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Loss of mass in deterministic and random fragmentations

Bénédicte Haas

Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, et C.N.R.S. UMR 7599, 175, rue du Chevaleret, F-75013 Paris, France

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Abstract

We consider a linear rate equation, depending on three parameters, that model fragmentation. For each of these fragmentation equations, there is a corresponding stochastic model, from which we construct an explicit solution to the equation. This solution is proved unique. We then use this solution to obtain criteria for the presence or absence of loss of mass in the fragmentation equation, as a function of the equation parameters. Next, we investigate small and large times asymptotic behavior of the total mass for a wide class of parameters. Finally, we study the loss of mass in the stochastic models.

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1. Introduction

Fragmentation of particles appear in various physical processes, such as polymer degradation, grinding, erosion and oxidation. In the models we consider, there are only particles with mass one at the initial time. Those particles split independent of each other to give smaller particles and each obtained particle splits in turn, independent of the past and of other particles etc. The splitting of a particle of mass x gives rise to a sequence of smaller particles with masses $xs_1, xs_2, \dots$ where $s_1 \geqslant s_2 \geqslant \cdots \geqslant 0$. Thus, it is convenient to introduce the following set:

$$\mathscr{S}^{\downarrow} := \left\{ s = (s_i)_{i \in \mathbb{N}^*},\ s_1 \geqslant s_2 \geqslant \cdots \geqslant 0 : \sum_{i=1}^{\infty} s_i \leqslant 1 \right\}.$$

E-mail address: haas@ccr.jussieu.fr (B. Haas).

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Note that we take into account the case when $\sum_{i=1}^{\infty} s_i < 1$, which corresponds to the loss of a part of the initial mass during the splitting. The rate at which a particle with mass one splits is then described by a non-negative measure v on $\mathscr{S}^* = \mathscr{S}^{\downarrow} \setminus \{(1,0,0,\ldots)\}$, called the *splitting measure.*This measure is supposed to fit the requirement

$$\int_{\mathscr{S}^*} (1-s_1) v(\mathrm{d}s) < \infty \tag{1}$$

(see Appendix A for an explanation). Note that the case when $v(\mathscr{S}^*) = \infty$, which is often excluded from fragmentation studies, is here included.

A linear rate equation has been developed (see e.g. Edwards et al., 1990) to study the time evolution of the mass distribution of particles involved in a fragmentation phenomenon (see also Beysens et al., 1995 for physical studies on fragmentation). Here, we consider the special case when the splitting rate for a particle with mass x is proportional to that of a particle with mass one. More precisely, this splitting rate is equal to $\tau(x)v(\mathrm{d}s)$, where τ is a continuous and positive function on $]0,1]$ such that $\tau(1) = 1$. As we will see in the next section, τ should be seen as the speed of fragmentation. Our deterministic fragmentation model is the weak form of this linear rate equation and describes the evolution of the family $(\mu_t, t \geqslant 0)$ of non-negative Radon measures on $]0,1]$, where $\mu_t(\mathrm{d}x)$ corresponds to the average number per unit volume of particles with mass in the interval $(x, x+\mathrm{d}x)$ at time t. This so-called *fragmentation equation* is

$$\begin{cases} \partial_t \langle \mu_t, f \rangle = \displaystyle\int_0^1 \tau(x) \left(-cxf'(x) + \int_{\mathscr{S}^{\downarrow}} \left[\sum_{i=1}^{\infty} f(xs_i) - f(x) \right] v(\mathrm{d}s) \right) \mu_t(\mathrm{d}x), \\ \mu_0 = \delta_1(\mathrm{d}x) \end{cases} \tag{2}$$

for test-functions f belonging to $\mathscr{C}_c^1(]0,1])$, the set of differentiable functions with compact support in $]0,1]$. The second term between parentheses on the right-hand side of Eq. (2) corresponds to a growth in the number of particles of masses $xs_1, xs_2, \ldots$ and to a decrease in the number of particles of mass x, as a consequence of the splitting of particles of mass x. The first term between parentheses on the right-hand side of (2) represents a loss of particles of mass x, as a result of erosion. The constant c is non-negative and called the *erosion coefficient* of the fragmentation. The measure v, the constant c and the function τ are called the parameters of the fragmentation equation.

We next introduce a random fragmentation model, called *fragmentation process*. A fragmentation process $(X(t),\ t \geqslant 0)$ is a Markov process with values in $\mathscr{S}^{\downarrow}$ satisfying the *fragmentation property*, which will be defined rigorously in Section 2. Informally, this means that given the system at a time t, say $X(t) = (s_1, s_2, \ldots)$, then for each $i \in \mathbb{N}^*$, the fragmentation system stemming from the particle with mass s_i evolves independent of the other particles and with the same law as the process X starting from a unique particle with mass s_i. And then, if we denote by $(s_{i,j}(r))_{j \geqslant 1}$ the masses of the particles stemming from the one with mass s_i after a time r, the process $X(t+r)$ will consist in the non-increasing rearrangement of the masses $(s_{i,j}(r))_{i,j \geqslant 1}$. A family of fragmentation processes with a scaling property (namely the self-similar fragmentation processes) was studied in Bertoin (2001, 2002a, b). In Section 2, the main results on these processes are recalled and a larger set of fragmentation processes, characterized each by the three parameters ν, c and τ of a fragmentation equation, is constructed.

This set of fragmentation processes is used to study the fragmentation equation. More precisely, given the parameters ν, c and τ, we construct in Section 3 the unique solution to the fragmentation equation with parameters ν, c and τ, by following a specific fragment (the so-called size-biased picked fragment process) of the corresponding fragmentation process. Let X^τ denote this fragmentation process. The solution to the fragmentation equation is then given for each $t \geqslant 0$ by

$$\langle \mu_t, f \rangle = E\left[\sum_{i=1}^{\infty} f(X_i^\tau(t))\right] \quad \text{for } f \in \mathscr{C}_c^1(]0,1]), \tag{3}$$

where $(X_1^\tau(t), X_2^\tau(t), \ldots)$ is the sequence $X^\tau(t)$. As a general rule, given a fragmentation process X, we denote by $(X_1(t), X_2(t), \ldots)$ the sequences $X(t)$, $t \geqslant 0$.

The main purpose of our work is to study the possible loss of mass in these deterministic and stochastic fragmentation models. If the family $(\mu_t, t \geqslant 0)$ is a solution to the fragmentation equation (2), it is easy to see that the total mass $\langle \mu_t, \mathrm{id} \rangle$ is non-increasing in t. We say that there is *loss of mass* in the fragmentation equation if there exists a time t such that

$$\langle \mu_t, \mathrm{id} \rangle < \langle \mu_0, \mathrm{id} \rangle = 1.$$

We will see that this is equivalent to loss of mass in the corresponding fragmentation process, as a result of

$$\exists t \geqslant 0 : \langle \mu_t, \mathrm{id} \rangle < 1 \quad \Leftrightarrow \quad \text{a.s. } \exists t \geqslant 0 : \sum_{i=1}^{\infty} X_i^\tau(t) < 1.$$

There are three distinct ways to lose mass. The first two are intuitively obvious: there is loss of mass if the erosion coefficient is positive or if the splitting of a particle with mass x gives rise to a sequence of particles with total mass strictly smaller than x. However, there is also an unexpected loss of mass, due to the formation of dust (i.e. an infinite number of particles with mass zero). This latter is of course the most interesting and one of our purposes is to establish for which parameters ν and τ it occurs. This formation of dust has to be compared with *gelation* which may happen in the context of coagulation models and which corresponds to the creation of an infinite-mass particle in finite time (see for example Jeon (1998) and Norris (2000) for gelation studies). We mention also Aldous (1999) for a survey on coagulation and fragmentation phenomena. Concerning loss of mass studies, Bertoin (2002b) proves the occurrence of loss of mass to dust in fragmentation processes with function $\tau(x) = x^\alpha$ as soon as $\alpha < 0$ and, in that case, that the mass vanishes entirely in finite time. Filippov (1961) obtains some conditions for the presence or absence of loss of mass (to compare with Corollary 8 in this paper) in the special case where $\nu(\mathscr{S}^*) < \infty$. Let us also mention Fournier and Giet (2003). They investigate this appearance of dust in some coagulation-fragmentation equations, whose fragmentation part is rather different from ours (their fragmentations are binary, with absolutely continuous rates that are not necessarily proportional to the one-mass rate). See also Jeon (2002).

Formula (3) is the key point in the study of loss of mass, which is undertaken in Section 4. We get necessary (respectively, sufficient) conditions on the parameters v, c and τ for loss of mass to occur and when there is loss of mass, we obtain results on small times and large times behavior of the total mass $\langle \mu_t, \mathrm{id}\rangle$. Section 5 is devoted to loss of mass and total loss of mass for a fragmentation process X^τ with parameters v, c and τ. Define ζ to be the first time at which all the mass has disappeared, i.e.

$$\zeta := \inf\{t \geqslant 0 : X_1^\tau(t) = 0\}.$$

We state necessary (respectively, sufficient) conditions on (v, c, τ) for $P(\zeta < \infty)$ to be positive. Then, we look at connections between loss of mass and total loss of mass and study the asymptotic behavior of $P(\zeta > t)$ as $t \to \infty$, for a large class of parameters v, c and τ.

This paper ends with an appendix containing on the one hand some results on the mass behavior of a fragmentation model constructed from the Brownian excursion of length 1 and on the other hand a proof that (1) is a necessary condition for our fragmentation models to exist.

2. Preliminaries on fragmentation processes

Let $(X(t), t \geqslant 0)$ be a Markov process with values in $\mathcal{S}^\downarrow$ and denote by P_s the law of X starting from $(s, 0, \ldots)$. The process X is a *fragmentation process* if it satisfies the following *fragmentation property*: for each $t_0 \geqslant 0$, conditionally on $X(t_0) = (s_1, s_2, \ldots)$, the process $(X(t + t_0), t \geqslant 0)$ has the same law as the process obtained, for each $t \geqslant 0$, by ranking in the non-increasing order the components of the sequences $X^1(t)$, $X^2(t), \ldots$, where the r.v.'s X^i are independent with respective laws P_{s_i}.

In this section, we first recall some results on homogeneous and self-similar fragmentation processes. Then we construct a larger family of fragmentation processes, depending on the parameters v, c and τ of the fragmentation equation (2). Given a fragmentation process X, recall the notation $(X_1(t), X_2(t), \ldots)$ for the sequence $X(t)$, $t \geqslant 0$.

2.1. Homogeneous and self-similar fragmentation processes

A *self-similar fragmentation process* $(X(t), t \geqslant 0)$ with index α is a fragmentation process having the following *scaling property*: if P_s is the law of X starting from $(s, 0, \ldots)$, then the law of $(sX(s^\alpha t), t \geqslant 0)$ under P_1 is P_s. If $\alpha = 0$, the fragmentation process X is said to be *homogeneous*. Bertoin (2002a) shows that we may always consider a càdlàg version of a self-similar fragmentation, the state $\mathcal{S}^\downarrow$ being endowed with the topology of pointwise convergence. We now recall some results on those processes. For more details, see Bertoin (2001, 2002a) and Berestycki (2002).

Interval representation: Let X be a self-similar fragmentation process. It may be convenient, for technical reasons, to work with an interval representation of X. Roughly, consider a Markov process F with state space the open sets of $]0, 1[$ and such that $F(t') \subset F(t)$ if $t' \geqslant t \geqslant 0$. The process F is called *self-similar interval fragmentation* *process* if it satisfies a scaling and a fragmentation property (for a precise definition, we refer to Bertoin, 2002a). The interesting point is that given X, we can build a self-similar interval fragmentation process, denoted by F_X, with the same index of similarity as X and such that $X(t)$ is the non-increasing sequence of the lengths of the interval components of $F_X(t)$, $t \geqslant 0$. In the sequel, we call F_X the *interval representation of* X. For each $t \geqslant 0$, we call *fragments* the interval components of $F_X(t)$ and denote by $I_x(t)$ the fragment containing the point x at time t. If such a fragment does not exist, $I_x(t) := \emptyset$. The length $|I_x(t)|$ is called the mass of the fragment.

Characterization and Poisson point process description of homogeneous fragmentation processes: The law of a homogeneous fragmentation process starting from $(1,0,\ldots)$ is characterized by two parameters: a non-negative real number c (the *erosion coefficient*) and a non-negative measure v on $\mathscr{S}^* = \mathscr{S}^{\downarrow} \setminus \{(1,0,\ldots)\}$ (the *splitting measure*) satisfying requirement (1). The erosion coefficient corresponds to the continuous part of the process, whereas the splitting measure describes the jumps of the process. More precisely, consider such a measure v and a Poisson point process $((\Delta(t),k(t)),t \geqslant 0)$ with values in $\mathscr{S}^* \times \mathbb{N}^*$ and whose characteristic measure is $v \otimes \#$, $\#$ denoting the counting measure on $\mathbb{N}^*$. As proved in Berestycki (2002), there is a pure jump càdlàg homogeneous fragmentation process X starting from $(1,0,\ldots)$, whose jumps are the times of occurrence of the Poisson point process and are described as follows: let t be a jump time, then the $k(t)$th term of $X(t^-)$, namely $X_{k(t)}(t^-)$, is removed and "replaced" by the sequence $X_{k(t)}(t^-)\Delta(t)$, i.e. $X(t)$ is obtained by ranking in the non-increasing order the components of sequences $(X_i(t^-)_{i \in \mathbb{N}^* \setminus \{k(t)\}}$ and $X_{k(t)}(t^-)\Delta(t)$. Now, consider a real number $c \geqslant 0$. The process $(\mathrm{e}^{-ct}X(t), t \geqslant 0)$ is also a càdlàg homogeneous fragmentation process. The point is that each homogeneous fragmentation process can be described like this for a constant $c \geqslant 0$ and a splitting measure v and then is called a homogeneous (v,c)-fragmentation process. Remark that when $v(S^*) = \infty$, each particle splits a.s. immediately.

Size-biased picked fragment process: Let X denote a homogeneous (v,c)-fragmentation process starting from $(1,0,\ldots)$ and let F_X be the interval representation of X. Consider a point picked at random in $]0,1[$ according to the uniform law on $]0,1[$ and independent of X and note $\lambda(t)$ the length of the fragment of F_X containing this point at time t. We call the process $(\lambda(t), t \geqslant 0)$ the *size-biased picked fragment process* of X. An important part of our work relies on the following property (see Bertoin (2001) for a proof): the process

$$(\xi(t), t \geqslant 0) := (-\log(\lambda(t)), t \geqslant 0) \tag{4}$$

is a subordinator (i.e. a right-continuous non-decreasing process with values in $[0,\infty]$, started from 0 and with independent and stationary increments on $[0,\varsigma[$, where ς is the first time when the process reaches ∞). We refer to Bertoin (1999) for background on subordinators. The distribution of ξ is then characterized by its Laplace exponent ϕ which is determined by

$$E[\exp(-q\xi_t)] = \exp(-t\phi(q)), \quad t \geqslant 0, \; q \geqslant 0$$

and which can be expressed here as a function of the parameters ν and c. More precisely,

$$\phi(q) = c(q+1) + \int_{\mathscr{S}^*} \left(1 - \sum_{i=1}^{\infty} s_i^{q+1}\right) \nu(\mathrm{d}s), \qquad q \geqslant 0. \tag{5}$$

In others words, the subordinator ξ has the following characteristics and will often be used in this work:

- killing rate $k = c + \int_{S^*} \left(1 - \sum_{i=1}^{\infty} s_i\right) \nu(\mathrm{d}s)$,
- drift coefficient $d = c$, (6)
- Lévy measure $L(\mathrm{d}x) = \mathrm{e}^{-x} \sum_{i=1}^{\infty} \nu(-\log(s_i) \in \mathrm{d}x), \quad x \in]0, \infty[$.

Recall that the first time ς when the process ξ reaches ∞ has an exponential law with parameter k and that there exists a subordinator η independent of ς, with the same drift coefficient and Lévy measure as ξ but with killing rate 0, such that $\xi_t = \eta_t$ when $t < \varsigma$.

We should point out that two different homogeneous fragmentation processes may lead to subordinators having the same distribution. For example, consider X^1 and X^2, two homogeneous fragmentation processes with erosion coefficient 0 and with respective splitting measures ν_1 and ν_2, where

$$\nu_1(\mathrm{d}s) = \tfrac{1}{2}\delta_{(1/2,1/2,0\ldots)}(\mathrm{d}s) + \tfrac{1}{2}\delta_{(1/2,1/4,1/4,0\ldots)}(\mathrm{d}s)$$

and

$$\nu_2(\mathrm{d}s) = \tfrac{3}{4}\delta_{(1/2,1/2,0\ldots)}(\mathrm{d}s) + \tfrac{1}{4}\delta_{(1/4,1/4,1/4,1/4,0\ldots)}(\mathrm{d}s).$$

Then in both cases, the Laplace exponent ϕ is given by

$$\phi(q) = 1 - \tfrac{3}{2}(\tfrac{1}{2})^{q+1} - (\tfrac{1}{4})^{q+1}.$$

Characterization of self-similar fragmentation processes: We have seen that a homogeneous fragmentation process is characterized by the two parameters ν and c. This property extends to self-similar fragmentation processes, which are characterized by three parameters: a splitting measure ν, an erosion coefficient c and the index of self-similarity α (this follows from a combination of results of Bertoin, 2001; Berestycki, 2002).

2.2. *Fragmentation processes* (ν, c, τ)

The purpose is to build fragmentation processes depending on the parameters ν, c and τ of the fragmentation equation (2). Recall that the function τ is continuous and positive on $]0,1]$ and such that $\tau(1)=1$. Throughout this paper, we will use the convention $\tau(0):=\infty$. Now, consider X a homogeneous (v,c)-fragmentation process and $(I_x(t), x\in]0,1[, t\geqslant 0)$ its interval representation. We introduce the time-change functions

$$T_x^{\tau}(t) := \inf\left\{u \geqslant 0: \int_0^u \frac{\mathrm{d}r}{\tau(|I_x(r)|)} > t\right\}, \quad t\geqslant 0,\ x\in]0,1[$$

with the convention $\inf\{\emptyset\}:=\infty$. Then, for each $t\geqslant 0$, consider the family of open intervals

$$\tilde{I}_x(t) := I_x(T_x^{\tau}(t)), \quad x\in]0,1[,$$

and remark that if $y\neq x$, either $\tilde{I}_x(t)=\tilde{I}_y(t)$ or $\tilde{I}_x(t)\cap\tilde{I}_y(t)=\emptyset$. Let $X^{\tau}(t)$ denote the non-increasing sequence of the lengths of the disjoint intervals of $(\tilde{I}_x(t), x\in]0,1[)$. Then, following the proof of Theorem 2 in Bertoin (2002a), we get:

Proposition 1. *The process $(X^{\tau}(t), t\geqslant 0)$ is a fragmentation process.*

We call the process X^{τ} a (v,c,τ)-*fragmentation process*. Note that if $\tau(x)=x^{\alpha}$ on $]0,1]$, $\alpha\in\mathbb{R}$, Theorem 2 in Bertoin (2002a) states that X^{τ} is a self-similar fragmentation process with parameters v, c and α.

If X^{τ_1} and X^{τ_2} are, respectively, (v,c,τ_1) and (v,c,τ_2)-fragmentation processes with $\tau_1\leqslant\tau_2$, the time-change functions T^{τ_1} and T^{τ_2} satisfy

$$T_x^{\tau_1}(t)\leqslant T_x^{\tau_2}(t) \quad \text{for } x\in]0,1[\text{ and } t\geqslant 0.$$

Then, at each time t and for each point $x\in]0,1[$, the fragment $I_x(T_x^{\tau_1}(t))$ is larger than $I_x(T_x^{\tau_2}(t))$. Informally, fragmentation is faster in the process X^{τ_2} than in X^{τ_1}.

As in the homogeneous case, consider the process

$$(\lambda^{\tau}(t), t\geqslant 0) := (|\tilde{I}_U(t)|, t\geqslant 0),$$

where U is a random variable uniformly distributed on $]0,1[$, independent of the fragmentation process X^{τ}. In other words, $\lambda^{\tau}(t)$ represents the mass at time t of the fragment containing a point picked at random uniformly in $]0,1[$ at time 0. It is easy to see that for each $t\geqslant 0$, if $X^{\tau}(t)=(X_1^{\tau}(t), X_2^{\tau}(t),\ldots)$, the law of $\lambda^{\tau}(t)$ is obtained as follows: consider $i(t)$ an integer-valued random variable such that

$$P(i(t)=i|X^{\tau}(t)) = X_i^{\tau}(t), \quad i\in\mathbb{N}^*,$$

$$P(i(t)=0|X^{\tau}(t)) = 1-\sum_{i=1}^{\infty} X_i^{\tau}(t).$$

Then,

$$\lambda^{\tau}(t) \overset{\text{law}}{\sim} X_{i(t)}^{\tau}(t), \tag{7}$$

where $X_0^\tau(t) := 0$. We call $(\lambda^\tau(t),\ t \geqslant 0)$ the *size-biased picked fragment process* of X^τ. The following proposition will be essential in the sequel. Its proof is straightforward.

Proposition 2. *If* $X^\tau(0) = (1, 0, \ldots)$, *the process* $(\lambda^\tau(t), t \geqslant 0)$ *has the same distribution as* $(\exp(-\xi_{\rho^\tau(t)}), t \geqslant 0)$, *where* ξ *is subordinator* (4) *constructed from the homogeneous process* X *and* ρ^τ *the time change*:

$$\rho^\tau(t) := \inf\left\{u \geqslant 0 : \int_0^u \frac{\mathrm{d}r}{\tau(\exp(-\xi_r))} > t\right\}. \tag{8}$$

It is then easy to see that $X_i^\tau(t) \overset{\text{a.s.}}{\to} 0$ as $t \to \infty$ for each $i \geqslant 0$ when the fragmentation process X^τ does not remain constant.

3. Existence and uniqueness of the solution to the fragmentation equation

Consider the fragmentation equation (2) with parameters v, c and τ and recall that a solution to this equation is a family of non-negative Radon measures on $]0, 1]$, satisfying (2) at least for test functions f belonging to $\mathscr{C}_c^1(]0, 1])$. Let X^τ be a (v, c, τ)-fragmentation process starting from $X^\tau(0) = (1, 0, \ldots)$. From X^τ, we build a solution to this fragmentation equation starting from $\mu_0 = \delta_1$ and prove that this solution is unique. More precisely, we have:

Theorem 3. *The fragmentation equation* (2) *has a unique solution* $(\mu_t, t \geqslant 0)$, *which is given for all* $t \geqslant 0$ *by*

$$\langle \mu_t, f\rangle = E\left[\sum_{i=1}^{\infty} f(X_i^\tau(t))\right] \quad \textit{for } f \in \mathscr{C}_c^1(]0, 1]).$$

Remark the following consequence of (7): for all $t \geqslant 0$ and all $f \in \mathscr{C}_c^1(]0, 1])$,

$$E\left[\sum_{i=1}^{\infty} f(X_i^\tau(t))\right] = E[\bar{f}(\lambda^\tau(t))], \tag{9}$$

where λ^τ is the size-biased picked fragment process related to X^τ and $\bar{f}$ the function defined from f by $\bar{f}(x) := f(x)/x$, $x \in]0, 1]$. This will be a key point of the proof of Theorem 3. In this proof, the notation C_K^1 refers to the set of differentiable functions on $]0, 1]$ with support in K.

Proof. (i) First, we turn the problem into an existence and uniqueness problem for an equation involving non-negative measures on $K = [a, 1]$, $0 < a \leqslant 1$. The advantage is that τ is bounded on K. Now, consider $(\pi_t, t \geqslant 0)$ a family of measures on $]0, 1]$ and set $\Pi_t(\mathrm{d}x) := x\pi_t(\mathrm{d}x)$, $t \geqslant 0$. It is easy to see that $(\pi_t, t \geqslant 0)$ solves Eq. (2) if and only if $(\Pi_t, t \geqslant 0)$ satisfies

$$\begin{cases} \partial_t \langle \Pi_t, f \rangle = \langle \Pi_t, \tau A(f) \rangle, & f \in C_c^1(]0,1]), \\ \Pi_0(\mathrm{d}x) = \delta_1(\mathrm{d}x), \end{cases} \tag{10}$$

where A is the linear operator on $C_c^1(]0,1])$ defined by

$$A(f)(x) = -cxf'(x) - cf(x) + \int_{\mathscr{S}^\downarrow} \left[\sum_{i=1}^{\infty} f(xs_i)s_i - f(x) \right] \nu(\mathrm{d}s), \quad x \in]0,1].$$

Note that if f is equal to 0 on $]0,a]$, so is $A(f)$. Then, $\tau A(f)$ is well defined on $[0,1]$ for functions $f \in C_c^1(]0,1])$. Moreover, this implies that the family $(\Pi_t, t \geqslant 0)$ is a solution to Eq. (10) if and only if, for each $0 < a \leqslant 1$, the family $(1_{[a,1]}\Pi_t, t \geqslant 0)$ is a solution to

$$\begin{cases} \partial_t \langle \nu_t, f \rangle = \langle \nu_t, \tau A(f) \rangle, & f \in C^1_{[a,1]}, \\ \nu_0(\mathrm{d}x) = \delta_1(\mathrm{d}x). \end{cases} \tag{11}$$

Then consider formula (9) and write l_t for the distribution of $\lambda^\tau(t)$, $t \geqslant 0$. Proving Theorem 3 is equivalent to prove that $(l_t, t \geqslant 0)$ is the unique solution to (10), which is true if and only if, for each $0 < a \leqslant 1$, the family $(1_{[a,1]}l_t,\ t \geqslant 0)$ is the unique family of non-negative measures on $[a,1]$ satisfying (11).

(ii) In the sequel, $K = [a,1]$, $0 < a \leqslant 1$. Consider the subordinator ξ such that $\lambda^\tau = \exp(-\xi_{\rho^\tau})$ where ρ^τ is the time change

$$\rho^\tau(t) = \inf \left\{ u \geqslant 0 : \int_0^u \frac{\mathrm{d}r}{\tau(\exp(-\xi_r))} > t \right\}$$

(see Proposition 2). As a subordinator, ξ is a Feller process on $[0,\infty]$ and its generator G^ξ has a domain containing the set of differentiable functions with compact support in $[0,\infty[$. It is well known that for every function f belonging to this set, the function $G^\xi(f)$ is given by

$$G^\xi(f)(x) = -kf(x) + \mathrm{d}f'(x) + \int_{]0,\infty[} (f(x+y) - f(x))L(dy), \quad x \in]0,1],$$

where k is the killing rate, d the drift coefficient and L the Lévy measure of ξ. From this and (6), we deduce that the generator $G^{\exp(-\xi)}$ of the Feller process $\exp(-\xi)$ has a domain $\mathscr{D}$ containing $\mathscr{C}_c^1(]0,1])$ and is given by

$$\begin{aligned} G^{\exp(-\xi)}(f)(x) &= -kf(x) - \mathrm{d}x\, f'(x) + \int_{]0,\infty[} (f(x\exp(-y)) - f(x))L(\mathrm{d}y) \\ &= A(f)(x) \end{aligned}$$

at least for $f \in \mathscr{C}_c^1(]0,1])$. Then, introduce the function

$$\tilde{\tau}(x) = \begin{cases} \tau(x) & \text{if } x \in K, \\ \tau(a) & \text{if } 0 \leqslant x \leqslant a \end{cases}$$

and consider the time-changed process $\exp(-\xi_{\rho^{\tilde{\tau}}(\cdot)})$, where

$$\rho^{\tilde{\tau}}(t) = \inf\left\{u \geqslant 0 : \int_0^u \frac{\mathrm{d}r}{\tilde{\tau}(\exp(-\xi_r))} > t\right\}.$$

Observing that $\tilde{\tau}$ is bounded away from 0 and ∞ on $[0,1]$, we apply Theorem 1 and its corollary in Lamperti (1967) to conclude that $\exp(-\xi_{\rho^{\tilde{\tau}}})$ is a Feller process and that its generator $G^{\exp(-\xi_{\rho^{\tilde{\tau}}})}$ has the same domain $\mathscr{D}$ as $G^{\exp(-\xi)}$ and is given by

$$G^{\exp(-\xi_{\rho^{\tilde{\tau}}})}(f) = \tilde{\tau} G^{\exp(-\xi)}(f), \quad f \in \mathscr{D}. \tag{12}$$

This formula can also be found in Section III.21 of Rogers and Williams (1994) (however, they do not consider the Feller property for the time-changed process). For each $t \geqslant 0$, denote by $\tilde{l}_t$ the law of the random variable $\exp(-\xi_{\rho^{\tilde{\tau}}(t)})$. The family $(\tilde{l}_t, t \geqslant 0)$ is then a solution to the Kolmogorov's forward equation:

$$\begin{cases} \partial_t \langle v_t, f\rangle = \langle v_t, G^{\exp(-\xi_{\rho^{\tilde{\tau}}})}(f)\rangle, & f \in \mathscr{D}, \\ v_0(\mathrm{d}x) = \delta_1(\mathrm{d}x). \end{cases} \tag{13}$$

Note that if the test-functions set is reduced to C_K^1, (13) is the same as Eq. (11), since $G^{\exp(-\xi_{\rho^{\tilde{\tau}}})} = \tau A$ on C_K^1. In particular, $(1_K l_t, t \geqslant 0)$ is a solution to (11), since for each $t \geqslant 0$ and each function f supported in K, the following identity holds:

$$E[f(\exp(-\xi_{\rho^{\tau}(t)})] = E[f(\exp(-\xi_{\rho^{\tilde{\tau}}(t)})].$$

This is due to the equality

$$\{t \geqslant 0 : \xi_{\rho^{\tau}(t)} \leqslant -\log a\} \overset{\text{a.s.}}{=} \{t \geqslant 0 : \xi_{\rho^{\tilde{\tau}}(t)} \leqslant -\log a\}$$

and the fact that $\rho^{\tau}(t) \overset{\text{a.s.}}{=} \rho^{\tilde{\tau}}(t)$ on this set. All this follows easily from the definitions of ρ^{τ} and $\rho^{\tilde{\tau}}$.

(iii) Now, it remains to prove that a non-negative solution to Eq. (13) is uniquely determined on K if the test-functions set is C_K^1. To prove this, it is sufficient to show that for each $\gamma > 0$, the image of C_K^1 by the operator $(\gamma\,\mathrm{id} - G^{\exp(-\xi_{\rho^{\tilde{\tau}}})})$ is dense in C_K^0 (the set of continuous functions with support in K) endowed with the uniform norm—see for instance the proof of Proposition 9.18 of Chapter 4 in Ethier and Kurtz (1986) and note that if $(v_t, t \geqslant 0)$ is a solution to (13), the functions $t \mapsto \langle v_t, f\rangle$ are continuous on $[0,\infty)$ for each $f \in C_K^1$. Thus, we just have to prove this density. To that end, observe that if $x < a$ and if $f \in C_K^0$,

$$E_x[f(\exp(-\xi_t))] := E[f(\exp(-\xi_t)) \mid \exp(-\xi_0) = x] = E_1[f(x\exp(-\xi_t))] = 0.$$

Therefore, the function $x \mapsto E_x[f(\exp(-\xi_t))]$ belongs to C_K^0 if $f \in C_K^0$. This allows us to consider the restriction of the generator $G^{\exp(-\xi)}$ to C_K^0, denoted by $G^{\exp(-\xi)}/C_K^0$. This operator is the generator of the strongly continuous contraction semigroup on C_K^0 defined by

$$T(t) : f \in C_K^0 \mapsto T(t)(f) \in C_K^0,$$
$$T(t)(f)(x) = E_1[f(x\exp(-\xi_t))], \quad x \in]0,1].$$

Its domain is $C_K^0 \cap \mathscr{D}$. The same remark holds for the process $\exp(-\xi_{\rho^{\tilde\tau}})$ (because we know that it is a Feller process and then the function $x \mapsto E_x[f(\exp(-\xi_{\rho^{\tilde\tau}(t)}))]$ is continuous if f is continuous). We denote by $G^{\exp(-\xi_{\rho^{\tilde\tau}})}/C_K^0$ the restriction of $G^{\exp(-\xi_{\rho^{\tilde\tau}})}$ to C_K^0. Its domain is $C_K^0 \cap \mathscr{D}$ as well. Now, to conclude, we just have to apply the forthcoming Lemma 4 to

$$E = K, \quad \mathscr{B} = C_K^0, \quad G = G^{\exp(-\xi)}/C_K^0, \quad \tilde{G} = G^{\exp(-\xi_{\rho^{\tilde\tau}})}/C_K^0 \quad \text{and} \quad D = C_K^1.$$

Indeed, generators G and $\tilde{G}$ satisfy (12), with $\tilde\tau$ bounded away from 0. The set C_K^1 is dense in C_K^0 and it is clear that the function $x \mapsto E_1[f(x\exp(-\xi_t))]$ belongs to C_K^1 as soon as f does. □

Lemma 4. *Let E be a metric space and $\mathscr{B}$ the Banach space of real-valued continuous bounded functions on E, endowed with the uniform norm. Let G be the generator of a strongly continuous contraction semigroup $(T(t), t \geqslant 0)$ on $\mathscr{B}$, with domain $\mathscr{D}(G)$. Consider $D \subset \mathscr{D}(G)$, a dense subspace of $\mathscr{B}$ such that $T(t): D \to D$ for all $t \geqslant 0$, and $\tilde\tau \in \mathscr{B}$ such that $\tilde\tau \geqslant m$ on E for some positive constant m. If $\tilde{G}$ is the generator of a strongly continuous contraction semigroup on $\mathscr{B}$ such that $\mathscr{D}(\tilde{G}) = \mathscr{D}(G)$ and $\tilde{G}(f) = \tilde\tau G(f)$ on $\mathscr{D}(G)$, then for every $\gamma > 0$, $(\gamma\,\mathrm{id} - \tilde{G})(D)$ is dense in $\mathscr{B}$.*

Proof. We need the notion of core. If A is a closed linear operator on $\mathscr{B}$, a subspace C of $\mathscr{D}(A)$ is a *core* for A if the following equivalence holds:

$$f \in \mathscr{D}(A) \text{ and } g = A(f)$$

$$\Leftrightarrow$$

$$\text{there is a sequence } (f_n) \in C \text{ such that } f_n \to f \text{ and } A(f_n) \to g.$$

The assumptions on D and $(T(t), t \geqslant 0)$ and Proposition 3.3 of Chapter 1 in Ethier and Kurtz (1986) ensure that D is a core for G. But then, D is also a core for $\tilde{G}$: if (f_n) is a sequence in D such that $f_n \to f$ and $\tilde{G}(f_n) \to g$, then, since $\tilde{G}(f_n) = \tilde\tau G(f_n)$ and $\tilde\tau \geqslant m > 0$ on E, $G(f_n) \to g/\tilde\tau$. Thus $f \in \mathscr{D}(G) = \mathscr{D}(\tilde{G})$ and $\tilde{G}(f) = \tilde\tau G(f) = g$. Conversely, given f belonging to $\mathscr{D}(\tilde{G}) = \mathscr{D}(G)$ and $g = \tilde{G}(f)$, there is a sequence $(f_n) \in D$ such that $f_n \to f$ and $G(f_n) \to G(f)$. But $\tilde\tau$ is bounded on E and then $\tilde{G}(f_n) \to \tilde{G}(f)$. At last, we conclude by using Proposition 3.1 of Chapter 1 in Ethier and Kurtz (1986). This proposition states that since D is a core for the generator $\tilde{G}$, then $(\gamma\,\mathrm{id} - \tilde{G})(D)$ is dense in $\mathscr{B}$ for *some* $\gamma > 0$, but it is easy to see with Lemma 2.11 (Chapter 1 in Ethier and Kurtz (1986)) that it holds for all $\gamma > 0$. □

Remark. As shown in Section 2, two homogeneous fragmentation processes with different laws may lead to subordinators with the same laws. Therefore, it may happen that two different fragmentation equations (i.e. with different parameters) have the same solution.

From Theorem 3, we deduce that the unique solution $(\mu_t, t \geqslant 0)$ to the fragmentation equation (2) is the hydrodynamic limit of stochastic fragmentation models. More precisely:

Corollary 5. *For each* $n \in \mathbb{N}^*$, *let* $X^{\tau,n}$ *be a* (ν, c, τ)*-fragmentation process starting from* $X^{\tau,n}(0) = (\underbrace{1, 1, \ldots, 1}_{n \ \text{term}}, 0, \ldots)$. *Then for each* $t \geqslant 0$, *with probability one*,

$$\frac{1}{n} \sum_{i=1}^{\infty} \delta_{X_i^{\tau,n}(t)}(\mathrm{d}x) \overset{\text{vaguely on }]0,1]}{\underset{n \to \infty}{\to}} \mu_t.$$

Proof. For each $k \in \{1, \ldots, n\}$, we denote by $((X_{k,1}^{\tau,n}(t), \ldots, X_{k,i}^{\tau,n}(t), \ldots), t \geqslant 0)$ the fragmentation process stemming from the kth fragment of $X^{\tau,n}(0)$. These processes are independent and identically distributed, with the distribution of a (ν, c, τ)-fragmentation process starting from $(1, 0, \ldots)$. Then fix $t \geqslant 0$. Using the strong law of large numbers for each $f \in \mathscr{C}_c^1(]0,1])$, we get

$$\frac{1}{n} \left(\sum_{i=1}^{\infty} f(X_i^{\tau,n}(t)) \right) = \frac{1}{n} \sum_{k=1}^{n} \left(\sum_{i=1}^{\infty} f(X_{k,i}^{\tau,n}(t)) \right) \overset{\text{a.s.}}{\underset{n \to \infty}{\to}} \langle \mu_t, f \rangle. \tag{14}$$

With probability one, this convergence holds for each function f such that for a $n \in \mathbb{N}^*$

$$f(x) = \begin{cases} 0 & \text{on }]0, \frac{1}{n}], \\ \left(x - \dfrac{1}{n} \right)^2 P(x) & \text{on }]\frac{1}{n}, 1], \\ \text{where } P \text{ is a polynomial with rational coefficients,} \end{cases}$$

since this set of functions—denoted $\mathscr{T}$—is countable. Observe that this set is dense in $\mathscr{C}_c^1(]0,1])$ for the uniform norm and for each $f \in \mathscr{C}_c^1(]0,1])$ consider a sequence $(g_k)_{k \geqslant 0}$ of functions of $\mathscr{T}$ such that $g_k \underset{k \to \infty}{\to} f/\mathrm{id}$. Since $\sum_{i=1}^{\infty} X_i^{\tau,n}(t) \leqslant n$,

$$\frac{1}{n} \left(\sum_{i=1}^{\infty} X_i^{\tau,n}(t) g_k(X_i^{\tau,n}(t)) \right) \overset{\text{uniformly in } n}{\underset{k \to \infty}{\to}} \frac{1}{n} \left(\sum_{i=1}^{\infty} f(X_i^{\tau,n}(t)) \right) \quad \text{a.s.}$$

and then it is easily seen that with probability one convergence (14) holds for each $f \in \mathscr{C}_c^1(]0,1])$. $\square$

Note that the question whether a similar result holds for the Smoluchowski's coagulation equation or not is still open (see Aldous, 1999). The problem is that the Smoluchowski's coagulation equation is non-linear and then the mean frequencies of the stochastic models do not evolve as the Smoluchowski's coagulation equation, contrary to what happens for the fragmentation equation. Nonetheless, Norris (1999) proved that under suitable assumptions on the coagulation kernel, the solution to Smoluchowski's coagulation equation may be obtained as the hydrodynamic limit of stochastic systems of coagulating particles.

4. Loss of mass in the fragmentation equation

Let $(\mu_t, t \geqslant 0)$ be the unique solution to the fragmentation equation (2) with parameters ν, c and τ and consider for each $t \geqslant 0$ the total mass of the system at time t

$$m(t) = \int_0^1 x\mu_t(\mathrm{d}x).$$

In this section, we give necessary (resp. sufficient) conditions on the parameters ν, c and τ for the occurrence of loss of mass (i.e. the existence of a time t such that $m(t) < m(0)$). Then, when loss of mass occurs, we study the asymptotic behavior of $m(t)$ as $t \to 0$ or $t \to \infty$ for a large class of parameters (ν, c, τ). This loss of mass study relies on the fact that the solution $(\mu_t, t \geqslant 0)$ can be constructed from a $(\nu,\ c, \tau)$-fragmentation process, denoted by X^τ (see the previous section). In particular, by monotone convergence, one can extend formula (9) to the pair of functions $(f, \bar{f}) = (\mathrm{id}, 1_{x>0})$. Hence,

$$m(t) = E\left[\sum_{i=1}^{\infty} X_i^\tau(t)\right] = P(\lambda^\tau(t) > 0), \qquad t \geqslant 0,$$

where $(\lambda^\tau(t), t \geqslant 0)$ is the size-biased picked fragment process related to X^τ. Then recall Proposition 2 and introduce the random variable

$$I_\tau := \int_0^\infty \frac{\mathrm{d}r}{\tau(\exp(-\xi_r))}. \tag{15}$$

Since $\tau(0) = \infty$, it is clear that I_τ is the first time when λ^τ is equal to 0. This leads to another expression of the mass

$$m(t) = P(I_\tau > t) \tag{16}$$

which will be useful in this section. Note that for self-similar fragmentations, i.e. $\tau(x) = x^\alpha$ on $]0, 1]$, $\alpha \in \mathbb{R}$, I_τ is the well-known *exponential functional of the Lévy process* $\alpha\xi$ (for background, we refer e.g. to Bertoin and Yor, 2002; Carmona et al., 1997).

At last, we recall that ϕ denotes the Laplace exponent of the subordinator ξ and can be expressed as a function of ν and c (see (5)) and that k, c and L are the characteristics of ξ (see (6)).

From now on, we exclude the degenerate case when the splitting measure ν and the erosion rate c are 0, for which there is obviously no loss of mass.

4.1. A criterion for loss of mass

If $k > 0$, either the erosion coefficient c is positive or a part of the mass of a particle may be lost during its splitting (i.e. $\nu(\sum_{i=1}^{\infty} s_i < 1) > 0$). Therefore, it is intuitively clear that if $k > 0$, there is loss of mass. Nevertheless, loss of mass may occur even when $k = 0$, as some particles may be reduced to dust in finite time. This phenomenon can be explained as follows when τ decreases near 0. Small fragments split even faster since their mass is smaller. Therefore, particles split faster and faster as time passes and so they may be reduced to dust in finite time. We now present a qualitative criterion for loss of mass.

Proposition 6. (i) *If* $k>0$, *there is loss of mass and* $\inf\{t \geqslant 0 : m(t) < m(0)\} = 0$.
(ii) *If* $k=0$, *then*

$$\int_{0^+} \frac{\phi'(x)}{\tau_{\inf}(\exp(-1/x))\phi^2(x)}\,\mathrm{d}x < \infty \quad \Rightarrow \quad \textit{there is loss of mass},$$

$$\int_{0^+} \frac{\phi'(x)}{\tau_{\sup}(\exp(-1/x))\phi^2(x)}\,\mathrm{d}x = \infty \quad \Rightarrow \quad \textit{there is no loss of mass},$$

where $\tau_{\inf}$ *and* $\tau_{\sup}$ *are the continuous non-increasing functions defined on* $]0,1]$ *by*

$$\tau_{\inf}(x) = \inf_{y\in]0,x]} \tau(y) \quad \textit{and} \quad \tau_{\sup}(x) = \sup_{y\in[x,1]} \tau(y).$$

Remark.

- If τ is bounded on $]0,1]$, we have that

$$\int_{0^+} \frac{\phi'(x)}{\tau_{\sup}(\exp(-1/x))\phi^2(x)}\,\mathrm{d}x = \infty,$$

since $\tau_{\sup}$ is then bounded on $]0,1]$ and $\int_{0^+} \phi'(x)\phi^{-2}(x)\,\mathrm{d}x = \infty$ (recall that $\phi(0)=0$). Thus, if τ is bounded on $]0,1]$ and $k=0$, there is no loss of mass. In particular, when $k=0$, there is no loss of mass in the homogeneous case (i.e. $\tau=1$).
- If τ is non-increasing near 0 and $k=0$, either $\lim_{x\to 0^+}\tau(x)<\infty$ and then there is no loss of mass or $\lim_{x\to 0^+}\tau(x)=\infty$ and then the functions $\tau_{\inf}$, τ and $\tau_{\sup}$ coincide on some neighborhood of 0. In both cases, the following equivalence holds:

$$\int_{0^+} \frac{\phi'(x)}{\tau(\exp(-1/x))\phi^2(x)}\,\mathrm{d}x < \infty \quad \Leftrightarrow \quad \text{there is loss of mass.}$$

In order to prove Proposition 6, observe that loss of mass occurs if and only if $P(I_\tau < \infty) > 0$, which justifies the use of the forthcoming lemma (see Lemma 3.6 in Bertoin, 1999):

Lemma 7. *Let* σ *be a subordinator with killing rate* 0 *and* U *its potential measure, which means that for each measurable function* f, $\int_0^\infty f(x)U(\mathrm{d}x) = E[\int_0^\infty f(\sigma_t)\,\mathrm{d}t]$. *Let* $h:[0,\infty) \to [0,\infty)$ *be a non-increasing function. Then the following are equivalent*:

(i) $\int_0^\infty h(x)U(\mathrm{d}x) < \infty$,
(ii) $P(\int_0^\infty h(\sigma_t)\,\mathrm{d}t < \infty) = 1$,
(iii) $P(\int_0^\infty h(\sigma_t)\,\mathrm{d}t < \infty) > 0$.

Proof of Proposition 6. (i) Let $e(k)$ denote the exponential random variable with parameter k at which the subordinator ξ is killed and η the subordinator with killing rate 0, independent of $e(k)$ and such that $\xi_t = \eta_t$ if $t < e(k)$ and $\xi_t = \infty$ if $t \geqslant e(k)$. Then, set

$$T^\tau(t) := \inf\left\{u \geqslant 0 : \int_0^u \frac{\mathrm{d}r}{\tau(\exp(-\eta_r))} > t\right\}.$$

This random variable is independent of $e(k)$ and using that for each time t

$$P(I_\tau > t) \quad \Leftrightarrow \quad T^\tau(t) \leqslant e(k),$$

we get

$$m(t) = E[\mathrm{e}^{-kT^\tau(t)}].$$

Note that this is true even if $k = 0$, with the convention $0 \times \infty := \infty$. Now if $k > 0$ and $t > 0$, $kT^\tau(t) > 0$ with probability one and then $m(t) < 1$.

(ii) Let U denote the potential measure of the subordinator ξ. It is straightforward that

$$\int_0^\infty \frac{U(\mathrm{d}x)}{\tau_{\inf}(\exp(-x))} < \infty \quad \Rightarrow \quad P(I_\tau < \infty) = 1$$

and it follows from Lemma 7 that

$$\int_0^\infty \frac{U(\mathrm{d}x)}{\tau_{\sup}(\exp(-x))} = \infty \quad \Rightarrow \quad P(I_\tau < \infty) = 0.$$

Thus, we just have to prove that for each continuous positive and non-increasing function f on $]0,1]$

$$\int_0^\infty \frac{U(\mathrm{d}x)}{f(\exp(-x))} < \infty \quad \Leftrightarrow \quad \int_{0^+} \frac{\phi'(x)}{f(\exp(-1/x))\phi^2(x)}\,\mathrm{d}x < \infty. \tag{17}$$

To that end, recall that the repartition function $U(x) = \int_0^x U(\mathrm{d}y)$ satisfies

$$U \asymp \frac{1}{\phi(1/\cdot)}, \tag{18}$$

where the notation $g \asymp h$ indicates that there are two positives constants C and C' such that $Cg \leqslant h \leqslant C'g$ (see Proposition 1.4 in Bertoin, 1999). Then if $\lim_{x\to 0^+} f(x) < \infty$,

$$\int_0^\infty \frac{U(\mathrm{d}x)}{f(\exp(-x))} = \int_{0^+} \frac{\phi'(x)}{f(\exp(-1/x))\phi^2(x)}\,\mathrm{d}x = \infty$$

since $U(\infty) = \infty$ and $\int_{0^+} \phi'(x)\phi^{-2}(x)\,\mathrm{d}x = \infty$. Next, if $\lim_{x\to 0^+} f(x) = \infty$, introduce the non-negative finite measure V defined on $[0,\infty[$ by

$$\int_0^x V(\mathrm{d}y) = \frac{1}{f(1)} - \frac{1}{f(\exp(-x))}.$$

Note that

$$\int_0^\infty \frac{U(\mathrm{d}x)}{f(\exp(-x))} = \int_0^\infty \int_x^\infty V(\mathrm{d}y)U(\mathrm{d}x) = \int_0^\infty U(y)V(\mathrm{d}y).$$

Combining this with (18) leads to the following equivalences:

$$\begin{aligned}\int_0^\infty \frac{U(\mathrm{d}x)}{f(\exp(-x))} < \infty &\Leftrightarrow \int_0^\infty \frac{V(\mathrm{d}y)}{\phi(1/y)} < \infty \\ &\Leftrightarrow \int_0^\infty \int_{1/y}^\infty \frac{\phi'(z)}{\phi^2(z)}\,\mathrm{d}z\, V(\mathrm{d}y) < \infty \\ &\Leftrightarrow \int_0^\infty \frac{\phi'(z)}{f(\exp(-1/z))\phi^2(z)}\,\mathrm{d}z < \infty\end{aligned}$$

and then to equivalence (17), since

$$\int_.^\infty \frac{\phi'(z)}{f(\exp(-1/z))\phi^2(z)}\,\mathrm{d}z < \infty \quad \text{(the case when } \xi = 0 \text{ is excluded).} \qquad \square$$

Provided that τ is non-increasing near 0 and $\phi'(0^+) < \infty$, the following corollary gives a simple necessary and sufficient condition on τ for loss of mass to occur. This result may be found in Filippov's (1961) paper in the special case when $v(S^*) < \infty$. Recall the notations $\tau_{\inf}$ and $\tau_{\sup}$ introduced in Proposition 6.

Corollary 8. *Suppose that $k = 0$. Then,*

(i) $\int_{0^+} \mathrm{d}x/x\tau_{\inf}(x) < \infty \Rightarrow$ *loss of mass.*
(ii) *If* $\phi'(0^+) < \infty$ (i.e. $\int_{\mathscr{S}^\downarrow} (\sum_{i=1}^\infty |\log(s_i)|s_i)v(\mathrm{d}s) < \infty$),

$$\textit{loss of mass} \Rightarrow \int_{0^+} \frac{\mathrm{d}x}{x\tau_{\sup}(x)} < \infty.$$

If τ is non-increasing in a neighborhood of $0, \tau_{\inf}$ and $\tau_{\sup}$ can be replaced by τ.

In particular, as soon as $\tau(x) \geqslant |\log x|^\alpha$ near 0 for some $\alpha > 1$, there is loss of mass.

Proof. The assumption $k = 0$ leads to

$$\frac{\phi(q)}{q} \underset{q\to 0}{\to} \int_0^\infty xL(\mathrm{d}x) = \int_{\mathscr{S}^\downarrow} \left(\sum_{i=1}^\infty (-\log(s_i))s_i\right) v(\mathrm{d}s).$$

Remark that $\int_0^\infty xL(\mathrm{d}x) \neq 0$, since $L \neq 0$ and then $\phi'(0^+) > 0$. If moreover $\phi'(0^+) < \infty$, we have

$$\frac{\phi'(x)}{\tau_{\sup}(\exp(-1/x))\phi^2(x)} \underset{x\to 0^+}{\sim} \frac{1}{\phi'(0^+)x^2\tau_{\sup}(\exp(-1/x))}.$$

Combining this with Proposition 6(ii) leads to result (ii). Now, if $\phi'(0^+) = \infty$, the function $x \mapsto x^2\phi'(x)\phi^{-2}(x)$ is still bounded near 0 and then we deduce (i) in the same way. $\square$

4.2. *Asymptotic behavior of the mass*

Our purpose is to study the asymptotic behavior of the mass $m(t) = \langle \mu_t, \mathrm{id} \rangle$ as $t \to 0$ or $t \to \infty$.

4.2.1. *Small times asymptotic behavior*

Proposition 9. *Assume that* $\phi'(0^+) < \infty$ *and* $\tau(x) \leqslant Cx^\alpha$, $0 < x \leqslant 1$, *with* $C > 0$ *and* $\alpha < 0$. *Then,* m *is differentiable at* 0^+ *and* $m'(0^+) = -k$.

Remark. We will see in the proof that the upper bound

$$\limsup_{t \to 0^+} \frac{m(t) - 1}{t} \leqslant -k$$

remains valid without any assumption on τ and ϕ.

Proof. As shown in the first part of the proof of Proposition 6

$$m(t) = E[\mathrm{e}^{-kT^\tau(t)}], \quad t \geqslant 0,$$

where T^τ is the time change

$$T^\tau(t) = \inf\left\{u \geqslant 0 : \int_0^u \frac{\mathrm{d}r}{\tau(\exp(-\eta_r))} > t\right\}$$

and η a subordinator with killing rate 0, drift coefficient c and Lévy measure L. Hence,

$$\frac{1 - m(t)}{t} = E\left[\frac{1 - \mathrm{e}^{-kT^\tau(t)}}{t}\right]. \tag{19}$$

Observe that it is sufficient to prove the statement for functions τ bounded on $]0, 1]$ or non-increasing and such that $\tau(x) \leqslant Cx^\alpha$ on $]0, 1]$ for some $C > 0$ and $\alpha < 0$. Indeed, for each continuous positive function τ such that $\tau(x) \leqslant Cx^\alpha$ on $]0, 1]$ with $C > 0$ and $\alpha < 0$, there are two continuous positive functions τ_1 and τ_2 such that $\tau_1 \leqslant \tau \leqslant \tau_2$ on $]0, 1]$ and

- τ_1 is bounded on $]0, 1]$ and $\tau_1(1) = 1$;
- τ_2 is non-increasing, $\tau_2(x) \leqslant Cx^\alpha$ on $]0, 1]$ and $\tau_2(1) = 1$

(we may take for example $\tau_2(x) := \sup_{y \in [x,1]} \tau(y)$). Then combine this with the fact that

$$\frac{1 - m_{\tilde{\tau}}(t)}{t} \leqslant \frac{1 - m_{\bar{\tau}}(t)}{t} \quad \forall t \geqslant 0 \tag{20}$$

when $\tilde{\tau} \leqslant \bar{\tau}$ on $]0,1]$ (here $m_{\tilde{\tau}}$ and $m_{\bar{\tau}}$ denote the respective masses of a $(v,c,\tilde{\tau})$-fragmentation equation and a $(v,c,\bar{\tau})$-fragmentation equation).

(i) For t such that $T^{\tau}(t) < \infty$, the time change T^{τ} can be expressed as follows:

$$\begin{aligned} T^{\tau}(t) &= \int_0^t \tau(\exp(-\eta_{T^{\tau}(r)}))\,\mathrm{d}r \\ &= t\int_0^1 \tau(\exp(-\eta_{T^{\tau}(tr)}))\,\mathrm{d}r. \end{aligned} \tag{21}$$

Note that the first time when T^{τ} reaches ∞ is positive with probability one. Then if τ is bounded (resp. non-increasing), we get by the dominated convergence theorem (resp. monotone convergence theorem), that

$$\frac{1-\mathrm{e}^{-kT^{\tau}(t)}}{t} \underset{t\to 0^+}{\overset{\text{a.s.}}{\to}} k.$$

If τ is bounded the dominated convergence theorem applies and gives

$$\frac{1-m(t)}{t} \underset{t\to 0^+}{\to} k.$$

(ii) To conclude when τ is non-increasing and smaller than the function $x \mapsto Cx^{\alpha}$, it remains to show that $(1-\mathrm{e}^{-kT^{\tau}(t)})/t$ is dominated—independently of t—by a random variable with finite expectation. To see this, first note that it is sufficient to prove the domination for $(1-\mathrm{e}^{-kT^{\alpha}(t)})/t$, where

$$T^{\alpha}(t) = \inf\left\{u \geqslant 0 : \int_0^u \exp(\alpha\eta_r)\,\mathrm{d}r > Ct\right\}$$

(since $T^{\tau}(t) \leqslant T^{\alpha}(t)$ for $t \geqslant 0$). Next, remark that if η^1 is a subordinator such that $\eta^1 \geqslant \eta$, the following inequality between time changes holds:

$$T_1^{\alpha}(t) := \inf\left\{u \geqslant 0 : \int_0^u \exp(\alpha\eta_r^1)\,\mathrm{d}r > Ct\right\} \geqslant T^{\alpha}(t)$$

and then

$$\frac{1-\mathrm{e}^{-kT^{\alpha}(t)}}{t} \leqslant \frac{1-\mathrm{e}^{-kT_1^{\alpha}(t)}}{t} \quad \text{for each } t \geqslant 0.$$

Thus it is sufficient to prove the domination for a subordinator bigger than η and so we can (and will) assume that the subordinator η has a drift coefficient $c \geqslant k/|\alpha|$. Now introduce the exponential functional

$$I_{\alpha} := \int_0^{\infty} \exp(\alpha\eta_r)\,\mathrm{d}r.$$

Observe that

$$C^{-1}I_{\alpha} = \inf\{t \geqslant 0 : T^{\alpha}(t) = \infty\}.$$

If $t < C^{-1}I_\alpha$, we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-kT^\alpha(t)}) = -kC\mathrm{e}^{-\alpha\eta_{T^\alpha(t)}}\mathrm{e}^{-kT^\alpha(t)} \tag{22}$$

(by using (21) for the function $\tau = Cx^\alpha$). But the (random) function $t \mapsto -\alpha\eta_t - kt$ is non-decreasing, since $c \geqslant k/|\alpha|$ and the process $(\eta_t - ct, t \geqslant 0)$ is a (pure jump) non-decreasing process (according to the Lévy–Itô decomposition of a subordinator—see Proposition 1.3 in Bertoin, 1999). Thus derivative (22) is non-increasing and $t \mapsto \mathrm{e}^{-kT^\alpha(t)}$ is a concave function on $[0, C^{-1}I_\alpha[$. From this, it follows that the slope $(1 - \mathrm{e}^{-kT^\alpha(t)})/t$ is non-decreasing on $[0, C^{-1}I_\alpha[$ and it is straightforward that it is decreasing on $[C^{-1}I_\alpha, \infty[$. This leads to the upper bound

$$\frac{1 - \mathrm{e}^{-kT^\tau(t)}}{t} \leqslant \frac{C}{I_\alpha} \quad \forall t \geqslant 0.$$

By Proposition 3.1 (iv) in Carmona et al. (1997), the expectation

$$E[I_\alpha^{-1}] = (-\alpha)\phi'(0^+) < \infty,$$

and this ends the proof. $\square$

If $k = 0$, there is a more precise result. Recall (16) and set $A := \sup\{a \geqslant 0 : E[I_\tau^{-a}] < \infty\}$. Then for each $\varepsilon > 0$ such that $A - \varepsilon > 0$,

$$t^{\varepsilon - A}(1 - m(t)) \leqslant \int_0^t x^{\varepsilon - A} P_{I_\tau}(\mathrm{d}x) \underset{t \to 0^+}{\to} 0,$$

since $E[I_\tau^{\varepsilon - A}] < \infty$. (Actually, it is easy to see that

$$\liminf_{t \to 0^+} \frac{\log(1 - m(t))}{\log(t)} = A.)$$

For self-similar fragmentation processes, this points out the influence of α on the loss of mass behavior near 0. Indeed, consider a family of self-similar fragmentation processes such that the subordinator ξ is fixed (with killing rate $k = 0$) and α varies, $\alpha < 0$. Then introduce the set

$$\mathcal{Q} := \left\{ q \in \mathbb{R} : \int_{x > 1} \mathrm{e}^{qx} L(\mathrm{d}x) < \infty \right\}.$$

This set is convex and contains 0. Let $\underline{p}$ be its right-most point. According to Theorem 25.17 in Sato (1999),

$$q \in \mathcal{Q} \;\Leftrightarrow\; E[\mathrm{e}^{q\xi_t}] < \infty \quad \forall t \geqslant 0$$

and in that case $E[\mathrm{e}^{q\xi_t}] = \mathrm{e}^{-t\phi(-q)}$. Then, following the proof of Proposition 2 in Bertoin and Yor (2002), we get

$$E[I_\alpha^{-q-1}] = \frac{-\phi(\alpha q)}{q} E[I_\alpha^{-q}] \quad \text{for } q < \frac{\underline{p}}{|\alpha|},$$

which leads to

$$\frac{p}{|\alpha|} \leqslant \sup\{q : E[I_\alpha^{-q-1}] < \infty\}.$$

And then

$$\liminf_{t\to 0^+} \frac{\log(1-m(t))}{\log(t)} \geqslant 1 + \frac{p}{|\alpha|}.$$

4.2.2. Large times asymptotic behavior

The main result of this subsection is the existence of exponential bounds for the mass $m(t)$ when t is large enough and when the parameters v, c and τ satisfy conditions (i) and (ii) of the following Proposition 11. Before proving this, we point out the following intuitive result, which is valid for all parameters v, c and τ.

Proposition 10. *When loss of mass occurs*, $m(t) \underset{t\to\infty}{\to} 0$.

Proof. From formula (16), we get that $m(t) \underset{t\to\infty}{\to} P(I_\tau = \infty)$. When $k > 0$, the subordinator ξ is killed at a finite time $e(k)$ and then

$$I_\tau \leqslant e(k) \sup_{[\exp(-\xi_{e(k)-}),1]} (1/\tau)$$

which is a.s. finite. When $k = 0$, our goal is to prove that the probability $P(I_\tau = \infty)$ is either 0 or 1. To that end, we introduce the family of i.i.d. random variables, defined for all $n \in \mathbb{N}$ by

$$X_n := (\xi_{n+t} - \xi_n)_{0 \leqslant t \leqslant 1}.$$

It is clear that I_τ can be expressed as a function of the random variables X_n. Then, since for all $n \in \mathbb{N}$

$$\{I_\tau = \infty\} = \left\{ \int_n^\infty \frac{\mathrm{d}r}{\tau(\exp(-\xi_r))} = \infty \right\},$$

it is easily seen that the set $\{I_\tau = \infty\}$ is invariant under finite permutations of the r.v. X_n, $n \in \mathbb{N}$. Hence, we can conclude by using the Hewitt-Savage 0-1 law (see e.g. Theorem 3, Section IV in Feller, 1971). □

Our sharper study of the asymptotic behavior of the mass $m(t)$ as $t \to \infty$ relies on the moments properties of the random variable I_τ. If $\tau(x) = x^\alpha$, $\alpha < 0$, it is well known that the entire moments of I_τ are given by

$$E[I_\tau^n] = \frac{n!}{\phi(-\alpha)\ldots\phi(-\alpha n)}, \quad n \in \mathbb{N}^*, \tag{23}$$

and then, that

$$E[\exp(rI_\tau)] < \infty \quad \text{for } r < \phi(\infty) := \lim_{q\to\infty} \phi(q).$$

(see Proposition 3.3 in Carmona et al., 1997). From this and formula (16) we deduce that the mass $m(t)$ decays at an exponential rate as $t \to \infty$, since for a positive $r < \phi(\infty)$,

$$m(t) = P(I_\tau > t) \leqslant \exp(-rt)E[\exp(rI_\tau)], \quad t \geqslant 0. \tag{24}$$

This result is still valid for a function $\tau(x) \geqslant Cx^\alpha$, where $\alpha < 0$ and $C > 0$, because $I_\tau \leqslant C^{-1}\int_0^\infty \exp(\alpha\xi_r)\,\mathrm{d}r$. Remark that until now, we have made no assumption on ϕ. We now state deeper results when ϕ behaves like a regularly varying function. Recall that a real function f *varies regularly with index* $a \geqslant 0$ *at* ∞ if

$$\frac{f(rx)}{f(x)} \underset{x\to\infty}{\to} r^a \quad \forall r > 0.$$

If $a = 0$, f is said to be slowly varying. Recall also that the notation $f \asymp g$ indicates that there exist two positives constants C and C' such that $Cg \leqslant f \leqslant C'g$.

Proposition 11. *Assume that*

(i) $C_2x^\beta \leqslant \tau(x) \leqslant C_1x^\alpha, 0 < x \leqslant 1, \alpha \leqslant \beta < 0, C_1 > 0, C_2 > 0.$
(ii) $\phi \asymp f$ *on* $[1,\infty)$, *where* f *varies regularly at* ∞ *with index* $a \in]0,1[$.

Denote by ψ *the inverse of the function* $t \mapsto t/\phi(t)$, *which is a bijection from* $[1,\infty)$ *to* $[1/\phi(1),\infty)$. *Then there exist two positive constants* A *and* B *such that for* t *large enough*

$$\exp(-B\psi(t)) \leqslant m(t) \leqslant \exp(-A\psi(t)). \tag{25}$$

Actually, if ϕ satisfies (ii), it is sufficient to suppose that $C_2x^\beta \leqslant \tau(x)$ with $\beta < 0$ and $C_2 > 0$ to obtain the upper bound $m(t) \leqslant \exp(-A\psi(t))$ and conversely, if $\tau(x) \leqslant C_1x^\alpha$ with $\alpha < 0$ and $C_1 > 0$, the lower bound $\exp(-B\psi(t))$ holds.

Remark. If $\tau(x) = x^\alpha$ for $x \in]0,1]$, $\alpha < 0$, and ϕ varies regularly at ∞ with index $a \in]0,1[$, it follows from a result in Rivero (2002) that

$$\log(m(t)) \underset{t\to\infty}{\sim} \frac{(1-a)a^{a/(a-1)}}{\alpha}\,\psi\left(\frac{-\alpha t}{a^a}\right).$$

We should also point out that there are some homogeneous fragmentation processes such that the associated Laplace exponent ϕ satisfies assumption (ii) without varying regularly.

Proof. The proof relies on Theorems 1 and 2 in Kôno (1977), which we now recall. Let σ be a non-decreasing and "nearly regularly varying function with index b", $b \in]0,1[$, which means that there exist two positive constants $r_1 \geqslant r_2$ and a slowly varying function s such that

$$r_2 x^b s(x) \leqslant \sigma(x) \leqslant r_1 x^b s(x) \quad \text{for } x \geqslant 1. \tag{26}$$

Let Y be a positive random variable such that, for n large enough,

$$c_2^{2n} \prod_{k=1}^{2n} \sigma(k) \leqslant E[Y^{2n}] \leqslant c_1^{2n} \prod_{k=1}^{2n} \sigma(k), \tag{27}$$

where c_1 and c_2 are positive constants. Then, there exist three positive constants A, B and C such that for x large enough,

$$\exp(-Bx) \leqslant P(Y \geqslant C\sigma(x)) \leqslant \exp(-Ax).$$

Coming back to the proof, we set

$$\sigma(x) := \frac{x}{\phi(x)}, \quad x \geqslant 1.$$

This is an increasing continuous function (by the concavity of ϕ) such that $\lim_{x\to\infty} \sigma(x) = \infty$ (by assumption (ii)). In particular, its inverse ψ is well defined and increasing on $[\sigma(1), \infty)$. Since f varies regularly with index $a \in]0,1[$, there exists a slowly varying function g such that $f(q) = q^a g(q)$ for $q \geqslant 1$. Then it follows from assumption (ii) that σ satisfies (26) with $b = 1 - a$ and $s = 1/g$ (note that g is a positive function on $[1, \infty)$). On the other hand, recall that if $\tau(x) = x^\alpha$, $\alpha < 0$, the entire moments of the random variable I_τ are given by (23). Thus, for each function τ satisfying assumption (i), we have

$$C_1^{-n} \prod_{k=1}^{n} \frac{k}{\phi(-\alpha k)} \leqslant E[I_\tau^n] \leqslant C_2^{-n} \prod_{k=1}^{n} \frac{k}{\phi(-\beta k)}.$$

Moreover, assumption (ii) implies that for each $C > 0$, $\phi(Ct) \asymp \phi(t)$ at least for $t \in [1, \infty)$. Therefore, the moments of I_τ satisfy (27). Then, by applying the theorems recalled at the beginning of the proof, we get

$$\exp(-B\psi(t/C)) \leqslant m(t) = P(I_\tau > t) \leqslant \exp(-A\psi(t/C)) \quad \text{for } t \text{ large enough.} \tag{28}$$

It remains to remove the constant C. To that end, introduce $h(x) := x/f(x)$ on $[1, \infty)$ and consider the generalized inverse of h:

$$h^{\leftarrow}(x) := \inf\{y \in [1,\infty) : h(y) > x\}, \quad x \in [1/f(1), \infty).$$

The function h varies regularly with index $1 - a$ and so, according to Theorem 1.5.12 in Bingham et al. (1987), $h^{\leftarrow}$ varies regularly with index $1/(1-a)$ and $h(h^{\leftarrow}(x)) \underset{x\to\infty}{\sim} x$. From this latter and assumption (ii), we deduce the existence of two positive constant D_1 and D_2 such that

$$D_1 x \leqslant \sigma(h^{\leftarrow}(x)) \leqslant D_2 x \quad \text{for } x \text{ large enough.}$$

And since ψ is increasing, we have

$$\psi(D_1 x) \leqslant h^{\leftarrow}(x) \leqslant \psi(D_2 x) \quad \text{for } x \text{ large enough.}$$

But then, since $h^{\leftarrow}$ varies regularly, the function $x \mapsto \psi(x/C)/\psi(x)$ is bounded away from 0 and ∞ when $x \to \infty$. Then combine this with (28) to obtain (25). $\square$

Note that assumption (ii) in Proposition 11 implies that the erosion rate c is equal to 0. Now, if $c > 0$ and if $\tau(x) \geqslant Ax^{\alpha}$ on $]0,1]$, with $\alpha < 0$ and $A > 0$, we observe that the mass $m(t)$ is equal to 0, as soon as $t \geqslant 1/|A\alpha c|$. Indeed, recall that

$$k \geqslant c \text{ and } \xi_t \geqslant ct \quad \text{for each } t \geqslant 0.$$

Then,

$$I_\tau \leqslant \frac{(1 - \exp(\alpha c e(k)))}{|A\alpha c|}$$

which leads to

$$\begin{cases} m(t) = 0 & \text{if } t \geqslant 1/|A\alpha c|, \\ m(t) \leqslant (1 + A\alpha ct)^{k/|\alpha c|} & \text{if } t \leqslant 1/|A\alpha c|. \end{cases}$$

In the same way, we obtain that $m(t) \leqslant e^{-kat}$ if $\tau \geqslant a$ on $]0,1]$ (before that, we had exponential upper bounds only when $\tau(x) \geqslant Ax^{\alpha}$, with $\alpha < 0$ and $A > 0$).

5. Loss of mass in fragmentation processes

Let X^{τ} be a (v, c, τ)-fragmentation process starting from $(1, 0, \ldots)$. We say that there is loss of mass in this random fragmentation if

$$P\left(\exists t \geqslant 0 : \sum_{i=1}^{\infty} X_i^{\tau}(t) < 1\right) > 0.$$

The results on the occurrence of this (stochastic) loss of mass as a function of the parameters v, c and τ are exactly the same as those on the occurrence of loss of mass for the corresponding deterministic model (constructed from X^{τ} by formula (3)). Indeed, the point is that, as shown in the proof of Proposition 10, the probability $P(I_\tau < \infty)$ is either 0 or 1 and then that the events $\{\exists t \geqslant 0 : \sum_{i=1}^{\infty} X_i^{\tau}(t) < 1\}$ and $\{I_\tau < \infty\}$ coincide apart from an event of probability 0. Thus, Proposition 6 and its corollary are still valid for the loss of mass in the fragmentation process X^{τ} and when there is loss of mass, it occurs with probability one.

When there is loss of mass, one may wonder if there exists a finite time at which all the mass has disappeared, i.e. if

$$\zeta := \inf\{t \geqslant 0 : X_1^{\tau}(t) = 0\} < \infty.$$

In the sequel, we will say that there is *total loss of mass* if $P(\zeta < \infty) > 0$. Bertoin (2002b) proves that total loss of mass occurs with probability one for a self similar fragmentation process with a negative index. Here, we give criteria on the parameters v, c and τ for the presence or absence of total loss of mass. From this we deduce that even if $k = 0$ there is no equivalence in general between loss of mass and total loss of mass. Eventually, we study the asymptotic behavior of $P(\zeta > t)$ as $t \to \infty$, when the parameters ν, c and τ satisfy the same assumptions as in Proposition 11.

The following remark will be useful in this section: if X^{τ} and $X^{\tau'}$ are two fragmentation processes constructed from the same homogeneous one and if $\tau \leqslant \tau'$ on $]0,1]$, then

$$\inf\{t \geqslant 0 : X_1^{\tau'}(t) = 0\} \leqslant \inf\{t \geqslant 0 : X_1^{\tau}(t) = 0\}. \tag{29}$$

5.1. A criterion for total loss of mass

Proposition 12. *Consider the continuous non-increasing functions* $\tau_{\inf}$ *and* $\tau_{\sup}$ *constructed from* τ *as in the statement of Proposition* 6.

(i) *If* $\int_{0^+} \mathrm{d}x/x\tau_{\inf}(x) < \infty$, *then* $P(\zeta < \infty) = 1$.
(ii) *If* $k = 0$ *and* $\int_{\mathscr{S}^{\downarrow}} |\log(s_1)| \nu(\mathrm{d}s) < \infty$, *then*

$$P(\zeta < \infty) > 0 \;\Rightarrow\; \int_{0^+} \frac{\mathrm{d}x}{x\tau_{\sup}(x)} < \infty.$$

If τ *is non-increasing in a neighborhood of* 0, $\tau_{\inf}$ *and* $\tau_{\sup}$ *can be replaced by* τ.

Remark.

- This should be compared with Corollary 8 which states similar connections between loss of mass and the integrability near 0 of functions $x \mapsto 1/x\tau_{\inf}(x)$ and $x \mapsto 1/x\tau_{\sup}(x)$.
- The condition $\int_{\mathscr{S}^{\downarrow}} |\log(s_1)| \nu(\mathrm{d}s) < \infty$ is satisfied as soon as $\nu(s_1 \leqslant \varepsilon) = 0$ for a positive ε, since $|\log(s_1)| \leqslant \varepsilon^{-1}(1 - s_1)$ when s_1 belongs to $]\varepsilon, 1]$. In particular, this last condition on the measure ν is satisfied for fragmentation models where $k = 0$ and such that the splitting of a particle gives at most n fragments (i.e. $\nu(s_{n+1} > 0) = 0$). Indeed, we have then that $\nu(s_1 < 1/n) = 0$, since $\nu(\sum_{i=1}^{\infty} s_i < 1) = 0$ when $k = 0$.

Proof. We just have to prove these assertions for a non-increasing function τ and then use remark (29). Thus in this proof τ is supposed to be non-increasing on $]0,1]$.

As shown in Section 2.2, the interval representation $(\tilde{I}_x(t), x \in]0,1[, t \geqslant 0)$ of X^{τ} is constructed from the interval representation $(I_x(t), x \in]0,1[, t \geqslant 0)$ of a homogeneous (ν, c) fragmentation process X in the following way:

$$\tilde{I}_x(t) = I_x(T_x^{\tau}(t)),$$

where

$$T_x^{\tau}(t) = \inf\left\{u \geqslant 0 : \int_0^u \frac{\mathrm{d}r}{\tau(|I_x(r)|)} > t\right\}.$$

For every x in $]0,1[$, set $\zeta_x := \inf\{t : I_x(t) = 0\}$. Then,

$$T_x^\tau(t) < \zeta_x \quad \text{if and only if } t < \int_0^\infty \frac{\mathrm{d}r}{\tau(|I_x(r)|)},$$

which leads to

$$\zeta = \sup_{x\in]0,1[} \int_0^\infty \frac{\mathrm{d}r}{\tau(|I_x(r)|)}. \tag{30}$$

(i) This part is merely adapted from the proof of Proposition 2(i) in Bertoin (2002b). In particular, as mentioned there,

$$\limsup_{r\to\infty} r^{-1} \log X_1(r) < 0.$$

Thus there exists a random positive number C such that

$$\frac{1}{\tau(|I_x(r)|)} \leqslant \frac{1}{\tau(\exp(-Cr))} \quad \text{for all } x \in]0,1[\text{ and all } r \geqslant 0,$$

since moreover τ is non-increasing. Now, we just have to combine this with equality (30) and the fact that the function $x \mapsto 1/x\tau(x)$ is integrable near 0 to conclude that $\zeta < \infty$ a.s.

(ii) Since $k = 0$, the drift coefficient c is equal to 0 and then the homogeneous fragmentation process X is a pure jump process constructed from a Poisson point process $((\Delta(t), k(t)), t \geqslant 0) \in S^* \times N^*$, with characteristic measure $\nu \otimes \#$ (see Section 2.1). From this process, we build another jump process Y which we first describe informally: $Y(0) = 1$ and for each time t, $Y(t)$ is an element of the sequence $X(t)$. When the fragment with mass Y splits, we keep the largest fragment and Y jumps to the mass of this new fragment, etc. Note that generally, the jump times may accumulate. Now, we give a rigorous construction of Y, by induction. To that end, we build simultaneously a sequence of particular times $(t_n)_{n\in\mathbb{N}}$. Set $t_0 := 0$ and $Y(t_0) := 1$. Suppose that t_{n-1} is known, that it is a randomized stopping time, and that Y is constructed until t_{n-1}. Let $k(n-1)$ be such that $Y(t_{n-1}) = X_{k(n-1)}(t_{n-1})$ and consider the fragmentation process stemming from $X_{k(n-1)}(t_{n-1})$. Since X is homogeneous, there exists a homogeneous (ν, c)-fragmentation process independent of $(X(t), t \leqslant t_{n-1})$, denoted by X^{n-1}, such that the fragmentation process stemming from $X_{k(n-1)}(t_{n-1})$ is equal to $Y(t_{n-1})X^{n-1}$. Let λ^{n-1} and $((\Delta^{n-1}(t), k^{n-1}(t)), t \geqslant 0)$ be respectively the size-biased picked fragment process and the Poisson point process related to X^{n-1}. Then set

$$t_n := t_{n-1} + \inf\{t : \lambda^{n-1}(t) < \tfrac{1}{2}\},$$

$$Y(t) := Y(t_{n-1})X_1^{n-1}(t - t_{n-1}), \quad t_{n-1} \leqslant t < t_n,$$

$$Y(t_n) := \begin{cases} \Delta_1^{n-1}(t_n - t_{n-1})Y(t_{n-1})X_1^{n-1}((t_n - t_{n-1})^-) & \text{if } k^{n-1}(t_n - t_{n-1}) = 1, \\ Y(t_{n-1})X_1^{n-1}(t_n - t_{n-1}) & \text{otherwise.} \end{cases}$$

Time t_n is a randomized stopping time. Note that the random variables $(t_n - t_{n-1})$ are iid with a positive expectation. So $t_n \to \infty$ and Y is then well defined on $[0, \infty)$.

Call σ the non-decreasing process $(-\log(Y))$ and consider the jumps $\tilde{\Delta}(t) := \sigma(t) - \sigma(t^-)$, $t \geqslant 0$. It is easily seen that $(\tilde{\Delta}(t), t \geqslant 0)$ is a Poisson point process on $]0,\infty[$ with characteristic measure $\nu(-\log s_1 \in \mathrm{d}x)$. In other words, σ is a subordinator with Laplace exponent

$$\varphi(q) = \int_{\mathscr{S}^\downarrow} (1 - s_1^q)\nu(\mathrm{d}s), \quad q \geqslant 0.$$

It can be shown that for each $t \geqslant 0$ there exists a (random) point $x_t \in]0,1[$ such that $Y(r) = |I_{x_t}(r)|$ for $r \leqslant t$. Combine this with equality (30) to conclude that

$$\zeta \geqslant \int_0^t \frac{\mathrm{d}r}{\tau(\exp(-\sigma(r))} \quad \text{for all } t \geqslant 0$$

and then

$$\zeta \geqslant \int_0^\infty \frac{\mathrm{d}r}{\tau(\exp(-\sigma(r))}.$$

Therefore, the assumption $P(\zeta < \infty) > 0$ implies that

$$P\left(\int_0^\infty \frac{\mathrm{d}r}{\tau(\exp(-\sigma(r))} < \infty\right) > 0$$

and so, following the proof of Proposition 6(ii), we conclude that

$$\int_{0^+} \frac{\varphi'(x)}{\tau(\exp(-1/x))\varphi^2(x)}\,\mathrm{d}x < \infty.$$

Together with the assumption

$$\varphi'(0^+) = \int_{\mathscr{S}^\downarrow} |\log(s_1)|\nu(\mathrm{d}s) < \infty,$$

this implies that $\int_{0^+} (1/x\tau(x))\,\mathrm{d}x < \infty$. $\square$

5.2. *Does loss of mass imply total loss of mass?*

If the killing rate k is positive, loss of mass always occurs, but in general total loss of mass does not. Think for example of a pure erosion process. Now, we focus on what happens when $k = 0$, i.e. when the loss of mass corresponds only to particles reduced to dust. First, if the Laplace exponent ϕ has a finite right-derivative at 0 and if τ is non-increasing near 0, loss of mass is equivalent to total loss of mass and both occur with probability zero or one. This just follows from a combination of Corollary 8(ii) and Proposition 12(i). However, without this assumption on ϕ there may be loss of mass but no total loss of mass. Here is an example: fix $a \in]0,1[$ and take the parameters ν, c and τ as follows:

- $\nu(\mathrm{d}s) = \sum_{n=1}^{\infty} \left(\frac{1}{n^a} - \frac{1}{(n+1)^a}\right) \delta_{\left(\frac{1}{2}, \underbrace{\frac{1}{2^{n+1}}, \ldots, \frac{1}{2^{n+1}}}_{2^n \text{ terms}}, 0, \ldots\right)}(\mathrm{d}s)$,
- $c = 0$,
- $\tau(x) = \begin{cases} 1 & \text{if } x \geqslant \mathrm{e}^{-1}, \\ (-\log x) & \text{if } 0 < x \leqslant \mathrm{e}^{-1}. \end{cases}$

It is clear that τ is decreasing on $]0, \mathrm{e}^{-1}]$ and $k = 0$.

Lemma 13. *Let ϕ be the Laplace exponent specified by (5) for the parameters above. Then $\phi(q) \geqslant Cq^a$ for some $C > 0$ and for all $q \in [0, 1]$.*

Proof. Consider the function

$$f(q) = \int_1^{\infty} (1 - \mathrm{e}^{-(\log 2)qx}) x^{-1-a} \, \mathrm{d}x$$
$$= (q \log 2)^a \int_{q \log 2}^{\infty} (1 - \mathrm{e}^{-x}) x^{-1-a} \, \mathrm{d}x.$$

The integral $\int_0^{\infty} (1 - \mathrm{e}^{-x}) x^{-1-a} \, \mathrm{d}x$ is positive and finite since $a \in]0, 1[$. Then there exists a positive real number C such that

$$f(q) \geqslant Cq^a \quad \forall q \in [0, 1].$$

On the other hand, remark that

$$f(q) = \sum_{n=1}^{\infty} \int_n^{n+1} (1 - \mathrm{e}^{-(\log 2)qx}) x^{-1-a} \, \mathrm{d}x$$
$$\leqslant \sum_{n=1}^{\infty} (1 - \mathrm{e}^{-(\log 2)q(n+1)}) \int_n^{n+1} x^{-1-a} \, \mathrm{d}x$$
$$\leqslant \frac{1}{a} \sum_{n=1}^{\infty} (1 - \mathrm{e}^{-(\log 2)q(n+1)}) \left(\frac{1}{n^a} - \frac{1}{(n+1)^a}\right).$$

As a consequence, the following inequality holds:

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{2^{q(n+1)}}\right) \left(\frac{1}{n^a} - \frac{1}{(n+1)^a}\right) \geqslant aCq^a \quad \forall q \in [0, 1].$$

This leads to:

$$\begin{aligned}
\phi(q) &= \int_{\mathcal{S}^*} \left(1 - \sum_{i=1}^{\infty} s_i^{q+1}\right) \nu(\mathrm{d}s) \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{n^a} - \frac{1}{(n+1)^a}\right) \left(1 - \left(\frac{1}{2}\right)^{q+1} - 2^n \times \frac{1}{2^{(n+1)(q+1)}}\right) \\
&\geqslant \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{n^a} - \frac{1}{(n+1)^a}\right) \left(1 - \frac{1}{2^{(n+1)q}}\right) \\
&\geqslant \frac{a}{2} Cq^a \quad \forall q \in [0,1]. \qquad \square
\end{aligned}$$

From this we deduce that there is loss of mass. Indeed, $\phi'(x) \leqslant x^{-1}\phi(x)$ for positive x since ϕ is a concave function. Then combine this with Lemma 13 to obtain that

$$\frac{1}{\tau(\exp(-1/x))} \times \frac{\phi'(x)}{\phi^2(x)} \leqslant \frac{1}{Cx^a} \quad \text{for } 0 < x \leqslant 1$$

and conclude with Proposition 6(ii). On the other hand, there is no total loss of mass since the equalities

$$\int_{\mathcal{S}^{\downarrow}} |\log(s_1)| \nu(\mathrm{d}s) = \log 2 \quad \text{and} \quad \int_0^1 \frac{\mathrm{d}x}{x\tau(x)} = \infty$$

imply with Proposition 12(ii) that $P(\zeta < \infty) = 0$.

5.3. *Asymptotic behavior of $P(\zeta > t)$ as $t \to \infty$*

In this subsection, we consider functions τ such that $C_2 x^{\beta} \leqslant \tau(x) \leqslant C_1 x^{\alpha}$ for $x \in]0,1]$, where $\alpha \leqslant \beta < 0$ and C_1 and C_2 are positive constants. Thus there is total loss of mass with probability one. The following proposition states that $P(\zeta > t)$ and $m(t)$ have then the same type of behavior as $t \to \infty$ (see also Proposition 11). More precisely, we have

Proposition 14. *Suppose that $C_2 x^{\beta} \leqslant \tau(x) \leqslant C_1 x^{\alpha}$ for $x \in]0,1]$, where $\alpha \leqslant \beta < 0$, $C_1 > 0$ and $C_2 > 0$. Then,*

(i) *$\exists C > 0$ such that $P(\zeta > t) \leqslant \exp(-Ct)$ for t large enough.*
(ii) *If $\phi \asymp f$ on $[1,\infty)$, for a function f varying regularly with index $a \in]0,1[$ at ∞, there are two positive constants A and B such that for t large enough*

$$\exp(-B\psi(t)) \leqslant P(\zeta > t) \leqslant \exp(-A\psi(t)),$$

where ψ is the inverse of the bijection $t \in [1,\infty) \mapsto t/\phi(t) \in [1/\phi(1),\infty)$.

Actually, the upper bounds hold as soon as $\tau(x) \geqslant C_2 x^{\beta}$, with $\beta < 0$ and $C_2 > 0$ and the lower bound holds for functions τ satisfying only $\tau(x) \leqslant C_1 x^{\alpha}$, with $\alpha < 0$ and $C_1 > 0$.

To prove the proposition we need the following lemma:

Lemma 15. *Let X be a self-similar fragmentation process with index $\alpha < 0$ and ζ the first time at which the entire mass has disappeared. Fix $\alpha' \geqslant \alpha$. Then, there exists a self-similar fragmentation process with the same parameters (v,c) as X and with index α', denoted* by *X', such that*

$$\zeta \leqslant \int_0^\infty (X_1'(r))^{\alpha'-\alpha}\,\mathrm{d}r.$$

Proof. Consider $(I_x(t), x \in]0,1[, t \geqslant 0)$ the interval representation of X. There exists a self-similar interval representation process with parameters (v,c) and with index α', denoted by $(I_x'(t), x \in]0,1[, t \geqslant 0)$, such that

$$I_x(t) = I_x'(T_x(t)),$$

where

$$T_x(t) = \inf\left\{u \geqslant 0 : \int_0^u |I_x'(r)|^{\alpha'-\alpha}\,\mathrm{d}r > t\right\}$$

(see Section 3.2. in Bertoin, 2002a). For each $t \geqslant 0$, call $X'(t)$ the non-increasing rearrangement of the lengths of the disjoint intervals components of $(I_x'(t), x \in]0,1[)$. Then X' is the required self-similar fragmentation process with index α'. Let x be in $]0,1[$. Since $|I_x'(r)| \leqslant X_1'(r)$ for each $r \geqslant 0$, we have that

$$T_x\left(\int_0^\infty (X_1'(r))^{\alpha'-\alpha}\,\mathrm{d}r\right) = \infty.$$

Then,

$$\zeta \leqslant \int_0^\infty (X_1'(r))^{\alpha'-\alpha}\,\mathrm{d}r,$$

because $I_x'(\infty) = 0$ for every x in $]0,1[$. $\square$

Proof of Proposition 14. If $\tau' = K\tau$ for a positive constant K and if X^τ and $X^{\tau'}$ are two fragmentation processes constructed from the same homogeneous one, it is easily seen that $X_1^{\tau'}(t) = X_1^\tau(Kt)$ for each $t \geqslant 0$. Recall moreover remark (29). Since it is supposed that $C_2 x^\beta \leqslant \tau(x) \leqslant C_1 x^\alpha$ on $]0,1]$, where $\alpha \leqslant \beta < 0$, it is then enough to prove results (i) and (ii) for a self-similar fragmentation process with a negative index. Thus, consider X a self-similar fragmentation process with index $\alpha < 0$. Applying the previous lemma to X and $\alpha' = \alpha/2$, we get

$$\begin{aligned} P(\zeta > 2t) &\leqslant P\left(\int_0^\infty (X_1'(rt))^{-\alpha/2}\,\mathrm{d}r > 2\right) \\ &\leqslant P\left(\int_1^\infty (X_1'(rt))^{-\alpha/2}\,\mathrm{d}r > 1\right) \\ &\leqslant \int_1^\infty E[(X_1'(rt))^{-\alpha/2}]\,\mathrm{d}r, \end{aligned} \tag{31}$$

since $X_1'(t) \leqslant 1,\ \forall t \geqslant 0$. Now, recall that

$$E[X_1'(t)] \leqslant E\left[\sum_{i=1}^{\infty} X_i'(t)\right] = m_{\tau'}(t),$$

where $m_{\tau'}$ is the total mass of the fragmentation equation with the same parameters (ν, c) as X and with parameter $\tau'(x) = x^{\alpha/2}$. This leads to

$$E[(X_1'(t))^{-\alpha/2}] \leqslant \begin{cases} m_{\tau'}(t) & \text{if } (-\alpha/2) \geqslant 1, \\ (m_{\tau'}(t))^{-\alpha/2} & \text{if } (-\alpha/2) < 1 \quad \text{(by Jensen's inequality)}. \end{cases} \tag{32}$$

(i) Combining (31), (32) and (24), we obtain that for t large enough

$$P(\zeta > 2t) \leqslant \int_1^{\infty} \exp(-C'rt)\,\mathrm{d}r = \frac{1}{C't}\exp(-C't),$$

where C' is a positive constant.

(ii) As stated in Proposition 11, since $\phi \asymp f$, with f a regularly varying function with index $a \in]0,1[$, and since $\tau'(x) = x^{\alpha/2}$, the function

$$\sigma : t \in [1,\infty) \mapsto t/\phi(t) \in [1/\phi(1),\infty)$$

is an increasing bijection and its inverse ψ satisfies $m_{\tau'}(t) \leqslant \exp(-A_1\psi(t))$ for a constant $A_1 > 0$ and t large enough. From this and inequalities (31) and (32), we deduce the existence of a positive constant A_2 so that for t large enough,

$$P(\zeta > 2t) \leqslant \int_1^{\infty} \exp(-A_2\psi(rt))\,\mathrm{d}r.$$

Moreover, σ is differentiable and its derivative is positive and smaller than $1/\phi$ (recall that ϕ' is positive) and then σ' is bounded on $[1,\infty)$. Thus for t large enough,

$$\begin{aligned} P(\zeta > 2t) &\leqslant t^{-1}\int_{\psi(t)}^{\infty} \exp(-A_2 r)\sigma'(r)\,\mathrm{d}r \\ &\leqslant A_2 \int_{\psi(t)}^{\infty} \exp(-A_2 r)\,\mathrm{d}r \\ &= \exp(-A_2\psi(t)). \end{aligned}$$

Then, as in the proof of Proposition 11 the constant 2 can be removed by using assumption (ii).

Eventually, introduce the r.v. I_τ (see definition (15)) to conclude for the lower bound. This random variable is the first time when the size-biased picked fragment vanishes and so $I_\tau \leqslant \zeta$ a.s. Then, we get the desired lower bound from Proposition 11 (recall that $m(t) = P(I_\tau > t)$). $\square$

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Appendix A

A.1. An example

Let us consider the self-similar fragmentation process constructed from the Brownian excursion of length 1. This process was introduced in Bertoin (2002a) and can be constructed as follows. Write $e=(e(t), 0 \leqslant t \leqslant 1)$ for the Brownian excursion of length 1 and introduce the family of random open sets of $]0,1[$ defined by

$$F(t) = \{s \in]0,1[: e(s) > t\}, \quad t \geqslant 0.$$

Then the process F is a self-similar interval fragmentation process with index $-1/2$. For each $t \geqslant 0$, define by $X(t)$ the non-increasing sequence of the lengths of the interval components of $F(t)$. The required fragmentation process is this process $(X(t), t \geqslant 0)$, which is obviously self-similar with index $-1/2$. Consider then the deterministic fragmentation model constructed from X and especially its mass, which is denoted by $m(t)$ for all time t. Since the process X is self-similar with a negative index, there is loss of mass. Moreover, as shown in Bertoin (2002a), the Laplace exponent of the associated subordinator ξ is given by

$$\phi(q) = 2q\sqrt{\frac{2}{\pi}}\, B\left(q+\frac{1}{2}, \frac{1}{2}\right),$$

and this leads to the following equivalence:

$$\phi(q) \underset{q\to\infty}{\sim} 2\sqrt{2}q^{1/2}$$

(B denotes here the beta function). Hence, the remark following Proposition 11 ensures that

$$\log m(t) \underset{t\to\infty}{\sim} -2t^2$$

and Proposition 14 gives exponential bounds for $P(\zeta > t)$ as $t \mapsto \infty$.

However, we may obtain sharper results. First, we know from Bertoin (2002a) that the first time when the size-biased picked fragment of X is equal to 0 (namely I_τ) has the same law as $e(U)$, where U is uniformly distributed on $[0,1]$ and independent of the Brownian excursion. Thus the distribution of I_τ is given by

$$P(2I_\tau \in \mathrm{d}r) = r\exp(-r^2/2)\,\mathrm{d}r,$$

and then the mass is explicitly known:

$$m(t) = P(I_\tau > t) = \exp(-2t^2), \quad t \geqslant 0.$$

On the other hand, the random variable ζ is obviously the maximum of the Brownian excursion with length 1. And then, as proved in Kennedy (1976), the tail distribution of this random variable is given by

$$P(\zeta > t) = 2\sum_{n=1}^{\infty}(4t^2n^2 - 1)\exp(-2t^2n^2) \quad t > 0.$$

This implies that

$$P(\zeta > t) \underset{t\to\infty}{\sim} 8t^2\exp(-2t^2).$$

A.2. Necessity of condition (1)

We discuss here the necessity of assumption (1) for the splitting measure v in the fragmentation equation (2) (that (1) is needed to construct a random fragmentation was pointed out in Bertoin (2001)).

Suppose that $\int_{\mathscr{S}^*}(1 - s_1)v(\mathrm{d}s) = \infty$ and that there exists a solution $(\mu_t, t \geqslant 0)$ to (2). Let f be a function of $C_c^1(]0,1])$ whose support is exactly $[3/4, 1]$ and such that $f(1) \neq 0$. Since the function $t \mapsto \langle \mu_t, f\rangle$ is continuous on $\mathbb{R}^+$ and $\mu_0 = \delta_1$, there exists a positive time t_0 such that

$$\operatorname{supp}\mu_t \cap [3/4, 1] \neq \emptyset \tag{A.1}$$

for $t < t_0$. Then define by g an non-decreasing non-negative function on $]0, 1]$, smaller than id, belonging to $C_c^1(]0,1])$ and such that

$$g(x) = \begin{cases} 0 & \text{on }]0, 1/2], \\ x & \text{on } [3/4, 1]. \end{cases}$$

Take x in $[3/4, 1]$. For each $s \in \mathscr{S}^{\downarrow}$ and each $i \geqslant 2$, $g(xs_i) = 0$ since $s_i \leqslant 1/2$ for $i \geqslant 2$. Thus

$$\int_{\mathscr{S}^{\downarrow}}\left[\sum_{i=1}^{\infty} g(xs_i) - g(x)\right]v(\mathrm{d}s) = \int_{\mathscr{S}^{\downarrow}}(g(xs_1) - g(x))v(\mathrm{d}s)$$
$$\leqslant x\int_{\mathscr{S}^{\downarrow}}(s_1 - 1)v(\mathrm{d}s) = -\infty.$$

By combining this with (A.1), we conclude that the derivative $\partial_t\langle\mu_t, g\rangle = -\infty$ on $[0, t_0[$ and then that the fragmentation equation (2) has no solution.

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