(z)e_i) = \sum \bar{\partial}(f_i(z))e_i,$$

where, in local coordinates of X, $\bar{\partial}(f) = \frac{\partial f}{\partial \bar{z}} d\bar{z}$. By chosing another trivialization so that $e'_j = g_{ji}e_i$ with g_{ji} holomorphic, we observe the the definition does not depend on the trivialization.

9.3 Divisors and line bundles

In this section we describe holomorphic line bundles by divisors.

9.3.1 Divisors on Riemann surfaces

Definition 9.12 *Let X be a Riemann surface. A divisor on X is a locally finite linear combination*

$$D = \sum s_i z_i$$

where $s_i \in \mathbb{Z}$ and $z_i \in X$.

Locally finite meaning that each point in the Riemann surface has a neighborhood intersecting only a finite number of points z_i. Another way of saying it is that the set of points $\{z_i\}$ is discrete and closed in X.

The set of divisors $Div(X)$ on a fixed Riemann surface forms an abelian group generated by its points.

The divisor is said to be effective if $s_i \geq 0$ (we write $D \geq 0$). This defines a partial order by writing $D_1 \geq D_2$ if $D_1 - D_2 \geq 0$.

To a meromorphic section s of a holomorphic vector bundle over a Riemann surface one can associate the divisor

$$div(s) = \sum ord_p(s)p$$

where the sum is over all zeros and poles of the section. In particular if s is holomorphic the divisor is effective.

Definition 9.13 *Two divisors are linearly equivalent if their difference is the divisor of a meromorphic function.*

The divisor of a meromorphic function f is called a principal divisor and is denoted by (f).

Definition 9.14 *The degree of a divisor $D = \sum s_i z_i$ is $degD = \sum s_i$.*

The degree defines a homomorphism $deg : Div(X) \to \mathbb{Z}$. In the next paragraph we show that from a divisor we obtain a line bundle.

9.3.2 Linebundles from divisors

Proposition 9.15 *Given a divisor $D = \sum s_i p_i$, $1 \leq i \leq n$ over a Riemann surface X one can associate a line bundle $L(D)$ and a meromorphic section s such that $div(s) = D$.*

Proof. Chose charts $z_i : U_i \to \mathbb{C}$ for $1 \leq i \leq n$ such that $p_i \in U_i$, $z_i(p_i) = 0$, and $U_i \cap U_j = \emptyset$ for for $i > 0$, $i \neq j$. Let $U_0 = X \setminus \{p_1, \cdots p_n\}$. Let $f_i(z) = z_i^{s_i}(z)$, for $z \in U_i$, $1 \leq i \leq n$ and $f_0(z) = 1$ for $z \in U_0$. We define transition functions by $g_{ij} = f_i/f_j$ which are clearly holomorphic in the intersections $U_i \cap U_j$. They also satisfy the cocycle condition and therefore define a holomorphic bundle which we denote $L(D)$. Moreover, the functions f_i defined on U_i match up to form a global meromorphic section f, as $f_i = g_{ij} f_j$ on the intersections. Observe then that $div(s) = D$. One can check that using a different choice of charts one gets an isomorphic line bundle.

□

Example 9.16 *Let $D = n0$ be the divisor defined on $\mathbb{C}P^1$ with support on the point $0 \in \mathbb{C} \subset \mathbb{C}P^1$. Given the covering $U_0 = \mathbb{C}$ and $U_1 = \mathbb{C}^* \cup \{\infty\}$ (we changed notations with respect to the proposition above) consider the meromorphic functions z^n on U_0 and 1 on U_1. We obtain transition functions*

$$g_{01} = \frac{z^n}{1} = z^n.$$

We conclude that $L(n0) = \mathcal{O}(n)$.

Observe that if s_1 and s_2 are two meromorphic sections of a line bundle over X, there exists a meromorphic function f defined on X such that $s_2 = fs_1$. Therefore the divisors defined by them differ by a principal divisor. That gives the motivation for the next proposition.

Proposition 9.17 *Two divisors are linearly equivalent if and only if their associated line bundles are isomorphic.*

Proof. Let D and D' be two divisors. If $L(D)$ is isomorphic to $L(D')$ then there exists $\varphi : L(D) \to L(D')$ and therefore $\varphi(s_D) = fs_{D'}$ for a meromorphic function f. As $div(\varphi(s_D)) = div(s_D))$ we obtain that $div(D) - div(D') = (f)$.

Conversely, suppose $D - D' = (f)$ for a meromorphic function f. Consider the line bundles $L(D)$ and $L(D')$ with meromorphic sections s_D and $s_{D'}$ constructed as above. Define the biholomorphic map

$$F : L(D) \setminus \pi^{-1}(D \cup D') \to L(D') \setminus \pi'^{-1}(D \cup D')$$

by $F(cs_D) = cfs_{D'}$. This is well defined because $div(f) = D - D'$. By the same reason, we can extend F to an isomorphism between the line bundles.

□

In order to show that the correspondence between classes of divisors and classes of line bundles is one to one we need to show that there exists a non-trivial meromorphic section of a given line bundle.

10 Differential calculus on a surface

10.1 Exterior differentiation

Recall the exterior differentiation of a 0-form f defined on a surface is, in local coordinates (x_1, x_2), given

$$df = \frac{\partial f}{\partial \tilde{x}_1} dx_1 + \frac{\partial f}{\partial \tilde{x}_2} dx_2.$$

For a 1-form $\alpha = \varphi_1 dx_1 + \varphi_2 dx_2$ it is

$$d\alpha = (\frac{\partial \varphi_2}{\partial x_1} - \frac{\partial \varphi_1}{\partial x_2}) dx_1 \wedge dx_2.$$

Let $\alpha = \varphi_1 dx_1 + \varphi_2 dx_2$ be a 1-form with φ_1 complex functions. Writting $dz = dx + idy$ and $d\bar{z} = dx - idy$ one can write

$$\alpha = adz + bd\bar{z}.$$

$T^{*1,0}$ are the forms which can be written as adz in one chart and therefore in all charts. We call them forms of type $(1,0)$. Analogously the forms of type $(0,1)$ are written as $ad\bar{z}$. $T^{*1,0}$ is a holomorphic line bundle over X which coincides with the canonical bundle K. We write $\mathcal{E}^{1,0}(U)$ the space of 1-forms on $U \subset X$ of type $(1,0)$.

10.2 Integration

Given a differential form α on a surface X and a piece-wise smooth curve $c : [0,1] \to X$ we define the integral

$$\int_c \alpha$$

using local charts $\varphi : U \to \mathbb{C}$ with coordinates (x,y). That is, suppose $Im(c) \subset U$ and $\alpha = \varphi_1 dx + \varphi_2 dy$ then

$$\int_c \alpha = \int (\varphi_1 \dot{x} + \varphi_2 \dot{y})\, dt.$$

If $Im(c)$ is not contained in a single coordinate chart we use a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$ so that each $c([t_i, t_{i+1}])$ is contained in a coordinate chart. Clearly this definition does not depend on the chart because if $(\tilde{x},\tilde{y})$ are different coordinates then $\alpha = \varphi_1 dx + \varphi_2 dy = \left(\varphi_1 \frac{\partial x}{\partial \tilde{x}} + \varphi_2 \frac{\partial y}{\partial \tilde{x}}\right) d\tilde{x} + \left(\varphi_1 \frac{\partial x}{\partial \tilde{y}} + \varphi_2 \frac{\partial y}{\partial \tilde{y}}\right) d\tilde{y}$ and therefore by the chain rule

$$\varphi_1 \dot{x} + \varphi_2 \dot{y} = \varphi_1 dx + \varphi_2 dy = \left(\varphi_1 \frac{\partial x}{\partial \tilde{x}} + \varphi_2 \frac{\partial y}{\partial \tilde{x}}\right) \dot{d\tilde{x}} + \left(\varphi_1 \frac{\partial x}{\partial \tilde{y}} + \varphi_2 \frac{\partial y}{\partial \tilde{y}}\right) \dot{d\tilde{y}}$$

and the integrals are the same.

Proposition 10.1 *Let α be a closed form and c, c' be homotopic curves between two points x_0, x_1 on a surface. Then $\int_c \alpha = \int_{c'} \alpha$.*

Proof. By Stokes theorem (see next section). □

Theorem 10.2 *On a simply connected surface every closed 1-form α is exact. That is, there exists a function F (called a primitive of α) such that $\alpha = dF$. Two primitives differ by a constant.*

Proof. It follows from the previous proposition by defining $F(x) = \int_{x_0}^{x} \alpha$ as the integral does not depend on the path of integration. □

In general, if X is a Riemann surface and $\pi : \tilde{X} \to X$ is its universal cover, then $\int_{\tilde{c}} \pi^* \alpha = \int_{\pi \tilde{c}} \alpha$. So if α is a 1-form on a Riemann surface X we can compute its integral

$$\int_c \alpha = F(\tilde{c}(1)) - F(\tilde{c}(0))$$

where $\tilde{c}$ is a lift of c to the universal cover of X and F is a primitive of $\pi^*\alpha$.

Remark: Let Γ be the group of Deck transformations of the cover $\pi : \tilde{X} \to X$. If F is a primitive of the form $\pi^*\alpha$ then $F \circ \gamma$ is also a primitive because $d(F \circ \gamma) = d\gamma^* F = \gamma^* dF = \gamma^* \pi^* \alpha = (\pi\gamma)^* \alpha = \pi^* \alpha$. As two primitives differ by a constant we obtain that $F \circ \gamma = F + a_\gamma$.

Definition 10.3 *Let α be a closed one-form defined on a surface X. The period map associated to α is the homomorphism*

$$\pi_1(X, x_0) \to \mathbb{C} \qquad \textit{given by} \quad c \to \int_c \alpha.$$

Let Γ be the group of Deck transformations of the cover $\pi : \tilde{X} \to X$ and F a primitive of α defined on $\tilde{X}$, then the image of the period map is given by the set $\{ a_\gamma \mid \gamma \in \Gamma \}$ where a_γ are defined in the remark above. This can be seen easily if we interpret an element of Γ as a closed curve c with lift $\tilde{c}$. Then

$$\int_c \omega = F(\tilde{c}(1)) - F(\tilde{c}(0)) = F(\gamma\tilde{c}(0)) - F(\tilde{c}(0)) = a_\gamma.$$

Theorem 10.4 *Suppose a closed differential form has all periods zero. Then it has a primitive.*

Proof. Construct explicitly the primitive as $F(z) = \int_{z_0}^{z} \alpha$ where z_0 is a point in X. This function is well defined as the periods are null. □

Corollary 10.5 *If ω is a closed holomorphic form on a compact Riemann surface such that the associated period map is zero then $\omega = 0$.*

Proof. By the previous theorem the form ω has a primitive. It is holomorphic on a compact Riemann surface therefore constant. □

10.2.1 Differential 2-forms

Using local coordinates $z = x + iy$ we may write a differential 2-form as

$$\alpha = f dx \wedge dy = \frac{i}{2} f dz \wedge d\bar{z},$$

where f is a function.

If $\varphi : V \to U$ is a diffeomorphism, recall the change of variable formula

$$\int\int_U f dxdy = \int\int_V \varphi^* f dudv$$

which can be written more explicitly as

$$\int\int_U f dxdy = \int\int_V f \circ \varphi |\frac{\partial(x,y)}{\partial(u,v)}| dudv$$

where $\frac{\partial(x,y)}{\partial(u,v)}$ is the Jacobian determinant. In the case φ is a biholomorphism we have

$$\int\int_U f dz \wedge d\bar{z} = \int\int_V f \circ \varphi |\frac{dz}{dw}|^2 dw \wedge d\bar{w}$$

To define the integral of a 2-form on a Riemann surface we use a partition of unit subordinated to a cover by charts. The fundamental theorem we will use is the following version of Stokes theorem.

Theorem 10.6 (Stokes Theorem) *Let α be a smooth 1-form defined on a neighborhood of a domain Ω with piecewise smooth boundary $\partial\Omega$ contained in a surface.*

$$\int_{\partial\Omega} \alpha = \int_{\Omega} d\alpha.$$

10.2.2 Cauchy's integral formula

We will admit the following integral formula (for a proof see [Hörmander]).

Theorem 10.7 *Let $\Omega \subset \mathbb{C}$ be a connected open domain whose boundary is a union of finitely many C^1 Jordan curves. Let $f \in C^1(\bar{\Omega})$. Then, for $z \in \Omega$,*

$$2\pi i f(z) = \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta + \int_{\Omega} \frac{\partial f(\zeta)/\partial\bar{\zeta}}{\zeta - z} d\zeta \wedge d\bar{\zeta}.$$

10.2.3 The residue theorem

Let ω be a meromorphic 1-form which is not identically null. Let $p \in X$ and $z : U \to \mathbb{C}$ be a chart such that ω is holomorphic on $U \setminus \{p\}$. We define the residue of ω at p as

$$res_p(\omega) = \frac{1}{2\pi i} \int_{\gamma} \omega$$

where γ is a curve with winding number 1 around p contained in U. It is easy to see that this integral is well defined. It can be computed using a Taylor expansion; write, using local coordinates, $\omega = f(z)dz$ where $f(z)$ has a pole at p and the residue is simply the coeficient of the term $\frac{1}{z}$ in the Taylor expansion. If we change the local chart then $\omega = g(w)dw = f(z)\frac{dw}{dz}dz$ and the residue is the same.

Proposition 10.8 *If X is compact then*

$$\sum_{p\in X} res_p(\omega) = 0$$

Proof. Stokes theorem. Suppose $D = \{p_i\}_{1\leq i\leq n}$ are the poles of ω. Choose non-intersecting neighborhoods U_i containing each p_i with boundary γ_i and compute

$$\sum_i \int_{\gamma_i} \omega = -\int_{X-\bigcup U_i} d\omega = 0$$

because $d\omega = \bar{\partial}\omega + \partial\omega = 0$. □

Proposition 10.9 *If X is compact and f is a meromorphic function, then the degree of the divisor $div(f)$ is zero.*

Proof. This follows from the proposition above and the fact that $deg(f) = \sum_{p\in X} res_p(\omega)$ for $\omega = df/f$. □

11 Homology and Cohomology

11.1 The de Rham complex

The de Rham complex over a surface X is

$$0 \to \mathbb{R} \to \mathcal{E}^0(X) \xrightarrow{d} \mathcal{E}^1(X) \xrightarrow{d} \mathcal{E}^2(X) \xrightarrow{d} 0.$$

where $\mathcal{E}^0(X) = \mathcal{C}^\infty(X)$ is the space of C^∞ functions on X, $\mathcal{E}^i(X)$ is the space of i-forms on X and d is the exterior differentiation. Poincaré's lemma says that the sequence is locally exact. The cohomology groups measure how much the sequence is far from being exact. Observe that the space of closed or exact forms are vector spaces.

Definition 11.1 *The i-th cohomology group $H^i(X, \mathbb{R})$, of the surface X is the quotient of the space of closed i-forms by the space of exact i-forms.*

Observe that $dimH^0(X, \mathbb{R})$ is the number connected components of X. In fact the space of exact 0-forms is formed by the trivial vector space of null functions.

In order to compute $H^1(X, \mathbb{R})$ we will introduce the singular homology. A singular p-simplex is a differential map from a p-simplex to X. We will write sometimes (P) for a singular 0-simplex, (P_1, P_2) for a singular 1-simplex and (P_1, P_2, P_3) for a singular 2-simplex. Fix now an abelian group G (we will mostly use $\mathbb{Z}$ or $\mathbb{R}$). A p-chain is a finite linear combination of singular p-simplices with coefficients in G. The space of p-chains will be noted C_p (with a convention that $C_{-1} = \{0\}$). There exists a boundary operator $\partial : C_p \to C_{p-1}$ satifying $\partial^2 c = 0$ for any chain c. It is defined on singular simplices by the formulas (using the obvious notation for the restriction of maps to the boundary of a simplex)

$$\partial(P) = 0 \quad \partial(P_1, P_2) = (P_2) - (P_1) \quad \partial(P_1, P_2, P_3) = (P_2, P_3) - (P_1, P_3) + (P_1, P_2)$$

and extended by linearity to all chains.

A chain c is called a cycle if $\partial c = 0$ and a boundary if there exists a chain $\tilde{c}$ such that $\partial\tilde{c} = c$. We define

Definition 11.2 *The p-th homology group, $H_p(X, G)$ is the quotient of the space of cycles, Z_n, by the space of boundaries, B_n.*

If the surface X is connected $dim\, H_0(X, \mathbb{R}) = 1$. If X is compact, orientable and connected then $dim\, H_2(X, \mathbb{R}) = 1$. To compute $H_1(X, \mathbb{Z})$, we will invoke van Kampen theorem, that describes the first homology as the abelianization of the fundamental group:

$$H_1(X, \mathbb{Z}) = \frac{\pi_1(X, z)}{\{\langle [a, b] \rangle | a, b \in \pi_1(X, z)\}}.$$

Using the generators a_i, b_i, $1 \leq i \leq g$ for a compact surface of genus g we obtain that $H_1(X, \mathbb{Z}) = \mathbb{Z}^{2g}$. The generators a_i, b_i, viewed as a basis of $H_1(X, \mathbb{Z})$ are also called a canonical basis for the homology. It follows from general theorems on the homology that we also have $H_1(X, \mathbb{R}) = \mathbb{R}^{2g}$.

The relation between homology and cohomology is essentially given by Stokes theorem on a chain c:

$$\int_{\partial c} \omega = \int_c d\omega.$$

Lemma 11.3 *If ω is closed and c_1 and c_2 are two homologous chains then*

$$\int_{c_1} \omega = \int_{c_2} \omega.$$

Proof. By hypothesis $c_2 - c_1 = \partial C$. Apply Stokes theorem. $\square$

This lemma shows that the bilinear map in the following theorem is well defined.

Theorem 11.4 *Let X be a compact orientable surface of genus p. The bilinear map $H_1 \times H^1 \to \mathbb{R}$ defined by*

$$(c, \omega) \to \int_a \omega$$

is non-degenerate.

Proof. The fact that $(\cdot, \omega)$ is non-zero follows from the fact that if all periods are null, the form ω is null. On the other hand, given an element $c \in H_1$ we construct a form such that $(c, \omega) \neq 0$ in the following two lemmas. $\square$

Suppose X is orientable. Let γ be simple closed curve in X. We consider an annulus A containing γ and let A^- be the left side and A^+ the right side. Let f be a function with compact support on A^- which is one on A^- intersected with a neighborhood of γ. Define then $\eta_\gamma = df$. Even if f is not continuous, η_γ is C^∞ 1-form. On the other hand η_γ is not exact in general. The form η_γ is dual to γ in the sense of the following lemma.

Lemma 11.5 *Let ω be a closed 1-form. Then*

$$\int_\gamma \omega = \int_X \eta_\gamma \wedge \omega.$$

Proof. We compute

$$\int_X \eta_\gamma \wedge \omega = \int_{A^-} df \wedge \omega = \int_{A^-} d(f\omega) - \int_{A^-} f d\omega = \int_\gamma f\omega = \int_\gamma \omega.$$

$\square$

Remark: Using notation of the next section we write $\int_\gamma \omega = (\omega, *\eta_\gamma)$.

Lemma 11.6 *Let a_i, b_i be an homology basis. Then*

$$\int_{a_i} \eta_{a_j} = \int_{b_i} \eta_{b_j} = 0 \quad \int_{a_i} \eta_{b_j} = -\int_{b_i} \eta_{a_i} = \delta_{ij}.$$

Proof. The first equality follows from the previous lemma. For the second one, we compute in the case that a, b are two loops intersecting once at a point with orientation given by the tangent vectors to a and b at the point of intersection in that order. we denote by f_b a function associated to the loop b with support in A_b^- as before. We obtain

$$= \int_a \eta_b = \int_a df_b = 1.$$

The last equality follows from the explicit form of the function f_b at the intersection point; it corresponds to the integration on a closed interval $[0, 1]$ of the derivative of a function such that $f(0) = 0$ and $f(1) = 1$. $\square$

11.2 The Dolbeault complex

Recall the Cauchy-Riemann operator $\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ defined on functions on an open subset $U \subset \mathbb{C}$. It is better understood in the guise of an operator:

$$\bar{\partial} : \mathcal{C}^\infty(U) \to \mathcal{E}^{0,1}(U)$$

given by $f \to \frac{\partial f}{\partial \bar{z}} d\bar{z}$.

Local solvability of the Cauchy-Riemann equation: for each $g \in C^\infty(U)$ there exists $V \subset U$ and $f \in C^\infty(V)$ such that

$$\frac{\partial f}{\partial \bar{z}} = g.$$

on V. A stronger result is true:

Proposition 11.7 (Dolbeault's lemma) *Let $\Omega \subset \mathbb{C}$ be an open subset and $g \in C^\infty(\Omega)$. Then there exists a function $f \in C^\infty(\Omega)$ such that*

$$\frac{\partial f}{\partial \bar{z}} = g.$$

Proof.

There are two cases:

1. In the first case we suppose g of compact support. An explicit solution is given in terms of the integral formula

$$f(z) = \frac{1}{2\pi i} \int\int_{\mathbb{C}} \frac{g(w)}{w-z} dw \wedge d\bar{w}.$$

The integral is well defined as can be seen by using polar coordinates $w - z = re^{i\theta}$ so that $\frac{1}{w-z} dw \wedge d\bar{w} = \frac{-2ir}{re^{i\theta}} dr \wedge d\theta$. Because g is of compact support, the integration is made in a sufficiently large rectangle and therefore we may differentiate under the integral sign. We obtain making the change w for $w - z$

$$\frac{\partial f(z)}{\partial \bar{z}} = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int\int_{|w|>\epsilon} \frac{\partial g(z+w)}{\partial \bar{z}} \frac{1}{w} dw \wedge d\bar{w}.$$

So

$$\frac{\partial f(z)}{\partial \bar{z}} = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int\int_{|w|>\epsilon} \frac{\partial}{\partial \bar{w}} \left(\frac{g(z+w)}{w} \right) dw \wedge d\bar{w} = - \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int\int_{|w|>\epsilon} d \left(\frac{g(z+w)}{w} dw \right)$$

$$= \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{g(z+w)}{w} dw = g(z).$$

2. If *suppg* $\subset \Omega$ is not compact we construct an exhaustion sequence of compact sets K_n ($K_n \subset Int(K_{n+1})$ with $\Omega \setminus K_n$ having no relatively compact component) and cut-off functions φ_n with $\varphi_{n|K_n} = 1$ and $\varphi_{n|K_{n+1}} = 0$. We solve

$$\frac{\partial f_n}{\partial \bar{z}} = \varphi_n g.$$

We would like to make sense of

$$f = f_n + (f_{n+1} - f_n) + (f_{n+2} - f_{n+1}) + \cdots$$

As this sum might not converge we modify each term by a holomorphic function using Runge's theorem: As $f_{m+1} - f_m$, $m \geq 1$, is holomorphic on a neighborhood of K_n there exists a holomorphic function h_m on Ω such that

$$|f_{m+1} - f_m - h_m| < \frac{1}{2^m}$$

on K_m. We redefine the sum to be

$$f = f_n + (f_{n+1} - f_n - h_n) + (f_{n+2} - f_{n+1} - h_{n+1}) + \cdots$$

Now the sum is uniformly convergent on K_m for each $m \geq n$ so f is well defined on Ω. Moreover we imediately see that on each K_m f solves the equation.

□

Remark 11.8 *On an n-dimensional complex manifold we have the following exact sequence*

$$0 \to \mathcal{O} \longrightarrow \mathcal{C}^\infty \xrightarrow{\bar\partial} \mathcal{E}^{0,1} \xrightarrow{\bar\partial} \mathcal{E}^{0,2} \cdots,$$

and more generally

$$0 \to \Omega^{p,q} \longrightarrow \mathcal{E}^{p,q} \xrightarrow{\bar\partial} \mathcal{E}^{p,q+1} \xrightarrow{\bar\partial} \mathcal{E}^{p,q+2} \cdots.$$

where the vector spaces in the exact sequence are germs of of differential forms. A better formulation is obtained using sheaf theory.

11.2.1 Weyl's lemma

Weyl's lemma is a regularity result for the solutions (in the distribution sense) of the Cauchy-Riemann equation.

Theorem 11.9 *Let $\Omega \subset \mathbb{C}$ be an open set and $F : C_0^\infty(\Omega) \to \mathbb{C}$ be a linear map such that for all $\varphi \in C_0^\infty(\Omega)$*

$$F(\frac{\partial \varphi}{\partial \bar z}) = 0.$$

Suppose that for each sequence φ_i, with support in a fixed compact subset in Ω, converging in the C^∞ norm to φ we have that

$$\lim_{i\to\infty} F(\varphi_i) = F(\varphi).$$

Then, there exists a holomorphic function $f : \Omega \to \mathbb{C}$ such that for all $\varphi \in C_0^\infty(\Omega)$

$$F(\varphi) = \int_\Omega f(\zeta)\varphi(\zeta) d\zeta \wedge d\bar\zeta.$$

Proof. F is a distribution. We want to show that it can be given, for $\varphi \in C_0^\infty(\Omega)$ by

$$F(\varphi) = \int_\Omega f(\zeta)\varphi(\zeta)d\zeta \wedge d\bar{\zeta}$$

with f smooth. The holomorphy of f will follow easily by integration by parts.

Observe that by Dolbeault lemma, we can solve $\frac{\partial \psi}{\partial \bar{z}}$ on Ω, so $F(\varphi) = F(\frac{\partial \psi}{\partial \bar{z}})$. But this is not zero in general when $\psi \notin C_0^\infty(\Omega)$. We deform the solution of the $\bar{\partial}$-equation as follows. Let $\chi(z)$ be a function with support in a ϵ-disc with value 1 on a $\epsilon/2$-disc. As in the proof of Dolbeault lemma (up to the cut off term $\psi(w-z)$) we write for φ with compact support in $\Omega_\epsilon = \{ z \in \Omega \mid d(z, \partial\Omega)) > \epsilon \}$,

$$\psi(z) = \frac{1}{2\pi i} \int\int_{\mathbb{C}} \frac{\varphi(w)}{w-z} \psi(w-z) dw \wedge d\bar{w}.$$

Clearly that is a deformation of the solution of the Cauchy-Riemann equation which has a compact support (to see that, make the change of coordinate w for $w-z$). We obtain, as in Dolbeault lemma,

$$\begin{aligned}\frac{\partial \psi(z)}{\partial \bar{z}} &= \varphi(z) + \frac{1}{2\pi i} \int\int_{\mathbb{C}} g(z+w) \frac{\partial}{\partial \bar{w}} \left(\frac{\chi(w)}{w} \right) dw \wedge d\bar{w}. \\ &= \varphi(z) + \frac{1}{2\pi i} \int\int_{\mathbb{C}} g(\zeta)\rho(\zeta - z) d\zeta \wedge d\bar{\zeta}.\end{aligned}$$

where we wrote $\rho(w) = \frac{\partial}{\partial \bar{w}} \left(\frac{\chi(w)}{w} \right)$.

Now we can write

$$F(\varphi) = F(\frac{\partial \psi}{\partial \bar{z}}) + \frac{1}{2\pi i} \int\int_{\mathbb{C}} g(\zeta) F(\rho(\zeta - \bullet)) d\zeta \wedge d\bar{\zeta}.$$

The first term in the right vanish and calling $f(\zeta) = F(\rho(\zeta - \bullet))$ we obtain

$$F(\varphi) = \frac{1}{2\pi i} \int\int_{\mathbb{C}} g(\zeta) f(\zeta) d\zeta \wedge d\bar{\zeta}$$

as desired. □

11.3 Hodge theory

We defined the cohomology space $H^1(X, \mathbb{R})$ as the quotient of closed $\mathbb{R}$-valued 1-forms by exact 1-forms. The same definition with $\mathbb{C}$ valued 1-forms gives $H^1(X, \mathbb{C})$. That is a $\mathbb{C}$-vector space of complex dimension g, the genus of X. Hodge theory identifies the cohomology group to the space of harmonic forms. We will work with complex valued forms in order to relate the de Rham and Dolbeault complexes. We write Λ^1 as the bundle of $\mathbb{C}$-valued 1-forms. If X is a Riemann surface we defined the canonical bundle K. An element $\alpha \in \Lambda^1$ can be decomposed as a sum $\alpha = a_1 dz + a_2 d\bar{z}$. Observe that $\Lambda^1 = K \oplus \bar{K}$ as complex line bundles.

Definition 11.10 *Let $\alpha \in \Lambda^1$ and write $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1 \in K$, $\alpha_2 \in \bar{K}$. Define*

$$\star\alpha = i\bar{\alpha}_1 - i\bar{\alpha}_2.$$

By a straight computation one can verify that the Hodge star operator defined above satisfies the following properties:

Proposition 11.11 *Let $\alpha \in \Lambda^1$. Then*

1. $\star\star\alpha = -\alpha$
2. $\star\bar{\alpha} = \overline{\star\alpha}$

Proposition 11.12 *Let $\alpha_1 \in K(X)$, $\alpha_2 \in \bar{K}(X)$ and $f \in C^\infty(X, \mathbb{C})$. Then*

1. $d \star \alpha_1 = i\partial\bar{\alpha}_1$
2. $d \star \alpha_2 = -i\bar{\partial}\bar{\alpha}_2$
3. $\star\partial f = i\bar{\partial}\bar{f}$
4. $\star\bar{\partial} f = -i\partial\bar{f}$
5. $d \star df = 2i\partial\bar{\partial}\bar{f}$

Using the star operator we define a hermitian product on 1-forms over a compact Riemann surface:

Definition 11.13 *Let X be a compact Riemann surface and α_1, α_2 1-forms in $\Lambda^1(X)$. Define*

$$\langle\alpha_1, \alpha_2\rangle = \int_X \alpha_1 \wedge \star\alpha_2.$$

Clearly $\langle\alpha_1, \alpha_2\rangle = \overline{\langle\alpha_2, \alpha_1\rangle}$. To show that $\langle\alpha, \alpha\rangle > 0$ for non-vanishing α, write in local coordinates $\alpha = adz + bd\bar{z}$. Then $\alpha \wedge \star\alpha = i(|a|^2 + |b|^2)dz \wedge d\bar{z} = 2(|a|^2 + |b|^2)dx \wedge dy$. Therefore the integrand is a positive form and the product is 0 if and only if $\alpha = 0$.

Proposition 11.14 *Let X be a compact Riemann surface. Then*

1. *$\partial C^\infty(X)$, $\bar{\partial} C^\infty(X)$, $H^0(X, K)$ and $\overline{H^0(X, K)}$ are pairwise orthogonal.*
2. *$dC^\infty(X)$ and $\star dC^\infty(X)$ are orthogonal*
3. *$dC^\infty(X) \oplus \star dC^\infty(X) = \partial C^\infty(X) \oplus \bar{\partial} C^\infty(X)$.*

The most important theorem in Hodge theory is the following.

Theorem 11.15 *For a compact Riemann surface X,*

$$\Lambda^1(X) = dC^\infty(X) \oplus \star dC^\infty(X) \oplus H^0(X, K) \oplus \overline{H^0(X, K)}.$$

One can identify $H^1(X, \mathbb{C})$ to harmonic forms:

Definition 11.16 *A harmonic 1-form $\alpha \in \Lambda^1$ is a form satisfying $d\alpha = d \star \alpha = 0$*

Proposition 11.17 *The following are equivalent:*

1. *α is harmonic*

2. $\partial\alpha = \bar{\partial}\alpha = 0$

3. $\alpha = \alpha_1 + \alpha_2$ *with* $\alpha_1 \in H^0(X, K)$ *and* $\alpha_2 \in \overline{H^0(X, K)}$

Defining $\Lambda_{harm}(X)$ as the space of harmonic forms, we can state Hodge theorem as

$$H^1(X, \mathbb{C}) \simeq \Lambda_{harm}(X).$$

Exercise 11.18 *1. Prove that α is harmonic if and only if, locally, $\alpha = df$ with f harmonic.*

2. Prove that α is holomorphic if and only if, locally, $\alpha = df$ with f holomorphic.

11.4 Harmonic and Holomorphic differentials

On a compact Riemann surface, a harmonic form which is exact vanishes identically. On the other hand, exact harmonic forms can be obtained if we allow singularities. In this section we state existence theorems for harmonic and holomorphic differentials with prescribed singularities.

Let $U \subset X$ be a neighborhood in a Riemann surface defined in local coordinates by $|z| < 1$. With a slight abuse of notation, we write a function on U using the local coordinate. For instance, we say that $1/z^n$ is a function defined on U with a singularity at $z_0 = 0$.

Theorem 11.19 *Let X be a Riemann surface and $z_0 \in X$ any point. Fix $n \geq 1$. There exists a 1-form ω, satisfying*

1. *ω is harmonic and exact on $X \setminus \{z_0\}$.*

2. *$\omega - d(\frac{1}{z^n})$ is harmonic on $|z| < 1/2$.*

Proof. Let $\rho \in C_0^\infty(X)$ with support in U and such that it is the identity on $|z| < 1/2$. We define the differential $\psi = d(\rho/z^n) \in X \setminus \{z_0\}$. Observe that ψ is of type $(1, 0)$ on $|z| < 1/2$. Therefore the form $\psi - i * \psi$ is smooth on X. By the Hodge theorem there exist smooth functions f and g such that

$$\psi - i * \psi = \omega_h + df + *dg.$$

Define now $\omega = \psi - df = \omega_h + *dg + *i\psi$.

1. Clearly, ω is smooth and exact on $X \setminus \{z_0\}$. To show that it is harmonic, we compute

$$d\omega = d(\psi - df) = 0$$

and

$$d * \omega = d * (\omega_h + *dg + *i\psi) = -id\psi = 0.$$

2. To show that we have the prescribed singularity, observe that on $|z| < 1/2$, $\omega - d(\frac{1}{z^n}) = \omega - \psi = -df$, therefore smooth.

□

Remarks:

1. If X is compact, it follows imediately that ω is unique. In fact if ω' satisfies the same conditions we obtain that $\omega - \omega'$ is harmonic and exact and therefore null.

2. Taking real and imaginary parts of ω we obtain real harmonic differentials with singularities $\mathrm{Re}\, 1/z^n$ and $\mathrm{Im}\, 1/z^n$ respectively.

3. From a harmonic differential ω we obtain a meromorphic differential $\omega + i * \omega$. It has $d(\frac{1}{z^n})$ as singularity but only its real part is in general exact.

The differentials obtained so far have null residue. In order to obtain differentials with a prescribed residue, consider as before $U \subset X$ a neighborhood in a Riemann surface defined in local coordinates by $|z| < 1$. Let z_1, z_2 such that $|z_i| < 1/2$. Define the meromorphic differential

$$d(\log \frac{z - z_1}{z - z_2}) = \frac{dz}{z - z_1} - \frac{dz}{z - z_2}.$$

Observe that, in fact,$\log \frac{z-z_2}{z-z_1}$ is well defined for $z > 1/2$ (prove that). Now, the following theorem follows with the same proof as the theorem above.

Theorem 11.20 *Let X be a Riemann surface and $z_i \in U \subset X$ two points with $|z_i| < 1/2$. There exists a 1-form ω, satisfying*

1. *ω is harmonic (or holomorphic) on $X \setminus \{z_1, z_2\}$ and exact on $X \setminus \{|z| < 1/2\}$.*

2. *$\omega - d(\log \frac{z-z_1}{z-z_2})$ is harmonic (or holomorphic) on $\{|z| < 1/2\}$.*

Now we can consider any two points z_1, z_2 on a Riemann surface X. Take a path joining the two points and a finite covering U_α by discs of radius one such that the discs of radius $1/2$ contained in them also cover the path. We can apply the theorem above for a sequence of points starting at z_1 and ending at z_2 such that each adjacent pair is contained in a disc of radius $1/2$. By construction we obtain the following

Corollary 11.21 *Let X be a Riemann surface and $z_i \in X$ two points. Then there exists a 1-form ω satisfying*

1. *ω is harmonic (or holomorphic) on $X \setminus \{z_1, z_2\}$.*

2. *ω has singularities $\frac{dz}{z}$ and $-\frac{dz}{z}$ around z_1 and z_2 respectively.*

A further construction gives

Corollary 11.22 *Let X be a Riemann surface and $z_i \in X$ any chosen n points. Suppose $c_i \in \mathbb{C}$ satisfy $c_1 + \cdots c_n = 0$. Then there exists a 1-form ω satisfying*

1. *ω is harmonic (or holomorphic) on $X \setminus \{z_1, \cdots, z_n\}$.*

2. *ω has singularities $c_i \frac{dz}{z}$ around z_i.*

Proof. To prove the corollary, we choose another point z_0 (distinct from the first sequence) and apply the previous corollary for each pair (z_i, z_0) with singularities $c_i \frac{dz}{z}$ and $-c_i \frac{dz}{z}$ around z_i and z_0 respectively. We obtain a form ω_i for each pair and $\omega = \omega_1 + \cdots + \omega_n$ gives the 1-form we need. □

11.5 Meromorphic functions

Meromorphic functions are obtained as quotients of meromorphic differentials. In fact, consider $\omega_1 = g_1(z)dz$ and $\omega_2 = g_2(z)dz$ two meromorphic differentials written in local coordinates. Then $f = g_1(z)/g_2(z)$ is a meromorphic function.

Theorem 11.23 *There exists a non-constant meromorphic function on any Riemann surface.*

Proof. We have to construct two meromorphic forms which are not proportional. To this end, let z_0, z_1, z_2 be three points and construct

1. ω_1 with residue 1 at z_1 and residue -1 at z_0,

2. ω_2 with residue 1 at z_2 and residue -1 at z_0.

The quotient ω_1/ω_2 has certainly a pole at z_1 (maybe of order greater than one) and a zero at z_2 (maybe of higher order). In any case it cannot be constant! □

11.6 Periods and Bilinear Relations

Let φ be a closed C^∞ 1-form defined on X and let a_i, b_i, $1 \leq i \leq g$, be a homology basis.

Definition 11.24 *The a-periods and b-periods of a closed C^∞ 1-form φ are respectively*

$$A_i(\varphi) = \int_{a_i} \varphi \quad \textit{and} \quad B_i(\varphi) = \int_{b_i} \varphi$$

In the last section we showed that $H^1(X, \Omega^{1,0})$ has dimension g. Let α_p , $1 \leq p \leq g$, be a basis of $H^1(X, \Omega^{1,0})$.

We define the period matrix of X as the $g \times 2g$ matrix with columns (called vector periods)

$$P_i = (A_i(\alpha_1), \cdots, A_i(\alpha_g))^T, \quad P_{i+g} = (B_{i+g}(\alpha_1), \cdots, B_{i+g}(\alpha_g))^T.$$

Lemma 11.25 *The vectors $P_j \in \mathbb{C}^g$ are linearly independent over $\mathbb{R}$.*

Proof. Otherwise, there would exist a real linear combination $\sum c_i \int_{a_i} \alpha_j + \sum c_{i+g} \int_{b_i} \alpha_j = 0$ for each fixed j. We take real and imaginary parts of the basis of holomorphic forms. They form a basis of the space of hamonic forms. We have that, for each harmonic form, the map $\sum c_i \int_{a_i} \alpha + \sum c_{i+g} \int_{b_i} \alpha = 0$. But this implies that all c_i vanish as the pairing bewen homology and cohomology is non degenerate. □

This implies that the set of vector periods P_i define a lattice $\Lambda \subset \mathbb{C}^g$.

Definition 11.26 *The Jacobian variety $J(X)$ is the complex torus $\mathbb{C}^g/\Lambda$.*

The Jacobian map

$$j_{z_0} : X \to J(X)$$

is given by

$$j_{z_0}(z) = \left(\int_{z_0}^{z} \alpha_i, \cdots, \int_{z_0}^{z} \alpha_g \right)$$

where z_0 is a chosen point in X and the integrals are computed using any path. The map is well defined because different choices of paths lead to equal vectors modulo Λ.

Proposition 11.27 *Let φ_1 and φ_2 be two closed 1-forms defined on X. Then*

$$\int\int_X \varphi_1 \wedge \varphi_2 = \sum_i (A_i(\varphi_1)B_i(\varphi_2) - B_i(\varphi_1)A_i(\varphi_2))$$

Proof. Let Δ be the polygon whose boundary is a homology basis. Fix a point $z_0 \in int(\Delta)$. If α is closed and defined on Δ we can define for each $P \in \Delta$,

$$u(P) = \int_{z_0}^{P} \alpha$$

The proposition follows imediately from the following lemma.

Lemma 11.28 *Suppose that φ is a 1-form defined on a neighborhood of $\partial\Delta$. Then, with α and u defined as above,*

$$\int_{\partial\Delta} u\varphi = \sum_i (A_i(\alpha)B_i(\varphi) - B_i(\alpha)A_i(\varphi))$$

Proof. Write

$$\int_{\partial\Delta} u\varphi = \int_{a_i} u\varphi + \int_{-a_i} u\varphi + \int_{b_i} u\varphi + \int_{-b_i} u\varphi$$

Observe now that, for corresponding points in z and z' in a_i and $-a_i$,

$$u(z') - u(z) = \int_{z_0}^{z'} \alpha - \int_{z_0}^{z} \alpha = \int_{z}^{z'} \alpha = \int_{b_i} \alpha$$

and therefore

$$\int_{a_i} u\varphi + \int_{-a_i} u\varphi = \int_{a_i} (u(z) - u(z'))\varphi(z) = -\int_{b_i} \alpha \int_{a_i} \varphi.$$

Analogously, we have

$$\int_{b_i} u\varphi + \int_{-b_i} u\varphi = \int_{a_i} \alpha \int_{b_i} \varphi.$$

□

□

Observe, from the proof, that φ_2 need to be defined only on a neighborhood of the homology basis. Another way to write this is with the help of the matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and the vector of periods $P(\varphi) = (A_1(\varphi), \ldots, B_1(\varphi), \cdots)$, so that

$$\int_X \varphi_1 \wedge \varphi_2 = P(\varphi_1)JP^T(\varphi_2)$$

We apply this result to the case of φ_1 holomorphic and φ_2 meromorphic.

Theorem 11.29 *Let φ_1 (holomorphic) and φ_2 (meromorphic) be two 1-forms defined on X. Suppose φ_2 is nonsigular along the homology basis and let Let $u = \int_{z_0}^{z} \varphi_1$ for $z_0 \in \Delta$. Then*

$$2\pi i \sum Res(u\varphi_2) = \sum_i (A_i(\varphi_1)B_i(\varphi_2) - B_i(\varphi_1)A_i(\varphi_2))$$

Proof. The proof follows from the previous proposition and the identity:

$$2\pi i \sum Res(u\varphi_2) = \int_{\partial\Delta} u\varphi_2 = \int\int_X \varphi_1 \wedge \varphi_2$$

□

Remark. Suppose that at $z = 0$ we have

$$\varphi_1(z) = (a_0 + \cdots)dz$$

and

$$\varphi_2(z) = (b_{-m}z^{-m} + \cdots + b_0 + \cdots)dz.$$

Then

$$u(z) = a_0 z + \cdots$$

and at $z = 0$

$$Res_0(u\varphi_2) = \sum_{k=2}^{m} \frac{a_{k-2}}{k-1} b_{-k}.$$

If both forms φ_i are holomorphic we obtain

Lemma 11.30 *Let φ_1 and φ_2 be two holomorphic 1-forms defined on X. Then*

$$\sqrt{-1}\int\int_X \varphi_1 \wedge \bar{\varphi}_2 = \sqrt{-1}\sum_i (A_i(\varphi_1)B_i(\bar{\varphi}_2) - B_i(\varphi_1)A_i(\bar{\varphi}_2))$$

Observe that, if $\varphi_1 = \varphi_2 = \varphi$, the lemma implies that

$$0 < (\varphi, \varphi) = \sqrt{-1}\int\int_X \varphi \wedge \bar{\varphi} = \sqrt{-1}\sum_i (A_i(\varphi)B_i(\bar{\varphi}) - B_i(\varphi)A_i(\bar{\varphi})).$$

Theorem 11.31 (Riemann's bilinear relations) *Let X be a compact surface of genus g. Then there exists a basis of holomorphic differentials such that the period matrix has the form*

$$(I, Z)$$

where I is the identity matrix of order g and Z a symmetric matrix satisfying $ImZ > 0$.

Proof. A change of basis of holomorphic forms is given by a matrix $g = (g_j^i)$ so that $\varphi_j = \sum_i g_j^i \varphi_i$. In that case, the new period matrix π' is $\pi' = g\pi$. We need to prove that π_{ij}, $1 \leq i, j \leq g$ is invertible. But if this is not the case there would exists a linearly combination of holomorphic forms, say φ with zero a-periods. From the formula before the theorem this is impossible.

Now, in that basis, $A_i(\varphi_l) = \delta_{il}$ and by Lemma 11.29 we obtain

$$0 = \sum_i (A_i(\varphi_l)B_i(\varphi_m) - B_i(\varphi_l)A_i(\varphi_m))$$

which implies that $B_l(\varphi_m) = B_m(\varphi_l)$ so that Z is symmetric.

Lastly, using Lemma 11.30 we get

$$0 < (\varphi_i, \varphi_j) = \sqrt{-1} \int\int_X \varphi_i \wedge \bar{\varphi}_j = \sqrt{-1}\,(B_i(\bar{\varphi}_j) - B_j(\varphi_i)) = 2\mathrm{Im}\, B_j(\varphi_i).$$

□

Exercises. Let X be a compact Riemann surface.

1. (normalized abelian differentials of the third kind) Prove that there exists a unique meromorphic 1-form, ω_{z_1,z_2} with only two simple poles with residues +1 and -1 at two points z_1, z_2 such that its a-periods are null.

2. (normalized abelian differentials of the second kind) Prove that there exists a unique meromorphic 1-form, ω_{z_0} with only one pole at z_0 such that at a local coordinate neighborhood (U, z) around z_0, $\omega - \frac{dz}{z^n}$ is holomorphic and such that its a-periods are null.

3. Any meromorphic 1-form is a combination of holomorphic 1-forms, abelian differentials of the third kind and abelian differentials of the second kind.

4. Prove the reciprocity relations (φ_k is a normalized basis of holomorphic forms)

$$\int_{b_k} \omega_{z_1,z_2} = 2\pi i \int_{z_2}^{z_1} \varphi_k$$

and ($\varphi_k = f_k(z)dz$ on U)

$$\int_{b_k} \omega_{z_0} = 2\pi i \frac{f_k^{n-1}(z_0)}{n!}.$$

12 Sheaves

Sheaves are the appropriate language to pass from local to global. In particular, to understand analytic continuation of holomorphic functions defined on open sets of a Riemann surface and to define cohomology. Perhaps one of the simplest questions in the theory of Riemann surfaces which introduces cohomology into play is the Mittag-Leffler problem: Find a meromorphic function on a Riemann surface X with given principal parts at a discrete set of points. A natural formulation of this problem is given by fixing a covering $\mathcal{U} = \{U_i\}$ of X by open sets (each open set containing at most one pole) and meromorphic functions $f_i \in \mathcal{M}(U_i)$ with fixed principal parts. We suppose that, in the intersections $U_i \cap U_j$, the functions $g_{ij} = f_i - f_j$ are holomorphic. The goal is to find holomorphic functions $g_i \in \mathcal{O}(U_i)$ such that $g_{ij} = g_i - g_j$ so that the functions $f_i - g_i \in \mathcal{M}(U_i)$ are well defined globaly as, on the intersections, $f_i - g_i = f_j - g_j$ and have the same principal part as before. This amounts to study the group

$$H^1(\mathcal{U}, \mathcal{O}) = \frac{\{g_{ij} \in \mathcal{O}(U_i \cap U_j) \mid g_{ij} + g_{jk} + g_{ki} = 0 \}}{\{g_{ij} = g_i - g_j \in \mathcal{O}(U_i \cap U_j) \mid \text{for } g_i \in \mathcal{O}(U_i) \}}$$

which is the first *Čech* cohomology group.

Equivalently we could use functions ρ_i subordinated to $\mathcal{U}$ (that is, having support on U_i) such that $\rho_i = 1$ near the poles. The closed $(0,1)$-form $\alpha = \sum \bar{\partial}(\rho_i f_i)$ is then globaly defined. The goal, in this formulation, is to find a solution $g \in C^\infty(X)$ of the equation $\bar{\partial} g = \alpha$. In that case the function $\sum \rho_i f_i - g$ is meromorphic, globaly defined and has the same principal part as the original data. The relevant group in this case is

$$H^{0,1}_{\bar{\partial}}(X) = \frac{\{ \bar{\partial} - \text{closed 1-forms} \}}{\{\bar{\partial} C^\infty(X) \}},$$

the first Dolbeault's cohomology group. The isomorphism between the two groups will be proved in due course.

To motivate further the introduction of sheaves and their associated local informations gathered in arbitrary small neighborhoods around every point we treat first a special case in the following section.

12.1 Germs of holomorphic functions and Analytic continuation

Let X be a Riemann surface and $x \in X$ a point. We define an equivalence relation between pairs (U_i, f_i), $i = 1, 2$, where $f_i : U_i \to \mathbb{C}$ are two holomorphic functions defined on neighborhoods containing x. We say $(U_1, f_1) \equiv (U_2, f_2)$ if $U_1 \cap U_2$ contains x and both functions restricted to the intersection coincide. We denote the equivalence class of (U, f) by f_x and call it a germ of a holomorphic function at $x \in X$.

We let $\mathcal{O}_x$ be the set of all germs at x. Using the usual sum and product of two functions one can easily verify that it is a $\mathbb{C}$-algebra. The set

$$m_x = \{ f_x \in \mathcal{O}_x \mid f_x(x) = 0 \}$$

is a maximal ideal in $\mathcal{O}_x$ as each element $f_x \in \mathcal{O}_x$ which is not in m_x is invertible. If we choose a chart $z : U \to \mathbb{C}$ with $x \in U$ we can describe $\mathcal{O}_x$ using the isomorphism between the $\mathbb{C}$-algebra $\mathcal{O}_x$ and $\mathbb{C}\{z\}$ (of power series which are convergent in some neighborhood of

z):

$$f_x \to \sum_0^\infty \frac{1}{n!}(f \circ z^{-1})^n(0)z^n.$$

We let $\mathcal{O}_X$ be the disjoint union $\cup_x \mathcal{O}_x$ and define a topology by exhibiting a fundamental system of neighborhoods at f_x, namely, the sets $N(U, f) = \{ f_y \mid y \in U \}$ for each pair (U, f) whose equivalence class coincides with f_x. The following proposition is a simple exercise using the definitions.

Proposition 12.1 *$\mathcal{O}_X$ is Hausdorff and the mapping $\pi : \mathcal{O}_X \to X$ given by $\pi(f_x) = x$ is a local homeomorphism.*

Using the local homeomorphism we define holomorphic charts on $\mathcal{O}_X$ by composition with the charts defined on X and therefore $\mathcal{O}_X$ is also a Riemann surface. This proposition shows that given an holomorphic function $f : U \to \mathbb{C}$ defined on an open set $U \subset \mathbb{C}$ there exists a "maximal" holomorphic function defined by analytic continuation. It is defined on a Riemann surface which is spread over an open subset of $\mathbb{C}$, namely, the component of $\mathcal{O}_\mathbb{C}$ containing a germ f_x with $x \in U$.

12.2 Sheaves

Definition 12.2 *Let X be a topological space. A presheaf $(\mathcal{F}, \rho_V^U)$ of abelian groups on X is a family of abelian groups $\mathcal{F}(U)$ (sections over U) and a family of group homomorphisms $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ (restriction maps), where $V \subset U \subset X$ are open sets, satisfying*

1. $\mathcal{F}(\emptyset) = \{0\}$
2. $\rho_U^U = Id$ *for any open* U
3. $\rho_W^V \circ \rho_V^U = \rho_W^U$ *for any* $W \subset V \subset U$.

If X is a Riemann surface and $\mathcal{F}(U) = \mathcal{O}(U)$ is the $\mathbb{C}$-algebra of holomorphic functions on U and ρ_V^U is the restriction map, we obtain the presheaf of holomorphic functions. It is also sheaf (see the next definition) as well as all the following presheaves.

1. $\mathcal{E}^p(U)$ of p-forms on an open set U in a manifold M. Those are sections of the bundle $\Lambda^p T^*(U)$. In local coordinates $(x^1, \cdots, x^n)$ they are described by

$$\sum \varphi^x_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

such that in the intersection with another coordinate chart $(y^1, \cdots, y^n)$ we have

$$\sum \varphi^y_{i_1 \cdots i_p} dy^{i_1} \wedge \cdots \wedge dy^{i_p}$$

with

$$\varphi^y_{i_1 \cdots i_p} = \sum \varphi^x_{j_1 \cdots j_p} \frac{\partial x^{j_1}}{\partial y^{i_1}} \cdots \frac{\partial x^{j_p}}{\partial y^{i_p}}.$$

The de Rham operator is defined in local coordinates as

$$d\left(\sum \varphi^x_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}\right) = \sum_{j=1}^n \sum \frac{\partial \varphi^x_{i_1 \cdots i_p}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

On a surface a 1-form is written

$$\varphi = \varphi_1 dx^1 + \varphi_2 dx^2,$$

and

$$d\varphi = \left(\frac{\partial\varphi_2}{\partial x^1} - \frac{\partial\varphi_1}{\partial x^2}\right) dx^1 \wedge dx^2.$$

Observe that on a surface $\mathcal{E}^p = 0$ for $p \geq 3$.

2. $\mathcal{Z}^p(U)$ of closed p-forms on an open set U in a manifold M, that is, forms $\varphi \in \mathcal{E}^p(U)$ such that $d\varphi = 0$.

3. $\mathcal{E}^{p,q}(U)$ of (p,q)-forms on an open set U in a complex manifold M. In local coordinates $(z^1, \cdots, z^n)$ they are given by

$$\sum \varphi^z_{i_1\cdots i_p, j_1\cdots j_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$$

The Dolbeault operator is defined in local coordinates as

$$\bar{\partial}\left(\sum \varphi^z_{i_1\cdots i_p, j_1\cdots j_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}\right)$$

$$= \sum_{k=1}^{n} \sum \frac{\partial \varphi^z_{i_1\cdots i_p, j_1\cdots j_q}}{\partial \bar{z}^k} d\bar{z}^k \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}.$$

On a Riemann surface a 1-form in $\varphi \in \mathcal{E}^1(U)$ can be written using a complex coordinate $z : U \to \mathbb{C}$:

$$\varphi = \varphi_z dz + \varphi_{\bar{z}} d\bar{z},$$

A form in $\mathcal{E}^{1,0}$ is written in local coordinates as

$$\varphi = \varphi_z dz$$

and the Dolbeault operator is written in local coordinates as

$$\bar{\partial}\varphi = -\frac{\partial \varphi_z}{\partial \bar{z}} dz \wedge d\bar{z}.$$

Observe that on a surface $\mathcal{E}^{p,q} = 0$ for $p \geq 2$ or $q \geq 2$.

4. $\mathcal{Z}^{p,q}(U)$ of $\bar{\partial}$-closed (p,q)-forms on an open set U in a complex manifold M. That is, $\varphi \in \mathcal{Z}^{p,q}(U)$ if $\bar{\partial}\varphi = 0$. On a Riemann surface, because of the previous item, $\mathcal{Z}^{p,q}(U) = \mathcal{E}^{p,q}(U)$ if $q = 1$. On the other hand, a closed form in $\mathcal{E}^{1,0}(U)$ satisfies

$$\bar{\partial}(\varphi_z dz) = \frac{\partial \varphi_z}{\partial \bar{z}} d\bar{z} \wedge dz = 0.$$

That means that φ_z is holomorphic.

5. $\Omega^p(U)$ of holomorphic p-forms on an open set U in a complex manifold M. Those are p forms which in local coordinates are written

$$\varphi = \sum \varphi^z_{i_1\cdots i_p} dz^{i_1} \wedge \cdots \wedge dz^{i_p}$$

with $\varphi^z_{i_1\cdots i_p}$ holomorphic.

A presheaf lacks a condition which is necessary in order to reconstruct global data from local information:

Definition 12.3 *A presheaf $(\mathcal{F}, \rho_V^U)$ is a sheaf if for any open subsets cover $U = \cup U_i$*

1. *if $f, g \in \mathcal{F}(U)$ and $\rho_{U_i}^U(f) = \rho_{U_i}^U(g)$ for all i, then $f = g$*
2. *given $f_i \in \mathcal{F}(U_i)$ such that $\rho_{U_i \cap U_j}^{U_i}(f_i) = \rho_{U_i \cap U_j}^{U_j}(f_j)$, there exists $f \in \mathcal{F}(U)$ such that $\rho_{U_i}^U(f) = f_i$.*

Example 12.4 *Let X be a topological space and for each open set $U \subset X$ let $\mathcal{F}(U)$ be the ring of continuous functions defined on U. Together with the usual restriction maps $\rho_V^U(f) = f_{|V}$, $(\mathcal{F}, \rho_V^U)$ is a sheaf.*

Exercise 12.5 *Let X be a Riemann surface and $\mathcal{B}(U)$ the $\mathbb{C}$-algebra of bounded holomorphic functions on U. Then $\mathcal{B}$ is a presheaf but not a sheaf.*

Example 12.6 *Let X be a topological space and for each open set $U \subset X$ let $\mathcal{F}(U)$ be the ring of constant functions defined on U. Together with the usual restriction maps $\rho_V^U(f) = f_{|V}$, $(\mathcal{F}, \rho_V^U)$ is a presheaf but not a sheaf. The problem is that if U has two components U_1 and U_2, in general, a constant function f_1 on U_1 and a constant function f_2 on U_2 do not match together to define a constant function on U. To obtain a sheaf we consider the ring of locally constant functions.*

Example 12.7 (The skyscraper sheaf) *Let $z \in X$ be a point on a Riemann surface. The skyscraper sheaf supported on z is the sheaf $\mathcal{S}_z$ defined by $\mathcal{S}_z(U) = 0$ if $z \notin U$ and $\mathcal{S}_z(U) = \mathbb{C}$ if $z \in U$.*

Example 12.8 *The sheaf of holomorphic functions with values in $\mathbb{C}^*$ is denoted by $\mathcal{O}_X^*$.*

Definition 12.9 *Let $(\mathcal{F}, \rho_V^U)$ and $(\mathcal{G}, \sigma_V^U)$ be two presheaves over a topological manifold X. A morphism $\alpha : \mathcal{F} \to \mathcal{G}$ is a family of group homomorphisms*

$$\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

satisfying $\alpha_V \circ \rho_V^U = \sigma_V^U \circ \alpha_U$. In the case the homomorphisms are isomorphisms we say the presheaves are isomorphic.

Example 12.10 *The following are morphisms between sheaves:*

1. *$d : \ \mathcal{E}^p \to \mathcal{E}^{p+1}$*
2. *$\bar{\partial} : \ \mathcal{E}^{p,q} \to \mathcal{E}^{p,q+1}$*

Example 12.11 *Let X be a Riemann surface. We define a morphism $\alpha : \mathcal{O}_X \to \mathcal{O}_X^*$, for each $U \subset X$ and $f : U \to \mathbb{C}$, by $\alpha(f) = e^{2\pi i f} : U \to \mathbb{C}^*$.*

Observe that the image $\alpha(\mathcal{F}) \subset \mathcal{G}$ is not necessarily a sheaf although it is always a presheaf.

Example 12.12 *For $X = \mathbb{C}^*$ let $U_1 = \mathbb{C}^* \setminus \{ \ z < 0 \ \}$ and $U_2 = \mathbb{C}^* \setminus \{ \ z > 0 \ \}$. The function $f(z) = z$ is well defined on $\mathbb{C}^*$ but it is not in the image of α. On the other hand, its restriction to U_i is in the image of $\alpha_{|U_i}$.*

Example 12.13 *Show that the Kernel $K(\alpha)$ of a morphism of sheaves, defined by $K(U) = Ker(\alpha_U)$, is a sheaf. Show that $K(\mathcal{O}_X \to \mathcal{O}_X^*) = \mathbb{Z}$.*

12.2.1 The sheaf associated to a presheaf

Images of sheaf morphisms, although presheaves, are not necessarily sheaves. But from a presheaf we can always form the associated sheaf by constructing a sheaf of germs.

Let X be a topological space and $\mathcal{F}$ a presheaf. We define an equivalence relation between f_1 and f_2, where $f_i \in \mathcal{F}(U_i)$ with U_i containing x. We say $f_1 \equiv f_2$ if $U_1 \cap U_2$ contains x and both elements restricted to the intersection coincide. We denote the equivalence class of f by f_x and call it a germ of $\mathcal{F}$ at $x \in X$. The stalk of $\mathcal{F}$ at x is the set of equivalence classes which is written $\mathcal{F}_x$.

Let $|\mathcal{F}| = \bigcup_{x \in X} \mathcal{F}_x$ be the disjoint union of all stalks and $p : |\mathcal{F}| \to X$ the obvious projection. The topology of $|\mathcal{F}|$ is defined by giving as basis of open sets the following subsets $N(U, f) = \{ f_y \mid y \in U \}$ for each open $U \in X$ and $f \in \mathcal{F}(U)$.

Exercise 12.14 *Check that $N(U, f)$ form a basis of a topology on $|\mathcal{F}|$ and that $p : |\mathcal{F}| \to X$ is a local homeomorphism.*

Exercise 12.15 *Prove that for a Riemann surface X and $\mathcal{O}_X$ the sheaf of holomorphic functions, $|\mathcal{O}_X|$ is Hausdorff.*

12.2.2 Exact sequences

The local behavior of a morphism $\alpha : \mathcal{F} \to \mathcal{G}$ between sheaves is better described using the induced homomorphism $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$ between stalks. But one should be careful as, although injectivity of α_x for all x implies injectivity of $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$, surjectivity of α_x for all x does not imply surjectivity of $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$.

Definition 12.16 *A sequence of morphisms between sheaves*

$$\mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_{n-1} \to \mathcal{F}_n$$

is exact if, for each $x \in X$, the sequence of induced homomorphisms between stalks

$$\mathcal{F}_{1_x} \to \mathcal{F}_{2_x} \to \cdots \to \mathcal{F}_{n-1_x} \to \mathcal{F}_{n_x}$$

is exact.

A first relation between local and global behavior is given by the following

Lemma 12.17 *Let*

$$0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G}$$

be an exact sequence os sheaves. Then, for every open set $U \subset X$,

$$0 \longrightarrow \mathcal{E}(U) \xrightarrow{\alpha_U} \mathcal{F}(U) \xrightarrow{\beta_U} \mathcal{G}(U)$$

is exact.

Proof.

1. The injectivity of $\mathcal{E}(U) \to \mathcal{F}(U)$ follows immediately from the injectivity of $\mathcal{E}_x \to \mathcal{F}_x$.

2. To prove that $\alpha_U(\mathcal{E}(U)) \subset Ker\beta_U$ we observe that $\alpha_x(\mathcal{E}_x) \subset Ker\beta_x$ means that for each $f \in \mathcal{E}(U)$ and $x \in U$, there exists a sufficiently small neighborhood V of x such that $\alpha_V(f_{|V}) \subset Ker\beta_V$ and we conclude because $\mathcal{E}$ is a sheaf.

3. To prove that $Ker\beta_U \subset \alpha_U(\mathcal{E}(U))$ we start with $f \in Ker\beta_U$ therefore $f_x \in Ker\beta_x$ for each $x \in U$. We obtain, from exactness, germs $g_x \in \mathcal{E}_x$ such that $\alpha_x(g_x) = f_x$. That means that there exists a covering by open subsets V_i with $g_i \in \mathcal{E}(V_i)$ such that $\alpha_{V_i}(g_i) = f_{|V_i}$. In the intersection $V_i \cap V_j$ we obtain $\alpha_{V_i \cap V_j}(g_j - g_i) = 0$ and therefore by the first item $g_j = g_i$. We conclude using the second sheaf axiom to obtain a $g \in \mathcal{E}(U)$.

□

Example 12.18 *On any Riemann surface X, the sequence*

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

is exact.

12.3 The de Rham and Dolbeault complexes.

12.3.1 The de Rham complex

The de Rham complex over a surface is

$$0 \to \mathbb{R} \to \mathcal{C}^\infty \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \to 0.$$

where $\mathcal{C}^\infty$ is the sheaf of C^∞ functions, $\mathcal{E}^i$ is the sheaf of i-forms and d is the exterior differentiation. Poincaré's lemma says that the sequence is exact.

Remark 12.19 *On any real manifold the following sequence is exact:*

$$0 \to \mathbb{R} \to \mathcal{C}^\infty \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \xrightarrow{d} \cdots$$

12.3.2 The Dolbeault complex

Recall the Cauchy-Riemann operator $\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ defined on functions on an open subset $U \subset \mathbb{C}$. It is better understood in the guise of a morphism between sheaves:

$$\bar{\partial} : \mathcal{C}^\infty \to \mathcal{E}^{0,1}$$

which fits in the following complex

$$0 \to \mathcal{O} \longrightarrow \mathcal{C}^\infty \xrightarrow{\bar{\partial}} \mathcal{E}^{0,1} \to 0.$$

In order to show that the sequence is exact we observe that $\bar{\partial} : \mathcal{C}^\infty \to \mathcal{E}^{0,1}$ is the operator $f \to \frac{\partial f}{\partial \bar{z}} d\bar{z}$ so that we need the local solvability of the Cauchy-Riemann equation: for each $g \in C^\infty(U)$ there exists $V \subset U$ and $f \in C^\infty(V)$ such that

$$\frac{\partial f}{\partial \bar{z}} = g.$$

on V. This follows from Dolbeault's lemma.

Remark 12.20 *On an n-dimensional complex manifold we have the following exact sequence*

$$0 \to \mathcal{O} \longrightarrow \mathcal{C}^\infty \xrightarrow{\bar{\partial}} \mathcal{E}^{0,1} \xrightarrow{\bar{\partial}} \mathcal{E}^{0,2} \cdots,$$

and more generally

$$0 \to \Omega^{p,q} \longrightarrow \mathcal{E}^{p,q} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,q+1} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,q+2} \cdots.$$

12.4 Cohomology

In this section we define sheaf cohomology but as we will deal only uo to first cohomology groups we state most results in a simplified form. We use Cech cohomology instead of using resolutions.

Let X be a topological space and $\mathcal{F}$ a sheaf of abelian groups on X. Let $\mathcal{U} = \{U_i\}$ be an open covering of X. Define the group of (Cech) q-cochains ($q \geq 0$), with respect to $\mathcal{U}$, as

$$C^q(\mathcal{U}, \mathcal{F}) = \Pi_{(i_0,\cdots,i_q)}\mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_q}),$$

that is, a q-cochain is a family of sections $c_{(i_0,\cdots,i_q)} \in \mathcal{F}(U_{i_0} \cap \cdots U_{i_q})$. The group operations between cochains are obviously defined using the group structure of the sheaf.

We define the coboundary:

$$\delta : C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$$

by $\delta((c_i)) = (c_{ij})$, where $c_{ij} = c_i - c_j$ is defined on $U_i \cap U_j$ and

$$\delta : C^1(\mathcal{U}, \mathcal{F}) \to C^2(\mathcal{U}, \mathcal{F})$$

by $\delta((c_{ij})) = (c_{ijk})$, where $c_{ijk} = c_{jk} - c_{ik} + c_{ij}$ is defined on $U_i \cap U_j \cap U_k$. In general we define

$$(\delta c)_{i_0 \cdots c_{q+1}} = c_{i_1 \cdots i_{q+1}} - c_{i_0 i_2 \cdots i_{q+1}} + \cdots + (-1)^{q+1} c_{i_0 i_2 \cdots i_q},$$

which defines a complex of abelian groups

$$0 \to C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F}) \cdots$$

Observe that the kernel of $\delta : C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$ is given by (c_i) with $c_i - c_j = 0$ on $U_i \cap U_j$, that is, the 0-cochains which are global sections. In general we define

$$Z^q(\mathcal{U}, \mathcal{F}) = Ker\left(\delta : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F})\right)$$

to be the q-cocycles and, for $q \geq 1$,

$$B^q(\mathcal{U}, \mathcal{F}) = Im\left(\delta : C^{q-1}(\mathcal{U}, \mathcal{F}) \to C^q(\mathcal{U}, \mathcal{F})\right)$$

the q-coboundaries. A simple computation shows that $B^q(\mathcal{U}, \mathcal{F}) \subset Z^q(\mathcal{U}, \mathcal{F})$.

Definition 12.21 *The q-cohomology group with coefficients in $\mathcal{F}$, with respect to $\mathcal{U}$ is the quotient*

$$H^q(\mathcal{U}, \mathcal{F}) = \frac{Z^q(\mathcal{U}, \mathcal{F})}{B^q(\mathcal{U}, \mathcal{F})}$$

Example 12.22 *Observe that $H^0(\mathcal{U}, \mathcal{F}) = Z^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$, that is, $H^0(\mathcal{U}, \mathcal{F})$ does not depend on the covering $\mathcal{U}$ and is isomorphic to $\mathcal{F}(X)$, the global sections.*

Example 12.23 *If $(c_{ij}) \in Z^1(\mathcal{U}, \mathcal{F})$ is a cocycle then $c_{jk} - c_{ik} + c_{ij} = 0$ (cocycle relation) on $U_i \cap U_j \cap U_k$. Clearly, if $(c_{ij}) = \delta((c_i))$ we obtain $c_{ij} = c_i - c_j$ and therefore the cocycle relation is satisfied.*

In order to define cohomology groups independently of the covering we observe first that a finer covering defines cohomology groups which are related to the original ones. In fact, a covering is finer if there exists a map $\tau_{\mathcal{U},\mathcal{W}}$ (a refining map), which we will write τ when there is no doubt about the concerned coverings, defined on the indexing set of $\mathcal{V}$ such that $V_i \subset U_{\tau(i)}$. This map induces a map on the cohomology

$$\tau^*_{\mathcal{U},\mathcal{V}} : H^q(\mathcal{U},\mathcal{F}) \to H^q(\mathcal{V},\mathcal{F})$$

by first defining the map on cochains: for $c \in Z^q(\mathcal{U},\mathcal{F})$ we define $\tau^*(c) \in Z^q(\mathcal{V},\mathcal{F})$ by $\tau^*(c)_{i_0\cdots i_q} = c_{\tau(i_0)\cdots\tau(i_q)}$, and observing that coboundaries are mapped to coboundaries.

In the following we will only deal with $H^i(\mathcal{V},\mathcal{F})$ with $i = 0, 1$, so we will state simplified versions of more general results.

Lemma 12.24 *The map $\tau^* : H^1(\mathcal{U},\mathcal{F}) \to H^1(\mathcal{V},\mathcal{F})$ does not depend on the refining map.*

Proof. Let σ and τ be two refining maps. Then $(\tau^*(c) - \sigma^*(c))_{ij} = c_{\tau(i)\tau(j)} - c_{\sigma(i)\sigma(j)}$ which by the cocycle relation is equal to $c_{\tau(i)\sigma(j)} + c_{\sigma(j)\tau(j)} - c_{\sigma(i)\tau(i)} - c_{\tau(i)\sigma(j)} = c_{\sigma(j)\tau(j)} - c_{\sigma(i)\tau(i)}$. Clearly the functions $c_{\sigma(j)\tau(j)}$ give a splitting of the cocycle $\tau^*(c) - \sigma^*(c)$.

□

Lemma 12.25 *The map $\tau^* : H^1(\mathcal{U},\mathcal{F}) \to H^1(\mathcal{V},\mathcal{F})$ is injective.*

Proof. Suppose $c \in Z^1(\mathcal{U},\mathcal{F})$ is a cocycle such that $\tau^*(c) \in B^1(\mathcal{V},\mathcal{F})$. That is, there exists $f_i \in \mathcal{F}(V_i)$ such that $\tau^*(c)_{ij} = f_i - f_j$. We want to find a splitting of c, that is, find functions $g_\alpha \in \mathcal{F}(U_\alpha)$ such that $c_{\alpha\beta} = g_\alpha - g_\beta$. In order to define g_α, we define it first over $U_\alpha \cap V_i$. The essential point is to understand what happens on the intersection $(U_\alpha \cap V_i) \cap (U_\alpha \cap V_j) = U_\alpha \cap V_i \cap V_j$.

On $U_\alpha \cap V_i \cap V_j$ we may write $f_i - f_j = c_{\tau(i)\tau(j)} = c_{\tau(i)\alpha} + c_{\alpha\tau(j)}$. Therefore

$$f_i - c_{\tau(i)\alpha} = f_j - c_{\tau(j)\alpha}$$

so by the second sheaf axiom we obtain an element $g_\alpha \in \mathcal{F}(U_\alpha)$. Now it is easy to check that the coboundary of this 0-cochain is c. In fact, we compute in $U_\alpha \cap U_\beta \cap V_j$ and use the first sheaf axiom.

□

The previous lemma allows us to define the cohomology of X by using arbitrarily fine coverings.

Definition 12.26 *The cohomology of X with coefficients in $\mathcal{F}$, $H^1(X,\mathcal{F})$, is the direct limit of the system $(H^1(\mathcal{U},\mathcal{F}), \tau^*)$.*

The direct limit is the quotient of the disjoint union $\bigcup H^1(\mathcal{U},\mathcal{F})$ by the equivalence relation which identifies two classes $\xi \in H^1(\mathcal{U},\mathcal{F})$ and $\eta \in H^1(\mathcal{V},\mathcal{F})$ if there exists a common refinement $\mathcal{W}$ such that $\tau^*_{\mathcal{U},\mathcal{W}}(\xi) = \tau^*_{\mathcal{V},\mathcal{W}}(\eta)$.

Remark 12.27 *From the definition and the injectivity of the refinement maps on the cohomology, we observe that* $H^1(\mathcal{U},\mathcal{F}) \to H^1(X,\mathcal{F})$ *is injective so* $H^1(X,\mathcal{F}) = 0$ *if and only if* $H^1(\mathcal{U},\mathcal{F}) = 0$ *for all coverings.*

Cohomology computations are greatly simplifyed by the following theorem.

Theorem 12.28 (Leray) *Let* $\mathcal{F}$ *be a sheaf of abelian groups over a manifold* X *and let* $\mathcal{U}$ *be a covering such that* $H^1(U_i,\mathcal{F}) = 0$ *for every open set of the covering. Then,*

$$H^1(\mathcal{U},\mathcal{F}) \to H^1(X,\mathcal{F})$$

is an isomorphism.

12.5 Computations

Example 12.29 (The skyscraper sheaf) *Let* $z \in X$ *be a point on a Riemann surface. Let* $\mathcal{S}_z$ *be the skyscraper sheaf supported on* z*. Then* $H^0(X,\mathcal{S}_z) = \mathbb{C}$ *and* $H^1(X,\mathcal{S}_z) = 0$*.*

Remark 12.30 *In the next example we make use of a partition of unity. For that sake, the definition of manifold usually includes second contability which is necessary for partition of unity arguments (see Warner). But in the case of Riemann surfaces second contability is automatic (Radó's theorem). A second countable space is a topological space with a countable basis of open subsets (each open set being a union of those open subsets).*

Remark 12.31 *A partition of unity subordinated to an open cover* $X = \cup U_i$ *of a differentiable manifold* X *is a family of* $\mathbb{C}^\infty$ *functions* φ_i *satisfying:*

1. *The support of each* φ_i *is contained in one of the* U_i*.*
2. *The supports of* φ_i *are locally finite, that is, for each* $x \in X$*, the set* $\{i \mid \varphi_i(x) \neq 0\}$ *is finite.*
3. *For each* $x \in X$*,* $\sum_i \varphi_i(x) = 1$*.*

A basic theorem is that any open cover of a differentiable manifold (Hausdorff and second countable) has a subordinated partition of unity with compact supports.

Example 12.32 *Let* X *be a manifold. Then* $H^1(X,\mathcal{C}^\infty) = 0$ *where* $\mathcal{C}^\infty$ *is the sheaf of differential functions. The same result is valid for sheaves of differential forms.*

Proof. This follows from the existence of a partition of unity subordinate to any open covering $\mathcal{U}$. Take $c_{ij} \in Z^1(\mathcal{U},\mathcal{F})$ any 1-cocycle. For any open set U_i consider the sum

$$f_i = \sum_j \psi_j c_{ij}$$

where the domain of each $\psi_j c_{ij}$ is considered to be U_i by extending it to zero in the complement of its support in $U_i \cap U_j$ (observe that $\psi_j = 0$ on a neighborhood of $\partial U_j \cap U_i$). Now on $U_i \cap U_j$ we have

$$f_i - f_j = \sum_k \psi_k c_{ik} - \sum_k \psi_k c_{jk} = \sum_k \psi_k (c_{ik} - c_{jk}) = \sum_k \psi_k c_{ij} = c_{ij}$$

□

Example 12.33 *For a simply connected Riemann surface X the first cohomology groups $H^1(X, \mathbb{C})$ and $H^1(X, \mathbb{Z})$ are trivial.*

Proof. We prove that $H^1(X, \mathbb{C}) = 0$ and leave the second result as an exercise. Take $c_{ij} \in Z^1(\mathcal{U}, \mathbb{C})$ any 1-cocycle. By the previous example there exists splitting functions f_i such that $c_{ij} = f_i - f_j$. Therefore, as $0 = dc_{ij} = df_i - df_j$, there exists a closed differential form ω defined on X. As X is simply connected we obtain that $\omega = df$ for a certain function. Define now $c_i = f_i - f$ on $U_i \cap U_j$. Then c_i is locally constant and it is also a splitting for c_{ij}. □

Example 12.34 $H^1(\mathbb{C}^*, \mathbb{Z}) = \mathbb{Z}$.

Proof. Use Leray's theorem and the previous example with the covering $\mathcal{U}$ by the simply connected open sets $U_1 = \mathbb{C}^* - \mathbb{R}^+$, $U_1 = \mathbb{C}^* - \mathbb{R}^-$. It suffices to compute $H^1(\mathcal{U}, \mathbb{Z})$. $U_1 \cap U_2$ has two components so c_{12} has two values. In fact all values are allowed for a 1-cocycle. Therefore $Z^1(\mathcal{U}, \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$. To compute $B^1(\mathcal{U}, \mathbb{Z})$ observe that $C^0(\mathcal{U}, \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$ (there are two connected open sets in the covering) and if $f \in C^0(\mathcal{U}, \mathbb{Z})$, then $\delta(f)_{12} = f_1 - f_2$. We conclude that $B^1(\mathcal{U}, \mathbb{Z}) = \mathbb{Z}$ contained as a diagonal in $Z^1(\mathcal{U}, \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$. □

Theorem 12.35 (Mittag-Leffler's theorem) *Let $\Omega \subset \mathbb{C}$ be an open set and $\mathcal{O}$ be the sheaf of germs of holomorphic functions on Ω. Then $H^1(\Omega, \mathcal{O}) = 0$.*

Proof.
Let $\mathcal{U} = \bigcup U_i$ be a covering and φ_i a partition of unity subordinated to $\mathcal{U}$. Given a holomorphic 1-cocycle c_{ij} there exists (by 12.32) C^∞ functions f_i defined on U_i such that $c_{ij} = f_i - f_j$ on $U_i \cap U_j$. As $0 = \bar{\partial} c_{ij} = \bar{\partial} f_i - \bar{\partial} f_j$ there exists a globally defined function g which is locally defined by $g = \bar{\partial} f_i$. By Dolbeault's lemma, we can solve the equation

$$\frac{\partial f}{\partial \bar{z}} = g.$$

We conclude, as in 12.33, that $f_i - f$ is a splitting of the 1-cocycle. □

12.6 The long exact sequence

Theorem 12.36 *Let*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

be an exact sequence of sheaves of abelian groups over a topological space X. Then, the following sequence is exact.

$$0 \to H^0(X, \mathcal{E}) \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^1(X, \mathcal{E}) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G})$$

Remark 12.37 *In fact when X is paracompact (for instance, a manifold), we can extend that sequence to include the higher cohomology groups:*

$$0 \to H^0(X, \mathcal{E}) \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^1(X, \mathcal{E}) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G})$$
$$\to H^2(X, \mathcal{E}) \to H^2(X, \mathcal{F}) \to H^2(X, \mathcal{G}) \to \cdots$$

Proof. The only non-obvious map is $H^0(X,\mathcal{G}) \to H^1(X,\mathcal{E})$. We will make it explicit and leave the rest of the proof for the reader. We start with an element $g \in \mathcal{G}(X)$. Since $\mathcal{F} \to \mathcal{G}$ is surjective there exists an open covering $\mathcal{U}$ and elements $f_i \in \mathcal{F}(U_i)$ such that, for all i, $f_i \to g_{|U_i}$. Therefore $f_i - f_j$ are in the kernel of $\mathcal{F}(U_i \cap U_j) \to \mathcal{G}(U_i \cap U_j)$. By Lemma 12.17 there exists $f_{ij} \in \mathcal{E}(U_i \cap U_j)$ such that $f_{ij} \to f_i - f_j$. Again using 12.17, we conclude that $f_{ij} \in Z^1(\mathcal{U},\mathcal{E})$ is a 1-cocycle and therefore defines an element in $H^1(X,\mathcal{E})$.

Exercise 12.38 *Verify that the element does not depend on the choice of covering neither the choice of the* f_i.

□

12.7 The abstract de Rham theorem

A resolution of a sheaf $\mathcal{F}$ of abelian groups over a topological space X is an exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{L}^0 \xrightarrow{d} \mathcal{L}^1 \xrightarrow{d} \mathcal{L}^2 \xrightarrow{d} \cdots .$$

The de Rham and Dolbeault complexes are resolutions of $\mathbb{C}$ and $\mathcal{O}$ respectively. Associated to a resolution is the complex $\mathcal{L}$ of global sections of sheaves

$$0 \to \mathcal{F}(X) \to \mathcal{L}^0(X) \xrightarrow{d} \mathcal{L}^1(X) \xrightarrow{d} \mathcal{L}^2(X) \xrightarrow{d} \cdots .$$

The cohomology of that complex is defined as

$$H^i(\mathcal{L}) = \frac{Ker\left(\mathcal{L}^i(X) \xrightarrow{d} \mathcal{L}^{i+1}(X)\right)}{d\left(\mathcal{L}^{i-1}(X)\right)}.$$

On a Riemann surface X, for the Dolbeault complex

$$0 \to \mathcal{O} \longrightarrow \mathcal{C}^\infty \xrightarrow{\bar\partial} \mathcal{E}^{0,1} \to 0.$$

we obtain

$$H^0(\mathcal{L}) = \{\ f \in \mathcal{C}^\infty(X) \mid \bar\partial f = 0\ \}$$

$$H^1(\mathcal{L}) = \frac{\mathcal{E}^{0,1}(X)}{\{\ \bar\partial f \mid f \in \mathcal{C}^\infty(X)\ \}}.$$

The relation between the cohomology of the resolution and Čech cohomology is given by the following

Theorem 12.39 *If the resolution satifies* $H^i(X,\mathcal{L}^j) = 0$ *for each* $i \geq 1$ *and* $j \geq 0$, *then there exists an isomorphism*

$$H^i(\mathcal{L}) \longrightarrow H^i(X,\mathcal{F}).$$

In particular, in the case of a Riemann surface X, we obtain the isomorphism

$$H^1(\mathcal{L}) = \frac{\mathcal{E}^{0,1}(X)}{\{\ \bar\partial f \mid f \in \mathcal{C}^\infty(X)\ \}} \longrightarrow H^1(X,\mathcal{O}).$$

Proof. We prove the theorem in the case of the Dolbeault complex on a Riemann surface X:

$$0 \to \mathcal{O} \longrightarrow \mathcal{C}^\infty \xrightarrow{\bar{\partial}} \mathcal{E}^{0,1} \to 0.$$

From the long exact sequence we obtain

$$0 \to H^0(X,\mathcal{O}) \to H^0(X,\mathcal{C}^\infty) \to H^0(X,\mathcal{E}^{0,1}) \to H^1(X,\mathcal{O}) \to H^1(X,\mathcal{C}^\infty) \to H^1(X,\mathcal{E}^{0,1}) \to 0.$$

Now, using that $H^1(X,\mathcal{C}^\infty) = 0$ we obtain

$$H^1(X,\mathcal{O}) = \frac{H^0(X,\mathcal{E}^{0,1})}{\bar{\partial} H^0(X,\mathcal{C}^\infty)}.$$

□

12.8 Line bundles and Sheaves

Line bundles are rank one vector bundles. The transition functions g_{ij} associated to trivializations over a covering $\mathcal{U}$, being with values in $\mathbb{C}^*$ and satisfying the cocycle relation, we can consider $g_{ij} \in Z^1(\mathcal{U},\mathcal{E}^*)$. A holomorphic line bundle defines transition functions in $Z^1(\mathcal{U},\mathcal{O}^*)$. To be more precise we state the following

Proposition 12.40 *There exists a bijection between isomorphism classes of line bundles over a manifold M and $H^1(M,\mathcal{E}^*)$.*

Proof. We defined above that map from a line bundle to a 1-cocycle of the sheaf $\mathcal{E}^*$. We have shown before that one can reconstruct from any cocycle a line bundle so this shows that the map is surjective.

If two line bundles are isomorphic $\varphi : E \to E'$, we can find a covering and common trivializations subordinated to that covering. Then we can see that there exist functions φ_i such that the transition functions of E' are given by $g'_{ij} = \varphi_i g_{ij} {\varphi_j}^{-1}$. But $\varphi_i {\varphi_j}^{-1}$ is a coboundary. □

The classification of complex line bundles over an orientable surface X up to isomorphism can be obtained using the long exact sequence associated to the exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{E}_X \to \mathcal{E}_X^* \to 1$$

where $\mathcal{E}$ is the sheaf of smooth functions over X and $\mathcal{E}^*$ is the sheaf of never vanishing smooth functions. The long exact sequence is

$$0 \to H^0(X,\mathbb{Z}) \to H^0(X,\mathcal{E}) \to H^0(X,\mathcal{E}^*) \to H^1(X,\mathbb{Z}) \to$$
$$\to H^1(X,\mathcal{E}) \to H^1(X,\mathcal{E}^*) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{E}) \to ...$$

As $H^1(X,\mathcal{E}) = H^2(X,\mathcal{E}) = 0$ we obtain that

$$H^1(X,\mathcal{E}^*) \to H^2(X,\mathbb{Z})$$

is an isomorphism. $H^2(X,\mathbb{Z}) \equiv \mathbb{Z}$ classifies complex line bundles up to isomorphisms. The element in $H^2(X,\mathbb{Z})$ associated to a line bundle is called its Chern class.

Analogously, there exists a bijection between isomorphism classes of holomorphic line bundles over a complex manifold M and $H^1(M,\mathcal{O}^*)$.

Exercise 12.41 *Prove that two line bundles L and L' given by transition functions $g_{ij}, g'_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ are isomorphic if there exist functions $\varphi_i \in \mathcal{O}^*(U_i)$ such that $g'_{ij} = \varphi_i g_{ij} \varphi_j^{-1}$.*

Recall the exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 1$$

where $\mathcal{O}$ is the sheaf of holomorphic functions over X and $\mathcal{O}^*$ is the sheaf of never vanishing holomorphic functions. The long exact sequence associated is

$$0 \to H^0(X,\mathbb{Z}) \to H^0(X,\mathcal{O}) \to H^0(X,\mathcal{O}^*) \to H^1(X,\mathbb{Z}) \to$$

$$\to H^1(X,\mathcal{O}) \to H^1(X,\mathcal{O}^*) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}) \to ...$$

Observe that $H^2(X,\mathcal{O}) = 0$ and that the first maps of the sequence are

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^*.$$

As the last map, $\mathbb{C} \to \mathbb{C}^*$, is surjective we have the exact sequence

$$0 \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}) \to H^1(X,\mathcal{O}^*) \to H^2(X,\mathbb{Z}) \to 0.$$

In particular, up to isomorphisms, the holomorphic line bundles with trivial Chern class is the abelian group

$$H^1(X,\mathcal{O})/H^1(X,\mathbb{Z}).$$

12.8.1 Sections of line bundles and sheaves

If L is a holomorphic line bundle the sheaf of holomorphic sections will be denoted $\mathcal{O}(L)$.

Given a divisor D and an open set $U \subset S$ of a Riemann surface we define

$$\mathcal{O}_D(U) = \{ \text{ f meromorphic on } U \mid div(f) \geq -D \text{ on } U \}.$$

It is easy to show that $\mathcal{O}_D(U)$ defines a sheaf which we denote $\mathcal{O}_D$. In this way we associate to a divisor a sheaf.

An element $f \in \mathcal{O}_D(a)$ in the stalk at $a \in M$ is given by a convergent Laurent series (written in a chart centred at a)

$$f = \sum_{n \geq -D(a)} c_n z^n.$$

Suppose now that $D_1 \leq D_2$. Then $\mathcal{O}_{D_1}(U) \subset \mathcal{O}_{D_2}(U)$. There exists an exact sequence

$$0 \to \mathcal{O}_{D_1} \to \mathcal{O}_{D_2} \to \mathcal{S}_{D_1}^{D_2} \to 0$$

The quotient sheaf $\mathcal{S}_{D_1}^{D_2}$ is such that the stalk at a point $a \notin D_1 \cup D_2$ is 0 and at a point $a \in D_1 \cup D_2$ it is a finite vector space isomorphic to the vector space $\sum_{-D_2(a) \leq n < -D_1(a)} c_n z^n$.

Proposition 12.42 *Let D be a divisor on a Riemann surface. Then, the sheaf of holomorphic sections of $L(D)$ constructed above is identified with $\mathcal{O}_D$.*

Proof. Define the map $\mathcal{O}_D(U) \to H^0(U, L(D))$ by $f \to fs$ where s is the standard section of $L(D)$ defined above. This is an isomorphism because $div(f) \geq -D$ if and only if $div(fs) \geq 0$.

□

12.8.2 Examples

Proposition 12.43 *$dimH^0(X, \mathcal{S}_{D_1}^{D_2}) = deg(D_2) - deg(D_1)$.*

Proof. The sheaf $\mathcal{S}_{D_1}^{D_2}$ is supported at $supp(D_1) \cup supp(D_2)$ and therefore

$$dimH^0(X, \mathcal{S}_{D_1}^{D_2}) = \sum dim\mathcal{O}_{D_2}/\mathcal{O}_{D_1}(a) = \sum (D_2(a) - D_1(a)).$$

□

Proposition 12.44 *$dimH^1(X, \mathcal{S}_{D_1}^{D_2}) = 0$.*

Proof. It suffices to chose a covering U_i such that $U_i \cap U_j$ does not intersect the divisors, for $i \neq j$. A cochain defined on those intersections is zero. □

12.8.3 Chern class of a line bundle

Consider a line bundle L over a surface X. A Hermitian metric on L is a C^∞ function $H : L \times L \to \mathbb{C}$ which fiberwise is a Hermitian metric, that is, $H(v, w)$ is linear on the first variable, anti-linear on the second variable and $H(v, w) = \overline{H(w, v)}$.

In terms of trivializations $U_i \times \mathbb{C}$ we can write $v = a_i$ and therefore $H(v, v) = \lambda_i^2 a_i \bar{a}_i$. Changing coordinates, we have $a_j = g_{ji} a_i$ and $\lambda_j^2 = \lambda_i^2 g_{ij} \bar{g}_{ij}$.

Definition 12.45 *For a Hermitian metric H on a holomorphic line bundle over a surface X, define the Chern form*

$$c_1(L, H) = \frac{1}{2\pi i} \frac{\partial^2}{\partial z \partial \bar{z}} log\lambda_i^2 \; dz \wedge d\bar{z}.$$

Exercise: Verify that the form is well defined and that its cohomology class does not depend on the chosen Hermitian metric.

Proposition 12.46 *Let D be a divisor on a Riemann surface X and $L(D)$ its associated holomorphic line bundle. Then*

$$\int_X c_1(L(D), H) = \; deg(D)$$

Proof. The map $D \to L(D)$ modulo linear equivalence of divisors and isomorphism of line bundles is a homomorphism. Moreover $c_1(L(D-D'), H) = c_1(L(D), H) - c_1(L(D'), H)$. Therefore, it suffices to prove the theorem for $D = z$.

Using the holomorphic section $s : X \setminus B(r, z) \to L$ defined above we obtain

$$\int_X c_1(L(D), H) = lim_{r \to 0} \frac{1}{2\pi i} \int_{X \setminus B(r,z)} \frac{\partial^2}{\partial z \partial \bar{z}} logH(s, s) \; dz \wedge d\bar{z} = 1.$$

□

12.9 Serre's duality

We consider a holomorphic line bundle $\pi : L \to X$ over a compact Riemann surface. Define $\mathcal{E}_L^{0,1}(X)$ to be the holomorphic sections of $\mathcal{E}_L^{0,1} = \mathcal{E}^{0,1} \otimes L$. There exist a bilinear form

$$\langle , \rangle = H^0(X, K_X \otimes L^*) \times \mathcal{E}_L^{0,1}(X) \to \mathbb{C}$$

defined by writing locally

$$s \in H^0(X, K_X \otimes L^*) \quad \text{and} \quad \varphi \in \mathcal{E}_L^{0,1}(X)$$

as $s(z) = \omega(z)\lambda(z)$, where $\omega(z)$ is a local section of K_X, $\lambda(z)$ is a local section of L^* and $\varphi(z) = \bar{\omega}(z)\alpha(z)$, where $\alpha(z)$ is a local section of L.

We define first a 2-form locally

$$(s, \varphi) = \lambda(z)\alpha(z)\omega(z) \wedge \bar{\omega}(z)$$

Exercice: Verify that the definition does not depend on the choice of $\omega(z), \lambda(z)$ and $\alpha(z)$.

We can therefore integrate the 2-form on X to obtain

$$\langle s, \varphi \rangle = \int_X (s, \varphi).$$

Proposition 12.47 *The bilinear map $\langle , \rangle$ passes to a well defined bilinear map*

$$H^0(X, K_X \otimes L^*) \times H^1(X, L) \to \mathbb{C}$$

.

Proof. By Dolbeault's theorem we have that $H^1(X, L)$ is isomorphic to $\mathcal{E}_L^{0,1}(X)/\bar{\partial}\mathcal{E}_L^0$. We need to show that $\langle s, \bar{\partial}\beta \rangle = 0$ for all $\beta \in \mathcal{E}_L^0$. For that sake, write in local trivializations and a partition of unity. Over an open set $U_i \subset X$, $\beta_i(z) = f_i(z)t_i(z)$ with $f_i(z)$ of compact support. At each U_i we have $\bar{\partial}(f_i(z)t_i(z)) = \frac{\partial f_i(z)}{\partial z} d\bar{z} t_i(z)$. Therefore

$$\langle s, \bar{\partial} t_i \rangle = \int_X s_i(z) \frac{\partial f_i(z)}{\partial \bar{z}} dz \wedge d\bar{z}.$$

The result follows by Stokes theorem. □

In the next section we prove that for any holomorphic line bundle L over a compact Riemann surface, the cohomology groups $H^0(X, L)$ and $H^1(X, L)$ are finite. We state the following important duality theorem which establishes the isomorphism between $H^0(X, K_X \otimes L^*)$ and $H^1(X, L)^*$.

Theorem 12.48 *(Serre's duality theorem) The bilinear map*

$$H^0(X, K_X \otimes L^*) \times H^1(X, L) \to \mathbb{C}$$

is non-degenerate.

The proof uses Weyl's lemma.

13 Finiteness theorem

We have seen that $H^0(X, L)$ is finite dimensional for any holomorphic line bundle $L \to X$ over a Riemann surface. In this section we will show that for a compact Riemann surface X and any holomorphic line bundle $L \to X$ the cohomology group $H^1(X, L)$ also is finite dimensional. Cf. [Narasimhan] for that proof.

Theorem 13.1 *For any holomorphic line bundle $\pi : L \to X$ over a compact Riemann surface X, the cohomology group $H^1(X, L)$ is finite dimensional.*

Proof. As in the theorem stating finiteness of $H^0(X, L)$, we consider the coverings $\mathcal{V}, \mathcal{U}$ and $\mathcal{W}$. Spaces of sections over an open set with the topology of uniform convergence on compact subsets are naturally Fréchet spaces. In order to simplify the proof we work with bounded holomorphic sections over an open set. A bounded section over U meaning that in the particular trivialization over W $s_i(z)$ is bounded. This defines a Banach space using the sup norm. We can define bounded cocycles $C_b^0(\mathcal{U}, L)$ and $Z_b^1(\mathcal{U}, L)$ with norm $||.||_{\mathcal{U}}$ which also are Banach spaces.

Define $H_b^1(\mathcal{U}, L) = Z_b^1(\mathcal{U}, L)/C_b^0(\mathcal{U}, L)$ and recall that from Mittag-Leffler's theorem $H^1(U_i, L) = H^1(W_i, L) = 0$. Therefore, by Leray's theorem $H^1(\mathcal{U}, L) = H^1(\mathcal{W}, L) = H^1(X, L)$. Consider the map

$$H^1{}_b(\mathcal{U}, L) \to H^1(\mathcal{U}, L) = H^1(X, L).$$

We claim it is an isomorphism. This follows from Leray's theorem as the composition

$$H^1(\mathcal{W}, L) \to H_b^1(\mathcal{U}, L) \to H^1(\mathcal{U}, L) = H^1(X, L)$$

is an isomorphism.

Lemma 13.2 (exercise) *There exists a constant $C > 0$ such that*

$$||s||_{\mathcal{U}} \leq ||\delta s||_{\mathcal{U}} + C||s||_{\mathcal{V}}$$

Now, we claim that $H_b^1(\mathcal{U}, L)$ is a Banach space. To see that, we prove first that $\delta C_b^0(\mathcal{U}, L)$ is closed in $Z_b^1(\mathcal{U}, L)$. We prove that in two steps:

1. As in the previous theorem we make profit of bounded sections s vanishing to order k at $a_i \in U_i$. Call this space $C_k^0(\mathcal{U}, L)$. As before,

   $$||s||_{\mathcal{V}} \leq \frac{1}{2^k}||s||_{\mathcal{U}}$$

   so by the lemma for a sufficiently large k we have

   $$||s||_{\mathcal{U}} \leq 2||\delta s||_{\mathcal{U}}.$$

   This implies that $\delta C_k^0(\mathcal{U}, L) \subset Z_b^1(\mathcal{U}, L)$ is closed. We conclude that

   $$Z_b^1(\mathcal{U}, L)/\delta C_k^0(\mathcal{U}, L)$$

   is Banach.

2. Now, $\delta : C^0_b(\mathcal{U}, L)/C^0_k(\mathcal{U}, L) \to Z^1_b(\mathcal{U}, L)/\delta C^0_k(\mathcal{U}, L)$ has finite dimensional image because $C^0_b(\mathcal{U}, L)/C^0_k(\mathcal{U}, L)$ is finite dimensional. We use the following

   Lemma 13.3 *Let $u : E \to F$ be a continuous map between Banach spaces such that the dimension of $F/u(E)$ is finite. Then $u(E) \subset F$ is closed.*

   to conclude that
$$\delta(C^0_b(\mathcal{U}, L)) \subset Z^1_b(\mathcal{U}, L)$$
   is closed.

The map between Banach spaces $Z^1_b(\mathcal{W}, L) \to H^1_b(\mathcal{U}, L)$ is then surjective and compact (by Montel's theorem). Therefore by the open mapping theorem $H^1_b(\mathcal{U}, L)$ has a relatively compact neighborhood of the origin. This implies that the dimension of $H^1_b(\mathcal{U}, L)$ is finite.

□

A consequence of the finiteness of the cohomology is the following

Theorem 13.4 *Let $L \to X$ be a line bundle over a compact Riemann surface and $a \in X$ any point. Suppose $dim H^1(X, L) = d$. Then L has a meromorphic section s with only one pole at a such that $1 \leq ord(s)_a \leq d + 1$. In particular any line bundle is obtained as the line bundle associated to a non trivial divisor.*

Proof. Consider a trivialization over U and the chart $\varphi : U \to \Delta(1)$ with $\varphi(a) = 0$. Define the covering $\mathcal{U}$ of X by two open sets U and $X - a$. To define an element in $Z^1(\mathcal{U}, L)$ it suffices to define a section over $U \cap X - a = U - a$. Using the trivialization, define the sequence
$$s_k(z) = (z, \frac{1}{z^k}) \in Z^1(\mathcal{U}, L).$$
As $dim H^1(X, L) = d$ and $H^1(\mathcal{U}, X) \to H^1(X, L)$ is injective, $d + 1$ of those sections are linearly dependent over coboundaries, that is, on $U - a$ there are non simultaneously vanishing constants c_i and holomorphic sections s_U and s_{X-a} over U and $X - a$ respectively, such that
$$c_1 s^1 + \cdots c_d s^d + c_{d+1} s^{d+1} = s_U - s_{X-a}.$$
We conclude therefore that the section s_{X-a} is meromorphic of order less than $d + 1$ but not holomorphic as one of the constants is non vanishing. □

Corollary 13.5 *Let X be a compact Riemann surface of genus g and $a \in X$ any point. Then there exists a non-constant meromorphic function having only one pole at a of order less than $g + 1$.*

In fact, the order of the pole has to be at least g.

A consequence of this corollary is the completion of the proof of the following theorem started in theorem 4.7

Theorem 13.6 (The field of meromorphic functions) *The field of meromorphic functions on a compact Riemann surface S is a finite extension of the field $\mathbb{C}(f)$ of rational functions in f, a meromorphic function on S.*

14 Riemann-Roch.

We would like to understand if we can prescribe a particular polar structure for a meromorphic function on X. If $X = \mathbb{C}P^1$ this is always possible (Exercise: construct any meromorphic with prescribed polar structure) but in general there are constraints.

To understand the compatibility conditions, suppose f is meromorphic and there exists a holomorphic 1-form ω. Then $\sum_{p\in X} res_p(f\omega) = 0$ should hold. That gives $g = dimH^1(X, \mathcal{O})$ conditions, corresponding to the dimension of the space of holomorphic forms.

Definition 14.1 *1. Let $g = dimH^1(X, \mathcal{O})$ be the arithmetic genus.*

2. Let $h^i(D) = dimH^i(X, \mathcal{O}_D)$.

3. $h^1(D) = i(D)$ is called the index of speciality of D.

4. $\chi(D) = h^0(D) - h^1(D)$.

Theorem 14.2 (Riemann-Roch theorem) *If D is a divisor on a compact Riemann surface X, then*

$$dimH^0(X, \mathcal{O}_D) - dimH^1(X, \mathcal{O}_D) = degD + 1 - g$$

where g is the genus of X.

Proof. We use induction on the divisors. For $D = 0$ the theorem is true as $dimH^0(X, \mathcal{O}) = 1$, $dimH^1(X, \mathcal{O}) = g$ and $degD = 0$. We assume the theorem true for D and show that it remains true for $D + z$ or $D - z$. Recall that

$$0 \to \mathcal{O}_D \to \mathcal{O}_{D+z} \to \mathcal{S}_z \to 0$$

which implies the long exact sequence

$$0 \to H^0(X, \mathcal{O}_D) \to H^0(X, \mathcal{O}_{D+z}) \to H^0(X, \mathcal{S}_z) \to H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D+z}) \to 0.$$

Therefore

$$dimH^0(X, \mathcal{O}_D) - dimH^0(X, \mathcal{O}_{D+z}) + dimH^0(X, \mathcal{S}_z) - dimH^1(X, \mathcal{O}_D) + dimH^1(X, \mathcal{O}_{D+z}) = 0,$$

which can be written (using $dimH^0(X, \mathcal{S}_z) = 1$) as

$$dimH^0(X, \mathcal{O}_D) - dimH^1(X, \mathcal{O}_D) = dimH^0(X, \mathcal{O}_{D+z}) - dimH^1(X, \mathcal{O}_{D+z}) - dimH^0(X, \mathcal{S}_z) - 1.$$

Observe now that $deg(D + z) = deg(D) + 1$ and we obtain the formula for $\mathcal{O}_{D+z}$ in terms of the formula for $\mathcal{O}_D$ and conversely.

□

Using Serre duality:

Theorem 14.3 *Let $L \to X$ be a holomorphic line bundle over a Riemann surface. Then $H^1(X, \mathcal{O}(L)) = H^0(X, \mathcal{O}(K \otimes L^*))^*$.*

we can write the Riemann-Roch theorem as

$$dimH^0(X, \mathcal{O}_D) - dimH^0(X, \mathcal{O}_{K-D}) = degD + 1 - g.$$

The sheaf of sections of the line bundle associated to $K - D$ is denoted Ω_{-D} so we can also write

$$dimH^0(X, \mathcal{O}_D) - dimH^0(X, \Omega_{-D}) = degD + 1 - g$$

14.1 Consequences

Corollary 14.4 *The space of holomorphic 1-forms is of dimension g, that is, $dim H^0(X,K) = dim H^1(X,\mathcal{O}) = g$.*

Proof. This follows from Serre duality as $H^0(X,K) = H^1(X, K \otimes K^*)^* = H^1(X,\mathcal{O})^*$. □

Corollary 14.5 *The degree of the canonical bundle is $2g-2$.*

Proof. This follows from $dim H^0(X,K) - dim H^1(X,K) = deg K + 1 - g$ and using Serre duality. □

Corollary 14.6

- *If $dim H^0(X,L) > 0$ then $deg L \geq 0$.*
- *If $dim H^0(X,L) > 0$ and $deg L = 0$ then L is trivial.*

Proof. The first assertion is obvious. For the second, it suffices to show the existence of a nonvanishing holomorphic section s. There exists holomorphic sections by hypothesis. If s is a section then $deg(div(s)) = 0$, as $div(s) \geq 0$ this implies that $div(s) = 0$, that is s is non-vanishing.

□

Proposition 14.7 (Vanishing theorem) *Let D be a divisor with $deg D > 2g-2$. Then $H^1(X,D) = 0$. In this case $h^0 = deg D + 1 - g$.*

Proof. By Serre duality $h^1(D) = h^0(K-D)$ and by the previous proposition if the dimension is positive then $deg(K-D) = deg(K) - deg(D) = 2g-2-deg D > 0$. □

Lemma 14.8 *Let D be a divisor on a Riemann surface X such that $deg(D) > 2g-1$. Then for any $z \in X$ there exists a holomorphic section $s \in H^0(X, L(D))$ with $s(z) \neq 0$.*

Proof. Observe that $L(D) = L(D-z) \otimes L(z)$ so we may consider the map $H^0(X, L(D-z)) \to H^0(X, L(D))$ defined as $s \to s.s_z$, where s_z is a holomorphic section of $L(z)$ vanishing only at z. Observe that the map is injective and its image consists of the sections of $L(D)$ vanishing at z. In fact, if s is a section of $H^0(X,L(D))$ vanishing at z we may divide by the section s_z to obtain a section of $H^0(X, L(D-z))$. By Riemann Roch and the vanishing theorem $dim H^0(X, L(D-z)) < dim H^0(X, L(D))$. Therefore there exists a section of $H^0(X, L(D))$ which does not vanish at z. □

If the degree is higher one can do better:

Lemma 14.9 *Let D be a divisor on a Riemann surface X such that $deg(D) > 2g$. Then for any $z_1 \neq z_2 \in X$ there exists a holomorphic section $s \in H^0(X, L(D))$ with $s(z_1) = 0$ and $s(z_2) \neq 0$.*

Proof. By the previous lemma $H^0(X, L(D - z_1))$ has a section which does not vanish at z_2. Multipliying by s_{z_1} we obtain a section of $H^0(X, L(D))$ which vanishes at z_1 and does not vanish at z_2. □

Lemma 14.10 *Let D be a divisor on a Riemann surface X such that $deg(D) > 2g$. Then, for any $z \in X$ there exists a holomorphic section $s \in H^0(X, L(D))$ with $ord_z(s) = 1$.*

Proof. As in the previous lemma $H^0(X, L(D - z_1))$ has a section which does not vanish at z_1. Multipliying by s_{z_1} we obtain a section of $H^0(X, L(D))$ which vanishes at z_1 with order one. □

14.1.1 The embedding theorem

Suppose L is a holomorphic line bundle over a Riemann surface X. Let $n = dimH^0(X, L) - 1$. Let $s_0, \cdots, s_n$ be a basis of $H^0(X, L)$. Let $A = \{ x \in X \mid s_i(x) = 0, 0 \leq i \leq n \}$. If $z \in X \setminus A$, chose a section $s \in H^0(U, L)$ over a neighborhood of z such that s never vanishes.

Definition 14.11 *We define the holomorphic map*

$$\varphi_L : X \setminus A \to \mathbb{C}P^n$$

by $\varphi_L(z) = [s_0/s, \cdots, s_n/s]$

The definition does not depend on the choice of $s \in H^0(U, L)$. In fact, if $s' \in H^0(U, L)$ is another non-vanishing section, $s' = hs$ for a certain non-vanishing holomorphic function. So $[s_0/s', \cdots, s_n/s'] = [s_0/hs, \cdots, s_n/hs]$.

Theorem 14.12 *If $degL > 2g$ then φ_L is an embedding.*

Proof. By Lemma 14.8 $A = 0$ so φ_L is defined on X. By Lemma 14.9 the map is injective. It remains to show that the rank of φ_L is maximal. For that sake, chose at $z \in X$ $\sigma \in H^0(X, L)$ such that $ord_z(\sigma) = 1$. We write

$$s = \sum_1^n c_i s_i$$

and observe that for k such that $s_k(z) \neq 0$

$$\frac{s}{s_k} = \sum_1^n c_i s_i / s_k$$

has order 1 at z, so that at least one of s_i/s_k has order 1 at z. Its differential is injective. □

Using the theorem of Chow (any embedded projective manifold is algebraic) this proves that there exists an equivalence between Riemann surfaces and smooth projective curves.

Definition 14.13 *A line bundle L is ample if for some m, L^m embeds X in projective space. A line bundle L is very ample if it embeds X in projective space.*

The embedding theorem implies the following

Corollary 14.14 *A holomorphic line bundle is ample if and only if its degree is positive.*

14.1.2 The Riemann-Hurwitz formula again

Proposition 14.15 *Let X be a compact Riemann surface such that $g = H^1(X, \mathcal{O}) = 0$. Then it is biholomorphic to $\mathbb{C}P^1$.*

Proof. We have that $degK = -2$ in this case. If $z \in \mathbb{C}P^1$ then $h^0(z) = dim(K - z) + deg(z) + 1 - g = 1 + 1 - 0 = 2$. Therefore there exists a non-constant meromorphic function having a simple pole at z. That implies that X is a (unbranched) covering of $\mathbb{C}P^1$. □

Recall that if $f : X \to Y$ be a holomorphic map between Riemann surfaces and $a \in X$. Locally, we can find charts centred at a and $f(a)$ such that $w \circ f \circ z^{-1}(x) = x^n$. The number $b(a, f) = n - 1 \geq 0$ is called the branching order of the map at a.

If X and Y are compact we can define the total branching index $b = \sum_{a \in X} b(a, f)$. Also let d be the degree of the map.

Theorem 14.16 *Let $f : X \to Y$ be a holomorphic map between compact Riemann surfaces. Then*

$$2g_X - 2 = d(2g_Y - 2) + b$$

where b is the total branching index, d is the degree and g_X, g_Y are the arithmetic genera.

Proof. To prove the theorem we start with a meromorphic 1-form ω on Y. At a point $a \in X$ and using local coordinates as above we obtain $\omega = h(w)dw$ with $h(w)$ meromorphic.

$$f^*(\omega) = h(f(z))\frac{dw}{dz}dz = h(z^n)nz^{n-1}dz$$

so that $ord_a(f^*(\omega)) = nord_a(\omega) + n - 1$ with $n = ord_a(f)$. That is

$$\begin{aligned} deg(f^*\omega) &= \sum_{b \in Y}\left(\sum_{a \in f^{-1}(b)} ord_a(f)\right) ord_a(\omega) + \sum_{b \in Y}\left(\sum_{a \in f^{-1}(b)} ord_a(f) - 1\right) \\ &= ddeg(\omega) + b. \end{aligned}$$

The result follows. □

From this formula which is the same as the formula for the topological genus of a branched covering we obtain

Proposition 14.17 *The topological genus is equal to the arithmetical genus*

Proof. We take $Y = \mathbb{C}P^1$. For $\mathbb{C}P^1$ both genera coincide. By the existence of non-trivial meromorphic functions on any Riemann surface we obtain the result as the genera are obtained by the same formula from the genus of Y. □