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# ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE FRAGMENTATION EQUATION WITH SHATTERING: AN APPROACH VIA SELF-SIMILAR MARKOV PROCESSES

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The subject of this paper is a fragmentation equation with nonconservative solutions, some mass being lost to a dust of zero-mass particles as a consequence of an intensive splitting. Under some assumptions of regular variation on the fragmentation rate, we describe the large time behavior of solutions. Our approach is based on probabilistic tools: the solutions to the fragmentation equation are constructed via nonincreasing self-similar Markov processes that continuously reach 0 in finite time. Our main probabilistic result describes the asymptotic behavior of these processes conditioned on nonextinction and is then used for the solutions to the fragmentation equation.

We note that two parameters significantly influence these large time behaviors: the rate of formation of "nearly-1 relative masses" (this rate is related to the behavior near 0 of the Lévy measure associated with the corresponding self-similar Markov process) and the distribution of large initial particles. Correctly rescaled, the solutions then converge to a nontrivial limit which is related to the quasi-stationary solutions of the equation. Besides, these quasi-stationary solutions, or, equivalently, the quasi-stationary distributions of the self-similar Markov processes, are fully described.

**1. Introduction and main results.** Fragmentation processes occur in a variety of natural phenomena, including polymer degradation, mineral grinding and droplet break-up, but also in the analysis of algorithms, phylogeny, etc. The kinetic equation used in the physics literature to describe the time-evolution of masses of particles prone to fragmentation has the form

$$\partial_t n_t(x) = \int_x^\infty a(y)b(y,x)n_t(y)\,dy - a(x)n_t(x), \tag{1}$$

where $n_t(x)$ is the concentration of particles of mass $x$ at time $t$, $a(x)$ is the overall rate at which a particle with mass $x$ splits and $b(x,y)$ describes the distribution of particles of mass $y$ produced by the fragmentation of a particle of mass $x$. It is assumed that no mass is lost when a particle breaks up, that is, $\int_0^x yb(x,y)\,dy = x$. The integral in the right-hand side of (1) models the increase of particles of mass

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$x$ due to the fragmentation of particles of masses $y > x$, whereas the negative term $-a(x)n_t(x)$ models the loss of particles of mass $x$, due to their fragmentation into smaller particles. This fragmentation equation has been intensively studied by both physicists and mathematicians. Among the first papers on the topic, we may cite, for example, [24, 25].

In both the physics and mathematics literature, particular attention has been paid to models with the following self-similar dynamic:

- $a(x) = Cx^\alpha$, for some fixed $C > 0$ and $\alpha \in \mathbb{R}$;
- $b(x, y) = h(y/x)/x$ [with $h$ such that $\int_0^1 uh(u)\,du = 1$]. This means that the distribution of the ratios of daughter masses to parent mass is only determined by a function of these ratios (and not by the parent mass).

There are two reasons for this. These self-similar assumptions are relevant for applications, for example, for polymer degradation [30], mineral crushing in the mining industry ([7] and the references therein) and the construction of phylogenetic trees [1]. But they are also more mathematically tractable. For the same reasons, there is also a significant literature on probabilistic models for the microscopic mechanism of fragmentation with a self-similar dynamic. We refer to the book by Bertoin [5] for an overview and to the papers [13] and [19] for discussions of the relations between the probabilistic models and the above equation.

The goal of this paper is to contribute to the understanding of solutions of the self-similar fragmentation equation, by describing their large time behavior. The cases where $\alpha > 0$ are treated in [13] and we will be concerned here only with the negative cases $\alpha < 0$.

We will actually consider the following generalization of the weak form of the above fragmentation equation (1) with a self-similar dynamic:

$$\partial_t \langle \mu_t, f\rangle = \int_0^\infty x^\alpha \left( \int_0^1 \big(f(yx) - f(x)y\big) B(dy)\right) \mu_t(dx), \tag{2}$$

where $(\mu_t, t \geq 0)$ denotes a family of measures on $]0, \infty[$, $\alpha \in \mathbb{R}$, $B$ is a measure on $]0, 1[$ such that

$$\int_0^1 y(1-y)B(dy) < \infty \quad \text{and} \quad B(]0,1[) > 0, \tag{3}$$

and $f$ denotes any test function. When $B(dy) = Ch(y)\,dy$ with $\int_0^1 yh(y)\,dy = 1$ and $\mu_t(dx) = n_t(x)\,dx$, we recover the weak form of (1) with $a(x) = Cx^\alpha$ and $b(x, y) = h(y/x)/x$. Informally, (2) corresponds to models in which particles with mass $xy$, $0 < y < 1$, are produced from the splitting of a particle with mass $x$ at rate $x^\alpha B(dy)$. Note that the overall rate at which a particle with mass $x$ splits is $x^\alpha \int_0^1 yB(dy)$, which may be infinite here. Let us add that the physical interpretation of the fragmentation equation imposes some constraints on the measure $B$. However, other interpretations are possible and, in the following, we will be concerned with all measures $B$ satisfying (3).

We focus on solutions of (2) with finite and nonzero initial total mass. The fragmentation equation being linear, we suppose, without loss of generality, that $\int_0^\infty x\mu_0(dx) = 1$. To be precise, we call *a solution of* (2) *starting from* $\mu_0$ any family of measures $(\mu_t, t \geq 0)$ on $]0,\infty[$ starting from $\mu_0$ such that:

- $(\mu_t, t \geq 0)$ satisfies (2) for any test function $f \in C_c^1$, the set of real-valued continuously differentiable functions on $]0,\infty[$ with compact support;
- the natural "physical properties"

$$m(t) := \langle \mu_t, \mathrm{id}\rangle \leq m(0) = 1 \qquad \forall t \geq 0, \tag{4}$$

and

$$\mu_0([M,\infty[) = 0 \qquad \text{for some } M > 0 \quad \Rightarrow \quad \mu_t([M,\infty[) = 0 \qquad \forall t \geq 0, \tag{5}$$

are respected ("id" denotes the identity function).

Note the *self-similarity of solutions*: if $(\mu_t, t \geq 0)$ is a solution of (2), then so is $(\gamma^{-1}\mu_{t\gamma^\alpha} \circ (\gamma\,\mathrm{id})^{-1})$ for all $\gamma > 0$. Also, note that if $(\mu_t, t \geq 0)$ is a solution of the equation with parameters $(\alpha, B)$, then for all $c > 0$, $(\mu_{ct}, t \geq 0)$ is a solution of the equation (2) with parameters $(\alpha, cB)$.

Many results on the existence and uniqueness of solutions of (1) are available in the literature; see, for example, [2, 13, 23] and the references therein. With the definition above, we have the following result on the existence and uniqueness of solutions of (2), which is a generalization of Theorem 1 of [19] (see also [17] for a similar approach). We recall that a *subordinator* is a nondecreasing Lévy process and that its distribution is characterized by two parameters: a nonnegative drift coefficient and a so-called Lévy measure on $]0,\infty[$ that governs the jumps of the process. See Section 2 for background on this topic.

THEOREM 1.1. *Let* $\mu_0$ *be a measure on* $]0,\infty[$ *such that* $\int_0^\infty x\mu_0(dx) = 1$ *and let* $\xi$ *be a subordinator with zero drift and Lévy measure* $\Pi$ *given, for any measurable function* $g : ]0,\infty[ \to [0,\infty[$, *by*

$$\int_0^\infty g(x)\Pi(dx) = \int_0^1 g(-\ln(x))xB(dx). \tag{6}$$

*Then, for each* $t > 0$, *define a measure* $\mu_t$ *on* $]0,\infty[$ *by*

$$\int_0^\infty f(x)x\mu_t(dx) := \int_0^\infty \mathbb{E}\big[f\big(x\exp\big(-\xi_{\rho(x^\alpha t)}\big)\big)\big]x\mu_0(dx) \tag{7}$$

*for all measurable* $f : [0,\infty[ \to [0,\infty[$, $f(0) = 0$, *where* $\rho$ *is the time-change*

$$\rho(s) := \inf\left\{u \geq 0 : \int_0^u \exp(\alpha\xi_r)\,dr > s\right\} \qquad \forall s \geq 0.$$

(i) *The family* $(\mu_t, t \geq 0)$ *is a solution of* (2), *provided that*

$$\alpha \leq 0$$

*or*

$$\alpha > 0 \quad \text{and either} \quad \int_1^\infty x \ln(x)\mu_0(dx) < \infty$$

(*or*)

$$x \in ]0, 1[ \to x^{|\alpha|} \int_0^x yB(dy) \text{ is bounded near } 0.$$

(ii) *This solution is unique, provided that* $\mu_0([M, \infty[) = 0$ *for some* $M > 0$.

When the family $(\mu_t, t \geq 0)$ is constructed via a subordinator by (7), some conditions on $\mu_0$ and $B$ for the existence of a density for $\mu_t$, $t > 0$, can be stated explicitly; see, for example, [18], Proposition 3.10. We also recall that there may exist multiple solutions of the fragmentation equation when the assumption (4) is dropped. We refer to [2] for some explicit examples.

The proof of Theorem 1.1, based on that of Theorem 1 in [19], is postponed to the Appendix.

The main purpose of this paper is to use the construction (7) of solutions of the fragmentation equation to describe the large time behavior of these solutions when $\alpha < 0$. From another, but equivalent, point of view, our main results describe the large time behavior of exponentials of minus time-changed subordinators, as defined in Theorem 1.1, conditioned on nonextinction. These processes belong to the family of so-called *self-similar Markov processes*. We refer to Section 3 for a statement of our results in that context.

The study of the large time behavior of solutions of the fragmentation equation when $\alpha > 0$ is investigated in detail in [13]. We point out that some results of [13] can be redemonstrated using a probabilistic approach: it consists mainly of combining the subordinator construction of solutions of the fragmentation equation with the description of large time behavior of time-changed subordinators when $\alpha > 0$ investigated in [6].

From now on, we consider $\alpha < 0$. It is well known that in such a case, small particles split so quickly that they are reduced to a dust of zero-mass particles, so that the total mass of nonzero particles

$$m(t) = \langle \mu_t, \mathrm{id} \rangle$$

decreases as time passes. This phenomenon, sometimes called "shattering," has been studied in, for example, [2, 4, 16, 19, 24, 29]. More precisely, one can check that the total mass $m$ is strictly decreasing and strictly positive on $[0, \infty[$, and that $m(t) \to 0$ as $t \to \infty$; see the forthcoming Proposition 3.3 for a proof in our framework.

In order to describe the behavior of $m(t)$ as $t \to \infty$ more accurately, we introduce the function defined for all $t \geq 0$ by

$$\phi(t) := \int_0^1 (1 - x^t) x B(dx). \tag{8}$$

It is not hard to check that the function $t \to t/\phi(t)$ is continuous and strictly increasing on $]0,\infty[$, and that its range is $](\int_0^1 |\ln(x)| x B(dx))^{-1}, \infty[$. Note that the integral $\int_0^1 |\ln(x)| x B(dx)$ may be finite or infinite. Then, introduce

$$\varphi, \qquad \text{the inverse of } t \to t/\phi(t), \tag{9}$$

which is well defined in a neighborhood of $\infty$. This function will play a key role in the description of the long-time behavior of solutions of the fragmentation equation.

Most of our main results rely on the following hypothesis on the measure $B$:

$$\text{(H)} \qquad \begin{array}{l} \text{the function } u : ]0,1[ \to \displaystyle\int_0^{1-u} x B(dx) \\ \text{varies regularly at 0 with an index } -\beta \in ]-1,0], \end{array}$$

which, in particular, ensures that $\phi$ and $\varphi$ are regularly varying functions at $\infty$ with respective indices $\beta$ and $1/(1-\beta)$. See Section 2.2 for details and background on regular variation.

Finally, we mention that the large time behavior of solutions of the fragmentation equation will depend strongly on the structure of the initial measure $\mu_0$, mainly on the manner in which it distributes weight near $\infty$. The statements of our results are therefore split into two parts, according as to whether the initial measure has a bounded support (Section 1.1) or not (Section 1.2). Section 1.3 deals with the quasi-stationary solutions.

1.1. *Initial measure $\mu_0$ with bounded support.* In this subsection, we adopt the following hypotheses and notation:

- $\alpha < 0$;
- the measure $\mu_0$ has a bounded support, that is, $\mu_0([M,\infty[) = 0$ for some $M > 0$;
- $(\mu_t, t \geq 0)$ denotes the unique solution of the fragmentation equation (2) starting from $\mu_0$.

The supremum of the support of $\mu_0$ is the real number $s$ such that $\mu_0(]s,\infty[) = 0$ and $\mu_0(]s-\varepsilon, s]) > 0$ for all $\varepsilon < s$. Thanks to the self-similarity of solutions, we can, and will, always suppose that *this supremum is equal to* 1. In such a framework, we have the following results.

PROPOSITION 1.2. *For all $\lambda < \phi(\infty) := \lim_{x\to\infty} \phi(x)$, there exists a constant $C_\lambda < \infty$ such that*

$$m(t) \leq C_\lambda \exp(-\lambda t) \qquad \forall t \geq 0.$$

*More precisely, under the hypothesis* (H),

$$-\ln(m(t)) \underset{t\to\infty}{\sim} \frac{(1-\beta)}{|\alpha|} \varphi(|\alpha| t).$$

*In particular, $t \to -\ln(m(t))$ is regularly varying at $\infty$ with index $1/(1-\beta)$.*

Together with the following theorem, this gives a complete description of the large time behavior of $(\mu_t, t \geq 0)$. Here, two positive functions $g$ and $h$ are said to be asymptotically equivalent if $g(x)/h(x) \to 1$ as $x \to \infty$.

THEOREM 1.3. *Suppose that* (H) *holds and* $\int_0 |\ln(x)| x B(dx) < \infty$. *Then, for all continuous bounded test functions* $f : ]0, \infty[ \to \mathbb{R}$,

$$\frac{1}{m(t)} \int_0^\infty f\left(\left(\frac{\varphi(|\alpha|t)}{|\alpha|t}\right)^{1/|\alpha|} x\right) x\mu_t(dx) \underset{t\to\infty}{\to} \int_0^\infty f(x) x\mu_\infty(dx),$$

*where* $x\mu_\infty(dx)$ *is a probability distribution on* $]0, \infty[$ *that is characterized by its moments*

$$\int_0^\infty x^{|\alpha|n} x\mu_\infty(dx) = \phi(|\alpha|)\phi(2|\alpha|)\cdots\phi(n|\alpha|), \qquad n \geq 1. \tag{10}$$

*The function* $t \to \varphi(|\alpha|t)/(|\alpha|t)$ *can be replaced by any asymptotically equivalent function.*

It is interesting to compare this result with that obtained by Escobedo, Mischler and Rodriguez Ricard [13] when the parameter $\alpha$ is positive. As already mentioned, part of their result can be rediscovered and completed by using results of Bertoin and Caballero [6]. With our notation, and under the assumptions $\int_0 |\ln(x)| x B(dx) < \infty$ and $\alpha > 0$, the asymptotic behavior of the solution $(\mu_t, t \geq 0)$ of the fragmentation equation $(\alpha, B)$ starting from $\mu_0 = \delta_1$ can be described as follows:

$$\int_0^\infty f(t^{1/\alpha} x) x\mu_t(dx) \underset{t\to\infty}{\to} \int_0^\infty f(x) x\eta_\infty(dx)$$

for all continuous bounded functions $f : ]0, \infty[ \to \mathbb{R}$. The measure $x\eta_\infty(dx)$ is a probability measure on $]0, \infty[$. Interestingly, the measure $B$ is then involved only in the description of the limit measure $\eta_\infty$, not in the "shape" of the speed of decrease of masses to 0.

We return to the case $\alpha < 0$. Note that when $\int_0^{1-u} x B(dx) \sim u^{-\beta}$ as $u \to 0$ for some $\beta \in [0, 1[$, we have $\phi(t) \sim \Gamma(1-\beta)t^\beta$ and therefore $(\varphi(|\alpha|t)/|\alpha|t)^{1/|\alpha|} \sim C_{\alpha,\beta} t^{\beta/((1-\beta)|\alpha|)}$ as $t \to \infty$, where $C_{\alpha,\beta} = (|\alpha|^\beta \Gamma(1-\beta))^{1/((1-\beta)|\alpha|)}$. When we also have $\int_0 |\ln(x)| x B(dx) < \infty$, Theorem 1.3 then reads

$$\frac{1}{m(t)} \int_0^\infty f\left(C_{\alpha,\beta} t^{\beta/((1-\beta)|\alpha|)} x\right) x\mu_t(dx) \underset{t\to\infty}{\to} \int_0^\infty f(x) x\mu_\infty(dx)$$

for all continuous bounded test functions $f : ]0, \infty[ \to \mathbb{R}$.

The existence and uniqueness of a measure $\mu_\infty$ on $]0, \infty[$ satisfying (10) actually hold without any assumption of regular variation on the measure $B$ or assumptions on its behavior near 0; see the discussion near equation (14) in Section 3 for details. Some properties of the measure $\mu_\infty$ (tail behavior near 0 and near $\infty$) are given in Section 5. In Section 1.3, we discuss its links with the *quasi-stationary solutions* of the fragmentation equation.

The proof of Theorem 1.3 consists of describing the behavior of the mass of a typical random nondust particle, defined as follows: at each time $t$, choose a particle at random among the particles with a *strictly positive mass*, with a probability proportional to its mass. That is, if $M(t)$ denotes the mass of this random particle, then the distribution of $M(t)$ is given by

$$M(t) \overset{d}{\sim} \frac{x\mu_t(dx)}{m(t)}.$$

In other words, in terms of the subordinator $\xi$ related to the equation by (7), $M(t)$ is distributed as $M(0)\exp(-\xi_{\rho(M(0)^\alpha t)})$, conditioned to be strictly positive, with $M(0)$ independent of $\xi$. In terms of $M$, the statement of Theorem 1.3 can be rephrased as follows:

$$\left(\frac{\varphi(|\alpha|t)}{|\alpha|t}\right)^{1/|\alpha|} M(t) \overset{d}{\to} M_\infty,$$

where $M_\infty$ is a random variable with distribution $x\mu_\infty(dx)$. Note the special case $\int_0^1 xB(dx) < \infty$, where $\varphi(t)/t \to \int_0^1 xB(dx) < \infty$. We then have that $M(t)$ converges in distribution to a nontrivial limit. In the other cases satisfying the assumptions of Theorem 1.3, $\varphi(t)/t \to \infty$ and therefore $M(t) \overset{\mathbb{P}}{\to} 0$.

Using this random approach, we can also specify the behavior of masses that decrease at different speeds to 0, as follows.

PROPOSITION 1.4. *Assume that* (H) *holds and let* $\kappa := \int_0^1 |\ln(x)| xB(dx) < \infty$.

(i) *Suppose, moreover, that the support of $B$ is not included in a set of the form $\{a^n, n \in \mathbb{N}\}$ for some $a \in ]0,1[$. Then, for all measurable functions $g : [0,\infty[ \to ]0,\infty[$ converging to 0 at $\infty$,*

$$\frac{g(t)^\alpha}{m(t)} \int_0^{g(t)(\varphi(|\alpha|t)/|\alpha|t)^{1/\alpha}} x\mu_t(dx) \underset{t\to\infty}{\to} \frac{1}{|\alpha|\kappa}.$$

(ii) *For all measurable functions $g : [0,\infty[ \to ]0,\infty[$ converging to $\infty$ at $\infty$:*

- *if $g^{|\alpha|}(t)t/\varphi(t)$ converges to $\infty$ at $\infty$, then*

$$\int_{g(t)(\varphi(|\alpha|t)/|\alpha|t)^{1/\alpha}}^\infty x\mu_t(dx) = 0$$

*for all $t$ sufficiently large;*

- *if $g^{|\alpha|}(t)t/\varphi(t)$ converges to 0 at $\infty$ and $0 < \beta < 1$, then*

$$\limsup_{t\to\infty} \frac{1}{\phi^{-1}(g(t)^{|\alpha|})} \ln\left(\frac{\int_{g(t)(\varphi(|\alpha|t)/|\alpha|t)^{1/\alpha}}^\infty x\mu_t(dx)}{m(t)}\right) \le -\frac{\beta}{|\alpha|},$$

*where $\phi^{-1}$ denotes the inverse of $\phi$.*

Note that the first assertion of (ii) is obvious since $g(t)(\varphi(|\alpha|t)/|\alpha|t)^{1/\alpha} \to \infty$ (which means that for $t$ sufficiently large, it is larger than 1, the supremum of the support of $\mu_t$).

We conclude this section with the following result on the remaining mass at time $t$ of particles of mass 1 when $\mu_0(\{1\}) > 0$. The measure $\mu_\infty$ is that introduced in Theorem 1.3.

PROPOSITION 1.5. *Suppose that* $\mu_0(\{1\}) > 0$ *and set* $\phi(\infty) := \int_0^1 xB(dx) \in ]0, \infty]$. *Then, for all* $t \geq 0$,

$$\mu_t(\{1\}) = \exp(-t\phi(\infty))\mu_0(\{1\}).$$

*If, further,* (H) *is satisfied,* $\int_0 |\ln(x)|xB(dx) < \infty$ *and* $\phi(\infty) < \infty$, *then*

$$\frac{\mu_t(\{1\})}{m(t)} \underset{t\to\infty}{\to} \phi(\infty)^{1/|\alpha|}\mu_\infty(\{\phi(\infty)^{1/|\alpha|}\})$$

*and this limit is nonzero if and only if* $\int^1 \frac{B(dx)}{1-x} < \infty$.

This means that under the assumptions of Proposition 1.5, for large times, the remaining total mass of mass-1 particles is proportional to the total mass of nonzero particles when $\int^1 (1-x)^{-1}B(dx) < \infty$, whereas it is negligible compared to the total mass of nonzero particles when $= \int^1 (1-x)^{-1}B(dx) = \infty$. We point out that the convergence of Proposition 1.5 is *not* necessarily true when $\mu_0(\{1\}) = 0$ [since then $\mu_t(\{1\}) = 0$ for all $t \geq 0$, whereas the term in the limit may be strictly positive].

1.2. *Initial measure* $\mu_0$ *with unbounded support*. We still suppose that $\alpha < 0$ and we denote by $(\mu_t, t \geq 0)$ the solution of the fragmentation equation (2) starting from $\mu_0$ and constructed via a subordinator by formula (7). The asymptotic behavior of the mass $m(t)$ is then strongly modified by the presence of large masses and depends on the behavior as $t \to \infty$ of both $\phi(t)$ and $\mu_0([t, \infty[)$. We investigate two particular cases: exponential and power decreases of $\mu_0([t, \infty[)$ as $t \to \infty$.

THEOREM 1.6. *Assume that* (H) *holds and that* $\mu_0$ *possesses a density, say* $u_0$, *in a neighborhood of* $\infty$ *such that*

$$\ln(u_0(x)) \underset{\infty}{\sim} -Cx^\gamma$$

*for some* $\gamma > 0$.

(i) *Then*,

$$-\ln(m(t)) \underset{\infty}{\sim} C_{\alpha,\beta,\gamma} C^{(1+(1-\beta)\gamma/|\alpha|)^{-1}} h(t),$$

*where $h$ is the inverse, well defined in the neighborhood of $\infty$, of $t \to t^{1+|\alpha|/\gamma}/\phi(t)$ and*

$$C_{\alpha,\beta,\gamma} = \left(1 + |\alpha|^{-1}\gamma(1-\beta)\right)\left(\frac{|\alpha|^{1/(1-\beta)}}{\gamma}\right)^{\gamma(1-\beta)/(\gamma(1-\beta)+|\alpha|)}.$$

*In particular, $-\ln(m(t))$ varies regularly at $\infty$ with index $1/(1-\beta+|\alpha|/\gamma)$.*

(ii) *Suppose, moreover, that $\int_0 |\ln(x)| x B(dx) < \infty$, which ensures that the function $\ln(m)$ is differentiable on $]0,\infty[$. Then, if the derivative $(\ln(m))'$ is regularly varying at $\infty$, one has, for all continuous bounded test functions $f : ]0,\infty[ \to \mathbb{R}$,*

$$\frac{1}{m(t)} \int_0^\infty f\left(\left(\frac{h(t)}{C_{\alpha,\beta,\gamma,C}\, t}\right)^{1/|\alpha|} x\right) x \mu_t(dx) \underset{t\to\infty}{\to} \int_0^\infty f(x) x \mu_\infty(dx),$$

*where $\mu_\infty(dx)$ is the measure introduced in Theorem 1.3 and*

$$C_{\alpha,\beta,\gamma,C} = \frac{C_{\alpha,\beta,\gamma} C^{(1+(1-\beta)\gamma/|\alpha|)^{-1}}}{1-\beta+|\alpha|/\gamma}.$$

Assuming that the derivative $(\ln(m))'$ is regularly varying at $\infty$ may seem overly demanding. In actual fact, this assumption is also needed to obtain Theorem 1.3, but we are able to show that it is always satisfied under the hypotheses of this theorem (see Lemma 3.9). Unfortunately, it seems difficult to adapt this proof to the case where the measure $\mu_0$ has unbounded support. However, according to a classical result on regular variation (the monotone density theorem), $(\ln(m))'$ varies regularly at $\infty$ provided that $\ln(m)$ varies regularly at $\infty$ and $(\ln(m))'$ is monotone near $\infty$, which can be checked in some particular cases.

There is also the following result on the decrease of the mass $m$ when the density $u_0$ of $\mu_0$ has a power decrease near $\infty$.

PROPOSITION 1.7. *Assume that $\mu_0$ possesses a density $u_0$ in a neighborhood of $\infty$ such that*

$$u_0(x) \underset{\infty}{\sim} C x^{-\gamma}$$

*for some $\gamma > 2$. Then,*

$$m(t) \underset{\infty}{\sim} C' t^{(\gamma-2)/\alpha}$$

*with $C' = |\alpha|^{-1} C \int_0^\infty \overline{m}(u) u^{(2-\gamma)/\alpha - 1}\, du < \infty$, where $\overline{m}$ denotes the total mass of the solution of the fragmentation equation with the same parameters $\alpha$, $B$ as that considered here, and with initial distribution $\delta_1$, the Dirac mass at 1.*

1.3. *Quasi-stationary solutions*. A *quasi-stationary solution* of the fragmentation equation (2) is a solution $(\mu_t, t \geq 0)$ such that

$$\mu_t = m(t)\mu_0 \qquad \forall t \geq 0,$$

with $m(t) = \langle \mu_t, \mathrm{id}\rangle$. These quasi-stationary solutions are closely related to the measure $\mu_\infty$ introduced in the statement of Theorem 1.3. We have already mentioned that existence and uniqueness of such a measure $\mu_\infty$ satisfying (10) hold without any assumption of regular variation on the measure $B$ or on its behavior near 0. The interesting fact is that, whatever the conditions on $B$, this measure and its self-similar counterparts

$$\mu_\infty^{(\lambda)} := \lambda^{-1}\mu_\infty \circ (\lambda\, \mathrm{id})^{-1},$$

$\lambda > 0$, are the only initial measures leading to quasi-stationary solutions of the fragmentation equation (2).

THEOREM 1.8. *For all $\lambda > 0$, let $(\mu_{\infty,t}^{(\lambda)}, t \geq 0)$ denote the solution of the fragmentation equation* (2) *starting from $\mu_\infty^{(\lambda)}$ and constructed via a subordinator by* (7). *Then, for all $t \geq 0$,*

$$\mu_{\infty,t}^{(\lambda)} = \exp(-\lambda^\alpha t)\mu_\infty^{(\lambda)} = m(t)\mu_\infty^{(\lambda)}.$$

*Reciprocally, if $(\mu_t, t \geq 0)$ is a quasi-stationary solution of the fragmentation equation, then there exists a $\lambda > 0$ such that $(\mu_t, t \geq 0) = (\mu_{\infty,t}^{(\lambda)}, t \geq 0)$.*

*Organization of the paper.* In Section 2, we begin with some background on subordinators and regular variation. Section 3 is the core of this paper: our main results on large time behavior of self-similar Markov processes conditioned on nonextinction are stated and proved there. Together with Theorem 1.1, these results imply Theorems 1.3, 1.6 and 1.8, as well as Propositions 1.2 and 1.7. Section 4 is devoted to the proof of Proposition 1.4. Some properties of the limit measure $\mu_\infty$ are given in Section 5 and used to prove Proposition 1.5. Finally, some specific examples are discussed in Section 6 and the proof of Theorem 1.1 is given in the Appendix.

## 2. Background on subordinators and regular variation.

2.1. *Subordinators*. A *subordinator* is a nondecreasing Lévy process, that is, a nondecreasing càdlàg process with stationary and independent increments. We recall here the main properties we need in this paper and refer to Chapter 3 of [3] for a more complete introduction to the subject.

The distribution of a subordinator $(\xi_t, t \geq 0)$ starting from $\xi_0 = 0$ is characterized by its so-called *Laplace exponent* $\phi : [0, \infty[ \to [0, \infty[$ via the identity

$$\mathbb{E}[\exp(-\lambda\xi_t)] = \exp(-t\phi(\lambda)) \qquad \forall \lambda, t \geq 0.$$

According to the Lévy–Khintchine formula [3], Theorem 1, Chapter 1, there exists a real number $d \geq 0$ and a measure $\Pi$ on $]0,\infty[$, $\int_0^\infty (1 \wedge x)\Pi(dx) < \infty$ such that

$$\phi(\lambda) = d\lambda + \int_0^\infty \bigl(1 - \exp(-\lambda x)\bigr)\Pi(dx) \qquad \forall \lambda \geq 0.$$

The measure $\Pi$ governs the jumps of the subordinator: the jumps process of $\xi$ is a Poisson point process with intensity $\Pi$.

We will need the strong Markov property of subordinators ([3], Proposition 6, Chapter 1): given a subordinator $\xi$ and a stopping time $T$ with respect to the filtration $(\mathcal{F}_t, t \geq 0)$ generated by $\xi$, then, conditionally on $\{T < \infty\}$, the process $(\xi_{t+T} - \xi_T, t \geq 0)$ is independent of $\mathcal{F}_T$ and is distributed as $\xi$. Finally, we recall that the semigroup of a subordinator possesses the Feller property ([3], Proposition 5, Chapter 1).

*Hereafter, all subordinators considered in this paper start from* 0 *and have drift* $d = 0$. Their distribution is therefore completely determined by their Lévy measure $\Pi$. Note that when $\Pi$ is related to a measure $B$ on $]0,1[$ via the formula (6), the above expression for $\phi$ coincides with that given by equation (8), that is,

$$\phi(\lambda) = \int_0^\infty \bigl(1 - \exp(-\lambda x)\bigr)\Pi(dx) = \int_0^1 (1 - x^\lambda) x B(dx) \qquad \forall \lambda \geq 0.$$

2.2. *Regular variation*. A function $f : ]0,\infty[ \to ]0,\infty[$ is said to vary regularly at $\infty$ (resp., 0) with index $\gamma \in \mathbb{R}$ if, for all $a > 0$,

$$\frac{f(ax)}{f(x)} \to a^\gamma \qquad \text{as } x \to \infty \text{ (resp., 0)}.$$

We refer to [9] for background on this topic. In particular, we have already implicitly used the fact that the inverse, when it exists, of a function regularly varying at $\infty$ with index $\gamma > 0$ is also regularly varying at $\infty$, with index $1/\gamma$ (see Section 1.5.7 of [9]).

Note that when the Lévy measure $\Pi$ is related to the fragmentation measure $B$ by the formula (6), our main assumption (H) reads "$u \in ]0,\infty[ \to \int_u^\infty \Pi(dx)$ varies regularly at 0 with index $-\beta$." It is classical that this is equivalent to the fact that

the function $\phi$ varies regularly at $\infty$ with index $\beta$.

This can be easily proven using the Karamata Abelian–Tauberian theorems (see, in particular, Chapters 1.6 and 1.7 of [9]). We will often use this form of the assumption (H).

To prove Theorem 3.1 below, which will then imply Theorems 1.3 and 1.6(ii), we will need the following technical lemma, which is taken from Chow and Cuzick [12].

LEMMA 2.1 (Chow and Cuzick [12], Lemma 3). *Let $f$ be regularly varying at infinity with index $\gamma > 0$ and suppose that for all $\varepsilon > 0$, there exists some $x(\varepsilon)$ such that*

$$\lambda^{\gamma-\varepsilon} \le \frac{f(\lambda x)}{f(x)} \le \lambda^{\gamma+\varepsilon} \qquad \forall \lambda \ge 1, \forall x \ge x(\varepsilon). \tag{11}$$

*Then, for all $\theta > -1$,*

$$e^{f(t)}\left(\frac{f(t)}{t}\right)^{\theta+1} \int_t^\infty (x-t)^\theta e^{-f(x)}\,dx \underset{t\to\infty}{\to} \gamma^{-1-\theta}\Gamma(1+\theta).$$

We point out that Chow and Cuzick state their result for all regularly varying functions with a positive index, but that their proof strongly relies on the key point (11), which is not true for any regularly varying function (counterexamples can easily be constructed). However, the functions we are interested in, that is, $-\ln(m)$, and to which we will apply this result, will, in general, satisfy (11). In particular, see Lemma 3.6 below.

**3. Asymptotic behavior of self-similar Markov processes.** Given the construction (7) via subordinators of solutions of the fragmentation equation, the issue of characterizing the large time asymptotics of these solutions is equivalent to characterizing large time behavior of distributions of time-changed subordinators.

So, let $\xi$ be a subordinator started from 0 with Lévy measure $\Pi$ and no drift. We denote by $\phi$ its Laplace exponent. Now, consider $\alpha < 0$ and let $X(0)$ be a strictly positive random variable, independent of $\xi$. Our goal is to specify the asymptotic behavior as $t \to \infty$ of the distributions of the random variables

$$X(t) := X(0)\exp\bigl(-\xi_{\rho(X(0)^\alpha t)}\bigr), \tag{12}$$

conditional on $\{X(t) > 0\}$, where $\rho$ is given by

$$\rho(t) = \inf\left\{u \ge 0 : \int_0^u \exp(\alpha\xi_r)\,dr > t\right\}.$$

Following Lamperti [22], the process $X$ belongs to the so-called family of *self-similar Markov processes*. This means that it is strongly Markovian and that for all $x > 0$, if $\mathbb{P}_x$ denotes the distribution of $X$ started from $x$, then, for all $a > 0$,

$$\text{the distribution of } \bigl(aX(a^\alpha t), t \ge 0\bigr) \text{ under } \mathbb{P}_x \text{ is } \mathbb{P}_{ax}.$$

Moreover, $X$ reaches 0 a.s. and it does so continuously. Conversely, Lamperti [22] also shows that any nonincreasing càdlàg self-similar Markov processes on $[0, \infty[$ that reaches 0 continuously in finite time a.s. can be constructed in this way via a time-changed subordinator.

Note that the moment at which $X$ reaches 0 is $X(0)^{|\alpha|}I$, where $I$ is the *exponential functional* defined by

$$I := \int_0^\infty \exp(\alpha\xi_r)\,dr, \tag{13}$$

which is clearly a.s. finite. The distribution of the random variable $I$ was first studied in detail in [11]. In particular, it is well known that for all integers $n \geq 1$,

$$\mathbb{E}[I^n] = \frac{n!}{\phi(|\alpha|)\phi(2|\alpha|)\cdots\phi(n|\alpha|)},$$

and that the distribution of $I$ is characterized by these moments ([11], Proposition 3.3). It will also be essential for us (see [8], Propositions 1 and 2) that there exists a unique probability measure $\mu_R$ on $]0,\infty[$ whose entire positive moments are given by

$$\int_0^\infty x^n \mu_R(dx) = \phi(|\alpha|)\phi(2|\alpha|)\cdots\phi(n|\alpha|), \qquad n \geq 1, \tag{14}$$

and that, moreover, if $R$ denotes a random variable with distribution $\mu_R$ independent of $I$, then

$$RI \stackrel{d}{=} \mathbf{e}(1), \tag{15}$$

where $\mathbf{e}(1)$ has an exponential distribution with parameter 1.

We now have the material necessary to state the main result of this section. To be consistent with the notation used for the fragmentation equation, we denote by $x\mu_0(dx)$, $x > 0$, the distribution of $X(0)$. Also, we recall the definition of the function $\varphi$ as the inverse, well defined in a neighborhood of $\infty$, of $t \to t/\phi(t)$.

THEOREM 3.1. *Suppose that* $\int_u^\infty \Pi(dx)$ *varies regularly at* 0 *with index* $-\beta$, $\beta \in [0,1[$, *and* $\int^\infty x\Pi(dx) < \infty$.

(i) *If the support of* $\mu_0$ *is bounded with a supremum equal to* 1, *then, for all bounded continuous functions* $f : ]0,\infty[ \to \mathbb{R}$,

$$\mathbb{E}\left[f\left(\left(\frac{\varphi(|\alpha|t)}{|\alpha|t}\right)^{1/|\alpha|} X(t)\right)\Big| X(t) > 0\right] \underset{t\to\infty}{\to} \mathbb{E}[f(R^{1/|\alpha|})],$$

*where* $R$ *is the random variable with distribution* $\mu_R$ *defined by* (14).

(ii) *If* $\mu_0$ *possesses a density* $u_0$ *in a neighborhood of* $\infty$ *such that*

$$\ln(u_0(x)) \underset{\infty}{\sim} -Cx^\gamma$$

*for some* $\gamma > 0$, *then the function* $t \in ]0,\infty[ \to \mathbb{P}(X(t) > 0)$ *is continuously differentiable. If, moreover, the derivative of* $t \to \ln(\mathbb{P}(X(t) > 0))$ *is regularly varying at* $\infty$—*which is true when, for example, this derivative is monotone near* $\infty$—*then, for all bounded continuous functions* $f : ]0,\infty[ \to \mathbb{R}$, *as* $t \to \infty$,

$$\mathbb{E}\left[f\left(\left(\frac{h(t)}{C_{\alpha,\beta,\gamma,C}\,t}\right)^{1/|\alpha|} X(t)\right)\Big| X(t) > 0\right] \underset{t\to\infty}{\to} \mathbb{E}[f(R^{1/|\alpha|})],$$

*where the function* $h$ *is the inverse, defined in the neighborhood of* $\infty$, *of* $t \to t^{1+|\alpha|/\gamma}/\phi(t)$ *and* $C_{\alpha,\beta,\gamma,C}$ *is the constant defined in the statement of Theorem* 1.6.

We will see in the proof of this result that the function $t \to \varphi(|\alpha|t)/|\alpha|t$ in assertion (i) can be replaced by any asymptotically equivalent function and likewise for $h$ in the second assertion.

Now, let $B$ be the fragmentation measure related to $\Pi$ by (6). If $(\mu_t, t \geq 0)$ refers to the solution of the $(\alpha, B)$-fragmentation equation constructed from $\xi$ by the formula (7), then we have

$$m(t) = \int_0^\infty x\mu_t(dx) = \mathbb{P}\big(X(t) > 0\big)$$

and the distribution of $X(t)$ conditional on $X(t) > 0$ is $x\mu_t(dx)/m(t)$. The above theorem then leads directly to the statements of Theorems 1.3 and 1.6(ii) [note that $\int^\infty x\Pi(dx) < \infty$ is equivalent to $\int_0 |\ln(x)|xB(dx) < \infty$]. The limit distribution $x\mu_\infty(dx)$ mentioned in these theorems is therefore the distribution of $R^{1/|\alpha|}$. The large time behavior of $m(t) = \mathbb{P}(X(t) > 0)$ is studied in Section 3.1 below, whereas Theorem 3.1 is established in Section 3.2.

We finish with the following result on the *quasi-stationary distributions* of $X$, which will be proven in Section 3.3 and which, in terms of the fragmentation equation, will lead to Theorem 1.8. We recall that the quasi-stationary distributions of $X$ are the distributions $\varsigma$ on $]0, \infty[$ such that

$$X(0) \overset{d}{\sim} \varsigma \quad \Rightarrow \quad \mathbb{E}[f(X(t))|X(t) > 0] = \mathbb{E}[f(X(0))]$$

for all $t \geq 0$ and all test functions $f$ defined on $]0, \infty[$.

THEOREM 3.2. *Let $\mu_R^{(\lambda)}$ denote the law of $\lambda R^{1/|\alpha|}$, $\lambda > 0$. Then, a probability measure $\varsigma$ on $]0, \infty[$ is a quasi-stationary distribution of $X$ if and only if $\varsigma = \mu_R^{(\lambda)}$ for some $\lambda > 0$. Moreover, if $X(0) \overset{d}{\sim} \mu_R^{(\lambda)}$, then*

$$\mathbb{P}\big(X(t) > 0\big) = \exp(-\lambda^\alpha t) \qquad \forall t \geq 0.$$

We point out that this theorem does not lead directly to the reciprocal assertion of Theorem 1.8. However, easy manipulations of the fragmentation equation will lead to it; see Section 3.3 for details.

3.1. *Total mass behavior.* This section is devoted to the description of the behavior of the total mass

$$m(t) = \int_0^\infty x\mu_t(dx) = \mathbb{P}\big(X(t) > 0\big) = \mathbb{P}\big(I > X(0)^\alpha t\big).$$

The notation is that introduced above in the introduction of Section 3. We start with the following result, which holds for all fragmentation equations with parameters $\alpha < 0$, $B$ and all initial measures $\mu_0$ such that $\int_0^\infty x\mu_0(dx) = 1$.

PROPOSITION 3.3. *The total mass $m$ is strictly positive and strictly decreasing on $[0, \infty[$. Moreover, $m(t) \to 0$ as $t \to \infty$.*

PROOF. Since

$$m(t) = \int_0^\infty \mathbb{P}(I > x^\alpha t) x \mu_0(dx),$$

it is sufficient to show that the function $t \in [0, \infty[ \to \mathbb{P}(I > t)$ is strictly positive, strictly decreasing and converges to 0 as $t \to \infty$. This last point is obvious since $I < \infty$ a.s. Next, suppose that $\mathbb{P}(I \le t) = 1$ for some $t > 0$. This would imply that for all $n \ge 1$,

$$\frac{n!}{\phi(|\alpha|)\phi(2|\alpha|)\cdots\phi(n|\alpha|)} = \mathbb{E}[I^n] \le t^n.$$

However, we saw in the Introduction that $x/\phi(x) \to \infty$ as $x \to \infty$. In particular, $2t \le n/\phi(n|\alpha|)$ for large enough $n$, say $n > n_0$. Hence, we would have

$$\frac{n_0!}{\phi(|\alpha|)\phi(2|\alpha|)\cdots\phi(n_0|\alpha|)}(2t)^{n-n0} \le t^n$$

for all $n > n_0$, which is impossible. Therefore, $\mathbb{P}(I > t) > 0$ for all $t > 0$.

Finally, for all $t > 0$, using the Markov property of subordinators, we get

$$\begin{aligned} I &= \int_0^t \exp(\alpha\xi_r)\,dr + \exp(\alpha\xi_t)\int_0^\infty \exp\bigl(\alpha(\xi_{r+t} - \xi_t)\bigr)\,dr \\ &\le t + \exp(\alpha\xi_t)\tilde{I}, \end{aligned}$$

where $\tilde{I}$ is distributed as $I$ and is independent of $\xi_t$. Consider $a$ such that $\mathbb{P}(I \le a) > 0$ and note, using the Poisson point process construction of the subordinator, that $\mathbb{P}(\exp(\alpha\xi_t) \le t/a) > 0$ for all $t > 0$. Then,

$$0 < \mathbb{P}\bigl(\exp(\alpha\xi_t) \le t/a, \tilde{I} \le a\bigr) \le \mathbb{P}(I \le 2t) \qquad \forall t > 0.$$

This leads to the fact that $\mathbb{P}(t \ge I > s) > 0$ for all $0 \le s < t$. Indeed, the event $\{I > s\}$ coincides with $\{\rho(s) < \infty\}$ and when $I > s$,

$$I = s + \exp\bigl(\alpha\xi_{\rho(s)}\bigr)\int_0^\infty \exp\bigl(\alpha\bigl(\xi_{r+\rho(s)} - \xi_{\rho(s)}\bigr)\bigr)\,dr.$$

Using the strong Markov property of the subordinator at the stopping time $\rho(s)$, we get, with probability 1,

$$(I - s)^+ = \exp\bigl(\alpha\xi_{\rho(s)}\bigr)\tilde{I}$$

with $\tilde{I}$ independent of $\xi_{\rho(s)}$ and distributed as $I$. Hence, for all $0 \le s < t$,

$$\begin{aligned} \mathbb{P}(I > s) - \mathbb{P}(I > t) &= \mathbb{P}(s < I \le t) \\ &= \mathbb{P}\bigl(\exp\bigl(\alpha\xi_{\rho(s)}\bigr) > 0, \tilde{I} \le (t-s)\exp\bigl(|\alpha|\xi_{\rho(s)}\bigr)\bigr) \end{aligned}$$

and this last probability is strictly positive since $\mathbb{P}(\exp(\alpha\xi_{\rho(s)}) > 0) = \mathbb{P}(I > s) > 0$ and $\mathbb{P}(\tilde{I} \le a) > 0$ for all $a > 0$. □

We now turn to the proofs of the more precise descriptions of the behavior of $m$ stated in Proposition 1.2, Theorem 1.6(i) and Proposition 1.7. The crucial point is the following lemma, which is basically a consequence of Rivero [28], Proposition 2, and König and Mörters [21], Lemma 2.3.

LEMMA 3.4. *Assume that* (H) *holds or, equivalently, that* $\phi$ *varies regularly at* $\infty$ *with index* $\beta \in [0, 1[$. *Then*,

$$-\ln\big(\mathbb{P}(I > t)\big) \underset{\infty}{\sim} \frac{(1-\beta)}{|\alpha|}\varphi(|\alpha|t) \underset{\infty}{\sim} (1-\beta)|\alpha|^{\beta/(1-\beta)}\varphi(t),$$

*where* $\varphi$ *is the inverse of* $t \to t/\phi(t)$, *which is well defined in the neighborhood of* $\infty$. *In particular*, $-\ln(\mathbb{P}(I > t))$ *is regularly varying at* $\infty$ *with index* $1/(1-\beta)$.

PROOF. Note that the Laplace exponent of the subordinator $|\alpha|\xi$ is $\phi(|\alpha|\cdot)$ and that the inverse of $t \to t/\phi(|\alpha|t)$ is $\varphi(|\alpha|\cdot)/|\alpha|$. Using these facts, we can restrict our proof to the case $|\alpha| = 1$, which is supposed in the following.

When $\beta \in ]0, 1[$, the statement of the lemma is exactly Proposition 2 of Rivero [28]. When $\beta = 0$ and $\phi(\infty) < \infty$,

$$\frac{1}{n}\ln\left(\frac{\mathbb{E}(I^n)}{n!}\right) = -\frac{1}{n}\sum_{i=1}^{n}\ln(\phi(i)) \xrightarrow[n\to\infty]{} -\ln\phi(\infty).$$

Then, by Lemma 2.3. of König and Mörters [21],

$$\lim_{t\to\infty}\frac{1}{t}\ln\big(\mathbb{P}(I > t)\big) = -\phi(\infty).$$

Finally, when $\beta = 0$ and $\phi(\infty) = \infty$, we can adapt König and Mörters' proof of [21], Lemma 2.3, to obtain the expected result. Indeed, first note that

$$\text{(16)}\qquad \frac{1}{n}\ln\left(\mathbb{E}\left[\frac{I^n\phi(n)^n}{n^n}\right]\right) = \frac{1}{n}\ln\left(\frac{n!}{n^n}\right) + \ln(\phi(n)) - \frac{1}{n}\sum_{i=1}^{n}\ln(\phi(i)) \underset{n\to\infty}{\to} -1$$

as a consequence of Stirling's formula and of the fact that

$$\ln(\phi(n)) - \frac{1}{n}\sum_{i=1}^{n}\ln(\phi(i)) \underset{n\to\infty}{\to} 0$$

since $\phi$ is a slowly varying function (see Section 3.2 of Rivero [28] for a proof of this last point). It is then easy, using Markov's inequality, to show that

$$\limsup_{n\to\infty}\frac{1}{n}\ln\big(\mathbb{P}\big(I > n/\phi(n)\big)\big) \le -1.$$

To get a lower bound for the limit inferior, set $Y_n := \ln(I\phi(n)/n)$. For every $\varepsilon > 0$ and every integer $m$, we have that

$$\frac{1}{\mathbb{E}[I^n]}\mathbb{E}\big[I^n\mathbf{1}_{\{Y_n\ge\varepsilon\}}\big] \le \exp(-\varepsilon m)\frac{\mathbb{E}[I^{n+m}]\phi(n)^m}{\mathbb{E}[I^n]n^m} \underset{n\to\infty}{\to} \exp(-\varepsilon m).$$

Letting $m \to \infty$, this gives

$$\frac{1}{\mathbb{E}[I^n]}\mathbb{E}\big[I^n \mathbf{1}_{\{Y_n \geq \varepsilon\}}\big] \underset{n\to\infty}{\to} 0. \tag{17}$$

Besides, for all $\varepsilon > 0$ and all $n \geq 1$,

$$\frac{1}{n}\ln\big(\mathbb{P}\big(I > n\exp(-\varepsilon)/\phi(n)\big)\big) \geq \frac{1}{n}\ln\big(\mathbb{P}(|Y_n| < \varepsilon)\big).$$

However, $I^{-n} > \exp(-n\varepsilon)n^{-n}\phi(n)^n$ on $\{|Y_n| < \varepsilon\}$, which gives

$$\begin{aligned}\frac{1}{n}\ln\big(\mathbb{P}(|Y_n| < \varepsilon)\big) &= \frac{1}{n}\ln\left(\frac{\mathbb{E}[I^{-n}I^n\mathbf{1}_{\{|Y_n|<\varepsilon\}}]}{\mathbb{E}[I^n]}\mathbb{E}[I^n]\right)\\ &\geq \frac{1}{n}\ln\left(\exp(-n\varepsilon)\frac{\mathbb{E}[I^n\mathbf{1}_{\{|Y_n|<\varepsilon\}}]}{\mathbb{E}[I^n]}n^{-n}\phi(n)^n\mathbb{E}[I^n]\right).\end{aligned}$$

By (16) and (17), the last line of this inequality converges to $-\varepsilon - 1$ as $n \to \infty$. Thus, since the function $t \to t/\phi(t)$ is increasing and $\varphi(t) \to \infty$ as $t \to \infty$, we have proven that

$$\limsup_{t\to\infty}\frac{1}{\varphi(t)}\ln\big(\mathbb{P}(I > t)\big) \leq -1,$$

$$\liminf_{t\to\infty}\frac{1}{\varphi(t\exp(\varepsilon))}\ln\big(\mathbb{P}(I > t)\big) \geq -\varepsilon - 1.$$

Using the regular variation of $\varphi$, we get the expected

$$\lim_{t\to\infty}\frac{1}{\varphi(t)}\ln\big(\mathbb{P}(I > t)\big) = -1.$$

□

3.1.1. $\mu_0$ *with bounded support*: *Proof of Proposition* 1.2. We recall that, with no loss of generality, the supremum of the support of $\mu_0$ is supposed to be equal to 1. Thus,

$$m(t) = \int_0^1 \mathbb{P}(I > tx^\alpha)x\mu_0(dx) \leq \mathbb{P}(I > t)\int_0^1 x\mu_0(dx) = \mathbb{P}(I > t).$$

According to Proposition 3.3 in [11], $C_\lambda := \mathbb{E}[\exp(\lambda I)] < \infty$ provided that $\lambda < \phi(\infty)$. Hence, for such $\lambda$'s,

$$m(t) \leq \mathbb{P}(I > t) \leq C_\lambda \exp(-\lambda t) \qquad \forall t \geq 0,$$

which gives the first part of the statement.

Now, assume that (H) holds. Then, on the one hand, since $m(t) \leq \mathbb{P}(I > t)$, we get, by Lemma 3.4,

$$\liminf_{t\to\infty}\frac{-\ln(m(t))}{\varphi(|\alpha|t)} \geq \frac{1-\beta}{|\alpha|}.$$

On the other hand, for all $0 < \varepsilon < 1$,

$$m(t) \geq \mathbb{P}\big(I > t(1-\varepsilon)^{\alpha}\big) \int_{1-\varepsilon}^{1} x\mu_0(dx).$$

By assumption, $\int_{1-\varepsilon}^{1} x\mu_0(dx) > 0$, hence

$$\limsup_{t\to\infty} \frac{-\ln(m(t))}{\varphi(|\alpha|t)} \leq \limsup_{t\to\infty} \frac{-\ln(\mathbb{P}(I > t(1-\varepsilon)^{\alpha}))}{\varphi(|\alpha|t)} = \frac{1-\beta}{|\alpha|}(1-\varepsilon)^{\alpha/(1-\beta)}.$$

Then, let $\varepsilon \downarrow 0$ to get the expected result.

3.1.2. $\mu_0$ *with unbounded support*: *Proofs of Theorem* 1.6(i) *and Proposition* 1.7.

PROOF OF THEOREM 1.6(i). First, suppose that $\mu_0(dx) = \exp(-Cx^{\gamma})\,dx$, $\gamma > 0$. We have

$$\begin{aligned} m(t) &= \int_0^{\infty} \mathbb{P}(I > tx^{\alpha})x\exp(-Cx^{\gamma})\,dx \\ &= \frac{t^{-2/\alpha}}{\gamma} \int_0^{\infty} \mathbb{P}(I > u^{\alpha/\gamma})u^{2/\gamma-1}\exp(-Cut^{-\gamma/\alpha})\,du, \end{aligned} \tag{18}$$

using the change of variable $u = (xt^{1/\alpha})^{\gamma}$. Now, use Lemma 3.4 and Theorem 4.12.10(iii) of [9] to get

$$\begin{aligned} -\ln\biggl(\int_0^{x} \mathbb{P}(I > u^{\alpha/\gamma})u^{2/\gamma-1}\,du\biggr) &\underset{x\to 0}{\sim} -\ln\bigl(\mathbb{P}(I > x^{\alpha/\gamma})\bigr) \\ &\underset{x\to 0}{\sim} (1-\beta)|\alpha|^{\beta/(1-\beta)}\varphi(x^{\alpha/\gamma}), \end{aligned}$$

which varies regularly at 0 with index $\alpha/(\gamma(1-\beta))$. Note that in a neighborhood of 0, $x \to 1/\varphi(x^{\alpha/\gamma})$ is the inverse of

$$x \to \biggl(x\phi\biggl(\frac{1}{x}\biggr)\biggr)^{-\gamma/\alpha}.$$

Hence, by de Bruijn's Tauberian theorem ([9], Theorem 4.12.9) we have

$$-\ln\biggl(\int_0^{\infty} \mathbb{P}(I > u^{\alpha/\gamma})u^{2/\gamma-1}\exp(-ut)\,du\biggr) \underset{t\to\infty}{\sim} C_{\alpha,\beta,\gamma}/h_0(t),$$

where $h_0$ is the inverse, well defined in the neighborhood of $\infty$, of $x \to x^{-1}(x\phi(1/x))^{\gamma/\alpha}$ and $C_{\alpha,\beta,\gamma}$ is the constant defined in the statement of Theorem 1.6(i). Together with (18), this leads to

$$-\ln(m(t)) \underset{\infty}{\sim} C_{\alpha,\beta,\gamma} C^{(1+(1-\beta)\gamma/|\alpha|)^{-1}} / h_0\bigl(t^{\gamma/|\alpha|}\bigr).$$

In other words,

$$-\ln(m(t)) \underset{\infty}{\sim} C_{\alpha,\beta,\gamma} C^{(1+(1-\beta)\gamma/|\alpha|)^{-1}} h(t),$$

where $h$ is the inverse of $t^{1+|\alpha|/\gamma}/\phi(t)$.

Now, suppose that $\mu_0$ possesses a density $u_0$ in a neighborhood of $\infty$ such that $\ln(u_0(x)) \sim_\infty -Cx^\gamma$, $\gamma > 0$. Fix $\varepsilon > 0$ and let $C_\varepsilon$ be such that $u_0(x)$ exists for $x \geq C_\varepsilon$ and

$$\exp\bigl(-(1+\varepsilon)Cx^\gamma\bigr) \leq u_0(x) \leq \exp\bigl(-(1-\varepsilon)Cx^\gamma\bigr) \qquad \forall x \geq C_\varepsilon. \tag{19}$$

Then, write

$$m(t) = \int_0^{C_\varepsilon} \mathbb{P}(I > tx^\alpha) x \mu_0(dx) + \int_{C_\varepsilon}^\infty \mathbb{P}(I > tx^\alpha) x u_0(x)\, dx.$$

On the one hand, following the argument developed in Section 3.1.1, we get

$$\limsup_{t\to\infty} \frac{\ln(\int_0^{C_\varepsilon} \mathbb{P}(I > tx^\alpha) x \mu_0(dx))}{\varphi(t)} \leq -(1-\beta)|\alpha|^{\beta/(1-\beta)} C_\varepsilon^{\alpha/(1-\beta)},$$

which actually holds for any initial measure $\mu_0$. Note that $\varphi(t)/h(t) \to \infty$ as $t \to \infty$, where $h$ is the function defined above in the first part of this proof.

On the other hand, inequalities (19) and the results of the first part of this proof imply that

$$\limsup_{t\to\infty} \frac{-\ln(\int_{C_\varepsilon}^\infty \mathbb{P}(I > tx^\alpha) x u_0(x)\, dx)}{h(t)} \leq C_{\alpha,\beta,\gamma}\bigl((1+\varepsilon)C\bigr)^{(1+(1-\beta)\gamma/|\alpha|)^{-1}}$$

and

$$\liminf_{t\to\infty} \frac{-\ln(\int_{C_\varepsilon}^\infty \mathbb{P}(I > tx^\alpha) x u_0(x)\, dx)}{h(t)} \geq C_{\alpha,\beta,\gamma}\bigl((1-\varepsilon)C\bigr)^{(1+(1-\beta)\gamma/|\alpha|)^{-1}}.$$

We have therefore proven that

$$\begin{aligned} C_{\alpha,\beta,\gamma}\bigl((1-\varepsilon)C\bigr)^{(1+(1-\beta)\gamma/|\alpha|)^{-1}} &\leq \liminf_{t\to\infty} \frac{-\ln(m(t))}{h(t)} \leq \limsup_{t\to\infty} \frac{-\ln(m(t))}{h(t)} \\ &\leq C_{\alpha,\beta,\gamma}\bigl((1+\varepsilon)C\bigr)^{(1+(1-\beta)\gamma/|\alpha|)^{-1}} \end{aligned}$$

for all $\varepsilon > 0$. The result follows by letting $\varepsilon \downarrow 0$. □

PROOF OF PROPOSITION 1.7. Suppose that $u_0(x) = Cx^{-\gamma}$ on $[a, \infty[$ for some $a > 0$ and $\gamma > 2$. Then,

$$m(t) = C\int_a^\infty \mathbb{P}(I > x^\alpha t) x^{1-\gamma}\, dx + \int_0^a \mathbb{P}(I > x^\alpha t) x \mu_0(dx).$$

With the change of variables $u = x^\alpha t$,

$$\int_a^\infty \mathbb{P}(I > x^\alpha t) x^{1-\gamma}\, dx = \frac{t^{(\gamma-2)/\alpha}}{|\alpha|} \int_0^{a^\alpha t} \mathbb{P}(I > u) u^{(2-\gamma)/\alpha - 1}\, du$$

and this last integral converges to a finite limit as $t \to \infty$ since $\mathbb{P}(I > u) \le C_\lambda \exp(-\lambda u)$ for all $u \ge 0$ and some $\lambda > 0$ sufficiently small (see the proof of Proposition 1.2 for this last point). Using the same upper bound for $\mathbb{P}(I > x^\alpha t)$, we get that

$$\int_0^a \mathbb{P}(I > x^\alpha t) x \mu_0(dx) \le C_\lambda \exp(-\lambda a^\alpha t) \int_0^a x \mu_0(dx).$$

Thus,

$$m(t) \underset{t\to\infty}{\sim} \frac{C}{|\alpha|} t^{(\gamma-2)/\alpha} \int_0^\infty \mathbb{P}(I > u) u^{(2-\gamma)/\alpha - 1}\, du.$$

It is not hard to extend this proof to the case where $u_0(x) \sim_\infty C x^{-\gamma}$, for some $\gamma > 2$. This is left to the reader. $\square$

3.2. *Proof of Theorem* 3.1. We start with the following lemma.

LEMMA 3.5. *Suppose that* $-\ln(m)$ *varies regularly at* $\infty$ *with a positive index* $\gamma$ *and satisfies* (11). *Then, for any function* $g : [0, \infty[ \to ]0, \infty[$ *such that* $g(t)/(-\ln(m(t))) \to 1$ *as* $t \to \infty$, *we have*

$$\mathbb{E}\left[f\left(\left(\frac{\gamma g(t)}{t}\right)^{1/|\alpha|} X(t)\right) \Big| X(t) > 0\right] \underset{t\to\infty}{\to} \mathbb{E}[f(R^{1/|\alpha|})]$$

*for all continuous bounded test functions* $f$ *on* $]0, \infty[$.

PROOF. First, note that when $X(0)^{|\alpha|} I > t$, we have

$$\begin{aligned} X(0)^{|\alpha|} I &= X(0)^{|\alpha|} \int_0^{\rho(X(0)^\alpha t)} \exp(\alpha \xi_r)\, dr \\ &\quad + X(0)^{|\alpha|} \exp(\alpha \xi_{\rho(X(0)^\alpha t)}) \int_0^\infty \exp(\alpha(\xi_{r+\rho(X(0)^\alpha t)} - \xi_{\rho(X(0)^\alpha t)}))\, dr \\ &= t + X(0)^{|\alpha|} \exp(\alpha \xi_{\rho(X(0)^\alpha t)}) \int_0^\infty \exp(\alpha(\xi_{r+\rho(X(0)^\alpha t)} - \xi_{\rho(X(0)^\alpha t)}))\, dr. \end{aligned}$$

Now, use the strong Markov property of $\xi$ at the (randomized) stopping time $\rho(X(0)^\alpha t)$ to get

$$(X(0)^{|\alpha|} I - t)^+ = X(0)^{|\alpha|} \exp(\alpha \xi_{\rho(X(0)^\alpha t)}) \tilde{I} = X(t)^{|\alpha|} \tilde{I}, \tag{20}$$

where $\tilde{I}$ is distributed as $I$ and is independent of $X(t)$. This gives, for all $n \in \mathbb{N}^*$,

$$\begin{aligned} m(t)^{-1}\mathbb{E}\biggl[\biggl(\biggl(\frac{\gamma g(t)}{t}\biggr)^{1/|\alpha|} X(t)\biggr)^{|\alpha|n}\biggr]\mathbb{E}[I^n] &= m(t)^{-1}\biggl(\frac{\gamma g(t)}{t}\biggr)^{n}\mathbb{E}[X(t)^{|\alpha|n}]\mathbb{E}[I^n] \\ &= m(t)^{-1}\biggl(\frac{\gamma g(t)}{t}\biggr)^{n}\mathbb{E}[((X(0)^{|\alpha|}I - t)^+)^n]. \end{aligned}$$

Then, recall that

$$m(t) = \mathbb{P}(X(0)^{|\alpha|}I > t), \qquad t \geq 0.$$

Integrating by parts, we have

$$\begin{aligned} m(t)^{-1}\biggl(\frac{\gamma g(t)}{t}\biggr)^{n}\mathbb{E}[((X(0)^{|\alpha|}I - t)^+)^n] &= nm(t)^{-1}\biggl(\frac{\gamma g(t)}{t}\biggr)^{n}\int_t^\infty (x-t)^{n-1} m(x)\,dx, \end{aligned}$$

which, according to Lemma 2.1 and the assumptions we have made on $-\ln(m)$ and $g$, converges as $t \to \infty$ to $n!$. Next, note that $\mathbb{E}[R^n]\mathbb{E}[I^n] = n!$, using the factorization property (15) of the exponential random variable with parameter 1. Putting all of the pieces together, we have proven that for all integers $n \geq 1$,

$$\mathbb{E}\biggl[\biggl(\biggl(\frac{\gamma g(t)}{t}\biggr)^{1/|\alpha|} X(t)\biggr)^{|\alpha|n} \bigg| X(t) > 0\biggr] \underset{t\to\infty}{\to} \mathbb{E}[R^n].$$

*Summary.* Let $\nu_t$ denote the distribution of $\gamma t^{-1} g(t) X(t)^{|\alpha|}$ conditional on $X(t) > 0$ ($\nu_t$ is a probability measure on $]0, \infty[$). We have shown that for all $n \geq 1$,

$$\int_0^\infty x^n \nu_t(dx) \to \int_0^\infty x^n \mu_R(dx),$$

where $\mu_R$ is the distribution of $R$. Of course, this still holds for $n = 0$, but the distribution of $R$ is characterized by its moments. It is then well known ([15], Chapter VIII, page 269) that this implies that $\nu_t$ converges in distribution to $\mu_R$. □

3.2.1. *Proof of Theorem* 3.1(i). By Proposition 1.2, under the hypothesis (H), $-\ln(m)$ varies regularly at $\infty$ with index $1/(1-\beta)$ and, more precisely,

$$-\ln(m(t)) \underset{t\to\infty}{\sim} \frac{(1-\beta)}{|\alpha|}\varphi(|\alpha|t).$$

Together with Lemma 3.5, this implies the statement of Theorem 3.1, provided that $-\ln(m)$ satisfies (11). The goal of this section is to prove this last point when $\mu_0$ has bounded support.

LEMMA 3.6. *Let*

$$f(x) = -\ln(m(x)), \qquad x \geq 0,$$

*and assume that* (H) *holds*, $\int^\infty x\,\Pi(dx) < \infty$ *and* $\mu_0$ *has bounded support*. *Then*, *for all* $\varepsilon > 0$, *there exists some* $x(\varepsilon)$ *such that*

$$\lambda^{1/(1-\beta)-\varepsilon} \leq \frac{f(\lambda x)}{f(x)} \leq \lambda^{1/(1-\beta)+\varepsilon} \qquad \forall \lambda \geq 1 \text{ and } \forall x \geq x(\varepsilon).$$

This lemma is a direct consequence of Lemmas 3.7 and 3.10 below.

LEMMA 3.7. *Let* $g : ]0,\infty[ \to ]0,\infty[$ *be a continuously differentiable function such that*

$$\frac{xg'(x)}{g(x)} \to c > 0 \qquad \text{as } x \to \infty.$$

*Then, for all* $\varepsilon > 0$, *there exists some* $x(\varepsilon)$ *such that*

$$\lambda^{c-\varepsilon} \leq \frac{g(\lambda x)}{g(x)} \leq \lambda^{c+\varepsilon} \qquad \forall \lambda \geq 1 \text{ and } \forall x \geq x(\varepsilon).$$

PROOF. For $\varepsilon > 0$, let $x(\varepsilon)$ be such that

$$c - \varepsilon \leq \frac{xg'(x)}{g(x)} \leq c + \varepsilon \qquad \text{for all } x \geq x(\varepsilon).$$

For such $x$'s and all $\lambda \geq 1$,

$$\begin{aligned}(c-\varepsilon)\ln(\lambda) &= (c-\varepsilon)\int_x^{\lambda x} y^{-1}\,dy \leq \int_x^{\lambda x} \frac{g'(y)}{g(y)}\,dy \\ &\leq (c+\varepsilon)\int_x^{\lambda x} y^{-1}\,dy = (c+\varepsilon)\ln(\lambda).\end{aligned}$$

Since $\int_x^{\lambda x} \frac{g'(y)}{g(y)}\,dy = \ln(g(\lambda x)) - \ln(g(x))$, the result is proved. □

LEMMA 3.8. *Suppose that* $\phi$ *is regularly varying at* $\infty$ *with index* $\beta \in [0, 1[$ *and that* $\phi(x) \to \infty$ *as* $x \to \infty$. *Let* $\beta' \in ]\beta, 1[$. *There then exists some* $x_1(\beta')$ *such that for* $x \geq x_1(\beta')$ *and all* $\lambda \geq 1$,

$$1 \leq \phi(x) \leq \phi(\lambda x) \leq \lambda^{\beta'}\phi(x)$$

*and*

$$\phi(x) \leq x^{\beta'}.$$

PROOF. Note that $\phi$ is infinitely differentiable on $]0,\infty[$ with derivative

$$\phi'(x)=\int_0^\infty v\exp(-xv)\Pi(dv),$$

which is nonincreasing. It is then a classical result on regular variation (see the monotone density theorem, [9], Theorem 1.7.2) that $\phi'$ is regularly varying with index $\beta-1$ and

$$\frac{x\phi'(x)}{\phi(x)}\underset{x\to\infty}{\to}\beta.$$

The first part of the lemma is then a consequence of the above Lemma 3.7 and of the fact that $\phi$ is increasing and converges to $\infty$. We also have that $\phi(x)/x^{\beta'}$ converges to 0 at $\infty$ (since $\beta'>\beta$), hence the second assertion holds for sufficiently large $x$. □

LEMMA 3.9. *Let*

$$f(x)=-\ln(\mathbb{P}(I>x)),\qquad x\geq 0,$$

*which, as proved in Lemma* 3.4, *is regularly varying with index* $1/(1-\beta)$, *under the assumption* (H). *Suppose, moreover, that* $\int^\infty x\Pi(dx)<\infty$. *Then* $f$ *is infinitely differentiable and*

$$\frac{xf'(x)}{f(x)}\to\frac{1}{1-\beta}\qquad \text{as } x\to\infty.$$

PROOF. According to [11], Proposition 2.1, when $\int^\infty x\Pi(dx)<\infty$, there exists an infinitely differentiable function $k:]0,\infty[\to[0,\infty[$ such that $k(x)\,dx$ is the distribution of $I$. Moreover,

$$\begin{aligned}k(x)&=\int_x^\infty\overline{\Pi}(|\alpha|^{-1}\ln(u/x))k(u)\,du\\&=\int_0^\infty\left(\int_x^{xe^{v|\alpha|}}k(u)\,du\right)\Pi(dv).\end{aligned}$$

To simplify notation, we suppose in the following that $|\alpha|=1$. The proof is identical for $|\alpha|\neq 1$. In particular, we have

$$\mathbb{P}(I>x)=\int_x^\infty k(u)\,du$$

and

$$f'(x)=\frac{k(x)}{\mathbb{P}(I>x)}=\int_0^\infty(1-\exp(f(x)-f(xe^v)))\Pi(dv),\qquad x>0. \tag{21}$$

Note that since $f$ is regularly varying with a positive index, we have that $f(x)\to\infty$ as $x\to\infty$ and, therefore, for all $v>0$,

$$f(x)-f(xe^v)=f(x)\left(1-\frac{f(xe^v)}{f(x)}\right)\underset{\infty}{\sim}f(x)(1-e^{v/(1-\beta)})\underset{x\to\infty}{\to}-\infty.$$

- When $\Pi(]0,\infty[) < \infty$, this implies the expected result since, by dominated convergence,

$$f'(x) \underset{x\to\infty}{\to} \Pi(]0,\infty[) = \lim_{x\to\infty} \frac{f(x)}{x}.$$

- The proof is much more technical when $\Pi(]0,\infty[) = \infty$, which is supposed for the rest of this proof. We proceed in two steps.

*Step* 1. The goal of this step is to prove that

$$\liminf_{x\to\infty} \frac{xf'(x)}{f(x)} \geq \frac{1}{1-\beta}.$$

First, suppose that there exists some $x_0$ and some *nondecreasing* positive function $g$ such that $f'(x) \geq g(x)$ for all $x \geq x_0$. Then, for $x \geq x_0$ and $v > 0$,

$$f(xe^v) - f(x) = \int_x^{xe^v} f'(u)\,du \geq g(x)x(e^v - 1) \geq g(x)xv.$$

Using (21), this gives

$$f'(x) \geq \phi(g(x)x), \qquad x \geq x_0. \tag{22}$$

Now, note that $f'(x) \to \infty$ as $x \to \infty$ since, for all $a > 0$,

$$\liminf_{x\to\infty} f'(x) \geq \liminf_{x\to\infty} \int_a^\infty \bigl(1 - \exp\bigl(f(x) - f(xe^v)\bigr)\bigr)\Pi(dv) = \int_a^\infty \Pi(dv)$$

(by dominated convergence) and the right-hand side converges to $\infty$ as $a \to 0$. In particular, $f'(x) \geq 1$ for $x$ sufficiently large (say $x \geq x_0$). Replacing $g$ by 1 in (22), we get

$$f'(x) \geq \phi(x) \qquad \forall x \geq x_0.$$

Recall that $\phi$ is nondecreasing and then iterate the procedure to get, for all $n \geq 0$,

$$f'(x) \geq h_n(x) \qquad \forall x \geq x_0, \tag{23}$$

where the functions $h_n : ]0,\infty[ \to ]0,\infty[$ are defined by induction by

$$h_0(x) = 1 \qquad \text{for all } x \geq 0;$$

$$h_n(x) = \phi(h_{n-1}(x)x) \qquad \text{for all } x \geq 0.$$

Now, the interesting fact is that for $x$ large enough, $h_n(x) \to \varphi(x)/x$ as $n \to \infty$. Indeed, let $\beta' \in ]\beta, 1[$. With the notation of Lemma 3.8, we have, for $x \geq x_1(\beta')$, $1 \leq \phi(x) \leq x^{\beta'}$, that is, $h_0(x) \leq h_1(x) \leq x^{\beta'}$. Using the fact that $\phi$ is nondecreasing, we easily have, by induction, that

$$1 \leq h_n(x) \leq h_{n+1}(x) \leq x^{\beta' + \cdots + \beta'^{n+1}} \leq x^{\beta'/(1-\beta')} < \infty$$

for all $n \geq 1$. Let $l(x) := \lim_{n\to\infty} h_n(x)$. We have shown that $0 < l(x) < \infty$. Then, necessarily, $l(x) = \phi(l(x)x)$ [in other words, $l(x)x/\phi(l(x)x) = x$] and, finally, $l(x)x = \varphi(x)$, $\forall x \geq x_1(\beta')$. To conclude, for $x$ large enough, letting $n \to \infty$ in (23), we get $f'(x) \geq \varphi(x)/x$, which, combined with Lemma 3.4, gives the expected lim inf.

*Step* 2. The proof of the lim sup is similar, but more technical. First, note that for all $\varepsilon > 0$ and all $a < \ln(1+\varepsilon)$, $a > 0$,

$$\begin{aligned}
\liminf_{x\to\infty} \int_0^{\ln(1+\varepsilon)} & \big(1 - \exp\big(f(x) - f(xe^v)\big)\big)\Pi(dv) \\
&\geq \liminf_{x\to\infty} \int_a^{\ln(1+\varepsilon)} \big(1 - \exp\big(f(x) - f(xe^v)\big)\big)\Pi(dv) \\
&= \int_a^{\ln(1+\varepsilon)} \Pi(dv) \qquad \text{(by dominated convergence)}, \\
&\underset{a\to 0}{\to} \infty,
\end{aligned}$$

whereas

$$\lim_{n\to\infty} \int_{\ln(1+\varepsilon)}^{\infty} \big(1 - \exp\big(f(x) - f(xe^v)\big)\big)\Pi(dv) = \int_{\ln(1+\varepsilon)}^{\infty} \Pi(dv) < \infty.$$

Hence, there exists some $x_1(\varepsilon)$ such that for $x \geq x_1(\varepsilon)$,

$$f'(x) \leq (1+\varepsilon) \int_0^{\ln(1+\varepsilon)} \big(1 - \exp\big(f(x) - f(xe^v)\big)\big)\Pi(dv). \tag{24}$$

Next, fix some $\beta' \in ]\beta, 1[$ and consider some $\delta > 0$ and $\varepsilon > 0$ such that $(1+\delta)(1+\varepsilon)^{1/(\beta-1)}\beta' < 1$. Since $f$ is regularly varying with index $1/(1-\beta)$, there exists some $x_2(\delta, \varepsilon)$ such that

$$f\big(x(1+\varepsilon)\big) \leq (1+\delta)(1+\varepsilon)^{1/(1-\beta)} f(x) \qquad \forall x \geq x_2(\delta, \varepsilon). \tag{25}$$

We will need this later. For the moment, let $x_0 = \max(x_1(\beta'), x_1(\varepsilon), x_2(\delta, \varepsilon))$, with $x_1(\beta')$ as introduced in Lemma 3.8. Next, suppose that for all $x \geq x_0$,

$$f'(x) \leq g(x)$$

for some *nondecreasing* function $g$ such that $g(x) \geq 1$ for all $x \geq x_0$. Note that this implies that

$$f(xe^v) - f(x) = \int_x^{xe^v} f'(u)\, du \leq g(xe^v)x(e^v - 1).$$

The function $v \to v^{-1}(e^v - 1)$ is increasing on $[0, \infty[$, hence $e^v - 1 \leq v\gamma(\varepsilon)$ for all $v \leq \ln(1+\varepsilon)$, where $\gamma(\varepsilon) = \varepsilon/(\ln(1+\varepsilon))$. Together with (24), this leads to

$$\begin{aligned}
f'(x) &\leq (1+\varepsilon) \int_0^{\ln(1+\varepsilon)} \big(1 - \exp\big(-g\big(x(1+\varepsilon)\big)xv\gamma(\varepsilon)\big)\big)\Pi(dv) \\
&\leq (1+\varepsilon)\phi\big(g\big(x(1+\varepsilon)\big)x\gamma(\varepsilon)\big)
\end{aligned} \tag{26}$$

for all $x \geq x_0$.

We then claim that for all $n \geq 1$ and all $x \geq x_0$,

$$f'(x) \leq (1+\varepsilon)^{1+\beta'+2\beta'^2+\cdots+n\beta'^n} \gamma(\varepsilon)^{\beta'+\beta'^2+\cdots+\beta'^n} g\big(x(1+\varepsilon)^n\big)^{\beta'^n} h_n(x), \tag{27}$$

where the sequence of functions $h_n$ is that introduced in step 1 of this proof. We will prove this by induction on $n$. First, though, let us mention that, by a simple application of induction, using Lemma 3.8,

$$h_n\big(x(1+\varepsilon)\big) \leq (1+\varepsilon)^{\beta'+\cdots+\beta'^n} h_n(x) \qquad \text{for all } x \geq x_0 \text{ and } n \geq 1.$$

We now turn to the proof of (27). For $n = 1$, we can use (26) and Lemma 3.8 to get [note that $\gamma(\varepsilon) \geq 1$, hence $\gamma(\varepsilon)g \geq 1$]

$$f'(x) \leq (1+\varepsilon) g\big(x(1+\varepsilon)\big)^{\beta'} \gamma(\varepsilon)^{\beta'} \phi(x), \qquad x \geq x_0,$$

which leads to (27) for $n = 1$. Now, assume that (27) is true for some integer $n$. Note that the function on the right-hand side of this inequality, which we call $g_1$, is larger than 1 for all $x \geq x_0$. Also, note that it is nondecreasing. Hence, we get, replacing $g$ by $g_1$ in (26), for $x \geq x_0$,

$$\begin{aligned} f'(x) &\leq (1+\varepsilon)\phi\big((1+\varepsilon)^{1+\beta'+2\beta'^2+\cdots+n\beta'^n} \gamma(\varepsilon)^{\beta'+\beta'^2+\cdots+\beta'^n} \\ &\qquad \times g\big(x(1+\varepsilon)^{n+1}\big)^{\beta'^n} h_n\big(x(1+\varepsilon)\big) x\gamma(\varepsilon)\big) \\ &\leq (1+\varepsilon)\phi\big((1+\varepsilon)^{1+2\beta'+3\beta'^2+\cdots+(n+1)\beta'^n} \\ &\qquad \times \gamma(\varepsilon)^{1+\beta'+\beta'^2+\cdots+\beta'^n} g\big(x(1+\varepsilon)^{n+1}\big)^{\beta'^n} h_n(x)x\big) \\ &\leq (1+\varepsilon)^{1+\beta'+2\beta'^2+\cdots+(n+1)\beta'^{n+1}} \gamma(\varepsilon)^{\beta'+\beta'^2+\cdots+\beta'^{n+1}} \\ &\qquad \times g\big(x(1+\varepsilon)^{n+1}\big)^{\beta'^{n+1}} \phi(h_n(x)x), \end{aligned}$$

where, for the last inequality, we have used Lemma 3.8. Hence, we have (27) for all $n \geq 1$.

Now, thanks to the assumptions (H) and $\int^\infty x\Pi(dx) < \infty$ and to Lemma 1 of [20], we know that the function $k$ is bounded from above on $]0,\infty[$, say by some constant $C \geq 1$. Hence, $f'(x) = k(x)/\mathbb{P}(I > x) \leq C\exp(f(x))$ for all $x > 0$. Since $f$ is nondecreasing and nonnegative, the function $x \to C\exp(f(x))$ is nondecreasing and greater than 1, hence we can replace $g$ by this function in (27) to get, for all $n \geq 1$ and all $x \geq x_0$,

$$\begin{aligned} f'(x) &\leq (1+\varepsilon)^{1+\beta'+2\beta'^2+\cdots+n\beta'^n} \gamma(\varepsilon)^{\beta'+\beta'^2+\cdots+\beta'^n} C^{\beta'^n} \\ &\qquad \times \exp\big(\beta'^n f\big(x(1+\varepsilon)^n\big)\big) h_n(x). \end{aligned} \tag{28}$$

Our goal now is to let $n \to \infty$ in this inequality. Iterating inequality (25), we get, for $x \geq x_0$ and for all $n \geq 1$,

$$f\big(x(1+\varepsilon)^n\big) \leq (1+\delta)^n (1+\varepsilon)^{n/(1-\beta)} f(x).$$

Since

$$(1+\delta)(1+\varepsilon)^{1/(\beta-1)}\beta' < 1,$$

this leads, for $x \geq x_0$, to

$$\exp(\beta'^n f(x(1+\varepsilon)^n)) \underset{n\to\infty}{\to} 1.$$

As $n \to \infty$, we also have

$$C^{\beta'^n} \to 1 \quad \text{and} \quad (1+\varepsilon)^{1+\beta'+2\beta'^2+\cdots+n\beta'^n} \to (1+\varepsilon)^{1+\beta'/(1-\beta')^2}$$

and

$$\gamma(\varepsilon)^{\beta'+\beta'^2+\cdots+\beta'^n} \to \gamma(\varepsilon)^{\beta'/(1-\beta')}.$$

Last, recall that for $x$ large enough, $h_n(x) \to \varphi(x)/x$ as $n \to \infty$. Letting $n \to \infty$ in (28), we therefore have, for $x$ large enough,

$$f'(x) \leq C_\varepsilon \varphi(x)/x,$$

where $C_\varepsilon \to 1$ as $\varepsilon \to 0$. This gives

$$\limsup_{x\to\infty} \frac{xf'(x)}{f(x)} \leq \frac{1}{1-\beta}. \qquad \square$$

LEMMA 3.10. *Let*

$$f(x) := -\ln(m(x)), \qquad x \geq 0,$$

*and suppose that* (H) *holds*, $\int^\infty x\Pi(dx) < \infty$ *and* $\mu_0$ *has bounded support. Then* $f$ *is differentiable on* $]0,\infty[$ *and*

$$\frac{xf'(x)}{f(x)} = -\frac{xm'(x)}{m(x)f(x)} \to \frac{1}{1-\beta} \qquad \text{as } x \to \infty.$$

PROOF. With no loss of generality, we suppose that the supremum of the support of $\mu_0$ is equal to 1. Under the assumptions of the lemma, we know (see the proof of the previous lemma) that $x \to \mathbb{P}(I > x)$ is differentiable on $]0,\infty[$, with derivative $-k$. By Lemma 1 in [20], we also know that the function $x \in ]0,\infty[ \to xk(x)$ is bounded. Let $M$ denote an upper bound. Recall, then, that

$$m(x) = \int_0^1 \mathbb{P}(I > xy^\alpha) y\mu_0(dy)$$

and note that for all $x > a > 0$ and all $y \in ]0,1[$,

$$|\partial_x(\mathbb{P}(I > xy^\alpha))| = k(xy^\alpha)y^\alpha \leq \frac{M}{a}.$$

Hence, by dominated convergence, $m$ is continuously differentiable on $]0,\infty[$, with derivative

$$m'(x) = -\int_0^1 k(xy^\alpha) y^\alpha y \mu_0(dy), \qquad x > 0.$$

Now, fix $\delta > 0$. By Lemma 3.9, there exists some $x(\delta)$ such that for $x \ge x(\delta)$,

$$\frac{1-\delta}{1-\beta} \le \frac{-xk(x)}{\mathbb{P}(I > x)\ln(\mathbb{P}(I > x))} \le \frac{1+\delta}{1-\beta}.$$

Then, for $x \ge x(\delta)$,

$$\begin{aligned} &\frac{1-\delta}{(1-\beta)x}\int_0^1 \mathbb{P}(I > xy^\alpha)\ln\bigl(\mathbb{P}(I > xy^\alpha)\bigr) y\mu_0(dy) \\ &\quad \le m'(x) \le \frac{1+\delta}{(1-\beta)x}\int_0^1 \mathbb{P}(I > xy^\alpha)\ln\bigl(\mathbb{P}(I > xy^\alpha)\bigr) y\mu_0(dy). \end{aligned} \tag{29}$$

Now, let $\varepsilon > 0$. On the one hand, we claim that

$$\begin{aligned} &\int_{1-\varepsilon}^1 \mathbb{P}(I > xy^\alpha)\ln\bigl(\mathbb{P}(I > xy^\alpha)\bigr) y\mu_0(dy) \\ &\quad \underset{x\to\infty}{\sim} \int_0^1 \mathbb{P}(I > xy^\alpha)\ln\bigl(\mathbb{P}(I > xy^\alpha)\bigr) y\mu_0(dy). \end{aligned} \tag{30}$$

Indeed, for all $0 < y < 1-\varepsilon$,

$$\frac{\mathbb{P}(I > xy^\alpha)\ln(\mathbb{P}(I > xy^\alpha))}{\mathbb{P}(I > x(1-\varepsilon)^\alpha)\ln(\mathbb{P}(I > x(1-\varepsilon)^\alpha))} \to 0 \qquad \text{as } x \to \infty$$

since $x \to -\ln(\mathbb{P}(I > x))$ is regularly varying at $\infty$ with a positive index and $\alpha < 0$. It is then not hard to see, using Lemmas 3.7 and 3.9, that for $x$ large enough, this function is bounded from above by

$$\exp\biggl(1 - \Bigl(\frac{y}{1-\varepsilon}\Bigr)^{\alpha(1-\varepsilon)/(1-\beta)}\biggr)\Bigl(\frac{y}{1-\varepsilon}\Bigr)^{\alpha(1+\varepsilon)/(1-\beta)},$$

which, in turn, is bounded for $y \in ]0, 1-\varepsilon[$. Hence, by dominated convergence, we see that

$$\begin{aligned} &\left|\int_0^{1-\varepsilon} \mathbb{P}(I > xy^\alpha)\ln\bigl(\mathbb{P}(I > xy^\alpha)\bigr) y\mu_0(dy)\right| \\ &\quad \underset{x\to\infty}{\ll} \bigl|\mathbb{P}\bigl(I > x(1-\varepsilon)^\alpha\bigr)\ln\bigl(\mathbb{P}\bigl(I > x(1-\varepsilon)^\alpha\bigr)\bigr)\bigr| \\ &\quad \le \frac{1}{\varepsilon}\left|\int_{1-\varepsilon}^1 \mathbb{P}(I > xy^\alpha)\ln\bigl(\mathbb{P}(I > xy^\alpha)\bigr) y\mu_0(dy)\right|, \end{aligned}$$

where, for the last inequality, we have used the fact that the function $x \to -x\ln(x)$ is increasing in a neighborhood of 0. Hence (30). A similar, but simpler, argument leads to the result

$$\int_{1-\varepsilon}^{1} \mathbb{P}(I > xy^{\alpha}) y\mu_0(dy) \underset{x\to\infty}{\sim} \int_0^1 \mathbb{P}(I > xy^{\alpha}) y\mu_0(dy). \tag{31}$$

On the other hand, using the fact that $x \to \ln(\mathbb{P}(I > x))$ is regularly varying with index $1/(1-\beta)$, we have, for $1-\varepsilon \le y \le 1$ and $x$ sufficiently large [say $x \ge x(\varepsilon)$],

$$\begin{aligned}(1+\varepsilon)(1-\varepsilon)^{\alpha/(1-\beta)} \ln(\mathbb{P}(I > x)) &\le \ln(\mathbb{P}(I > x(1-\varepsilon)^{\alpha})) \le \ln(\mathbb{P}(I > xy^{\alpha})) \\ &\le \ln(\mathbb{P}(I > x)).\end{aligned}$$

Thus,

$$\begin{aligned}\int_{1-\varepsilon}^{1} \mathbb{P}(I > xy^{\alpha}) y\mu_0(dy) &\le \frac{\int_{1-\varepsilon}^{1} \mathbb{P}(I > xy^{\alpha}) \ln(\mathbb{P}(I > xy^{\alpha})) y\mu_0(dy)}{\ln(\mathbb{P}(I > x))} \\ &\le (1+\varepsilon)(1-\varepsilon)^{\alpha/(1-\beta)} \int_{1-\varepsilon}^{1} \mathbb{P}(I > xy^{\alpha}) y\mu_0(dy),\end{aligned}$$

which, taking $x(\varepsilon)$ larger if necessary and using (30) and (31), gives, for $x \ge x(\varepsilon)$,

$$\begin{aligned}(1-\delta)m(x) &\le \frac{\int_0^1 \mathbb{P}(I > xy^{\alpha}) \ln(\mathbb{P}(I > xy^{\alpha})) y\mu_0(dy)}{\ln(\mathbb{P}(I > x))} \\ &\le (1+\delta)(1+\varepsilon)(1-\varepsilon)^{\alpha/(1-\beta)} m(x).\end{aligned}$$

Plugging this into (29) and letting first $\varepsilon \to 0$ and then $\delta \to 0$, we get the expected convergence since $f(x) \sim -\ln(\mathbb{P}(I > x))$ as $x \to \infty$. □

3.2.2. *Proof of Theorem* 3.1(ii). The fact that the function

$$x \in ]0,\infty[ \to f(x) := -\ln(m(x)) = -\ln(\mathbb{P}(X(x) > 0))$$

is continuously differentiable on $]0,\infty[$ can be proven in exactly the same way as when the support of $\mu_0$ is compact; see the beginning of the proof of Lemma 3.10. Next, by Karamata's theorem (Theorem 1.5.11 of [9]), if $f$ varies regularly at $\infty$ with index $\lambda > 0$ and if its derivative is also regularly varying at $\infty$, then

$$\frac{xf'(x)}{f(x)} \to \lambda \qquad \text{as } x \to \infty.$$

Together with Theorem 1.6(i), Lemma 3.7 and Lemma 3.5, this implies Theorem 3.1(ii).

3.3. *Quasi-stationary distributions.*

PROOF OF THEOREM 3.2. When $X(0) \sim \mu_R^{(\lambda)}$, the distribution of $X(0)^{|\alpha|} I$ is that of $\lambda^{|\alpha|} RI$, with $R$ independent of $I$, that is, that of an exponential random variable with parameter $\lambda^\alpha$. We then immediately have that for $n \geq 1$ and $t \geq 0$,

$$\mathbb{E}\big[\big((X(0)^{|\alpha|} I - t)^+\big)^n\big] = \lambda^{|\alpha| n} n! \exp(-\lambda^\alpha t)$$

and

$$\mathbb{P}\big(X(t) > 0\big) = \mathbb{P}\big(X(0)^{|\alpha|} I > t\big) = \exp(-\lambda^\alpha t).$$

Following the beginning of the proof of Lemma 3.5, this gives

$$\mathbb{E}\big[(X(t))^{|\alpha| n}\big]\mathbb{E}[I^n] = \mathbb{E}\big[\big((X(0)^{|\alpha|} I - t)^+\big)^n\big] = \lambda^{|\alpha| n} n! \exp(-\lambda^\alpha t)$$

and then

$$\mathbb{E}\big[(X(t))^{|\alpha| n} | X(t) > 0\big] = \mathbb{E}\big[\lambda^{|\alpha| n} R^n\big] = \mathbb{E}\big[X(0)^{|\alpha| n}\big].$$

Hence, $\mu_R^{(\lambda)}$ is a quasi-stationary distribution since the distribution of $R$ is characterized by its entire positive moments. Note that there is no other quasi-stationary distribution. Indeed, let $\varsigma$ be a quasi-stationary distribution and suppose that $X(0) \sim \varsigma$. Then, necessarily, by the Markov property of $X$, $\mathbb{P}(X(t+s) > 0) = \mathbb{P}(X(t) > 0)\mathbb{P}(X(s) > 0)$, which implies that $X(0)^{|\alpha|} I$ has an exponential distribution, say with parameter $\ell$, that is, $\ell X(0)^{|\alpha|} I$ has an exponential distribution with parameter 1. Since the factorization (15) characterizes the distribution of $R$, we get that $\varsigma = \mu_R^{(\ell^{1/\alpha})}$. □

PROOF OF THEOREM 1.8. The first part of this theorem is an obvious consequence of Theorem 3.2. The reverse cannot be directly deduced from Theorem 3.2 since we do not know if uniqueness holds for the fragmentation equation when the initial measure has an unbounded support.

So, consider $(\mu_t, t \geq 0)$, a quasi-stationary solution of the fragmentation equation (2). We want to prove that this solution belongs to the family of solutions $((\mu_{\infty,t}^{(\lambda)}, t \geq 0), \lambda > 0)$, as defined in Theorem 1.8. Replacing $\mu_t$ by $m(t)\mu_0$ in equation (2), we get that

$$\big(1 - m(t)\big)\langle \mu_0, f\rangle = -\int_0^t m(s)\, ds \langle \mu_0, G(f)\rangle \qquad \forall f \in C_c^1,$$

where $G(f)(x) = x^\alpha \int_0^1 (f(xy) - f(x)y) B(dy)$. In other words, there exists some constant $C > 0$ such that

$$m(t) = \exp(-Ct) \qquad \forall t \geq 0,$$

and

$$\langle \mu_0, f\rangle = -C^{-1} \langle \mu_0, G(f)\rangle \qquad \forall f \in C_c^1.$$

When $f \in C_c^1$, the function $x \to xf(x)$ is also in $C_c^1$. Hence, the above identity can be rewritten

$$\langle x\mu_0, f\rangle = -C^{-1}\langle x\mu_0, \tilde{A}(f)\rangle \qquad \forall f \in C_c^1, \tag{32}$$

where $\tilde{A}(f)(x) = x^\alpha \int_0^1 (f(xy) - f(x))yB(dy)$.

To show that this characterizes $\mu_0$, we need the following fact: for all $\beta > 0$, there exists a nondecreasing sequence of functions $f_{\beta,n} : ]0,\infty[ \to [0,\infty[$ such that $f_{\beta,n}(x) \to x^\beta$ as $n \to \infty$, $\forall x > 0$, $f_{\beta,n} \in C_c^1$ and $|f'_{\beta,n}(x)| \le \beta x^{\beta-1}$ for all $x > 0$ and all $n \ge 1$. This sequence can, for example, be constructed by first considering a nondecreasing sequence of continuous functions $g_{\beta,n} : ]0,\infty[ \to [0,\infty[$ such that $g_{\beta,n}(x) \le \beta x^{\beta-1}$, $\forall x > 0, n \ge 1$, $g_{\beta,n}(x) = \beta x^{\beta-1}$ for $x \in [n^{-1}, n]$ and $g_{\beta,n}(x) = 0$ for $x \in ]0,(2n)^{-1}] \cup [2n,\infty[$. Then, set $f_{\beta,n}(x) := \int_0^x g_{\beta,n}(u)\,du$ for $x \in ]0, 2n]$ and extend these functions to $]2n,\infty[$ so that $f_{\beta,n} \in C_c^1$ and $|f'_{\beta,n}(x)| \le \beta x^{\beta-1}$, for all $x > 0$ and all $n \ge 1$, and the sequence $(f_{\beta,n}, n \ge 1)$ is nondecreasing. For all $\beta > 0$, this implies that for all $x > 0$,

$$\tilde{A}(f_{\beta,n})(x) \underset{n\to\infty}{\to} x^{\alpha+\beta}\int_0^1 (y^\beta - 1)yB(dy) = -x^{\alpha+\beta}\phi(\beta), \tag{33}$$

together with

$$|\tilde{A}(f_{\beta,n})(x)| \le (2+\beta)x^{\alpha+\beta}\int_0^1 (1-y)yB(dy). \tag{34}$$

Indeed, this is obvious when $\beta \ge 1$: we just need that

$$\begin{aligned} |f_{\beta,n}(xy) - f_{\beta,n}(x)| &\le \sup_{z\in[xy,x]} |f'_{\beta,n}(z)|x(1-y) \\ &\le \beta x^\beta(1-y) \qquad \text{for } y \in ]0,1[, x > 0, \end{aligned}$$

and then use the dominated convergence theorem. The case $0 < \beta < 1$ needs more care. Using the aforementioned properties of $f_{\beta,n}$ and also the fact that $f_{\beta,n}(x) \le x^\beta$, we obtain, for $x > 0$ and $y \in ]0,1[$,

$$\begin{aligned} &x|f_{\beta,n}(xy) - f_{\beta,n}(x)| \\ &\quad \le xf_{\beta,n}(xy)(1-y) + |xyf_{\beta,n}(xy) - xf_{\beta,n}(x)| \\ &\quad \le x^{1+\beta}(1-y) + \sup_{z\in[xy,x]} |(\mathrm{id} f_{\beta,n})'(z)|x(1-y) \\ &\quad \le x^{1+\beta}(1-y) + (1+\beta)x^{1+\beta}(1-y), \end{aligned}$$

which leads to (33) and (34).

Now, take $\beta = |\alpha|$. Then use (33), (34) and the dominated convergence theorem on the right-hand side of (32) [recall that $x\mu_0(dx)$ is a probability measure], together with the monotone convergence theorem on the left-hand side of (32), to get

$$\int_0^\infty x^{|\alpha|} x\mu_0(dx) = C^{-1}\phi(|\alpha|) < \infty.$$

Then, by an obvious induction, taking successively $\beta = 2|\alpha|$, $\beta = 3|\alpha|$, etc., we get, for all $n \geq 1$, that

$$\begin{aligned}\int_0^\infty x^{n|\alpha|} x\mu_0(dx) &= C^{-1}\phi(n|\alpha|)\int_0^\infty x^{(n-1)|\alpha|} x\mu_0(dx)\\ &= C^{-n}\phi(n|\alpha|)\cdots\phi(|\alpha|).\end{aligned}$$

We recognize the moments formula (14). Hence, $x\mu_0(dx) = \mu_R^{(C^{1/\alpha})}(dx) = x\mu_\infty^{(C^{1/\alpha})}(dx)$ and for all $t \geq 0$, $\mu_t = m(t)\mu_0 = \exp(-Ct)\mu_\infty^{(C^{1/\alpha})} = \mu_{\infty,t}^{(C^{1/\alpha})}$. $\square$

## 4. Different speeds of decrease: proof of Proposition 1.4.

4.1. *Proof of Proposition* 1.4(i). Recall that the support of $\mu_0$ is supposed to be bounded with supremum 1. The goal of this section is to prove the forthcoming Corollary 4.3, which is the statement of Proposition 1.4(i) translated in terms of the process $X$ defined by (12), provided the Lévy measure $\Pi$ of the subordinator $\xi$ involved in the construction of $X$ is related to the fragmentation measure $B$ by (6) and $X(0)$ is distributed according to $x\mu_0(dx)$. We recall that the distribution of $X(t)$ conditional on $X(t) > 0$ is then $x\mu_t(dx)/m(t)$, $t \geq 0$, where $(\mu_t, t \geq 0)$ denotes the solution of the fragmentation equation starting from $\mu_0$. We start with some preliminary lemmas.

LEMMA 4.1. *Suppose that* (H) *holds and that* $\int_0 |\ln(x)| x B(dx) < \infty$. *Consider some random variable* $I$ *independent of* $X$, *with distribution that of* $\int_0^\infty \exp(\alpha\xi_r)\,dr$. *Then*:

(i) *there exists some* $t_0 > 0$ *such that*

$$\sup_{t\geq t_0, a>0} a^\alpha \mathbb{P}\left(\left(\frac{\varphi(|\alpha|t)}{|\alpha|t}\right)^{1/|\alpha|} X(t) I^{1/|\alpha|} \leq a \,\middle|\, X(t) > 0\right) < \infty;$$

(ii) *for all positive functions* $g : [0,\infty[ \to ]0,\infty[$ *converging to* 0 *at* $\infty$, *we have*, *as* $t \to \infty$,

$$g(t)^\alpha \mathbb{P}\left(\left(\frac{\varphi(|\alpha|t)}{|\alpha|t}\right)^{1/|\alpha|} X(t) I^{1/|\alpha|} \leq g(t) \,\middle|\, X(t) > 0\right) \to 1.$$

PROOF. To simplify notation, suppose that $\alpha = -1$ (the proof is identical for all $\alpha < 0$). Recall, then, the key equality in law (20), which leads to the following identities for all $a > 0$:

$$
\begin{aligned}
& a^{-1}\mathbb{P}\left(\frac{\varphi(t)}{t}X(t)I \le a | X(t) > 0\right) \\
& \quad = \frac{1}{am(t)}\mathbb{P}\left(0 < X(0)I - t \le \frac{at}{\varphi(t)}\right) \\
& \quad = \frac{m(t) - m(t + at/\varphi(t))}{am(t)} \\
& \quad = \frac{1}{a}\left(1 - \exp\left(-\ln(m(t))\left(1 - \frac{\ln(m(t(1 + a/\varphi(t))))}{\ln(m(t))}\right)\right)\right).
\end{aligned} \tag{35}
$$

We then use the regular variation of $-\ln(m)$ with index $1/(1-\beta)$ (Proposition 1.2) and Lemma 3.6 to see that for all $\varepsilon > 0$, there exists a real number $t(\varepsilon)$ such that for all $t \ge t(\varepsilon)$ and all $a > 0$,

$$
\begin{aligned}
1 - \bigl(1 + a/\varphi(t)\bigr)^{1/(1-\beta)+\varepsilon} &\le 1 - \frac{\ln(m(t(1 + a/\varphi(t))))}{\ln(m(t))} \\
&\le 1 - \bigl(1 + a/\varphi(t)\bigr)^{1/(1-\beta)-\varepsilon}.
\end{aligned} \tag{36}
$$

Now, let $0 < \varepsilon < 1 - \beta$. Since

$$
1 - (1 + x)^{1/(1-\beta)+\varepsilon} \ge -x\left(\frac{1}{1-\beta} + 2\varepsilon\right)
$$

for all $x > 0$ sufficiently small and since, further, $\varphi(t) \to \infty$ as $t \to \infty$ and $-\ln(m) \sim_\infty (1 - \beta)\varphi$, we have that for all $0 < a \le 1$ and all $t \ge t'(\varepsilon)$ [for some $t'(\varepsilon)$ depending on $\varepsilon$ but not on $0 < a \le 1$],

$$
\begin{aligned}
& -\ln(m(t))\bigl(1 - \bigl(1 + a/\varphi(t)\bigr)^{1/(1-\beta)+\varepsilon}\bigr) \\
& \quad \ge \frac{\ln(m(t))}{\varphi(t)} a\left(\frac{1}{1-\beta} + 2\varepsilon\right) \\
& \quad \ge -a(1 - \beta + \varepsilon)\left(\frac{1}{1-\beta} + 2\varepsilon\right).
\end{aligned}
$$

Together with identities (35) and inequalities (36), this implies that for all $t \ge \max(t(\varepsilon), t'(\varepsilon))$,

$$
\sup_{0<a\le 1} a^{-1}\mathbb{P}\left(\frac{\varphi(t)}{t}X(t)I \le a \,\middle|\, X(t) > 0\right) < \infty.
$$

This is enough to get (i) since for $a \ge 1$, $a^{-1}$ multiplied by a probability is bounded by 1.

The proof of (ii) relies on the same idea. Since $g(t)/\varphi(t) \to 0$ as $t \to \infty$,

$$
-\ln(m(t))\bigl(1 - \bigl(1 + g(t)/\varphi(t)\bigr)^{1/(1-\beta)+\varepsilon}\bigr) \underset{\infty}{\sim} -g(t)(1 - \beta)\left(\frac{1}{1-\beta} + \varepsilon\right)
$$

and a similar result holds by replacing $\varepsilon$ by $-\varepsilon$. Together with the inequalities (36) and the identities (35) [there replacing $a$ by $g(t)$], also using the fact that $g(t) \to 0$ as $t \to \infty$, we get (ii). □

LEMMA 4.2. *Suppose that* $\kappa := \int_0^1 |\ln(x)| x B(dx) < \infty$. *Then*:

(i) *$I$ possesses a density* $k \in \mathcal{C}^\infty(]0, \infty[)$;
(ii) $\mathbb{E}[I^{-1}] = \kappa|\alpha| < \infty$;
(iii) *if, further, the support of $B$ is not included in a set of the form* $\{a^n, n \in \mathbb{N}\}$ *for some* $a \in ]0, 1[$, *then the function*

$$x \in \mathbb{R} \to \mathbb{E}[I^{ix-1}] = \int_0^\infty y^{ix-1} k(y)\, dy$$

*is well defined and nonzero for all real numbers $x$.*

PROOF. If $\Pi$ is the Lévy measure associated with the fragmentation equation, then the assumption $\kappa < \infty$ is equivalent to $\int_0^\infty x\Pi(dx) < \infty$, which, by Propositions 3.1 and 2.1 of [11] implies (i) and (ii). Next, it was proven in the proof of Theorem 2 of [20] that $\mathbb{E}[I^{ix-1}] \neq 0$ for all $x \in \mathbb{R}$ under the additional assumption that the support of $\Pi$ is not included in a set of the form $\{rn, n \geq 0\}$ for some $r > 0$. □

COROLLARY 4.3. *Suppose that* (H) *holds*, $\kappa = \int_0^1 |\ln(x)| x B(dx) < \infty$ *and the support of $B$ is not included in a set of the form* $\{a^n, n \in \mathbb{N}\}$ *for some* $a \in ]0, 1[$. *Then, for all measurable functions* $g : [0, \infty[ \to ]0, \infty[$ *converging to* $\to 0$ *at* $\infty$,

$$g(t)^\alpha \mathbb{P}\left(\left(\frac{\varphi(|\alpha|t)}{|\alpha|t}\right)^{1/|\alpha|} \frac{X(t)}{g(t)} \leq 1 \middle| X(t) > 0\right) \to \frac{1}{|\alpha|\kappa}.$$

PROOF. For $x \geq 0$, let

$$U_t(x) := g(t)^\alpha \mathbb{P}\left(\left(\frac{\varphi(|\alpha|t)}{|\alpha|t}\right)^{1/|\alpha|} X(t) \leq x g(t) \middle| X(t) > 0\right)$$

and note that this quantity increases in $x$ when $t$ is fixed. Then consider some random variable $I$, independent of $X$, with distribution that of $\int_0^\infty \exp(\alpha\xi_r)\, dr$. Consider $b$ such that $\mathbb{P}(I \leq b) > 0$. Then

$$U_t(x)\mathbb{P}(I \leq b) \leq g(t)^\alpha \mathbb{P}\left(\left(\frac{\varphi(|\alpha|t)}{|\alpha|t}\right)^{1/|\alpha|} X(t) I^{1/|\alpha|} \leq b^{1/|\alpha|} x g(t) \middle| X(t) > 0\right),$$

which, according to Lemma 4.1(i), is bounded from above by some constant (independent of $t$ and $x$) times $bx^{|\alpha|}$ for all $x \geq 0$ and $t \geq t_0$. That is, there exists some finite constant $C$ such that for all $t$ sufficiently large and all $x \geq 0$,

$$x^\alpha U_t(x) \leq C. \tag{37}$$

Now, consider an increasing function $l:\mathbb{N}\to\mathbb{N}$. For all $x\geq 0$, the sequence $(U_{l(n)}(x), n\geq 0)$ is bounded. Hence, there exists some nondecreasing right-continuous function $U:[0,\infty[\,\to[0,\infty[$, with $U(0)=0$, and a subsequence $(U_{\tilde{l}(n)}, n\geq 0)$ of $(U_{l(n)}, n\geq 0)$ such that $U_{\tilde{l}(n)}(x)\to U(x)$ for a.e. $x>0$; see, for example, [15], Theorem 2, Section VIII.7. Hence, if we prove that the limit $U$ is given by

$$U(x)=\frac{x^{|\alpha|}}{|\alpha|\kappa}\qquad \forall x\geq 0, \tag{38}$$

for all sequences $(l(n), n\geq 0)$, $(\tilde{l}(n), n\geq 0)$ as defined above, then we will have the expected result [note that the continuity of the function involved in (38) implies that the convergence will hold for every $x\geq 0$].

To prove (38), recall that by Lemma 4.2(ii), $\int_0^\infty x^{-1}k(x)\,dx<\infty$. Hence, by dominated convergence, for all $a>0$, $\int_0^\infty U_{l(n)}(ax^{1/\alpha})k(x)\,dx\to\int_0^\infty U(ax^{1/\alpha})\times k(x)x$. By Lemma 4.1(ii), we therefore have

$$\int_0^\infty U(ax^{1/\alpha})k(x)\,dx=a^{|\alpha|}\qquad \forall a>0. \tag{39}$$

We claim that this equation characterizes $U$ under the additional assumption that the support of $B$ is not included in a set of the form $\{a^n, n\in\mathbb{N}\}$ for some $a\in\,]0,1[$. Indeed, note first that by setting $V(x):=\exp(x)U(\exp(x/\alpha))$ and $\overline{k}(x):=k(\exp(-x))$ for all $x\in\mathbb{R}$, the above equation can be rewritten

$$\int_{-\infty}^{\infty}V(x)\overline{k}(y-x)\,dx=1\qquad \forall y\in\mathbb{R}.$$

However, the function $V$ is bounded a.e. on $\mathbb{R}$, by (37). Moreover, by Lemma 4.2(ii), $\overline{k}\in L^1(\mathbb{R})$ and by Lemma 4.2(iii), the Fourier transform of $\overline{k}$ is nonzero on $\mathbb{R}$. We conclude, using the Wiener approximation theorem for $L^1(\mathbb{R})$ ([9], Theorem 4.8.4), that the above equation in $V$ has a unique bounded solution (in the sense that two solutions are equal a.e.). This determines $V$, hence $U$, almost everywhere. Since $U$ is right continuous, it is determined for all $x\geq 0$. Finally, it is not hard to check that the expression for $U$ given by (38) indeed satisfies (39). $\square$

4.2. *Proof of Proposition* 1.4(ii). We need only prove the second part of Proposition 1.4(ii), the first part being obvious since $\mu_t(]1,\infty[)=0$ for all $t\geq 0$ and $g(t)(\varphi(|\alpha|t)/|\alpha|t)^{1/\alpha}\to\infty$ as $t\to\infty$. We keep the notation from the previous section and recall that we work under the assumption (H). From the proof of Lemma 4.1, we get that

$$\ln\left(\frac{m(t(1+|\alpha|h(t)^{|\alpha|}/\varphi(|\alpha|t)))}{m(t)}\right)\underset{t\to\infty}{\sim}-h(t)^{|\alpha|}$$

for all positive functions $h$ such that $h(t)^{|\alpha|}/\varphi(t) \to 0$ as $t \to \infty$. In other words, for such functions $h$,

$$\ln\Big(m(t)^{-1}\mathbb{P}\Big(\Big(\frac{\varphi(|\alpha|t)}{|\alpha|t}\Big)^{1/|\alpha|} X(t) I^{1/|\alpha|} \geq h(t)\Big)\Big) \underset{t\to\infty}{\sim} -h(t)^{|\alpha|}.$$

Note that for all $t \geq 0$ and all $c > 0$, since $X$ is independent of $I$,

$$\begin{aligned}
&\ln\Big(m(t)^{-1}\mathbb{P}\Big(\Big(\frac{\varphi(|\alpha|t)}{|\alpha|t}\Big)^{1/|\alpha|} X(t) \geq \big(c\phi\big(h(t)^{|\alpha|}\big)\big)^{1/|\alpha|}\Big)\Big)\\
&\quad + \ln\big(\mathbb{P}\big(I^{1/|\alpha|} \geq h(t)/\big(c\phi\big(h(t)^{|\alpha|}\big)\big)^{1/|\alpha|}\big)\big)\\
&\quad \leq \ln\Big(m(t)^{-1}\mathbb{P}\Big(\Big(\frac{\varphi(|\alpha|t)}{|\alpha|t}\Big)^{1/|\alpha|} X(t) I^{1/|\alpha|} \geq h(t)\Big)\Big).
\end{aligned}$$

Suppose, moreover, that $h(t) \to \infty$ as $t \to \infty$ and that $\beta < 1$, which implies that $h(t)^{|\alpha|}/\phi(h(t)^{|\alpha|}) \to \infty$. By Lemma 3.4, we have, for all real numbers $c > 0$,

$$\begin{aligned}
-\ln\big(\mathbb{P}\big(I \geq h(t)^{|\alpha|}/c\phi\big(h(t)^{|\alpha|}\big)\big)\big) &\underset{\infty}{\sim} \frac{1-\beta}{|\alpha|}\varphi\Big(\frac{|\alpha|h(t)^{|\alpha|}}{c\phi(h(t)^{|\alpha|})}\Big)\\
&\underset{\infty}{\sim} \frac{(1-\beta)|\alpha|^{\beta/(1-\beta)}}{c^{1/(1-\beta)}} h(t)^{|\alpha|},
\end{aligned}$$

using both the regular variation of $\varphi$ and the fact that $\varphi$ is the inverse of $t \to t/\phi(t)$ near $\infty$. Now, let $\varepsilon \in ]0, 1[$ and $c$ be such that $c^{1/(1-\beta)} > (1-\beta)|\alpha|^{\beta/(1-\beta)}$. We have proven that

$$\begin{aligned}
&\ln\Big(m(t)^{-1}\mathbb{P}\Big(\Big(\frac{\varphi(|\alpha|t)}{|\alpha|t}\Big)^{1/|\alpha|} X(t) \geq \big(c\phi\big(h(t)^{|\alpha|}\big)\big)^{1/|\alpha|}\Big)\Big)\\
&\quad \leq -(1-\varepsilon)\Big(1 - \frac{(1-\beta)|\alpha|^{\beta/(1-\beta)}}{c^{1/(1-\beta)}}\Big) h(t)^{|\alpha|}
\end{aligned}$$

for $t$ large enough. Next, let $g_{h,c}(t) = (c\phi(h(t)^{|\alpha|}))^{1/|\alpha|}$, $t \geq 0$, and suppose that $\beta > 0$ (hence the existence of the inverse of $\phi$ near $\infty$). We have, for $t$ large enough,

$$\begin{aligned}
&\ln\Big(m(t)^{-1}\mathbb{P}\Big(\Big(\frac{\varphi(|\alpha|t)}{|\alpha|t}\Big)^{1/|\alpha|} X(t) \geq g_{h,c}(t)\Big)\Big)\\
&\quad \leq -(1-\varepsilon)\Big(1 - \frac{(1-\beta)|\alpha|^{\beta/(1-\beta)}}{c^{1/(1-\beta)}}\Big)\phi^{-1}\big(g_{h,c}(t)^{|\alpha|}/c\big)\\
&\quad \underset{\infty}{\sim} -(1-\varepsilon)\Big(1 - \frac{(1-\beta)|\alpha|^{\beta/(1-\beta)}}{c^{1/(1-\beta)}}\Big) c^{-1/\beta}\phi^{-1}\big(g_{h,c}(t)^{|\alpha|}\big).
\end{aligned}$$

It is not hard to check that the maximum of

$$\Big\{\Big(1 - \frac{(1-\beta)|\alpha|^{\beta/(1-\beta)}}{c^{1/(1-\beta)}}\Big) c^{-1/\beta}, c > (1-\beta)^{1-\beta}|\alpha|^{\beta}\Big\}$$

is equal to $\beta/|\alpha|$ and is reached at $c=|\alpha|^{\beta}$. Finally, if $g_h = g_{h,|\alpha|^{\beta}}$, letting $\varepsilon \to 0$, we have proven that

$$\limsup_{t\to\infty} \frac{1}{\phi^{-1}(g_h(t)^{|\alpha|})} \ln\biggl(m(t)^{-1}\mathbb{P}\biggl(\biggl(\frac{\varphi(|\alpha|t)}{|\alpha|t}\biggr)^{1/|\alpha|} X(t) \geq g_h(t)\biggr)\biggr) \leq -\frac{\beta}{|\alpha|}. \tag{40}$$

To conclude, to get the second part of the statement of Proposition 1.4(ii), suppose that $0<\beta<1$ and consider some positive function $g$ that converges to $\infty$ at $\infty$, such that $g(t)^{|\alpha|}t/\varphi(t) \to 0$. Set $h(t) = (\phi^{-1}(g(t)^{|\alpha|}/|\alpha|^{\beta})^{1/|\alpha|}, t \geq 0$. Then, $h(t)$ converges to $\infty$ as $t\to\infty$ and it is easily seen that $h(t)^{|\alpha|}/\varphi(t) \to 0$ as $t\to\infty$. Since $g = g_h$ with the notation above, the result follows from (40).

**5. Some properties of the limit measure $\mu_\infty$.** Recall that the distribution $x\mu_\infty(dx)$ on $]0,\infty[$ is that of $R^{1/|\alpha|}$, where $R$ denotes a random variable with entire positive moments

$$\mathbb{E}[R^n] = \phi(|\alpha|)\cdots\phi(n|\alpha|), \qquad n \geq 1, \tag{41}$$

that characterize its distribution. Using this particular moments' shape, we get the following description of the measure $\mu_\infty$ near 0 and $\infty$. Some of these properties are then used at the end of this section to prove Proposition 1.5.

PROPOSITION 5.1 (Behavior at $\infty$).

(i) *Suppose that* (H) *holds for some* $\beta \in ]0,1[$. *Then*

$$-\ln\biggl(\int_t^\infty x\mu_\infty(dx)\biggr) = -\ln(\mathbb{P}(R > t^{|\alpha|})) \underset{\infty}{\sim} \frac{\beta}{|\alpha|}\phi^{-1}(t^{|\alpha|}),$$

*where* $\phi^{-1}$ *denotes the inverse of* $\phi$ (*and is therefore a function regularly varying at* $\infty$ *with index* $1/\beta$).

(ii) *Suppose that* $\phi(\infty) := \int_0^1 xB(dx) < \infty$. *Then* $\mu_\infty$ *has a bounded support with supremum* $\phi(\infty)^{1/|\alpha|}$ *and*

$$\mu_\infty(\{\phi(\infty)^{1/|\alpha|}\}) > 0 \quad \Leftrightarrow \quad \int^1 \frac{B(dx)}{1-x} < \infty.$$

PROPOSITION 5.2 (Behavior at 0). *Suppose that* $\int_0^{\exp(-u)} xB(dx)$ *varies regularly at* $\infty$ *with index* $-\gamma$, $\gamma \in [0,1]$. *Then, as* $s \to 0$,

$$\int_s^\infty x^{1+\alpha}\mu_\infty(dx) = \mathbb{E}[\mathbf{1}_{\{R>s^{|\alpha|}\}}R^{-1}] \sim \frac{1}{(|\alpha|^{\gamma}\Gamma(1+\gamma)\phi(-1/\ln(s^{|\alpha|})))}$$

*and*

$$\int_0^s x\mu_\infty(dx) = \mathbb{P}(R < s^{|\alpha|}) = \circ\biggl(\frac{s^{|\alpha|}}{\phi(-1/\ln(s^{|\alpha|}))}\biggr).$$

PROOF. This is a direct consequence of Corollary 1 of Caballero and Rivero [10], which gives these results in terms of the random variable $R$. □

PROOF OF PROPOSITION 5.1. (i) Our proof strongly relies on the proof of Proposition 2 of Rivero [28]. Rivero shows there that if a positive random variable $Y$ has entire moments satisfying

$$\mathbb{E}[Y^n] = \prod_{i=1}^{n} \psi(i)$$

for some function $\psi$ regularly varying at $\infty$ with index $\gamma \in ]0, 1[$, then

$$-\ln(\mathbb{P}(Y > t)) \underset{\infty}{\sim} \gamma \psi^{\leftarrow}(t),$$

where $\psi^{\leftarrow}$ is the right inverse of $\psi$. We apply this result to the random variable $R$, by taking $\psi = \phi(|\alpha|\cdot)$ and $\gamma = \beta$.

(ii) Using (41) and the fact that $\phi$ is increasing, we get, for all $n \geq 0$,

$$\mathbb{E}\left[\left(\frac{R}{\phi(\infty)}\right)^n\right] \leq 1. \tag{42}$$

Besides, writing

$$\mathbb{E}\left[\left(\frac{R}{\phi(\infty)}\right)^n\right] = \mathbb{E}\left[\left(\frac{R}{\phi(\infty)}\right)^n \mathbf{1}_{\{R>\phi(\infty)\}}\right] + \mathbb{E}\left[\left(\frac{R}{\phi(\infty)}\right)^n \mathbf{1}_{\{R<\phi(\infty)\}}\right] + \mathbb{P}(R = \phi(\infty))$$

and using the monotone and dominated convergence theorems, we see that

$$\mathbb{E}\left[\left(\frac{R}{\phi(\infty)}\right)^n\right] \underset{n\to\infty}{\to} \begin{cases} \infty, & \text{if } \mathbb{P}(R > \phi(\infty)) > 0, \\ \mathbb{P}(R = \phi(\infty)), & \text{otherwise.} \end{cases}$$

In particular, from (42), we see that $\mathbb{P}(R > \phi(\infty)) = 0$. Similarly, it is easy to show, using (41), that for all $0 < \varepsilon < \phi(\infty)$,

$$\mathbb{E}\left[\left(\frac{R}{\phi(\infty) - \varepsilon}\right)^n\right] \to \infty,$$

which implies that $\mathbb{P}(R > \phi(\infty) - \varepsilon) > 0$. Finally, to get the remaining part of the statement, note that

$$\ln\left(\frac{\phi(n|\alpha|)}{\phi(\infty)}\right) \underset{n\to\infty}{\sim} \frac{\phi(n|\alpha|)}{\phi(\infty)} - 1 = \frac{-1}{\phi(\infty)} \int_0^1 x^{n|\alpha|+1} B(dx).$$

Therefore,

$$\ln\left(\mathbb{E}\left[\left(\frac{R}{\phi(\infty)}\right)^n\right]\right) = \sum_{i=1}^{n} \ln\left(\frac{\phi(i|\alpha|)}{\phi(\infty)}\right)$$

converges to $-\infty$ as $n \to \infty$ if and only if

$$\int_0^1 \sum_{i=1}^{\infty} x^{i|\alpha|+1} B(dx) = \int_0^1 \frac{x^{1+|\alpha|}}{1-x^{|\alpha|}} B(dx) = \infty.$$

Since $\int_0 x B(dx) < \infty$,

$$\mathbb{E}\biggl[\biggl(\frac{R}{\phi(\infty)}\biggr)^n\biggr] \underset{n\to\infty}{\to} 0 \quad \text{iff} \quad \int^1 \frac{1}{1-x} B(dx) = \infty,$$

which ends the proof. □

PROOF OF PROPOSITION 1.5. From the construction (7) of $\mu_t$, we see that

$$\mu_t(\{1\}) = \mu_0(\{1\})\mathbb{P}\bigl(\xi(\rho(t)) = 0\bigr) = \mu_0(\{1\})\mathbb{P}\bigl(\xi(t) = 0\bigr)$$

and from the Poisson point process construction of a pure jump subordinator with Lévy measure $\Pi$, we have that $\mathbb{P}(\xi(t) = 0) = \exp(-t\Pi(]0,\infty[)) = \exp(-t\phi(\infty))$. Next, we get, from the factorization (15), that

$$\exp(-t\phi(\infty)) = \mathbb{P}\bigl(RI > t\phi(\infty)\bigr) \geq \mathbb{P}(I > t)\mathbb{P}\bigl(R \geq \phi(\infty)\bigr).$$

On the one hand, from the proof of Proposition 5.1, we see that when $\phi(\infty) < \infty$, $\mathbb{P}(R \geq \phi(\infty)) = \mathbb{P}(R = \phi(\infty))$ and that this quantity is nonzero iff $\int^1 (1-x)^{-1} B(dx) < \infty$. On the other hand, under (H), we get from the regular variation of $-\ln(\mathbb{P}(I > t))$ that $\mathbb{P}(I > x^\alpha t)/\mathbb{P}(I > t) \to 0$, for all $0 < x < 1$, as $t \to \infty$ and then, from the dominated convergence theorem that

$$\frac{m(t)}{\mathbb{P}(I > t)} = \int_0^1 \frac{\mathbb{P}(I > x^\alpha t)}{\mathbb{P}(I > t)} x\mu_0(dx) \to \mu_0(\{1\}) \qquad \text{as } t \to \infty.$$

In other words, we have proven that under the hypothesis (H), when $\mu_0(\{1\}) > 0$ and $\phi(\infty) < \infty$,

$$\liminf_{t\to\infty} \frac{\mu_t(\{1\})}{m(t)} \geq \mathbb{P}\bigl(R = \phi(\infty)\bigr) = \phi(\infty)^{1/|\alpha|}\mu_\infty\bigl(\{\phi(\infty)^{1/|\alpha|}\}\bigr).$$

Next, suppose that (H) holds, $\int_0 |\ln(x)| x B(dx) < \infty$ and $\phi(\infty) < \infty$. According to Theorem 1.3, for all $\varepsilon \in ]0,1[$ such that $(1-\varepsilon)\phi(\infty)^{1/|\alpha|}$ is not an atom of $\mu_\infty$,

$$\frac{\mu_t(\{1\})}{m(t)} \leq \frac{\int_{1-\varepsilon}^1 x\mu_t(dx)}{m(t)} \underset{t\to\infty}{\to} \int_{(1-\varepsilon)\phi(\infty)^{1/|\alpha|}}^{\phi(\infty)^{1/|\alpha|}} x\mu_\infty(dx).$$

Letting $\varepsilon \to 0$, we get

$$\limsup_{t\to\infty} \frac{\mu_t(\{1\})}{m(t)} \leq \phi(\infty)^{1/|\alpha|}\mu_\infty\bigl(\{\phi(\infty)^{1/|\alpha|}\}\bigr). \qquad \square$$

**6. Examples.** Below is a list of standard examples where the main quantities involved in our results can be computed explicitly. More precisely, for each of these examples, we specify the distributions of $I$ [defined in (13)] and $R$ [defined in (14)], which leads to explicit expressions of the limit measure $\mu_\infty$ [since $R^{1/|\alpha|} \overset{d}{\sim} x\mu(dx)$] and of the mass

$$m_1(t) = \mathbb{P}(I > t), \qquad t \geq 0,$$

which is the mass of the solution of the fragmentation equation starting from $\mu_0 = \delta_1$. We also specify the behavior as $t \to \infty$ of the quantity $\varphi(|\alpha|t)/|\alpha|t$ involved in the statement of Theorem 1.3. For all of these examples, we give the main tools to get the distributions of $I$ and $R$, but we leave the calculation details to the reader. We recall that $\beta$ denotes the index of regular variation of the hypothesis (H) and that when $(\mu_t, t \geq 0)$ is a solution of the equation with parameters $(\alpha, B)$, $(\mu_{ct}, t \geq 0)$ is a solution of the equation with parameters $(\alpha, cB)$. For this reason, in the examples below, given a measure $B$, we choose its "representative" among the measures $cB$, $c > 0$, which is the most convenient for the statement of the results.

The first four examples concern absolutely continuous measures $B(du) = b(u)\,du$, where $b$ is a function defined on $]0, 1[$. The Lévy measure is therefore also absolutely continuous and we denote by $\pi$ its density. It turns out that the limit distribution $\mu_\infty$ is also absolutely continuous. We denote by $u_\infty$ its density.

EXAMPLE 1 [$b(u) = bu^{b-2}$, $b > 0$; $\alpha < 0$].

- $\beta = 0$;
- $\varphi(t) \sim t$ as $t \to \infty$;
- $I \overset{d}{\sim} \Gamma(b/|\alpha| + 1, 1)$;
- $m_1(t) = \frac{1}{\Gamma(b/|\alpha|+1)} \int_t^\infty x^{b/|\alpha|} \exp(-x)\,dx$, $t \geq 0$;
- $R \overset{d}{\sim} \beta(1, b/|\alpha|)$;
- $u_\infty(x) = bx^{|\alpha|-2}(1 - x^{|\alpha|})^{b/|\alpha|-1}$, $0 < x < 1$.

The notation $\Gamma(x, y)$ [resp., $\beta(x, y)$] refers to the classical Gamma distribution with parameters $x, y > 0$ (resp., Beta distribution). In these examples, the density of the Lévy measure associated with $B$ is $\pi(x) = b\exp(-bx)$, $x > 0$, hence the Lévy measure associated with the subordinator $|\alpha|\xi$ (where $\xi$ has Lévy measure $\Pi$) has a density given by $b\exp(-bx/|\alpha|)/|\alpha|$, $x > 0$. According to Example B, page 5 of [11], the density of $I$ is then proportional to $x^{b/|\alpha|}\exp(-x)$, $x > 0$. Finally, we refer to formula (4), Section 3 of [8], to get the distribution of $R$.

We point out that the solutions of the fragmentation equation with this measure $B$ are studied in [24]. In particular, when $\alpha = -b/2$ and $\mu_0 = \delta_1$, the solutions $(\mu_t, t \geq 0)$ have the explicit expression

$$\mu_t(dx) = \exp(-t)\big(\delta_1(dx) + bx^{b-2}\big(t - \tfrac{1}{2}t^2(1 - x^{-b/2})\big)\mathbf{1}_{\{0<x<1\}}\,dx\big),$$

which gives

$$m_1(t) = \exp(-t)\left(1 + t + \frac{t^2}{2}\right)$$

and, for all bounded test functions $f : ]0, \infty[ \to \mathbb{R}$,

$$\frac{1}{m_1(t)} \int_0^1 f(x) x \mu_t(dx) \underset{t\to\infty}{\to} \int_0^1 f(x) b x^{b-1} (x^{-b/2} - 1)\, dx,$$

which is consistent with the above expressions for $m_1$ and $u_\infty$.

EXAMPLE 2 $[b(u) = |\alpha|\Gamma(1-\gamma)^{-1} u^{|\alpha|/\gamma - 2}(1 - u^{|\alpha|/\gamma})^{-\gamma-1};\ 0 < \gamma < 1;\ \alpha < 0]$.

- $\beta = \gamma$;
- $\varphi(t) \sim (\frac{\gamma}{|\alpha|})^{\gamma/(1-\gamma)} t^{1/(1-\gamma)}$ as $t \to \infty$;
- $I \overset{d}{\sim} \tau_\gamma^{-\gamma}$;
- $m_1(t) = \int_t^\infty g_\gamma(x)\, dx$, $t > 0$;
- $R \overset{d}{\sim} \mathbf{e}(1)^\gamma$;
- $\mu_\infty(x) = |\alpha|\gamma^{-1} x^{|\alpha|/\gamma - 2} \exp(-x^{|\alpha|/\gamma})$, $x > 0$.

Here, $\mathbf{e}(1)$ denotes a random variable with exponential distribution with parameter 1 and $\tau_\gamma$ denotes a $\gamma$-stable random variable, that is, with Laplace transform $t \in [0, \infty[ \to \exp(-t^\gamma)$. Hence, $\tau_\gamma^{-\gamma}$ has the so-called Mittag–Leffler distribution. We recall that it possesses a density given by

$$g_\gamma(x) = \frac{1}{\pi\gamma} \sum_{i=0}^{\infty} \frac{(-x)^{i-1}}{i!} \Gamma(\gamma i + 1) \sin(\pi\gamma i), \qquad x > 0,$$

and its entire positive moments are equal to $n!/\Gamma(\gamma n + 1)$, $\forall n \geq 1$ (see, e.g., [27], Section 0.3). The Lévy measure associated with $B$ has a density for $x > 0$ given by

$$\pi(x) = \frac{|\alpha| \exp(-|\alpha| x/\gamma)}{\Gamma(1-\gamma)(1 - \exp(-|\alpha| x/\gamma))^{\gamma+1}}.$$

Using formula (5) and the following discussion in [8], we get that $I \overset{d}{\sim} \tau_\gamma^{-\gamma}$ and $R \overset{d}{\sim} \mathbf{e}(1)^\gamma$.

EXAMPLE 3 $[b(u) = |\alpha|\gamma^2((1-\gamma)\Gamma(2-\gamma))^{-1} u^{\gamma|\alpha|/(1-\gamma) - 2}(1 - u^{|\alpha|/(1-\gamma)})^{-\gamma-1};\ 0 < \gamma < 1;\ \alpha < 0]$.

- $\beta = \gamma$;
- $\varphi(t) \sim (1-\gamma)^{-1} |\alpha|^{\gamma/(\gamma-1)} t^{1/(1-\gamma)}$, as $t \to \infty$;
- $I \overset{d}{\sim} \mathbf{e}(1)^{1-\gamma}$;

- $m_1(t) = \exp(-t^{1/(1-\gamma)})$, $t \geq 0$;
- $R \overset{d}{\sim} \tau_{1-\gamma}^{\gamma-1}$;
- $\mu_\infty(x) = |\alpha| x^{|\alpha|-2} g_{1-\gamma}(x^{|\alpha|})$, $x > 0$,

where $g_{1-\gamma}$ is the Mittag–Leffler density given in the previous example. Note the duality with this previous example. In the present example,

$$\pi(x) = \frac{|\alpha|\gamma^2 \exp(|\alpha|x/(1-\gamma))}{(1-\gamma)\Gamma(2-\gamma)(\exp(|\alpha|x/(1-\gamma))-1)^{1+\gamma}}, \qquad x > 0,$$

and we again refer to formula (5) and the discussion which follows in [8] to get the distributions of $I$ and $R$.

EXAMPLE 4 [$b(u) = |\alpha|\Gamma(2+\alpha)^{-1} u^{|\alpha|-2}(1-u)^{\alpha-1}$; $-1 < \alpha < 0$].

- $\beta = |\alpha|$;
- $\varphi(t) \sim (\frac{t}{1+\alpha})^{1/(1-|\alpha|)}$ as $t \to \infty$;
- $I/(1+\alpha)$ is a size-biased version of the Mittag–Leffler distribution with parameter $|\alpha|$, that is, for all test functions $f$,

$$\mathbb{E}[f(I)] = \frac{\mathbb{E}[f((1+\alpha)\tau_{|\alpha|}^{-|\alpha|})\tau_{|\alpha|}^{-|\alpha|}]}{\mathbb{E}[\tau_{|\alpha|}^{-|\alpha|}]};$$

- $m_1(t) = \Gamma(|\alpha|+1) \int_{t/(1+\alpha)}^\infty x g_{|\alpha|}(x)\, dx$, $t > 0$;
- $((1+\alpha)R)^{1/|\alpha|} \overset{d}{\sim} \Gamma(|\alpha|, 1)$;
- $\mu_\infty(x) = \frac{(1+\alpha)}{\Gamma(|\alpha|)} x^{|\alpha|-2} \exp(-(1+\alpha)^{1/|\alpha|} x)$, $x > 0$.

Indeed, here,

$$\pi(x) = \frac{|\alpha| \exp(x)}{\Gamma(2+\alpha)(\exp(x)-1)^{1-\alpha}}, \qquad x > 0.$$

Following the end of the proof of Lemma 4 of Miermont [26], we get that $I$ has its moment of order $k$ equal to

$$\frac{k!(1+\alpha)^k \Gamma(|\alpha|)}{\Gamma((k+1)|\alpha|)}$$

for all $k \in \mathbb{N}$. Hence,

$$\begin{aligned}
\mathbb{E}[R^k] &= \frac{k!}{\mathbb{E}[I^k]} \\
&= \frac{\Gamma((k+1)|\alpha|)}{(1+\alpha)^k \Gamma(|\alpha|)} \\
&= \frac{1}{(1+\alpha)^k \Gamma(|\alpha|)} \int_0^\infty x^{(k+1)|\alpha|-1} \exp(-x)\, dx.
\end{aligned}$$

REMARK. Note that Examples 2, 3 and 4 give, for all $0 < \gamma < 1$:

- if $b(u) = u^{-1}(1-u)^{-\gamma-1}$ and $\alpha = -\gamma$, then

$$x\mu_\infty(dx) \overset{d}{\sim} c_1(\gamma)\mathbf{e}(1);$$

- if $b(u) = u^{\gamma-2}(1-u)^{-\gamma-1}$ and $\alpha = \gamma - 1$, then

$$x\mu_\infty(dx) \overset{d}{\sim} c_2(\gamma)\tau_{1-\gamma}^{-1};$$

- if $b(u) = u^{\gamma-2}(1-u)^{-\gamma-1}$ and $\alpha = -\gamma$, then

$$x\mu_\infty(dx) \overset{d}{\sim} c_3(\gamma)\Gamma(\gamma, 1),$$

where $c_1(\gamma), c_2(\gamma)$ and $c_3(\gamma)$ are real numbers that depend on $\gamma$. Hence, both $\alpha$ and the behavior of $b$ near 0 have a significant influence on the shape of the limit measure $\mu_\infty$.

Finally, we turn to the case where $B$ is a Dirac measure.

EXAMPLE 5 ($B = a^{-1}\delta_a$ for some $a \in ]0, 1[$; $\alpha < 0$).

- $\beta = 0$;
- $\varphi(t) \sim t$ as $t \to \infty$;
- $I$ has a density $k$ on $]0, \infty[$ given by

$$k(x) = \sum_{i \geq 0} \exp\bigl(\alpha \ln(a)i - x \exp(\alpha \ln(a)i)\bigr) \prod_{p \neq i} \bigl(1 - \exp\bigl(\alpha \ln(a)(i-p)\bigr)\bigr)^{-1};$$

- $m_1(t) = \int_t^\infty k(x)\, dx$, $t \geq 0$.

In this case, $\Pi = \delta_{-\ln(a)}$, that is, the associated subordinator is a Poisson process. We then refer to [11], Proposition 6.5(ii), for the expression of the density $k$. Note that $\phi(t) = (1 - a^t)$ for all $t \geq 0$, hence $\mathbb{E}[R^n] = \prod_{i=1}^n (1 - (|\alpha| a)^i)$ for all $n \geq 1$.

## APPENDIX: EXISTENCE AND UNIQUENESS OF SOLUTIONS

This appendix is devoted to the proof of Theorem 1.3 on the existence and uniqueness of solutions of the fragmentation equation (2). Therefore, in this section, $\alpha \in \mathbb{R}$. The proof follows the main lines of that of Theorem 1 in [19], which gives existence and uniqueness of solutions of a slightly restricted form of the fragmentation equation (2) and which concentrates on solutions starting from $\mu_0 = \delta_1$. We note that it was implicit in the statement of this theorem that a solution should satisfy assumptions (4) and (5).

Let $\xi$ denote a subordinator with Lévy measure $\Pi$ and zero drift, such that $\xi_0 = 0$. We recall that its semigroup possesses the Feller property and that the domain of its infinitesimal generator contains at least all functions $f$ that are continuously differentiable on $\mathbb{R}$ and such that $f$ and $f'$ tend to 0 at infinity; see, for example, Chapter 1 of [3]. As a consequence, the domain of the infinitesimal generator of $\exp(-\xi)$ contains continuously differentiable functions $f$ on $[0,\infty[$ with compact support and null near 0.

One can easily check that when $f:[0,\infty[\to\mathbb{R}$ is bounded and continuous, the function

$$x\to\mathbb{E}\big[f\big(x\exp\big(-\xi_{\rho(x^\alpha t)}\big)\big)\big]$$

is also bounded and continuous on $[0,\infty[$. This mainly relies on the càdlàg and quasi-left-continuity ([3], Proposition 7, Chapter 1) of subordinators.

Now, for every $0<a<b$, let $\mathcal{C}_{a,b}$ be the set of continuous functions $f:[0,b]\to\mathbb{R}$ that are null on $[0,a]$, and let $\mathcal{C}^1_{a,b}$ be the set of continuously differentiable functions $f:[0,b]\to\mathbb{R}$ that are null on $[0,a]$. It is clear from the remark above that for all $0<a<b$, the linear operators $T_t$ and $\tilde{T}_t$, $t\ge 0$, defined by

$$T_t(f)(x)=\mathbb{E}[f(x\exp(-\xi_t))]$$

and

$$\tilde{T}_t(f)(x)=\mathbb{E}\big[f\big(x\exp\big(-\xi_{\rho(x^\alpha t)}\big)\big)\big]$$

send $\mathcal{C}_{a,b}$ into $\mathcal{C}_{a,b}$. Following the proof of Theorem 1 of [19] (see also [22]), we see that both families of operators define strongly continuous contraction semigroups on $\mathcal{C}_{a,b}$ and that the domains of their infinitesimal generators are identical and contain $\mathcal{C}^1_{a,b}$. These generators are, respectively, given, for $f\in\mathcal{C}^1_{a,b}$ and $x\in[0,b]$, by

$$A(f)(x)=\int_0^\infty\big(f(x\exp(-y))-f(x)\big)\Pi(dy)$$

and

$$\tilde{A}(f)(x)=x^\alpha A(f)(x),\qquad x>0, \tilde{A}(f)(0)=0.$$

Note that when $B$ is a measure on $]0,1[$ defined from $\Pi$ by (6), we have

$$\tilde{A}(f)(x)=x^\alpha\int_0^1\big(f(xy)-f(x)\big)yB(dy).$$

**Existence of solutions to (2).** With the above remarks and Kolmogorov's backward equation (see Proposition 15, page 9 of [14]), we have that

$$\tilde{T}_t(f)(x)=f(x)+\int_0^t\tilde{T}_s(\tilde{A}(f))(x)\,ds,\tag{43}$$

$\forall x\in[0,b]$, $\forall f\in\mathcal{C}^1_{a,b}$, $\forall 0<a<b$, $\forall b>0$. In other words, if we let $f:[0,\infty[\to\mathbb{R}$ be null near 0 and continuously differentiable, then, considering its restriction to $[0,b]$ and $x\le b$, we have that $f$ and $x$ satisfy (43).

Now, consider $\nu_0$, a probability measure on $]0,\infty[$, and define for all $t > 0$ a measure $\nu_t$ on $]0,\infty[$ by

$$\langle \nu_t, g\rangle := \langle \nu_0, \tilde{T}_t(g)\rangle$$

for all bounded, measurable functions $g$ on $[0,\infty[$ such that $g(0) = 0$. Note that for all $t \geq 0$, $\nu_t(]0,\infty[) \leq 1$ and $\nu_t(x \geq M) = 0$ provided that $\nu_0(x \geq M) = 0$ for some $M > 0$. Then let $f$ be some continuously differentiable function on $[0,\infty[$, null near 0 and with compact support. It is clear that $\tilde{A}(f)$ is null near 0 and it is easy to see, using Fubini's theorem, that there exist some constants $b, c > 0$ such that $|\tilde{A}(f)|(x) \leq cx^{\alpha}\overline{\Pi}(\ln(x/b))$ for large enough $x$ [here, $\overline{\Pi}(y) = \int_y^{\infty} \Pi(dx)$]. In particular, $\tilde{A}(f)$ is bounded on $[0,\infty[$ when $x^{\alpha}\overline{\Pi}(\ln(x))$ is bounded near $\infty$ (hence when $\alpha \leq 0$). It is then clear that in such a case, we can apply Fubini's theorem when integrating (43) with respect to $\nu_0$ to get

$$\langle \nu_t, f\rangle = \langle \nu_0, f\rangle + \int_0^t \langle \nu_s, \tilde{A}(f)\rangle\, ds.$$

This holds for all continuously differentiable functions $f$ on $[0,\infty[$, null near 0 and with compact support. Therefore, defining the measures $\mu_t$ on $]0,\infty[$ by $\langle \mu_t, g\rangle := \langle \nu_t, \tilde{g}\rangle$, where $g$ denotes any test function on $]0,\infty[$ and $\tilde{g}(x) = g(x)/x$, $x > 0$, we have proven that $(\mu_t, t \geq 0)$ is a solution of the fragmentation equation, as defined in the Introduction. To summarise: provided that the function $x \to x^{\alpha}\overline{\Pi}(\ln(x))$ is bounded near $\infty$, for all measures $\mu_0$ on $]0,\infty[$ such that $\int_0^{\infty} x\mu_0(dx) = 1$, there exists a solution, constructed via subordinators, of the fragmentation equation.

When $\alpha > 0$, the function $x \to x^{\alpha}\overline{\Pi}(\ln(x))$ may not be bounded near $\infty$. Another way to tackle the problem in this case is to use the definition of $\rho$ to get that

$$\begin{aligned}\int_0^{\infty}\int_0^t \tilde{T}_s(|\tilde{A}(f)|)(x)\, ds\, \nu_0(dx)\\ = \int_0^{\infty} \mathbb{E}\left[\int_0^{\rho(x^{\alpha}t)} |A(f)|(x\exp(-\xi_u))\, du\right]\nu_0(dx),\end{aligned}$$

the function $f$ still being supposed continuously differentiable on $[0,\infty[$, null near 0 and with compact support. For such $f$, the function $A(f)$ is bounded on $[0,\infty[$. Hence, the double integral involved in the identity above is bounded by a constant times $\int_0^{\infty} \mathbb{E}[\rho(x^{\alpha}t)]\nu_0(dx)$, which is finite provided that $\int^{\infty} \ln(x)\nu_0(dx) < \infty$: indeed, according to Proposition 2 of [8], for all $x \geq 0$, $\mathbb{E}[\rho(x)] = \int_0^x \mathbb{E}[\exp(-s \times R)]\, ds$, where $R$ is a random variable with distribution $\mu_R$ defined by (14). Now, let $I$ be a random variable defined by (13), independent of $R$, and consider a real number $a$ such that $\mathbb{P}(I \le a) > 0$. Using the factorization property (15), we get

$$\mathbb{E}[\rho(x)]\mathbb{P}(I \le a) = \int_0^x \mathbb{E}\big[\exp(-sR)\mathbf{1}_{\{I\le a\}}\big]\,ds$$
$$\le \int_0^x \mathbb{E}[\exp(-sa^{-1}\mathbf{e}(1))]\,ds = a\ln(1+a^{-1}x).$$

It this then possible to apply Fubini's theorem when integrating (43) with respect to $\nu_0$ and we conclude, as above, that there exists a solution of (2).

**Uniqueness of solutions of (2).** Let $\nu_0$ be a probability measure with support included in $]0,b]$ for some $b>0$ and suppose that $(\nu_t, t\ge 0)$ is a family of measures with support included in $]0,b]$ such that

$$\langle \nu_t, f\rangle = \langle \nu_0, f\rangle + \int_0^t \langle \nu_s, \tilde{A}(f)\rangle\,ds, \qquad \forall f \in \bigcup_{0<a<b} \mathcal{C}^1_{a,b}.$$

Suppose, moreover, that $\nu_t(]0,\infty[) \le 1$, $\forall t \ge 0$. Our goal is to prove that $(\nu_t, t\ge 0)$ is uniquely determined. Using the fact that the total weight of $\nu_t$ is less than or equal to 1, we get that $\sup_{t\ge 0}\langle \nu_t, |f|\rangle < \infty$ and $\sup_{t\ge 0}\langle \nu_t, |A(f)|\rangle < \infty$ for each $f \in \bigcup_{0<a<b}\mathcal{C}^1_{a,b}$. It is then possible to follow the proof of Proposition 18, Section 4.9 of [14] to deduce that uniqueness holds, provided that, for all $\lambda > 0$, $(\lambda\,\mathrm{id} - \tilde{A}(f))(\mathcal{C}^1_{a,b})$ is dense (for the uniform norm) in $\mathcal{C}_{a,b}$ for all $0<a<b$. Following the proof of Theorem 1 in [19], we see that $\mathcal{C}^1_{a,b}$ is a core for the strongly contraction semigroup $\tilde{T}_t : \mathcal{C}_{a,b} \to \mathcal{C}_{a,b}$, $t \ge 0$. Hence the result.

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