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Fragmentation Processes with an Initial Mass Converging to Infinity

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Abstract We consider a family of fragmentation processes where the rate at which a particle splits is proportional to a function of its mass. Let $F_1^{(m)}(t), F_2^{(m)}(t), \ldots$ denote the decreasing rearrangement of the masses present at time t in a such process, starting from an initial mass m. Let then $m \to \infty$. Under an assumption of regular variation type on the dynamics of the fragmentation, we prove that the sequence $(F_2^{(m)}, F_3^{(m)}, \ldots)$ converges in distribution, with respect to the Skorohod topology, to a fragmentation with immigration process. This holds jointly with the convergence of $m - F_1^{(m)}$ to a stable subordinator. A continuum random tree counterpart of this result is also given: the continuum random tree describing the genealogy of a self-similar fragmentation satisfying the required assumption and starting from a mass converging to ∞ will converge to a tree with a spine coding a fragmentation with immigration.

Keywords Fragmentation · Immigration · Weak convergence · Regular variation · Continuum random tree

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1 Introduction and Main Results

We consider Markovian models for the evolution of systems of particles that undergo splitting, so that each particle evolves independently of others with a splitting rate proportional to a function of its mass. In [8], Bertoin obtains such fragmentation

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model with some self-similarity property by cutting the Brownian Continuum Random Tree (CRT) of Aldous [1, 2] as follows: for all $t \geq 0$, remove all the vertices of the Brownian CRT that are located under height t and consider the connected components of the remaining vertices. Next, set $F^{B_r,(1)}(t) := (F_1^{B_r,(1)}(t), F_2^{B_r,(1)}(t), \ldots)$ for the decreasing sequence of masses of these connected components: $F^{B_r,(1)}$ is then a fragmentation process starting from $(1, 0, \ldots)$ where fragments split with a rate proportional to their mass to the power $-1/2$.

On the other hand, Aldous [1] shows that the Brownian CRT rescaled by a factor $1/\varepsilon$ converges in distribution to an infinite CRT composed by an infinite baseline $[0, \infty)$ on which are attached compact CRT's distributed, up to a scaling factor, as the Brownian CRT. In terms of fragmentations, his result implies that

$$\varepsilon^{-2}(F_2^{B_r,(1)}(\varepsilon\cdot), F_3^{B_r,(1)}(\varepsilon\cdot), \ldots) \overset{\text{law}}{\to} FI^{B_r} \quad \text{as } \varepsilon \to 0,$$

where FI^{B_r} is some fragmentation with immigration process constructed from the infinite Brownian CRT of Aldous. Equivalently, if $F^{B_r,(m)}$ denotes the Brownian fragmentation starting from $(m, 0, \ldots)$, $m \geq 0$,

$$(F_2^{B_r,(m)}, F_3^{B_r,(m)}, \ldots) \overset{\text{law}}{\to} FI^{B_r} \quad \text{as } m \to \infty.$$

Motivated by this example, our goal is to characterized in terms of fragmentation with immigration processes the limiting behavior of

$$(m - F_1^{(m)}, F_2^{(m)}, F_3^{(m)}, \ldots) \quad \text{as } m \to \infty$$

for some general fragmentations $F^{(m)}$ where the rates at which particles split are proportional to a function τ of their mass. In cases where τ is a power function, this will give the asymptotic behavior of $(1 - F^{(1)}(\varepsilon\cdot), F_2^{(1)}(\varepsilon\cdot), \ldots)$ as $\varepsilon \to 0$.

This paper is organized as follows. In the remainder of this section, we first introduce the fragmentation and fragmentation with immigration processes we will work with (Subsect. 1.1) and then state the main results on the limiting behavior of $F^{(m)}$ (Subsect. 1.2). These results are proved in Sect. 2. Sections 3, 4 and 5 are devoted to fragmentations with a power function τ. Section 3 concerns the behavior near 0 of such fragmentations starting from $(1, 0, \ldots)$. Section 4 deals with the asymptotic behavior as $m \to \infty$ of some CRT representations of the fragmentations $F^{(m)}$. Section 5 is an application of these results to a family of fragmentations, namely the "stable fragmentations", introduced by Miermont [26, 27]. Last, an Appendix contains some technical proofs and some generalization of our results to fragmentations with erosion.

1.1 Fragmentation and Fragmentation with Immigration Processes

1.1.1 $(\boldsymbol{\tau}, \boldsymbol{\nu})$-Fragmentations

For us, the only distinguishing feature of a particle is its mass, so that the fragmentation system is characterized at a given time by the decreasing sequence $s_1 \geq s_2 \geq$

$\cdots \geq 0$ of masses of particles present at that time. We shall then work in the state space

$$l_1^{\downarrow} := \left\{ \mathbf{s} = (s_i)_{i \geq 1} : s_1 \geq s_2 \geq \cdots \geq 0 : \sum_{i \geq 1} s_i < \infty \right\}$$

which is equipped with the distance

$$d(\mathbf{s}, \mathbf{s}') := \sum_{i \geq 1} |s_i - s_i'|.$$

The *dust* state $(0, 0, \ldots)$ is rather denoted by $\mathbf{0}$. Consider then $(F(t), t \geq 0)$, a càdlàg $l_1^{\downarrow}$-valued Markov process, and denote by $F^{(m)}$ a version of F starting from $(m, 0, \ldots)$, $m \geq 0$.

Definition 1 The process F is called a *fragmentation process* if

- for all $m, t \geq 0$, $\sum_{i \geq 1} F_i^{(m)}(t) \leq m$
- for all $t_0 \geq 0$, conditionally on $F(t_0) = (s_1, s_2, \ldots)$, $(F(t_0 + t), t \geq 0)$ is distributed as the process of the decreasing rearrangements of $F^{(s_1)}(t)$, $F^{(s_2)}(t), \ldots$ where the $F^{(s_i)}$'s are independent versions of F starting respectively from $(s_i, 0, 0, \ldots)$, $i \geq 1$.

When $F^{(m)} \overset{\text{law}}{=} mF^{(1)}$ for all $m \geq 0$, the fragmentation is usually called *homogeneous*. Such homogeneous processes have been studied by Bertoin [7] and Berestycki [4]. In particular, one knows that when the process is *pure-jump*, its law is characterized by a so-called *dislocation measure* ν on

$$l_{1,\leq 1}^{\downarrow} := \left\{ \mathbf{s} \in l_1^{\downarrow} : \sum_{i \geq 1} s_i \leq 1, s_1 < 1 \right\}$$

that integrates $(1 - s_1)$ and that describes the jumps of the process. Informally, each mass s will split into masses $ss_1, ss_2, \ldots$, $\sum_{i \geq 1} s_i \leq 1$, at rate $\nu(\mathrm{d}\mathbf{s})$. We call such process a *ν-homogeneous fragmentation*. To be more precise, the papers [4, 7] give a construction of the fragmentation based on a Poisson point process $(t_i, (\mathbf{s}(t_i), k(t_i)))_{i \geq 1}$ on $l_{1,\leq 1}^{\downarrow} \times \mathbb{N}$ ($\mathbb{N} = \{1, 2, \ldots\}$) with intensity measure $\nu \otimes \#$, where $\#$ denotes the counting measure on $\mathbb{N}$. The construction is so that, at each time t_i, the $k(t_i)$-th mass $F_{k(t_i)}^{(m)}(t_i-)$ splits in masses $s_1(t_i) F_{k(t_i)}^{(m)}(t_i-)$, $s_2(t_i) F_{k(t_i)}^{(m)}(t_i-), \ldots$, the other masses being unchanged. The sequence $F^{(m)}(t_i)$ is then the decreasing rearrangement of these new masses and of the unchanged masses $F_k^{(m)}(t_i-)$, $k \neq k(t_i)$. When ν is finite, this means that each particle with mass s waits an exponential time with parameter $\nu(l_{1,\leq 1}^{\downarrow})$ before splitting, and when it splits, it divides into particles with masses sS_1, $sS_2, \ldots$, where $(S_1, S_2, \ldots)$ is independent of the splitting time and is distributed according to $\nu(\cdot)/\nu(l_{1,\leq 1}^{\downarrow})$. When ν is infinite, the particles split immediately.

General Setting In this paper, we are more generally interested in pure-jump fragmentation processes where particles with mass s split at rate $\tau(s)\nu(\mathrm{d}\mathbf{s})$, where

– τ is a continuous strictly positive function on $(0,\infty)$
– τ is monotone near 0
– ν is a dislocation measure on $l^{\downarrow}_{1,\leq 1}$, i.e. $\int_{l^{\downarrow}_{1,\leq 1}}(1-s_1)\nu(\mathrm{d}\mathbf{s})<\infty$ (H)
– $\nu(l^{\downarrow}_{1,\leq 1})=\infty$
– $\nu(\sum_{i\geq 1}s_i<1)=0$.

The latter hypothesis on ν means that the particles do not lose mass within sudden dislocations. The fact that ν is infinite will be essential in the assumptions of our main theorems (forthcoming Theorems 5 and 7; see also Lemma 6).

Construction The distribution of each (τ,ν)-fragmentation is constructed through time-changes of a ν-homogeneous fragmentation starting from $(1,0,\ldots)$ in the following manner (see [17] for details): let $F^{(1),\mathrm{hom}}$ be a ν-homogeneous fragmentation starting from $(1,0,\ldots)$ and consider a family $(I^{\mathrm{hom}}(t),t\geq 0)$ of nested random open sets of $(0,1)$ such that $F^{(1),\mathrm{hom}}(t)$ is the decreasing sequence of the lengths of interval components of $I^{\mathrm{hom}}(t)$, for all $t\geq 0$. One knows ([4, 8]) that such *interval representation* of the fragmentation always exists. For $x\in(0,1)$, $t\geq 0$, call $I_x^{\mathrm{hom}}(t)$ the connected component of $I^{\mathrm{hom}}(t)$ that contains x, with the convention $I_x^{\mathrm{hom}}(t):=\emptyset$ if $x\notin I^{\mathrm{hom}}(t)$. Moreover, it is easily seen that $F^{(1),\mathrm{hom}}(t)\to\mathbf{0}$ a.s. as $t\to\infty$, since the dislocation measure $\nu(l^{\downarrow}_{1,\leq 1})\neq 0$. This implies that a.s. for all $x\in(0,1)$, $|I_x^{\mathrm{hom}}(t)|\to 0$ as $t\to\infty$, where $|I_x^{\mathrm{hom}}(t)|$ denotes the length of the interval $I_x^{\mathrm{hom}}(t)$. We therefore set $I_x^{\mathrm{hom}}(\infty):=\emptyset$, which makes the function $t\mapsto|I_x^{\mathrm{hom}}(t)|$ continuous at ∞.

Introduce then for $m>0$ the time-changes

$$T_x^m(t):=\inf\left\{u\geq 0:\int_0^u\frac{\mathrm{d}r}{\tau(m|I_x^{\mathrm{hom}}(r)|)}>t\right\},\tag{1}$$

where, by convention, $\tau(0):=\infty$ and $\inf\{\emptyset\}:=\infty$. Clearly, the open sets of $(0,1)$

$$I^{\tau}(t):=\bigcup_{x\in(0,1)}I_x^{\mathrm{hom}}(T_x^m(t)),\quad t\geq 0,$$

are nested and we call $F^{(m)}(t)$ the decreasing rearrangement of m times the lengths of the intervals components of $I^{\tau}(t)$, $t\geq 0$. The process $F^{(m)}$ is then the required fragmentation process starting from $(m,0,\ldots)$ with splitting rates $\tau(s)\nu(\mathrm{d}\mathbf{s})$ (Proposition 1, [17]). When $m=0$, $F^{(m)}(t)=\mathbf{0}$ for all $t\geq 0$, by definition.

Self-Similar Fragmentations When $\tau(s)=s^{\alpha}$ for some $\alpha\in\mathbb{R}$, the fragmentation is called *self-similar* with index α, since $F^{(m)}\overset{\mathrm{law}}{=}mF^{(1)}(m^{\alpha}\cdot)$ for all $m\geq 0$. These self-similar fragmentations processes have been extensively studied by Bertoin [7–9].

Two Classical Examples The *Brownian fragmentation* is a self-similar fragmentation process constructed from a normalized Brownian excursion $e^{(m)}$ with length m as follows: for each t, $F^{B_r,(m)}(t)$ is the decreasing rearrangement of lengths of connected components of $\{x \in (0,m) : 2e^{(m)}(x) > t\}$. Equivalently it can be constructed from the Brownian continuum random tree of Aldous by removing vertices under height t, as explained in the introduction (precise definition of continuum random trees are given in Sect. 4). The index of self-similarity is then $-1/2$ and Bertoin [8] proves that the dislocation measure is given by

$$\begin{aligned} &\nu_{B_r}(s_1 \in \mathrm{d}x) = (2\pi x^3(1-x)^3)^{-1/2}\mathrm{d}x, \quad x \in [1/2, 1), \quad \text{and} \\ &\nu_{B_r}(s_1 + s_2 < 1) = 0, \end{aligned} \tag{2}$$

this second property meaning that each fragment splits into two pieces when dislocating.

On the other hand, by logging the Brownian continuum random tree along its skeleton, Aldous and Pitman [3] have introduced a self-similar fragmentation F^{AP} with index $1/2$ which is transformed by an exponential time-reversal into the standard additive coalescent. This *Aldous-Pitman fragmentation* is in some sense dual to the Brownian one: its dislocation measure is also ν_{B_r} (see [8]).

Loss of Mass Consider the total mass $M^{(m)}(t) = \sum_{i\geq 1} F_i^{(m)}(t)$ of macroscopic particles present at time t in a fragmentation $F^{(m)}$. When the fragmentation rate of small particles is sufficiently high, some mass may be lost to dust (i.e. a large quantity of microscopic—or 0-mass—particles arises in finite time), so that the mass $M^{(m)}(t)$ decreases to 0 as $t \to \infty$. Such phenomenon does not depend on the initial mass $m > 0$ and more precisely either occurs with probability one simultaneously for all $m > 0$ or does not occur, for any $m > 0$, with probability one. For example, one knows that there is loss of mass as soon as $\int_{0+} \mathrm{d}x/(x\tau(x)) < \infty$. We refer to [17] for details and some necessary and sufficient condition. An interesting fact is that the mass $M^{(m)}$ decreases continuously:

Proposition 2 *The function* $t \mapsto M^{(m)}(t)$ *is a.s. continuous on* $[0,\infty)$.

This will be useful for some forthcoming proofs. A proof is given in the Appendix.

1.1.2 $(\boldsymbol{\tau},\boldsymbol{\nu},\mathbf{I})$-Fragmentations with Immigration

Let $\mathcal{I}$ be the set of measures on $l_1^{\downarrow}$ that integrate $(\sum_{j\geq 1} s_j) \wedge 1$. Two such measures I, J are considered to be equivalent if their difference $I - J$ puts mass only on $\{\mathbf{0}\}$. Implicitly, we always identify a measure with its equivalence class. In particular, in the following, we will often do the assumption $I(l_1^{\downarrow}) \neq 0$, which means that I puts mass on some non-trivial sequences. Endow then $\mathcal{I}$ with the distance

$$D(I,J) = \sup_{f\in\mathcal{F}} \left| \int_{l_1^{\downarrow}} f(\mathbf{s})(I-J)(\mathrm{d}\mathbf{s}) \right|, \tag{3}$$

where $\mathcal{F}$ is the set of non-negative continuous functions on $l_1^{\downarrow}$ such that $f(\mathbf{s}) \leq (\sum_{j\geq 1} s_j) \wedge 1$. The function $\mathbf{s} \mapsto (\sum_{j\geq 1} s_j) \wedge 1$ belongs to $\mathcal{F}$ and therefore $\mathcal{I}$ is closed. It is called the set of *immigration measures*.

Definition 3 Let $((r_i, \mathbf{u}^i), i \geq 1)$ be a Poisson point process (PPP) with intensity $I \in \mathcal{I}$ and, conditionally on this PPP, let $F^{(u_j^i)}$, $i, j \geq 1$, be independent (τ, ν) fragmentations starting respectively from u_j^i, $i, j \geq 1$. Then consider for each $t \geq 0$, the decreasing rearrangement

$$FI(t) := \left\{F_k^{(u_j^i)}(t - r_i), r_i \leq t, j, k \geq 1\right\}^{\downarrow} \in l_1^{\downarrow}.$$

The process FI is called a fragmentation with immigration process with parameters (τ, ν, I).

When there is no fragmentation ($\nu(l_{1,\leq 1}^{\downarrow}) = 0$), we rather call such process a pure immigration process with parameter I and we denote it by $(I(t), t \geq 0)$.

This means that at time r_i, particles with masses $u_1^i, u_2^i, \ldots$ immigrate and then start to fragment independently of each other (conditionally on their masses), according to a (τ, ν) fragmentation. The initial state is $\mathbf{0}$. Note that the total mass of immigrants until time t

$$\sigma_I(t) := \sum_{r_i \leq t, j \geq 1} u_j^i \tag{4}$$

is a.s. finite and therefore that the decreasing rearrangement $FI(t)$ indeed exists and is in $l_1^{\downarrow}$. The process σ_I is a *subordinator,* i.e. an increasing Lévy process. We refer to the lecture [6] for backgrounds on subordinators. In particular, we recall that a subordinator σ is characterized by its Laplace exponent, which is a function ϕ_σ such that $E[\exp(-q\sigma(t))] = \exp(-t\phi_\sigma(q))$, for all $q, t \geq 0$.

Note also that FI is càdlàg, since the $F^{(u_j^i)}$ are càdlàg, since dominated convergence applies and since, clearly, the following result holds.

Lemma 4 *For all integers* $1 \leq n \leq \infty$, *let* $x^n = (x_i^n, i \geq 1)$ *be a sequence of non-negative real numbers such that* $\sum_{i\geq 1} x_i^n < \infty$ *and let* $x^{n\downarrow}$ *denotes its decreasing rearrangement. If* $\sum_{i\geq 1} |x_i^n - x_i^\infty| \to 0$, *then* $\sum_{i\geq 1} |x_i^{n\downarrow} - x_i^{\infty\downarrow}| \to 0$, *i.e.* $x^{n\downarrow} \to x^{\infty\downarrow}$ *in* $l_1^{\downarrow}$.

Equilibrium for such fragmentation with immigration processes has been studied in [18] in a slightly less general context.

1.2 Main Results: Asymptotics of $F^{(m)}$

Introduce for all $m > 0$, the measure $\nu_m \in \mathcal{I}$ defined for all non-negative measurable functions f on $l_1^{\downarrow}$ by

$$\int_{l_1^{\downarrow}} f(\mathbf{s})\nu_m(\mathrm{d}\mathbf{s}) := \int_{l_{1,\leq 1}^{\downarrow}} f(s_2 m, s_3 m, \ldots)\nu(\mathrm{d}\mathbf{s}).$$

Set also

$$\varphi_\nu(m) := (\nu(s_1 < 1 - m^{-1}))^{-1} = \langle \nu_m, \mathbf{1}_{\{\sum_{i\geq 1} s_i > 1\}}\rangle^{-1}$$

which is finite for m large enough and converges to 0 as $m \to \infty$, since $\nu(l^{\downarrow}_{1,\leq 1}) = \infty$.

We are now ready to state our main result. We remind that the distance on $\mathcal{I}$ is defined by (3). Also, the set of càdlàg paths in $\mathbb{R}^+ \times l_1^{\downarrow}$ is endowed with the *Skorohod topology.*

Theorem 5 *Let F be a (τ, ν) fragmentation and suppose that $\tau(m)\nu_m \to I \in \mathcal{I}$, $I(l_1^{\downarrow}) \neq 0$, as $m \to \infty$. Then,*

$$(m - F_1^{(m)}, (F_2^{(m)}, F_3^{(m)}, \ldots)) \overset{\text{law}}{\to} (\sigma_I, FI) \quad \textit{as } m \to \infty,$$

where FI is a fragmentation with immigration with parameters (τ, ν, I) starting from $\mathbf{0}$, and σ_I is the process (4) corresponding to the total mass of particles that have immigrated until time t, $t \geq 0$.

In some sense, letting $m \to \infty$ in $F^{(m)}$ creates an infinite amount of mass that regularly injects into the system some groups of finite masses which then undergo fragmentation. A similar phenomenon has been observed in the study of some different processes conditioned on survival (see e.g. [11, 13, 15, 25]).

Example Recall the characterization (2) of the Brownian dislocation measure ν_{B_r}. Clearly, $m^{-1/2}\nu_{B_r,m} \to I_{B_r}$ where the measure I_{B_r} is defined by

$$I_{B_r}(s_1 \in \mathrm{d}x) = (2\pi x^3)^{-1/2}\mathrm{d}x, \quad x > 0, \quad \text{and} \quad I_{B_r}(s_2 > 0) = 0. \tag{5}$$

So the previous theorem applies to the Brownian fragmentation and the fragmentation with immigration appearing in the limit has parameters $(\tau : x \mapsto x^{-1/2}, \nu_{B_r}, I_{B_r})$. The Lévy measure of the subordinator $\sigma_{I_{B_r}}$ is simply $I_{B_r}(s_1 \in \mathrm{d}x)$. Informally, this corresponds to the convergence, mentioned in the introduction, of the Brownian CRT to a tree with a spine on which are branched rescaled Brownian CRTs. This tree with a spine codes (see Sect. 4 for precise statements) the above $(\tau : x \mapsto x^{-1/2}, \nu_{B_r}, I_{B_r})$ fragmentation with immigration.

Other examples are given in Sect. 5.1.

The assumption on the convergence of $\tau(m)\nu_m$ may seem demanding and, clearly, is not always satisfied. A moment of thought, using test-functions of type $f_a(\mathbf{s}) = \mathbf{1}_{\{\sum_{i\geq 1} s_i > a\}}$, $a > 0$, leads to the following result.

Lemma 6 *Suppose that $\tau(m)\nu_m$ converges to some measure $I \in \mathcal{I}$, $I(l_1^{\downarrow}) \neq 0$, as $m \to \infty$. Then both τ and φ_ν vary regularly at ∞ with some index $-\gamma_\nu$, $\gamma_\nu \in (0, 1)$ and $\tau(m) \sim C\varphi_\nu(m)$ as $m \to \infty$, with $C = I(\sum_{i\geq 1} s_i > 1) > 0$. As a consequence, the limit I is γ_ν-self-similar, that is*

$$\int_{l_1^{\downarrow}} f(as_1, as_2, \ldots) I(\mathrm{d}\mathbf{s}) = a^{\gamma_\nu} \int_{l_1^{\downarrow}} f(\mathbf{s}) I(\mathrm{d}\mathbf{s}) \quad \textit{for all } a > 0, \; f \in \mathcal{F},$$

which in turn implies that σ_I is a stable subordinator with index γ_ν and Laplace exponent $C\Gamma(1-\gamma_\nu)q^{\gamma_\nu}$, $q \geq 0$.

Note that $\gamma_\nu < 1$ since I integrates $(\sum_{i\geq 1} s_i) \wedge 1$ and that $\gamma_\nu > 0$ since $I(\sum_{i\geq 1} s_i > a) \to 0$ as $a \to \infty$. Note also that $\gamma_\nu > 0$ is equivalent to $\varphi_\nu(m) \to 0$ as $m \to \infty$, which occurs if and only if ν is infinite. That is why our general hypotheses (H) include the fact that ν is infinite.

Lemma 6 therefore implies that Theorem 5 applies to measures ν such that $\varphi_\nu(m)\nu_m$ converges, coupled together with functions τ whose behavior at ∞ is proportional to that of φ_ν. In particular, the speed of fragmentation of small particles plays no role in the existence of a limit for $F^{(m)}$.

Remark then that it is possible to construct from any γ-self-similar immigration measure I, $I(l_1^\downarrow) \neq 0$, $\gamma \in (0,1)$, some dislocation measures ν such that $\varphi_\nu(m)\nu_m$ converge[1] to I, which gives a large class of measures ν to which Theorem 5 applies. Also, note that when the fragmentation is binary (i.e. when $\nu(s_1+s_2<1)=0$), the convergence of $\varphi_\nu(m)\nu_m$ holds as soon as φ_ν varies regularly at ∞ with some index in $(-1,0)$.

We now turn to functions τ such that $(\varphi_\nu/\tau)(m)$ converges to 0 or ∞, in which cases Theorem 5 does not apply. One way to avoid trivial limits in such situation is to consider the process $F^{(m)}$ up to a time change:

Theorem 7 *Suppose that τ varies regularly at ∞, and that $\varphi_\nu(m)\nu_m \to I \in \mathcal{I}$ as $m \to \infty$ (in particular, by Lemma 6, φ_ν varies regularly at infinity with an index in $(-1,0)$).*

(i) *If, as $m \to \infty$, $(\varphi_\nu/\tau)(m) \to 0$, then,*

$$\begin{aligned}&\left(\left(m - F_1^{(m)}((\varphi_\nu/\tau)(m)\cdot)\right), F_2^{(m)}((\varphi_\nu/\tau)(m)\cdot), F_3^{(m)}((\varphi_\nu/\tau)(m)\cdot), \ldots\right)\\ &\quad \overset{\text{law}}{\to} (\sigma_I, (I(t), t \geq 0)),\end{aligned}$$

where $(I(t), t\geq 0)$ is a pure immigration process with parameter I.

(ii) *If $(\varphi_\nu/\tau)(m) \to \infty$ as $m \to \infty$ and the fragmentations $F^{(m)}$ lose mass to dust, then the following finite-dimensional convergence holds as $m \to \infty$,*

$$\left(\left(m - F_1^{(m)}((\varphi_\nu/\tau)(m)\cdot)\right), F_2^{(m)}((\varphi_\nu/\tau)(m)\cdot), F_3^{(m)}((\varphi_\nu/\tau)(m)\cdot), \ldots\right) \underset{\text{f.d.}}{\overset{\text{law}}{\to}} (\sigma_I, \mathbf{0}).$$

We recall that the case when $(\varphi_\nu/\tau)(m) \to \ell \in (0,\infty)$ is given by Theorem 5.

The assertion (ii) is not valid when the fragmentations $F^{(m)}$ do not lose mass, since the quantities $m - \sum_{i\geq 1} F_i^{(m)}((\varphi_\nu/\tau)(m))$ are then equal to 0 and so cannot converge to $\sigma_I(1)$. However, a result similar to that stated in (ii) holds for fragmentations that do not lose mass, provided that the distance d is replaced by the distance of uniform convergence on $l_1^\downarrow$. Also, the reason why the limit in this statement (ii) holds only in

[1] For example, define ν by $\int_{l_{1,\leq 1}^\downarrow} f(\mathbf{s})\nu(\mathrm{d}\mathbf{s}) := \int_{l_1^\downarrow} f(1-\sum_{j\geq 1} s_j, s_1, s_2, \ldots)\mathbf{1}_{\{s_1 \leq 1 - \sum_{j\geq 1} s_j\}} I(\mathrm{d}\mathbf{s})$. Clearly, $\nu(\sum_{j\geq 1} s_j \neq 1) = 0$, ν integrates $(1-s_1)$ and $m^{-\gamma}\nu_m \to I$.

the finite dimensional sense and not with respect to the Skorohod topology, is that the functional limit of $(F_2^{(m)}((\varphi_\nu/\tau)(m)\cdot), F_3^{(m)}((\varphi_\nu/\tau)(m)\cdot), \ldots)$ cannot be càdlàg (see the last paragraph of Sect. 2.2 for an explanation).

Another remark is that under the assumptions of assertion (i), the processes $F_i^{(m)}((\varphi_\nu/\tau)(m)\cdot)$, $i \geq 2$, although not increasing, converge as $m \to \infty$ to some increasing processes. In particular, $F_2^{(m)}((\varphi_\nu/\tau)(m)\cdot)$ converges to $(\Delta_1^a(t), t \geq 0)$ where $\Delta_1^a(t)$ is the largest jump before time t of some stable subordinator with Laplace exponent $a\Gamma(1-\gamma_\nu)q^{\gamma_\nu}$, $q \geq 0$, and $a = \lim_{m\to\infty} \varphi_\nu(m)\nu(s_2 > m^{-1})$ (this limit exists, although $\mathbf{s} \mapsto \mathbf{1}_{\{s_1>1\}} \notin \mathcal{F}$, because $I(s_1 \in dx)$ is absolutely continuous, as a consequence of the self-similarity). In case ν is binary, one more precisely has:

Corollary 8 *Suppose that ν is binary, that φ_ν varies regularly at ∞ with some index $-\gamma_\nu$, $\gamma_\nu \in (0,1)$ and that τ varies regularly at ∞. Then, if $(\varphi_\nu/\tau)(m) \to 0$,*

$$\begin{aligned}&((m - F_1^{(m)}((\varphi_\nu/\tau)(m)\cdot)), F_2^{(m)}((\varphi_\nu/\tau)(m)\cdot), F_3^{(m)}((\varphi_\nu/\tau)(m)\cdot), \ldots)\\ &\quad\stackrel{\text{law}}{\to} (\sigma, \Delta_1, \Delta_2, \ldots),\end{aligned}$$

where σ is a stable subordinator with Laplace exponent $\Gamma(1-\gamma_\nu)q^{\gamma_\nu}$ and $(\Delta_1(t), \Delta_2(t), \ldots)$ the decreasing sequence of its jumps before time t, $t \geq 0$.

Example This can be applied to the Aldous-Pitman fragmentation, since $\varphi_{\nu_{B_r}}(m) \sim \pi^{1/2}(2m)^{-1/2}$. We get that

$$\begin{aligned}&((m - F_1^{AP,(m)}(m^{-1}\cdot)), F_2^{AP,(m)}(m^{-1}\cdot), F_3^{AP,(m)}(m^{-1}\cdot), \ldots)\\ &\quad\stackrel{\text{law}}{\to} (\sigma_{AP}, \Delta_1^{AP}, \Delta_2^{AP}, \ldots),\end{aligned} \tag{6}$$

where σ_{AP} is a stable subordinator with Laplace exponent $(2q)^{1/2}$ and $(\Delta_1^{AP}(t), \Delta_2^{AP}(t), \ldots)$ the decreasing sequence of its jumps before time t, $t \geq 0$. To see this, use that $\varphi_{\nu_{B_r}}(m)m^{-1/2} \sim (\pi/2)^{1/2}m^{-1}$ to obtain from Corollary 8 the convergence of the process in the left hand side of (6) to $(\sigma((2/\pi)^{1/2}\cdot), \Delta_1((2/\pi)^{1/2}\cdot), \Delta_2((2/\pi)^{1/2}\cdot), \ldots)$ where σ is a stable subordinator with Laplace exponent $\Gamma(1/2)q^{1/2} = (\pi q)^{1/2}$ and $(\Delta_1(t), \Delta_2(t), \ldots)$ the decreasing sequence of its jumps before time t, $t \geq 0$. It then clear that the Laplace exponent of $\sigma_{AP} = \sigma((2/\pi)^{1/2}\cdot)$ is $(2q)^{1/2}$. Aldous and Pitman [3], Corollary 13, obtained this result by studying size-biased permutations of their fragmentation.

Other explicit (and non-binary) examples are studied in Sect. 5.2.

2 Proofs

The main lines of the proofs of Theorems 5 and 7 are quite similar. We first give a detailed proof of Theorem 5 and then explain how it can be adapted to prove Theorem 7. We will need the following classical result on Skorohod convergence (see Proposition 3.6.5, [14]).

Lemma 9 *Consider a metric space (E,d_E) and let f_n, f be càdlàg paths with values in E. Then $f_n \to f$ with respect to the Skorohod topology if and only if the three following assertions are satisfied for all sequences $t_n \to t$, $t_n, t \geq 0$:*

(a) $\min(d_E(f_n(t_n), f(t)), d_E(f_n(t_n), f(t-))) \to 0$

(b) $d_E(f_n(t_n), f(t)) \to 0 \Rightarrow d_E(f_n(s_n), f(t)) \to 0$ *for all sequences* $s_n \to t$, $s_n \geq t_n$

(c) $d_E(f_n(t_n), f(t-)) \to 0 \Rightarrow d_E(f_n(s_n), f(t-)) \to 0$ *for all sequences* $s_n \to t$, $s_n \leq t_n$.

2.1 Proof of Theorem 5

In this section it is supposed that $\tau(m)\nu_m \to I$, $I(l_1^{\downarrow}) \neq 0$, as $m \to \infty$. Our goal is then to prove Theorem 5, which is a corollary of the forthcoming Lemma 11. In order to state and prove this lemma, we first introduce some notations and give some heuristic geometric description of what is happening. There is no loss of generality in supposing that the (τ,ν) fragmentations $F^{(m)}$, $m \geq 0$, are constructed from the same ν-homogeneous one, which is done in the following.

2.1.1 Heuristic Description

We first give a geometric description of the fragmentation $F^{(m)}$, which may be viewed as a baseline $\mathcal{B} = [0,\infty)$ on which fragmentation processes are attached.

Let $\Lambda^{(m)}$ be the process obtained by following at each dislocation the largest sub-fragment. According to the Poissonian construction of homogeneous fragmentation processes and the time-change between ν-homogeneous and (τ,ν)-fragmentations (see Sect. 1.1.1), the process $\Lambda^{(m)}$ is constructed from some Poisson point process $((t_i, \mathbf{s}^i), i \geq 1)$ (independent of m) with intensity measure ν as follows: if ξ denotes the subordinator defined by

$$\xi(t) := \sum_{t_i \leq t} (-\log(s_1^i)), \quad t \geq 0, \tag{7}$$

and $\rho^{-(m)}$ the integral

$$\rho^{-(m)}(t) := \int_0^t \mathrm{d}r/\tau(m\exp(-\xi(r))),$$

then

$$\Lambda^{(m)}(t) = m\exp(-\xi(\rho^{(m)}(t))), \quad t \geq 0, \tag{8}$$

where

$$\rho^{(m)}(t) := \inf\{u : \rho^{-(m)}(u) > t\} \tag{9}$$

$(\inf\{\emptyset\} = \infty)$. The set of jump times of $\Lambda^{(m)}$ is then $\{t_i^m := \rho^{-(m)}(t_i), i \geq 1\}$.

The evolution of the fragmentation $F^{(m)}$ then relies on the point process $((t_i^m, \mathbf{s}^i), i \geq 1)$: at time t_i^m, the fragment with mass $\Lambda^{(m)}(t_i^m-)$ splits to give a fragment with mass $\Lambda^{(m)}(t_i^m) = \Lambda^{(m)}(t_i^m-)s_1^i$ and smaller fragments with masses $\Lambda^{(m)}(t_i^m-)s_j^i$, $j \geq 2$. For $j \geq 2$, call $F^{(\Lambda^{(m)}(t_i^m-)s_j^i)}$ the fragmentation describing the evolution of the mass $\Lambda^{(m)}(t_i^m-)s_j^i$ and consider that it is branched at height t_i^m on the baseline $\mathcal{B}$. Then the process $F^{(m)}$ is obtained by considering for each $t \geq 0$ all fragmentations branched at height $t_i^m \leq t$ and by ordering in the decreasing order the terms of sequences $F^{(\Lambda^{(m)}(t_i^m-)s_j^i)}(t - t_i^m)$, $t_i^m \leq t$, $j \geq 2$, and $\Lambda^{(m)}(t)$. In some sense, there is then a tree structure under this baseline with "fragmentation" leaves. This will be discussed in Sect. 4.

Similarly, a (τ, ν, I) fragmentation with immigration FI can be viewed as the baseline $\mathcal{B}$ with fragmentations leaves $F^{(u_j^i)}$, $j \geq 1$, attached at time r_i, where $((r_i, \mathbf{u}^i), i \geq 1)$ is a Poisson point process with intensity I and $F^{(u_j^i)}$, $i, j \geq 1$, some (τ, ν) fragmentations starting respectively from u_j^i, $i, j \geq 1$, that are independent conditionally on $((r_i, \mathbf{u}^i), i \geq 1)$.

Now, to see the connection between these descriptions and the result we want to prove on the convergence of $(F_2^{(m)}, F_3^{(m)}, \ldots)$ to FI, note that the processes $\Lambda^{(m)}$ and $F_1^{(m)}$, although different, coincide at least when $\Lambda^{(m)}(t) \geq m/2$, since $\Lambda^{(m)}(t)$ is then the largest fragment of $F^{(m)}(t)$. Fix $t_0 < \infty$. It is easily seen that under the assumption $\tau(m)\nu_m \to I$ (which in particular implies that $\tau(m) \to 0$ as $m \to \infty$), a.s. $\rho^{(m)}(t_0) \to 0$ as $m \to \infty$, which in turn implies that for large m's and all $t \leq t_0$, $\Lambda^{(m)}(t) \geq m/2$, and therefore $\Lambda^{(m)}(t) = F_1^{(m)}(t)$. In particular $(F_2^{(m)}(t), F_3^{(m)}(t), \ldots)$ is then the decreasing rearrangement of the terms of sequences $F^{(\Lambda^{(m)}(t_i^m-)s_j^i)}(t - t_i^m)$, $t_i^m \leq t$, $j \geq 2$.

Hence, informally, one may expect that the process $(F_2^{(m)}, F_3^{(m)}, \ldots)$ converges in law to FI as soon as $(\Lambda^{(m)}(t_i^m-)s_j^i, i \geq 1, j \geq 2)$ converges to $(u_j^i, i, j \geq 1)$, and $(t_i^m, i \geq 1)$ to $(r_i, i \geq 1)$. The statement of these convergences is made rigorous in the forthcoming Lemma 10, which is then used to prove the required Lemma 11.

2.1.2 Convergence of the Point Processes

Consider the set $[0, \infty) \times l_1^{\downarrow} \times l_1^{\downarrow}$ endowed with the distance

$$d_{[0,\infty)\times l_1^{\downarrow}\times l_1^{\downarrow}}((t_1, \mathbf{s}_1, \mathbf{s}_1'), (t_2, \mathbf{s}_2, \mathbf{s}_2')) := |t_1 - t_2| + \sum_{i\geq 1}\left(|s_{1,i} - s_{2,i}| + |s_{1,i}' - s_{2,i}'|\right)$$

(which makes it Polish) and introduce the set $\mathcal{R}_{[0,\infty)\times l_1^{\downarrow}\times l_1^{\downarrow}}$ of Radon point measures on $[0, \infty) \times l_1^{\downarrow} \times l_1^{\downarrow}$ that integrate $\mathbf{1}_{\{t\leq t_0\}} \times \sum_{j\geq 1}(s_j + s_j')$, for all $t_0 \geq 0$. Two such measures are considered to be equivalent if their difference puts mass only on $[0, \infty) \times \{(\mathbf{0}, \mathbf{0})\}$. Again, we shall implicitly identify a measure with its equivalence class. Introduce then $\mathcal{F}_{[0,\infty)\times l_1^{\downarrow}\times l_1^{\downarrow}}$, the set of $\mathbb{R}^+$-valued continuous functions f on $[0, \infty) \times l_1^{\downarrow} \times l_1^{\downarrow}$ such that $f(t, \mathbf{s}, \mathbf{s}') \leq \mathbf{1}_{\{t\leq t_0\}} \sum_{j\geq 1}(s_j + s_j')$ for some $t_0 \geq 0$ (we shall denote by t_0^f such t_0's) and equip $\mathcal{R}_{[0,\infty)\times l_1^\downarrow\times l_1^\downarrow}$ with the topology induced by the convergence $\mu_n\to\mu \Leftrightarrow \langle\mu_n, f\rangle\to\langle\mu, f\rangle$ for all $f\in\mathcal{F}_{[0,\infty)\times l_1^\downarrow\times l_1^\downarrow}$.

Lemma 10 *Set*

$$\mu_m := \sum_{i\ge 1}\delta_{(t_i^m,(\Lambda^{(m)}(t_i^m-)s_j^i)_{j\ge 2},(ms_j^i)_{j\ge 2})} \quad \textit{and} \quad \mu := \sum_{i\ge 1}\delta_{(r_i,\mathbf{u}^i,\mathbf{u}^i)}.$$

These measures belong to $\mathcal{R}_{[0,\infty)\times l_1^\downarrow\times l_1^\downarrow}$, *and with respect to the above topology on* $\mathcal{R}_{[0,\infty)\times l_1^\downarrow\times l_1^\downarrow}$,

$$\mu_m \overset{\text{law}}{\to} \mu \quad \textit{as } m\to\infty.$$

Proof According to Theorems 4.2 and 4.9 of Kallenberg [22], the convergence in distribution of μ_m to μ is equivalent to the convergence of all Laplace transforms $E[\exp(-\langle\mu_m, f\rangle)]$ to $E[\exp(-\langle\mu, f\rangle)]$, $f\in\mathcal{F}_{[0,\infty)\times l_1^\downarrow\times l_1^\downarrow}$. The proof is split into two parts.

(1) In this first part, we will show that for all $f\in\mathcal{F}_{[0,\infty)\times l_1^\downarrow\times l_1^\downarrow}$ and all $t\ge 0$, $E[\exp(-\langle\mu_m^{(t)}, f\rangle)]$ converges to $E[\exp(-\langle\mu^{(t)}, f\rangle)]$, where

$$\mu_m^{(t)} := \sum_{t_i\le t\tau(m)}\delta_{(t_i^m,(\Lambda^{(m)}(t_i^m-)s_j^i)_{j\ge 2},(ms_j^i)_{j\ge 2})},$$
$$\mu^{(t)} := \sum_{r_i\le t}\delta_{(r_i,\mathbf{u}^i,\mathbf{u}^i)}.$$

We recall that $t_i^m=\rho^{-(m)}(t_i)$ and $\Lambda^{(m)}(t_i^m-)=m\exp(-\xi(t_i-))$ and set

$$g_f(t,\mathbf{s},\mathbf{s}') := 1-\exp(-f(t,\mathbf{s},\mathbf{s}'))$$

for all $f\in\mathcal{F}_{[0,\infty)\times l_1^\downarrow\times l_1^\downarrow}$. Note that $g_f\in\mathcal{F}_{[0,\infty)\times l_1^\downarrow\times l_1^\downarrow}$ and $g_f(t,\mathbf{s},\mathbf{s}')\le 1\wedge(\sum_{j\ge 1}(s_j+s_j'))$. Now, to start with, we claim that for all $t\ge 0$,

$$E\big[\exp\big(-\langle\mu_m^{(t)}, f\rangle + I_{m,t}(f)\big)\big] = 1, \tag{10}$$

where $I_{m,t}(f)$ is defined by

$$\begin{aligned} I_{m,t}(f) &:= \int_{[0,t\tau(m)]\times l_1^\downarrow} g_f(\rho^{-(m)}(v), \exp(-\xi(v-))\mathbf{s},\mathbf{s})\mathrm{d}v\otimes\nu_m(\mathrm{d}\mathbf{s}) \\ &= \int_{[0,t]\times l_1^\downarrow} g_f(\rho^{-(m)}(v\tau(m)), \exp(-\xi(v\tau(m)-))\mathbf{s},\mathbf{s})\mathrm{d}v\otimes\tau(m)\nu_m(\mathrm{d}\mathbf{s}). \end{aligned}$$

Indeed, the Change of Variables Formula for right-continuous processes of finite variation gives

$$
\begin{aligned}
&\exp\big(-\langle\mu_m^{(t)}, f\rangle + I_{m,t}(f)\big) - 1 \\
&\quad = -\sum_{0<t_i\le t\tau(m)} \exp\big(-\langle\mu_m^{(t_i-)}, f\rangle + I_{m,t_i}(f)\big) \\
&\qquad \times g_f(\rho^{-(m)}(t_i), (m\exp\big(-\xi(t_i-)\big)s_j^i)_{j\ge2}, (ms_j^i)_{j\ge2}) \\
&\qquad + \int_{[0,t\tau(m)]\times l_1^{\downarrow}} \exp\big(-\langle\mu_m^{(v-)}, f\rangle + I_{m,v}(f)\big) \\
&\qquad \times g_f(\rho^{-(m)}(v), \exp(-\xi(v-))\mathbf{s}, \mathbf{s})\mathrm{d}v \otimes \nu_m(\mathrm{d}\mathbf{s}).
\end{aligned}
$$

Then, using the Master Formula for Poisson Point Processes (see e.g. [31], Chap. XII), we get that the expectation of the sum in the right-hand side of the equality is equal to the expectation of the integral. Hence (10).

(a) We now claim that a.s. for all $f \in \mathcal{F}_{[0,\infty)\times l_1^{\downarrow}\times l_1^{\downarrow}}$, the integral $I_{m,t}(f)$ converges to $I_t(f) := \int_{[0,t]\times l_1^{\downarrow}} g_f(v,\mathbf{s},\mathbf{s})\mathrm{d}v\otimes I(\mathrm{d}\mathbf{s})$ as $m\to\infty$. Indeed,

$$
\begin{aligned}
I_{m,t}(f) - I_t(f) &= \int_{[0,t]\times l_1^{\downarrow}} g_f(v,\mathbf{s},\mathbf{s})\mathrm{d}v\otimes(\tau(m)\nu_m(\mathrm{d}\mathbf{s}) - I(\mathrm{d}\mathbf{s})) \\
&\quad + \int_{[0,t]\times l_1^{\downarrow}} (g_f(\rho^{-(m)}(v\tau(m)), \exp(-\xi(v\tau(m)-))\mathbf{s}, \mathbf{s}) \\
&\quad - g_f(v,\mathbf{s},\mathbf{s}))\mathrm{d}v\otimes\tau(m)\nu_m(\mathrm{d}\mathbf{s}).
\end{aligned}
$$

The first integral in the right-hand side of this equality converges to 0, since $\tau(m)\nu_m \to I$ and the function $\mathbf{s}\mapsto \int_{[0,t]} g_f(v,\mathbf{s},\mathbf{s})\mathrm{d}v$ is continuous and bounded by $2t(\sum_{i\ge1} s_i\wedge 1)$ on $l_1^{\downarrow}$. It remains to prove that the second integral converges to 0, a.s. for all $f\in\mathcal{F}_{[0,\infty)\times l_1^{\downarrow}\times l_1^{\downarrow}}$. This will be proved if we check that a.s. for all *1-Lipschitz* functions $g\in\mathcal{F}_{[0,\infty)\times l_1^{\downarrow}\times l_1^{\downarrow}}$ such that $g_f(t,\mathbf{s},\mathbf{s})\le 1\wedge(\sum_{i\ge1} s_i)$,

$$
\int_{[0,t]\times l_1^{\downarrow}} (g(\rho^{-(m)}(v\tau(m)), \exp(-\xi(v\tau(m)-))\mathbf{s},\mathbf{s}) - g(v,\mathbf{s},\mathbf{s}))\mathrm{d}v\otimes\tau(m)\nu_m(\mathrm{d}\mathbf{s}) \to 0.
$$

The absolute value of this last integral is bounded from above by

$$
\begin{aligned}
&\int_{[0,t]\times l_1^{\downarrow}} \left(\left(|\rho^{-(m)}(v\tau(m)) - v| + \sum_i s_i(1-\exp(-\xi(t\tau(m)-)))\right)\wedge 1\wedge\sum_i s_i\right) \\
&\quad \times \mathrm{d}v\otimes\tau(m)\nu_m(\mathrm{d}\mathbf{s}).
\end{aligned} \tag{11}
$$

Fix then some $\varepsilon>0$ and recall that τ varies regularly at ∞ with some negative index $-\gamma_\nu$ (cf. Lemma 6). In particular, $\tau(m)\to 0$ and therefore, with probability one, for all m sufficiently large (depending on ε and t), $1-\exp(-\xi(t\tau(m)-))\le\varepsilon$. On the other hand, still thanks to the regular variation of τ, one knows (see Potter's Theorem, Th. 1.5.6 [10]) that there exists for all $A>1$, some constant $M(A)\ge 0$

such that

$$A^{-1}\exp((-3\gamma_\nu/2)\xi(r)) \le \frac{\tau(m)}{\tau(m\exp(-\xi(r)))} \le A\exp((-\gamma_\nu/2)\xi(r)), \tag{12}$$

for all m, r such that $m\exp(-\xi(r)) \ge M(A)$. When $m\exp(-\xi(t\tau(m))) \ge M(A)$, this implies that

$$A^{-1}\int_0^v \exp((-3\gamma_\nu/2)\xi(\tau(m)r))\mathrm{d}r \le \rho^{-(m)}(v\tau(m)) \le A\int_0^v \exp((-\gamma_\nu/2)\xi(\tau(m)r))\mathrm{d}r,$$

hence

$$A^{-1}v\exp((-3\gamma_\nu/2)\xi(\tau(m)t)) \le \rho^{-(m)}(v\tau(m)) \le Av$$

for all $v \in [0,t]$. Taking $A = (1+\varepsilon)$, it is then easy to get that with probability one,

$$\left|\rho^{-(m)}(v\tau(m)) - v\right| \le \varepsilon t \tag{13}$$

for all m large enough and all $0 \le v \le t$. Coming back to the integral (11), we see that for m large enough, it is smaller than

$$\int_{[0,t]\times l_1^\downarrow}\left(\left(\varepsilon\left(t+\sum_{i\ge1}s_i\right)\right)\wedge 1\wedge\sum_{i\ge1}s_i\right)\mathrm{d}v\otimes\tau(m)\nu_m(\mathrm{d}\mathbf{s}),$$

which converges to $t\int_{l_1^\downarrow}((\varepsilon(t+\sum_{i\ge1}s_i))\wedge 1\wedge\sum_{i\ge1}s_i)I(\mathrm{d}\mathbf{s})$ as $m\to\infty$. At last, letting $\varepsilon\to0$, we see that the integral (11) converges to 0 as $m\to\infty$, which implies that a.s. for all functions $f\in\mathcal{F}_{[0,\infty)\times l_1^\downarrow\times l_1^\downarrow}$, $I_{m,t}(f)$ converges to $I_t(f)$.

(b) To finish this first part of the proof, fix $f\in\mathcal{F}_{[0,\infty)\times l_1^\downarrow\times l_1^\downarrow}$ and note that

$$E\left[\exp(-\langle\mu_m^{(t)},f\rangle)(\exp(I_{m,t}(f))-\exp(I_t(f)))\right]\to0$$

by dominated convergence, since for all m, $I_{m,t}(f)\le t\int_{l_1^\downarrow}((2\sum_{i\ge1}s_i)\wedge1)\times\tau(m)\nu_m(\mathrm{d}\mathbf{s})$, which is deterministic and converges to $t\int_{l_1^\downarrow}((2\sum_{i\ge1}s_i)\wedge1)I(\mathrm{d}\mathbf{s})$ as $m\to\infty$. On the other hand, by (10),

$$\begin{aligned}&E\left[\exp(-\langle\mu_m^{(t)},f\rangle)(\exp(I_{m,t}(f))-\exp(I_t(f)))\right]\\&\quad=1-E\left[\exp(-\langle\mu_m^{(t)},f\rangle)\right]\exp(I_t(f)),\end{aligned}$$

and therefore, $E[\exp(-\langle\mu_m^{(t)},f\rangle)]\to\exp(-I_t(f))$, which, applying Campbell formula (see e.g. [23]) to the Poisson Point Process $(r_i,\mathbf{u}^i)_{i\ge1}$, is equal to $E[\exp(-\langle\mu^{(t)},f\rangle)]$.

(2) In particular, $E[\exp(-\langle\mu_m^{(t_0^f+1)}, f\rangle)]$ converges to $E[\exp(-\langle\mu^{(t_0^f+1)}, f\rangle)] = E[\exp(-\langle\mu, f\rangle)]$. Write then

$$\begin{aligned}
&E[\exp(-\langle\mu_m, f\rangle)]\\
&\quad = E\Big[\exp(-\langle\mu_m^{(t_0^f+1)}, f\rangle)\Big]\\
&\qquad -E\Big[\exp(-\langle\mu_m^{(t_0^f+1)}, f\rangle)\Big(1-\exp\Big(-\sum_{(t_0^f+1)\tau(m)<t_i} f(t_i^m, (\Lambda^{(m)}(t_i^m-)s_j^i)_{j\geq 2}, (ms_j^i)_{j\geq 2})\Big)\Big)\Big]
\end{aligned} \tag{14}$$

and note that the integrand in the second expectation in the right-hand side is null as soon as $\rho^{-(m)}((t_0^f+1)\tau(m)) > t_0^f$. Since $\rho^{-(m)}((t_0^f+1)\tau(m)) \to t_0^f+1$ a.s. (see (13)), this integrand is null for m large enough and we conclude by dominated convergence that the second expectation in the right-hand side of (14) converges to 0. Hence we have that $E[\exp(-\langle\mu_m, f\rangle)] \to E[\exp(-\langle\mu, f\rangle)]$. □

2.1.3 A.s. Convergence of Versions of $(m-F_1^{(m)}, (F_2^{(m)}, \dots))$ to a Version of (σ_I, FI)

Using Skorohod's representation theorem (our set of point measures is Polish, see e.g. Appendix A7 of Kallenberg [22]), one can consider versions μ_m and μ such that $\mu_m \to \mu$ a.s. We will work with these versions in this section, and, to simplify, we will consider the representation of the measure μ (resp. μ_m, $m>0$) that does not put mass on $[0,\infty)\times\{(\mathbf{0},\mathbf{0})\}$. We then call σ^m, $m>0$, a (random) family of permutations such that, a.s. $\forall i\geq 1$,

$$\begin{aligned}
r_i^m &:= t_{\sigma^m(i)}^m \to r_i,\\
\mathbf{u}^{i,m} &:= (\Lambda^{(m)}(t_{\sigma^m(i)}^m-)s_j^{\sigma^m(i)})_{j\geq 2} \to \mathbf{u}^i,\\
\mathbf{z}^{i,m} &:= (ms_j^{\sigma^m(i)})_{j\geq 2} \to \mathbf{u}^i.
\end{aligned} \tag{15}$$

Consider next some i.i.d. family of ν-homogeneous fragmentations issued from $(1,0,\dots)$, say $F^{\mathrm{hom},(i,j)}$, $i,j\geq 1$, and for each pair (i,j), construct from $F^{\mathrm{hom},(i,j)}$ some (τ,ν)- fragmentations $F^{(u_j^{i,m})}$, $m>0$, and $F^{(u_j^i)}$, starting respectively from $u_j^{i,m}$, $m>0$, and u_j^i. Extend the definition of these processes to $t\in\mathbb{R}^{*-}$ by setting $F^{(u_j^{i,m})}(t) = F^{(u_j^i)}(t) := \mathbf{0}$. Then for $t\geq 0$, let

$$F^{(i,j),m}(t) := F^{(u_j^{i,m})}(t-r_i^m) \tag{16}$$

and set

$$\overline{\Lambda}^{(m)}(t) := m\prod_{r_i^m\leq t}\Big(1-m^{-1}\sum_{j\geq 1} z_j^{i,m}\Big). \tag{17}$$

The process $\overline{\Lambda}^{(m)}$ is distributed as $\Lambda^{(m)}$, since $\sum_{j\geq 1} s^i_j = 1$ ν-a.e. for all $i \geq 1$ (but note that the law of $(\overline{\Lambda}^{(m)}, m > 0)$ is a priori not the same as that of $(\Lambda^{(m)}, m > 0)$, since the distribution of $(\mu_m, m > 0)$ has been transformed when considering versions of μ_m that converge a.s. to μ). The point is then that the process $\overline{F}^{(m)}$ obtained by considering for each $t \geq 0$ the decreasing rearrangement of the terms $\overline{\Lambda}^{(m)}(t)$, $F^{(i,j),m}_k(t)$, $i, j, k \geq 1$, is distributed as $F^{(m)}$. Furthermore, if $t_0 < \infty$ is fixed, then a.s. for m large enough and all $0 \leq t \leq t_0$, $\overline{F}^{(m)}_1(t) = \overline{\Lambda}^{(m)}(t)$ and

$$L^{(m)}(t) := (\overline{F}^{(m)}_2(t), \overline{F}^{(m)}_3(t), \ldots) = \{F^{(i,j),m}_k(t), i, j, k \geq 1\}^{\downarrow}, \tag{18}$$

as noticed in the heuristic description.

Define similarly

$$F^{(i,j),I}(t) := F^{(u^i_j)}(t - r_i), \quad t \geq 0.$$

The process of the decreasing rearrangements of terms of $F^{(i,j),I}(t)$, $i, j \geq 1$, which we still denote by FI, is a (τ, ν, I)-fragmentation with immigration starting from $\mathbf{0}$. Also, we still call $\sigma_I(t) := \sum_{r_i \leq t, j \geq 1} u^i_j$, $t \geq 0$. Theorem 5 is thus a direct consequence of the following convergence:

Lemma 11 $(m - \overline{\Lambda}^{(m)}, L^{(m)}) \overset{a.s.}{\to} (\sigma_I, FI)$ *as* $m \to \infty$.

To prove this convergence, we shall prove that a.s. for all $t \geq 0$, the following assertions hold whenever $m_n \to \infty$, and $t_n \to t$, $t_n \geq 0$:

(A_a) if t is not a jump time of (σ_I, FI), then $(m_n - \overline{\Lambda}^{(m_n)}(t_n), L^{(m_n)}(t_n)) \to (\sigma_I(t), FI(t))$

(A_b) if t is a jump time of (σ_I, FI), there exist two increasing integer-valued sequences (one of them may be finite) $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ such that $\mathbb{N} = \{\varphi_n, \psi_n, n \geq 1\}$ and

(i) if $\boldsymbol{\varphi}$ is infinite, then for all sequences $s_{\varphi_n} \to t$ s.t. $s_{\varphi_n} \geq t_{\varphi_n}$,

$$(m_{\varphi_n} - \overline{\Lambda}^{(m_{\varphi_n})}(s_{\varphi_n}), L^{(m_{\varphi_n})}(s_{\varphi_n})) \to (\sigma_I(t), FI(t))$$

(ii) if $\boldsymbol{\psi}$ is infinite, then for all sequences $s_{\psi_n} \to t$ s.t. $s_{\psi_n} \leq t_{\psi_n}$,

$$(m_{\psi_n} - \overline{\Lambda}^{(m_{\psi_n})}(s_{\psi_n}), L^{(m_{\psi_n})}(s_{\psi_n})) \to (\sigma_I(t-), FI(t-)).$$

According to Lemma 9, this is sufficient to conclude that $(m - \overline{\Lambda}^{(m)}, L^{(m)}) \overset{\text{a.s.}}{\to} (\sigma_I, FI)$ with respect to the Skorohod topology. In order to prove these assertions, we will first show two preliminary lemmas.

Lemma 12 *Let* F^{hom} *be a* ν*-homogeneous fragmentation starting from* $(1, 0, \ldots)$ *and fix some* $a \geq 0$. *Then the following statement holds a.s. for all sequences* $a_n \to a$, $a_n \geq 0$. *Consider* $F^{(a_n)}$ *and* $F^{(a)}$ *the* (τ, ν)*-fragmentations constructed from* F^{hom} *starting respectively from* a_n, $n \geq 0$, *and* a, *and extend these processes to* $t \in \mathbb{R}^{*-}$ *by setting* $F^{(a_n)}(t) = F^{(a)}(t) = \mathbf{0}$. *Then, whenever* $v_n \to v$, $v_n, v \in \mathbb{R}$, *one has,*

(a) *if* v *is not a jump time of* $F^{(a)}$, $F^{(a_n)}(v_n) \to F^{(a)}(v)$
(b) *if* v *is a jump time of* $F^{(a)}$ *there exist two increasing sequences* $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ *such that* $\mathbb{N} = \{\varphi_n, \psi_n, n \geq 1\}$ *and*
 (i) *if* $\boldsymbol{\varphi}$ *is infinite and if* $w_{\varphi_n} \to v$, $w_{\varphi_n} \geq v_{\varphi_n}$, *then* $F^{(a_{\varphi_n})}(w_{\varphi_n}) \to F^{(a)}(v)$
 (ii) *if* $\boldsymbol{\psi}$ *is infinite and if* $w_{\psi_n} \to v$, $w_{\psi_n} \leq v_{\psi_n}$,*then* $F^{(a_{\psi_n})}(w_{\psi_n}) \to F^{(a)}(v-)$.

In particular, when $v = 0$, $\boldsymbol{\varphi}$ *is the increasing rearrangement of* $\{k : v_k \geq 0\}$ *and* $\boldsymbol{\psi}$ *is that of* $\{k : v_k < 0\}$.

This implies that a.s. for all sequences $a_n \to a$, $a_n \geq 0$, $F^{(a_n)} \to F^{(a)}$ with respect to the Skorohod topology.

Proof Note that the statement is obvious when $a = 0$, since $\sup_v \sum_{k\geq 1} F_k^{(a_n)}(v) \leq a_n \to 0$. It is also obvious when $v_n \to v < 0$, since $F^{(a_n)}(v_n) = F^{(a)}(v) = 0$ for large n. So we suppose in the following that $a, a_n > 0$, and $v \geq 0$.

To start with, we point out two convergence results when $w_n \to v$, $w_n \geq 0$. The notations I^{hom}, T_x^m, $m \geq 0$, $x \in (0,1)$, were introduced in Sect. 1.1.1. First, we claim that $T_x^{a_n}(w_n) \to T_x^a(v) \in [0,\infty]$, provided $|I_x^{\mathrm{hom}}(r)| > 0$ for all $r \geq 0$, which occurs a.s. for a.e. $x \in (0,1)$, since $\nu(\sum_{j\geq 1} s_j < 1) = 0$ (in the rest of this proof, we omit the "a.s."). Indeed, consider such x. If there were a subsequence $(k_n)_{n\geq 0}$ s.t. $\lim_{n\to\infty} T_x^{a_{k_n}}(w_{k_n}) > T_x^a(v)$ with $T_x^a(v) < \infty$ (the limit may be infinite), then one would have

$$w_{k_n} \geq \int_0^{T_x^{a_{k_n}}(w_{k_n})} \mathrm{d}r/\tau(a_{k_n}|I_x^{\mathrm{hom}}(r)|) > \int_0^{T_x^a(v)+\varepsilon} \mathrm{d}r/\tau(a_{k_n}|I_x^{\mathrm{hom}}(r)|)$$

for some $\varepsilon > 0$ and all n large enough. The latter integral would then converge to $\int_0^{T_x^a(v)+\varepsilon} \mathrm{d}r/\tau(a|I_x^{\mathrm{hom}}(r)|) > v$, by dominated convergence (note that under our assumptions, for n_0 large enough, the set $\{a_{k_n}|I_x^{\mathrm{hom}}(r)|, r \leq T_x^a(v)+\varepsilon, n \geq n_0\}$ belongs to some compact of $(0,\infty)$). This would lead to $\liminf_{n\to\infty} w_{k_n} > v$, which is impossible. Similarly, it is not possible that $T_x^{a_{k_n}}(w_{k_n}) \to b < T_x^a(v)$. Hence

$$T_x^{a_n}(w_n) \to T_x^a(v) \quad \text{for a.e. } x \in (0,1). \tag{19}$$

Next, we want to check that

$$M^{(a_n)}(w_n) \to M^{(a)}(v) \quad \text{as } n \to \infty. \tag{20}$$

If there is no loss of mass, this is trivial, since $M^{(a_n)}(w_n) = a_n$ and $M^{(a)}(v) = a$. If there is loss of mass, write $M^{(a)}(v) = a\int_0^1 \mathbf{1}_{\{T_x^a(v)<\infty\}}\mathrm{d}x$ and note that $\int_0^1 \mathbf{1}_{\{T_x^a(v)=\infty; T_x^a(v-)<\infty\}}\mathrm{d}x = 0$ (by definition $\{T_x^a(v-) < \infty\} = \{\int_0^\infty \mathrm{d}r/\tau(a|I_x^{\mathrm{hom}}(r)|) \geq v\}$), since the function $v \mapsto M^{(a)}(v)$ is (a.s.) continuous (Proposition 2). Then write

$$M^{(a_n)}(w_n) = a_n \int_0^1 \mathbf{1}_{\{T_x^{a_n}(w_n)<\infty; T_x^a(v)<\infty\}}\mathrm{d}x + a_n \int_0^1 \mathbf{1}_{\{T_x^{a_n}(w_n)<\infty; T_x^a(v)=\infty\}}\mathrm{d}x.$$

By (19), the first term in the right hand side converges to $M^{(a)}(v)$ as $n\to\infty$. It remains to check that the integral $\int_0^1 \mathbf{1}_{\{T_x^{a_n}(w_n)<\infty; T_x^a(v)=\infty\}}\mathrm{d}x$ converges to 0. To see this, we use that for all $x\in(0,1)$ s.t. $|I_x^{\mathrm{hom}}(r)|>0$ for all $r\geq 0$, one has

$$\limsup_{n\to\infty} \mathbf{1}_{\{T_x^{a_n}(w_n)<\infty\}} = \limsup_{n\to\infty} \mathbf{1}_{\{\int_0^\infty \mathrm{d}r/\tau(a_n|I_x^{\mathrm{hom}}(r)|)>w_n\}} \leq \mathbf{1}_{\{\int_0^\infty \mathrm{d}r/\tau(a|I_x^{\mathrm{hom}}(r)|)\geq v\}}. \tag{21}$$

The latter inequality is obtained by splitting the integrals into two $\int_0^\infty=\int_0^M+\int_M^\infty$, where M is such that $b\mapsto\tau(b|I_x^{\mathrm{hom}}(r)|)$ is monotone in a neighborhood of a, $\forall r\geq M$ (we recall that τ is monotone near 0). Then $\int_0^M \mathrm{d}r/\tau(a_n|I_x^{\mathrm{hom}}(r)|)$ converges to $\int_0^M \mathrm{d}r/\tau(a|I_x^{\mathrm{hom}}(r)|)$ by dominated convergence and $\int_M^\infty \mathrm{d}r/\tau(a_n|I_x^{\mathrm{hom}}(r)|)$ converges to $\int_M^\infty \mathrm{d}r/\tau(a|I_x^{\mathrm{hom}}(r)|)$ by monotone convergence (consider monotone subsequences of $(a_n)_{n\geq 1}$; also note that all integrals involved here are finite, as the $F^{(a_n)}, n\geq 1$, and $F^{(a)}$ lose mass to dust). Hence (21). Then, by dominatedconvergence, $\limsup_{n\to\infty}\int_0^1 \mathbf{1}_{\{T_x^{a_n}(w_n)<\infty; T_x^a(v)=\infty\}}\mathrm{d}x$ is smaller than $\int_0^1 \mathbf{1}_{\{T_x^a(v-)<\infty; T_x^a(v)=\infty\}}\mathrm{d}x$, which is equal to 0.

We are now ready to prove assertion (a). Suppose that v is not a jump time of $F^{(a)}$. Then for all $x\in(0,1)$, $s\mapsto|I_x^{\mathrm{hom}}(s)|$ is continuous at $T_x^a(v)$ (even when $T_x^a(v)=\infty$, see the construction of the process, Sect. 1.1.1) and since v is necessarily strictly positive, we may suppose that $v_n\geq 0$ and then apply (19). Therefore, $F_k^{(a_n)}(v_n)\to F_k^{(a)}(v)$ for all $k\geq 1$. On the other hand, $M^{(a_n)}(v_n)\to M^{(a)}(v)$ by (20). Hence $F^{(a_n)}(v_n)\to F^{(a)}(v)$.

We now turn to (b) and first suppose that $v>0$. That v is a jump time of $F^{(a)}$ means that there exists a *unique* interval component of its interval representation that splits at time v. More precisely, it means that there exists a unique interval component, say $I_{v,a}^{\mathrm{hom}}$, of the interval representation of F^{hom} that splits at some time $T^a(v)$ such that $T^a(v)=T_x^a(v)$ for all $x\in I_{v,a}^{\mathrm{hom}}$. Moreover, for all $s<T^a(v)$ and all $x,y\in I_{v,a}^{\mathrm{hom}}$, $I_x^{\mathrm{hom}}(s)=I_y^{\mathrm{hom}}(s)$, which implies that for all $b,u>0$ and $x\in I_{v,a}^{\mathrm{hom}}$, $T_x^b(u)<T^a(v)\Rightarrow T_y^b(u)=T_x^b(u)\ \forall y\in I_{v,a}^{\mathrm{hom}}$. This allows us to introduce the increasing sequence $\boldsymbol{\psi}$, *independent of* $x\in I_{v,a}^{\mathrm{hom}}$, of all integers k s.t. $T_x^{a_k}(v_k)<T_x^a(v)$ for some (hence all) $x\in I_{v,a}^{\mathrm{hom}}$. The increasing sequence $\boldsymbol{\varphi}$ is then that of all integers k s.t. $T_x^{a_k}(v_k)\geq T_x^a(v)$ for some (all) $x\in I_{v,a}$.

Suppose then that $\boldsymbol{\varphi}$ is infinite and let $w_{\varphi_n}\to v$, $w_{\varphi_n}\geq v_{\varphi_n}$. On the one hand, $T_x^{a_{\varphi_n}}(w_{\varphi_n})\geq T_x^a(v)$, $n\geq 1$, and the functions $s\mapsto|I_x^{\mathrm{hom}}(s)|$ are right-continuous at $T_x^a(v)$ when $x\in I_{v,a}^{\mathrm{hom}}$. On the other hand, the functions $s\mapsto|I_x^{\mathrm{hom}}(s)|$ are continuous at $T_x^a(v)$ when $x\notin I_{v,a}^{\mathrm{hom}}$. Therefore, the convergences (19) imply that $F_k^{(a_{\varphi_n})}(w_{\varphi_n})$ converges to $F_k^{(a)}(v)$, $\forall k\geq 1$. Moreover, $M^{a_{\varphi_n}}(w_{\varphi_n})$ converges to $M^a(v)$ by (20) and then, $F^{(a_{\varphi_n})}(w_{\varphi_n})$ converges to $F^{(a)}(v)$. Hence (b)(i).

Suppose next that $\boldsymbol{\psi}$ is infinite and let $w_{\psi_n}\to v$, $w_{\psi_n}\leq v_{\psi_n}$. One has $T_x^{a_{\psi_n}}(w_{\psi_n})<T_x^a(v)$ for all $x\in I_{v,a}^{\mathrm{hom}}$, $n\geq 1$. By (19), this implies that $F_k^{(a_{\psi_n})}(w_{\psi_n})$ converges to

$F_k^{(a)}(v-)$, $\forall k \geq 1$. Then, using (20), we get that $F^{(a_{\psi_n})}(w_{\psi_n})$ converges to $F^{(a)}(v-)$. Hence (b)(ii).

Last, it remains to prove (b) when $v=0$. Let here $\boldsymbol{\varphi}$ be the increasing rearrangement of $\{k : v_k \geq 0\}$ and $\boldsymbol{\psi}$ the increasing rearrangement of $\{k : v_k < 0\}$. If $\boldsymbol{\varphi}$ is infinite, let $w_{\varphi_n} \to 0$, $w_{\varphi_n} \geq v_{\varphi_n}$. Then $w_{\varphi_n} \geq 0$, and so, according to (19) and (20), $F^{(a_{\varphi_n})}(w_{\varphi_n})$ converges to $F^{(a)}(0)$. If $\boldsymbol{\psi}$ is infinite, let $w_{\psi_n} \to 0$, $w_{\psi_n} \leq v_{\psi_n}$. Then $F^{(a_{\psi_n})}(w_{\psi_n}) = 0 = F^{(a)}(0-)$. □

The results and convergences stated in the rest of this subsection hold, simultaneously, a.s. for all $t \geq 0$, and all sequences $m_n \to \infty$ and $t_n \to t$, $t_n \geq 0$. Therefore we drop the "a.s." from notations. Also, we fix $t \geq 0$, as well as sequences $m_n \to \infty$ and $t_n \to t$, $t_n \geq 0$.

Lemma 13 (i) *If $t \notin \{r_i, i \geq 1\}$, then $m_n - \overline{\Lambda}^{(m_n)}(t_n) \to \sigma_I(t)$.*

(ii) *If $t = r_{i_0}$ for some i_0, then $m_n - \overline{\Lambda}^{(m_n)}(t_n)$ converges to $\sigma_I(t)$ when $t_n \geq r_{i_0}^{m_n}$ for large n's and it converges to $\sigma_I(t-)$ when $t_n < r_{i_0}^{m_n}$ for large n's.*

Proof Recall that $\mu_m \to \mu$ and set $Z^{i,m_n} := \sum_{j\geq 1} z_j^{i,m_n}$, $U^i := \sum_{j \geq 1} u_j^i$.

(i) Take $t' > t$ s.t. $t' \notin \{r_i, i \geq 1\}$ and fix $0 < \eta < 1/2$. One has $\sum_{i>k, r_i \leq t'} U^i < \eta$ for some k large enough and then $\sum_{i>k, r_i^{m_n} \leq t'} Z^{i,m_n} < \eta$ for all n large enough. In particular, all components of these sums are then smaller than η. Taking n larger if necessary (so that $m_n \geq 1$) one gets that for all $i > k$, $m_n^{-1} Z^{i,m_n} \mathbf{1}_{\{r_i^{m_n} \leq t'\}} < \eta < 1/2$, which implies (using $|\ln(1-x)| \leq 2x$ for $0 < x \leq 1/2$) that

$$\left| m_n \ln(1 - m_n^{-1} Z^{i,m_n}) \mathbf{1}_{\{r_i^{m_n} \leq t'\}} \right| \leq 2 Z^{i,m_n} \mathbf{1}_{\{r_i^{m_n} \leq t'\}} \leq 2\eta.$$

When moreover $t_n \leq t'$,

$$\begin{aligned} &\left| \sum_{i \geq 1} (U^i \mathbf{1}_{\{r_i \leq t\}} + m_n \ln(1 - m_n^{-1} Z^{i,m_n}) \mathbf{1}_{\{r_i^{m_n} \leq t_n\}}) \right| \\ &\quad \leq \left| \sum_{i \leq k} (U^i \mathbf{1}_{\{r_i \leq t\}} + m_n \ln(1 - m_n^{-1} Z^{i,m_n}) \mathbf{1}_{\{r_i^{m_n} \leq t_n\}}) \right| \\ &\qquad + \sum_{i>k} (U^i \mathbf{1}_{\{r_i \leq t'\}} + 2 Z^{i,m_n} \mathbf{1}_{\{r_i^{m_n} \leq t'\}}) \\ &\quad \leq \left| \sum_{i \leq k} (U^i \mathbf{1}_{\{r_i \leq t\}} + m_n \ln(1 - m_n^{-1} Z^{i,m_n}) \mathbf{1}_{\{r_i^{m_n} \leq t_n\}}) \right| + 3\eta. \end{aligned} \tag{22}$$

On the other hand, since $t \notin \{r_i, i \geq 1\}$, $Z^{i,m_n} \mathbf{1}_{\{r_i^{m_n} \leq t_n\}} \to U^i \mathbf{1}_{\{r_i \leq t\}}$ for all $i \geq 1$, or equivalently,

$$-m_n \ln(1 - m_n^{-1} Z^{i,m_n}) \mathbf{1}_{\{r_i^{m_n} \leq t_n\}} \to U^i \mathbf{1}_{\{r_i \leq t\}}.$$

Hence the upper bound of (22) is bounded from above by 4η for n large enough. Therefore $-m_n \sum_{r_i^{m_n} \leq t_n} \ln(1 - m_n^{-1} Z^{i,m_n})$ converges to $\sigma_I(t)$ $(= \sum_{r_i \leq t} U^i)$, which implies that

$$m_n\Bigg(1-\prod_{r_i^{m_n}\le t_n}\big(1-m_n^{-1}Z^{i,m_n}\big)\Bigg)\to\sigma_I(t).$$

(ii) If $t=r_{i_0}$ and $t_n\ge r_{i_0}^{m_n}$ for n large enough, then, for all $i\ge 1$, $Z^{i,m_n}\mathbf{1}_{\{r_i^{m_n}\le t_n\}}$ converges to $U^i\mathbf{1}_{\{r_i\le t\}}$ and one concludes exactly as above. Now, if $t_n<r_{i_0}^{m_n}$ for large n's, $Z^{i_0,m_n}\mathbf{1}_{\{r_{i_0}^{m_n}\le t_n\}}$ converges to $U^{i_0}\mathbf{1}_{\{r_{i_0}<t\}}$ and still, $Z^{i,m_n}\mathbf{1}_{\{r_i^{m_n}\le t_n\}}$ converges to $U^i\mathbf{1}_{\{r_i\le t\}}$ for $i\ne i_0$. The conclusion then follows by replacing $\mathbf{1}_{\{r_{i_0}\le t\}}$ by $\mathbf{1}_{\{r_{i_0}<t\}}$ in the proof above. □

We are ready to prove assertions (A_a) and (A_b).

Proof of Lemma 11 The proof is split into two parts, according to whether t is, or not, a jump time of FI. It strongly relies on the convergence $\mu_m\to\mu$.

(1) If t is not a jump time of FI, then $t-r_i$ is not a jump time of $F^{(u_j^i)}$, $\forall i,j\ge 1$ (in particular, $t\notin\{r_i,i\ge 1\}$). When $t-r_i>0$, applying Lemma 12(a) to the sequences $a_n=u_j^{i,m_n}$, $a=u_j^i$, $v_n=t_n-r_i^{m_n}$ and $v=t-r_i$, one gets that $F^{(i,j),m_n}(t_n)\to F^{(i,j),I}(t)$, $\forall j\ge 1$. Clearly, such convergence also holds when $t<r_i$, since then $F^{(i,j),m_n}(t_n)=\mathbf{0}$ for n large enough. Fix next some $\eta,\varepsilon>0$ and then some k s.t. $\sum_{i+j>k}u_j^i\mathbf{1}_{\{r_i\le t+\varepsilon\}}<\eta$. For n large enough, $\sum_{i+j>k}u_j^{i,m_n}\mathbf{1}_{\{r_i^{m_n}\le t_n\}}<\eta$, and therefore

$$\sum_{i,j\ge 1}d(F^{(i,j),m_n}(t_n),F^{(i,j),I}(t))\le\sum_{i+j\le k}d(F^{(i,j),m_n}(t_n),F^{(i,j),I}(t))+2\eta.$$

So, the right hand side of this inequality is smaller than 3η for n large enough, i.e. $\sum_{i,j\ge 1}d(F^{(i,j),m_n}(t_n),F^{(i,j),I}(t))\to 0$. Then, by Lemma 4, one concludes that $L^{(m_n)}(t_n)\to FI(t)$. On the other hand, Lemma 13(i) implies that $m_n-\overline{\Lambda}^{(m_n)}(t_n)\to\sigma_I(t)$. Hence we have assertion (A_a).

(2) Now assume that t is a jump time of FI. Our goal is to construct some increasing sequences $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$, $\mathbb{N}=\{\varphi_n,\psi_n,n\ge 1\}$, such that assertions (A_b)(i) and (A_b)(ii) hold. For all $i,j\ge 1$, let $\mathcal{J}^{(i,j)}$ denote the set of *strictly positive* jump times of $F^{(u_j^i)}$. Such process $F^{(u_j^i)}$ only jumps when a fragment splits, since its total mass is continuous (Proposition 2). Then, since the $F^{\text{hom},(i,j)}$ are constructed from independent Poisson point processes, independent of $((r_i,\mathbf{u}^i),i\ge 1)$, (a.s.) the $\mathcal{J}^{(i,j)}$'s are pairwise disjoint and disjoint from $\{r_i,i\ge 1\}$. Also, every $F^{(u_j^i)}$ jumps at 0 (we recall that these processes are defined on $\mathbb{R}$) and therefore the set of jump times of $F^{(u_j^i)}$ is $\mathcal{J}^{(i,j)}\cup\{0\}$, $i,j\ge 1$. So, if t is a jump time of FI:

- either $t-r_{i_0}\in\mathcal{J}^{(i_0,j_0)}$ for some (unique) pair (i_0,j_0). Then, one can apply Lemma 12(b) to $a_n=u_{j_0}^{i_0,m_n}$, $a=u_{j_0}^{i_0}$, $v_n=t_n-r_{i_0}^{m_n}$ and $v=t-r_{i_0}$. Let $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ be the sequences that appear in this statement and first, suppose that $\boldsymbol{\varphi}$ is infinite. Consider then some sequence $s_{\varphi_n}\to t$, $s_{\varphi_n}\ge t_{\varphi_n}$, and apply Lemma 12(b)(i) to $w_{\varphi_n}=s_{\varphi_n}-r_{i_0}^{m_{\varphi_n}}$. One obtains that $F^{(i_0,j_0),m_{\varphi_n}}(s_{\varphi_n})$ converges to $F^{(i_0,j_0),I}(t)$. On the other hand, by Lemma 12(a), $F^{(i,j),m_{\varphi_n}}(s_{\varphi_n})$ converges to $F^{(i,j),I}(t)$ for all $(i,j)\neq(i_0,j_0)$ since $t-r_i\notin\mathcal{J}^{(i,j)}\cup\{0\}$. Hence, as in (1), we get that $\sum_{i,j\geq 1} d(F^{(i,j),m_{\varphi_n}}(s_{\varphi_n}), F^{(i,j),I}(t))$ tends to 0, and then, by Lemma 4, that $L^{m_{\varphi_n}}(s_{\varphi_n})$ converges to $FI(t)$. Moreover, $m_{\varphi_n}-\overline{\Lambda}^{(m_{\varphi_n})}(s_{\varphi_n})$ converges to $\sigma_I(t)$, by Lemma 13(i) since $t\notin\{r_i, i\geq 1\}$. Hence (A_b)(i).

Similarly, supposing that $\boldsymbol{\psi}$ is infinite and $s_{\psi_n}\to t$, $s_{\psi_n}\leq t_{\psi_n}$, one gets, by applying Lemma 12(b)(ii), that $L^{m_{\psi_n}}(s_{\psi_n})\to FI(t-)$. Moreover, $m_{\psi_n}-\overline{\Lambda}^{(m_{\psi_n})}(s_{\psi_n})$ converges to $\sigma_I(t)=\sigma_I(t-)$, still by Lemma 13(i) since $t\notin\{r_i, i\geq 1\}$. Hence (A_b)(ii) and then (A_b).

• or $t\in\{r_i, i\geq 1\}$, say $t=r_{i_0}$. For $i\neq i_0$, $t-r_i\notin\mathcal{J}^{(i,j)}\cup\{0\}$ and therefore, as explain above, $F^{(i,j),m_n}(s_n)$ converges to $F^{(i,j),I}(t)=F^{(i,j),I}(t-)$, for all sequences $s_n\to t$. Let then $\boldsymbol{\varphi}$ be the increasing sequence of integers k such that $t_k\geq r_{i_0}^{m_k}$ and $\boldsymbol{\psi}$ be the increasing sequence of integers k such that $t_k<r_{i_0}^{m_k}$. When $\boldsymbol{\varphi}$ is infinite and $s_{\varphi_n}\to t=r_{i_0}$, $s_{\varphi_n}\geq t_{\varphi_n}$, one has, by Lemma 12(b)(i), that $F^{(i_0,j),m_{\varphi_n}}(s_{\varphi_n})$ converges to $F^{(i_0,j),I}(t)$, $\forall j\geq 1$. Together with the fact that $F^{(i,j),m_{\varphi_n}}(s_{\varphi_n})$ converges to $F^{(i,j),I}(t)$ for $i\neq i_0$, $j\geq 1$, we obtain, as in (1), that $L^{(m_{\varphi_n})}(s_{\varphi_n})$ converges to $FI(t)$. On the other hand, $m_{\varphi_n}-\overline{\Lambda}^{(m_{\varphi_n})}(s_{\varphi_n})$ converges to $\sigma_I(t)$, by Lemma 13(ii). Hence assertion (A_b)(i). Now, if $\boldsymbol{\psi}$ is infinite, let $s_{\psi_n}\to t$, $s_{\psi_n}\leq t_{\psi_n}$. Clearly, $F^{(i_0,j),m_{\psi_n}}(s_{\psi_n})=0=F^{(i_0,j),I}(t-)$, $\forall j,n\geq 1$. Moreover $F^{(i,j),m_{\psi_n}}(s_{\psi_n})$ converges to $F^{(i,j),I}(t-)$ for $i\neq i_0$, $j\geq 1$, and therefore $L^{(m_{\psi_n})}(s_{\psi_n})$ tends to $FI(t-)$. At last, $m_{\psi_n}-\overline{\Lambda}^{(m_{\psi_n})}(s_{\psi_n})$ converges to $\sigma_I(t-)$ by Lemma 13(ii). Hence assertion (A_b)(ii). □

Remark The convergence in law of $m-\Lambda^{(m)}$ (and a fortiori of $m-F_1^{(m)}$) to some γ-stable subordinator σ, $\gamma\in(0,1)$, actually holds as soon as φ_ν varies regularly at ∞ with index $-\gamma$ and $\tau(m)\sim C\varphi_\nu(m)$, $C>0$. Very roughly, the point is

- either to check that the regular variation assumptions imply that the measures $\sum_{i\geq 1}\delta_{(r_i^m,\sum_{j\geq 1}u_j^{i,m})}$ converge in distribution to some Poisson point measure $\sum_{i\geq 1}\delta_{(r_i,x^i)}$ on $[0,\infty)\times\mathbb{R}^+$, where $((r_i,x^i), i\geq 1)$ is a PPP with intensity $C'x^{-\gamma-1}\mathrm{d}x$, $x>0$. This will lead to some result identical to Lemma 13 (replacing there σ_I by σ)
- or to use classical results on convergence of subordinators (see e.g. [21]) and, again, regular variation theorems.

2.2 Proof of Theorem 7

We still use the notations $\Lambda^{(m)}$, $((t_i^m,\mathbf{s}^i), i\geq 1)$ and $((r_i,\mathbf{u}^i, i\geq 1)$ introduced in the previous subsection, and we suppose that τ varies regularly at ∞, and that $\varphi_\nu(m)\nu_m\to I$ as $m\to\infty$.

Lemma 14 *Define*

$$\widetilde{\mu}_m:=\sum_{i\geq 1}\delta_{(t_i^m(\tau/\varphi_\nu)(m),(\Lambda^{(m)}(t_i^m-)s_j^i)_{j\geq 2},(ms_j^i)_{j\geq 2})},\quad m>0.$$

Then, as $m \to \infty$,

$$\widetilde{\mu}_m \overset{\text{law}}{\to} \mu = \sum_{i \geq 1} \delta_{(r_i, \mathbf{u}^i, \mathbf{u}^i)}.$$

Proof As in the proof of Lemma 10, the goal is to prove the convergence of Laplace transforms $E[\exp(-\langle \widetilde{\mu}_m, f\rangle)]$ to $E[\exp(-\langle \mu, f\rangle)]$, for all $f \in \mathcal{F}_{[0,\infty)\times l_1^{\downarrow}\times l_1^{\downarrow}}$. The argument is very similar to that of Lemma 10, so we leave it to the reader. Roughly, the main changes consist in replacing there $\tau(m)$ by $\varphi_\nu(m)$ and $\rho^{-(m)}(t)$ by $\rho^{-(m)}(t)(\tau/\varphi_\nu)(m)$. As an example, the first stage consists in proving the convergences of $E[\exp(-\langle \widetilde{\mu}_m^{(t)}, f\rangle)]$ to $E[\exp(-\langle \mu^{(t)}, f\rangle)]$, $\forall t \geq 0$, where

$$\widetilde{\mu}_m^{(t)} := \sum_{t_i \leq t\varphi_\nu(m)} \delta_{(t_i^m(\tau/\varphi_\nu)(m), (\Lambda^{(m)}(t_i^m -) s_j^i)_{j\geq 2}, (m s_j^i)_{j \geq 2})},$$

$$\mu^{(t)} = \sum_{r_i \leq t} \delta_{(r_i, \mathbf{u}^i, \mathbf{u}^i)}.$$

The main tools are the convergence of $\varphi_\nu(m)\nu_m$ to I and the regular variation of τ, which still gives the Potter's bounds (12). □

In the rest of this section, we consider versions of $\widetilde{\mu}_m$, μ such that $\widetilde{\mu}_m \to \mu$ a.s. Let then $\widetilde{\sigma}^m$, $m > 0$, be a family of permutations such that, a.s.,

$$\widetilde{r}_i^m(\tau/\varphi_\nu)(m) := t^m_{\widetilde{\sigma}^m(i)}(\tau/\varphi_\nu)(m) \to r_i,$$

$$\widetilde{\mathbf{u}}^{i,m} := (\Lambda^{(m)}(t^m_{\widetilde{\sigma}^m(i)} -) s_j^{\widetilde{\sigma}^m(i)})_{j \geq 2} \to \mathbf{u}^i,$$

$$\widetilde{\mathbf{z}}^{i,m} := (m s_j^{\widetilde{\sigma}^m(i)})_{j\geq 2} \to \mathbf{u}^i.$$

Define next $\widetilde{F}^{(i,j),m}$, $\widetilde{\Lambda}^{(m)}$ and $\widetilde{L}^{(m)}$ from $\widetilde{r}_i^m, \widetilde{\mathbf{u}}^{i,m}, \widetilde{\mathbf{z}}^{i,m}$, $i \geq 1$, $m \geq 0$, by formulas similar to (16), (17) and (18). Also, let $\widetilde{F}^{(m)}$ be obtained by considering for each $t \geq 0$ the decreasing rearrangement of the terms $\widetilde{\Lambda}^{(m)}(t)$, $\widetilde{F}_k^{(i,j),m}(t)$, $i, j, k \geq 1$, and note that, for all $m \geq 0$, $\widetilde{\Lambda}^{(m)} \overset{\text{law}}{=} \Lambda^{(m)}$ and $\widetilde{F}^{(m)} \overset{\text{law}}{=} F^{(m)}$. We should point out that contrary to what happens when $\tau(m)\nu_m$ converges to a non-trivial limit, $\widetilde{\Lambda}^{(m)}$ and $\widetilde{F}_1^{(m)}$ do not necessarily coincide on $[0, t_0]$ for large m's under the assumptions of Theorem 7. However $\widetilde{\Lambda}^{(m)}((\varphi_\nu/\tau)(m)\cdot)$ and $\widetilde{F}_1^{(m)}((\varphi_\nu/\tau)(m)\cdot)$ do coincide on $[0, t_0]$ for large m's and that is all we need for the proof of Theorem 7.

Then, by imitating the proof of Lemma 13, one easily obtains

Lemma 15 *With probability one, for all* $t \geq 0$, *and all sequences* $m_n \to \infty$ *and* $t_n \to t$, $t_n \geq 0$,

(i) *if* $t \notin \{r_i, i \geq 1\}$, *then* $m_n - \widetilde{\Lambda}^{(m_n)}((\varphi_\nu/\tau)(m_n)t_n) \to \sigma_I(t)$,
(ii) *if* $t = r_{i_0}$, *then* $m_n - \widetilde{\Lambda}^{(m_n)}((\varphi_\nu/\tau)(m_n)t_n)$ *converges to* $\sigma_I(t)$ *when* $(\varphi_\nu/\tau)(m_n)t_n \geq \widetilde{r}_{i_0}^{m_n}$ *for n large enough and it converges to* $\sigma_I(t-)$ *when* $(\varphi_\nu/\tau)(m_n)t_n < r_{i_0}^{m_n}$ *for n large enough.*

To prove Theorem 7(ii), we also need the following lemma.

Lemma 16 *Suppose that the parameters* (τ,ν) *are such that* (τ,ν)*-fragmentations lose mass to dust and let* F^{hom} *be a homogeneous* ν*-fragmentation starting from* $(1,0,\dots)$. *Fix then some* $a\geq 0$ *and for all sequences* $a_n\to a$, $a_n\geq 0$, *let* $F^{(a_n)}$, $F^{(a)}$ *be the* (τ,ν)*-fragmentations constructed from* F^{hom}, *starting respectively from* a_n, $n\geq 0$, *and* a. *Then, a.s. for all sequences* $a_n\to a$, $a_n\geq 0$, *and all sequences* $t_n\to\infty$, $F^{(a_n)}(t_n)\to\mathbf{0}$.

Proof As in the proof of Lemma 12 we may suppose that $a>0$. Then, with probability one, since the fragmentation $F^{(a)}$ loses mass, every x falls into the dust after a finite time, i.e. $\int_0^\infty 1/\tau(aI_x^{\mathrm{hom}}(r))\mathrm{d}r<\infty$. It is then easily seen, using dominated convergence and the fact that τ is monotone near 0, that there exists some (random) C such that $\int_0^\infty 1/\tau(a_nI_x^{\mathrm{hom}}(r))\mathrm{d}r\leq C<\infty$ for all n large enough (this has already be detailed in the proof of Lemma 12). Hence $T_x^{a_n}(t_n)=\infty$ for large n's, and therefore, $M^{(a_n)}(t_n)=a_n\int_0^1\mathbf{1}_{\{T_x^{a_n}(t_n)<\infty\}}\mathrm{d}x$ converges to 0. □

Proof of Theorem 7 (i) Let $I(t)$ be the decreasing rearrangement $\{u_j^i, j\geq 1, r_i\leq t\}^{\downarrow}$, $t\geq 0$. Our goal is to show that a.s.

$$\big(m-\widetilde{\Lambda}^{(m)}((\varphi_\nu/\tau)(m)\cdot),\widetilde{L}^{(m)}((\varphi_\nu/\tau)(m)\cdot)\big)\to(\sigma_I,(I(t),t\geq 0)) \tag{23}$$

when $(\varphi_\nu/\tau)(m)\to 0$. Under this assumption, a.s. for all $t\geq 0$ and all sequences $m_n\to\infty$, $t_n\to t$, with $t_n\geq 0$, one has that $(\varphi_\nu/\tau)(m_n)(t_n-\widetilde{r}_i^{m_n}(\tau/\varphi_\nu)(m_n))$ converges to 0, since $\widetilde{r}_i^{m_n}(\tau/\varphi_\nu)(m_n)\to r_i$. Hence $\widetilde{F}^{(i,j),m_n}((\varphi_\nu/\tau)(m_n)t_n)$ converges to u_j^i when $(\varphi_\nu/\tau)(m_n)t_n\geq\widetilde{r}_i^{m_n}$ for large n's and it reaches 0 when $(\varphi_\nu/\tau)(m_n)t_n<\widetilde{r}_i^{m_n}$ for large n's. Recalling Lemma 15, it is then easy to adapt the proof of Lemma 11 to obtain the required convergence (23). Note that the only jump times of the limit process are the r_i's, which makes the proof shorter than that of Lemma 11.

(ii) The following hold a.s. Suppose that $(\varphi_\nu/\tau)(m)\to\infty$ and fix $t\geq 0$. When $t>r_i$, $(\varphi_\nu/\tau)(m)(t-(\tau/\varphi_\nu)(m)\widetilde{r}_i^m)\to\infty$ and then, according to Lemma 16, $\widetilde{F}^{(i,j),m}((\varphi_\nu/\tau)(m)t)\to\mathbf{0}$. When $t<r_i$, $\widetilde{F}^{(i,j),m}((\varphi_\nu/\tau)(m)t)=\mathbf{0}$ for m large enough. From this, we deduce that for all $t\notin\{r_i,i\geq 1\}$, $\widetilde{L}^{(m)}((\varphi_\nu/\tau)(m)t)\to\mathbf{0}$. Furthermore, $m-\widetilde{\Lambda}^{(m)}((\varphi_\nu/\tau)(m)t)\to\sigma_I(t)$ according to Lemma 15. So, if we consider some finite sequence of deterministic times $t_1,\dots,t_k$, we know that (a.s.) these times are not in $\{r_i,i\geq 1\}$, and therefore that the convergences of $(m-\widetilde{\Lambda}^{(m)}((\varphi_\nu/\tau)(m)t_l),\widetilde{L}^{(m)}((\varphi_\nu/\tau)(m)t_l))$ to $(\sigma_I(t_l),\mathbf{0})$, $1\leq l\leq k$, hold simultaneously. Hence the convergence in the finite dimensional sense. □

Let us point out that the convergence of $\widetilde{L}^{(m)}((\varphi_\nu/\tau)(m)\cdot)$ to $\mathbf{0}$ in the Skorohod sense does not hold when $(\varphi_\nu/\tau)(m)\to\infty$. Indeed, consider some i such that $u_1^i>0$ (such i exists since $I(l_1^{\downarrow})\neq 0$) and set $t_m:=(\tau/\varphi_\nu)(m)\widetilde{r}_i^m$, $m\geq 0$. Then $\widetilde{F}^{(i,1),m}((\varphi_\nu/\tau)(m)t_m)$ converges to $u_1^i\neq 0$ and consequently $\widetilde{L}^{(m)}((\varphi_\nu/\tau)(m)t_m)\nrightarrow\mathbf{0}$. Therefore, assertion (a) of Lemma 9 is not satisfied.

3 Small Times Asymptotics in the Self-Similar Cases

We are now looking at the small times asymptotics of $F^{(1)}$ when $\tau(m)=m^{\alpha}$, $\alpha\in\mathbb{R}$. The assumptions on τ and ν are still those made in (H) (to complete our study, remark that when $\nu(l^{\downarrow}_{1,\leq 1})<\infty$, the initial particle waits a positive time before splitting and therefore $F^{(1)}(\varepsilon)\overset{\text{a.s.}}{=}(1,0,\dots)$ for ε small enough). Introduce then the function

$$\varphi_{\nu}^{-1}(\varepsilon):=\inf\{m:\varphi_{\nu}(m)<\varepsilon\},$$

which is well defined in a neighborhood of 0 since $\nu(l^{\downarrow}_{1,\leq 1})=\infty$, and recall that under the assumption $\varphi_{\nu}(m)\nu_m\to I$, the function φ_{ν} is regularly varying at ∞ (with index $-\gamma_{\nu}$). Classical results on regular variation (see [10]) then implies that φ_{ν}^{-1} is also regularly varying (at 0) and $\varphi_{\nu}\circ\varphi_{\nu}^{-1}(\varepsilon)\sim\varepsilon$ when $\varepsilon\to 0$.

In the following, we say that a fragmentation with immigration process with parameters $\tau(m)=m^{\alpha}$, $\ell\nu$, I, where $\ell\in\mathbb{R}^{+}$, is a $(\alpha,\ell\nu,I)$ fragmentation with immigration. By convention, when $\ell=\infty$, a $(\alpha,\ell\nu,I)$ fragmentation with immigration FI is a process constantly null, $FI(t)=\mathbf{0}$, $\forall t\geq 0$, but the subordinator σ_I of its total mass of immigrants is still non-trivial and constructed from the measure I. Roughly, this corresponds to the case where particles immigrate and vanish immediately.

Corollary 17 *Suppose that $\varphi_{\nu}(m)\nu_m\to I$ and $m^{-\alpha}\varphi_{\nu}(m)\to\ell\in[0,\infty]$ as $m\to\infty$, and let FI be a self-similar fragmentation with immigration with parameters $(\alpha,\ell\nu,I)$, starting from $\mathbf{0}$. Then,*

$$\varphi_{\nu}^{-1}(\varepsilon)(1-F_1^{(1)}(\varepsilon\cdot),(F_2^{(1)}(\varepsilon\cdot),F_3^{(1)}(\varepsilon\cdot),\dots))\overset{\text{law}}{\to}(\sigma_I,FI)\quad \text{as } \varepsilon\to 0,\tag{24}$$

where the convergence holds with respect to the Skorohod topology when $\ell<\infty$ and in the finite-dimensional sense when $\ell=\infty$.

Proof Thanks to the self-similarity of F, the convergence (24) is obtained by:

- applying Theorem 5 when $m^{-\alpha}\varphi_{\nu}(m)\to\ell\in(0,\infty)$ to the process $F^{((\varepsilon\ell^{-1})^{1/\alpha})}$, and then using that a fragmentation with immigration process with parameters $(\alpha,\ell\nu,I)$ is distributed as $FI^{\ell^{-1}}(\ell\cdot)$ where $FI^{\ell^{-1}}$ denotes a fragmentation with immigration $(\alpha,\nu,\ell^{-1}I)$
- applying Theorem 7 when $m^{-\alpha}\varphi_{\nu}(m)\to\ell\in\{0,\infty\}$ to the process $F^{(\varphi_{\nu}^{-1}(\varepsilon))}$. □

Remark that the fragmentation with immigration process that arises in this limit is γ_{ν}*-self-similar* (as a consequence of the γ_{ν}-self-similarity of I stated in Lemma 6), i.e.

$$(FI(at),t\geq 0)\overset{\text{law}}{=}(a^{1/\gamma_{\nu}}FI(t),t\geq 0)\quad \text{for all } a>0.$$

Bertoin [9] proves that large times behavior of self-similar fragmentations differs significantly according as $\alpha<0$, $\alpha=0$ or $\alpha>0$. The above corollary shows that the rules are quite different for small times behavior: the convergence rate only depends on ν and then the form of the limit only depends on the position of α with respect to γ_ν. The index $\alpha = -\gamma_\nu$ is the only one for which the limit may be a non-trivial fragmentation with immigration and this occurs if and only if $\varphi_\nu(m)$ behaves as a power function as $m \to \infty$. This suggests that this index is in some sense more natural than the others.

However, the limit is also non-trivial when $\alpha > -\gamma_\nu$. In particular, Corollary 8, in this self-similar setting, says that if ν is binary and if φ_ν varies regularly at ∞ with some index $-\gamma_\nu \in (-1, 0)$, then, as soon as $\alpha > -\gamma_\nu$,

$$\varphi_\nu^{-1}(\varepsilon)(1 - F_1^{(1)}(\varepsilon\cdot), F_2^{(1)}(\varepsilon\cdot), F_3^{(1)}(\varepsilon\cdot), \dots) \overset{\text{law}}{\to} (\sigma, \Delta_1, \Delta_2, \dots) \quad \text{as } \varepsilon \to 0,$$

where σ is a stable subordinator with Laplace exponent $\Gamma(1 - \gamma_\nu)q^{\gamma_\nu}$ and $(\Delta_1(t), \Delta_2(t), \dots)$ the decreasing sequence of its jumps before time t, $t \geq 0$. This completes a result of Berestycki [5] who shows that

$$\varphi_\nu^{-1}(\varepsilon)(F_2^{(1)}(\varepsilon), F_3^{(1)}(\varepsilon), \dots) \overset{\text{law}}{\to} (\Delta_1(1), \Delta_2(1), \dots)$$

when $\alpha \geq 0$, ν is binary and φ_ν varies regularly at ∞. He also investigates the behavior of $F_2^{(1)}(\varepsilon)$ near 0 for all measures ν and $\alpha \geq 0$, and obtains that $F_2^{(1)}(\varepsilon) \sim R(\varepsilon)$ a.s. where R is the record process of a PPP with intensity $\nu(s_2 \in \mathrm{d}x)$.

We also refer to Miermont and Schweinsberg [28] for some specific examples.

Total Mass Behavior In the self-similar setting, the total mass $M^{(1)}(t) = \sum_{i \geq 1} F_i^{(1)}(t)$ of macroscopic particles present at time t is non-constant if and only if $\alpha < 0$. A consequence of Corollary 17 is that the behavior near 0 of the mass $1 - M^{(1)}$ is then specified as follows.

Corollary 18 *Under the assumptions of Corollary 17, as* $\varepsilon \to 0$,

$$\varphi_\nu^{-1}(\varepsilon)(1 - M^{(1)}(\varepsilon\cdot)) \overset{\text{law}}{\to} \sigma_I - M_{FI},$$

where $M_{FI}(t) = \sum_{j \geq 1} FI_j(t)$, $t \geq 0$ (*again, the convergence holds with respect to the Skorohod topology when* $\ell < \infty$ *and in the finite-dimensional sense when* $\ell = \infty$). *In particular, the limit is equal to* σ_I *when* $\ell = \infty$, *is* $\mathbf{0}$ *when* $\ell = 0$, *and is non-trivial when* $0 < \ell < \infty$.

Note that when $\alpha > -\gamma_\nu$, the limit ℓ equals 0 and so the speed of convergence of $1 - M^{(1)}(\varepsilon)$ to 0 is faster than $1/\varphi_\nu^{-1}(\varepsilon)$. When $-\gamma_\nu < \alpha < 0$, one can obtain a lower bound for this speed by using Theorem 4 of [20], which implies that for all $\gamma < -\alpha$, there exists a positive constant C_γ such that $1 - M^{(1)}(\varepsilon) \geq C_\gamma \varepsilon^{1/\gamma}$, $\forall \varepsilon > 0$.

4 Underlying Continuum Random Trees

In this section $\tau(m) = m^\alpha$ with $\alpha < 0$, so that the fragmentation loses mass to dust and reaches $\mathbf{0}$ in finite time a.s. As noticed in [20], the genealogy of the fragmentation can then be described in terms of a continuum random tree.

The definition of CRT we are considering here is the one given by Aldous [2], to which we refer for background and precise definitions. Let $l_1 := \{\mathbf{x} = (x_1, x_2, \ldots), \sum_{k\geq 1} |x_k| < \infty\}$ be endowed with the norm $\|x\|_1 := \sum_{k\geq 1} |x_k|$, and let $\{\mathbf{e}_k, k \geq 1\}$ be its usual basis. Roughly, a CRT is a pair $(\mathcal{T}, \varkappa)$ where $\mathcal{T}$ is a closed subset of l_1 that possesses the "tree" property: for all $v, w \in \mathcal{T}$, there exists a unique (injective) path connecting v to w, denoted by $[[v, w]]$. This tree is *rooted*, that is one vertex is distinguished as being the root $\emptyset_{\mathcal{T}}$. It is moreover equipped with a σ-finite mass measure $\varkappa$, which is non-atomic and puts mass only on the set of leaves, a leaf of $\mathcal{T}$ being a vertex that does not belong to $[[\emptyset, v[[$, $\forall v \in \mathcal{T}$.

According to Theorem 1 of [20], since $\alpha < 0$, the fragmentation $F^{(1)}$ can be constructed from some random compact CRT $(\mathcal{T}^1, \varkappa^1)$ rooted at $\mathbf{0}$ as follows: for each $t \geq 0$, $F^{(1)}(t)$ is the decreasing rearrangement of the $\varkappa^1$-masses of connected components of $\mathcal{T}^1$ obtained by removing the vertices with a distance from the root smaller than t. We shall say that $(\mathcal{T}^1, \varkappa^1)$ *codes* the fragmentation $F^{(1)}$. Note that the measure $\varkappa^1$ is here a (random) probability measure.

Now, for $m > 0$, let $\mathcal{T}^m$ denote the tree $\mathcal{T}^1$ rescaled by a factor $m^{-\alpha}$ and let $\varkappa^m$ be m times the image measure of $\varkappa^1$ by this scaling. Then, according to the self-similarity property, $(\mathcal{T}^m, \varkappa^m)$ codes a fragmentation $F^{(m)}$ starting from $(m, 0, \ldots)$, with parameters (α, ν) $(:= (\tau, \nu)$ with $\tau(m) = m^\alpha)$.

In the remainder of this section we assume that

$$m^\alpha \nu_m \to I \in \mathcal{I},\ I(l_1^{\downarrow}) \neq 0, \quad \text{as } m \to \infty. \tag{25}$$

Given Theorem 5, one can then expect that the sequence of CRTs $(\mathcal{T}^m, \varkappa^m)$ converges in distribution to some "(α, ν, I) fragmentation with immigration CRT" $(\mathcal{T}_{FI}, \varkappa_{FI})$, which should be seen as an infinite baseline $\mathcal{B} := \{x\mathbf{e}_1, x \geq 0\}$ on which compact CRTs are branched. A version of this tree with a spine is constructed below.

We first specify the notion of convergence of trees we are using here. Two trees are considered to be equivalent if there exists an isometry that maps one onto the other and that preserves the root. Implicitly, we always identify a tree with its equivalence class. A natural distance to consider then is the so-called *Gromov-Hausdorff* distance, which is a distance measuring how far two metric spaces are from being isometric (see [16] for a precise definition and properties). Restricted to compact trees of l_1, this distance is given by

$$D_{\mathcal{GH}}(\mathcal{T}, \mathcal{T}') := \inf(D_{\mathcal{H}}^E(\varphi(\mathcal{T}), \varphi'(\mathcal{T}')) \vee d_E(\varphi(\emptyset_{\mathcal{T}}), \varphi'(\emptyset_{\mathcal{T}'}))),$$

where the infimum is taken over all isometric embeddings $\varphi, \varphi' : l_1 \to E$ into a same metric space (E, d_E) and $D_{\mathcal{H}}^E$ denotes the usual Hausdorff distance on the set of compact subsets of E. However, the trees that appear as limit of $\mathcal{T}^m$ are not compact (but their restrictions to closed balls are). Hence we have to truncate them, by introducing, for every tree $\mathcal{T}$ and every integer n, $\mathcal{T}|_n := \{\mathbf{x} \in \mathcal{T} : \|\mathbf{x}\|_1 \leq n\}$. We then consider that a sequence $\mathcal{T}_k$ converges to $\mathcal{T}$ as $k \to \infty$ i.f.f. $D_{\mathcal{GH}}(\mathcal{T}_k|_n, \mathcal{T}|_n) \to 0$ for all $n \geq 0$.

Let us now construct, for each positive m, a "nice version" of the CRT $(\mathcal{T}^m, \varkappa^m)$ by using the geometric description of $F^{(m)}$ of Sect. 2.1. Instead of branching fragmentations on a baseline, we here branch CRTs. To do so, write $\mathbb{N} \backslash \{1\} = \biguplus_{i,j\geq 1} J_{i,j}$ where $\mathrm{Card}(J_{i,j}) = \infty$ and let $f_{i,j}$ be a bijection between $\mathbb{N}$ and $J_{i,j}$. Remind then that $((r_i, \mathbf{u}^i), i \geq 1)$ is a PPP with intensity I and that there exist some versions of the measures $\mu_m = \sum_{i\geq 1} \delta_{(r_i^m, \mathbf{u}^{i,m}, \mathbf{z}^{i,m})}$ (r_i^m, $\mathbf{u}^{i,m}$, and $\mathbf{z}^{i,m}$ were introduced in formula (15) Sect. 2.1) and $\mu = \sum_{i\geq 1} \delta_{(r_i, \mathbf{u}^i, \mathbf{u}^i)}$ such that μ_m converges a.s. to μ, and $r_i^m \to r_i$, $\mathbf{u}^{i,m} \to \mathbf{u}^i$, $\mathbf{z}^{i,m} \to \mathbf{u}^i$. Define then the maps

$$m_{i,j}^m : \sum_{k\geq 1} x_k \mathbf{e}_k \mapsto r_i^m \mathbf{e}_1 + (u_j^{i,m})^{-\alpha} \sum_{k\geq 1} x_k \mathbf{e}_{f_{i,j}(k)},$$

$$m_{i,j}^I : \sum_{k\geq 1} x_k \mathbf{e}_k \mapsto r_i \mathbf{e}_1 + (u_j^i)^{-\alpha} \sum_{k\geq 1} x_k \mathbf{e}_{f_{i,j}(k)}.$$

Introduce next a family $(\mathcal{T}_{i,j}, \varkappa_{i,j}), i, j \geq 1$, of independent copies of $(\mathcal{T}^1, \varkappa^1)$, independent of $(r_i^m, \mathbf{u}^{i,m}, i \geq 1)$ and $((r_i, \mathbf{u}^i), i \geq 1)$. The tree $\mathcal{T}^{(u_j^{i,m})} := m_{i,j}^m(\mathcal{T}_{i,j})$, endowed with the measure $u_j^{i,m} \varkappa_{i,j} \circ (m_{i,j}^m)^{-1}$, codes a fragmentation $F^{(u_j^{i,m})}$ branched on $\mathcal{B}$ at height r_i^m and the required "nice version" of $(\mathcal{T}^m, \varkappa^m)$, which is denoted by $(\overline{\mathcal{T}}^m, \overline{\varkappa}^m)$, is defined by

$$\begin{aligned} \overline{\mathcal{T}}^m &:= \{x\mathbf{e}_1, 0 \leq x \leq t_\infty^m\} \bigcup_{i,j\geq 1} \mathcal{T}^{(u_j^{i,m})}, \\ \overline{\varkappa}^m &:= \sum_{i,j\geq 1} u_j^{i,m} \varkappa_{i,j} \circ (m_{i,j}^m)^{-1}, \end{aligned} \tag{26}$$

where t_∞^m is the first time at which $\overline{\Lambda}^{(m)}$ (defined by (17)) reaches 0.

Similarly, a nice version of the (α, ν, I) fragmentation with immigration CRT is defined by

$$\begin{aligned} \mathcal{T}_{FI} &:= \mathcal{B} \bigcup_{i,j\geq 1} \mathcal{T}^{(u_j^i)}, \\ \varkappa_{FI} &:= \sum_{i,j\geq 1} u_j^i \varkappa_{i,j} \circ (m_{i,j}^I)^{-1}, \end{aligned} \tag{27}$$

where $\mathcal{T}^{(u_j^i)} := m_{i,j}^I(\mathcal{T}_{i,j})$. To obtain a version of the (α, ν, I) fragmentation with immigration from this tree, just set $FI(t)$ for the decreasing sequence of $\varkappa_{FI}$-masses of connected components of $\{\mathbf{x} \in \mathcal{T}_{FI} : \|\mathbf{x}\|_1 \geq t, x_1 \leq t\}$. At last, note that since I is $(-\alpha)$-self-similar (by Lemma 6), the CRT is also self-similar, i.e.

$$(\mathcal{T}_{FI}^a, a^{-1/\alpha} \varkappa_{FI}^a) \overset{\text{law}}{=} (\mathcal{T}_{FI}, \varkappa_{FI}) \quad \text{for all } a > 0,$$

where $\mathcal{T}_{FI}^a$ is the tree $\mathcal{T}_{FI}$ rescaled by the factor a and $\varkappa_{FI}^a$ is the image measure of $\varkappa_{FI}$ by this scaling.

We are now ready to state the counterpart, in term of trees, of Theorem 5, assuming that (25) holds. The topology on the set of measures on l_1 is the topology of vague convergence.

Theorem 19 *As* $m \to \infty$,

$$(\mathcal{T}^m, \varkappa^m) \stackrel{\text{law}}{\to} (\mathcal{T}_{FI}, \varkappa_{FI}).$$

For the proof, we need the following lemma, where $h_{i,j} := \sup\{\|\mathbf{x}\|_1, \mathbf{x} \in \mathcal{T}_{i,j}\}$ is the height of the tree $\mathcal{T}_{i,j}$. It is known (see [17]) that those random variables have exponential moments.

Lemma 20 *For all* $n \in \mathbb{N}$,

$$\sum_{r_i^m \le n, j \ge 1} u_j^{i,m} h_{i,j}^{-1/\alpha} \stackrel{\mathrm{P}}{\to} \sum_{r_i \le n, j \ge 1} u_j^i h_{i,j}^{-1/\alpha} \quad \text{as } m \to \infty. \tag{28}$$

As a consequence, one can extract from any increasing integer-valued sequence $\boldsymbol{\kappa}$ *a subsequence* $\overline{\boldsymbol{\kappa}}$ *such that for all* $n, p \in \mathbb{N}$, *as* $m \to \infty$,

$$\sum_{r_i^{\overline{\kappa}m} \le n, j \ge 1} \mathbf{1}_{\{(u_j^{i,\overline{\kappa}m})^{-\alpha} h_{i,j} p > 1\}} \stackrel{a.s.}{\to} \sum_{r_i \le n, j \ge 1} \mathbf{1}_{\{(u_j^i)^{-\alpha} h_{i,j} p > 1\}} < \infty. \tag{29}$$

Proof (i) Fix $n \in \mathbb{N}$ and recall that a.s. μ_m converges to μ and $r_i \notin \mathbb{N}$, $i \ge 1$. Consequently, $u_j^{i,m} \mathbf{1}_{\{r_i^m \le n\}} \to u_j^i \mathbf{1}_{\{r_i \le n\}}$ for all $i, j \ge 1$ a.s., and a.s. for all $\eta > 0$, there exists a $k \in \mathbb{N}$ such that for m large enough,

$$\sum_{i+j \ge k} (u_j^{i,m} \mathbf{1}_{\{r_i^m \le n\}} + u_j^i \mathbf{1}_{\{r_i \le n\}}) \le \eta. \tag{30}$$

We want to prove that $X_m := \sum_{r_i^m \le n, j \ge 1} u_j^{i,m} h_{i,j}^{-1/\alpha}$ converges to $X := \sum_{r_i \le n, j \ge 1} u_j^i h_{i,j}^{-1/\alpha}$ in probability. Remark that $X < \infty$ a.s. since $E[X \mid (r_i, \mathbf{u}^i), i \ge 1] = E[h_{1,1}^{-1/\alpha}] \sum_{r_i \le n, j \ge 1} u_j^i$ is finite a.s. Similarly, $X_m < \infty$ a.s. Then, since

$$P(|X_m - X| > \varepsilon) = E[E[\mathbf{1}_{\{|X_m - X| > \varepsilon\}} \mid (r_i^m, \mathbf{u}^{i,m}), (r_i, \mathbf{u}^i), i, m \ge 1]],$$

it is sufficient, by dominated convergence, to prove that the conditional expectation converges a.s. to 0, $\forall \varepsilon > 0$. For large m's, one has

$$\begin{aligned}
&E[\mathbf{1}_{\{|X_m - X| > \varepsilon\}} \mid (r_i^m, \mathbf{u}^{i,m}), (r_i, \mathbf{u}^i), i, m \ge 1] \\
&\quad \le \varepsilon^{-1} E[|X_m - X| \mid (r_i^m, \mathbf{u}^{i,m}), (r_i, \mathbf{u}^i), i, m \ge 1] \\
&\quad \le \varepsilon^{-1} E[h_{1,1}^{-1/\alpha}] \sum_{i,j \ge 1} |u_j^{i,m} \mathbf{1}_{\{r_i^m \le n\}} - u_j^i \mathbf{1}_{\{r_i \le n\}}| \\
&\quad \le \varepsilon^{-1} E[h_{1,1}^{-1/\alpha}] \left(\sum_{i+j < k} |u_j^{i,m} \mathbf{1}_{\{r_i^m \le n\}} - u_j^i \mathbf{1}_{\{r_i \le n\}}| + \eta \right),
\end{aligned}$$

the last inequality coming from (30). So for all $\eta > 0$, we have a upper bound smaller than $2\eta\varepsilon^{-1} E[h_{1,1}^{-1/\alpha}]$ for all m sufficiently large, a.s. Hence the conclusion.

(ii) The measure I is self-similar (by Lemma 6) and consequently atomless on $l_1^{\downarrow}\backslash\{\mathbf{0}\}$. As $(r_i, \mathbf{u}^i)_{i\geq 1}$ is a PPP with intensity I, independent of the $h_{i,j}$'s, this implies that a.s. $(u_j^i)^{-\alpha} h_{i,j} p \neq 1$, $\forall i, j, p \geq 1$, which in turn leads to the convergence of $\mathbf{1}_{\{(u_j^{i,m})^{-\alpha} h_{i,j} p > 1\}} \mathbf{1}_{\{r_i^m \leq n\}}$ to $\mathbf{1}_{\{(u_j^i)^{-\alpha} h_{i,j} p > 1\}} \mathbf{1}_{\{r_i \leq n\}}$ a.s. $\forall i, j, p, n \geq 1$. Then for all $k \geq 1$,

$$\begin{aligned}
&\left| \sum_{r_i^m \leq n, j \geq 1} \mathbf{1}_{\{(u_j^{i,m})^{-\alpha} h_{i,j} p > 1\}} - \sum_{r_i \leq n, j\geq 1} \mathbf{1}_{\{(u_j^i)^{-\alpha} h_{i,j} p > 1\}} \right| \\
&\quad \leq \sum_{i+j<k} |\mathbf{1}_{\{(u_j^{i,m})^{-\alpha} h_{i,j} p > 1\}} \mathbf{1}_{\{r_i^m \leq n\}} - \mathbf{1}_{\{(u_j^i)^{-\alpha} h_{i,j} p > 1\}} \mathbf{1}_{\{r_i \leq n\}}| \\
&\qquad + p^{-1/\alpha} \sum_{i+j\geq k} (u_j^{i,m} h_{i,j}^{-1/\alpha} \mathbf{1}_{\{r_i^m \leq n\}} + u_j^i h_{i,j}^{-1/\alpha} \mathbf{1}_{\{r_i \leq n\}}). \qquad (31)
\end{aligned}$$

So if we prove that each sequence $\boldsymbol{\kappa}$ possesses a subsequence $\overline{\boldsymbol{\kappa}}$ *independent of* $n \in \mathbb{N}$ such that, a.s. for all $\varepsilon > 0$ there exists a k such that

$$\sum_{i+j\geq k} (u_j^{i,\overline{\kappa}_m} h_{i,j}^{-1/\alpha} \mathbf{1}_{\{r_i^{\overline{\kappa}_m} \leq n\}} + u_j^i h_{i,j}^{-1/\alpha} \mathbf{1}_{\{r_i \leq n\}}) \leq \varepsilon \quad \text{for all } m \text{ large enough}, \qquad (32)$$

then we will have the statement (using also that the first term in the right hand side of the inequality (31) is composed by a finite number of terms that all converge to 0 as $m \to \infty$). Clearly, to get (32), it is sufficient to show that there is a subsequence $\overline{\boldsymbol{\kappa}}$ such that $\forall n$,

$$\sum_{r_i^{\overline{\kappa}_m} \leq n, j \geq 1} u_j^{i,\overline{\kappa}_m} h_{i,j}^{-1/\alpha} \to \sum_{r_i \leq n, j \geq 1} u_j^i h_{i,j}^{-1/\alpha} \quad \text{a.s.}$$

To construct this subsequence, we use the convergence in probability (28). It implies that for all n, there is a subsequence $\overline{\boldsymbol{\kappa}}^{(n)}$ such that the above a.s. convergence holds. We want a sequence $\overline{\boldsymbol{\kappa}}$ independent of n and to do so, use a diagonal extraction argument: extract $\overline{\boldsymbol{\kappa}}^{(1)}$ from $\boldsymbol{\kappa}$ and then recursively $\overline{\boldsymbol{\kappa}}^{(n+1)}$ from $\overline{\boldsymbol{\kappa}}^{(n)}$. Then set $\overline{\boldsymbol{\kappa}}_m := \overline{\boldsymbol{\kappa}}^{(m)}(m)$. □

Proof of Theorem 19 The goal is to prove that the version $(\overline{\mathcal{T}}^m, \overline{\varkappa}^m)$, defined in (26), of the fragmentation CRT with total weight m converges in probability to the version $(\mathcal{T}_{FI}, \varkappa_{FI})$, defined in (27), of the fragmentation with immigration CRT. Or, equivalently, that for any increasing integer-valued sequence $\boldsymbol{\kappa}$, one can extract a subsequence $\overline{\boldsymbol{\kappa}}$ such that $(\overline{\mathcal{T}}^{\overline{\kappa}_m}, \overline{\varkappa}^{\overline{\kappa}_m})$ converges a.s. to $(\mathcal{T}_{FI}, \varkappa_{FI})$. So, fix such a sequence $\boldsymbol{\kappa}$ and consider its subsequence $\overline{\boldsymbol{\kappa}}$ introduced in Lemma 20, so that the a.s. convergences (29) hold. In the rest of the proof, all the assertions hold a.s., so we drop the "a.s." from the notations.

(i) A first remark is that for all $i, j \geq 1$,

$$D_{\mathcal{H}}^{l_1}(\mathcal{T}^{(u_j^{i,m})}, \mathcal{T}^{(u_j^i)}) \leq |r_i^m - r_i| + \left|(u_j^{i,m})^{-\alpha} - (u_j^i)^{-\alpha}\right| h_{i,j} \to 0 \quad \text{as } m \to \infty. \quad (33)$$

Fix then $n, p \in \mathbb{N}$. As a consequence of (29), the number of trees among $\{\mathcal{T}^{(u_j^{i,\overline{\kappa}_m})}$, $i, j \geq 1$, $r_i^{\overline{\kappa}_m} \leq n\}$, which are not entirely contained in $\{\mathbf{x} : \|\mathbf{x}-x_1\mathbf{e}_1\|_1 \leq p^{-1}\}$ is constant (finite) for m large enough. Let $\mathcal{K}$ be the finite set of (i, j) s.t. $\mathcal{T}^{(u_j^i)}$ is not entirely contained in $\{\mathbf{x} : \|\mathbf{x}-x_1\mathbf{e}_1\|_1 \leq p^{-1}\}$. Then for large m's,

$$D_{\mathcal{H}}^{l_1}(\overline{\mathcal{T}}^{\overline{\kappa}_m}|_n, \mathcal{T}_{FI}|_n) \leq p^{-1} + \max_{i,j\in\mathcal{K}} D_{\mathcal{H}}^{l_1}(\mathcal{T}^{(u_j^{i,\overline{\kappa}_m})}, \mathcal{T}^{(u_j^i)}).$$

Considering (33) and taking m larger if necessary, one sees that this upper bound is in turn bounded by $2p^{-1}$. This holds for all $p \in \mathbb{N}$. Hence $D_{\mathcal{GH}}(\overline{\mathcal{T}}^{\overline{\kappa}_m}|_n, \mathcal{T}_{FI}|_n) \to 0$, $\forall n \in \mathbb{N}$.

(ii) Next, for all $\mathbb{R}^+$-valued continuous function f with compact support on l_1, $\langle u_j^{i,m}\varkappa_{i,j} \circ (m_{i,j}^m)^{-1}, f\rangle$ converges to $\langle u_j^i\varkappa_{i,j} \circ (m_{i,j}^I)^{-1}, f\rangle$, since $m_{i,j}^m(\mathbf{x}) \to m_{i,j}^I(\mathbf{x})$ for all $\mathbf{x} \in l_1$. To deduce from this that the sum over $i, j \geq 1$ of these measures converges, fix some $\eta > 0$ and let $C_f := \sup_{l_1}|f(\mathbf{x})|$. Again we use the argument that there exists some $k \in \mathbb{N}$ such that $\sum_{i+j\geq k} u_j^i \mathbf{1}_{\{r_i \leq C_f\}} < \eta$ and $\sum_{i+j\geq k} u_j^{i,m}\mathbf{1}_{\{r_i^m \leq C_f\}} < \eta$ for all m large enough, which leads to

$$\begin{aligned}&|\langle \varkappa^m, f\rangle - \langle \varkappa_{FI}, f\rangle| \\ &\quad\leq 2C_f\eta + \sum_{i+j<k} \left|\langle u_j^{i,m}\varkappa_{i,j} \circ (m_{i,j}^m)^{-1}, f\rangle - \langle u_j^i\varkappa_{i,j} \circ (m_{i,j}^I)^{-1}, f\rangle\right|\end{aligned}$$

which is bounded by $(2C_f + 1)\eta$ for large m's. □

5 Stable Fragmentations

In this section, we apply our results to two specific families of fragmentations constructed from the so-called stable tree $(\mathcal{T}^\beta, \varkappa^\beta)$ with index β, $1 < \beta < 2$. This object is a CRT introduced by Duquesne and Le Gall [12], to which we refer for a rigorous construction. Roughly, $\mathcal{T}^\beta$ arises as the limit in distribution of rescaled critical Galton-Watson trees T_n, conditioned to have n vertices and edge-lengths $n^{\beta^{-1}-1}$, and an offspring distribution $(\eta_k, k \geq 0)$ such that $\eta_k \sim Ck^{-1-\beta}$ as $k \to \infty$. It is endowed with a probability measure $\varkappa^\beta$ which is the limit as $n \to \infty$ of the empiric measure on the vertices of T_n.

5.1 Stable Fragmentations with a Negative Index of Self-Similarity

Let $F^{\beta-}(t)$ denotes the decreasing sequence of $\varkappa^\beta$-masses of connected components obtained by removing in $\mathcal{T}^\beta$ all vertices at distance less than t from the root, $t \geq 0$. Miermont [26] shows that $F^{\beta-}$ is a self-similar fragmentation with index $1/\beta - 1$, and with a dislocation measure ν^β given by

$$\int_{l_{1,\leq 1}^{\downarrow}} f(\mathbf{s})\nu^\beta(\mathrm{d}\mathbf{s}) = C_\beta E\big[T_1^\beta f((T_1^\beta)^{-1}(\Delta_1^\beta, \Delta_2^\beta, \ldots))\big], \quad f \in \mathcal{F},$$

where $C_\beta = \beta^2\Gamma(2-\beta^{-1})/\Gamma(2-\beta)$. The process T^β is a stable subordinator with Laplace exponent $q^{1/\beta}$, i.e.

$$E[\exp(-qT_r^\beta)] = \exp(-rq^{1/\beta}), \quad q, r \geq 0, \tag{34}$$

and $(\Delta_1^\beta, \Delta_2^\beta, \ldots)$ denotes the sequence of jumps in the decreasing order of T^β before time 1. In order to apply Theorem 5 to these fragmentations, we state the following lemma.

Lemma 21 *As $m \to \infty$, $m^{1/\beta-1}\nu_m^\beta \to I^\beta$, where I^β is defined by*

$$\int_{l_1^\downarrow} f(\mathbf{s})I^\beta(\mathrm{d}\mathbf{s}) = \beta(\beta-1)(\Gamma(2-\beta))^{-1}\int_0^\infty E\big[f(x^\beta(\Delta_1^\beta, \Delta_2^\beta, \ldots))\big]x^{-\beta}\mathrm{d}x,$$
$$f \in \mathcal{F}.$$

Using (34), one sees that I^β integrates $(1-\exp(-\sum_{i\geq 1}s_i))$ and therefore that it is an immigration measure (it is also a consequence of the above convergence).

Proof In all the proof, $T_1^\beta, \Delta_1^\beta, \Delta_2^\beta, \ldots$ are rather denoted by $T_1, \Delta_1, \Delta_2, \ldots$. A classical idea is to use a size-biased permutation $(\Delta_1^*, \Delta_2^*, \ldots)$ of $(\Delta_1, \Delta_2, \ldots)$ to obtain some results on the latter. To do so, we first recall that T_1 has a density (see e.g. formula (40) in [30]), that we denote by q. One then obtains, using Palm measures theory (see e.g. [29]), the following equality:

$$\begin{aligned}
&E[f(T_1, \Delta_1^*, \Delta_2^*, \Delta_3^*, \ldots, \Delta_{k+1}^*)] \\
&\quad = c_\beta^{k+1}\int_0^\infty\int_0^{s_0}\int_0^{s_0-s_1} \\
&\qquad \ldots\int_0^{s_0-s_1-\cdots-s_k}\frac{f(s_0, s_1, \ldots, s_{k+1})q(s_0-s_1-\cdots-s_{k+1})\mathrm{d}s_{k+1}\ldots\mathrm{d}s_1\mathrm{d}s_0}{s_{k+1}^{1/\beta}s_k^{1/\beta}\ldots s_1^{1/\beta}(s_0-s_1-\cdots-s_k)\ldots(s_0-s_1)s_0}
\end{aligned} \tag{35}$$

for all non-negative measurable function f on $(\mathbb{R}^+)^{k+2}$, where $c_\beta = (\beta\Gamma(1-1/\beta))^{-1}$.

Let then g be a non-negative measurable function on $(\mathbb{R}^+)^k$. One has

$$\begin{aligned}
&m^{1/\beta-1}E\left[T_1 g\left(\frac{m\Delta_2^*}{T_1}, \frac{m\Delta_3^*}{T_1}, \ldots, \frac{m\Delta_{k+1}^*}{T_1}\right)\right] \\
&\quad = c_\beta^{k+1}\int_0^\infty\int_0^u\int_0^{u-s_2} \\
&\qquad \ldots\int_0^{u-\cdots-s_k}\left(m^{1/\beta-1}\int_u^\infty\frac{g(ms_2/s_0, ms_3/s_0, \ldots, ms_{k+1}/s_0)}{(s_0-u)^{1/\beta}}\mathrm{d}s_0\right)
\end{aligned}$$

$$\times \frac{q(u-s_2-\cdots-s_{k+1})\mathrm{d}s_{k+1}\ldots\mathrm{d}s_2\mathrm{d}u}{s_{k+1}^{1/\beta}s_k^{1/\beta}\ldots s_2^{1/\beta}(u-s_2-\cdots-s_k)(u-s_2-\cdots-s_{k-1})\ldots(u-s_2)u}$$

$$= \beta c_\beta E\left[\int_0^\infty \frac{g(\Delta_1^*/(v^{-\beta}+m^{-1}T_1),\ldots,\Delta_k^*/(v^{-\beta}+m^{-1}T_1))}{v^\beta}\mathrm{d}v\right], \qquad (36)$$

where for the first equality we use formula (35), the change of variables $s_1 \mapsto s_0 - u$ and Fubini's Theorem, and for the second equality the change of variables $s_0 \mapsto u + mv^{-\beta}$ and again formula (35). This holds in particular for $g(x_1,\ldots,x_k) = f \circ d^\downarrow(x_1,\ldots,x_k,0,\ldots)$ when $f \in \mathcal{F}$ and $d^\downarrow$ is the function that associates to $(x_1,x_2,\ldots) \in (\mathbb{R}^+)^\infty$, $\sum_{i\geq 1} x_i < \infty$, its decreasing rearrangement in $l_1^\downarrow$ (this function is measurable). Our aim now is to let $k \to \infty$ in equality (36) for such functions g. To do so, first note that $d^\downarrow(x_1,\ldots,x_k,0,\ldots) \to d^\downarrow(x_1,x_2,\ldots)$ in $l_1^\downarrow$ as $k \to \infty$, for all $(x_1,x_2,\ldots) \in (\mathbb{R}^+)^\infty$, $\sum_{i\geq 1} x_i < \infty$. We then claim that dominated convergence applies in both sides of the equality. Indeed, for the left hand side, since $f(\mathbf{s}) \leq (\sum_{i\geq 1} s_i) \wedge 1$, one has for all k,

$$T_1 g\left(\frac{m\Delta_2^*}{T_1}, \frac{m\Delta_3^*}{T_1}, \ldots, \frac{m\Delta_{k+1}^*}{T_1}\right) \leq T_1\left(1 \wedge m\left(\frac{T_1-\Delta_1^*}{T_1}\right)\right).$$

It is therefore sufficient to prove that $E[T_1 \wedge m(T_1-\Delta_1^*)] < \infty$, which, clearly, holds if $E[T_1-\Delta_1^*] < \infty$. It is not hard to see, using the joint distribution (35), that the last expectation is proportional to $E[(T_1)^{1-1/\beta}]$, which, according to formula (43) in [30], is finite. Hence dominated convergence applies in the left hand side of (36). Now, for the right hand side, one uses that

$$g(\Delta_1^*/(v^{-\beta}+m^{-1}T_1),\ldots,\Delta_k^*/(v^{-\beta}+m^{-1}T_1)) \leq (T_1 v^\beta \wedge 1)$$

which is integrable with respect to $\mathrm{d}\mathbb{P}\otimes v^{-\beta}\mathrm{d}v$, because $(1-\exp(-T_1 v^\beta))$ is. At last, letting $k \to \infty$, one obtains

$$m^{1/\beta-1}E\left[T_1 f \circ d^\downarrow\left(\frac{m\Delta_2^*}{T_1}, \frac{m\Delta_3^*}{T_1}, \ldots\right)\right]$$

$$= \beta c_\beta E\left[\int_0^\infty \frac{f \circ d^\downarrow(\Delta_1^*/(v^{-\beta}+m^{-1}T_1), \Delta_2^*/(v^{-\beta}+m^{-1}T_1),\ldots)}{v^\beta}\mathrm{d}v\right]$$

$$= \beta c_\beta E\left[\int_0^\infty \frac{f(\Delta_1/(v^{-\beta}+m^{-1}T_1), \Delta_2/(v^{-\beta}+m^{-1}T_1),\ldots)}{v^\beta}\mathrm{d}v\right].$$

This latter term converges as $m \to \infty$ to $\beta c_\beta \int_0^\infty E[f(v^\beta(\Delta_1,\Delta_2,\ldots))]v^{-\beta}\mathrm{d}v$, again by dominated convergence. Hence we would have the required convergence $\langle m^{1/\beta-1}\nu_m^\beta, f\rangle \to \langle I^\beta, f\rangle$ for all continuous non-negative functions $f \in \mathcal{F}$ if we could replace in the left hand side of the above formula the sequence $d^\downarrow(m\Delta_2^*/T_1, m\Delta_3^*/T_1,\ldots)$ by $(m\Delta_2/T_1, m\Delta_3/T_1,\ldots)$. Of course, this is not possible. However, when $\Delta_1^* > T_1/2$, one has $\Delta_1^* \overset{\text{a.s.}}{=} \Delta_1$ (equivalently $d^\downarrow(\Delta_2^*,\Delta_3^*,\ldots) \overset{\text{a.s.}}{=} (\Delta_2,\Delta_3,\ldots)$), since the size-biased pick Δ_1^*/T_1 is then necessarily equal to the largest mass Δ_1/T_1. Therefore, one can write

$$m^{1/\beta-1}E\left[T_1 f\left(\frac{m\Delta_2}{T_1},\frac{m\Delta_3}{T_1},\ldots\right)\right]$$
$$=m^{1/\beta-1}E\left[T_1 f\circ d^{\downarrow}\left(\frac{m\Delta_2^*}{T_1},\frac{m\Delta_3^*}{T_1},\ldots\right)\right]$$
$$+m^{1/\beta-1}E\left[\left(T_1 f\left(\frac{m\Delta_2}{T_1},\frac{m\Delta_3}{T_1},\ldots\right)\right.\right.$$
$$\left.\left.-T_1 f\circ d^{\downarrow}\left(\frac{m\Delta_2^*}{T_1},\frac{m\Delta_3^*}{T_1},\ldots\right)\right)\mathbf{1}_{\{\Delta_1^*\le T_1/2\}}\right]$$

and this converges to the required limit as $m\to\infty$, because the absolute value of the second term in the right hand side of the equality is bounded from above by $m^{1/\beta-1}E[2T_1\mathbf{1}_{\{\Delta_1^*\le T_1/2\}}]$ which in turn is bounded by $m^{1/\beta-1}E[4(T_1-\Delta_1^*)]$, which converges to 0 as $m\to\infty$. □

From this and Theorem 5, one deduces that

$$(m-(F_1^{\beta-})^{(m)},((F_2^{\beta-})^{(m)},(F_3^{\beta-})^{(m)},\ldots))\overset{\text{law}}{\to}(\sigma_{I^\beta},FI^\beta), \tag{37}$$

where FI^β is a fragmentation with immigration process $(1/\beta-1,\nu^\beta,I^\beta)$ and σ_{I^β} is the stable subordinator with index $1-1/\beta$ representing the total mass of immigrants. In terms of trees (Theorem 19), one has

$$(\mathcal{T}^{\beta,m},m\varkappa^{\beta,m})\overset{\text{law}}{\to}(\mathcal{T}_{FI^\beta},\varkappa_{FI^\beta}),$$

where $\mathcal{T}^{\beta,m}$ is the stable tree rescaled by a factor $m^{1-1/\beta}$ and $\varkappa^{\beta,m}$ is the image of $\varkappa^\beta$ by this scaling; $(\mathcal{T}_{FI^\beta},\varkappa_{FI^\beta})$ is a fragmentation with immigration CRT with parameters $(1/\beta-1,\nu^\beta,FI^\beta)$.

In Chap. 4.4.2 of [19], it is shown that (some version of) this fragmentation with immigration FI^β can be constructed from the height process H^β coding a continuous state branching process with immigration, with branching mechanism u^β and immigration mechanism $\beta u^{\beta-1}$ as follows: for all $t\ge 0$, $FI^\beta(t)$ is the decreasing rearrangement of the lengths of finite excursions of H^β above t. In a recent work, Duquesne [11] shows that the rescaled height function of some ordered version of the stable tree with index β converges to H^β, which corroborates our results.

Last, thanks to the self-similarity, the convergence (37) also specifies the behavior of $(F^{\beta-})^{(1)}(\varepsilon\cdot)$ as $\varepsilon\to 0$. In particular, the mass of dust $1-(M^\beta)^{(1)}$ behaves as follows.

Corollary 22 *As $\varepsilon\to 0$, $\varepsilon^{-\beta/(\beta-1)}(1-(M^\beta)^{(1)}(\varepsilon\cdot))\overset{\text{law}}{\to}\int_0^t L^\beta(u)du$, where L^β is a continuous state branching process with immigration starting from 0, with branching mechanism u^β and immigration mechanism $\beta u^{\beta-1}$.*

In a previous work, Miermont [26] obtained this convergence result on the mass of dust for one dimensional marginal.

Proof According to (37), $\varepsilon^{-\beta/(\beta-1)}(1-(M^\beta)^{(1)}(\varepsilon\cdot))$ converges in law to some non-trivial limit that corresponds to the total mass of microscopic particles produced until time t by the fragmentation with immigration FI^β. The construction of FI^β from H^β implies that this limit is equal to $\int_0^t L^\beta(u)\mathrm{d}u$, where $L^\beta(u)$ is the local time at u of H^β. Last, Lambert [24] proves that L^β is a continuous state branching process with immigration starting from 0, with the characteristics mentioned in the statement. □

Remark Using the same tools, one sees that the above corollary is also valid when replacing β by 2 and the fragmentation F^β by a self-similar fragmentation with index $-1/2$ and dislocation measure $\sqrt{2}\nu_{B_r}$.

5.2 Stable Fragmentations with a Positive Index of Self-Similarity

We here consider the self-similar fragmentations $F^{\beta+}$ with index $1/\beta$ and dislocation measure ν^β, $1<\beta<2$. Such fragmentations can also be constructed from the stable trees $\mathcal{T}^\beta$, by cutting them at nodes (see [27]). According to Lemma 21, $m^{1/\beta-1}\nu_m^\beta \to I^\beta$ as $m\to\infty$. It is then easy that $\varphi_\nu(m)\sim m^{1/\beta-1}\Gamma(1/\beta)/\beta$ as $m\to\infty$ and therefore $\varphi_\nu(m)\nu_m^\beta \to \Gamma(1/\beta)I^\beta/\beta$. On the other hand, the index of self-similarity is $1/\beta$ and $m^{-1/\beta}\varphi_\nu(m)\to 0$. Corollary 17 then ensures that letting $\varepsilon\to 0$,

$$\varepsilon^{-\beta/(\beta-1)}(1-F_1^{\beta+}(\varepsilon\cdot),(F_2^{\beta+}(\varepsilon\cdot),F_3^{\beta+}(\varepsilon\cdot),\ldots))\overset{\text{law}}{\to}((\sigma_{I^\beta}(t),I^\beta(t)),t\geq 0),$$

where $(I^\beta(t),t\geq 0)$ denotes a pure immigration process with intensity I^β and σ_{I^β} its $(1-1/\beta)$-stable subordinator of total mass of immigrants (here we have used that a pure immigration process with intensity $\Gamma(1/\beta)I^\beta/\beta$ is distributed as $((\Gamma(1/\beta)/\beta)^{\beta/(\beta-1)}I^\beta(t),t\geq 0))$. Let then ϱ be a $(\beta-1)$-stable subordinator with Laplace exponent $\beta q^{\beta-1}$, $q\geq 0$, independent of T^β and call $(\Delta_1^\beta(\varrho(t)),\Delta_2^\beta(\varrho(t)),\ldots)$ the decreasing sequence of jumps of T^β before time $\rho(t)$, $t\geq 0$. A moment of thought shows that $((\Delta_1^\beta(\varrho(t)),\Delta_2^\beta(\varrho(t)),\ldots),t\geq 0)$ is distributed as $(I^\beta(t),t\geq 0)$. Therefore,

Corollary 23 *As* $\varepsilon\to 0$,

$$\begin{aligned}&\varepsilon^{-\beta/(\beta-1)}(1-F_1^{\beta+}(\varepsilon\cdot),(F_2^{\beta+}(\varepsilon\cdot),F_3^{\beta+}(\varepsilon\cdot),\ldots))\\&\quad\overset{\text{law}}{\to}(T^\beta_{\varrho(\cdot)},(\Delta_1^\beta(\varrho(\cdot)),\Delta_2^\beta(\varrho(\cdot)),\ldots)).\end{aligned}$$

Miermont [27] obtained this result for one dimensional marginal.

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Appendix

Proof of Proposition 2

Our aim is to prove that under the general hypotheses (H) we have made on τ, ν, the mass $M^{(m)}(t) = \sum_{i\geq 1} F_i^{(m)}(t)$ is a.s. continuous in t. We point out that the condition $\nu(l^{\downarrow}_{1,\leq 1}) = \infty$ is actually not required, i.e. the result is available for any dislocation measure, provided that $\nu(\sum_{i\geq 1} s_i < 1) = 0$. So in the following ν can be finite or infinite, but the other hypotheses on τ, ν made in (H) have to be fulfiled. We exclude the trivial case $\nu(l^{\downarrow}_{1,\leq 1}) \neq 0$ and, since the proof is the same for all m, we suppose that $m = 1$ and use the notations M, F instead of $M^{(1)}$, $F^{(1)}$.

As often in the study of loss of mass, the problem can be tackled by considering the evolution of some fragments independently tagged at random. So, consider the interval representation I^{τ} from which F has been constructed in Sect. 1.1.1 and let U, U' be two independent r.v. uniformly distributed on $(0, 1)$, independent of I^{τ}. Let then D_{τ} (resp. D'_{τ}) be the first time, possibly infinite, at which U (resp. U') falls into the dust and note that with probability one, $P(D_{\tau} = D'_{\tau} = t \mid I^{\tau}) = (M(t-) - M(t))^2$ for all $t \geq 0$. Consequently, the mass M is a.s. continuous as soon as $P(D_{\tau} = D'_{\tau} < \infty) = 0$.

The goal now is to prove that this probability is equal to 0. To do so, note first, using the time changes (1), that

$$D_{\tau} = \int_0^{\infty} \mathrm{d}r/\tau(|I_U^{\mathrm{hom}}(r)|) = \int_0^{\infty} \mathrm{d}r/\tau(\exp(-\sigma(r))),$$

where, by definition, $\sigma = -\ln(|I_U^{\mathrm{hom}}|)$. A well-known result of [7] says that σ is a subordinator with *zero drift* and Lévy measure $L(\mathrm{d}x) = \sum_{i\geq 1} e^{-x}\nu(-\log s_i \in \mathrm{d}x)$.

Introduce then T, the first time at which U and U' do not belong to the same fragment and call $m(T)$ (resp. $m'(T)$) the length of the fragment containing U (resp. U') at that time. Since ν does not lose mass during sudden dislocations, the masses $m(T)$, $m'(T)$ are a.s. strictly positive. Let then, for $m > 0$, $\tau(m\cdot)$ denote the function $t \in [0, \infty) \mapsto \tau(mt)$. Using the fragmentation property, one sees that $D_{\tau} = T + \widetilde{D}_{\tau(m(T)\cdot)}$ and $D'_{\tau} = T + \widetilde{D}_{\tau(m'(T)\cdot)}$, where, conditionally on $m(T)$ and $m'(T)$, $\widetilde{D}_{\tau(m(T)\cdot)}$ and $\widetilde{D}_{\tau(m'(T)\cdot)}$ are independent and distributed as $D_{\tau(m(T)\cdot)}$ and $D_{\tau(m'(T)\cdot)}$ respectively. Therefore, $P(D_{\tau} = D'_{\tau} < \infty) = P(\widetilde{D}_{\tau(m(T)\cdot)} = \widetilde{D}_{\tau(m'(T)\cdot)} < \infty)$ is equal to 0 as soon as the point ∞ is the only possible atom of $D_{\tau(m\cdot)}$, $\forall m > 0$. The proof ends with the following lemma. We recall that σ has no drift component.

Lemma 24 *Let f be a locally integrable and strictly positive function on* $[0, \infty)$. *Suppose moreover that f is monotone near ∞. Then the integral $\int_0^{\infty} f(\sigma(r))\mathrm{d}r$ is either a.s. finite or a.s. infinite and when it is a.s. finite, its distribution is atomless.*

Proof The first assertion is a consequence of the Hewitt-Savage 0-1 law and is shown, e.g., in the proof of Proposition 10 of [17]. In the following we suppose that the integral $\int_0^{\infty} f(\sigma(r))\mathrm{d}r$ is a.s. finite. In particular, f is non-increasing near ∞ and converges to 0.

(i) The proof is easy when ν is finite. Indeed, let then T_1 be the first jump time of σ. It is well-known that T_1 and $\sigma(r+T_1)$ are independent and that T_1 has an exponential distribution. Therefore, splitting the integral at T_1, we see that $\int_0^\infty f(\sigma(r))\mathrm{d}r$ can be written as the sum of two independent r.v.:

$$\int_0^\infty f(\sigma(r))\mathrm{d}r = f(0)T_1 + \int_0^\infty f(\sigma(r+T_1))\mathrm{d}r,$$

the first one, $f(0)T_1$, being absolutely continuous. It is easy that $\int_0^\infty f(\sigma(r))\mathrm{d}r$ is then also absolutely continuous, hence atomless.

(ii) From now on, we suppose that ν is infinite. Introduce then for all $t \geq 0$ the stopping times

$$\theta(t) := \inf\left\{u : \int_0^u f(\sigma(r))\mathrm{d}r > t\right\},$$

with the convention $\inf\{\emptyset\} = \infty$. According to the strong Markov property, conditional on $\theta(t) < \infty$,

$$\int_0^\infty f(\sigma(r))\mathrm{d}r = t + \int_0^\infty f(\sigma(\theta(t)) + \sigma^{(t)}(r))\mathrm{d}r,$$

where $\sigma^{(t)}(r) := \sigma(r+\theta(t)) - \sigma(\theta(t))$, $r \geq 0$, is a subordinator distributed as σ and independent of $\sigma(\theta(t))$.

Now fix some $t > 0$ and to begin with, suppose that f is strictly decreasing on $[0,\infty)$. The function $x \in (0,\infty) \mapsto \int_0^\infty f(x+\sigma^{(t)}(r))\mathrm{d}r$ is then strictly decreasing, hence injective. Consequently, there is at most one point, say X_t, such that $\int_0^\infty f(X_t+\sigma^{(t)}(r))\mathrm{d}r = t$. If that point does not exist, $X_t := \infty$. Then,

$$\begin{aligned} P\left(\int_0^\infty f(\sigma(r))\mathrm{d}r = 2t\right) &= P\left(\int_0^\infty f(\sigma(\theta(t)) + \sigma^{(t)}(r))\mathrm{d}r = t, \theta(t) < \infty\right) \\ &= P(\sigma(\theta(t)) = X_t, \theta(t) < \infty) \end{aligned}$$

with X_t independent of $\sigma(\theta(t))$. This latter probability is then 0, because for all $0 < a < \infty$, $P(\sigma(\theta(t)) = a) \leq P(\exists s : \sigma(s) = a)$ and, by Kesten's Theorem (see e.g. Proposition 1.9 in [6]), since σ has 0 drift and ν is infinite, $P(\exists s : \sigma(s) = a) = 0$. Hence the conclusion holds when f is strictly decreasing on $[0,\infty)$.

Suppose next that f is only non-increasing on $[0,\infty)$ and that $P(\int_0^\infty f(\sigma(r))\mathrm{d}r = t') > 0$ for some $t' > 0$. Still because the random variables $\sigma(\theta(t))$ ($t > 0$) have no atom (except ∞), this implies that for each $t \in (0,t')$, the probability that the function $x \mapsto \int_0^\infty f(x+\sigma^{(t)}(r))\mathrm{d}r$ is equal to $t'-t$ on some non-void interval is strictly positive, which implies in turn that

$$\forall t \in (0,t'), \exists q_t \in \mathbb{Q}^+ : P\left(\int_0^\infty f(q_t+\sigma(r))\mathrm{d}r = t\right) > 0. \tag{38}$$

On the other hand, $\forall q \in \mathbb{Q}^+$, the set of $t \in \mathbb{R}^+$ such that $P(\int_0^\infty f(q+\sigma(r))\mathrm{d}r = t) > 0$ is at most countable, hence the set of $(q,t) \in \mathbb{Q}^+ \times \mathbb{R}^+$ s.t. $P(\int_0^\infty f(q+\sigma(r))\mathrm{d}r = t) > 0$ is at most countable. This contradicts (38).

At last, when f is non-increasing (only) in a neighborhood of ∞, say on $[b,\infty)$, we can turn down to the previous case as follows: let $T_b := \inf\{t : \sigma(t) > b\}$ and write

$$\int_0^\infty f(\sigma(r))\mathrm{d}r = \int_0^{T_b} f(\sigma(r))\mathrm{d}r + \int_0^\infty f(\sigma(T_b) + \widetilde{\sigma}(r))\mathrm{d}r, \qquad (39)$$

where $\widetilde{\sigma}$ is independent of $(\sigma(t), t \le T_b)$ and distributed as σ. Conditional on $(\sigma(t), t \le T_b)$, we know that $\int_0^\infty f(\sigma(T_b) + \widetilde{\sigma}(r))\mathrm{d}r$ is atomless since $f(\sigma(T_b) + \cdot)$ is non-increasing on $[0,\infty)$. Therefore, using (39) and still conditioning on $(\sigma(t), t \le T_b)$, we see that $\int_0^\infty f(\sigma(r))\mathrm{d}r$ is also atomless. □

Fragmentations with Erosion

Until now, we have considered pure-jump fragmentation processes. However it is well-known that a fragmentation may have a continuous part, and more precisely, that a general homogeneous fragmentation is characterized by its dislocation measure ν and by an *erosion coefficient* $c \ge 0$ that measures the melting of the particles. More precisely, any homogeneous fragmentation F^{hom} can be factorized as $F^{\mathrm{hom}}(t) = e^{-ct}\overline{F}^{\mathrm{hom}}(t)$, $t \ge 0$, for some $c \ge 0$ and some pure-jump ν-homogeneous fragmentation $\overline{F}^{\mathrm{hom}}$. Exactly as in Sect. 1.1.1, one can then construct from any (c,ν)-homogeneous fragmentation, some (τ, c, ν) fragmentation and (τ, c, ν, I) fragmentation with immigration.

We still work under the hypotheses (H). Theorems 5 and 7 can then be modified as follows:

- all the results concerning the convergence of $(F_2^{(m)}, F_3^{(m)}, \ldots)$ are still valid, provided that in Theorem 5 we replace the (τ, ν, I) fragmentation with immigration by some (τ, c, ν, I) fragmentation with immigration
- under the assumptions of Theorem 5, this convergence holds jointly with that of $(m - F_1^{(m)})/m\tau(m)$ to the deterministic process $(ct, t \ge 0)$. Under the assumptions of Theorem 7, it holds jointly with that of $(m - F_1^{(m)}((\varphi_\nu/\tau)(m)\cdot))/m\varphi_\nu(m)$ to $(ct, t \ge 0)$.

The main difference in the proofs is that the subordinator ξ introduced in (7) is here replaced by the subordinator ξ_c, $\xi_c(t) := \xi(t) + ct$, $t \ge 0$.

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