TD 3 - Quillen functors, Derived functors and (a bit of) homotopy colimits

Exercice 1. A model structure on Cat (Charles Rezk)

Denote Cat the category of small categories. We assume that it is complete and cocomplete. We let $\mathcal{W}$ denote equivalences of categories and $\mathcal{C}$ denote functors that are injective on objects. We let $\mathcal{F}$ denote functors $F : \mathcal{A} \to \mathcal{B}$ such that for every isomorphism $g : F(a) \to b$ of $\mathcal{B}$, there is a map $f : a \to a'$ with $g = F(f)$; such functors are called isofibrations.

1. Denote $*$ the category with one object and no non-trivial arrows, and I the category with two objects $0, 1$ and exactly one isomorphism in each direction. Let $i : * \to I$ be the inclusion at 0. Show that $\mathcal{F} = RLP(\{i\})$.
2. Show that $\mathcal{W}$ verifies 2-out-of-3, and that $\mathcal{W}, \mathcal{C}$ and $\mathcal{F}$ are stable under retracts.
3. (a) Show that every functor F of $\mathcal{C} \cap \mathcal{W}$ has a left inverse G which is also a quasi-inverse and such that the natural transformation $FG \simeq \mathrm{id}$ is equal to the identity on the image of F.
   (b) Deduce that $\mathcal{F} \subset RLP(\mathcal{C} \cap \mathcal{W})$.
4. Show that $\mathcal{C} \subset LLP(\mathcal{F} \cap \mathcal{W})$.
5. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Denote $\mathrm{Path}(F) := \mathcal{A} \times_{\mathcal{B}} \mathcal{B}^I$ where the map $\mathcal{B}^I \to \mathcal{B}$ is the source map, and $\mathrm{Cyl}(F) := \mathcal{A} \times (I \coprod_{\mathcal{A}} \mathcal{B})$. Show that F factors through $\mathrm{Path}(F)$ and $\mathrm{Cyl}(F)$; deduce that $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ is a model structure on Cat. What are the fibrant objects, the cofibrant objects?

Exercice 2 (Derived functors in homological algebra vs model categories)**.** The goal of this exercise is to understand how the model-categorical notion of derived functor generalizes what you have seen in homological algebra. Let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ be an additive functor between abelian categories. We suppose that F is right exact.

1. Consider the projective model structure on $Ch_{\geq 0}(\mathcal{A})$. Show that F sends quasi-isos between cofibrant objects to quasi-isos in $Ch_{\geq 0}\mathcal{B}$. Deduce that it has a total left derived functor in the model categorical sense.
2. Show that $\mathbb{L}F(V) \cong F(P)$ where P is a projective resolution of V.
3. What is the link between the homological-algebraic derived functors $L^iF(V)$ and $\mathbb{L}F(V)$?
4. Apply this to prove the existence and identify the total derived functors of $\mathrm{Hom}_R(-, M)$:
$$Ch_{\geq 0}(A)^{op} \to Ch_{\geq 0}(A).$$
   Identify them with the derived functors $\mathrm{Ext}^j(-,-)$ from the homological algebra course. Distinguish between the cases of projective and injective structures, and explain how this affects the computations.
5. Take R a commutative ring. Do the functors $- \otimes - : Ch_{\geq 0}(R) \times Ch_{\geq 0}(R) \to Ch_{\geq 0}(\mathcal{R})$ and $\mathrm{Hom}(-,-) : Ch_{\geq 0}(R)^{op} \times Ch_{\geq 0}(R) \to Ch_{\geq 0}(R)$ have total derived functors?

Exercice 3 (Composition of Derived Functors)**.** Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ and $G : \mathcal{C}_2 \to \mathcal{C}_3$ be functors and let $\mathcal{W}_i$ be a class of morphisms in $\mathcal{C}_i$.

1. Assuming all the relevant total left derived functors exist, use their universal properties to construct a natural transformation $\mathbb{L}G \circ \mathbb{L}F \to \mathbb{L}(G \circ F)$.

2. Suppose now that $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ are model categories and that F and G are left Quillen functors. Show that $G \circ F$ is a left Quillen functor.

3. Show that the arrow of 1) induces a natural equivalence $\mathbb{L}(G \circ F) \simeq (\mathbb{L}G) \circ (\mathbb{L}F)$.

4. Let (L, R) be an adjoint pair. Show that L is left Quillen if and only if R is right Quillen.

5. Suppose the restriction of a functor F to cofibrant objects sends acyclic cofibrations to weak equivalences, show that F is left derivable. (Hint: Ken Brown's lemma).

Exercice 4 (Slice categories II, by Victor Saunier)**.** Let $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ be a model structure on $\mathcal{A}$. Let $f : X \to Y$ be a morphism. Recall that we defined in the first exercise sheet a model structure on every slice category $\mathcal{A}/X$.

1. Show that the functor $f_! : \mathcal{A}/X \to \mathcal{A}/Y$ which postcomposes by f admits a right adjoint f^* and describe it.

2. Show that the pair $(f_!, f^*)$ is a Quillen pair of adjoints.

3. Suppose $\mathcal{A}$ is right proper, i.e. weak equivalences are stable under pullback by fibrations, and that $f \in \mathcal{W}$. Show that the pair $(f_!, f^*)$ is a Quillen equivalence.

4. (Rezk) Suppose that for every weak equivalence f, the pair $(f_!, f^*)$ is a Quillen equivalence. Show that $\mathcal{A}$ is right proper.

On homotopy (co)limits

Exercice 5 (Homotopy colimits)**.** In this exercise, we first deal with generalities on homotopy pushouts and then specialize to chain complexes with the projective model structure. Let $\mathcal{C}$ be a model category and let I be the category given by the diagram-shape

$$\begin{array}{ccc} b & \longrightarrow & c \\ \downarrow & & \\ a & & \end{array}$$

1. Let $f : X \to Y$ be a natural transformation of diagrams $X, Y \in \mathrm{Fun}(I, \mathcal{C})$. Show that f has the left lifting property with respect to all projective acyclic fibrations if and only if the the natural maps
$$X_a \bigsqcup_{X_b} Y_b \to Y_a, \quad X_b \to Y_b, \quad X_c \bigsqcup_{X_b} Y_b \to Y_c$$
are cofibrations in $\mathcal{C}$. (Here we mean the usual pushouts in $\mathcal{C}$.)
Deduce that a diagram $Y : I \to \mathcal{C}$ is cofibrant if and only if Y_b is cofibrant in $\mathcal{C}$ and the maps $Y_a \to Y_b$ and $Y_a \to Y_c$ are cofibrations. Moreover, show that $X \to Y$ has the left lifting property with respect to projective fibrations if and only the above three maps are acyclic cofibrations.

2. Show that the category of diagrams $\mathrm{Fun}(I, \mathcal{C})$ admits the projective model structure (without using the result seen in class that such a structure exists since I is very small).

3. Show that the colimit functor $\mathrm{colim} \colon \mathrm{Fun}(I, \mathcal{C}) \to \mathcal{C}$ is a left Quillen functor.

4. Assume that $\mathcal{C}$ is left proper (i.e. weak equivalences are stable under pushouts along cofibrations). Show that any pushout diagram
$$\begin{array}{ccc} B & \overset{f}{\rightarrowtail} & C \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \bigsqcup_B C \end{array}$$
where $f \colon B \longrightarrow C$ a cofibration, is also a homotopy pushout diagram.

5. **Case of Topological spaces.** Assume now that $\mathcal{C} = \mathbf{Top}$.

   (a) Using that Top is proper (as seen in exercise 3. from the sheet on Quillen model structure), show that there is a canonical isomorphism

$$\mathbb{L}\operatorname{colim}(X \leftarrow A \to Y) \cong X \coprod_A^{\mathrm{h}} Y = X \coprod_{A \times \{0\}} \mathrm{Cyl}(A \to Y)$$

   in Ho(Top) between the homotopy pushout computed by the projective model structure and the formula given by the mapping cylinder.

   (b) Give a formula for computing the homotopy colimit of a tower $(X_0 \to X_1 \to X_2 \to \dots)$ as well as the homotopy limit of a tower $(\dots \to Y_2 \to Y_1 \to Y_0)$.

6. **Case of chain complexes.** Assume now that $\mathcal{C}$ is the model category of chain complexes over a ring R.

   (a) Show that $\mathcal{C}$ is left proper.

   (b) Let $g : A \to B$ be a map of chain complexes. Recall that the *mapping cone* of g, denoted $C(g)$, is the chain complex given in level n by $B_n \oplus A_{n-1}$ and whose differential $B_{n+1} \oplus A_n \to B_n \oplus A_{n-1}$ is given $(b, a) \mapsto (\partial_B(b) + g(a), -\partial_A(a))$. Let I denote the chain complex given by $R \oplus R$ in degree 0 and R in degree 1 with differential given by $\partial_R : R \to R \oplus R$ given by $r \mapsto (-r, r)$. We define the *mapping cylinder* of g, denoted $\mathrm{Cyl}(g)$, as the pushout in chain complexes of

$$\begin{array}{ccc} A & \xrightarrow{\ g\ } & B \\ \downarrow{\scriptstyle i_0} & & \downarrow \\ I \otimes A & \longrightarrow & \mathrm{Cyl}(g) \end{array}$$

   where the vertical arrow $A \to I \otimes A$ is induced by the inclusion $i_0 : R \to I$ corresponding to the inclusion of the second factor $R \hookrightarrow R \oplus R$ in degree 0. The differential on $I \otimes A$ is given by $r \otimes a \mapsto \partial_R(r) \otimes a + (-1)^{\deg(r)} r \otimes \partial_A(a)$. Show that the mapping cone of g is the pushout of

$$\begin{array}{ccc} I \otimes A & \longrightarrow & \mathrm{Cyl}(g) \\ \downarrow & & \downarrow \\ C(\mathrm{Id}_A) & \longrightarrow & C(g). \end{array}$$

   (c) Let Δ^1 be the category with two objects and one non trivial morphism in between them. Show that the construction of the mapping cone defines a functor $C : \mathrm{Fun}(\Delta^1, \mathrm{Ch}(R)) \to \mathrm{Ch}(R)$ sending natural transformations objectwise given by quasi-isomorphisms to quasi-isomorphisms.

   (d) Let $Y := (0 \longleftarrow A \xrightarrow{g} B)$ be a diagram in $\mathcal{C}$. Show that there exists a diagram of the form $Y' := (0 \longleftarrow A' \xrightarrow{g'} B')$ with g' a cofibration and A' and B' cofibrant, together with a natural transformation $u : Y' \to Y$ which is objectwise a weak equivalence. Notice that by the previous question the induced map $C(g') \to C(g)$ is a weak equivalence.

   (e) Let $Y := (0 \longleftarrow A \xrightarrow{g} B)$ be a diagram in $\mathcal{C}$ with A and B cofibrant and g a cofibration. Show that $A \to I \otimes A$ is a weak equivalence and show that we can construct a zigzag of diagrams $Y \leftarrow Y' \rightarrow Y''$ of the form

$$\begin{array}{ccccc} 0 & \longleftarrow & A & \xrightarrow{g} & B \\ \uparrow & & \uparrow & & \uparrow \\ C(A) & \longleftarrow & A & \xrightarrow{g} & B \\ \downarrow & & \downarrow & & \downarrow \\ C(A) & \longleftarrow & I \otimes A & \xrightarrow{g} & \mathrm{Cyl}(g) \end{array}$$

where each vertical arrow is a weak equivalence and the map $I \otimes A \to \mathrm{Cyl}(g)$ is a cofibration.

(f) Let $Y := (0 \longleftarrow A \xrightarrow{g} B)$ be any diagram. Conclude that the mapping cone $C(g)$ is a model for the homotopy colimit of the diagram Y.

Exercice 6. We recall the definition of a coend : Let $F : I^{\mathrm{op}} \to \mathbf{sSet}$ be a functor and $G : I \to \mathbf{sSet}$ be another functor. We define their coend denoted $F \otimes_I G$ as the following coequalizer

$$\coprod_{f:i \to j} F(j) \times G(i) \rightrightarrows \coprod_{i \in I} F(i) \times G(i) \to F \otimes_I G$$

which is a simplicial set.

1. We admit that $* \otimes_I G$ computes the colimit of the functor G. Let X be a simplicial set, that we see as a functor $sSet^{op} \to sSet$ by seeing $X(n)$ as a discrete simplicial set. The *geometric realization* of X is defined as the coend $\Delta[-] \otimes X \in Fun(sSet^{op}, sSet)$. Convince yourself that it is well-named and compute it.

2. Fix a small category I. Recall why $Fun(I, sSet)$ has the projective model structure. Consider the coend pairing defined by
$$sSet^{D^{op}} \times sSet^D \to sSet$$
which sends (F, G) to $F \otimes G$. Our goal is to see how this pairing interacts with the model structure. Show that if we fix a cofibrant G,it preserves weak equivalences in the F variabl (hint : show it on the generating cofibrations + small object argument).

3. Deduce that if X is a simplicial set which is cofibrant for the projective model structure, then the geometric realization $\Delta[-] \otimes_{sSet^{op}} X$ is a model for the homotopy colimit of X seen as a functor $\Delta^{op} \to sSet$.

Actually, this fact is true even if X is not projectively cofibrant, but we need more work : the Reedy model structure on simplicial objects.

Exercice 7. We assume that there is a model structure on $\mathbf{sSet}^{\Delta^{op}}$, called the *Reedy model structure*, such that

(a) The weak equivalences are the objectwise weak equivalences.

(b) The cofibrations are the maps $f : F \to G$ such that for all n, the map
$$F[n] \sqcup_{L_n F} L_n G \to G[n]$$
is a cofibration. Here $L_n F$ denotes the n-th latching object defined by $L_n(F) = \mathrm{colim}_{f:k \to n, k<n} F(k)$

(c) The fibrations are the maps such that for all n, the map
$$F[n] \to G[n] \times_{M_n G} M_n F$$
is a fibration. Here $M_n F$ denotes the n-th matching object defined by $M_n(G) = \lim_{f:k \to n, k>n} G(k)$.

1. Show that the geometric realization functor $\mathbf{sSet}^{\Delta^{op}} \to \mathbf{sSet}$ is left Quillen for the Reedy structure. Hint : show that its right adjoint is right Quillen.

2. Deduce from the previous exercise that $|X| \simeq \operatorname{hocolim} X$ for any simplicial set X.

Exercice 8 (The fundamental theorem of homotopy theory, after Geoffroy Horel)**.** We assume the result of the previous exercise. Let $F : \mathbf{sSet} \to \mathbf{sSet}$ be a homotopical functor that preserves homotopy colimits. Then F is naturally weakly equivalent to the functor

$$X \mapsto X \otimes^{\mathbb{L}} F(*)$$

If you are familiar with simplicial cofibrantly generated model categories, you can try to do the exercise replacing the target of F by any such model category M.