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FRAGMENTATIONS ET PERTE DE MASSE

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Introduction

On s'intéresse à l'évolution de systèmes de particules se fragmentant au cours du temps. De tels systèmes apparaissent dans des processus physiques variés : on peut penser par exemple à la dégradation de polymères, à la fragmentation d'étoiles ou encore à l'industrie minière où des blocs de roche sont brisés de manière répétitive jusqu'à l'obtention de petits fragments qui sont ensuite traités chimiquement pour en extraire les minéraux.

Lorsque la fragmentation est intensive, on peut observer une perte de masse suite à l'apparition de particules microscopiques, la masse perdue étant celle de l'ensemble de ces particules. Cet ensemble, qui grossit avec le temps, est appelé *poussière*. Le thème principal de cette thèse est l'étude d'un point de vue le plus souvent probabiliste, parfois déterministe, des fragmentations qui perdent de la masse par apparition de poussière.

Ce travail est divisé en quatre chapitres. Le premier chapitre est consacré à une famille de modèles aléatoires et déterministes de fragmentation, et notamment à l'étude en fonction de la dynamique de la fragmentation de l'existence de poussière et des propriétés asymptotiques de sa masse. Le deuxième chapitre traite de la régularité de la masse de la poussière en fonction du temps dans le cadre de fragmentations aléatoires vérifiant une propriété d'auto-similarité. Ces mêmes fragmentations sont étudiées dans le troisième chapitre, qui est consacré à la description de leur généalogie à l'aide d'arbres continus aléatoires. Enfin, le dernier chapitre, qui ne concerne pas spécifiquement les fragmentations produisant de la poussière, porte sur l'étude de systèmes avec fragmentation et immigration de particules et de leurs états d'équilibre.

Ces chapitres sont autonomes et sont rédigés en anglais. Les trois premiers chapitres sont, à quelques modifications près, les versions d'articles publiés ([38],[39],[40]), le quatrième est une version longue d'un article soumis pour publication.

Cette introduction a pour but de présenter les modèles de fragmentations avec lesquels nous travaillons et de synthétiser les résultats de ce travail de thèse. Elle se compose de six parties : une première partie introductive au sujet, quatre parties correspondant chacune à un chapitre de la thèse et une conclusion.

0.1 Processus de fragmentation

En 1941, Kolmogorov [47] est le premier à considérer un modèle aléatoire pour la fragmentation. Son modèle est à temps discret $n = 0,1,2,...$ et décrit l'évolution de particules se scindant en un nombre fini de morceaux à chaque étape. Il se construit par récurrence à partir de la loi d'une suite finie aléatoire $s_1 \geq s_2 \geq ... \geq s_N$ qui représente les fractions des masses des morceaux obtenus. Les particules présentes à un temps n évoluent indépendamment les unes des autres et suivent toutes la même dynamique : une particule de masse m au temps n se scinde au temps $n+1$ en particules de masses $m\overline{s}_1,...,m\overline{s}_{\overline{N}}$ où $(\overline{s}_1,...,\overline{s}_{\overline{N}})$ est une suite aléatoire de même loi que $(s_1,...,s_N)$, indépendante de l'évolution du processus jusqu'au temps n. On obtient ainsi une chaîne de Markov *homogène,* dans le sens où la loi de cette chaîne issue d'une particule de masse m est la même que celle de m fois la chaîne issue d'une particule de masse 1.

Nous nous intéressons ici à des modèles de fragmentation à *temps continu* généralisant celui-ci. Ces processus sont à valeurs dans l'espace de suites décroissantes

$$\mathcal{S}^{\downarrow} = \left\{ \mathbf{s} = (s_1,s_2,...) : s_1 \geq s_2 \geq ... \geq 0, \sum\nolimits_{i\geq 1} s_i \leq 1 \right\},$$

muni de la topologie de la convergence terme à terme. Les termes d'une suite $\mathbf{s} \in \mathcal{S}^{\downarrow}$ représentent des masses de particules.

Définition 0.1 *Soit $(F(t),t \geq 0)$ un processus de Markov à valeurs dans $\mathcal{S}^{\downarrow}$, continu en probabilité. Pour tout $0 < m \leq 1$, on note P_m la loi de F partant de $(m,0,...)$. Le processus F est un processus de fragmentation si pour tout $t_0 \geq 0$, conditionnellement à $F(t_0) = (s_1,s_2,...)$, le processus $(F(t+t_0),t \geq 0)$ a même loi que le processus obtenu en rangeant par ordre décroissant les termes des suites $F^{(i)}(t)$, $i \geq 1$, où les processus $F^{(1)},F^{(2)},...$ sont indépendants, de lois respectives $P_{s_1},P_{s_2},...$.*

Cette propriété de fragmentation signifie simplement que les particules présentes au temps t_0, de masses $s_1,s_2,...$, vont évoluer indépendamment les unes des autres, chacune suivant une loi qui ne dépend que de sa masse, à savoir, respectivement, $P_{s_1},P_{s_2},...$.

On peut construire de manière simple des exemples de tels processus. Soit ν une mesure finie sur $\mathcal{S}^{\downarrow}$ et α un réel. On part initialement d'une particule de masse m. Elle se fragmente au bout d'un temps E de loi exponentielle de paramètre $m^{\alpha}\nu(\mathcal{S}^{\downarrow})$ en particules de masses $mS_1,mS_2,...$ où $\mathbf{S} = (S_1,S_2,...) \in \mathcal{S}^{\downarrow}$ est une variable aléatoire de loi $\nu(\cdot)/\nu(\mathcal{S}^{\downarrow})$, indépendante du temps E. Les particules obtenues se fragmentent à leur tour, suivant une dynamique similaire : conditionnellement à E et $\mathbf{S}$, soient $(E^{(i)},\mathbf{S}^{(i)})$, $i \geq 1$, des couples indépendants de variables aléatoires où $E^{(i)}$ est distribuée suivant une loi exponentielle de paramètre $(mS_i)^{\alpha}\nu(\mathcal{S}^{\downarrow})$ et est indépendante de $\mathbf{S}^{(i)}$, de loi $\nu(\cdot)/\nu(\mathcal{S}^{\downarrow})$. La particule de masse mS_i se fragmente alors au bout d'un temps $E^{(i)}$ en particules de masses $mS_i\mathbf{S}_1^{(i)},mS_i\mathbf{S}_2^{(i)},...$, ceci pour chaque $i \geq 1$. On construit ainsi par récurrence un système de particules où une particule de masse m présente à un temps t se fragmente indépendamment des autres particules présentes avec un taux $m^{\alpha}\nu(\mathrm{d}\mathbf{s})$. Le processus de fragmentation F correspondant s'obtient en considérant à chaque temps t la suite $F(t)$ des masses des particules présentes, rangées par ordre décroissant. Notons que ce processus possède une propriété d'auto-similarité : la loi de $(F(t),t \geq 0)$ sous P_m est la même que celle de $(mF(m^{\alpha}t),t \geq 0)$ sous P_1.

Fragmentations auto-similaires

D'une manière générale, si F est un processus de fragmentation et si la loi de $(F(t),t \geq 0)$ sous P_m est la même que celle de $(mF(m^\alpha t),t \geq 0)$ sous P_1 pour tout $m \leq 1$, on dit que la fragmentation est *auto-similaire* d'indice α, $\alpha \in \mathbb{R}$. Ce paramètre α influence la vitesse de fragmentation : lorsque α est positif, une particule se fragmente d'autant moins vite que sa masse est petite et par conséquent la vitesse de fragmentation des particules ralentit au cours du temps; tandis que si α est négatif une particule se fragmente d'autant plus vite que sa masse est petite et la fragmentation des particules s'accélère. Dans le cas particulier où $\alpha = 0$, le taux de fragmentation d'une particule ne dépend pas de sa masse et la fragmentation est dite *homogène*. Ces fragmentations auto-similaires ont été introduites et étudiées par Bertoin ([13],[14]) en 2001.

Bertoin [13] et Berestycki [9] montrent que la loi d'un processus de fragmentation homogène F est entièrement caractérisée par deux paramètres : un *coefficient d'érosion* $c \geq 0$ et une mesure *de dislocation* ν sur $\mathcal{S}^{\downarrow}$ qui ne charge pas (1,0,...) et telle que $\int_{\mathcal{S}^{\downarrow}}(1-s_1)\nu(\mathrm{d}\mathbf{s}) < \infty$. L'érosion est un phénomème déterministe : le processus F peut s'écrire sous la forme $F(t) = \exp(-ct)\overline{F}(t)$ pour tout $t \geq 0$ où $\overline{F}$ est un processus de fragmentation homogène sans érosion ($c = 0$) et de même mesure de dislocation ν. Cette mesure ν décrit, par l'intermédiaire d'un processus ponctuel de Poisson, la structure des sauts de $\overline{F}$: informellement, la mesure $\nu(\mathrm{d}\mathbf{s})$ représente le taux de fragmentation d'une particule de masse m en particules de masses $m\mathbf{s}$, $\mathbf{s} \in \mathcal{S}^{\downarrow}$.

Bertoin [14] montre qu'à l'aide d'un changement de temps aléatoire complexe - que nous ne détaillons pas ici - et bijectif, tout processus de fragmentation auto-similaire peut être transformé en un processus de fragmentation homogène. La loi d'un processus de fragmentation auto-similaire est donc caractérisée par trois paramètres : l'indice d'auto-similarité α, le coefficient d'érosion c et la mesure de dislocation ν du processus homogène associé. La structure des sauts d'un processus de fragmentation auto-similaire peut se résumer ainsi : une particule de masse m se disloque en particules de masses $m\mathbf{s}$, $\mathbf{s} \in \mathcal{S}^{\downarrow}$, à un taux $m^\alpha \nu(\mathrm{d}\mathbf{s})$. Lorsque ν est finie et $c = 0$, on retrouve les modèles décrits ci-dessus, où les particules attendent des temps de lois exponentielles avant de se fragmenter.

Dans la suite, sauf cas particulier, on considèrera toujours que **l'état initial d'un processus de fragmentation F est composé d'une seule particule de masse 1**, c'est-à-dire : $F(0) = (1,0,...)$.

Fragmentation d'intervalles

Une fragmentation d'intervalles est une famille d'ouverts aléatoires emboités $(I(t),t \geq 0)$ de (0,1) ($I(t) \subset I(t')$ lorsque $t' \leq t$) issue de $I(0) = (0,1)$ et vérifiant une propriété de fragmentation - et le cas échéant d'auto-similarité - semblable à celle d'une fragmentation à valeurs dans $\mathcal{S}^{\downarrow}$. Elle donne une structure généalogique de la fragmentation. Voici un exemple : si $(U_n,n \geq 1)$ est une famille de variables aléatoires indépendantes et uniformément distribuées sur (0,1) et si $(N(t),t \geq 0)$ est un processus de Poisson de paramètre 1 indépendant de cette famille, alors le processus I défini par $I(t) = (0,1) \setminus \{U_n, n \leq N(t)\}$, $t \geq 0$, est une fragmentation auto-similaire d'indice 1.

Il est possible d'associer à chaque fragmentation auto-similaire F un processus de fragmentation $(I_F(t), t \geq 0)$ à valeurs dans les ouverts de $(0,1)$, de même indice d'auto-similarité que F, de sorte que si $F'(t)$ désigne la suite décroissante des longueurs des composantes connexes de $I_F(t)$, $t \geq 0$, alors $F' \overset{\text{loi}}{=} F$ [14]. On dit alors que I_F est une fragmentation d'intervalles associée à F. Réciproquement, les suites rangées par ordre décroissant des longueurs des composantes connexes d'une fragmentation auto-similaire $(I(t), t \geq 0)$ à valeurs dans $(0,1)$, donnent une fragmentation auto-similaire à valeurs dans $\mathcal{S}^{\downarrow}$. Notons qu'il existe également une correspondance entre les lois des processus de fragmentation auto-similaires à valeurs dans $\mathcal{S}^{\downarrow}$ et celles de processus de fragmentation auto-similaires à valeurs dans les partitions de $\mathbb{N}^* = \{1,2,...\}$ ([9], [13], [14]). C'est par l'intermédiaire des ces fragmentations à valeurs dans les ouverts de $(0,1)$ et dans les partitions de $\mathbb{N}^*$ que Bertoin et Berestycki montrent leurs résultats sur la caractérisation des fragmentations auto-similaires par les triplets (α, c, ν).

Le processus du fragment marqué

La structure d'un processus de fragmentation est complexe et parfois difficilement exploitable pour obtenir des renseignements sur la fragmentation. Cette difficulté peut souvent être contournée en utilisant *le processus du fragment marqué*. Soit F un processus de fragmentation et I_F une fragmentation d'intervalles associée. Soit U (la marque) une variable aléatoire uniformément distribuée sur $(0,1)$, indépendante de I_F. On s'intéresse à l'évolution au cours du temps de l'intervalle de I_F contenant U et on note $\Lambda(t)$ sa longueur au temps t. Dans le cas homogène, la construction Poissonnienne de la fragmentation implique l'existence d'un subordinateur ξ [13], c'est-à-dire d'un processus croissant, càdlàg, à accroissements indépendants et stationnaires, tel que

$$\Lambda \overset{\text{loi}}{=} (\exp(-\xi(t)), t \geq 0).$$

Il est bien connu [10] qu'un subordinateur est caractérisé par son exposant de Laplace ϕ ($\forall t, q \geq 0$, $E[\exp(-q\xi(t)] = \exp(-t\phi(q)))$, et celui-ci s'exprime ici en termes du coefficient d'érosion et de la mesure de dislocation, par

$$\phi(q) = c(q+1) + \int_{\mathcal{S}^{\downarrow}} (1 - \sum\nolimits_{i \geq 1} s_i^{q+1}) \nu(\mathrm{d}\mathbf{s}), \; q \geq 0. \tag{1}$$

Dans le cas d'une fragmentation auto-similaire, le passage par changement de temps à une fragmentation homogène et le résultat ci-dessus impliquent ([14]) que

$$\Lambda \overset{\text{loi}}{=} (\exp(-\xi(\rho(t))), t \geq 0),$$

où $\rho(t) = \inf \left\{ u \geq 0 : \int_0^u \exp(\alpha\xi(r)) \mathrm{d}r > t \right\}$, $t \geq 0$.

Les subordinateurs sont des processus bien étudiés ([10],[11]) et l'expression du fragment marqué Λ comme fonctionnelle d'un subordinateur va nous apporter de précieuses informations sur la fragmentation. Il faut préciser cependant que la loi de F n'est pas caractérisée par celle de Λ, puisque deux fragmentations auto-similaires de paramètres différents peuvent avoir des processus du fragment marqué de même loi.

Fragmentation brownienne

On présente ici un exemple de processus de fragmentation construit à partir d'une excursion brownienne normalisée $(e(x), 0 \leq x \leq 1)$ (informellement, e est un mouvement brownien sur l'intervalle unité, conditionné à valoir 0 en $x = 0$ et en $x = 1$ et à être strictement positif sur $(0,1)$). Cet exemple a été introduit et étudié par Bertoin [14] et va illustrer et motiver les résultats des 4 chapitres de cette thèse. Pour tout $t \geq 0$, soit

$$I_e(t) = \{x \in (0,1) : e(x) > t\}$$

et $F_e(t)$ le réarrangement par ordre décroissant des longueurs des composantes connexes de $I_e(t)$. En utilisant la théorie des excursions browniennes, Bertoin montre que les processus $(I_e(t), t \geq 0)$ et $(F_e(t), t \geq 0)$ sont des processus de fragmentation auto-similaires d'indice $\alpha_e = -1/2$, sans érosion et de mesure de dislocation ν_e donnée par

$$\nu_e(s_1 \in \mathrm{d}x) = \frac{\sqrt{2}}{\sqrt{\pi x^3 (1-x)^3}} \mathrm{d}x, \quad x \in [1/2,1) \quad \text{et} \quad \nu_e(s_1 + s_2 < 1) = 0.$$

La fragmentation est *binaire* : à chaque dislocation une particule se scinde en deux morceaux. Ceci résulte du fait que les minima locaux du mouvement brownien sont presque sûrement disjoints.

Il est évident sur cet exemple que la masse totale $\sum_{i \geq 1} (F_e)_i(t)$ décroit et atteint 0 en un temps fini (égal à $\max_{x \in [0,1]} e(x)$). Pourtant, il n'y a pas d'érosion et au moment où un intervalle se disloque en sous-intervalles disjoints, aucune masse n'est perdue puisque la somme des longueurs des sous-intervalles obtenus est égale à celle de l'intervalle qui vient de se disloquer. La masse perdue au temps t, à savoir $1 - \sum_{i \geq 1} (F_e)_i(t)$, est la masse de la poussière (ensemble des particules de masse 0) qui s'est formée suite à une accélération de la fragmentation.

Formation de poussière

Dans un système de fragmentation, la poussière peut apparaître de trois façons : soit par érosion, soit au moment où une particule se disloque (la masse des morceaux obtenus est strictement plus petite que la masse de la particule qui vient de se disloquer), soit par accélération de la fragmentation. Ce dernier phénomène est le plus intéressant et on peut espérer l'observer lorsque les temps de fragmentation des particules s'accumulent, ce qui produit en temps fini des particules de masse 0. Ceci peut être vu comme le phénomène dual de la *gélification* (apparition d'une particule de masse infinie) qu'on observe dans certains systèmes de coagulation (voir par exemple [41], [58]).

On considèrera dans la suite qu'il y a formation de poussière pour la fragmentation F si la quantité de poussière produite est non négligeable, c'est-à-dire si elle occassionne une perte de masse. Ceci se traduit par l'existence d'un temps t tel que la masse $\sum_{i \geq 1} F_i(t)$, qui ne tient pas compte des particules de masse 0, soit strictement plus petite que sa valeur initiale, à savoir 1. La différence $1 - \sum_{i \geq 1} F_i(t)$ mesure la masse de poussière.

L'apparition de poussière dans certaines fragmentations a été observée pour la première fois par Filippov, un élève de Kolmogorov, en 1961 [35]. Dans le cas particulier d'une fragmentation auto-similaire d'indice α sans érosion et de mesure de dislocation ν finie telle que $\nu(\sum_{i\geq 1} s_i < 1) = 0$ (aucune poussière n'est formée au moment où une particule se disloque), son résultat s'énonce ainsi : il y a formation de poussière si et seulement si $\alpha < 0$. En 2003, Bertoin [15] généralise ce résultat au cas où ν est infinie : il y a formation de poussière si et seulement si $\alpha < 0$ et plus précisément, si $\alpha < 0$, la masse initiale est entièrement réduite à l'état de poussière en un temps presque sûrement fini.

La formation de poussière a intéressé également des physiciens (Edwards *et al.* [31], McGrady et Ziff [54]), dans les années 80. Ils ont abordé le problème d'un point de vue déterministe en étudiant *l'équation de fragmentation* suivante

$$\partial_t n_t(x) = \int_0^\infty (2F(y+x,x)n_t(x+y) - F(x,y)\mathbf{1}_{\{y<x\}}n_t(x))\mathrm{d}y. \tag{2}$$

La quantité $n_t(x)\mathrm{d}x$ correspond au nombre moyen de particules ayant une masse dans l'intervalle $[x,x+\mathrm{d}x)$ au temps t. Le taux de fragmentation d'une particule de masse x en particules de masses y et $x-y$ est donné par $F(x,y)\mathrm{d}y$ et la symétrie du problème impose que $F(x,y) = F(x,x-y)$. La partie positive de l'intégrale traduit alors l'augmentation de particules de masse x suite à la fragmentation de particules de masses plus grandes, tandis que la partie négative correspond à la diminution de particules de masse x suite à leur fragmentation en particules de masses plus petites. Les résultats obtenus dans les papiers [31] et [54] concernent des taux de fragmentation du type $F(x,y) = x^{\alpha-1}h(x/y)$ (ce qui correspond à l'auto-similarité du cas aléatoire) pour certaines fonctions h, et sont analogues à ceux de Filippov : il y a formation de poussière si et seulement si $\alpha < 0$. Plus récemment, Jeon [42] et Fournier et Giet [36] ont étudié l'apparition de poussière pour des familles de taux de fragmentation ne se factorisant pas nécessairement sous la forme $x^{\alpha-1}h(x/y)$. Nous renvoyons à leurs travaux pour des résultats précis.

0.2 Formation de poussière pour les fragmentations (τ,c,ν)

Dans ce premier travail, nous nous intéressons à des processus de fragmentation où une particule de masse m se disloque en particules de masses $m\mathbf{s}$, $\mathbf{s} \in \mathcal{S}^{\downarrow}$, à un taux $\tau(m)\nu(\mathrm{d}\mathbf{s})$, où τ est une fonction continue sur $(0,1]$ qui vaut 1 en 1. Lorsque $\tau(m) = m^\alpha$ on retrouve les fragmentations auto-similaires. Ces processus se construisent à partir des fragmentations homogènes à l'aide d'un changement de temps dépendant de τ, de manière analogue à la construction des fragmentations auto-similaires à partir des fragmentations homogènes. Ils sont donc caractérisés par 3 paramètres : τ, c et ν.

Le modèle déterministe correspondant est l'équation "(τ,c,ν)" suivante :

$$\partial_t \langle \mu_t,f\rangle = \int_0^1 \tau(x)\left(-cxf'(x) + \int_{\mathcal{S}^{\downarrow}} \left[\sum\nolimits_{i\geq 1} f(xs_i) - f(x)\right]\nu(\mathrm{d}\mathbf{s})\right)\mu_t(\mathrm{d}x) \tag{3}$$

et nous considérons qu'à l'état initial $\mu_0 = \delta_1$, c'est-à-dire qu'il n'y a que des particules de masse 1. Ici, l'ensemble des fonctions test f est l'ensemble noté $\mathcal{C}_c^1(0,1]$ des fonctions réelles à support compact dans $(0,1]$ et de dérivée continue. La mesure de Radon $\mu_t(\mathrm{d}x)$ correspond à la quantité moyenne de particules ayant une masse dans l'intervalle $[x,x+\mathrm{d}x)$ au temps t. L'intégrale impliquant $\sum_{i\geq 1} f(xs_i) - f(x)$ modélise le remplacement de particules de masse x, suite à leurs dislocations, par des particules de masses $x\mathbf{s}$, $\mathbf{s}=(s_1,s_2,...)\in\mathcal{S}^{\downarrow}$. Enfin, le terme impliquant c correspond à l'érosion. Dans le cas particulier où $c=0$, où $\nu(s_1+s_2<1)=0$ et où $\nu(s_1\in \mathrm{d}y)=2\mathbf{1}_{\{1/2\leq y<1\}}h(y)\mathrm{d}y$, on retrouve l'équation de fragmentation (2) avec $F(x,y)=\tau(x)x^{-1}h(y/x)$ où pour $z<1/2$, $h(z)$ est défini par $h(z)=h(1-z)$.

Comme dans le cas auto-similaire, on montre qu'un fragment marqué Λ dans le modèle aléatoire (τ,c,ν) peut être représenté sous la forme $\Lambda(t)=\exp(-\xi(\rho(t)))$ pour tout $t\geq 0$, où ξ est un subordinateur dont la transformée de Laplace ϕ est donnée par (1) et ρ un changement de temps dépendant de τ. En suivant ce fragment marqué, nous établissons le lien suivant entre le processus et l'équation de fragmentation (τ,c,ν).

Théorème 0.1 *Il existe une unique solution $(\mu_t,t\geq 0)$ à l'équation (τ,c,ν). Cette solution se construit à l'aide d'un processus de fragmentation F de paramètres (τ,c,ν) de la manière suivante : pour tout $t\geq 0$,*

$$\langle \mu_t,f\rangle = E\left[\sum\nolimits_{i\geq 1} f(F_i(t))\right],\ f\in\mathcal{C}_c^1\,(0,1]\,.$$

On cherche ensuite à savoir quelles fragmentations (τ,c,ν) produisent de la poussière. Dans le cas stochastique ceci se traduit par l'existence d'un temps t tel que $\sum_{i\geq 1} F_i(t)<1$ et dans le cas déterministe par l'existence d'un temps t tel que $\int_0^1 x\mu_t(\mathrm{d}x)<1$. Compte tenu du résultat précédent, ces deux notions sont étroitement liées.

Bien sûr, il y a formation de poussière dès qu'il y a de l'érosion ($c>0$) ou production de poussière au moment de la dislocation d'une particule ($\nu(\sum_{i\geq 1} s_i<1)>0$). L'intérêt de cette étude concerne les modèles où $c=0$ et $\nu(\sum_{i\geq 1} s_i<1)=0$ et nous supposerons dans la suite de ce chapitre que ces deux conditions sont toujours réalisées. En étudiant le premier instant où le fragment marqué est réduit à l'état de poussière, on obtient le résultat suivant (la fonction ϕ est définie à partir du coefficient d'érosion c et de la mesure de dislocation ν par la formule (1)).

Théorème 0.2 (i) *Il y a formation de poussière pour la fragmentation stochastique (τ,c,ν) avec probabilité* 0 *ou* 1 *et cette probabilité est égale à* 1 *si et seulement s'il y a formation de poussière pour la fragmentation déterministe (τ,c,ν).*

(ii) *Si τ est décroissante au voisinage de* 0*, il y a formation de poussière pour les fragmentations (τ,c,ν) si et seulement si*

$$\int_{0^+}\frac{\phi'(x)}{\tau(\exp(-1/x))\phi^2(x)}\mathrm{d}x<\infty.$$

Si $\tau\leq\widetilde{\tau}$, la fragmentation $(\widetilde{\tau},0,\nu)$ est plus rapide que la fragmentation $(\tau,0,\nu)$ et les modèles $(\widetilde{\tau},0,\nu)$ produisent donc de la poussière dès que les modèles $(\tau,0,\nu)$ le font. Par conséquent, lorsque τ n'est pas décroissante au voisinage de 0, il suffit de la comparer à des fonctions décroissantes et d'appliquer le résultat (ii) pour obtenir des conditions nécessaires et/ou suffisantes pour la formation de poussière.

On voit en particulier qu'il n'y a pas de poussière dès que τ est bornée près de 0 et on retrouve le résultat déjà connu du cas auto-similaire : il y a de la poussière si et seulement si $\alpha < 0$. Remarquons également que lorsque $\phi'(0^+) < \infty$, le résultat (ii) s'énonce plus simplement ainsi : si τ est décroissante au voisinage de 0, on a formation de poussière si et seulement si $\int_{0^+} \mathrm{d}x/x\tau(x) < \infty$. Filippov [35] avait établi ce critère dans le cas particulier où ν est finie.

On se place maintenant dans le cadre stochastique et on s'intéresse au premier instant ζ où toute la masse initiale est réduite à l'état de poussière, i.e.

$$\zeta = \inf \{t \geq 0 : F_1(t) = 0\}.$$

Lorsque la fragmentation est auto-similaire d'indice $\alpha < 0$, on sait (Proposition 2, [15]) que ζ est fini presque sûrement. Ce phénomène de perte de toute la masse initiale n'a pas toujours lieu pour une fragmentation (τ,c,ν) même s'il y a formation de poussière. Pour établir le critère caractérisant la formation de poussière, on a suivi le fragment marqué. Pour obtenir un critère caractérisant la perte de toute la masse, on raisonne de même en suivant un fragment particulier, qui est cette fois un peu plus gros, à savoir le processus du plus gros sous-fragment : ce processus part du fragment initial de masse 1 et à chaque fois qu'il y a une dislocation, il suit le plus gros sous-fragment obtenu. On établit ainsi le résultat suivant.

Proposition 0.1 *Supposons que τ soit décroissante au voisinage de 0 et que ν intègre $|\log s_1|$. Alors, la probabilité $P(\zeta < \infty)$ est soit égale à 0 soit égale à 1 et elle est égale à 1 si et seulement si $\int_{0^+} \mathrm{d}x/x\tau(x) < \infty$.*

En utilisant ce critère et le théorème précédent, on peut alors construire des exemples de processus de fragmentation tels qu'il y ait formation de poussière et que $\zeta = \infty$ presque sûrement (Chapitre 1.5.2).

En dehors de quelques cas particuliers, dont l'exemple de la fragmentation brownienne développé précédemment, on ne connait pas explicitement la loi de ζ. On peut cependant montrer que sa queue de distribution $P(\zeta > t)$ décroit exponentiellement lorsque $t \to \infty$ si $\tau(x) \geq C_\alpha x^\alpha$ pour un certain $\alpha < 0$ et une constante $C_\alpha > 0$. Ce taux de décroissance peut être précisé (Proposition 1.8) en fonction de la mesure ν. Il est à noter que la masse totale déterministe $m(t) = \int_0^1 x\mu_t(\mathrm{d}x)$ a le même comportement asymptotique lorsque $t \to \infty$ que $P(\zeta > t)$ (Proposition 1.6).

0.3 Régularité de la masse de poussière

On s'intéresse ici à la régularité de l'évolution de la masse de poussière $M(t) = 1 - \sum_{i\geq 1} F_i(t)$ d'une fragmentation auto-similaire F de paramètres (α,c,ν) sous l'hypothèse

$$\alpha < 0, \quad c = 0, \quad \text{et} \quad \nu(\sum_{i\geq 1} s_i < 1) = 0. \tag{4}$$

Commençons par regarder le cas de la fragmentation brownienne F_e où la masse totale de la poussière au temps t est donnée par $M_e(t) = \int_0^1 \mathbf{1}_{\{e(x)<t\}}\mathrm{d}x$. En utilisant la formule des temps d'occupation (voir par exemple [60]), on peut réécrire cette masse sous la forme $M_e(t) = \int_0^t L_e(u)\mathrm{d}u$ où L_e est le processus de temps local de l'excursion brownienne. Il est bien connu que le temps local $L_e(t)$ peut s'approximer par différentes fonctionnelles de l'excursion brownienne et en particulier que pour tout $t \geq 0$,

$$L_e(t) \overset{\text{p.s.}}{=} \lim_{\varepsilon\to 0} \sqrt{\frac{2\pi}{\varepsilon}} M_e(t,\varepsilon) \overset{\text{p.s.}}{=} \lim_{\varepsilon\to 0} \sqrt{2\pi\varepsilon} N_e(t,\varepsilon)$$

où $M_e(t,\varepsilon)$ est la somme des longueurs des excursions de e au-dessus de t de longueur inférieure à ε et $N_e(t,\varepsilon)$ le nombre d'excursions de e au-dessus de t de longueur supérieure à ε. Du point de vue de la fragmentation F_e, $M_e(t,\varepsilon)$ représente la masse totale de particules de masse inférieure à ε présentes au temps t et $N_e(t,\varepsilon)$ le nombre de particules ayant une masse supérieure à ε présentes au temps t.

On cherche à savoir dans quelle mesure ces résultats se généralisent à une fragmentation F de paramètres vérifiant l'hypothèse (4). On commence par étudier l'absolue continuité et la singularité par rapport à la mesure de Lebesgue de la mesure $\mathrm{d}M$. Sous une contrainte technique sur la mesure ν (on renvoie au Théorème 2.1 pour un énoncé précis) on obtient :

Théorème 0.3 (i) *Si* $\alpha > -1$ *et* $\int_{\mathcal{S}^\downarrow} \sum_{1\leq i<j<\infty} s_i^{1+\alpha} s_j \nu(\mathrm{d}x) < \infty$, *alors presque sûrement la mesure* $\mathrm{d}M$ *a une densité* L *par rapport à la mesure de Lebesgue et cette densité appartient à l'espace* $L^2(\mathrm{d}t \otimes \mathrm{d}\mathbb{P})$.

(ii) *Si* $\alpha \leq -1$, *la mesure* dM *est p.s. singulière par rapport à la mesure de Lebesgue.*

La preuve de la première assertion est plus technique que celle de la deuxième, qui repose sur le fait (cf. [15]) que les fragmentations d'indice $\alpha \leq -1$ n'ont qu'un nombre presque sûrement fini de masses non nulles à un temps t fixé, ce qui nous permet de conclure que $\varepsilon^{-1}(M(t+\varepsilon)-M(t))$ converge vers 0 quand $\varepsilon \to 0$ presque sûrement pour presque tout t. L'existence d'une densité se montre à l'aide du théorème de Plancherel. Le second moment de la transformée de Fourier de la mesure $\mathrm{d}M$ est estimé en suivant *deux* fragments marqués indépendamment et en évaluant le comportement de leurs masses au premier instant où ils sont disjoints.

Il est facile de vérifier que l'intégrale $\int_{\mathcal{S}^\downarrow} \sum_{1\leq i<j<\infty} s_i^{1+\alpha} s_j \nu(\mathrm{d}\mathbf{s})$ est toujours finie lorsque $\alpha > -1$ et lorsque $\nu(s_N > 0) = 0$ pour un entier positif N (ce qui signifie que chaque particule, à chaque étape, se disloque en $N-1$ morceaux au plus). Pour de telles mesures de dislocation, l'existence d'une densité pour le mesure $\mathrm{d}M$ ne dépend donc que de l'indice α et de sa position par rapport à -1.

Introduisons maintenant, par analogie avec l'exemple brownien, la fonction

$$M(t,\varepsilon) = \sum\nolimits_{i\geq 1} F_i(t)\mathbf{1}_{\{F_i(t)\leq\varepsilon\}},$$

qui mesure la masse totale des particules de masse inférieure à ε au temps t, et la fonction

$$N(t,\varepsilon) = \sum\nolimits_{i\geq 1} \mathbf{1}_{\{F_i(t)\geq\varepsilon\}},$$

qui compte le nombre de particules de masse supérieure à ε présentes au temps t. On pose $\mu = \phi'(0^+)$, ϕ étant la transformée de Laplace (1) et on suppose dans le théorème suivant que $\mu < \infty$ et que la fragmentation n'est pas géométrique, c'est-à-dire qu'il n'existe pas de réel $0 < r < 1$ tel que tous les $F_i(t)$, $t \geq 0$, $i \geq 1$, appartiennent à $\{r^k : k \in \mathbb{N}\}$.

Théorème 0.4 *Supposons que la mesure* $\mathrm{d}M$ *ait une densité* L *appartenant à* $L^p(\mathrm{d}t \otimes \mathrm{d}\mathbb{P})$ *pour un* $p > 1$. *Alors, pour presque tout* t,

$$\varepsilon^{\alpha} M(t,\varepsilon) \underset{\varepsilon \to 0}{\overset{\text{p.s.}}{\to}} L(t)/\left|\alpha\right| \mu$$

et

$$\varepsilon^{1+\alpha} N(t,\varepsilon) \underset{\varepsilon \to 0}{\overset{\text{p.s}}{\to}} L(t)\left(1 - |\alpha|\right)/\left|\alpha\right|^2 \mu.$$

Ce résultat se montre en deux étapes : tout d'abord, en utilisant la propriété d'auto-similarité de la fragmentation, on établit que $\varepsilon^{\alpha} E[M(t,\varepsilon D^{1/\alpha}) \mid F] \to L(t)$ p.s. quand $\varepsilon \to 0$, où D est une variable aléatoire indépendante de F, de même loi que $\inf\{t : \Lambda(t) = 0\}$ le premier instant où le fragment marqué a une masse nulle. On utilise ensuite un théorème taubérien qui nous permet d'"oublier" D dans l'espérance précédente et d'obtenir la limite quand $\varepsilon \to 0$ de $\varepsilon^{\alpha} M(t,\varepsilon)$. Le comportement de N se déduit de celui de M à l'aide de théorèmes abéliens-taubériens. Ce lien entre les comportements de M et N est montré par Bertoin dans [16], où il étudie le comportement de ces fonctions quand $\varepsilon \to 0$ dans le cas $\alpha > 0$. Il est intéressant de noter la différence de ses résultats avec ceux obtenus ci-dessus : quand $\alpha > 0$ et à condition que ν vérifie certaines propriétés de régularité, il existe une fonction $f(\varepsilon)$ ne dépendant que de ν telle que $f(\varepsilon) M(t,\varepsilon)$ et $\varepsilon f(\varepsilon) N(t,\varepsilon)$ convergent presque sûrement vers une limite non triviale. Ici, lorsque $\alpha < 0$, les vitesses de convergence dépendent de α, pas de ν.

Lorsque $\alpha \leq -1$, on a vu que la mesure $\mathrm{d}M$ est singulière par rapport à la mesure de Lebesgue. On peut préciser ce résultat en calculant sa dimension de Hausdorff $\dim_{\mathcal{H}}(\mathrm{d}M)$. On rappelle que la dimension de Hausdorff d'un sous-ensemble E d'un espace métrique est l'infimum des $\gamma > 0$ tel que $\sup_{\varepsilon > 0} \inf_{(B_i)_{i \geq 1} \in C_{\varepsilon}(E)} \sum_{i \geq 1} |B_i|^{\gamma} = 0$ où $C_{\varepsilon}(E)$ est l'ensemble des recouvrements de E par des boules de diamètre inférieur à ε. La dimension de Hausdorff de la mesure $\mathrm{d}M$ est alors définie par

$$\dim_{\mathcal{H}}(\mathrm{d}M) = \inf\left\{\dim_{\mathcal{H}}(E) : \mathrm{d}M(E) = 1\right\}.$$

Pour simplifier, on énonce ici le résultat dans le cas où $\nu(s_N > 0) = 0$ pour un certain $N \in \mathbb{N}$.

Proposition 0.2 *S'il existe un entier* N *tel que* $\nu(s_N > 0) = 0$, *alors* $\dim_{\mathcal{H}}(\mathrm{d}M) = 1 \wedge |\alpha|^{-1}$ *presque sûrement.*

La mesure $\mathrm{d}M$ est donc portée par des ensembles d'autant plus "fins" que l'indice α est négatif. La majoration $\dim_{\mathcal{H}}(\mathrm{d}M) \leq 1 \wedge |\alpha|^{-1}$ est obtenue à l'aide d'une famille explicite de recouvrements d'un ensemble E portant la mesure $\mathrm{d}M$. Pour la minoration, on utilise à nouveau un couple de fragments marqués, ce qui nous permet de montrer que

$$E\left[\int_0^\infty \int_0^\infty |u-t|^{-\gamma}\,\mathrm{d}M(u)\mathrm{d}M(t)\right] < \infty$$

dès que $\gamma < 1 \wedge |\alpha|^{-1}$. D'où la conclusion par application du lemme de Frostman.

Enfin, un dernier résultat sur la régularité de M concerne sa continuité höldérienne.

Proposition 0.3 *Supposons que $\nu(s_N > 0) = 0$ pour un certain $N \in \mathbb{N}$. Alors, il existe un paramètre C_ν ne dépendant que de ν tel que presque sûrement M est γ-höldérienne pour tout $\gamma < 1 \wedge (C_\nu / |\alpha|)$ et n'est pas γ-höldérienne pour tout $\gamma > 1 \wedge (1/|\alpha|)$.*

La masse M est donc d'autant moins régulière que l'indice α est négatif.

0.4 Généalogie des fragmentations auto-similaires d'indice négatif

Ce travail, réalisé en collaboration avec Grégory Miermont, a été motivé par des exemples de processus de fragmentation construits à partir d'arbres continus aléatoires tels qu'Aldous les a introduits dans [2],[3]. Commençons par définir ces arbres.

Arbres continus. Un *arbre réel* est un espace métrique complet $(\mathcal{T},d)$ ayant une structure d'arbre :

- $\forall\,(v,w) \in \mathcal{T}^2$, il existe une unique isométrie $f_{(v,w)} : [0,d(v,w)] \to \mathcal{T}$ telle que $f_{(v,w)}(0) = v$ et $f_{(v,w)}(d(v,w)) = w$; on note $[[v,w]]$ son image.

- si $f : [0,1] \to \mathcal{T}$ est une fonction continue injective telle que $f(0) = v$ et $f(1) = w$, alors $f([0,1]) = [[v,w]]$.

On considèrera toujours qu'un arbre réel est enraciné; on note $\emptyset$ la racine. Une feuille de $\mathcal{T}$ est un noeud de l'arbre qui n'appartient à aucun chemin de la forme $[[\emptyset,v[[$, $v \in \mathcal{T}$. On note $\mathcal{L}(\mathcal{T})$ l'ensemble des feuilles de $\mathcal{T}$. Son complémentaire $\mathcal{S}(\mathcal{T}) = \mathcal{T} \backslash \mathcal{L}(\mathcal{T})$ est appelé *squelette* de l'arbre.

Définition 0.2 *Un arbre continu est une paire $(\mathcal{T},\mu)$ où $\mathcal{T}$ est un arbre réel et μ une mesure de probabilité non-atomique sur $\mathcal{T}$ qui ne charge que les feuilles et telle que $\mu\{v \in \mathcal{T} : [[\emptyset,v]] \cap [[\emptyset,w]] = [[\emptyset,w]]\} > 0$ pour tout $w \in \mathcal{S}(\mathcal{T})$. La mesure μ est appelée mesure masse de l'arbre.*

Aldous [2] a introduit la notion d'arbres continus *aléatoires* (dont l'abréviation anglaise est *CRT*) en construisant le "CRT brownien" comme limite d'arbres de Galton-Watson renormalisés. Une autre façon de construire cet arbre ([3]) consiste à partir de l'excursion brownienne normalisée e : soit $d_e(x,y) = e(x) + e(y) - 2\inf_{z\in[x,y]} e(z)$ une pseudo-distance sur $[0,1]$ et soit $\mathcal{T}_e = [0,1]/\sim_e$ l'espace métrique quotient associé à la relation d'équivalence

$x \sim_e y \Leftrightarrow d_e(x,y) = 0$. Alors, l'espace $\mathcal{T}_e$ muni de la mesure μ_e induite par la mesure de Lebesgue sur [0,1] est un CRT. La racine de cet arbre est la classe d'équivalence du point 0. La fragmentation brownienne F_e se construit à partir du CRT $(\mathcal{T}_e,\mu_e)$ de la manière suivante : pour tout $t \geq 0$, $F_e(t)$ est le réarrangement par ordre décroissant des μ_e-masses des composantes connexes de $\{v \in \mathcal{T}_e : d_e(\emptyset,v) > t\}$. D'autres exemples de fragmentations auto-similaires se construisent de cette façon à partir de CRTs [56].

Il est naturel de vouloir généraliser ces exemples en associant à une fragmentation quelconque un CRT décrivant sa structure généalogique. Les CRTs $(\mathcal{T},\mu)$ ont leurs feuilles à distance finie de la racine. Par conséquent, $\mu(\{v \in \mathcal{T} : d(\emptyset,v) > t\})$ décroit vers 0 quand $t \to \infty$ et seules les fragmentations dont la masse totale diminue ont une chance de pouvoir être construites, de manière analogue au cas brownien, à partir d'un CRT. On se place donc dans le cas où $\alpha < 0$. On suppose également que $c = 0$ (pour éviter d'obtenir une mesure masse chargeant le squelette) et que $\nu(\sum_{i\geq 1} s_i < 1) = 0$ (pour éviter d'obtenir une mesure masse atomique). Sous ces hypothèses, nous montrons le résultat suivant.

Théorème 0.5 *Il existe un CRT $(\mathcal{T}_F,\mu_F)$ tel que si pour tout $t \geq 0$, $F'(t)$ désigne la suite décroissante des μ_F-masses des composantes connexes de $\{v \in \mathcal{T}_F : d(\emptyset,v) > t\}$, alors F' a même loi que F. De plus, l'arbre $\mathcal{T}_F$ est p.s. compact et lorsque la mesure de dislocation intègre la fonction $(s_1^{-1} - 1)$, $\dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F)) = |\alpha|^{-1}$ p.s.*

La dimension de Hausdorff du squelette, qui est une réunion dénombrable de segments, est égale à 1. Par conséquent, lorsque ν intègre la fonction $(s_1^{-1} - 1)$, la dimension de Hausdorff de l'arbre $\mathcal{T}_F$ est $1 \vee |\alpha|^{-1}$.

L'arbre $\mathcal{T}_F$ est la limite en loi d'une suite consistante d'arbres discrets non-ordonnés $(T_F^n,n \geq 1)$ que l'on construit de la manière suivante. Soit I_F une fragmentation d'intervalles associée à F et soit $(U_n,n \geq 1)$ une suite de variables aléatoires indépendantes uniformément distribuées sur (0,1), indépendantes de I_F. L'arbre T_F^1 est une branche de longueur D_1 où $D_1 = \sup\{t : U_1 \in I_F\}$. Soit ensuite $D_{\{1,2\}}$ le premier instant où U_1 et U_2 n'appartiennent plus à la même composante connexe de I_F. L'arbre T_F^2 s'obtient à partir de T_F^1 en ajoutant à distance $D_{\{1,2\}}$ de la racine une branche de longueur $D_2 - D_{\{1,2\}} = \sup\{t : U_2 \in I_F\} - D_{\{1,2\}}$. A la n-ième étape, on introduit $D_{\{(1,...,n-1),n\}}$ le premier instant où la composante connexe contenant U_n ne contient aucun des U_i, $i \leq n-1$, et on considère un $j \leq n-1$ tel qu'au temps $(D_{\{(1,...,n-1),n\}}-)$ U_n et U_j appartiennent à la même composante connexe. On ajoute sur le chemin reliant la j-ième feuille à la racine une nouvelle branche - de longueur $\sup\{t : U_n \in I_F\} - D_{\{(1,...,n-1),n\}}$ - à distance $D_{\{(1,...,n-1),n\}}$ de la racine. On construit ainsi par récurrence des arbres T_F^n à n feuilles et on utilise un résultat d'Aldous [3] pour conclure que ces arbres convergent en loi vers un arbre continu $\mathcal{T}_F$. La mesure μ_F est alors la limite des mesures empiriques associées aux feuilles des T_F^n.

La majoration $\dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F)) \leq |\alpha|^{-1}$ s'obtient à l'aide d'une famille de recouvrements adéquats de l'arbre. On obtient de la même façon la compacité. La preuve de la minoration est plus technique. Une première approche consiste à utiliser le lemme de Frostman, ce qui nous permet d'obtenir le minorant $|\alpha|^{-1}$ dans les cas où ν est finie et $\nu(s_N > 0) = 0$ pour un $N \in \mathbb{N}$. Pour une mesure de dislocation ν quelconque, l'idée est de se ramener au cas précédent en considérant le sous-arbre $\mathcal{T}_F^{N,\varepsilon} \subset \mathcal{T}_F$ construit à partir de $\mathcal{T}_F$ en ne gardant à chaque noeud :

- soit que le plus gros sous-arbre issu de ce noeud si la masse relative de ce sous-arbre par rapport à la masse totale des sous-arbres issus du noeud est supérieure à $1-\varepsilon$;

- soit, si ce n'est pas le cas, que les N plus gros sous-arbres.

On obtient ainsi un CRT $(\mathcal{T}_F^{N,\varepsilon},\mu_F^{N,\varepsilon})$. Le lemme de Frostman permet d'obtenir un minorant pour la dimension de Hausdorff de $\mathcal{L}(\mathcal{T}_F^{N,\varepsilon})$ et donc de $\mathcal{L}(\mathcal{T}_F)$. Ce minorant converge vers $|\alpha|^{-1}$ quand $\varepsilon \downarrow 0$ et $N \uparrow \infty$ dès que $\int_{\mathcal{S}^\downarrow}(s_1^{-1}-1)\nu(\mathrm{d}\mathbf{s}) < \infty$.

On a construit un CRT codant la fragmentation F. Dans le cas de l'exemple brownien, ce CRT est lui même codé par une fonction continue positive sur [0,1], s'annulant en 0 et en 1. A nouveau, on aimerait savoir si ces exemples se généralisent aux fragmentations. On sait qu'il n'est pas toujours possible de construire un CRT à partir d'une fonction continue. Aldous [3] montre que pour que ce soit possible il faut et il suffit que l'arbre soit compact et ordonné de façon à ce que les feuilles soient denses dans l'arbre en respectant l'ordre. Pour appliquer ce résultat à l'arbre $\mathcal{T}_F$, on commence par mettre un ordre par récurrence sur les arbres $\mathcal{T}_F^n$. On vérifie ensuite facilement que les feuilles sont denses si et seulement si $\nu(\mathcal{S}^\downarrow)=\infty$. Comme l'arbre est compact, on en déduit l'existence d'une fonction H_F (appelée *fonction de hauteur* de l'arbre) continue positive sur [0,1] s'annulant en 0 et en 1, telle que $\mathcal{T}_F=[0,1]\,/\sim_{H_F}$ où

$$x \sim_{H_F} y \Leftrightarrow d(x,y) = H_F(x)+H_F(y)-2\inf_{z\in[x,y]} H_F(z) = 0$$

et telle que μ_F soit la mesure image par la projection sur l'espace quotient de la mesure de Lebesgue. Ainsi, une version de F peut se construire à partir de la fonction continue H_F de manière identique à la construction de la fragmentation brownienne à partir de l'excursion normalisée brownienne : si $F'(t)$ est la suite décroissante des longueurs des composantes connexes de $\{x\in[0,1] : H_F(x)>t\}$, $t\geq 0$, alors $F' \overset{\text{loi}}{=} F$.

Théorème 0.6 *Supposons que la fonction $x \mapsto \nu(s_1<1-x)$ varie régulièrement quand $x\to 0$ avec indice $\vartheta \in (0,1)$. Alors presque sûrement, la fonction H_F est höldérienne d'indice γ, $\forall\gamma<\vartheta\wedge|\alpha|$, et n'est pas höldérienne d'indice γ, $\forall\gamma>\vartheta\wedge|\alpha|$.*

On obtient plus généralement, si la fonction $x\mapsto\nu(s_1<1-x)$ n'est pas à variation régulière en 0, un encadrement de l'indice de Hölder maximal de H_F (Théorème 3.4).

Ces résultats sur la dimension de Hausdorff de l'arbre $\mathcal{T}_F$ et sur la régularité höldérienne de sa fonction de hauteur s'appliquent en particulier à l'*arbre stable d'indice* β, $1<\beta\leq 2$. Lorsque $\beta=2$, cet arbre est le CRT brownien. Lorsque $1<\beta<2$, c'est un CRT qui est la limite en loi quand $n\to\infty$ d'arbres de Galton-Watson critiques ayant une loi de reproduction $(\eta(k),k\geq 0)$ telle que $\eta(k) \underset{k\to\infty}{\sim} Ck^{-1-\beta}$, et conditionnés à avoir n feuilles et des arêtes de longueur $n^{\beta^{-1}-1}$, [28], [29].

Soit $(\mathcal{T}_\beta,\mu_\beta)$ un arbre stable d'indice β et pour tout $t\geq 0$, soit $F_\beta(t)$ la suite décroissante des μ_β-masses des composantes connexes de $\{v\in\mathcal{T}_\beta : d(\emptyset,v)>t\}$. Miermont [56] montre que le processus $(F_\beta(t),t\geq 0)$ est un processus de fragmentation auto-similaire d'indice $1/\beta-1$ sans érosion et calcule explicitement sa mesure de dislocation ν_β. On vérifie que cette mesure intègre la fonction $(s_1^{-1}-1)$ et que $\nu_\beta(s_1<1-x)\sim Cx^{\beta^{-1}-1}$ au voisinage de 0. Il résulte alors des théorèmes ci-dessus que la dimension de Hausdorff de l'arbre stable $\mathcal{T}_\beta$ est presque sûrement égale à $\beta/(\beta-1)$ et que sa fonction de hauteur est presque sûrement höldérienne d'indice γ pour tout $\gamma < (\beta-1)/\beta$ et n'est pas höldérienne d'indice γ pour tout $\gamma > (\beta-1)/\beta$. Ces résultats ont été obtenus indépendamment par Duquesne et Le Gall [30].

0.5 Fragmentation avec immigration

On introduit ici des modèles aléatoires et déterministes qui décrivent l'évolution d'un système avec fragmentation et immigration (arrivée régulière) de particules. Ceci correspond à l'exemple sus-cité des industries minières où des blocs de roche sont amenés en permanence pour être fragmentés. On s'intéresse en particulier à l'existence d'un état d'équilibre pour de tels systèmes, ce qui peut être interprété comme un moyen de compenser grâce à l'immigration la perte de masse par formation de poussière, et plus généralement la fragmentation de particules.

Les *processus de fragmentation avec immigration* sont à valeurs dans l'espace des suites décroissantes tendant vers 0 à l'infini

$$\mathcal{D} = \{\mathbf{s} = (s_j)_{j\geq 1} : s_1 \geq s_2 \geq ... \geq 0, \lim_{j\to\infty} s_j = 0\},$$

muni de la distance $d(\mathbf{s},\mathbf{s}') = \sup_{j\geq 1} \left|s_j - s'_j\right|$. Ils modélisent des systèmes où l'immigration et la fragmentation se déroulent indépendamment. L'immigration est codée par un processus ponctuel de Poisson à valeurs dans $\mathcal{D}$ de mesure d'intensité I telle que

$$\int_{\mathcal{D}} \sum\nolimits_{j\geq 1} (s_j \wedge 1)\, I(\mathrm{d}\mathbf{s}) < \infty,$$

ce qui assure que la masse totale des particules immigrant dans un intervalle de temps fini est presque sûrement finie. La fragmentation est une fragmentation auto-similaire de paramètres (α,c,ν) .

Définition 0.3 *Soient* $\mathbf{u} = (u_1,u_2,...) \in \mathcal{D}$ *une suite aléatoire et* $((\mathbf{s}(t_i),t_i)\,,i \geq 1)$ *les atomes d'un processus ponctuel de Poisson d'intensité* I*, indépendante de* $\mathbf{u}$*. Soit* $(F^{(n)},F^{(i,j)},n,i,j, \geq 1)$ *une famille de fragmentations* (α,c,ν) *mutuellement indépendantes et indépendantes de* $\mathbf{u}$ *et* $((\mathbf{s}(t_i),t_i)\,,i \geq 1)$*. Alors, presque sûrement pour tout* $t \geq 0$*, le réarrangement par ordre décroissant*

$$FI^{(\mathbf{u})}(t) = \{u_n F^{(n)}(u_n^\alpha t), s_j(t_i) F^{(i,j)}(s_j^\alpha(t_i)(t-t_i)), n,j \geq 1, t_i \leq t\}^{\downarrow}$$

existe et appartient à $\mathcal{D}$*. Le processus* $FI^{(\mathbf{u})}$ *est appelé processus de fragmentation avec immigration de paramètres* (α,c,ν,I) *partant de* $\mathbf{u}$*.*

Autrement dit, la suite $FI^{(\mathbf{u})}(t)$ est la suite des masses de particules provenant d'une part de la fragmentation pendant un temps t des particules de masses $u_1,u_2,...$ présentes au temps 0 et, d'autre part, de la fragmentation pendant un temps $t-t_i$ de particules de masses $s_1(t_i),s_2(t_i),...$ ayant immigrées au temps t_i, $t_i \leq t$. Le processus $FI^{(\mathbf{u})}$ est fellérien.

Voici un exemple de processus de fragmentation avec immigration. Soit B un mouvement brownien réel issu de 0 et

$$B_{(d)}(x) = B(x) + dx, x \geq 0,$$

un mouvement brownien avec une dérive $d > 0$. Pour chaque $t \geq 0$, on note $FI_{(d)}(t)$ la suite rangée par ordre décroissant des longueurs des excursions finies de $B_{(d)}$ au dessus de t. Grâce à la théorie des excursions browniennes, on voit que $FI_{(d)}$ est un processus de fragmentation avec immigration : la fragmentation est celle construite à partir de l'excursion brownienne normalisée, de paramètres $(\alpha_e, 0, \nu_e)$; et l'immigration est caractérisée par $I_{(d)}(s_2 > 0) = 0$ (les particules arrivent une par une) et

$$I_{(d)}(s_1 \in \mathrm{d}x) = \sqrt{(2\pi)^{-1}} x^{-3/2} \exp(-xd^2/2)\mathrm{d}x, \quad x > 0.$$

Par ailleurs, la propriété de Markov forte du mouvement brownien implique que le processus $FI_{(d)}$ est stationnaire, c'est-à-dire que $FI_{(d)}(t) \overset{\text{loi}}{=} FI_{(d)}(0)$ pour tout $t \geq 0$. Le Théorème de Girsanov permet d'obtenir explicitement cette loi stationnaire (Proposition 4.2 (ii)).

Ceci nous amène à la question suivante : existe-t-il dans le cas général un état d'équilibre, c'est-à-dire une loi stationnaire pour le processus FI, et si oui, quelle est la vitesse de convergence vers cet équilibre ? Un candidat naturel pour une loi stationnaire est la limite en loi éventuelle du processus FI partant de $(0,0,...)$. Cette limite, si elle existe, se construit de la manière suivante : soient $F^{(i,j)}$, $i,j \geq 1$, des fragmentations (α, c, ν) indépendantes et $((\mathbf{s}(t_i), t_i), i \geq 1)$ un processus ponctuel de Poisson d'intensité I, indépendant des fragmentations $F^{(i,j)}$. Si les termes $s_j(t_i)F^{(i,j)}(s_j^\alpha(t_i)t_i)$, $i,j \geq 1$, peuvent être rangés par ordre décroissant de manière à former une suite de $\mathcal{D}$, alors la loi de cette suite

$$\mathbf{U}_{\text{stat}} = \{s_j(t_i)F^{(i,j)}(s_j^\alpha(t_i)t_i), i,j \geq 1\}^{\downarrow}$$

est la limite cherchée. En combinant des résultats sur les processus ponctuels de Poisson et sur les fragmentations auto-similaires, on obtient alors la caractérisation suivante pour l'existence d'une loi stationnaire. Soit

$$\alpha_I = -\sup \left\{ a \geq 0 : \int_{\mathcal{D}} s_1^a \mathbf{1}_{\{s_1 \geq 1\}} I(\mathrm{d}\mathbf{s}) < \infty \right\}. \tag{5}$$

Si $\alpha_I < 0$ et si $\alpha > \alpha_I$, alors $\mathbf{U}_{\text{stat}}$ existe presque sûrement et sa loi est l'unique loi stationnaire. Si $\alpha < \alpha_I$, il n'y a pas de loi stationnaire. Dans ce dernier cas, avec une probabilité non nulle les particules de masses supérieures à un (qui se fragmentent d'autant moins vite que l'indice α est négatif) s'accumulent et la suite $\mathbf{U}_{\text{stat}}$ n'existe pas. Des résultats concernant les cas critiques $\alpha_I = 0$ ou $\alpha = \alpha_I$ et la structure de $\mathbf{U}_{\text{stat}}$, c'est-à-dire son appartenance à certains espaces $l^p = \{\mathbf{s} \in \mathcal{D} : \sum_{i \geq 1} s_i^p < \infty\}$, $p \geq 0$, sont donnés dans les Théorèmes 4.1, 4.2 et 4.3.

En ce qui concerne la vitesse de convergence vers la loi stationnaire, on obtient des résultats très différents suivant que $\alpha > 0$, $\alpha = 0$ ou $\alpha < 0$. La distance considérée sur les mesures de probabilité sur $\mathcal{D}$ est la distance de Fortet-Mourier

$$\mathfrak{D}(\mu, \mu') = \sup_{\substack{f \text{ 1-Lipschitzienne,} \\ \sup_{\mathbf{s} \in \mathcal{D}} |f(\mathbf{s})| \leq 1}} \left| \int_{\mathcal{D}} f(\mathbf{s})\mu(\mathrm{d}\mathbf{s}) - \int_{\mathcal{D}} f(\mathbf{s})\mu'(\mathrm{d}\mathbf{s}) \right|.$$

On rappelle qu'une fonction 1-Lipschitzienne est une fonction telle que $|f(\mathbf{s})-f(\mathbf{s}')|\leq d(\mathbf{s},\mathbf{s}')$ pour tous $\mathbf{s},\mathbf{s}'\in\mathcal{D}$, et que la distance $\mathfrak{D}$ induit la topologie de la convergence faible. Dans l'énoncé suivant, $\mathrm{v}(t)=\left\|\mathcal{L}(FI^{(\mathbf{u})}(t))-\mathcal{L}(\mathbf{U}_{\mathrm{stat}})\right\|$ est la vitesse de convergence vers la loi stationnaire; $\mathcal{L}(FI^{(\mathbf{u})}(t))$ désigne la loi de $FI^{(\mathbf{u})}(t)$ et $\mathcal{L}(\mathbf{U}_{\mathrm{stat}})$ la loi stationnaire. Les suites initiales $\mathbf{u}$ sont déterministes.

Théorème 0.7 (i) *Supposons que $\alpha>0$, que $\int_{\mathcal{D}}\sum_{j\geq1}s_j^pI(\mathrm{d}\mathbf{s})<\infty$ pour un certain $p>0$ et que $\mathbf{u}\in l^p$. Alors, pour tout $a<1/\alpha$, $\mathrm{v}(t)=o(t^{-a})$ quand $t\to\infty$.*

(ii) *Supposons que $\alpha=0$, que $\int_{\mathcal{D}}\sum_{j\geq1}s_j^{1+\varepsilon}I(\mathrm{d}\mathbf{s})<\infty$ pour un certain $\varepsilon>0$ et que $\mathbf{u}\in l^{1+\varepsilon}$. Alors, pour tout $a<\phi(\varepsilon)/(2+\varepsilon)$, $\mathrm{v}(t)=o(\exp(-at))$ quand $t\to\infty$.*

(iii) *Supposons que $\alpha<0$, que $\int_{\mathcal{D}}\sum_{j\geq1}s_j^{-\alpha}\mathbf{1}_{\{s_j\geq1\}}I(\mathrm{d}\mathbf{s})<\infty$ et que $\sum_{j\geq1}\exp(-u_j^\alpha)<\infty$. Alors, il existe une constante $A>0$ telle que lorsque $t\to\infty$,*

$$\mathrm{v}(t)=O\Big(\int_{\mathcal{D}}\sum_{j\geq1}s_j^{-\alpha}\exp(-Ats_j^\alpha)I(\mathrm{d}\mathbf{s})+\exp(-Atu_1^\alpha)\Big)$$

Le résultat dans le cas où α est négatif peut-être précisé (Théorème 4.4) et rendu plus explicite lorsque la fonction $x\mapsto\int_{\mathcal{D}}\sum_{j\geq1}\mathbf{1}_{\{s_j\geq x\}}I(\mathrm{d}\mathbf{s})$ vérifie certaines propriétés de variation régulière (Corollaire 4.1). Le principe de la preuve de ce théorème est le même dans les trois cas $\alpha>0$, $\alpha=0$ et $\alpha<0$. Il repose sur une méthode de couplage : on considère $FI^{(\mathbf{u})}$ une version du processus partant de $\mathbf{u}$ et $FI^{(\mathbf{U}_{\mathrm{stat}})}$ une version partant de $\mathbf{U}_{\mathrm{stat}}$ et on arrête ces processus à un même temps T au-delà duquel seules les particules provenant de l'immigration jouent un rôle "non-négligeable", dans le sens où toutes les particules issues des états initiaux $\mathbf{u}$ et $\mathbf{U}_{\mathrm{stat}}$ ont une masse inférieure à une certaine quantité $r(t)$ pour tout $t>T$. Comme l'évolution des particules immigrées est la même (en loi) pour $FI^{(\mathbf{u})}$ et $FI^{(\mathbf{U}_{\mathrm{stat}})}$, il s'ensuit que $\mathrm{v}(t)\leq2(r(t)+P(T>t))$ et le résultat découle de la vitesse de convergence vers 0 de $P(T>t)$. La différence entre les trois cas $\alpha>0$, $\alpha=0$ et $\alpha<0$ est dans le choix du couple (r,T).

Pour finir, nous considérons un modèle déterministe pour la fragmentation avec immigration, à savoir l'équation "(α,c,ν,I)"

$$\begin{aligned}\partial_t\langle\mu_t,f\rangle&=\int_0^\infty x^\alpha\left(-cxf'(x)+\int_{\mathcal{S}^\downarrow}\Big[\sum_{j\geq1}f(xs_j)-f(x)\Big]\nu(\mathrm{d}\mathbf{s})\right)\mu_t(\mathrm{d}x)\\&\quad+\int_{\mathcal{D}}\sum_{j\geq1}f(s_j)I(\mathrm{d}\mathbf{s}),\end{aligned}\qquad(6)$$

qui ajoute un facteur d'immigration à l'équation (3) étudiée ci-dessus. L'ensemble de fonctions tests ici est l'ensemble des fonctions f définies sur $(0,\infty)$, à support compact et de dérivée continue. On le note $\mathcal{C}_c^1(0,\infty)$. Soit μ_0 une mesure de Radon sur $(0,\infty)$ telle que $\mu_0\left[1,\infty\right)<\infty$ et soit $(u(t_i),i\geq1)$ un processus ponctuel de Poisson d'intensité μ_0. On note $\mathbf{u}(\mu_0)$ le réordonnement décroissant des termes de cette suite et on considère $FI^{(\mathbf{u}(\mu_0))}$ un processus de fragmentation avec immigration partant de $\mathbf{u}(\mu_0)$. A l'aide du Théorème 0.1 ci-dessus on montre que la famille de mesures $(\mu_t,t\geq0)$ définies par

$$\langle\mu_t,f\rangle=E\left[\sum_{k\geq1}f(FI_k^{(\mathbf{u}(\mu_0))}(t))\right],\ \ f\in\mathcal{C}_c^1\left(0,\infty\right),$$

est l'unique solution à l'équation (6), *pourvu que les mesures* μ_t *soient de Radon*. On renvoie à la Proposition 4.4 pour des conditions suffisantes sur μ_0 et I pour que les mesures μ_t soient de Radon et également pour une extension de ce résultat à une mesure initiale μ_0 ne vérifiant pas nécessairement l'hypothèse $\mu_0\,[1,\infty) < \infty$.

On s'intéresse ensuite aux solutions stationnaires de l'équation (6), c'est-à-dire aux mesures de Radon μ_{stat} telles que la famille constante $\mu_t = \mu_{\text{stat}}$, $t \geq 0$, soit une solution à (6). Le subordinateur ξ intervenant dans le résultat suivant est toujours celui associé au fragment marqué de la fragmentation.

Proposition 0.4 *Supposons que* $\int_{\mathcal{D}} \sum_{j\geq 1} s_j I(\mathrm{d}\mathbf{s}) < \infty$. *Il y a alors une unique solution stationnaire* μ_{stat} *et pour tout* $\alpha \in \mathbb{R}$, $\mu_{\text{stat}}(\mathrm{d}x) = x^{-\alpha}\mu_{\text{stat}}^{(\text{hom})}(\mathrm{d}x)$ *où la mesure* $\mu_{\text{stat}}^{(\text{hom})}$ *est indépendante de* α *et est définie pour toute fonction* $f \in \mathcal{C}_c^1\,(0,\infty)$ *par*

$$\langle \mu_{\text{stat}}^{(\text{hom})}(\mathrm{d}x), f\rangle = \int_0^\infty \int_{\mathcal{D}} \sum\nolimits_{j\geq 1} E\left[f(s_j \exp(-\xi(t)))\exp(\xi(t))\right] I(\mathrm{d}\mathbf{s})\mathrm{d}t.$$

Si de plus $(\mu_t, t \geq 0)$ *est la solution à l'équation* (6) *partant d'une mesure* μ_0 *telle que* $\int_1^\infty x\mu_0(\mathrm{d}x) < \infty$, *alors* $\mu_t \overset{\text{vaguement}}{\to} \mu_{\text{stat}}$ *quand* $t \to \infty$.

Contrairement au cas stochastique, la condition d'existence de la loi stationnaire ne dépend pas ici de l'indice d'auto-similarité α. On montre réciproquement que si $E\left[\xi(1)\right] < \infty$ (ce qui est équivalent à $c = \nu(\sum_{i\geq 1} s_i < 1) = 0$ et $\phi'(0^+) < \infty$) et si $\int_{\mathcal{D}} \sum_{j\geq 1} s_j I(\mathrm{d}\mathbf{s}) = \infty$, il n'y a pas de solution stationnaire à l'équation (6). Sous ces hypothèses sur ξ et I, les masses des particules s'accumulent dans des compacts $[a,b]$, $0 < a < b$, et la mesure μ_{stat} définie dans la proposition ci-dessus (qui de toute façon est la seule mesure stationnaire possible) n'est pas une mesure de Radon.

0.6 Conclusion

Il est intéressant de noter que la plupart des résultats obtenus dans le cadre des fragmentations auto-similaires *sans érosion et sans production de poussière au moment de la dislocation d'une particule* (nous supposons dans cette conclusion que ces deux conditions sont toujours réalisées) dépendent essentiellement de l'indice d'auto-similarité α et de sa position par rapport à certains indices “critiques”. Ainsi, trois indices critiques apparaissent : $\alpha = 0$, $\alpha = -1$ et, pour les modèles avec immigration, $\alpha = \alpha_I$ (ce dernier indice étant défini par la formule (5)).

La condition $\alpha < 0$ caractérise l'existence de poussière et nous montrons ainsi que dès que $\alpha < 0$, la structure généalogique de la fragmentation aléatoire se décrit à l'aide d'un arbre continu aléatoire compact, dont la dimension de Hausdorff est égale à $1 \vee |\alpha|^{-1}$ (pourvu que le plus gros fragment obtenu lors d'une dislocation ne soit pas trop petit, i.e. que ν intègre $s_1^{-1} - 1$).

Cette dimension atteint donc un seuil critique en $\alpha = -1$, en dessous duquel l'arbre est très fin et la fragmentation très rapide. La même limite intervient dans les résultats sur la régularité de la masse (aléatoire) de poussière M puisque presque sûrement, la mesure $\mathrm{d}M$ est singulière lorsque $\alpha \leq -1$, tandis que si $\alpha > -1$ et si la fragmentation est N-aire (chaque particule se fragmente en au plus N morceaux), la mesure $\mathrm{d}M$ a une densité. Lorsque la fragmentation n'est pas N-aire, la condition $\alpha > -1$ n'est pas suffisante a priori pour établir l'existence d'une densité et le critère que nous obtenons dépend de manière plus significative de la mesure de dislocation ν.

Pour les systèmes aléatoires avec immigration, le paramètre α_I correspond à une limite en deçà de laquelle la fragmentation des grosses particules immigrées n'est pas assez rapide, ce qui entraîne l'accumulation de grosses particules et l'absence d'un état d'équilibre. Par contre, lorsque $\alpha > \alpha_I$, l'immigration "compense" la fragmentation et le système converge vers un état stationnaire à une vitesse qui dépend fortement de la position de α par rapport l'indice critique 0. Dans les modèles déterministes, l'indice α n'influence pas l'existence d'une loi stationnaire (il suffit pour cela que la masse moyenne immigrant par unité de temps soit finie).

Notons cependant qu'il y a quelques résultats qui dépendent significativement de la mesure ν. En particulier, nous avons vu que si l'indice α est strictement négatif, il existe une fonction continue codant la fragmentation si et seulement si $\nu(\mathcal{S}^{\downarrow}) = \infty$, et la continuité höldérienne de cette fonction dépend alors à la fois de l'indice α et du comportement au voisinage de 0 de la fonction $x \mapsto \nu(s_1 < 1 - x)$.

Dans le cadre plus général où les particules se fragmentent à un taux $\tau(x)\nu(\mathrm{d}\mathbf{s})$, nous avons vu que c'est essentiellement le comportement de τ en 0 qui caractérise l'existence de poussière : dans la mesure où τ est décroissante au voisinage de 0 et où les fragments produits par ν ne sont pas trop gros ($\phi'(0^+) < \infty$), l'existence de poussière est équivalente à $\int_{0^+} \mathrm{d}x/x\tau(x) < \infty$.

Il est naturel de se demander alors si les résultats des chapitres 2, 3 et 4 sur les fragmentations auto-similaires se généralisent aux fragmentations $(\tau,0,\nu)$. Pour la plupart la réponse est positive, et souvent ces résultats se déduisent des cas auto-similaires par comparaison, simplement parce que les particules se fragmentent plus vite dans le modèle $(\tau,0,\nu)$ que dans le modèle $(\tau',0,\nu)$ lorsque $\tau \geq \tau'$.

Ainsi, si τ décroît dans un voisinage de 0 et si $\int_{0^+} \mathrm{d}x/x\tau(x) < \infty$, on montre que la fragmentation peut être codée par un arbre aléatoire continu d'Aldous et, si de plus la mesure ν est infinie, par une fonction continue. Des encadrements de la dimension de Hausdorff de l'arbre et des coefficients de Hölder de la fonction continue associée peuvent être obtenus en fonction du comportement de τ en 0.

De même, les résultats sur l'immigration s'adaptent bien aux modèles dépendant de τ (dans ce cas il faut considérer des fonctions τ définies sur $(0,\infty)$) : si $\tau(m) \leq Cm^{\alpha}$ sur $(0,\infty)$ avec $\alpha < \alpha_I$, il n'y pas de loi stationnaire, tandis que si $\tau(m) \geq Cm^{\beta}$ sur $(0,\infty)$ avec $\beta > \alpha_I$, il y a une loi stationnaire et la vitesse de convergence vers la loi stationaire est plus rapide pour une fragmentation $(\tau,0,\nu)$ que pour une fragmentation $(\beta,0,\nu)$. Par ailleurs, les conditions d'existence d'un état d'équilibre pour l'équation déterministe associée (on remplace m^{α} par $\tau(m)$ dans l'équation (6)), sont exactement les mêmes que dans le cas auto-similaire et la solution stationnaire est alors donnée par $\mu_{\mathrm{stat}}(\mathrm{d}x) = (\tau(x))^{-1}\mu_{\mathrm{stat}}^{(\mathrm{hom})}(\mathrm{d}x)$ avec la même mesure $\mu_{\mathrm{stat}}^{(\mathrm{hom})}$ que dans la Proposition 0.4.

Il semble plus difficile de généraliser les résultats sur la régularité de la masse de poussière M aux cas $(\tau,0,\nu)$. Seule la preuve de la singularité de la masse $\mathrm{d}M$ s'adapte bien : par comparaison, on voit que dM est presque sûrement singulière dès que $\tau(m) \geq Cm^{-1}$ près de 0. On peut d'ailleurs améliorer ce critère et montrer que la masse dM est presque sûrement singulière dès que $\int_{0^+} \mathrm{d}x/x^2\tau(x) < \infty$ et que la fonction $m \mapsto m\tau(m)$ décroit dans un voisinage de 0. Dans le cas auto-similaire, les résultats sur l'existence d'une densité et sur l'approximation de cette densité par des fonctionnelles dépendant des petits fragments reposent sur les égalités en loi de $(F(t),t \geq 0)$ sous P_m et de $(mF(\tau(m)t),t \geq 0)$ sous P_1, $m \geq 0$. Ces égalités, et par suite nos preuves, ne sont plus valables si la fonction τ n'est pas proportionnelle à une puissance. Par comparaison, on obtient quand même des résultats sur le comportement asymptotique des fonctionnelles dépendant des petits fragments.

Enfin, pour compléter l'étude entreprise au chapitre 3 sur la généalogie des fragmentations auto-similaires, nous signalons qu'il est possible de décrire la généalogie des fragmentations d'indice α positif ou nul à l'aide d'arbres continus aléatoires dont toutes les feuilles sont à une même distance de la racine. Il serait intéressant d'étudier la structure de ces arbres, ainsi que celle de leurs feuilles.

Chapitre 1

Loss of mass in deterministic and random fragmentations

Abstract: We consider a linear rate equation, depending on three parameters, that modelizes fragmentation. For each of these fragmentation equations, there is a corresponding stochastic model, from which we construct an explicit solution to the equation. This solution is proved unique. We then use this solution to obtain criteria for the presence or absence of loss of mass in the fragmentation equation, as a function of the equation parameters. Next, we investigate small and large times asymptotic behavior of the total mass for a wide class of parameters. Finally, we study the loss of mass in the stochastic models.

1.1 Introduction

Fragmentation of particles appears in various physical processes, such as polymer degradation, grinding, erosion and oxidation. In the models we consider, there are only particles with mass one at the initial time. Those particles split independently of each others to give smaller particles and each obtained particle splits in turn, independently of the past and of others particles. And so on ... The splitting of a particle of mass x gives rise to a sequence of smaller particles with masses $xs_1, xs_2, ...$ where $s_1 \geq s_2 \geq ... \geq 0$. Thus, it is convenient to introduce the following set:

$$\mathcal{S}^{\downarrow} := \left\{ s = (s_i)_{i \in \mathbb{N}^*} \, , \; s_1 \geq s_2 \geq ... \geq 0 : \sum_{i=1}^{\infty} s_i \leq 1 \right\}.$$

Note that we take into account the case when $\sum_{i=1}^{\infty} s_i < 1$, which corresponds to the loss of a part of the initial mass during the splitting. The rate at which a particle with mass one splits is then described by a non-negative measure ν on $\mathcal{S}^* = \mathcal{S}^{\downarrow} \setminus \{(1, 0, 0, ...)\}$, called the *splitting measure.* This measure is supposed to fit the requirement

$$\int_{\mathcal{S}^*} (1 - s_1) \nu(ds) < \infty \tag{1.1}$$

(see the Appendix for an explanation). Note that the case when $\nu(\mathcal{S}^*) = \infty$, which is often excluded from fragmentation studies, is here included.

A linear rate equation has been developed (see e.g. [31]) to study the time evolution of the mass distribution of particles involved in a fragmentation phenomenon (see also [20] for physical studies on fragmentation). Here, we consider the special case when the splitting rate for a particle with mass x is proportional to that of a particle with mass one. More precisely, this splitting rate is equal to $\tau(x)\nu(ds)$, where τ is a continuous and positive function on $]0,1]$ such that $\tau(1) = 1$. As we will see in the next section, τ should be seen as the speed of fragmentation. Our deterministic fragmentation model is the weak form of this linear rate equation and describes the evolution of the family $(\mu_t, t \geq 0)$ of non-negative Radon measures on $]0,1]$, where $\mu_t(dx)$ corresponds to the average number per unit volume of particles with mass in the interval $(x, x+dx)$ at time t. This so-called *fragmentation equation* is

$$\begin{cases} \partial_t \langle \mu_t, f \rangle = \displaystyle\int_0^1 \tau(x) \left(-cxf'(x) + \int_{\mathcal{S}^\downarrow} \left[\sum_{i=1}^{\infty} f(xs_i) - f(x) \right] \nu(ds) \right) \mu_t(dx) \\ \mu_0 = \delta_1(dx) \end{cases} \tag{1.2}$$

for test-functions f belonging to $\mathcal{C}_c^1(]0,1])$, the set of differentiable functions with compact support in $]0,1]$. The second term between parentheses on the right side of equation (1.2) corresponds to a growth in the number of particles of masses $xs_1, xs_2, ...$ and to a decrease in the number of particles of mass x, as a consequence of the splitting of particles of mass x. The first term between parentheses on the right side of (1.2) represents a loss of particles of mass x, as a result of erosion. The constant c is non-negative and called the *erosion coefficient* of the fragmentation. The function τ, the constant c and the measure ν are called the parameters of the fragmentation equation.

We next introduce a random fragmentation model, called *fragmentation process*. A fragmentation process $(F(t), t \geq 0)$ is a Markov process with values in $\mathcal{S}^\downarrow$ satisfying the *fragmentation property*, which will be defined rigorously in Section 1.2. Informally, this means that given the system at a time t, say $F(t) = (s_1, s_2, ...)$, then for each $i \in \mathbb{N}^*$, the fragmentation system stemming from the particle with mass s_i evolves independently of the others particles and with the same law as the process F starting from a unique particle with mass s_i. And then, if we denote by $(s_{i,j}(r))_{j\geq 1}$ the masses of the particles stemming from the one with mass s_i after a time r, the sequence $F(t+r)$ will consist in the non-increasing rearrangement of the masses $(s_{i,j}(r))_{i,j\geq 1}$. A family of fragmentation processes with a scaling property (namely the self-similar fragmentation processes) was studied by Bertoin in [13], [14] and [15]. In Section 1.2, the main results on these processes are recalled and a larger set of fragmentation processes, characterized each by the three parameters τ, c and ν of a fragmentation equation, is constructed.

This set of fragmentation processes is used to study the fragmentation equation. More precisely, given the parameters τ, c and ν, we construct in Section 1.3 the unique solution to the fragmentation equation with parameters τ, c and ν, by following a specific fragment (the so-called size-biased picked fragment process) of the corresponding fragmentation process. Let F^τ denote this fragmentation process. The solution to the fragmentation equation is then given for each $t \geq 0$ by

$$\langle \mu_t, f \rangle = E\left[\sum_{i=1}^{\infty} f(F_i^{\tau}(t))\right] \text{ for } f \in \mathcal{C}_c^1\left(]0,1]\right), \tag{1.3}$$

where $(F_1^{\tau}(t), F_2^{\tau}(t), ...)$ is the sequence $F^{\tau}(t)$. As a general rule, given a fragmentation process F, we denote by $(F_1(t), F_2(t), ...)$ the sequences $F(t)$, $t \geq 0$.

The main purpose of our work is to study the possible loss of mass in these deterministic and stochastic fragmentation models. If the family $(\mu_t, t \geq 0)$ is a solution to the fragmentation equation (1.2), it is easy to see that the total mass $\langle \mu_t, id \rangle$ is non-increasing in t. We say that there is *loss of mass* in the fragmentation equation if there exists a time t such that

$$\langle \mu_t, id \rangle < \langle \mu_0, id \rangle = 1.$$

We will see that this is equivalent to loss of mass in the corresponding fragmentation process, as a result of:

$$\exists t \geq 0 : \langle \mu_t, id \rangle < 1 \Leftrightarrow \text{a.s. } \exists t \geq 0 : \sum_{i=1}^{\infty} F_i^{\tau}(t) < 1.$$

There are three distinct ways to lose mass. The first two are intuitively obvious: there is loss of mass if the erosion coefficient is positive or if the splitting of a particle with mass x gives rise to a sequence of particles with total mass strictly smaller than x. However, there is also an unexpected loss of mass, due to the formation of dust (i.e. an infinite number of particles with mass zero). This latter is of course the most interesting and one of our purposes is to establish for which parameters τ and ν it occurs. This formation of dust has to be compared with *gelation* which may happen in the context of coagulation models and which corresponds to the creation of an infinite-mass particle in finite time (see for example Jeon [41] and Norris [58] for gelation studies). We mention also Aldous [4] for a survey on coagulation and fragmentation phenomena. Concerning loss of mass studies, [15] proves the occurrence of loss of mass to dust in fragmentation processes with function $\tau(x) = x^{\alpha}$ as soon as $\alpha < 0$ and, in that case, that the mass vanishes entirely in finite time. Filippov [35] obtains some conditions for the presence or absence of loss of mass (to compare with Corollary 1.2 in this paper) in the special case where $\nu(\mathcal{S}^*) < \infty$. Let us also mention Fournier and Giet [36], who investigate this appearance of dust in some coagulation-fragmentation equations, whose fragmentation part is rather different than ours (their fragmentations are binary, with absolutely continuous rates that are not necessarily proportional to the one-mass rate). See also Jeon [42].

Formula (1.3) is the key point in the study of loss of mass, which is undertaken in Section 1.4. We get necessary (respectively, sufficient) conditions on the parameters τ, c and ν for loss of mass to occur and when there is loss of mass, we obtain results on small times and large times behavior of the total mass $\langle \mu_t, id \rangle$. Section 1.5 is devoted to loss of mass and total loss of mass for a fragmentation process F^{τ} with parameters τ, c and ν. Define ζ to be the first time at which all the mass has disappeared, i.e.

$$\zeta := \inf\left\{t \geq 0 : F_1^{\tau}(t) = 0\right\}.$$

We state necessary (respectively, sufficient) conditions on (τ, c, ν) for $P(\zeta < \infty)$ to be positive. Then, we look at connections between loss of mass and total loss of mass and study the asymptotic behavior of $P(\zeta > t)$ as $t \to \infty$, for a large class of parameters τ, c and ν.

This paper ends with an appendix containing on the one hand some results on the mass behavior of a fragmentation model constructed from the Brownian excursion of length 1 and on the other hand a proof that (1.1) is a necessary condition for our fragmentation models to exist.

1.2 Preliminaries on fragmentation processes

Let $(F(t), t \geq 0)$ be a Markov process with values in $\mathcal{S}^{\downarrow}$ and denote by P_s the law of F starting from $(s, 0, ...)$, $0 \leq s \leq 1$. The process F is a *fragmentation process* if it satisfies the following *fragmentation property:* for each $t_0 \geq 0$, conditionally on $F(t_0) = (s_1, s_2, ...)$, the process $(F(t + t_0), t \geq 0)$ has the same law as the process obtained, for each $t \geq 0$, by ranking in the non-increasing order the components of the sequences $F^1(t)$, $F^2(t), ...$, where the r.v. F^i are independent with respective laws P_{s_i}.

In this section, we first recall some results on homogeneous and self-similar fragmentation processes. Then we construct a larger family of fragmentation processes, depending on the parameters τ, c and ν of the fragmentation equation (1.2). Given a fragmentation process F, recall the notation $(F_1(t), F_2(t), ...)$ for the sequence $F(t)$, $t \geq 0$.

1.2.1 Homogeneous and self-similar fragmentation processes

A *self-similar fragmentation process* $(F(t), t \geq 0)$ with index α is a fragmentation process having the following *scaling property:* if P_s is the law of F starting from $(s, 0, ...)$, then the law of $(sF(s^{\alpha}t), t \geq 0)$ under P_1 is P_s. If $\alpha = 0$, the fragmentation process F is said to be *homogeneous*. We now recall some results on those processes. For more details, see [13], [14] and [9]. The state $\mathcal{S}^{\downarrow}$ is endowed with the topology of pointwise convergence.

• **Interval representation.** Let F be a self-similar fragmentation process. It may be convenient, for technical reasons, to work with an interval representation of F. Roughly, consider a Markov process $(I(t), t \geq 0)$ with state space the open sets of $]0, 1[$ and such that $I(t') \subset I(t)$ if $t' \geq t \geq 0$. The process $(I(t), t \geq 0)$ is called *self-similar interval fragmentation process* if it satisfies a scaling and a fragmentation property (for a precise definition, we refer to [14]). The interesting point is that there is a càdlàg version of F (which we also call F and which we implicitly consider in the following), and a self-similar interval fragmentation process I_F with the same index of similarity as F, such that $F(t)$ is the non-increasing sequence of the lengths of the interval components of $I_F(t)$, $t \geq 0$. In the sequel, we call I_F the *interval representation of* F. For each $t \geq 0$, we call *fragments* the interval components of $I_F(t)$ and denote by $I_x(t)$ the fragment containing the point x at time t. If such a fragment does not exist, $I_x(t) := \emptyset$. The length $|I_x(t)|$ is called the mass of the fragment.

• **Characterization and Poisson point process description of homogeneous fragmentation processes.** The law of a homogeneous fragmentation process starting from $(1, 0, ...)$ is characterized by two parameters: a non-negative real number c (the *erosion co-*

efficient) and a non-negative measure ν on $\mathcal{S}^* = \mathcal{S}^{\downarrow}\backslash\{(1,0,...)\}$ (the *splitting measure*) satisfying the requirement (1.1). The erosion coefficient corresponds to the continuous part of the process, whereas the splitting measure describes the jumps of the process. More precisely, consider such a measure ν and a Poisson point process $((\Delta(t),k(t)),t\geq 0)$ with values in $\mathcal{S}^*\times\mathbb{N}^*$ and whose characteristic measure is $\nu\otimes\#$, $\#$ denoting the counting measure on $\mathbb{N}^*$. As proved in [9], there is a pure jump càdlàg homogeneous fragmentation process F starting from $(1,0,...)$, whose jumps are the times of occurrence of the Poisson point process and are described as follows: let t be a jump time, then the $k(t)$-th term of $F(t^-)$, namely $F_{k(t)}(t^-)$, is removed and "replaced" by the sequence $F_{k(t)}(t^-)\Delta(t)$, that is $F(t)$ is obtained by ranking in the non-increasing order the components of sequences $(F_i(t^-))_{i\in\mathbb{N}^*\backslash\{k(t)\}}$ and $F_{k(t)}(t^-)\Delta(t)$. Now, consider a real number $c\geq 0$. The process $(e^{-ct}F(t),t\geq 0)$ is also a càdlàg homogeneous fragmentation process. The point is that the distribution of each homogeneous fragmentation process can be described like this for a constant $c\geq 0$ and a splitting measure ν. Such process is then called a homogeneous (c,ν)-fragmentation process. Remark that when $\nu(\mathcal{S}^*)=\infty$, each particle splits a.s. immediately.

• **Size-biased picked fragment process.** Let F denote a homogeneous (c,ν)- fragmentation process starting from $(1,0,...)$ and I_F be the interval representation of F. Consider a point picked at random in $]0,1[$ according to the uniform law on $]0,1[$ and independently of F and note $\lambda(t)$ the length of the fragment of I_F containing this point at time t. We call the process $(\lambda(t),t\geq 0)$ the *size-biased picked fragment process* of F. An important part of our work relies on the following property (see [13] for a proof): the process

$$(\xi(t),t\geq 0):=(-\log(\lambda(t)),t\geq 0) \tag{1.4}$$

is a subordinator (i.e. a right-continuous non-decreasing process with values in $[0,\infty]$, started from 0 and with independent and stationary increments on $[0,\varsigma[$, where ς is the first time when the process reaches ∞). We refer to [11] for background on subordinators. The distribution of ξ is then characterized by its Laplace exponent ϕ which is determined by

$$E\left[\exp\left(-q\xi_t\right)\right]=\exp(-t\phi(q)),\quad t\geq 0,\quad q\geq 0$$

and which can be expressed here as a function of the parameters ν and c. More precisely:

$$\phi(q)=c(q+1)+\int_{\mathcal{S}^*}\left(1-\sum_{i=1}^{\infty}s_i^{q+1}\right)\nu(ds),\quad q\geq 0. \tag{1.5}$$

In others words, the subordinator ξ has the following characteristics, which we will often refer to:

$$\begin{aligned}
&\bullet \text{ the killing rate } k=c+\textstyle\int_{\mathcal{S}^*}\left(1-\sum_{i=1}^{\infty}s_i\right)\nu(ds),\\
&\bullet \text{ the drift coefficient } d=c,\\
&\bullet \text{ the Lévy measure } L(dx)=e^{-x}\textstyle\sum_{i=1}^{\infty}\nu(-\log(s_i)\in dx),\ x\in\,]0,\infty[\,.
\end{aligned} \tag{1.6}$$

Recall that the first time ς when the process ξ reaches ∞ has an exponential law with parameter k and that there exists a subordinator η independent of ς, with the same drift coefficient and Lévy measure as ξ but with killing rate 0, such that $\xi_t=\eta_t$ when $t<\varsigma$.

We should point out that two different homogeneous fragmentation processes may lead to subordinators having the same distribution. For example, consider F^1 and F^2, two homogeneous fragmentation processes with erosion coefficient 0 and with respective splitting measures ν_1 and ν_2, where

$$\nu_1(ds) = \frac{1}{2}\delta_{\left(\frac{1}{2},\frac{1}{2},0\ldots\right)}(ds) + \frac{1}{2}\delta_{\left(\frac{1}{2},\frac{1}{4},\frac{1}{4},0\ldots\right)}(ds)$$

and

$$\nu_2(ds) = \frac{3}{4}\delta_{\left(\frac{1}{2},\frac{1}{2},0\ldots\right)}(ds) + \frac{1}{4}\delta_{\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},0\ldots\right)}(ds).$$

Then in both cases, the Laplace exponent ϕ is given by

$$\phi(q) = 1 - \frac{3}{2}\left(\frac{1}{2}\right)^{q+1} - \left(\frac{1}{4}\right)^{q+1}.$$

• **Characterization of self-similar fragmentation processes.** We have seen that the law of a homogeneous fragmentation process is characterized by the two parameters c and ν. This property extends to self-similar fragmentation processes, which are characterized by three parameters: an index of self-similarity α, an erosion coefficient c and a splitting measure ν (this follows from a combination of results of [14] and [9]).

1.2.2 Fragmentation processes (τ, c, ν)

The purpose is to build fragmentation processes depending on the parameters τ, c and ν of the fragmentation equation (1.2). Recall that the function τ is continuous and positive on $]0,1]$ and such that $\tau(1) = 1$. Throughout this paper, we will use the convention $\tau(0) := \infty$. Now, consider F a homogeneous (c,ν)-fragmentation process and $(I_x(t), x \in]0,1[, t \geq 0)$ its interval representation. We introduce the time-change functions

$$T_x^\tau(t) := \inf\left\{u \geq 0 : \int_0^u \frac{dr}{\tau(|I_x(r)|)} > t\right\},\ t \geq 0,\ x \in]0,1[,$$

with the convention $\inf\{\emptyset\} := \infty$. Then, for each $t \geq 0$, consider the family of open intervals

$$\widetilde{I}_x(t) := I_x(T_x^\tau(t)),\ x \in]0,1[,$$

and remark that if $y \neq x$, either $\widetilde{I}_x(t) = \widetilde{I}_y(t)$ or $\widetilde{I}_x(t) \cap \widetilde{I}_y(t) = \emptyset$. Let $F^\tau(t)$ denote the non-increasing sequence of the lengths of the disjoint intervals of $\left(\widetilde{I}_x(t), x \in]0,1[\right)$. Then, following the proof of Theorem 2 in [14], we get:

Proposition 1.1 *The process $(F^\tau(t), t \geq 0)$ is a fragmentation process.*

We call the process F^τ a (τ, c, ν)-*fragmentation process*. Note that if $\tau(x) = x^\alpha$ on $]0,1]$, $\alpha \in \mathbb{R}$, Theorem 2 in [14] states that F^τ is a self-similar fragmentation process with parameters α, c and ν.

If F^{τ_1} and F^{τ_2} are respectively (τ_1, c, ν) and (τ_2, c, ν)-fragmentation processes constructed from the same homogeneous interval fragmentation and such that $\tau_1 \leq \tau_2$, the time-change functions T^{τ_1} and T^{τ_2} satisfy

$$T_x^{\tau_1}(t) \leq T_x^{\tau_2}(t), \text{ for } x \in]0,1[\text{ and } t \geq 0.$$

Then, at each time t and for each point $x \in]0,1[$, the fragment $I_x(T_x^{\tau_1}(t))$ is larger than $I_x(T_x^{\tau_2}(t))$. Informally, fragmentation is faster in the process F^{τ_2} than in F^{τ_1}.

As in the homogeneous case, consider the process

$$(\lambda^\tau(t), t \geq 0) := \left(\left|\widetilde{I}_U(t)\right|, t \geq 0\right)$$

where U is a random variable uniformly distributed on $]0,1[$, independent of the fragmentation process F^τ. In other words, $\lambda^\tau(t)$ represents the mass at time t of the fragment containing a point picked at random uniformly in $]0,1[$ at time 0. It is easy to see that for each $t \geq 0$, if $F^\tau(t) = (F_1^\tau(t), F_2^\tau(t), ...)$, the law of $\lambda^\tau(t)$ is obtained as follows: consider $i(t)$ an integer-valued random variable such that

$$P(i(t) = i | F^\tau(t)) = F_i^\tau(t), \; i \in \mathbb{N}^*,$$
$$P(i(t) = 0 | F^\tau(t)) = 1 - \sum_{i=1}^{\infty} F_i^\tau(t).$$

Then,

$$\lambda^\tau(t) \overset{law}{\sim} F_{i(t)}^\tau(t), \tag{1.7}$$

where $F_0^\tau(t) := 0$. We call $(\lambda^\tau(t), t \geq 0)$ the *size-biased picked fragment process* of F^τ. The following proposition will be essential in the sequel. Its proof is straightforward.

Proposition 1.2 *If $F^\tau(0) = (1, 0, ...)$, the process $(\lambda^\tau(t), t \geq 0)$ has the same distribution as $\left(exp(-\xi_{\rho^\tau(t)}), t \geq 0\right)$, where ξ is the subordinator* (1.4) *constructed from the homogeneous process F and ρ^τ the time-change:*

$$\rho^\tau(t) := \inf\left\{u \geq 0 : \int_0^u \frac{dr}{\tau\left(\exp(-\xi_r)\right)} > t\right\}. \tag{1.8}$$

It is then easy to see that $F_i^\tau(t) \overset{a.s.}{\to} 0$ as $t \to \infty$ for each $i \geq 0$ when the fragmentation process F^τ does not remain constant.

1.3 Existence and uniqueness of the solution to the fragmentation equation

Consider the fragmentation equation (1.2) with parameters τ, c and ν and recall that a solution to this equation is a family of non-negative Radon measures on $]0,1]$, satisfying (1.2) at least for test-functions f belonging to $\mathcal{C}_c^1(]0,1])$. Let F^τ be a (τ, c, ν)-fragmentation process starting from $F^{\tau}(0)=(1,0,...)$. From F^{τ}, we build a solution to this fragmentation equation starting from $\mu_0=\delta_1$ and prove that this solution is unique. More precisely, we have:

Theorem 1.1 *The fragmentation equation* (1.2) *has a unique solution* $(\mu_t, t\geq 0)$*, which is given for all* $t\geq 0$ *by:*

$$\langle \mu_t, f\rangle = E\left[\sum_{i=1}^{\infty} f(F_i^{\tau}(t))\right] \quad \textit{for } f\in \mathcal{C}_c^1(]0,1]).$$

Remark the following consequence of (1.7): for all $t\geq 0$ and all $f\in \mathcal{C}_c^1(]0,1])$,

$$E\left[\sum_{i=1}^{\infty} f(F_i^{\tau}(t))\right] = E\left[\overline{f}\left(\lambda^{\tau}(t)\right)\right], \tag{1.9}$$

where λ^{τ} is the size-biased picked fragment process related to F^{τ} and $\overline{f}$ the function defined from f by $\overline{f}(x):=f(x)/x$, $x\in\,]0,1]$. This will be a key point of the proof of Theorem 1.1. In this proof, the notation C_K^1 refers to the set of differentiable functions on $]0,1]$ with support in K.

Proof. (i) First, we turn the problem into an existence and uniqueness problem for an equation involving non-negative measures on $K=[a,1]$, $0<a\leq 1$. The advantage is that τ is bounded on K. Now, consider $(\pi_t, t\geq 0)$ a family of measures on $]0,1]$ and set $\Pi_t(dx):=x\pi_t(dx)$, $t\geq 0$. It is easy to see that $(\pi_t,t\geq 0)$ solves equation (1.2) if and only if $(\Pi_t, t\geq 0)$ satisfies

$$\begin{cases} \partial_t \langle \Pi_t, f\rangle = \langle \Pi_t, \tau A(f)\rangle, & f\in C_c^1(]0,1]) \\ \Pi_0(dx)=\delta_1(dx), \end{cases} \tag{1.10}$$

where A is the linear operator on $C_c^1(]0,1])$ defined by

$$A(f)(x) = -cxf'(x) - cf(x) + \int_{\mathcal{S}^{\downarrow}} \left[\sum_{i=1}^{\infty} f(xs_i)s_i - f(x)\right]\nu(ds), \quad x\in\,]0,1].$$

Note that if f is equal to 0 on $]0,a]$, so is $A(f)$. Then, $\tau A(f)$ is well-defined on $[0,1]$ for functions $f\in C_c^1(]0,1])$. Moreover, this implies that the family $(\Pi_t,t\geq 0)$ is a solution to equation (1.10) if and only if, for each $0<a\leq 1$, the family $\left(1_{[a,1]}\Pi_t, t\geq 0\right)$ is a solution to

$$\begin{cases} \partial_t \langle \nu_t, f\rangle = \langle \nu_t, \tau A(f)\rangle, & f\in C_{[a,1]}^1 \\ \nu_0(dx)=\delta_1(dx). \end{cases} \tag{1.11}$$

Then consider formula (1.9) and write l_t for the distribution of $\lambda^{\tau}(t)$, $t\geq 0$. Proving Theorem 1.1 is equivalent to prove that $(l_t, t\geq 0)$ is the unique solution to (1.10), which is true if and only if, for each $0<a\leq 1$, the family $\left(1_{[a,1]}l_t, t\geq 0\right)$ is the unique family of non-negative measures on $[a,1]$ satisfying (1.11).

(ii) In the sequel, $K=[a,1]$, $0<a\leq 1$. Consider the subordinator ξ such that $\lambda^{\tau}=\exp(-\xi_{\rho^{\tau}})$ where ρ^{τ} is the time-change

$$\rho^{\tau}(t)=\inf\left\{u\geq 0 : \int_0^u \frac{dr}{\tau\left(\exp(-\xi_r)\right)} > t\right\}$$

(see Proposition 1.2). As a subordinator, ξ is a Feller process on $[0,\infty]$ and its generator G^{ξ} has a domain containing the set of differentiable functions with compact support in $[0,\infty[$. It is well-known that for every function f belonging to this set, the function $G^{\xi}(f)$ is given by

$$G^{\xi}(f)(x)=-kf(x)+df'(x)+\int_{]0,\infty[}(f(x+y)-f(x))L(dy),\quad x\in\,]0,1]\,,$$

where k is the killing rate, d the drift coefficient and L the Lévy measure of ξ. From this and (1.6), we deduce that the generator $G^{\exp(-\xi)}$ of the Feller process $\exp(-\xi)$ has a domain $\mathcal{D}$ containing $\mathcal{C}_c^1\,(]0,1])$ and is given by

$$\begin{aligned}G^{\exp(-\xi)}(f)(x)&=-kf(x)-dxf'(x)+\textstyle\int_{]0,\infty[}(f(x\exp(-y))-f(x))L(dy)\\&=A(f)(x)\end{aligned}$$

at least for $f\in\mathcal{C}_c^1\,(]0,1])$. Then, introduce the function

$$\widetilde{\tau}(x)=\begin{cases}\tau(x), & \text{if } x\in K\\ \tau(a), & \text{if } 0\le x\le a\end{cases}$$

and consider the time-changed process $\exp(-\xi_{\rho^{\tilde\tau}(\cdot)})$, where

$$\rho^{\tilde\tau}(t)=\inf\left\{u\ge 0:\int_0^u\frac{dr}{\widetilde{\tau}\left(\exp(-\xi_r)\right)}>t\right\}.$$

Observing that $\widetilde{\tau}$ is bounded away from 0 and ∞ on $[0,1]$, we apply Theorem 1 and its corollary in [50] to conclude that $\exp(-\xi_{\rho^{\tilde\tau}})$ is a Feller process and that its generator $G^{\exp(-\xi_{\rho^{\tilde\tau}})}$ has the same domain $\mathcal{D}$ as $G^{\exp(-\xi)}$ and is given by

$$G^{\exp(-\xi_{\rho^{\tilde\tau}})}(f)=\tilde\tau G^{\exp(-\xi)}(f),\ f\in\mathcal{D}.\tag{1.12}$$

This formula can also be found in Section III.21 of Rogers and Williams [63] (however they do not consider the Feller property for the time-changed process). For each $t\ge 0$, denote by $\widetilde{l}_t$ the law of the random variable $\exp(-\xi_{\rho^{\tilde\tau}(t)})$. The family $\left(\widetilde{l}_t,t\ge 0\right)$ is then a solution to the Kolmogorov's forward equation:

$$\begin{cases}\partial_t\left\langle\nu_t,f\right\rangle=\left\langle\nu_t,G^{\exp(-\xi_{\rho^{\tilde\tau}})}(f)\right\rangle, & f\in\mathcal{D}\\ \nu_0(dx)=\delta_1(dx).\end{cases}\tag{1.13}$$

Note that if the test-functions set is reduced to C_K^1, (1.13) is the same as equation (1.11), since $G^{\exp(-\xi_{\rho^{\tilde\tau}})}=\tau A$ on C_K^1. In particular, $(1_K l_t,t\ge 0)$ is a solution to (1.11), since for each $t\ge 0$ and each function f supported in K, the following identity holds:

$$E\left[f(\exp(-\xi_{\rho^{\tau}(t)})\right]=E\left[f(\exp(-\xi_{\rho^{\tilde\tau}(t)})\right].$$

This is due to the equality

$$\left\{t\ge 0:\xi_{\rho^{\tau}(t)}\le-\log a\right\}\overset{a.s.}{=}\left\{t\ge 0:\xi_{\rho^{\tilde\tau}(t)}\le-\log a\right\}$$

and the fact that $\rho^{\tau}(t) \overset{a.s.}{=} \rho^{\widetilde{\tau}}(t)$ on this set. All this follows easily from the definitions of ρ^{τ} and $\rho^{\widetilde{\tau}}$.

(iii) Now, it remains to prove that a non-negative solution to equation (1.13) is uniquely determined on K if the test-functions set is C_K^1. To prove this, it is sufficient to show that for each $\gamma > 0$, the image of C_K^1 by the operator $\left(\gamma id - G^{\exp(-\xi_{\rho^{\widetilde{\tau}}})}\right)$ is dense in C_K^0 (the set of continuous functions with support in K) endowed with the uniform norm - see for instance the proof of Proposition 9.18 of chapter 4 in [32] and note that if $(\nu_t, t \geq 0)$ is a solution to (1.13), the functions $t \mapsto \langle \nu_t, f \rangle$ are continuous on $[0, \infty)$ for each $f \in C_K^1$. Thus, we just have to prove this density. To that end, observe that if $x < a$ and if $f \in C_K^0$,

$$E_x\left[f(\exp(-\xi_t))\right] := E\left[f(\exp(-\xi_t)) \mid \exp(-\xi_0) = x\right] = E_1\left[f(x\exp(-\xi_t))\right] = 0.$$

Therefore, the function $x \mapsto E_x\left[f(\exp(-\xi_t))\right]$ belongs to C_K^0 if $f \in C_K^0$. This allows us to consider the restriction of the generator $G^{\exp(-\xi)}$ to C_K^0, denoted by $G^{\exp(-\xi)}/C_K^0$. This operator is the generator of the strongly continuous contraction semigroup on C_K^0 defined by

$$\begin{aligned} T(t) : f \in C_K^0 \mapsto \quad & T(t)(f) \in C_K^0, \\ & T(t)(f)(x) = E_1\left[f(x\exp(-\xi_t))\right]), \quad x \in \left]0,1\right]. \end{aligned}$$

Its domain is $C_K^0 \cap \mathcal{D}$. The same remark holds for the process $\exp(-\xi_{\rho^{\widetilde{\tau}}})$ (because we know that it is a Feller process and then the function $x \mapsto E_x\left[f(\exp(-\xi_{\rho^{\widetilde{\tau}}(t)}))\right]$ is continuous if f is continuous). We denote by $G^{\exp(-\xi_{\rho^{\widetilde{\tau}}})}/C_K^0$ the restriction of $G^{\exp(-\xi_{\rho^{\widetilde{\tau}}})}$ to C_K^0. Its domain is $C_K^0 \cap \mathcal{D}$ as well. Now, to conclude, we just have to apply the forthcoming Lemma 1.1 to

$$E = K, \quad \mathcal{B} = C_K^0, \quad G = G^{\exp(-\xi)}/C_K^0, \quad \widetilde{G} = G^{\exp(-\xi_{\rho^{\widetilde{\tau}}})}/C_K^0 \quad \text{and } D = C_K^1.$$

Indeed, generators G and $\widetilde{G}$ satisfy (1.12), with $\widetilde{\tau}$ bounded away from 0. The set C_K^1 is dense in C_K^0 and it is clear that the function $x \mapsto E_1\left[f(x\exp(-\xi_t))\right]$ belongs to C_K^1 as soon as f does. ■

Lemma 1.1 *Let E be a metric space and $\mathcal{B}$ the Banach space of real-valued continuous bounded functions on E, endowed with the uniform norm. Let G be the generator of a strongly continuous contraction semigroup $(T(t), t \geq 0)$ on $\mathcal{B}$, with domain $\mathcal{D}(G)$. Consider $D \subset \mathcal{D}(G)$, a dense subspace of $\mathcal{B}$ such that $T(t) : D \to D$ for all $t \geq 0$, and $\widetilde{\tau} \in \mathcal{B}$ such that $\widetilde{\tau} \geq m$ on E for some positive constant m. If $\widetilde{G}$ is the generator of a strongly continuous contraction semigroup on $\mathcal{B}$ such that $\mathcal{D}\left(\widetilde{G}\right) = \mathcal{D}(G)$ and $\widetilde{G}(f) = \widetilde{\tau}G(f)$ on $\mathcal{D}(G)$, then for every $\gamma > 0$, $\left(\gamma id - \widetilde{G}\right)(D)$ is dense in $\mathcal{B}$.*

Proof. We need the notion of core. If A is a closed linear operator on $\mathcal{B}$, a subspace C of $\mathcal{D}(A)$ is a *core* for A if the following equivalence holds:

$$\begin{gathered} f \in \mathcal{D}(A) \text{ and } g = A(f) \\ \Leftrightarrow \\ \text{there is a sequence } (f_n) \in C \text{ such that } f_n \to f \text{ and } A(f_n) \to g. \end{gathered}$$

The assumptions on D and $(T(t), t \geq 0)$ and Proposition 3.3 of chapter 1 in [32] ensure that D is a core for G. But then, D is also a core for $\widetilde{G}$: if (f_n) is a sequence in D such that $f_n \to f$ and $\widetilde{G}(f_n) \to g$, then, since $\widetilde{G}(f_n) = \widetilde{\tau} G(f_n)$ and $\widetilde{\tau} \geq m > 0$ on E, $G(f_n) \to g/\widetilde{\tau}$. Thus $f \in \mathcal{D}(G) = \mathcal{D}\left(\widetilde{G}\right)$ and $\widetilde{G}(f) = \widetilde{\tau} G(f) = g$. Conversely, given f belonging to $\mathcal{D}\left(\widetilde{G}\right) = \mathcal{D}(G)$ and $g = \widetilde{G}(f)$, there is a sequence $(f_n) \in D$ such that $f_n \to f$ and $G(f_n) \to G(f)$. But $\widetilde{\tau}$ is bounded on E and then $\widetilde{G}(f_n) \to \widetilde{G}(f)$. At last, we conclude by using Proposition 3.1 of chapter 1 in [32]. This proposition states that since D is a core for the generator $\widetilde{G}$, then $\left(\gamma id - \widetilde{G}\right)(D)$ is dense in $\mathcal{B}$ for *some* $\gamma > 0$, but it is easy to see with Lemma 2.11 (chapter 1 in [32]) that it holds for all $\gamma > 0$. ∎

Remark. As shown in Section 1.2, two homogeneous fragmentation processes with different laws may lead to subordinators with the same laws. Therefore, it may happen that two different fragmentation equations (i.e. with different parameters) have the same solution.

From Theorem 1.1, we deduce that the unique solution $(\mu_t, t \geq 0)$ to the fragmentation equation (1.2) is the hydrodynamic limit of stochastic fragmentation models. More precisely:

Corollary 1.1 *For each $n \in \mathbb{N}^*$, let $F^{\tau,n}$ be a (τ, c, ν)-fragmentation process starting from $F^{\tau,n}(0) = (\underbrace{1, 1, ..., 1}_{n \text{ terms}}, 0, ...)$. Then for each $t \geq 0$, with probability one,*

$$\frac{1}{n} \sum_{i=1}^{\infty} \delta_{F_i^{\tau,n}(t)}(dx) \underset{n \to \infty}{\overset{vaguely\ on\ ]0,1]}{\longrightarrow}} \mu_t.$$

Proof. For each $k \in \{1, ..., n\}$, we denote by $\left(\left(F_{k,1}^{\tau,n}(t), ..., F_{k,i}^{\tau,n}(t), ...\right), t \geq 0\right)$ the fragmentation process stemming from the k-th fragment of $F^{\tau,n}(0)$. These processes are independent and identically distributed, with the distribution of a (ν, c, τ)-fragmentation process starting from $(1, 0, ...)$. Then fix $t \geq 0$. Using the strong law of large numbers for each $f \in \mathcal{C}_c^1(]0,1])$, we get

$$\frac{1}{n}\left(\sum_{i=1}^{\infty} f(F_i^{\tau,n}(t))\right) = \frac{1}{n}\sum_{k=1}^{n}\left(\sum_{i=1}^{\infty} f(F_{k,i}^{\tau,n}(t))\right) \underset{n \to \infty}{\overset{a.s.}{\to}} \langle \mu_t, f \rangle. \tag{1.14}$$

With probability one, this convergence holds for each function f such that for a $n \in \mathbb{N}^*$

$$f(x) = \begin{cases} 0 \text{ on } \left]0, \frac{1}{n}\right] \\ \left(x - \frac{1}{n}\right)^2 P(x) \text{ on } \left]\frac{1}{n}, 1\right] \\ \text{where } P \text{ is a polynomial with rational coefficients} \end{cases}$$

since this set of functions - denoted by $\mathcal{T}$ - is countable. Observe that this set is dense in $\mathcal{C}_c^1(]0,1])$ for the uniform norm and for each $f \in \mathcal{C}_c^1(]0,1])$ consider a sequence $(g_k)_{k \geq 0}$ of functions of $\mathcal{T}$ such that $g_k \underset{k \to \infty}{\to} f/id$. Since $\sum_{i=1}^{\infty} F_i^{\tau,n}(t) \leq n$,

$$\frac{1}{n}\left(\sum_{i=1}^{\infty} F_i^{\tau,n}(t) g_k\left(F_i^{\tau,n}(t)\right)\right) \underset{k \to \infty}{\overset{\text{uniformly in } n}{\longrightarrow}} \frac{1}{n}\left(\sum_{i=1}^{\infty} f(F_i^{\tau,n}(t))\right) \text{ a.s.}$$

and then it is easily seen that with probability one the convergence (1.14) holds for each $f \in \mathcal{C}_c^1(]0,1])$. ■

Note that the question whether a similar result holds for the Smoluchowski's coagulation equation or not is still open (see [4]). The problem is that the Smoluchowski's coagulation equation is non-linear and then the mean frequencies of the stochastic models do not evolve as the Smoluchowski's coagulation equation, contrary to what happens for the fragmentation equation. Nonetheless, Norris [57] proved that under suitable assumptions on the coagulation kernel, the solution to Smoluchowski's coagulation equation may be obtained as the hydrodynamic limit of stochastic systems of coagulating particles.

1.4 Loss of mass in the fragmentation equation

Let $(\mu_t, t \geq 0)$ be the unique solution to the fragmentation equation (1.2) with parameters τ, c and ν and consider for each $t \geq 0$ the total mass of the system at time t

$$m(t) = \int_0^1 x\mu_t(dx).$$

In this section, we give necessary (resp. sufficient) conditions on the parameters τ, c and ν for the occurrence of loss of mass (i.e. the existence of a time t such that $m(t) < m(0)$). Then, when loss of mass occurs, we describe the asymptotic behavior of $m(t)$ as $t \to 0$ or $t \to \infty$ for a large class of parameters. This loss of mass study relies on the fact that the solution $(\mu_t, t \geq 0)$ can be constructed from a (τ, c, ν)-fragmentation process, denoted by F^τ (see the previous section). In particular, by monotone convergence, one can extend formula (1.9) to the pair of functions $\left(f, \overline{f}\right) = (id, 1_{x>0})$. Hence,

$$m(t) = E\left[\sum_{i=1}^{\infty} F_i^\tau(t)\right] = P(\lambda^\tau(t) > 0),\ t \geq 0,$$

where $(\lambda^\tau(t), t \geq 0)$ is the size-biased picked fragment process related to F^τ. Then recall Proposition 1.2 and introduce the random variable

$$I_\tau := \int_0^\infty \frac{dr}{\tau(\exp(-\xi_r))}. \tag{1.15}$$

Since $\tau(0) = \infty$, it is clear that I_τ is the first time when λ^τ is equal to 0. This leads to another expression of the mass

$$m(t) = P(I_\tau > t) \tag{1.16}$$

which will be useful in this section. Note that for self-similar fragmentations, i.e. $\tau(x) = x^\alpha$ on $]0,1]$, $\alpha \in \mathbb{R}$, I_τ is the well-known *exponential functional of the Lévy process* $\alpha\xi$ (for background, we refer e.g. to [19] and [25]).

At last, we recall that ϕ denotes the Laplace exponent of the subordinator ξ and can be expressed as a function of c and ν (see (1.5)) and that k, c and L are the characteristics of ξ (see (1.6)).

From now on, we exclude the degenerate case when the splitting measure ν and the erosion rate c are 0, for which there is obviously no loss of mass.

1.4.1 A criterion for loss of mass

If $k > 0$, either the erosion coefficient c is positive or a part of the mass of a particle may be lost during its splitting (i.e. $\nu\left(\sum_{i=1}^{\infty} s_i < 1\right) > 0$). Therefore, it is intuitively clear that if $k > 0$, there is loss of mass. Nevertheless, loss of mass may occur even when $k = 0$, as some particles may be reduced to dust in finite time. This phenomenon can be explained as follows when τ decreases near 0. Small fragments split even faster since their mass is smaller. Therefore, particles split faster and faster as time passes and so they may be reduced to dust in finite time. We now present a qualitative criterion for loss of mass.

Proposition 1.3 *(i) If $k > 0$, there is loss of mass and* $\inf\{t \geq 0 : m(t) < m(0)\} = 0$.

(ii) If $k = 0$, then

$$\int_{0^+} \frac{\phi'(x)}{\tau_{\inf}(\exp(-1/x))\phi^2(x)} dx < \infty \quad \Rightarrow \textit{there is loss of mass}$$
$$\int_{0^+} \frac{\phi'(x)}{\tau_{\sup}(\exp(-1/x))\phi^2(x)} dx = \infty \quad \Rightarrow \textit{there is no loss of mass}$$

where $\tau_{\inf}$ and $\tau_{\sup}$ are the continuous non-increasing functions defined on $]0,1]$ *by*

$$\tau_{\inf}(x) = \inf_{y \in]0,x]} \tau(y) \quad \textit{and} \quad \tau_{\sup}(x) = \sup_{y \in [x,1]} \tau(y).$$

Remarks. • If τ is bounded on $]0,1]$, we have that

$$\int_{0^+} \frac{\phi'(x)}{\tau_{\sup}(\exp(-1/x))\phi^2(x)} dx = \infty$$

since $\tau_{\sup}$ is then bounded on $]0,1]$ and $\int_{0^+} \phi'(x)\phi^{-2}(x)dx = \infty$ (recall that $\phi(0) = 0$). Thus, if τ is bounded on $]0,1]$ and $k = 0$, there is no loss of mass. In particular, when $k = 0$, there is no loss of mass in the homogeneous case (i.e. $\tau = 1$).

• If τ is non-increasing near 0 and $k = 0$, either $\lim_{x \to 0^+} \tau(x) < \infty$ and then there is no loss of mass or $\lim_{x \to 0^+} \tau(x) = \infty$ and then the functions $\tau_{\inf}$, τ and $\tau_{\sup}$ coincide on some neighborhood of 0. In both cases, the following equivalence holds:

$$\int_{0^+} \frac{\phi'(x)}{\tau(\exp(-1/x))\phi^2(x)} dx < \infty \Leftrightarrow \text{there is loss of mass.}$$

In order to prove Proposition 1.3, observe that loss of mass occurs if and only if $P(I_\tau < \infty) > 0$, which justifies the use of the forthcoming lemma (see Lemma 3.6 in [11]):

Lemma 1.2 *Let σ be a subordinator with killing rate 0 and U its potential measure, which means that for each measurable function f, $\int_0^\infty f(x)U(dx) = E\left[\int_0^\infty f(\sigma_t)dt\right]$. Let $h : [0,\infty) \to [0,\infty)$ be a non-increasing function. Then the following are equivalent*

$$\begin{array}{ll} (i) & \int_0^\infty h(x)U(dx) < \infty \\ (ii) & P\left(\int_0^\infty h(\sigma_t)dt < \infty\right) = 1 \\ (iii) & P\left(\int_0^\infty h(\sigma_t)dt < \infty\right) > 0. \end{array}$$

Proof of Proposition 1.3. (i) Let $e(k)$ denote the exponential random variable with parameter k at which the subordinator ξ is killed and η the subordinator with killing rate 0, independent of $e(k)$ and such that $\xi_t = \eta_t$ if $t < e(k)$ and $\xi_t = \infty$ if $t \geq e(k)$. Then, set

$$T^{\tau}(t) := \inf\left\{u \geq 0 : \int_0^u \frac{dr}{\tau\left(exp(-\eta_r)\right)} > t\right\}.$$

This random variable is independent of $e(k)$ and using that for each time t

$$P(I_\tau > t) \Leftrightarrow T^{\tau}(t) \leq e(k)$$

we get,

$$m(t) = E\left[e^{-kT^{\tau}(t)}\right].$$

Note that this is true even if $k = 0$, with the convention $0 \times \infty := \infty$. Now if $k > 0$ and $t > 0$, $kT^{\tau}(t) > 0$ with probability one and then $m(t) < 1$.

(ii) Let U denote the potential measure of the subordinator ξ. It is straightforward that

$$\int_0^{\infty} \frac{U(dx)}{\tau_{\inf}(\exp(-x))} < \infty \Rightarrow P\left(I_\tau < \infty\right) = 1$$

and it follows from Lemma 1.2 that

$$\int_0^{\infty} \frac{U(dx)}{\tau_{\sup}(\exp(-x))} = \infty \Rightarrow P\left(I_\tau < \infty\right) = 0.$$

Thus we just have to prove that for each continuous positive and non-increasing function f on $]0,1]$

$$\int_0^{\infty} \frac{U(dx)}{f(\exp(-x))} < \infty \Leftrightarrow \int_{0+} \frac{\phi'(x)}{f(\exp(-1/x))\phi^2(x)}dx < \infty. \tag{1.17}$$

To that end, recall that the repartition function $U(x) = \int_0^x U(dy)$ satisfies

$$U \asymp \frac{1}{\phi(1/\cdot)}, \tag{1.18}$$

where the notation $g \asymp h$ indicates that there are two positives constants C and C' such that $Cg \leq h \leq C'g$ (see Proposition 1.4 in [11]). Then if $\lim_{x\to 0^+} f(x) < \infty$,

$$\int_0^{\infty} \frac{U(dx)}{f(\exp(-x))} = \int_{0+} \frac{\phi'(x)}{f(\exp(-1/x))\phi^2(x)}dx = \infty$$

since $U(\infty) = \infty$ and $\int_{0+} \phi'(x)\phi^{-2}(x)dx = \infty$. Next, if $\lim_{x\to 0^+} f(x) = \infty$, introduce the non-negative finite measure V defined on $[0,\infty[$ by

$$\int_0^x V(dy) = \frac{1}{f(1)} - \frac{1}{f(\exp(-x))}.$$

Note that

$$\int_0^{\infty} \frac{U(dx)}{f(\exp(-x))} = \int_0^{\infty}\int_x^{\infty} V(dy)U(dx) = \int_0^{\infty} U(y)V(dy).$$

Combining this with (1.18) leads to the following equivalences

$$\begin{aligned}\int_0^\infty \frac{U(dx)}{f(\exp(-x))} < \infty \quad &\Leftrightarrow \int_0^\infty \frac{V(dy)}{\phi(1/y)} < \infty \\ &\Leftrightarrow \int_0^\infty \int_{1/y}^\infty \frac{\phi'(z)}{\phi^2(z)} dz V(dy) < \infty \\ &\Leftrightarrow \int_0^\infty \frac{\phi'(z)}{f(\exp(-1/z))\phi^2(z)} dz < \infty\end{aligned}$$

and then to equivalence (1.17) , since

$$\int_.^\infty \frac{\phi'(z)}{f(\exp(-1/z))\phi^2(z)} dz < \infty \quad \text{(the case when } \xi = 0 \text{ is excluded).}$$

■

Provided that τ is non-increasing near 0 and $\phi'(0^+) < \infty$, the following corollary gives a simple necessary and sufficient condition on τ for loss of mass to occur. This result may be found in Filippov's paper ([35]) in the special case when $\nu(S^*) < \infty$. Recall the notations τ_{inf} and τ_{sup} introduced in Proposition 1.3.

Corollary 1.2 *Suppose that $k = 0$. Then,*

(i) $\displaystyle\int_{0^+} \frac{dx}{x\tau_{\mathrm{inf}}(x)} < \infty \Rightarrow$ *loss of mass.*

(ii) If $\phi'(0^+) < \infty$ $\left(\textit{i.e.}\ \int_{\mathcal{S}^\downarrow} \left(\sum_{i=1}^\infty |\log(s_i)|\, s_i \right) \nu(ds) < \infty \right)$,

$$\textit{loss of mass} \Rightarrow \int_{0^+} \frac{dx}{x\tau_{\mathrm{sup}}(x)} < \infty.$$

If τ is non-increasing in a neighborhood of 0, τ_{inf} and τ_{sup} can be replaced by τ.

In particular, as soon as $\tau(x) \geq |\log x|^\alpha$ near 0 for some $\alpha > 1$, there is loss of mass.

Proof. The assumption $k = 0$ leads to

$$\frac{\phi(q)}{q} \underset{q \to 0}{\to} \int_0^\infty x L(dx) = \int_{\mathcal{S}^\downarrow} \left(\sum_{i=1}^\infty (-\log(s_i)) s_i \right) \nu(ds).$$

Remark that $\int_0^\infty x L(dx) \neq 0$, since $L \neq 0$ and then $\phi'(0^+) > 0$. If moreover $\phi'(0^+) < \infty$, we have

$$\frac{\phi'(x)}{\tau_{\mathrm{sup}}(\exp(-1/x))\phi^2(x)} \underset{x \to 0^+}{\sim} \frac{1}{\phi'(0^+) x^2 \tau_{\mathrm{sup}}(\exp(-1/x))}.$$

Combining this with Proposition 1.3 (ii) leads to result (ii). Now, if $\phi'(0^+) = \infty$, the function $x \mapsto x^2 \phi'(x) \phi^{-2}(x)$ is still bounded near 0 and then we deduce (i) in the same way. ■

1.4.2 Asymptotic behavior of the mass

Our purpose is to study the asymptotic behavior of the mass $m(t) = \langle \mu_t, id \rangle$ as $t \to 0$ or $t \to \infty$.

1.4.2.1 Small times asymptotic behavior

Proposition 1.4 *Assume that* $\phi'(0^+) < \infty$ *and* $\tau(x) \leq Cx^{\alpha}$, $0 < x \leq 1$, *with* $C > 0$ *and* $\alpha < 0$. *Then,* m *is differentiable at* 0^+ *and* $m'(0^+) = -k$.

Remark. We will see in the proof that the upper bound

$$\limsup_{t \to 0^+} \frac{m(t) - 1}{t} \leq -k$$

remains valid without any assumption on τ and ϕ.

Proof. As shown in the first part of the proof of Proposition 1.3

$$m(t) = E\left[e^{-kT^{\tau}(t)}\right], \ t \geq 0,$$

where T^{τ} is the time-change

$$T^{\tau}(t) = \inf\left\{u \geq 0 : \int_0^u \frac{dr}{\tau\left(\exp(-\eta_r)\right)} > t\right\}$$

and η a subordinator with killing rate 0, drift coefficient c and Lévy measure L. Hence,

$$\frac{1 - m(t)}{t} = E\left[\frac{1 - e^{-kT^{\tau}(t)}}{t}\right]. \tag{1.19}$$

Observe that it is sufficient to prove the statement for functions τ bounded on $]0,1]$ or non-increasing and such that $\tau(x) \leq Cx^{\alpha}$ on $]0,1]$ for some $C > 0$ and $\alpha < 0$. Indeed, for each continuous positive function τ such that $\tau(x) \leq Cx^{\alpha}$ on $]0,1]$ with $C > 0$ and $\alpha < 0$, there are two continuous positive functions τ_1 and τ_2 such that $\tau_1 \leq \tau \leq \tau_2$ on $]0,1]$ and

- τ_1 is bounded on $]0,1]$ and $\tau_1(1) = 1$
- τ_2 is non-increasing, $\tau_2(x) \leq Cx^{\alpha}$ on $]0,1]$ and $\tau_2(1) = 1$

(we may take for example $\tau_2(x) := \sup_{y \in [x,1]} \tau(y)$). Then combine this with the fact that

$$\frac{1 - m_{\widetilde{\tau}}(t)}{t} \leq \frac{1 - m_{\overline{\tau}}(t)}{t}, \quad \forall t \geq 0, \tag{1.20}$$

when $\widetilde{\tau} \leq \overline{\tau}$ on $]0,1]$ (here $m_{\widetilde{\tau}}$ and $m_{\overline{\tau}}$ denote the respective masses of a $(\widetilde{\tau}, c, \nu)$-fragmentation equation and a $(\overline{\tau}, c, \nu)$-fragmentation equation).

(i) For t such that $T^\tau(t) < \infty$, the time-change T^τ can be expressed as follows:

$$T^\tau(t) = \int_0^t \tau(\exp(-\eta_{T^\tau(r)}))dr \tag{1.21}$$
$$= t\int_0^1 \tau(\exp(-\eta_{T^\tau(tr)}))dr.$$

Note that the first time when T^τ reaches ∞ is positive with probability one. Then if τ is bounded (respectively, non-increasing), we get by the dominated convergence theorem (resp. monotone convergence theorem), that

$$\frac{1-e^{-kT^\tau(t)}}{t} \underset{t\to 0^+}{\overset{a.s.}{\to}} k.$$

If τ is bounded the dominated convergence theorem applies and gives

$$\frac{1-m(t)}{t} \underset{t\to 0^+}{\to} k.$$

(ii) To conclude when τ is non-increasing and smaller than the function $x \mapsto Cx^\alpha$, it remains to show that $\left(1-e^{-kT^\tau(t)}\right)/t$ is dominated - independently of t - by a random variable with finite expectation. To see this, first note that it is sufficient to prove the domination for $\left(1-e^{-kT^\alpha(t)}\right)/t$, where

$$T^\alpha(t) = \inf\left\{u \geq 0 : \int_0^u \exp(\alpha\eta_r)dr > Ct\right\}$$

(since $T^\tau(t) \leq T^\alpha(t)$ for $t \geq 0$). Next remark that if η^1 is a subordinator such that $\eta^1 \geq \eta$, the following inequality between time-changes holds

$$T_1^\alpha(t) := \inf\left\{u \geq 0 : \int_0^u \exp(\alpha\eta_r^1)dr > Ct\right\} \geq T^\alpha(t)$$

and then

$$\frac{1-e^{-kT^\alpha(t)}}{t} \leq \frac{1-e^{-kT_1^\alpha(t)}}{t} \text{ for each } t \geq 0.$$

Thus it is sufficient to prove the domination for a subordinator bigger than η and so we can (and will) assume that the subordinator η has a drift coefficient $c \geq k/|\alpha|$. Now introduce the exponential functional

$$I_\alpha := \int_0^\infty \exp(\alpha\eta_r)dr.$$

Observe that

$$C^{-1}I_\alpha = \inf\{t \geq 0 : T^\alpha(t) = \infty\}.$$

If $t < C^{-1}I_\alpha$, we get that

$$\frac{d}{dt}\left(e^{-kT^\alpha(t)}\right) = -kCe^{-\alpha\eta_{T^\alpha(t)}}e^{-kT^\alpha(t)} \tag{1.22}$$

(by using (1.21) for the function $\tau = Cx^{\alpha}$). But the (random) function $t \mapsto -\alpha\eta_t - kt$ is non-decreasing, since $c \geq k/|\alpha|$ and the process $(\eta_t - ct, t \geq 0)$ is a (pure jump) non-decreasing process (according to the Lévy-Itô decomposition of a subordinator - see Proposition 1.3 in [11]). Thus the derivative (1.22) is non-increasing and $t \mapsto e^{-kT^{\alpha}(t)}$ is a concave function on $[0, C^{-1}I_{\alpha}[$. From this, it follows that the slope $\left(1 - e^{-kT^{\alpha}(t)}\right)/t$ is non-decreasing on $[0, C^{-1}I_{\alpha}[$ and it is straightforward that it is decreasing on $[C^{-1}I_{\alpha}, \infty[$. This leads to the upper bound:

$$\frac{1 - e^{-kT^{\tau}(t)}}{t} \leq \frac{C}{I_{\alpha}} \quad \forall t \geq 0.$$

By Proposition 3.1 (iv) in [25], the expectation

$$E\left[I_{\alpha}^{-1}\right] = (-\alpha)\,\phi'(0^{+}) < \infty,$$

and this ends the proof. ■

If $k = 0$, there is a more precise result. Recall (1.16) and set $A := \sup\left\{a \geq 0 : E\left[I_{\tau}^{-a}\right] < \infty\right\}$. Then for each $\varepsilon > 0$ such that $A - \varepsilon > 0$,

$$t^{\varepsilon - A}\left(1 - m(t)\right) \leq \int_0^t x^{\varepsilon - A} P_{I_{\tau}}(dx) \underset{t \to 0^{+}}{\to} 0,$$

since $E\left[I_{\tau}^{\varepsilon - A}\right] < \infty$. (Actually, it is easy to see that

$$\liminf_{t \to 0^{+}} \frac{\log(1 - m(t))}{\log(t)} = A).$$

For self-similar fragmentation processes, this points out the influence of α on the loss of mass behavior near 0. Indeed, consider a family of self-similar fragmentation processes such that the subordinator ξ is fixed (with killing rate $k = 0$) and α varies, $\alpha < 0$. Then introduce the set

$$\mathcal{Q} := \left\{ q \in \mathbb{R} : \int_{x > 1} e^{qx} L(dx) < \infty \right\}.$$

This set is convex and contains 0. Let $\underline{p}$ be its right-most point. According to Theorem 25.17 in [64],

$$q \in \mathcal{Q} \Leftrightarrow E\left[e^{q\xi_t}\right] < \infty \ \forall t \geq 0$$

and in that case $E\left[e^{q\xi_t}\right] = e^{-t\phi(-q)}$. Then, following the proof of Proposition 2 in [19], we get

$$E\left[I_{\alpha}^{-q-1}\right] = \frac{-\phi(\alpha q)}{q} E\left[I_{\alpha}^{-q}\right] \text{ for } q < \frac{\underline{p}}{|\alpha|},$$

which leads to

$$\frac{\underline{p}}{|\alpha|} \leq \sup\left\{ q : E\left[I_{\alpha}^{-q-1}\right] < \infty \right\}.$$

And then

$$\liminf_{t \to 0^{+}} \frac{\log(1 - m(t))}{\log(t)} \geq 1 + \frac{\underline{p}}{|\alpha|}.$$

1.4.2.2 Large times asymptotic behavior

The main result of this subsection is the existence of exponential bounds for the mass $m(t)$ when t is large enough and when the parameters τ, c and ν satisfy the conditions (i) and (ii) of the following Proposition 1.6. Before proving this, we point out the following intuitive result, which is valid for all parameters τ, c and ν.

Proposition 1.5 *When loss of mass occurs,* $m(t) \underset{t\to\infty}{\to} 0$.

Proof. From formula (1.16), we get that $m(t) \underset{t\to\infty}{\to} P(I_\tau = \infty)$. When $k > 0$, the subordinator ξ is killed at a finite time $e(k)$ and then

$$I_\tau \leq e(k) \times \sup_{x \in \left[\exp(-\xi_{e(k)-}),1\right]} (1/\tau(x))$$

is a.s. finite. When $k = 0$, our goal is to prove that the probability $P(I_\tau = \infty)$ is either 0 or 1. To that end, we introduce the family of i.i.d. random variables, defined for all $n \in \mathbb{N}$ by

$$F_n := (\xi_{n+t} - \xi_n)_{0 \leq t \leq 1}.$$

It is clear that I_τ can be expressed as a function of the random variables F_n. Then, since for all $n \in \mathbb{N}$

$$\{I_\tau = \infty\} = \left\{ \int_n^\infty \frac{dr}{\tau\left(\exp(-\xi_r)\right)} = \infty \right\},$$

it is easily seen that the set $\{I_\tau = \infty\}$ is invariant under finite permutations of the r.v. F_n, $n \in \mathbb{N}$. Hence, we can conclude by using the Hewitt-Savage 0-1 law (see e.g. Th.3 Section IV in [34]). ■

Our sharper study of the asymptotic behavior of the mass $m(t)$ as $t \to \infty$ relies on the moments properties of the random variable I_τ. If $\tau(x) = x^\alpha$, $\alpha < 0$, it is well known that the entire moments of I_τ are given by

$$E\left[I_\tau^n\right] = \frac{n!}{\phi(-\alpha)...\phi(-\alpha n)},\ n \in \mathbb{N}^*, \tag{1.23}$$

and then, that

$$E\left[\exp(rI_\tau)\right] < \infty \text{ for } r < \phi(\infty) := \lim_{q\to\infty} \phi(q).$$

(See Proposition 3.3 in [25]). From this and formula (1.16) we deduce that the mass $m(t)$ decays at an exponential rate as $t \to \infty$, since for a positive $r < \phi(\infty)$,

$$m(t) = P(I_\tau > t) \leq \exp(-rt)E\left[\exp(rI_\tau)\right], \quad t \geq 0. \tag{1.24}$$

This result is still valid for a function $\tau(x) \geq Cx^\alpha$, where $\alpha < 0$ and $C > 0$, because $I_\tau \leq C^{-1}\int_0^\infty \exp(\alpha\xi_r)dr$. Remark that until now, we have made no assumption on ϕ. We now state deeper results when ϕ behaves like a regularly varying function. Recall that a real function f *varies regularly with index* $a \geq 0$ *at* ∞ if

$$\frac{f(rx)}{f(x)} \underset{x\to\infty}{\to} r^a \quad \forall r > 0.$$

If $a=0$, f is said to be slowly varying. Recall also that the notation $f \asymp g$ indicates that there exist two positives constants C and C' such that $Cg \leq f \leq C'g$.

Proposition 1.6 *Assume that*

$$(i)\ C_2 x^{\beta} \leq \tau(x) \leq C_1 x^{\alpha}, \quad 0<x\leq 1, \quad \alpha \leq \beta < 0, \quad C_1>0, \quad C_2>0.$$
$$(ii)\ \phi \asymp f \text{ on } [1,\infty), \text{ where } f \text{ varies regularly at } \infty \text{ with index } a \in \,]0,1[\,.$$

Denote by ψ *the inverse of the function* $t \mapsto t/\phi(t)$, *which is a bijection from* $[1,\infty)$ *to* $[1/\phi(1),\infty)$. *Then there exist two positive constants* A *and* B *such that for* t *large enough*

$$\exp(-B\psi(t)) \leq m(t) \leq \exp(-A\psi(t)). \tag{1.25}$$

Actually, if ϕ satisfies (ii), it is sufficient to suppose that $C_2x^{\beta} \leq \tau(x)$ with $\beta<0$ and $C_2>0$ to obtain the upper bound $m(t) \leq \exp(-A\psi(t))$ and conversely, if $\tau(x) \leq C_1x^{\alpha}$ with $\alpha<0$ and $C_1>0$, the lower bound $\exp(-B\psi(t))$ holds.

Remark. If $\tau(x)=x^{\alpha}$ for $x \in \,]0,1]$, $\alpha<0$, and ϕ varies regularly at ∞ with index $a \in \,]0,1[\,$, it follows from a result in [61] that

$$\log(m(t)) \underset{t\to\infty}{\sim} \frac{(1-a)a^{\frac{a}{a-1}}}{\alpha}\psi\left(\frac{-\alpha t}{a^a}\right).$$

We should also point out that there are some homogeneous fragmentation processes such that the associated Laplace exponent ϕ satisfies assumption (ii) without varying regularly.

Proof. The proof relies on Theorem 1 and Theorem 2 of Kôno [48], which we now recall. Let σ be a non-decreasing and "nearly regularly varying function with index b", $b \in \,]0,1[$, which means that there exist two positive constants $r_1 \geq r_2$ and a slowly varying function s such that

$$r_2 x^b s(x) \leq \sigma(x) \leq r_1 x^b s(x) \text{ for } x \geq 1. \tag{1.26}$$

Let Y be a positive random variable such that, for n large enough,

$$c_2^{2n}\prod_{k=1}^{2n}\sigma(k) \leq E\left[Y^{2n}\right] \leq c_1^{2n}\prod_{k=1}^{2n}\sigma(k) \tag{1.27}$$

where c_1 and c_2 are positive constants. Then, there exist three positive constants A, B and C such that for x large enough,

$$\exp(-Bx) \leq P(Y \geq C\sigma(x)) \leq \exp(-Ax).$$

Coming back to the proof, we set

$$\sigma(x) := \frac{x}{\phi(x)},\ x \geq 1.$$

This is an increasing continuous function (by the concavity of ϕ) such that $\lim_{x\to\infty}\sigma(x)=\infty$ (by assumption (ii)). In particular, its inverse ψ is well-defined and increasing on $[\sigma(1),\infty)$. Since f varies regularly with index $a\in\,]0,1[$, there exists a slowly varying function g such that $f(q)=q^a g(q)$ for $q\geq 1$. Then it follows from assumption (ii) that σ satisfies (1.26) with $b=1-a$ and $s=1/g$ (note that g is a positive function on $[1,\infty)$). On the other hand, recall that if $\tau(x)=x^\alpha$, $\alpha<0$, the entire moments of the random variable I_τ are given by (1.23). Thus, for each function τ satisfying assumption (i), we have

$$C_1^{-n}\prod_{k=1}^{n}\frac{k}{\phi(-\alpha k)}\leq E\left[I_\tau^n\right]\leq C_2^{-n}\prod_{k=1}^{n}\frac{k}{\phi(-\beta k)}.$$

Moreover, the assumption (ii) implies that for each $C>0$, $\phi(Ct)\asymp\phi(t)$ at least for $t\in\ [1,\infty)$. Therefore, the moments of I_τ satisfy (1.27). Then, by applying the theorems recalled at the beginning of the proof, we get

$$\exp(-B\psi\left(t/C\right))\leq m(t)=P\left(I_\tau>t\right)\leq\exp(-A\psi\left(t/C\right))\text{ for } t \text{ large enough.}\qquad(1.28)$$

It remains to remove the constant C. To that end, introduce $h(x):=x/f(x)$ on $[1,\infty)$ and consider the generalized inverse of h:

$$h^{\leftarrow}(x):=\inf\left\{y\in[1,\infty):h(y)>x\right\},\ x\in[1/f(1),\infty).$$

The function h varies regularly with index $1-a$ and so, according to Theorem 1.5.12 in [21], $h^{\leftarrow}$ varies regularly with index $1/\left(1-a\right)$ and $h(h^{\leftarrow}(x))\underset{x\to\infty}{\sim}x$. From this latter and assumption (ii), we deduce the existence of two positive constant D_1 and D_2 such that

$$D_1x\leq\sigma\left(h^{\leftarrow}(x)\right)\leq D_2x\text{ for } x \text{ large enough.}$$

And since ψ is increasing, we have

$$\psi\left(D_1x\right)\leq h^{\leftarrow}(x)\leq\psi\left(D_2x\right)\ \text{ for } x \text{ large enough.}$$

But then, since $h^{\leftarrow}$ varies regularly, the function $x\mapsto\psi\left(x/C\right)/\psi\left(x\right)$ is bounded away from 0 and ∞ when $x\to\infty$. Then combine this with (1.28) to obtain (1.25). ■

Note that the assumption (ii) in Proposition 1.6 implies that the erosion rate c is equal to 0. Now, if $c>0$ and if $\tau(x)\geq Ax^\alpha$ on $]0,1]$, with $\alpha<0$ and $A>0$, we observe that the mass $m(t)$ is equal to 0, as soon as $t\geq 1/|A\alpha c|$. Indeed, recall that

$$k\geq c\text{ and }\xi_t\geq ct\text{ for each }t\geq 0.$$

Then,

$$I_\tau\leq\frac{(1-\exp(\alpha ce(k)))}{|A\alpha c|}$$

which leads to

$$\begin{cases} m(t)=0\text{ if }t\geq 1/|A\alpha c|\\ m(t)\leq(1+A\alpha ct)^{k/|\alpha c|}\text{ if }t\leq 1/|A\alpha c|.\end{cases}$$

In the same way, we obtain that $m(t)\leq e^{-kat}$ if $\tau\geq a$ on $]0,1]$ (before that, we had exponential upper bounds only when $\tau(x)\geq Ax^\alpha$, with $\alpha<0$ and $A>0$).

1.5 Loss of mass in fragmentation processes

Let F^{τ} be a (τ, c, ν)-fragmentation process starting from $(1, 0, ...)$. We say that there is loss of mass in this random fragmentation if

$$P\left(\exists t \geq 0 : \sum_{i=1}^{\infty} F_i^{\tau}(t) < 1\right) > 0.$$

The results on the occurrence of this (stochastic) loss of mass as a function of the parameters τ, c and ν are exactly the same as those on the occurrence of loss of mass for the corresponding deterministic model (constructed from F^{τ} by formula (1.3)). Indeed, the point is that, as shown in the proof of Proposition 1.5, the probability $P(I_{\tau} < \infty)$ is either 0 or 1 and then that the events $\left\{\exists t \geq 0 : \sum_{i=1}^{\infty} F_i^{\tau}(t) < 1\right\}$ and $\{I_{\tau} < \infty\}$ coincide apart from an event of probability 0. Thus, Proposition 1.3 and its corollary are still valid for the loss of mass in the fragmentation process F^{τ} and when there is loss of mass, it occurs with probability one.

When there is loss of mass, one may wonder if there exists a finite time at which all the mass has disappeared, i.e. if

$$\zeta := \inf\{t \geq 0 : F_1^{\tau}(t) = 0\} < \infty.$$

In the sequel, we will say that there is *total loss of mass* if $P(\zeta < \infty) > 0$. Bertoin [15] proves that total loss of mass occurs with probability one for a self similar fragmentation process with a negative index. Here, we give criteria on the parameters τ, c and ν for the presence or absence of total loss of mass. From this we deduce that even if $k = 0$, there is no equivalence in general between loss of mass and total loss of mass. Then, we study the asymptotic behavior of $P(\zeta > t)$ as $t \to \infty$, when the parameters τ, c and ν satisfy the same assumptions as in Proposition 1.6. The following remark will be useful in this study of ζ: if F^{τ} and $F^{\tau'}$ are two fragmentation processes constructed from the same homogeneous one and if $\tau \leq \tau'$ on $]0, 1]$, then

$$\inf\left\{t \geq 0 : F_1^{\tau'}(t) = 0\right\} \leq \inf\{t \geq 0 : F_1^{\tau}(t) = 0\}. \tag{1.29}$$

Eventually, we investigate in the last subsection the behavior as $t \to 0$ of the random mass $M(t) = 1 - \sum_{i=1}^{\infty} F_i^{\tau}(t)$.

1.5.1 A criterion for total loss of mass

Proposition 1.7 *Consider the continuous non-increasing functions $\tau_{\inf}$ and $\tau_{\sup}$ constructed from τ as in the statement of Proposition 1.3.*

(i) If $\int_{0+} \frac{dx}{x\tau_{\inf}(x)} < \infty$, *then* $P(\zeta < \infty) = 1$.

(ii) If $k = 0$ *and* $\int_{\mathcal{S}^{\downarrow}} |\log(s_1)| \, \nu(ds) < \infty$, *then*

$$P(\zeta < \infty) > 0 \Rightarrow \int_{0+} \frac{dx}{x\tau_{\sup}(x)} < \infty.$$

If τ is non-increasing in a neighborhood of 0, $\tau_{\inf}$ and $\tau_{\sup}$ can be replaced by τ.

Remarks. • This should be compared with Corollary 1.2 which states similar connections between loss of mass and the integrability near 0 of functions $x \mapsto 1/x\tau_{\inf}(x)$ and $x \mapsto 1/x\tau_{\sup}(x)$.

• The condition $\int_{\mathcal{S}^{\downarrow}} |\log(s_1)| \, \nu(ds) < \infty$ is satisfied as soon as $\nu(s_1 \leq \varepsilon) = 0$ for a positive ε, since $|\log(s_1)| \leq \varepsilon^{-1}(1-s_1)$ when s_1 belongs to $]\varepsilon, 1]$. In particular, this last condition on the measure ν is satisfied for fragmentation models where $k=0$ and such that the splitting of a particle gives at most n fragments (i.e. $\nu(s_{n+1} > 0) = 0$). Indeed, we have then that $\nu(s_1 < 1/n) = 0$, since $\nu\left(\sum_{i=1}^{\infty} s_i < 1\right) = 0$ when $k = 0$.

Proof. We just have to prove these assertions for a non-increasing function τ and then use the remark (1.29). Thus in this proof τ is supposed to be non-increasing on $]0,1]$.

As shown in Section 1.2.2, the interval representation $\left(\widetilde{I}_x(t), x \in]0,1[\,, t \geq 0\right)$ of F^{τ} is constructed from the interval representation $(I_x(t), x \in]0,1[\,, t \geq 0)$ of a homogeneous (c,ν) fragmentation process F in the following way:

$$\widetilde{I}_x(t) = I_x(T_x^{\tau}(t)),$$

where

$$T_x^{\tau}(t) = \inf\left\{u \geq 0 : \int_0^u \frac{dr}{\tau(|I_x(r)|)} > t\right\}.$$

For every x in $]0,1[$, set $\zeta_x := \inf\{t : I_x(t) = 0\}$. Then,

$$T_x^{\tau}(t) < \zeta_x \text{ if and only if } t < \int_0^{\infty} \frac{dr}{\tau(|I_x(r)|)},$$

which leads to

$$\zeta = \sup_{x \in]0,1[} \int_0^{\infty} \frac{dr}{\tau(|I_x(r)|)}. \tag{1.30}$$

(i) This part is merely adapted from the proof of Proposition 2 (i) in [15]. In particular, as mentioned there,

$$\limsup_{r \to \infty} r^{-1} \log F_1(r) < 0.$$

Thus there exists a random positive number C such that

$$\frac{1}{\tau(|I_x(r)|)} \leq \frac{1}{\tau(\exp(-Cr))} \quad \text{for all } x \in]0,1[\text{ and all } r \geq 0,$$

since moreover τ is non-increasing. Now, we just have to combine this with equality (1.30) and the fact that the function $x \mapsto 1/x\tau(x)$ is integrable near 0 to conclude that $\zeta < \infty$ a.s.

(ii) Since $k = 0$, the drift coefficient c is equal to 0 and then the homogeneous fragmentation process F is a pure jump process constructed from a Poisson point process $((\Delta(t), k(t)), t \geq 0)$ $\in S^* \times N^*$, with characteristic measure $\nu \otimes \#$ (see Section 1.2.1). From this process, we build another jump process Y which we first describe informally: $Y(0) = 1$ and for each time t, $Y(t)$ is an element of the sequence $F(t)$. When the fragment with mass Y splits, we keep the largest fragment and Y jumps to the mass of this new fragment. And so on ... Note that generally, the jump times may accumulate. Now, we give a rigorous construction of Y, by induction. To that end, we build simultaneously a sequence of particular times $(t_n)_{n\in\mathbb{N}}$. Set $t_0 := 0$ and $Y(t_0) := 1$. Suppose that t_{n-1} is known, that it is a randomized stopping time, and that Y is constructed until t_{n-1}. Let $k(n-1)$ be such that $Y(t_{n-1}) = F_{k(n-1)}(t_{n-1})$ and consider the fragmentation process stemming from $F_{k(n-1)}(t_{n-1})$. Since F is homogeneous, there exists a homogeneous (c,ν)-fragmentation process independent of $(F(t), t \le t_{n-1})$, denoted by F^{n-1}, such that the fragmentation process stemming from $F_{k(n-1)}(t_{n-1})$ is equal to $Y(t_{n-1})F^{n-1}$. Let λ^{n-1} and $((\Delta^{n-1}(t), k^{n-1}(t)), t \ge 0)$ be respectively the size-biased picked fragment process and the Poisson point process related to F^{n-1}. Then set

$$t_n := t_{n-1} + \inf\left\{t : \lambda^{n-1}(t) < \tfrac{1}{2}\right\}$$
$$Y(t) := Y(t_{n-1})F_1^{n-1}(t - t_{n-1}), \quad t_{n-1} \le t < t_n$$

$$Y(t_n) := \begin{cases} \Delta_1^{n-1}(t_n - t_{n-1})Y(t_{n-1})F_1^{n-1}((t_n - t_{n-1})^-) & \text{if } k^{n-1}(t_n - t_{n-1}) = 1 \\ Y(t_{n-1})F_1^{n-1}(t_n - t_{n-1}) & \text{otherwise.} \end{cases}$$

Time t_n is a randomized stopping time. Note that the random variables $(t_n - t_{n-1})$ are iid with a positive expectation. So $t_n \to \infty$ and Y is then well-defined on $[0,\infty)$.

Call σ the non-decreasing process $(-\log(Y))$ and consider the jumps $\widetilde{\Delta}(t) := \sigma(t) - \sigma(t^-)$, $t \ge 0$. It is easily seen that $\left(\widetilde{\Delta}(t), t \ge 0\right)$ is a Poisson point process on $]0,\infty[$ with characteristic measure $\nu(-\log s_1 \in dx)$. In other words, σ is a subordinator with Laplace exponent

$$\varphi(q) = \int_{\mathcal{S}^{\downarrow}} (1 - s_1^q)\nu(ds), \ q \ge 0.$$

It can be shown that for each $t \ge 0$ there exists a (random) point $x_t \in]0,1[$ such that $Y(r) = |I_{x_t}(r)|$ for $r \le t$. Combine this with equality (1.30) to conclude that

$$\zeta \ge \int_0^t \frac{dr}{\tau(\exp(-\sigma(r)))} \quad \text{for all } t \ge 0$$

and then

$$\zeta \ge \int_0^\infty \frac{dr}{\tau(\exp(-\sigma(r)))}.$$

Therefore, the assumption $P(\zeta < \infty) > 0$ implies that

$$P\left(\int_0^\infty \frac{dr}{\tau(\exp(-\sigma(r)))} < \infty\right) > 0$$

and so, following the proof of Proposition 1.3 (ii), we conclude that

$$\int_{0^+} \frac{\varphi'(x)}{\tau(\exp(-1/x))\varphi^2(x)} dx < \infty.$$

Together with the assumption

$$\varphi'(0^+) = \int_{\mathcal{S}^{\downarrow}} |\log(s_1)|\,\nu(ds) < \infty,$$

this implies that $\int_{0^+} \frac{1}{x\tau(x)} dx < \infty$. ■

1.5.2 Does loss of mass imply total loss of mass ?

If the killing rate k is positive, loss of mass always occurs, but in general total loss of mass does not. Think for example of a pure erosion process. Now, we focus on what happens when $k = 0$, i.e. when the loss of mass corresponds only to particles reduced to dust. First, if the Laplace exponent ϕ has a finite right-derivative at 0 and if τ is non-increasing near 0, loss of mass is equivalent to total loss of mass and both occur with probability zero or one. This just follows from a combination of Corollary 1.2 (ii) and Proposition 1.7 (i). However, without this assumption on ϕ there may be loss of mass but no total loss of mass. Here is an example: fix $a \in]0,1[$ and take the parameters τ, c and ν as follows:

- $\tau(x) = \begin{cases} 1 \text{ if } x \geq e^{-1} \\ (-\log x) \text{ if } 0 < x \leq e^{-1}, \end{cases}$
- $c = 0$,
- $\nu(ds) = \sum_{n=1}^{\infty} \left(\frac{1}{n^a} - \frac{1}{(n+1)^a} \right) \delta_{\left(\frac{1}{2}, \underbrace{\frac{1}{2^{n+1}}, ..., \frac{1}{2^{n+1}}}_{2^n \text{ terms}}, 0, ... \right)}(ds).$

It is clear that τ is decreasing on $]0, e^{-1}]$ and $k = 0$.

Lemma 1.3 *Let ϕ be the Laplace exponent specified by* (1.5) *for the parameters above. Then $\phi(q) \geq Cq^a$ for some $C > 0$ and for all $q \in [0,1]$.*

Proof. Consider the function

$$\begin{aligned} f(q) &= \int_1^{\infty} (1 - e^{-(\log 2)qx}) x^{-1-a} dx \\ &= (q \log 2)^a \int_{q \log 2}^{\infty} (1 - e^{-x}) x^{-1-a} dx. \end{aligned}$$

The integral $\int_0^{\infty} (1 - e^{-x}) x^{-1-a} dx$ is positive and finite since $a \in]0,1[$. Then there exists a positive real number C such that

$$f(q) \geq Cq^a, \forall q \in [0,1].$$

On the other hand, remark that

$$\begin{aligned} f(q) &= \sum_{n=1}^{\infty} \int_n^{n+1} (1 - e^{-(\log 2)qx}) x^{-1-a} dx \\ &\leq \sum_{n=1}^{\infty} (1 - e^{-(\log 2)q(n+1)}) \int_n^{n+1} x^{-1-a} dx \\ &\leq \frac{1}{a} \sum_{n=1}^{\infty} (1 - e^{-(\log 2)q(n+1)}) \left(\frac{1}{n^a} - \frac{1}{(n+1)^a} \right). \end{aligned}$$

As a consequence, the following inequality holds

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{2^{q(n+1)}} \right) \left(\frac{1}{n^a} - \frac{1}{(n+1)^a} \right) \geq aCq^a, \forall q \in [0,1].$$

This leads to:

$$\begin{aligned}
\phi(q) &= \int_{S^*}\left(1-\sum_{i=1}^{\infty} s_i^{q+1}\right)\nu(ds)\\
&= \sum_{n=1}^{\infty}\left(\frac{1}{n^a}-\frac{1}{(n+1)^a}\right)\left(1-\left(\frac{1}{2}\right)^{q+1}-2^n\times\frac{1}{2^{(n+1)(q+1)}}\right)\\
&\geq \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n^a}-\frac{1}{(n+1)^a}\right)\left(1-\frac{1}{2^{(n+1)q}}\right)\\
&\geq \frac{a}{2}Cq^a,\ \forall q\in[0,1].
\end{aligned}$$

■

From this we deduce that there is loss of mass. Indeed, $\phi'(x) \leq x^{-1}\phi(x)$ for positive x since ϕ is a concave function. Then combine this with Lemma 1.3 to obtain that

$$\frac{1}{\tau(\exp(-1/x))}\times\frac{\phi'(x)}{\phi^2(x)}\leq\frac{1}{Cx^a}\quad\text{for } 0<x\leq 1$$

and conclude with Proposition 1.3 (ii). On the other hand, there is no total loss of mass since the equalities

$$\int_{\mathcal{S}^{\downarrow}}|\log(s_1)|\,\nu(ds)=\log 2\quad\text{and}\quad\int_0^1\frac{dx}{x\tau(x)}=\infty$$

imply with Proposition 1.7 (ii) that $P(\zeta<\infty)=0$.

1.5.3 Asymptotic behavior of $P(\zeta > t)$ as $t\to\infty$

In this subsection, we consider functions τ such that $C_2x^{\beta}\leq\tau(x)\leq C_1x^{\alpha}$ for $x\in\,]0,1]$, where $\alpha\leq\beta<0$ and C_1 and C_2 are positive constants. Thus there is total loss of mass with probability one. The following proposition states that $P(\zeta>t)$ and $m(t)$ have then the same type of behavior as $t\to\infty$ (see also Proposition 1.6). More precisely, we have

Proposition 1.8 *Suppose that $C_2x^{\beta}\leq\tau(x)\leq C_1x^{\alpha}$ for $x\in\,]0,1]$, where $\alpha\leq\beta<0$, $C_1>0$ and $C_2>0$. Then,*

(i) $\exists C>0$ such that $P(\zeta>t)\leq\exp(-Ct)$ for t large enough.

(ii) If $\phi\asymp f$ on $[1,\infty)$, for a function f varying regularly with index $a\in\,]0,1[$ at ∞, there are two positive constants A and B such that for t large enough

$$\exp(-B\psi(t))\leq P(\zeta>t)\leq\exp(-A\psi(t))$$

where ψ is the inverse of the bijection $t\in[1,\infty)\mapsto t/\phi(t)\in[1/\phi(1),\infty)$.

Actually, the upper bounds hold as soon as $\tau(x)\geq C_2x^{\beta}$, with $\beta<0$ and $C_2>0$ and the lower bound holds for functions τ satisfying only $\tau(x)\leq C_1x^{\alpha}$, with $\alpha<0$ and $C_1>0$.

To prove the proposition we need the following lemma:

Lemma 1.4 *Let F be a self-similar fragmentation process with parameters (α, c, ν), $\alpha < 0$, and ζ the first time at which the entire mass has disappeared. Fix $\alpha' \geq \alpha$. Then, there exists a self-similar fragmentation process with parameters (α', c, ν), denoted by F', such that*

$$\zeta \leq \int_0^\infty (F_1'(r))^{\alpha'-\alpha} \, dr.$$

Proof. Consider $(I_x(t), x \in]0,1[, t \geq 0)$ the interval representation of F. There exists a self-similar interval representation process with parameters (α', c, ν), denoted by $(I_x'(t), x \in]0,1[, t \geq 0)$, such that

$$I_x(t) = I_x'(T_x(t))$$

where

$$T_x(t) = \inf\left\{u \geq 0 : \int_0^u |I_x'(r)|^{\alpha'-\alpha} \, dr > t\right\}.$$

(See Section 3.2. in [14]). For each $t \geq 0$, call $F'(t)$ the non-increasing rearrangement of the lengths of the disjoint intervals components of $(I_x'(t), x \in]0,1[)$. Then F' is the required self-similar fragmentation process with index α'. Let x be in $]0,1[$. Since $|I_x'(r)| \leq F_1'(r)$ for each $r \geq 0$, we have that

$$T_x\left(\int_0^\infty (F_1'(r))^{\alpha'-\alpha} \, dr\right) = \infty.$$

Then,

$$\zeta \leq \int_0^\infty (F_1'(r))^{\alpha'-\alpha} \, dr,$$

because $I_x'(\infty) = 0$ for every x in $]0,1[$. ■

Proof of Proposition 1.8. If $\tau' = K\tau$ for a positive constant K and if F^τ and $F^{\tau'}$ are two fragmentation processes constructed from the same homogeneous one, it is easily seen that $F_1^{\tau'}(t) = F_1^\tau(Kt)$ for each $t \geq 0$. Recall moreover the remark (1.29). Since it is supposed that $C_2 x^\beta \leq \tau(x) \leq C_1 x^\alpha$ on $]0,1]$, where $\alpha \leq \beta < 0$, it is then enough to prove results (i) and (ii) for a self-similar fragmentation process with a negative index. Thus, consider F a self-similar fragmentation process with parameters (α, c, ν), $\alpha < 0$. Applying the previous lemma to F and $\alpha' = \alpha/2$, we get

$$\begin{aligned} P(\zeta > 2t) &\leq P(\textstyle\int_0^\infty (F_1'(rt))^{-\alpha/2} \, dr > 2) \\ &\leq P(\textstyle\int_1^\infty (F_1'(rt))^{-\alpha/2} \, dr > 1) \\ &\leq \textstyle\int_1^\infty E\left[(F_1'(rt))^{-\alpha/2}\right] dr, \end{aligned} \tag{1.31}$$

since $F_1'(t) \leq 1$, $\forall t \geq 0$. Now, recall that

$$E\left[F_1'(t)\right] \leq E\left[\sum_{i=1}^\infty F_i'(t)\right] = m_{\tau'}(t),$$

where $m_{\tau'}$ is the total mass of the fragmentation equation with parameters $\tau'(x) = x^{\alpha/2}$, c and ν. This leads to

$$E\left[(F_1'(t))^{-\alpha/2}\right] \leq \begin{cases} m_{\tau'}(t) \text{ if } (-\alpha/2) \geq 1 \\ (m_{\tau'}(t))^{-\alpha/2} \text{ if } (-\alpha/2) < 1 \text{ (by Jensen's inequality).} \end{cases} \tag{1.32}$$

(i) Combining (1.31), (1.32) and (1.24), we obtain that for t large enough

$$P(\zeta > 2t) \leq \int_1^\infty \exp(-C'rt)dr = \frac{1}{C't}\exp(-C't),$$

where C' is a positive constant.

(ii) As stated in Proposition 1.6, since $\phi \asymp f$, with f a regularly varying function with index $a \in]0,1[$, and since $\tau'(x) = x^{\alpha/2}$, the function

$$\sigma : t \in [1,\infty) \mapsto t/\phi(t) \in [1/\phi(1),\infty)$$

is an increasing bijection and its inverse ψ satisfies $m_{\tau'}(t) \leq \exp(-A_1\psi(t))$ for a constant $A_1 > 0$ and t large enough. From this and inequalities (1.31) and (1.32) , we deduce the existence of a positive constant A_2 so that for t large enough,

$$P(\zeta > 2t) \leq \int_1^\infty \exp(-A_2\psi(rt))dr.$$

Moreover, σ is differentiable and its derivative is positive and smaller than $\dfrac{1}{\phi}$ (recall that ϕ' is positive) and then σ' is bounded on $[1,\infty)$. Thus for t large enough,

$$\begin{aligned} P(\zeta > 2t) &\leq t^{-1}\textstyle\int_{\psi(t)}^\infty \exp(-A_2 r)\sigma'(r)dr \\ &\leq A_2 \textstyle\int_{\psi(t)}^\infty \exp(-A_2 r)dr \\ &= \exp(-A_2\psi(t)). \end{aligned}$$

Then, as in the proof of Proposition 1.6 the constant 2 can be removed by using the assumption (ii).

Eventually, introduce the r.v. I_τ (see definition (1.15)) to conclude for the lower bound. This random variable is the first time when the size-biased picked fragment vanishes and so $I_\tau \leq \zeta$ a.s. Then, we get the desired lower bound from Proposition 1.6 (recall that $m(t) = P(I_\tau > t)$). ■

1.5.4 Small times asymptotic behavior

We are interested in the asymptotic behavior as $t \to 0$ of the random mass $M(t) = 1 - \sum_{i=1}^\infty F_i^\tau(t)$ of the (τ,c,ν) fragmentation process F^τ starting from $(1,0,...)$.

Proposition 1.9 *One has,*

$$\frac{M(t)}{t} \overset{a.s.}{\to} c \text{ as } t \to 0.$$

More generally, when F^τ denotes a (τ,c,ν)-fragmentation process starting from $(s,0,...)$, $s>0$, one easily checks, by adapting the following proof, that $(s - \sum_{i=1}^\infty F_i^\tau(t))/t \overset{\text{a.s.}}{\to} cs\tau(s)$ as $t \to 0$.

Proof. On the one hand, among the $F_i^\tau(t)$'s, $i \geq 1$, there is the size-biased picked fragment, and therefore

$$\sum_{i=1}^{\infty} F_i^\tau(t) \geq \lambda^\tau(t) = \exp(-\xi_{\rho^\tau(t)}),\ t \geq 0,$$

where ξ is a subordinator with Laplace exponent (1.5) and ρ^τ the time change (1.8). It is easily seen that a.s. $\rho^\tau(t)/t \overset{t\to 0}{\to} \tau(1) = 1$ and it is well known that a.s. $\xi_t/t \overset{t\to 0}{\to} c$ (see chapter 3, [10]). Hence

$$\limsup_{t\to 0}(M(t)/t) \leq c \text{ a.s.}$$

On the other hand, in the homogeneous case ($\tau \equiv 1$), it is clear that $\sum_{i=1}^{\infty} F_i^\tau(t) \leq e^{-ct}$. This *a fortiori* holds when $\tau \geq 1$ on $]0,1]$, since the fragmentation is then faster than a homogeneous one. Therefore, in such cases,

$$\liminf_{t\to 0}(M(t)/t) \geq c \text{ a.s.} \tag{1.33}$$

and the conclusion follows. To prove that this lower limit holds for all functions τ, we study the mass lost by erosion by the size-biased picked fragment λ^τ until time t. First, in the homogeneous case (we write λ^{hom} for the size-biased picked fragment), $\lambda^{\text{hom}}(t) = e^{-ct}\lambda_0^{\text{hom}}(t)$, $t \geq 0$, where λ_0^{hom} is the size-biased picked fragment of some homogeneous fragmentation without erosion. If $\lambda_0^{\text{hom}}(t) \geq m$ for $t \in [t_1, t_2]$, the mass lost by erosion by λ^{hom} between times t_1 and t_2 is then larger than $m(\exp(-ct_1) - \exp(-ct_2))$. In particular, the mass lost by erosion by λ^{hom} between times 0 and t is larger than $\lambda_0^{\text{hom}}(t)(1-\exp(-ct))$. Now, for any function τ, $\lambda^\tau(t) = \lambda^{\text{hom}}(\rho^\tau(t))$ for some size-biased picked fragment λ^{hom} of some homogeneous fragmentation, and therefore, the mass lost by erosion by λ^τ until time t is larger than $\lambda_0^{\text{hom}}(\rho^\tau(t))(1-\exp(-c\rho^\tau(t))$ for some process λ_0^{hom}. Consequently, $M(t) \geq \lambda_0^{\text{hom}}(\rho^\tau(t))(1-\exp(-c\rho^\tau(t))$. Since, $\lambda_0^{\text{hom}}(u) \to 1$ a.s. as $u \to 0$ and since $\rho^\tau(t)/t \to 1$ a.s. as $t \to 0$, one gets

$$\liminf_{t\to 0}(M(t)/t) \geq c \text{ a.s.}$$

We point out that it is possible to check (by refining the above argument) that the mass lost by erosion by the size-biased picked fragment in the homogeneous case is equal to $\int_0^t c\lambda^{\text{hom}}(s)ds$ and therefore that it is equal to

$$\int_0^{\rho(t)} c\lambda^{\text{hom}}(s)ds = \int_0^t c\lambda^\tau(s)\tau(\lambda^\tau(s))ds$$

in the general case, which, from a physical view point, was intuitive. ■

1.6 Appendix

1.6.1 An example

Let us consider the self-similar fragmentation process constructed from the Brownian excursion of length 1. This process was introduced in [14] and can be constructed as follows. Write

$e = (e(t), 0 \leq t \leq 1)$ for the Brownian excursion of length 1 and introduce the family of random open sets of $]0,1[$ defined by

$$I(t) = \{s \in]0,1[: e(s) > t\}, \ t \geq 0.$$

Then the process $(I(t), t \geq 0)$ is a self-similar interval fragmentation process with index $-1/2$. For each $t \geq 0$, define by $F(t)$ the non-increasing sequence of the lengths of the interval components of $I(t)$. The required fragmentation process is this process $(F(t), t \geq 0)$, which is obviously self-similar with index $-1/2$. Consider then the deterministic fragmentation model constructed from F and especially its mass, which is denoted by $m(t)$ for all time t. Since the process F is self-similar with a negative index, there is loss of mass. Moreover, as shown in [14], the Laplace exponent of the associated subordinator ξ is given by

$$\phi(q) = 2q\sqrt{\frac{2}{\pi}} B\left(q + \frac{1}{2}, \frac{1}{2}\right),$$

and this leads to the following equivalence

$$\phi(q) \underset{q \to \infty}{\sim} 2\sqrt{2} q^{\frac{1}{2}}$$

(B denotes here the beta function). Hence, the remark following Proposition 1.6 ensures that

$$\log m(t) \underset{t \to \infty}{\sim} -2t^2$$

and Proposition 1.8 gives exponential bounds for $P(\zeta > t)$ as $t \mapsto \infty$.

However, we may obtain sharper results. First, recall that I_τ denotes the first time when the size-biased picked fragment of F is equal to 0. We know from [14] that $2I_\tau$ follows the Rayleigh distribution, that is

$$P(2I_\tau \in dr) = r \exp(-r^2/2) dr,$$

and then the mass is explicitly known:

$$m(t) = P(I_\tau > t) = \exp(-2t^2), \ t \geq 0.$$

On the other hand, the random variable ζ is obviously the maximum of the Brownian excursion with length 1. And then, as proved in [43], the tail distribution of this random variable is given by

$$P(\zeta > t) = 2\sum_{n=1}^{\infty} \left(4t^2n^2 - 1\right) \exp(-2t^2n^2), \quad t > 0.$$

This implies that

$$P(\zeta > t) \underset{t \to \infty}{\sim} 8t^2 \exp(-2t^2).$$

1.6.2 Necessity of condition (1.1)

We discuss here the necessity of assumption (1.1) for the splitting measure ν in the fragmentation equation (1.2) (that (1.1) is needed to construct a random fragmentation was pointed out in [13]).

Suppose that $\int_{\mathcal{S}^*}(1-s_1)\nu(ds)=\infty$ and that there exists a solution $(\mu_t, t\geq 0)$ to (1.2). Let f be a function of $C_c^1\left(]0,1]\right)$ whose support is exactly $[3/4,1]$ and such that $f(1)\neq 0$. Since the function $t\mapsto\langle\mu_t,f\rangle$ is continuous on $\mathbb{R}^+$ and $\mu_0=\delta_1$, there exists a positive time t_0 such that

$$\mathrm{supp}\mu_t\cap[3/4,1]\neq\emptyset \tag{1.34}$$

for $t<t_0$. Then define by g an non-decreasing non-negative function on $]0,1]$, smaller than id, belonging to $C_c^1\left(]0,1]\right)$ and such that

$$g(x)=\begin{cases} 0 \text{ on } \ ]0,1/2] \\ x \text{ on } \ [3/4,1]. \end{cases}$$

Take x in $[3/4,1]$. For each $s\in\mathcal{S}^{\downarrow}$ and each $i\geq 2$, $g(xs_i)=0$ since $s_i\leq 1/2$ for $i\geq 2$. Thus

$$\begin{aligned} \int_{\mathcal{S}^{\downarrow}}\left[\sum_{i=1}^{\infty}g(xs_i)-g(x)\right]\nu(ds) &= \int_{\mathcal{S}^{\downarrow}}\left(g(xs_1)-g(x)\right)\nu(ds) \\ &\leq x\int_{\mathcal{S}^{\downarrow}}\left(s_1-1\right)\nu(ds)=-\infty. \end{aligned}$$

By combining this with (1.34), we conclude that the derivative $\partial_t\left\langle\mu_t,g\right\rangle=-\infty$ on $[0,t_0[$ and then that the fragmentation equation (1.2) has no solution.

Chapitre 2

Regularity of formation of dust in self-similar fragmentations

Abstract: In self-similar fragmentations with a negative index, fragments split even faster as their mass is smaller, so that the fragmentation runs away and some mass is reduced to dust. Our purpose is to investigate the regularity of this formation of dust. Let $M(t)$ denote the mass of dust at time t. We give some sufficient and some necessary conditions for the measure dM to be absolutely continuous. In case of absolute continuity, we obtain an approximation of the density by functions of small fragments. We also study the Hausdorff dimension of dM and of its support, as well as the Hölder-continuity of the dust's mass M.

2.1 Introduction

Fragmentation processes are random models for the evolution of an object that splits as time goes on. These models, together with their deterministic counterparts, have been widely studied by both mathematicians and physicists. We mention Aldous' survey [4] of the literature on the subject and Les Houches proceedings [20] for physical view points.

The self-similar fragmentations processes we consider in this work are those studied by Bertoin in [13], [14], [15]. Informally, a self-similar fragmentation is a process that enjoys both a fragmentation property and a scaling property. By fragmentation property, we mean that the fragments present at a time t will evolve independently with break-up rates depending on their masses. The scaling property specifies these mass-dependent rates. More precisely, there is a real number α, called index of self-similarity, such that the process starting from a fragment with mass m has the same distribution as m times the process starting from a fragment with mass 1, up to the time change $t \mapsto tm^{\alpha}$. The definition will be made rigorous in Section 2.2.

Our interest is more specifically in self-similar fragmentations with negative indices of self-similarity, in which a *loss of mass* occurs (see e.g. [15]), corresponding to the appearance of dust - or microscopic fragments - whose total mass is non-zero. This phenomenon is a consequence of an intensive splitting that results from the scaling property: when $\alpha < 0$, small fragments split faster than large ones, so that the average speed of splitting increases as time goes on and the fragmentation runs away and produces some dust. Let us mention [35], [36], [38] and [42] for discussions on the appearance of dust for some different classes of random fragmentations and for some deterministic fragmentation models.

The purpose of this paper is to study the *regularity* of this formation of dust. To be more precise, let $M(t)$ be the dust's mass at time t, $t \geq 0$. It is a non-decreasing function that can be written as $M(t) = \int_0^t dM(u)$ for some non-negative measure dM. Our main point of interest is to investigate the existence of a Lebesgue density for the mass measure dM. We are also concerned with questions such as the approximation of the density (when it exists) by functions depending on small fragments, the Hausdorff dimensions of dM and dM's support when dM is singular and the Hölder-continuity of the dust's mass M.

This study is motivated and illustrated by the "Brownian excursion fragmentation" example, introduced first in [14] and that we now roughly present. Let $e = (e(x), 0 \leq x \leq 1)$ be the normalized Brownian excursion (informally, e is a Brownian motion on the unit interval, conditioned by $e(0) = e(1) = 0$ and $e(x) > 0$ for $0 < x < 1$) and consider the family of random nested open sets of $]0,1[$

$$I_e(t) = \{x \in]0,1[: e(x) > t\}, \ t \geq 0.$$

This family corresponds to a fragmentation of the interval $]0,1[$ as time passes (actually, one may prove that it is a self-similar fragmentation with index $\alpha = -1/2$ - see [14]). The interval components of $I_e(t)$ are the "fragments" present at time t with a positive mass (the mass of a fragment being the length of the corresponding interval) and their total mass is equal to $\int_0^1 1_{\{e(u)>t\}} du$. The dust's mass $M_e(t)$ is thus equal to $\int_0^1 1_{\{e(u)\leq t\}} du$, which is positive for all $t > 0$. According to the Brownian motion theory, there is a local time process $(L_e(t), t \geq 0)$ such that

$$M_e(t) = \int_0^t L_e(s) ds \ \text{ for all } t \geq 0, \text{ a.s.},$$

so that the mass measure dM_e has L_e for Lebesgue density a.s. It is further known that this density L_e can be approximated by functions of small interval components (i.e. fragments) as follows (see e.g. [60]): for every $t \geq 0$,

$$\lim_{\varepsilon \to 0} \sqrt{\frac{2\pi}{\varepsilon}} M_e(t, \varepsilon) \overset{a.s.}{=} \lim_{\varepsilon \to 0} \sqrt{2\pi\varepsilon} N_e(t, \varepsilon) \overset{a.s.}{=} L_e(t),$$

where $M_e(t,\varepsilon)$ denotes the total length of excursions intervals of e above t of length less or equal to ε (that is, in terms of fragments, the total mass of fragments present at time t having a mass in $]0,\varepsilon]$); and $N_e(t,\varepsilon)$ is the number of excursions of e above t of length greater then ε (i.e. the number of fragments present at time t of mass greater than ε). Another point we are interested in, as mentioned above, is the Hölder-continuity of the dust's mass M_e. It is well-known that that the local time L_e is bounded a.s.: the dust's mass M_e is therefore Lipschitz a.s.

Miermont [56] constructs similarly some fragmentations from the normalized excursions of some random continuous processes possessing a local time, which gives some more examples of fragmentations with absolutely continuous mass measure dM.

Our goal is to see how these regularity results extend to general self-similar fragmentations with negative indices. The paper is organized as follows. In Section 2.2, self-similar fragmentations are introduced and their main properties recalled. Section 2.3 concerns some preliminary results on the dust's mass M and on tagged fragments, a tagged fragment being a fragment containing a point tagged at random, independently of the fragmentation. The evolution of such fragments is well-known and is closely connected to the mass M as we shall see later. Following one or several tagged fragments as time passes will then be a key tool in the study of regularity.

There are some self-similar fragmentations for which the mass measure dM does not have a Lebesgue density. Section 2.4 presents some sufficient (respectively necessary) conditions for dM to be absolutely continuous. These conditions are stated in terms of the index of self-similarity α and of a dislocation measure, introduced in Section 2.2, that, roughly, describes the distribution of sudden dislocations. For a large class of fragmentations the critical value is $\alpha = -1$, in the sense that almost surely dM has a Lebesgue density if and only if $\alpha > -1$. The sufficient conditions' proofs are coarser than the necessary ones and rely on Fourier analysis.

For fragmentations with an absolutely continuous mass measure dM, the approximation of the density is discussed in Section 2.5. Let $L(t) := dM(t)/dt$. In most cases, we prove the existence of a finite deterministic constant C such that for a.e. t, the functions $\varepsilon^{\alpha} M(t,\varepsilon)$ and $\varepsilon^{1+\alpha} N(t,\varepsilon)$ converge a.s. to $CL(t)$ as $\varepsilon \to 0$. As in the Brownian excursion fragmentation, $M(t,\varepsilon)$ denotes the total mass of fragments of mass in $]0,\varepsilon]$ at time t and $N(t,\varepsilon)$ the number of fragments of mass greater than ε at time t.

Section 2.6 is devoted to the Hölder-continuity of the dust's mass M and, in cases where dM is singular, to its Hausdorff dimension and that of its support. The paper ends with an Appendix containing a technical proof of a result stated in Section 2.3.

2.2 Background on self-similar fragmentations

Since for us the only distinguishing feature of a fragment is its mass, the fragmentation system is characterized at a given time t by the ranked sequence $s_1 \geq s_2 \geq ... \geq 0$ of masses of fragments present at that time. Starting from a single object with mass one, the appropriate space for our models is then $\mathcal{S}^{\downarrow}$, the state of non-increasing non-negative sequences with total sum at most 1, i.e.

$$\mathcal{S}^{\downarrow} := \left\{ s = (s_i)_{i \in \mathbb{N}^*},\ s_1 \geq s_2 \geq ... \geq 0 : \sum_{i=1}^{\infty} s_i \leq 1 \right\},$$

endowed with the topology of pointwise convergence. The difference $1 - \sum_i s_i$ may be thought as the mass of dust.

Definition 2.1 *Let $(F(t), t \geq 0)$ be a $\mathcal{S}^{\downarrow}$-valued Markov process continuous in probability and denote by P_r, $0 < r \leq 1$, the law of F starting from $(r, 0, ...)$.*

(i) The process F is a fragmentation process if for each $t_0 \geq 0$, conditionally on $F(t_0) = (s_1, s_2, ...)$, the process $(F(t + t_0), t \geq 0)$ has the same law as the process obtained, for each

$t \geq 0$, *by ranking in the decreasing order the components of sequences* $F^1(t)$, $F^2(t), \ldots$, *where the r.v.* F^i *are independent with respective laws* P_{s_i}.

(ii) If further F *enjoys the scaling property, which means that there exists a real number* α, *called index of self-similarity, such that the law of* $(F(t), t \geq 0)$ *under* P_r *is the same as that of* $(rF(tr^\alpha), t \geq 0)$ *under* P_1, *then* F *is a self-similar fragmentation process with index* α. *When* $\alpha = 0$, F *is called a homogeneous fragmentation process.*

We consider fragmentation processes starting from $F(0) = (1, 0, 0, \ldots)$ and denote by $F_i(t)$, $i \geq 1$, the components of the sequence $F(t)$, $t \geq 0$, and by $\mathcal{F} = (\mathcal{F}(t), t \geq 0)$, the natural filtration generated by F, completed up to P-null sets. According to Berestycki [9] and Bertoin [14], a self-similar fragmentation is Feller (then possesses a càdlàg version which we may always consider) and its distribution is entirely characterized by three parameters: the index of self-similarity α, an erosion coefficient $c \geq 0$ and a *dislocation measure* ν, which is a sigma-finite measure on $\mathcal{S}^\downarrow$ that does not charge $(1, 0, \ldots)$ and such that

$$\int_{\mathcal{S}^\downarrow} (1 - s_1)\nu(ds) < \infty.$$

Roughly speaking, the erosion is a deterministic continuous phenomenon and the dislocation measure describes the rates of sudden dislocations: a fragment with mass x splits in fragments with mass $xs, s \in \mathcal{S}^\downarrow$, at rate $x^\alpha \nu(ds)$. Conversely, given α, c, ν satisfying the requirements above, one can construct a corresponding self-similar fragmentation. As a consequence of the Feller property, the fragmentation property holds for $\mathcal{F}$-stopping times and we shall refer to it as the *strong fragmentation property.*

For technical reasons, we may need to work with an interval representation of the fragmentation: by combination of results of [9] and [14], there is no loss of generality in assuming that a α-self-similar fragmentation F is constructed from a family $(I(t), t \geq 0)$ of nested random open sets of $]0,1[$ so that, for every $t \geq 0$, $F(t) = (F_1(t), \ldots)$ is the ordered sequence of the lengths of the interval components of $I(t)$. This process I possesses both the α-self-similarity and fragmentation properties (we refer to [14] for precise definitions). Moreover it is Fellerian and as such, satisfies a strong fragmentation property. From now on, we call I an *interval representation* of F. There is actually a one-to-one correspondence between the laws of $\mathcal{S}^\downarrow$-valued and interval-valued self-similar fragmentations.

The advantage of this interval's view point is the passage from homogeneous to self-similar fragmentations by appropriate time-changes: consider a homogeneous interval fragmentation $(I^0(t), t \geq 0)$ and define by $I_x(t)$ the interval component of $I^0(t)$ that contains x if $x \in I^0(t)$ and set $I_x(t) := \emptyset$ if $x \notin I^0(t)$, x in $]0,1[$. Then introduce the time-changed functions

$$T_x^\alpha(t) := \inf \left\{ u \geq 0 : \int_0^u |I_x(r)|^{-\alpha} \, dr > t \right\}, \tag{2.1}$$

and consider the family of nested open sets of $]0,1[$ defined by

$$I^\alpha(t) = \bigcup_{x \in]0,1[} I_x(T_x^\alpha(t)), \;\; t \geq 0.$$

As proved in [14], I^α is an α-self-similar interval fragmentation and each self-similar interval fragmentation can be constructed like this from a homogeneous one. This associated homogeneous fragmentation has the same dislocation measure and erosion coefficient as the self-similar fragmentation.

This interval setting is particularly appropriate to tag fragments at random as explained in detail in the following section.

2.3 Tagged fragments and dust's mass

From now on, we shall focus on self-similar fragmentations such that

$$\alpha < 0, \quad c = 0, \quad \nu \neq 0 \ \text{ and } \ \nu\left(\sum_i s_i < 1\right) = 0. \tag{H}$$

That $\nu\left(\sum_i s_i < 1\right) = 0$ means that no mass is lost within sudden dislocations and $c = 0$ means there is no erosion. In terms of the fragmentation F, the dust's mass at time t then writes

$$M(t) = 1 - \sum_{i=1}^{\infty} F_i(t). \tag{2.2}$$

The index α being negative, we know by Proposition 2 in [15], that with probability one M is càdlàg, non-decreasing and reaches 1 in finite time. It can then be viewed as the distribution function of some random probability measure, that we denote by dM:

$$M(t) = \int_0^t dM(u), \ t \geq 0.$$

A useful tool to study this mass of dust is to *tag* a fragment at random in the fragmentation. To do so, consider I an interval representation of F as recalled in the previous section and let U be a random variable uniformly distributed on $]0,1[$ and independent of I. At each time t, if $U \in I(t)$, denote by $\lambda(t)$ the length of the interval component of $I(t)$ containing U. If $U \notin I(t)$, set $\lambda(t) := 0$. Bertoin, in [13] and [14], has determined the law of the process λ :

$$\lambda \overset{law}{=} \exp(-\xi_{\rho(.)}) \tag{2.3}$$

where ξ is a subordinator with Laplace exponent ϕ given for all $q \geq 0$ by

$$\phi(q) = \int_{\mathcal{S}^\downarrow} \left(1 - \sum_{i=1}^{\infty} s_i^{1+q}\right) \nu(ds), \tag{2.4}$$

and ρ is the time-change

$$\rho(t) = \inf\left\{u \geq 0 : \int_0^u \exp(\alpha \xi_r) dr > t\right\}, \ t \geq 0.$$

We refer to [11] for background on subordinators and recall that $E\left[e^{-q\xi_r}\right] = e^{-r\phi(q)}$ for $r, q \geq 0$. Remark that formula (2.4) defines in fact a function ϕ on $\mathbb{R}$ such that $\phi(q) \in [0, \infty[$ for $q \geq 0$

and $\phi(q) \in [-\infty, 0[$ for $q < 0$. Let $\underline{p}$ be the largest q such that $\phi(-q) > -\infty$. Since ν integrates $(1 - s_1)$, this definition is equivalent to

$$\underline{p} = \sup\left\{ q \geq 0 : \int_{\mathcal{S}^{\downarrow}} \sum_{i=2}^{\infty} s_i^{1-q} \nu(ds) < \infty \right\}. \tag{2.5}$$

Here we use the convention $0^{-a} = \infty$ for $a > 0$. Hence, when $q > 1$ the series $\sum_{i=2}^{\infty} s_i^{1-q} = \infty$ for any sequence in $\mathcal{S}^{\downarrow}$ and consequently $\underline{p} \leq 1$. The Hölder-continuity of the dust's mass M, studied in Section 2.6.2, depends on this coefficient $\underline{p}$.

The law of the first time D at which the tagged fragment is reduced to dust, i.e.

$$D := \inf \{t \geq 0 : \lambda(t) = 0\},$$

can then be expressed as a function of α and ξ :

$$D \overset{law}{=} \int_0^{\infty} \exp(\alpha \xi_r) dr. \tag{2.6}$$

One first important example of the use of tagged fragments is that the dust's mass M then coincides with the distribution function of D conditional on F, that is, a.s.

$$M(t) = P(D \leq t \mid F), \ t \geq 0. \tag{2.7}$$

Indeed, $D \leq t$ if and only if $U \notin I(t)$ and the conditional probability of this event given F is the total length of $]0,1[\setminus I(t)$, i.e. $1 - F_1(t) - F_2(t) - ... = M(t)$. The point is that the law of D has been extensively studied (see e.g. [25],[18]) and it will therefore give us some information on M.

The rest of the section concerns some preliminary results that will be needed in the sequel. Subsection 2.3.1 deals with some regularity properties of D's distribution. The main results of Carmona et al. [25] are recalled and some other properties developed. In Subsection 2.3.2, we tag several fragments independently and study their masses at the first time at which some tagged fragments are different. Subsection 2.3.3 is devoted to the first time at which all the mass is reduced to dust.

2.3.1 On the regularity of D's distribution

By (2.6), D has the same law as $\int_0^{\infty} \exp(\alpha \xi_r) dr$. Carmona, Petit and Yor studied in [25] these exponential functionals. They showed (Prop. 3.1 iv, Prop. 3.3) that D has entire moments of all positive orders and that

$$\mu := E[\xi_1] = \frac{1}{|\alpha|} E\left[D^{-1}\right]. \tag{2.8}$$

Remark with (2.4), that

$$\mu = E[\xi_1] = \phi'(0^+) = \int_{\mathcal{S}^{\downarrow}} \left(\sum_{i=1}^{\infty} |\log(s_i)| \, s_i \right) \nu(ds).$$

In the sequel, we will often assume that $\mu < \infty$, because of the following lemma:

Lemma 2.1 *Suppose that* $\mu < \infty$ *and* $\int_{\mathcal{S}^\downarrow} (1-s_1)^\beta \nu(ds) < \infty$ *for some* $\beta < 1$. *Then, there is an infinitely differentiable function* $k :]0,\infty[\to [0,\infty[$ *such that*

(i) $P(D \in dx) = k(x)dx$

(ii) for all $a \geq 0$, *the function* $x \mapsto x^a k(x)$ *is bounded on* $]0,\infty[$.

We point out that the existence of some $\beta < 1$ such that $\int_{\mathcal{S}^\downarrow} (1-s_1)^\beta \nu(ds) < \infty$ is not necessary to prove the assertion (i).

Proof. (i) It is Proposition 2.1 of [25].

(ii) The point is to show that for all $a \geq 0$, the function $x \mapsto e^{ax}k(e^x)$ is bounded on $\mathbb{R}$. To that end, we need the following result of [25] (Prop. 2.1): the density k is a solution of the equation

$$k(x) = \int_x^\infty \overline{\pi}\left(\frac{1}{|\alpha|}\log\left(\frac{u}{x}\right)\right) k(u)du,\ x > 0,$$

where π denotes the Lévy measure of ξ and $\overline{\pi}(x) := \pi\left(]x,\infty[\right),\ x > 0$. This leads to

$$\begin{aligned} e^{ax}k(e^x) &= \int_{-\infty}^{\infty} 1_{\{u-x>0\}}\overline{\pi}((u-x)/|\alpha|)e^{a(x-u)}e^{(a+1)u}k(e^u)du \\ &= \left(1_{\{\cdot<0\}}\overline{\pi}(-\cdot/|\alpha|)e^{a\cdot} * e^{(a+1)\cdot}k(e^\cdot)\right)(x), \end{aligned} \tag{2.9}$$

where $*$ denotes the convolution product. It is well-known (by Hölder inequality) that for $p \geq 1$ the convolution product of a function of $L^p(dx)$ with a function of $L^{p/(p-1)}(dx)$ is bounded on $\mathbb{R}$. So if we prove that the functions $x \mapsto 1_{\{x<0\}}\overline{\pi}(-x/|\alpha|)e^{ax}$ and $x \mapsto e^{(a+1)x}k(e^x)$ respectively belong to $L^p(dx)$ and $L^{p/(p-1)}(dx)$ for some $p \geq 1$, the proof will be ended.

Let us first show that $\overline{\pi} \in L^\gamma(dx)$ for all $1 < \gamma < 1/\beta$ such that $\int_{\mathcal{S}^\downarrow} (1-s_1)^\beta \nu(ds) < \infty$ (such β exists by assumption). To see this, note that

$$\pi(dx) = e^{-x}\nu(-\log(s_1) \in dx) \text{ on }]0,\log 2[$$

(see e.g. the remarks at the end of [13]), which gives

$$\int_0^{\log 2} x^c \pi(dx) = \int_{\mathcal{S}^\downarrow} 1_{\{s_1>1/2\}} s_1 |\log s_1|^c \nu(ds),\ c \in \mathbb{R}.$$

Then combine this with $\int_0^\infty x\pi(dx) = \phi'(0^+) < \infty$ (which is a consequence of $\mu < \infty$ and (2.4)) to get that $\int_0^\infty \left(x^\beta \vee x\right)\pi(dx) < \infty$ for the $\beta < 1$ such that $\int_{\mathcal{S}^\downarrow} (1-s_1)^\beta \nu(ds) < \infty$. Therefore, there exists $C > 0$ such that $\overline{\pi}(x) \leq C\left(x^{-1} \wedge x^{-\beta}\right)$ for $x > 0$. Then $\overline{\pi}$, and a fortiori $x \mapsto 1_{\{x<0\}}\overline{\pi}(-x/|\alpha|)e^{ax}$, belongs to $L^\gamma(dx)$ for all $1 < \gamma < 1/\beta$.

It remains to prove that for all $a \geq 0$, the function $x \mapsto e^{(a+1)x}k(e^x)$ belongs to $L^{\gamma/(\gamma-1)}(dx)$ for some $\gamma \in]1,1/\beta[$. Fix such a γ and remark that it is sufficient to show that this function belongs to $L^{\gamma^n}(dx)$ for all $n \in \mathbb{N}$ (because $L^1 \cap L^{\gamma^n} \subset L^{\gamma/(\gamma-1)}$ when $\gamma^n \geq \gamma/(\gamma-1) \geq 1$). We prove this by induction on n. For $n = 0$, this is an immediate consequence of

$\int_{-\infty}^{\infty} e^{(a+1)u} k(e^u) du = E[D^a]$, which is finite for all $a \geq 0$ by Proposition 3.3 of [25]. For the next step, we need the following result: for all $p, q \geq 1$,

$$\text{if } f \in L^p(dx) \cap L^1(dx) \text{ and if } g \in L^q(dx), \text{ then } f * g \in L^{pq}(dx),$$

which we first prove. By applying Hölder inequality twice, first to the measure $|f(x-y)|\,dy$ and second to $|g(y)|^q\,dy$, we get

$$\begin{aligned} |f * g(x)| &\leq \left(\int_{-\infty}^{\infty} |g(y)|^q |f(x-y)|\,dy\right)^{1/q} \left(\int_{-\infty}^{\infty} |f(x-y)|\,dy\right)^{(q-1)/q} \\ &\leq \left(\int_{-\infty}^{\infty} |g(y)|^q |f(x-y)|^p\,dy\right)^{1/pq} \left(\int_{-\infty}^{\infty} |g(y)|^q\,dy\right)^{(p-1)/pq} \\ &\quad \times \left(\int_{-\infty}^{\infty} |f(x-y)|\,dy\right)^{(q-1)/q}. \end{aligned}$$

The last two integrals do not depend on x and are finite. The first integral, seen as a function of x, is integrable by Fubini's Theorem. So, $f * g \in L^{pq}(dx)$. Now we apply this result to functions $x \mapsto 1_{\{x<0\}}\overline{\pi}(-x/|\alpha|)e^{ax}$ and $x \mapsto e^{(a+1)x}k(e^x)$, which belong respectively to $L^\gamma(dx)$ and $L^1(dx)$, and this shows with (2.9) that $x \mapsto e^{ax}k(e^x) \in L^\gamma(dx)$ for $a \geq 0$. Applying this recursively, we get that the function $x \mapsto e^{ax}k(e^x) \in L^{\gamma^n}(dx)$ for all $a \geq 0$ and $n \in \mathbb{N}$.

2.3.2 Tagging n fragments independently

We consider the joint behavior of n fragments tagged independently. More precisely, let $U_1, ..., U_n$ be n independent random variables, uniformly distributed on $]0,1[$ and independent of the fragmentation process, and for $i = 1, ..., n$ and $t \geq 0$, let $\lambda_i(t)$ be the length of the interval component of $I(t)$ containing the point U_i if $U_i \in I(t)$ and set $\lambda_i(t) := 0$ if $U_i \notin I(t)$. The law of $(\lambda_1, \lambda_2, ..., \lambda_n)$ is exchangeable, but the processes $\lambda_1, \lambda_2, ..., \lambda_n$ are not independent. They coincide on $[0, T_n[$, where T_n denotes the first time at which the U_i's, $i = 1, ..., n$, do not all belong to the same fragment, that is

$$T_n := \sup\{t \geq 0 : U_1, ..., U_n \in \text{same interval component of } I(t)\}.$$

Note that $T_n > 0$ a.s., since, by independence of the U_i's, $P(T_n > t \mid \lambda_1) \overset{a.s.}{=} \lambda_1(t)^{n-1}$ which tends to 1 as $t \to 0$. At time T_n, there are L distinct tagged fragments - for some random $L \geq 2$ - which, according to the fragmentation and scaling properties, evolve independently and with a law depending on their masses. The aim of this subsection is to give some information on these masses.

Consider an integer $l \geq 2$. Conditionally on $L = l$, we may assume, by exchangeability, that $U_1, U_2, ..., U_l$ belong all to different fragments at time T_n, so that the masses of the l distinct tagged fragments at time T_n are $\lambda_1(T_n), \lambda_2(T_n), ..., \lambda_l(T_n)$. For each l-tuple $(n_1, n_2, ..., n_l) \in (\mathbb{N}\backslash\{0\})^l$ such that $n_1 + n_2 + ... + n_l = n$, define then by $A_{(n_1,...,n_l)}$ the event

$$A_{(n_1,...,n_l)} := \left\{ \begin{array}{l} L = l \text{ and at time } T_n, \text{ there are } n_k \text{ tagged points} \\ \text{in the fragment containing } U_k,\ 1 \leq k \leq l \end{array} \right\}.$$

The following lemma provides an integrability property of a function depending on the masses of tagged fragments at time T_n. It will be a key point in the study of regularity. More precisely, it will be used to prove the Hölder-continuity of the dust's mass M (see Section 2.6) and, in the special case where $n=2$, to show the absolute continuity of the mass measure dM for some (α,ν)-fragmentations (see Section 2.4).

Lemma 2.2 *For all $a_1,...,a_l$ in $\mathbb{R}$, the following assertions are equivalent*

(i) $E\left[\prod_{k=1}^{l}\lambda_k^{-a_k}(T_n)1_{\{\lambda_1(T_n)\geq\lambda_2(T_n)\geq...\geq\lambda_l(T_n)\}}1_{\left\{A_{(n_1,n_2,...,n_l)}\right\}}\right]<\infty$

(ii) $\sum_{k=1}^{l}a_k<n-1$ *and* $\int_{\mathcal{S}^{\downarrow}}\sum_{i_1<i_2<...<i_l}\prod_{k=1}^{l}s_{i_k}^{n_k-a_k}1_{\left\{s_{i_k}>0\right\}}\nu(ds)<\infty$.

The proof of this technical result is provided in the Appendix at the end of the paper.

2.3.3 First time at which all the mass is reduced to dust

The first time at which the mass is entirely reduced to dust, i.e.

$$\zeta:=\inf\{t\geq0:F_1(t)=0\} \tag{2.10}$$

is almost surely finite (see [15]). The asymptotic behavior of $P(\zeta>t)$ as $t\to\infty$ is discussed in [38] and leads us to

Lemma 2.3 $E[\zeta]<\infty$ *and* $P(\zeta>t)<1$ *for every* $t>0$.

Proof. According to Section 5.3 in [38], there exist two positive finite constants A and B such that

$$P(\zeta>t)\leq Ae^{-Bt},\text{ for all }t\geq0. \tag{2.11}$$

That $E[\zeta]<\infty$ is then immediate. To prove the second assertion, assume first that

$$\{t>0:P(\zeta<t)=0\}\neq\emptyset \tag{2.12}$$

and denote by t_0 its largest element. Define then u by $(t_0-u)/t_0=1/2^{|\alpha|}$. Since $u<t_0$, $\zeta\geq u$ a.s. Thus, applying the fragmentation and scaling properties at time u,

$$\zeta=u+\sup_{1\leq i<\infty}F_i^{|\alpha|}(u)\zeta^{(i)},$$

where the $\zeta^{(i)}$ are iid with the same law as ζ and independent of $\mathcal{F}(u)$. In other words, if (2.12) holds, then for all $\varepsilon\in\,]0,t_0-u[$,

$$\prod_i P\left(F_i^{|\alpha|}(u)\zeta^{(i)}\leq t_0-u-\varepsilon\mid\mathcal{F}(u)\right)=P\left(\zeta\leq t_0-\varepsilon\mid\mathcal{F}(u)\right)\overset{a.s.}{=}0. \tag{2.13}$$

To prove the statement, we therefore have to show that (2.13) is false. In that aim, suppose first that

$$P\left(F_1^{|\alpha|}(u)\zeta^{(1)}\leq t_0-u-\varepsilon\mid\mathcal{F}(u)\right)\overset{a.s.}{=}0\text{ for all }\varepsilon\in\,]0,t_0-u[\,. \tag{2.14}$$

By definition of t_0 and u, this implies that a.s. $(t_0-u)/F_1^{|\alpha|}(u) \leq t_0$ and then $F_1(u) \geq 1/2$. Using the connections between homogeneous fragmentations and self-similar ones as explained in Section 2.2, we see that this leads to the existence of a homogeneous fragmentation F^h with dislocation measure ν such that a.s. for all $t \geq 0$, $F_1^h(t) \geq F_1(t)$. In particular, $F_1^h(u) \geq 1/2$ a.s. From Proposition 12 in [9] and its proof, we know the existence of a subordinator σ with Laplace exponent given by (2.4) such that $F_1^h = \exp(-\sigma)$ on $[0,u]$. We then have $\sigma(u) \leq \ln 2$ a.s. However, it is well known that the jump process of σ is a Poisson point process with intensity the Lévy measure of σ and since here this Lévy measure is not trivial and $u>0$, the r.v. $\sigma(u)$ can not have a deterministic upper bound. Thus (2.14) can not be true and for some ε_0 in $]0,t_0-u[$, $P\left(F_1^{|\alpha|}(u)\zeta^{(1)} \leq t_0-u-\varepsilon_0 \mid \mathcal{F}(u)\right) > 0$ with a positive probability. Since $P\left(F_i^{|\alpha|}(u)\zeta^{(i)} \leq t_0-u-\varepsilon_0 \mid \mathcal{F}(u)\right) \nearrow 1$ as $i \nearrow \infty$, this would imply, if (2.13) holds, that the sum

$$\sum_i \left(1-P\left(F_i^{|\alpha|}(u)\zeta^{(i)} \leq t_0-u-\varepsilon_0 \mid \mathcal{F}(u)\right)\right) \tag{2.15}$$

diverges on the event $\left\{P\left(F_1^{|\alpha|}(u)\zeta^{(1)} \leq t_0-u-\varepsilon_0 \mid \mathcal{F}(u)\right) > 0\right\}$, which has positive probability. Yet, this is not possible: by (2.11),

$$\begin{aligned} \textstyle\sum_i P\left(F_i^{|\alpha|}(u)\zeta^{(i)} > t_0-u-\varepsilon_0 \mid \mathcal{F}(u)\right) &\leq A \textstyle\sum_i e^{-B(t_0-u-\varepsilon_0)F_i^{\alpha}(u)} 1_{\{F_i(u)>0\}} \\ &\leq AC \textstyle\sum_i F_i(u) \text{ a.s.,} \end{aligned}$$

where $C := \sup_{0 \leq x < \infty} x^{-1} e^{-B(t_0-u-\varepsilon_0)x^{\alpha}} < \infty$. Since $\sum_i F_i(t) \leq 1$ a.s., the sum (2.15) is then finite a.s. and consequently (2.13) is false.

2.4 Regularity of the mass measure dM

This section is devoted to the study of existence or absence of a Lebesgue density for the mass measure dM of a fragmentation F with parameters α, c and ν satisfying hypothesis (H). More precisely, we give some sufficient conditions on α and ν for the existence of a density in $L^2(dt \otimes dP)$ and some sufficient conditions for the measure dM to be singular a.s. In the sequel, we will often assume [1] that the constant μ introduced in (2.8) is finite, i.e.

$$\mu = \int_{\mathcal{S}^{\downarrow}} \left(\sum_{i=1}^{\infty} |\log(s_i)|\, s_i\right) \nu(ds) = \frac{1}{|\alpha|} E\left[D^{-1}\right] < \infty \tag{A1}$$

and that

$$\int_{\mathcal{S}^{\downarrow}} (1-s_1)^{\beta}\, \nu(ds) < \infty \text{ for some } \beta < 1. \tag{A2}$$

We recall that D is a random variable that corresponds to the first time at which a tagged fragment vanishes and that its distribution is given by (2.6). Here is our main result:

[1] These assumptions $(A1)$ and $(A2)$ hold as soon as $\underline{p} > 0$ ($\underline{p}$ defined by (2.5)). However, it is easy to find some fragmentations for which $\underline{p} = 0$ and $(A1)$ and $(A2)$ hold nonetheless.

Theorem 2.1 *Suppose* $(A1)$.

(i) If $(A2)$ *holds,* $\alpha > -1$ *and* $\int_{\mathcal{S}^{\downarrow}} \sum_{i<j} s_i^{1+\alpha} s_j \nu(ds) < \infty$, *then the measure* dM *is absolutely continuous a.s. and its density belongs to* $L^2(dt \otimes dP)$.

(ii) If $\alpha \leq -1$, *then* dM *is singular a.s.*

In (i), the criterion $\int_{\mathcal{S}^{\downarrow}} \sum_{i<j} s_i^{1+\alpha} s_j \nu(ds) < \infty$ is optimal in the sense that there are some fragmentations satisfying assumptions $(A1)$ and $(A2)$ on ν, with index $\alpha > -1$ and $\int_{\mathcal{S}^{\downarrow}} \sum_{i<j} s_i^{1+\alpha} s_j \nu(ds) = \infty$, and such that dM is not absolutely continuous with a density in $L^2(dt \otimes dP)$. Some illustrating examples are given after the proof of Theorem 2.1 (i).

In the special case where $\nu(s_{N+1} > 0) = 0$ for some given $N \geq 2$ (that is each dislocation gives rise to at most N fragments), note that when $\alpha > -1$,

$$\int_{\mathcal{S}^{\downarrow}} \sum_{i<j} s_i^{1+\alpha} s_j \nu(ds) \leq \int_{\mathcal{S}^{\downarrow}} (N-1) \sum_{2 \leq j \leq N} s_j \nu(ds) \leq (N-1) \int_{\mathcal{S}^{\downarrow}} (1 - s_1)\, \nu(ds) < \infty. \quad (2.16)$$

Both parts of Theorem 2.1 then complement each other and give the following result.

Corollary 2.1 *Assume that* $\nu(s_{N+1} > 0) = 0$ *for some integer* N *and that* $(A1)$ *and* $(A2)$ *hold. Then, with probability one, the measure* dM *is absolutely continuous if and only if* $\alpha > -1$. *When* $\alpha > -1$, *the density of* dM *is in* $L^2(dt \otimes dP)$ *and when* $\alpha \leq -1$, dM *is singular a.s.*

We now turn to the proofs. That of Theorem 2.1 (i) uses Fourier analysis.

Proof of Theorem 2.1 (i). Introduce the Fourier transform of dM, i.e.

$$\widehat{M}(\theta) = \int_0^{\infty} e^{i\theta t} dM(t), \quad \theta \in \mathbb{R}. \quad (2.17)$$

It is well-known that the measure dM is absolutely continuous with a density L in $L^2(dt)$ if and only if the integral $\int_{-\infty}^{\infty} \left|\widehat{M}(\theta)\right|^2 d\theta$ is finite and then that $\int_{-\infty}^{\infty} \left|\widehat{M}(\theta)\right|^2 d\theta = \int_0^{\infty} L^2(t) dt$. Consequently, taking the expected values, dM is absolutely continuous with a density in $L^2(dt \otimes dP)$ if and only if $E\left[\int_{-\infty}^{\infty} \left|\widehat{M}(\theta)\right|^2 d\theta\right]$ is finite. To see when the latter happens, let us first rewrite $\widehat{M}$ in a more convenient way. We know, by (2.7), that the dust's mass can be expressed a.s. as $M(t) = P(D \leq t \mid F)$, $t \geq 0$, where D corresponds to the first time at which a tagged fragment vanishes. In others words, dM is the conditional law of D given F and $\widehat{M}$ can be written as

$$\widehat{M}(\theta) = E\left[e^{i\theta D} \mid F\right], \quad \theta \in \mathbb{R}, \text{ a.s.} \quad (2.18)$$

Dealing with $\left|\widehat{M}(\theta)\right|^2$ suggests then to work with two fragments tagged independently. So, consider U_1 and U_2, two independent random variables uniformly distributed on $]0,1[$ and independent of F, and the corresponding tagged fragments, as explained in Section 2.3.2. Let D_1

(resp. D_2) denote the first time at which the tagged fragment containing U_1 (resp. U_2) vanishes. These random variables are not independent, however they are independent conditionally on F and then, by (2.18),

$$\begin{aligned} E\left[\left|\widehat{M}(\theta)\right|^2\right] &= E\left[E\left[e^{i\theta D_1} \mid F\right] E\left[e^{-i\theta D_2} \mid F\right]\right] \\ &= E\left[e^{i\theta(D_1-D_2)}\right], \quad \theta \in \mathbb{R}. \end{aligned}$$

Recall the notations of Section 2.3.2: T_2 is the first time at which the fragments containing the tagged points U_1 and U_2 are different and $\lambda_1(T_2)$ (resp. $\lambda_2(T_2)$) the mass of the fragment containing U_1 (resp. U_2) at that time T_2. An application of the scaling and strong fragmentation properties at this (randomized) stopping time T_2 leads to the existence of two independent random variables $\widetilde{D}_1$ and $\widetilde{D}_2$, independent of $\mathcal{F}(T_2)$ and $(\lambda_1(T_2), \lambda_2(T_2))$, and with the same distribution as D, such that

$$D_1 = T_2 + \lambda_1^{|\alpha|}(T_2)\widetilde{D}_1 \quad \text{and} \quad D_2 = T_2 + \lambda_2^{|\alpha|}(T_2)\widetilde{D}_2.$$

This yields to

$$E\left[\left|\widehat{M}(\theta)\right|^2\right] = E\left[e^{i\theta\left(\lambda_1^{|\alpha|}(T_2)\widetilde{D}_1 - \lambda_2^{|\alpha|}(T_2)\widetilde{D}_2\right)}\right]. \tag{2.19}$$

Our goal is then to show that the characteristic function of the random variable $\lambda_1^{|\alpha|}(T_2)\widetilde{D}_1 - \lambda_2^{|\alpha|}(T_2)\widetilde{D}_2$ belongs to $L^1(d\theta)$.

To prove this, we use the following result (see [22], p.20): if a function $f \in L^1(dx)$, is bounded in a neighborhood of 0 and has a non-negative Fourier transform $\widehat{f}$, then $\widehat{f} \in L^1(dx)$. We already know that the characteristic function of $\lambda_1^{|\alpha|}(T_2)\widetilde{D}_1 - \lambda_2^{|\alpha|}(T_2)\widetilde{D}_2$ is non-negative, since it is equal to $E\left[\left|\widehat{M}(\theta)\right|^2\right]$. Next, recall that $\widetilde{D}_1, \widetilde{D}_2$ and $(\lambda_1(T_2), \lambda_2(T_2))$ are independent and that D has a bounded density k, according to Lemma 2.1 and assumptions $(A1)$ and $(A2)$. Let C be an upper bound of k. Then, easy calculation shows that the random variable $\lambda_1^{|\alpha|}(T_2)\widetilde{D}_1 - \lambda_2^{|\alpha|}(T_2)\widetilde{D}_2$ has a density f given by

$$f(x) = \int_{x\vee 0}^{\infty} E\left[\lambda_1^{\alpha}(T_2)\lambda_2^{\alpha}(T_2)k\left(u\lambda_1^{\alpha}(T_2)\right)k\left((u-x)\lambda_2^{\alpha}(T_2)\right)\right]du, \ x \in \mathbb{R} \tag{2.20}$$

which is bounded by

$$\begin{aligned} 0 \le f(x) \quad &\le C\int_{x\vee 0}^{\infty} E\left[\lambda_1^{\alpha}(T_2)\lambda_2^{\alpha}(T_2)k\left((u-x)\lambda_2^{\alpha}(T_2)\right)1_{\{\lambda_1(T_2)\ge\lambda_2(T_2)\}}\right]du \\ &+C\int_{x\vee 0}^{\infty} E\left[\lambda_1^{\alpha}(T_2)\lambda_2^{\alpha}(T_2)k\left(u\lambda_1^{\alpha}(T_2)\right)1_{\{\lambda_2(T_2)\ge\lambda_1(T_2)\}}\right]du. \end{aligned}$$

The first integral is bounded from above by $E\left[\lambda_1^{\alpha}(T_2)1_{\{\lambda_1(T_2)\ge\lambda_2(T_2)\}}\right]$ (recall that $\int_0^{\infty} k(v)dv = 1$) and the second one by $E\left[\lambda_2^{\alpha}(T_2)1_{\{\lambda_2(T_2)\ge\lambda_1(T_2)\}}\right]$. These two expectations are equal. By applying Lemma 2.2 to $a_1 = |\alpha|$ and $a_2 = 0$, we see that there are finite as soon as $\alpha > -1$ and $\int_{\mathcal{S}^{\downarrow}} \sum_{i<j} s_i^{1+\alpha} s_j \nu(ds) < \infty$. Therefore f is bounded and the function $\theta \in \mathbb{R} \mapsto \widehat{f}(\theta) = E\left[\left|\widehat{M}(\theta)\right|^2\right]$ belongs to $L^1(d\theta)$. ■

Some examples. Let us now give some examples of fragmentation processes with parameters α, ν satisfying assumptions $(A1)$, $(A2)$, such that $\alpha > -1$ and $\int_{\mathcal{S}^{\downarrow}} \sum_{i<j} s_i^{1+\alpha} s_j \nu(ds) = \infty$, and such that the mass measure dM does not have a density in $L^2(dt \otimes dP)$. Specifically, fix $\alpha > -1$ and consider the dislocation measure

$$\nu(ds) = \sum_{n \geq 1} a_n \delta_{\Big(\underbrace{n^{-1}, n^{-1}, ..., n^{-1}}_{n \text{ times}}, 0, ...\Big)}(ds),$$

where $(a_n)_{n \geq 1}$ is a sequence of non-negative real numbers such that

$$\textstyle\sum_{n \geq 1} a_n \ln n < \infty \quad \text{and} \quad \sum_{n \geq 1} a_n n^{|\alpha|} = \infty.$$

The assumption $\sum_{n \geq 1} a_n \ln n < \infty$ leads both to the integrability of $\sum_{i \geq 1} |\log(s_i)| s_i$ with respect to ν and to the finiteness of $\int_{\mathcal{S}^{\downarrow}} (1 - s_1)^{\beta} \nu(ds)$ for $\beta \geq 0$. Hence both assumptions $(A1)$ and $(A2)$ are satisfied. The assumption $\sum_{n \geq 1} a_n n^{|\alpha|} = \infty$ implies $\int_{\mathcal{S}^{\downarrow}} \sum_{i<j} s_i^{1+\alpha} s_j \nu(ds) = \infty$ and this in turn will imply that dM has no density in $L^2(dt \otimes dP)$. To see this, note that the measure ν is constructed so that when a fragment splits, it splits into n fragments with same masses for some $1 \leq n < \infty$. Combined with (2.19), this remarks leads to

$$E\left[\left|\widehat{M}(\theta)\right|^2\right] = E\left[e^{i\theta\lambda_1^{|\alpha|}(T_2)\left(\tilde{D}_1 - \tilde{D}_2\right)}\right] = E\left[\left|\psi_D(\theta\lambda_1^{|\alpha|}(T_2))\right|^2\right],$$

where ψ_D denotes the characteristic function of D. This characteristic function is in $L^2(dx)$, since the density k of the law of D is in $L^2(dx)$ (see Lemma 2.1). Hence $\int_{-\infty}^{\infty} E\left[\left|\widehat{M}(\theta)\right|^2\right] d\theta$ is finite if and only if $E\left[\lambda_1^{\alpha}(T_2)\right] = E\left[\lambda_1^{\alpha}(T_2) 1_{\{\lambda_1(T_2) \geq \lambda_2(T_2)\}}\right]$ is finite. And according to Lemma 2.2, this last expectation is infinite when $\int_{\mathcal{S}^{\downarrow}} \sum_{i<j} s_i^{1+\alpha} s_j \nu(ds) = \infty$, which is the case here. Therefore, $\int_{-\infty}^{\infty} E\left[\left|\widehat{M}(\theta)\right|^2\right] d\theta$ is infinite and dM cannot be absolutely continuous with a density in $L^2(dt \otimes dP)$.

The proof of Theorem 2.1 (ii) relies essentially on the following lemma:

Lemma 2.4 *If $\alpha \leq -1$, for a.e. t, the number of fragments with positive mass present at time t is finite a.s.*

This has already been proved in the last section of [15] for $\alpha < -1$ and extends to $\alpha \leq -1$ as follows.

Proof. For fixed time t, by applying the fragmentation and scaling properties at that time, we see that we can rewrite the differences $M(t + \varepsilon) - M(t)$, $\varepsilon > 0$, as

$$M(t + \varepsilon) - M(t) = \sum_i F_i(t) 1_{\{F_i(t) > 0\}} M^{(i)}(\varepsilon F_i(t)^{\alpha}), \quad \text{for all } \varepsilon > 0, \tag{2.21}$$

where the processes $M^{(i)}$ are mutually independent and independent of $\mathcal{F}(t)$, and have the same law as M. Let then $\zeta^{(i)}$, $i \geq 1$, denote the first time at which the dust's mass $M^{(i)}$ reaches 1 and remark that for all $a > 0$,

$$M(t+\varepsilon) - M(t) \geq \sum_i F_i(t) 1_{\left\{0<F_i(t)^{|\alpha|}\leq \varepsilon/a\right\}} 1_{\left\{\zeta^{(i)}\leq a\right\}}, \quad \varepsilon > 0. \tag{2.22}$$

The Lebesgue differentiation theorem implies that a.s., for a.e. t, $\lim_{\varepsilon\to 0} \left(M(t+\varepsilon) - M(t)\right)/\varepsilon$ exists and is finite. By Fubini's theorem, the order of "almost surely" and "for almost every t" can be exchanged and therefore, for a.e. t, there exists a finite r.v. $L(t)$ such that

$$\frac{M(t+\varepsilon) - M(t)}{\varepsilon} \underset{\varepsilon\to 0}{\overset{a.s.}{\to}} L(t). \tag{2.23}$$

For such a time t, denote by E_t the event

"the number of macroscopic fragments at time t is infinite"

and take ω in E_t such that (2.23) holds. Given a positive a, we introduce the (random) sequence $\varepsilon_n = aF_n(t)(\omega)$. Since $|\alpha| \geq 1$ and $\varepsilon_n > 0$ for all $n \geq 1$, we deduce from (2.22) (ω being dropped from notations) that

$$\begin{aligned} L(t) &\geq \tfrac{1}{a} \limsup_{n\to\infty} \tfrac{1}{F_n(t)} \sum_i F_i(t) 1_{\{0<F_i(t)\leq F_n(t)\}} 1_{\left\{\zeta^{(i)}\leq a\right\}} \\ &\geq \tfrac{1}{a} \limsup_{n\to\infty} 1_{\left\{\zeta^{(n)}\leq a\right\}}. \end{aligned}$$

By Lemma 2.3, $P(\zeta^{(1)} \leq a) > 0$ and then, since the $\zeta^{(n)}$ are iid,

$$\limsup_{n\to\infty} 1_{\left\{\zeta^{(n)}\leq a\right\}} = 1 \text{ a.s.}$$

This holds for every $a > 0$. In other words, for a.e. $\omega \in E_t$, $L(t)(\omega) = \infty$. But $L(t) < \infty$ a.s, and so $P(E_t) = 0$.

Proof of Theorem 2.1 (ii). According to Proposition 1.9, Chapter 1, $M(\varepsilon)/\varepsilon \overset{\text{a.s.}}{\to} 0$ as $\varepsilon \to 0$. So, if t is a time such that the number of fragments with positive mass present at that time is a.s. finite, one sees with formula (2.21) that

$$\frac{M(t+\varepsilon) - M(t)}{\varepsilon} \overset{\text{a.s.}}{\to} 0 \text{ as } \varepsilon \to 0.$$

According to the previous lemma, this holds for a.e. $t \geq 0$ when $\alpha \leq -1$, and this implies the a.s. singularity of dM, by the Lebesgue differentiability theorem. ■

2.5 Approximation of the density

When the mass measure dM of some (α, ν)-fragmentation F (satisfying hypothesis (H)) possesses a Lebesgue density, a question that naturally arises, is to know if, as in the Brownian excursion fragmentation discussed in the Introduction, this density can be approximated by functions of small fragments. In most cases, the answer is positive. To see this, introduce for $t \geq 0$ and $\varepsilon > 0$

$$M(t,\varepsilon) := \sum_{i\geq 1} F_i(t) 1_{\{0<F_i(t)\leq \varepsilon\}},$$

the total mass at time t of macroscopic fragments with mass at most ε, and

$$N(t,\varepsilon) := \sum_{i\geq 1} 1_{\{F_i(t)>\varepsilon\}}$$

the number of fragments present at time t with mass greater than ε. We then have:

Theorem 2.2 *Consider a dislocation measure ν such that $(A1)$ holds and suppose that*

(a) the mass measure dM is absolutely continuous with a density L in $L^p(dx \otimes dP)$ for some $p > 1$,

(b) the fragmentation is not geometric, i.e. there exists no $r > 0$ such that the mass of every fragment at every time t belongs to the set $\{e^{-kr} : k \in \mathbb{N}\}$.

Then, for a.e. t,

$$\varepsilon^{\alpha} M(t,\varepsilon) \underset{\varepsilon\to 0}{\overset{a.s.}{\to}} L(t)/\left|\alpha\right| \mu$$

and

$$\varepsilon^{1+\alpha} N(t,\varepsilon) \underset{\varepsilon\to 0}{\overset{a.s.}{\to}} L(t)\left(1-\left|\alpha\right|\right)/\left|\alpha\right|^2 \mu.$$

The assumptions (a) and (b) are not so restrictive. First, recall that Theorem 2.1 (i), Section 2.4, gives sufficient conditions for the mass measure to have a density in $L^2(dx \otimes dP)$. Next, concerning assumption (b), it is easy to see that the fragmentation is not geometric as soon as $\nu\left(\mathcal{S}^{\downarrow}\right) = \infty$. This is a consequence of Corollary 24.6 in [64] and its proof (to see this, consider the subordinator ξ introduced in Section 2.3 and note that its Lévy measure is finite if and only if ν is finite).

To prove Theorem 2.2, we need the following lemma and the Wiener-Pitt Tauberian Theorem, which is recalled just after the proof of the Lemma.

Lemma 2.5 *Let D be a r.v. independent of F, with the same distribution as the first time of vanishing of a tagged fragment (given by (2.6)). If the mass measure dM is absolutely continuous with a density L in $L^p(dx \otimes dP)$ for some $p > 1$, then for a.e. t,*

$$\lim_{\varepsilon\to 0} \varepsilon^{\alpha} E\left[M(t,\varepsilon D^{-1/|\alpha|}) \mid F\right] \overset{a.s.}{=} L(t). \tag{2.24}$$

Proof. As in the proof of Lemma 2.4, we rewrite the difference $M(t+\varepsilon) - M(t)$, as

$$M(t+\varepsilon) - M(t) = \sum_i F_i(t) 1_{\{F_i(t)>0\}} \left(M^{(i)}(\varepsilon F_i(t)^{\alpha})\right), \text{ for all } \varepsilon > 0, \tag{2.25}$$

where the processes $M^{(i)}$ are independent copies of M and independent of $\mathcal{F}(t)$. If D denotes a r.v. independent of F and with the same distribution as (2.6), we get from (2.7) that $E[M(s)] = P(D \leq s)$, for $s \geq 0$, and then that

$$E\left[M^{(i)}(\varepsilon F_i(t)^\alpha) \mid \mathcal{F}(t)\right] \overset{a.s.}{=} P(D \leq \varepsilon F_i(t)^\alpha \mid \mathcal{F}(t)) \overset{a.s.}{=} P(D \leq \varepsilon F_i(t)^\alpha \mid F), \quad i \geq 1.$$

Hence, almost surely,

$$\begin{aligned} E\left[M(t+\varepsilon) - M(t) \mid \mathcal{F}(t)\right] &= \textstyle\sum_i F_i(t) 1_{\{F_i(t)>0\}} P(D \leq \varepsilon F_i(t)^\alpha \mid F) \\ &= E\left[\textstyle\sum_i F_i(t) 1_{\left\{0 < F_i(t)^{|\alpha|} \leq \varepsilon D^{-1}\right\}} \mid F\right] \\ &= E\left[M(t, \varepsilon^{1/|\alpha|} D^{-1/|\alpha|}) \mid F\right]. \end{aligned} \tag{2.26}$$

For a.e. t, $(M(t+\varepsilon) - M(t))/\varepsilon$ converges to $L(t)$ as $\varepsilon \to 0$, L being the density of dM. Since this density is supposed to belong to $L^p(dx \otimes dP)$ for some $p > 1$, we may apply the maximal inequality of Hardy-Littlewood (see e.g. [65], p.5), which yields

$$\int_0^\infty \sup_{\varepsilon > 0} \left(\frac{M(t) - M(t+\varepsilon)}{\varepsilon} \right)^p dt \leq C \int_0^\infty L^p(t) dt$$

for some deterministic constant C. Then, for a.e. t, the r.v. $\sup_{\varepsilon>0} (M(t+\varepsilon) - M(t))/\varepsilon$ has a moment of order p and the dominated convergence theorem can be applied in the left-hand side of (2.26). Therefore, for a.e. t,

$$\lim_{\varepsilon \to 0} \varepsilon^\alpha E\left[M(t, \varepsilon D^{-1/|\alpha|}) \mid F\right] \overset{a.s.}{=} E\left[L(t) \mid \mathcal{F}(t)\right] \overset{a.s.}{=} L(t),$$

since $L(t)$ is $\mathcal{F}(t)$ -measurable, $\mathcal{F}$ being a right-continuous filtration. This right-continuity of $\mathcal{F}$ is a classical consequence of the Feller property of F (proved in [9]).

The following Wiener-Pitt Tauberian Theorem is proved in [21], on page 227. We recall that a function g with values in $\mathbb{R}$ is said to be slowly decreasing if

$$\lim_{\lambda \searrow 1} \lim_{x \to \infty} \inf \inf_{l \in [1,\lambda]} (g(lx) - g(x)) \geq 0.$$

Hence a slowly decreasing function is a function whose decrease, if any, is slow. As example, an increasing function is slowly decreasing.

Theorem 2.3 *(Wiener-Pitt) Consider $f, g : (0, \infty) \to \mathbb{R}$ and let $\check{f}(z) := \int_0^\infty t^z f(1/t) dt/t$ for $z \in \mathbb{C}$ such that the integral converges. If $\check{f}(z)$ exists and is non-zero for $Re(z) = 0$ and if g is bounded, measurable and slowly decreasing, then*

$$\int_0^\infty f(x/t) g(t) dt/t \underset{x \to \infty}{\to} c\check{f}(0)$$

implies

$$g(x) \underset{x \to \infty}{\to} c.$$

By definition, a function g is slowly increasing if $(-g)$ is slowly decreasing. The Wiener-Pitt Theorem thus remains valid for slowly increasing functions g.

Proof of Theorem 2.2. Let us start with the convergence of $\varepsilon^a M(t,\varepsilon)$ as $\varepsilon \to 0$. In that aim, consider D a r.v. independent of F and with the same distribution as the first time of vanishing of a tagged fragment and fix $t \geq 0$ such that (2.24) holds. Then set

$$f(x) := k(1/x),\ x \in (0,\infty) \quad (k \text{ is the density of } D)$$

and

$$g(x) := xM(t, x^{-1/|\alpha|}),\ x \in (0,\infty)\,,$$

(g is a random function). The convergence (2.24) is equivalent to

$$\int_0^\infty f(x/u)g(u)du/u \underset{x\to\infty}{\overset{a.s.}{\to}} L(t),$$

so that, provided that the Wiener-Pitt Theorem applies,

$$g(x) \underset{x\to\infty}{\overset{a.s.}{\to}} L(t)/\check{f}(0).$$

This is equivalent to $\varepsilon^\alpha M(t,\varepsilon) \underset{\varepsilon\to 0}{\overset{a.s.}{\to}} L(t)/\left|\alpha\right|\mu$, since $\check{f}(0) = \int_0^\infty k(t)dt/t = E\left[D^{-1}\right] = \left|\alpha\right|\mu$ (by (2.8)). Thus, we just have to check that f and g satisfy the assumptions of the Wiener-Pitt Theorem.

Consider first f. For every x in $\mathbb{R}$, $\check{f}(ix) = E\left[D^{ix-1}\right]$ exists since $E\left[D^{-1}\right]$ is finite. We would like to show that $E\left[D^{ix-1}\right]$ is non-zero for all $x \in \mathbb{R}$. When $x = 0$, $E\left[D^{-1}\right] > 0$ since D is a positive random variable. Now for $x \neq 0$, consider the subordinator ξ introduced in Section 2.3.2 and related to the law of D by (2.6). As a consequence of assumption (b), the Lévy measure π_α of the subordinator $\left|\alpha\right|\xi$ is not supported by a set $r\mathbb{N}$, for some $r > 0$, so that the characteristic exponent $\psi(x) = \int_0^\infty (1-e^{ixu})\pi_\alpha(du)$ of this subordinator is non-zero when $x \neq 0$. Then, following the proof of Proposition 3 in [25], we get that $E\left[D^{ix-1}\right] = E\left[D^{ix}\right]\psi(x)/ix$ for $x \neq 0$. Thus we just have to prove that $E\left[D^{ix}\right]$ is non-zero. We know ([18]) that there exists a random variable R, independent of D, such that $DR \overset{law}{=} e$ where e denotes the exponential r.v. with parameter 1. Therefore,

$$E\left[D^{ix}\right]E\left[R^{ix}\right] = \int_0^\infty t^{ix}e^{-t}dt.$$

This last integral is equal to $\Gamma(1+ix)$, Γ being the analytic continuation of the Gamma function, and it is well-known (see e.g. [6]) that $\Gamma(z) \neq 0$ for all z in the complex plane. Thus $E\left[D^{ix}\right]$ is non-zero.

Now consider the function g. Since $x \mapsto M(t,x)$ is non-decreasing, g is bounded from above by

$$xE\left[M(t, x^{-1/|\alpha|}D^{-1/|\alpha|})1_{\{D\leq 1\}} \mid F\right]/P(D \leq 1),$$

which is a.s. bounded on $\mathbb{R}_+^*$ (by (2.24) and since $P(D \leq 1) \geq P(\zeta \leq 1) > 0$ by Lemma 2.3). The function $x \mapsto M(t,x)$ is a limit of step functions, thus it is measurable and g is measurable. It remains to show that g is slowly increasing, that is

$$\lim_{\lambda \searrow 1} \lim \inf_{x \to \infty} \inf_{l \in [1,\lambda]} (g(x) - g(lx)) \geq 0.$$

We have that

$$g(x) - g(lx) = x(1-l)M(t, x^{-1/|\alpha|}) + lx\left(M(t, x^{-1/|\alpha|}) - M(t, (lx)^{-1/|\alpha|})\right).$$

For all $l \geq 1$, the second term in the right-hand side of this identity is non-negative, which leads to

$$\inf_{l \in [1,\lambda]} (g(x) - g(lx)) \geq (1-\lambda)g(x).$$

Now, since g is a.s. bounded, there exists a positive random constant C such that a.s.

$$\lim \inf_{x \to \infty} \inf_{l \in [1,\lambda]} (g(x) - g(lx)) \geq C(1-\lambda),$$

and finally,

$$\lim_{\lambda \searrow 1} \lim \inf_{x \to \infty} \inf_{l \in [1,\lambda]} (g(x) - g(lx)) \geq 0.$$

The Wiener-Pitt Theorem therefore applies to f and g and the convergence of $\varepsilon^{\alpha} M(t,\varepsilon)$ to $L(t)/|\alpha|\,\mu$ as $\varepsilon \to 0$ is proved.

The last point to show, is the a.s. convergence of $\varepsilon^{1+\alpha} N(t,\varepsilon)$ to $L(t)\,(1-|\alpha|)\,/\,|\alpha|^2\,\mu$ as $\varepsilon \to 0$. Bertoin's proof, p.4. in [16], which relies on Abelian-Tauberian theorems, adapts easily here to give

$$N(t,\varepsilon) \underset{\varepsilon \to 0}{\sim} \left(\frac{1-|\alpha|}{|\alpha|}\right) \frac{M(t,\varepsilon)}{\varepsilon}. \tag{2.27}$$

The asymptotic behavior of $N(t,\varepsilon)$ as $\varepsilon \to 0$ can then be deduced from that of $M(t,\varepsilon)$. ■

Some remarks on small fragments behavior. Theorem 2.2 shows that for most of fragmentations with an index of self-similarity in $]-1,0[$, the small fragments functions $\varepsilon^{\alpha} M(t,\varepsilon)$ and $\varepsilon^{1+\alpha} N(t,\varepsilon)$ converge, for a.e. fixed time t, to non-degenerate limits as $\varepsilon \to 0$. Moreover, for negative-index fragmentations that are not taken into account in Theorem 2.2, one can see[2]

[2] With the notations of the proof of Lemma 2.4 and using (2.22) and (2.23), one gets that for a.e. t,

$$\sup_{\varepsilon > 0} \frac{1}{\varepsilon} \sum_i F_i(t) 1_{\{0 < F_i(t)^{|\alpha|} \leq \varepsilon/a\}} 1_{\{\zeta^{(i)} \leq a\}} \text{ is a.s. finite for all } a > 0.$$

Consider then $a_{1/2}$ such that $P\left(\zeta^{(1)} \leq a_{1/2}\right) \geq 1/2$. Since the r.v. $\zeta^{(i)}$ are iid and independent of $\mathcal{F}(t)$,

$$\begin{aligned} &P\left(\sup_{\varepsilon>0} \tfrac{1}{\varepsilon} \sum_i F_i(t) 1_{\{0<F_i(t)^{|\alpha|} \leq \varepsilon\}} 1_{\{\zeta^{(i)} > a_{1/2}\}} < \infty\right) \\ &\geq P\left(\sup_{\varepsilon>0} \tfrac{1}{\varepsilon} \sum_i F_i(t) 1_{\{0<F_i(t)^{|\alpha|} \leq \varepsilon\}} 1_{\{\zeta^{(i)} \leq a_{1/2}\}} < \infty\right) = 1. \end{aligned}$$

By taking the sum, we see that $\varepsilon^{\alpha} M(t,\varepsilon)$ is a.s. bounded for t such that (2.23) holds and so does $\varepsilon^{1+\alpha} N(t,\varepsilon)$ in view of equivalence (2.27).

that for a.e. $t \geq 0$, $\varepsilon^{\alpha} M(t,\varepsilon)$ and $\varepsilon^{1+\alpha} N(t,\varepsilon)$ are anyway bounded a.s. When $\alpha \leq -1$, we more precisely have that $M(t,\varepsilon) = 0$ and $N(t,\varepsilon)$ is constant for ε small enough, almost surely and for almost every t (it is Lemma 2.4).

This completes in some way the discussion on the asymptotic behavior of $M(t,\varepsilon)$ and $N(t,\varepsilon)$ as $\varepsilon \to 0$ undertaken by Bertoin in [16] for fragmentations with a positive index of self-similarity. The investigating methods (and the results) are completely different according whether the index of self-similarity is positive or negative. The positive case relies on a martingale approach (that cannot be shifted to the negative case) and gives, with suitable assumptions on ν, that

$$M(t,\varepsilon) \underset{\varepsilon \to 0}{\overset{a.s.}{\sim}} C(t,\omega) f(\varepsilon) \text{ and } N(t,\varepsilon) \underset{\varepsilon \to 0}{\overset{a.s.}{\sim}} C(t,\omega) C f(\varepsilon)/\varepsilon$$

for some constants $C(t,\omega)$, C and where $f(\varepsilon) = \int_{\mathcal{S}^{\downarrow}} \sum_i s_i 1_{\{s_i < \varepsilon\}} \nu(ds)$. Note that this function depends on ν but not on α, whereas in the negative case the convergence rate depends only on α.

Another remark when $\alpha < 0$ and $(A1)$ holds is that the measure dM is singular if and only if $\varepsilon^{\alpha} M(t,\varepsilon) \overset{a.s.}{\to} 0$ for a.e t. To see this, combine equations (2.22) and (2.26).

2.6 Hausdorff dimension and Hölder-continuity

When the measure dM is singular, it may be interesting to estimate the "size" of the support of dM (denoted here by $\mathrm{supp}(dM)$), which is the smallest closed set C of $\mathbb{R}_+$ such that $dM(\mathbb{R}_+ \backslash C) = 0$. An appropriate concept is then that of *Hausdorff dimension:*

$$\dim_H(E) := \inf\{\gamma > 0 : m_{\gamma}(E) = 0\}, \qquad E \subset \mathbb{R}_+, \tag{2.28}$$

where

$$m_{\gamma}(E) := \sup_{\varepsilon > 0} \inf \sum_i |B_i|^{\gamma}, \tag{2.29}$$

the infimum being taken over all collections of intervals with length $|B_i| < \varepsilon$, whose union covers E. For background on the subject, see e.g. [33]. In Subsection 2.6.1, we give some lower and upper bounds for $\dim_H(\mathrm{supp}(dM))$ and $\dim_H(dM)$, the latter being defined as

$$\dim_H(dM) := \inf\{\dim_H(E) : dM(E) = 1\}.$$

That $\dim_H(dM) \leq \dim_H(\mathrm{supp}(dM))$ holds anyway and we show below that when $\nu\left(\mathcal{S}^{\downarrow}\right) = \infty$ and $\alpha < -1$, these dimensions are different.

It is well known, since the dust's mass M is the distribution function of dM, that the Hausdorff dimension of dM is connected to the Hölder-continuity of M, in the sense that $\dim_H(dM) \geq \gamma$ as soon as M is Hölder-continuous of order γ. Subsection 2.6.2 is devoted to this Hölder-continuity of the mass.

For the sequel, we recall that $\underline{p}$ is defined as

$$\underline{p} = \sup\left\{q : \int_{\mathcal{S}^{\downarrow}} \sum_{i \geq 2} s_i^{1-q} \nu(ds) < \infty\right\}$$

and set

$$A := \sup\left\{a \leq 1 : \int_{\mathcal{S}^{\downarrow}} \sum_{i<j} s_i^{1-a} s_j \nu(ds) < \infty\right\}.$$

Remark that $0 \leq \underline{p} \leq A \leq 1$.

2.6.1 Hausdorff dimensions of dM and supp(dM)

Recall that ζ denotes the first time at which all the initial mass is reduced to dust, so that $\text{supp}(dM) \subset [0, \zeta]$.

Proposition 2.1 *(i) If* $(A1)$ *and* $(A2)$ *hold, then* $\dim_H(dM) \geq 1 \wedge (A/|\alpha|)$ *a.s.*

(ii) A.s., $\dim_H(dM) \leq 1 \wedge (1/|\alpha|)$.

(iii) If $\nu(\mathcal{S}^{\downarrow}) < \infty$, *then* $\dim_H(\mathit{supp}(dM)) \leq 1 \wedge (1/|\alpha|)$ *a.s.*

(iv) If $\nu(\mathcal{S}^{\downarrow}) = \infty$, *then the mass* M *is strictly increasing on* $[0, \zeta]$ *and* $\dim_H(\mathit{supp}(dM)) = 1$ *a.s.*

Let us make two remarks about these results. First, the difference between the above statements (iii) and (iv), can mainly be explained by the Poisson point process construction of homogeneous fragmentations (see [13] and [9]) and the passage from homogeneous to self-similar fragmentations. Indeed, this construction shows that when ν is finite the notion of "first splitting" is well-defined and that it occurs at an exponential time T with parameter $\nu(\mathcal{S}^{\downarrow})$, so that M is null near 0, whereas when ν is infinite the splitting times are dense in $\mathbb{R}_+$. This will be a key point in the proofs below.

Second, the parameter $A = 1$ as soon as $\nu(s_{N+1} > 0) = 0$ for some integer N (this was shown in (2.16)). Hence in that case, if moreover assumptions $(A1)$ and $(A2)$ hold, the results (i) and (ii) above give

$$\dim_H(dM) = 1 \wedge (1/|\alpha|) \text{ a.s.}$$

We now turn to the proofs. The upper bound stated in Proposition 2.1 (ii) was recently shown in [40] and we refer to this paper for the proof. Concerning statement (i), it is a standard result (see e.g. Theorem 4.13 of Falconer [33]) that the convergence of $\int_0^{\infty}\int_0^{\infty} |u-v|^{-a}\, dM(u)dM(v)$ for some real number $a \leq 1$ leads to $\dim_H(dM) \geq a$. Thus, the proof of Proposition 2.1 (i) is an immediate consequence of the following lemma:

Lemma 2.6 *Consider a positive real number* a *and suppose that assumptions* $(A1)$ *and* $(A2)$ *hold. Then*

$$E\left[\int_0^{\infty}\int_0^{\infty} \frac{dM(u)dM(v)}{|u-v|^a}\right] < \infty \Leftrightarrow a < 1 \wedge (A/|\alpha|).$$

We point out that the implication $\Rightarrow$ does not take into account the assumptions $(A1)$ and $(A2)$.

Proof. Using the same notations as in the proof of Theorem 2.1 (i), we have that

$$E\left[\int_0^\infty \int_0^\infty |u-v|^{-a}\, dM(u)dM(v)\right] = E\left[|D_1-D_2|^{-a}\right] = E\left[\left|\lambda_1^{|\alpha|}(T_2)\widetilde{D}_1 - \lambda_2^{|\alpha|}(T_2)\widetilde{D}_2\right|^{-a}\right]. \tag{2.30}$$

Suppose first that $a < 1\wedge(A/|\alpha|)$. By assumptions $(A1)$ and $(A2)$ and Lemma 2.1, we know that D has a density k such that $k(x)$ and $xk(x)$ are bounded on $\mathbb{R}_+^*$, say by C and C' and then that $\lambda_1^{|\alpha|}(T_2)\widetilde{D}_1 - \lambda_2^{|\alpha|}(T_2)\widetilde{D}_2$ has a density f (see (2.20) for an explicit expression). Our goal is to prove that $\int_{-\infty}^\infty |\theta|^{-a} f(\theta)d\theta$ is finite. From (2.20), we get that

$$\begin{aligned}&\textstyle\int_0^\infty \theta^{-a} f(\theta)d\theta \\ &\leq \textstyle\int_0^\infty \theta^{-a}\int_0^\infty E\left[\lambda_1^\alpha(T_2)\lambda_2^\alpha(T_2)k\left((u+\theta)\,\lambda_1^\alpha(T_2)\right)k\left(u\lambda_2^\alpha(T_2)\right)1_{\{\lambda_1(T_2)\geq\lambda_2(T_2)\}}\right]dud\theta \\ &+C\textstyle\int_0^\infty\theta^{-a}\int_\theta^\infty E\left[\lambda_1^\alpha(T_2)\lambda_2^\alpha(T_2)k\left(u\lambda_1^\alpha(T_2)\right)1_{\{\lambda_2(T_2)\geq\lambda_1(T_2)\}}\right]dud\theta.\end{aligned} \tag{2.31}$$

By Fubini's Theorem, the second term in the right-hand side of this inequality is proportional to

$$\left(\int_0^\infty u^{1-a}k(u)du\right)E\left[\lambda_1^{|\alpha|(1-a)}(T_2)\lambda_2^\alpha(T_2)1_{\{\lambda_2(T_2)\geq\lambda_1(T_2)\}}\right],$$

which is finite. Indeed, recall that D has positive moments of all orders and remark that the expectation is bounded from above by $E\left[\lambda_2^{\alpha a}(T_2)1_{\{\lambda_2(T_2)\geq\lambda_1(T_2)\}}\right]$, which is finite by Lemma 2.2, as $a\,|\alpha| < A \leq 1$. Next, in order to bound the first term in the right-hand side of (2.31), remark that

$$\int_0^\infty \theta^{-a}k\left((u+\theta)\,\lambda_1^\alpha(T_2)\right)\lambda_1^\alpha(T_2)d\theta = (\lambda_1(T_2))^{a\alpha}\int_0^\infty \theta^{-a}k\left(\theta+u\lambda_1^\alpha(T_2)\right)d\theta.$$

Using the upper bounds C of $k(x)$ and C' of $xk(x)$, one gets

$$\int_0^\infty \theta^{-a}k\left(\theta+u\lambda_1^\alpha(T_2)\right)d\theta \leq C\int_0^1\theta^{-a}d\theta + C'\int_1^\infty \theta^{-a-1}d\theta < \infty$$

and so, the first term in the right-hand side of (2.31) is bounded from above by

$$E\left[(\lambda_1(T_2))^{a\alpha}\,\lambda_2^\alpha(T_2)1_{\{\lambda_1(T_2)\geq\lambda_2(T_2)\}}\int_0^\infty k\left(u\lambda_2^\alpha(T_2)\right)du\right]$$

multiplied by a finite constant. Since $\lambda_2^\alpha(T_2)\int_0^\infty k\left(u\lambda_2^\alpha(T_2)\right)du = 1$, this expectation is bounded by $E\left[(\lambda_1(T_2))^{a\alpha}1_{\{\lambda_1(T_2)\geq\lambda_2(T_2)\}}\right]$, which is finite, according to Lemma 2.2 and the assumption on a. All this shows that $\int_0^\infty\theta^{-a}f(\theta)d\theta<\infty$ and then that $\int_{-\infty}^\infty|\theta|^{-a}f(\theta)d\theta<\infty$ since the random variable $\lambda_1^{|\alpha|}(T_2)\widetilde{D}_1-\lambda_2^{|\alpha|}(T_2)\widetilde{D}_2$ is symmetric.

To prove the converse implication, first note that

$$\begin{aligned}E\left[\left|\lambda_1^{|\alpha|}(T_2)\widetilde{D}_1-\lambda_2^{|\alpha|}(T_2)\widetilde{D}_2\right|^{-a}\right] &\geq E\left[1_{\{\lambda_1(T_2)\geq\lambda_2(T_2)\},\{\widetilde{D}_1\geq\widetilde{D}_2\}}\left|\lambda_1^{|\alpha|}(T_2)\widetilde{D}_1-\lambda_2^{|\alpha|}(T_2)\widetilde{D}_2\right|^{-a}\right] \\ &\geq E\left[1_{\{\lambda_1(T_2)\geq\lambda_2(T_2)\}}\lambda_1^{a\alpha}(T_2)\right]E\left[1_{\{\widetilde{D}_1\geq\widetilde{D}_2\}}\widetilde{D}_1^{-a}\right],\end{aligned}$$

since $(\lambda_1(T_2),\lambda_2(T_2))$ and $\left(\widetilde{D}_1,\widetilde{D}_2\right)$ are independent. Therefore, by identity (2.30),

$$E\left[\int_0^\infty\int_0^\infty|u-v|^{-a}\,dM(u)dM(v)\right]<\infty\Rightarrow E\left[1_{\{\lambda_1(T_2)\geq\lambda_2(T_2)\}}\lambda_1^{a\alpha}(T_2)\right]<\infty,$$

which is, by Lemma 2.2 and the definition of A, equivalent to $a < (A/|\alpha|)$. On the other hand, one can show that $v \mapsto \int_0^\infty |u-v|^{-a} dM(u) = \infty$ on

$$V = \left\{ v > 0 : \limsup_{\varepsilon \to 0} \varepsilon^{-a} \left(M(v+\varepsilon) - M(v-\varepsilon) \right) > 0 \right\}$$

and the Lebesgue theory implies $dM(V) = 1$ when $a \geq 1$. Hence $\int_0^\infty \int_0^\infty |u-v|^{-a} dM(u) dM(v) = \infty$ when $a \geq 1$.

Proof of Proposition 2.1 (iii). Consider an interval representation I of the fragmentation as explained in Section 2.2 and denote by ζ_x, $x \in]0,1[$, the time at which the fragment containing x vanishes, that is $\zeta_x = \inf\{t > 0 : x \notin I(t)\}$. Then set

$$\mathcal{A} := \{\zeta_x, x \in]0,1[\}.$$

By formula (2.7), $M(t) = P(D \leq t \mid F)$ for all $t \geq 0$ a.s., and since D is the first time at which a tagged fragment vanishes, we have $M(t) = \int_0^1 1_{\{\zeta_x \leq t\}} dx$, $t \geq 0$. Then the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ contains the support of the measure dM and it is sufficient to bound from above $\dim_H \left(\overline{\mathcal{A}}\right)$. Since $\nu(S^{\downarrow}) < \infty$, we may consider the first splitting time, denoted by T. It is a stopping time. Let $J_1, J_2, ...$ denote the non-empty disjoint intervals obtained after this first split so that $F_1(T) \geq F_2(T) \geq ...$ are their respective sizes and remark that

$$\mathcal{A} = \{T\} \bigcup_i \{\zeta_x, x \in J_i\}.$$

We first need to prove that

$$\overline{\mathcal{A}} = \{T\} \bigcup_i \overline{\{\zeta_x, x \in J_i\}}. \tag{2.32}$$

To that end, take a in $\overline{\cup_i \{\zeta_x, x \in J_i\}}$ and consider a sequence (x_n) in $\cup_i J_i$ such that $\zeta_{x_n} \to a$. Extracting a subsequence if necessary, we may assume that (x_n) converges. Call x its limit and J_{x_n} the interval that contains x_n, $n \geq 1$. Either $|J_{x_n}| \nrightarrow 0$ as $n \to \infty$ and then there is a subsequence $\left(x_{\varphi(n)}\right)$ such that the number of disjoint $J_{x_{\varphi(n)}}$, $n \geq 1$, is finite, so that there is at least one of these intervals containing an infinite number of $x_{\varphi(n)}$ and then $a \in \cup_i \overline{\{\zeta_x, x \in J_i\}}$. Or, $|J_{x_n}| \to 0$ as $n \to \infty$, which implies that $\zeta_{x_n} \overset{a.s.}{\to} T$ as $n \to \infty$. To see why this last point holds, introduce ζ_n the first time at which the fragment J_{x_n} vanishes during the fragmentation, $n \geq 1$. Of course, $T < \zeta_{x_n} \leq \zeta_n$. By application of the scaling and strong fragmentation properties at time T, we see that there exists a r.v. $\zeta^{(n)}$, independent of $\mathcal{F}(T)$ and with the same distribution as ζ (see (2.10)) such that $\zeta_n - T = |J_{x_n}|^{|\alpha|} \zeta^{(n)}$. Hence, using that $E[\zeta] < \infty$ (see Lemma 2.3) and extracting a subsequence if necessary,

$$0 \leq \zeta_{x_n} - T \leq |J_{x_n}|^{|\alpha|} \zeta^{(n)} \underset{n \to \infty}{\to} 0 \text{ a.s.}$$

So, in both cases, $\overline{\cup_i \{\zeta_x, x \in J_i\}} \subset \{T\} \cup_i \overline{\{\zeta_x, x \in J_i\}}$ and then $\overline{\mathcal{A}} \subset \{T\} \cup_i \overline{\{\zeta_x, x \in J_i\}}$. The opposite inclusion is obvious.

Now, for each $i \geq 1$ set $\mathcal{A}_i := \left(\{\zeta_x, x \in J_i\} - T\right) \left(F_i(T)\right)^{\alpha}$. It follows from the scaling and strong fragmentation properties that the sets $\mathcal{A}_i$ are iid with the same law as $\mathcal{A}$ and are independent of $\mathcal{F}(T)$. Combining this with (2.32) will lead us to $m_\gamma(\overline{\mathcal{A}}) = 0$ for $\gamma > 1/|\alpha|$, which in turn will imply that $\dim_H(\overline{\mathcal{A}}) \leq 1/|\alpha|$, by the definitions of m_γ and $\dim_H$ (see respectively (2.29) and (2.28)). To see this, fix $\gamma > 1/|\alpha|$ and $\varepsilon > 0$ and define for every subset E of $\mathbb{R}_+$

$$m_\gamma^\varepsilon(E) := \inf_{\substack{\text{coverings of } E \\ \text{by intervals } B_n \text{ of lengths} \leq \varepsilon}} \sum_n |B_n|^\gamma .$$

Using that

$$\overline{\mathcal{A}} = \{T\} \bigcup_i \left(T + (F_i(T))^{-\alpha} \overline{\mathcal{A}_i}\right)$$

we have

$$m_\gamma^\varepsilon(\overline{\mathcal{A}}) \leq \sum_i (F_i(T))^{-\alpha\gamma} m_\gamma^{\varepsilon(F_i(T))^\alpha}(\overline{\mathcal{A}_i}) \leq \sum_i (F_i(T))^{-\alpha\gamma} m_\gamma^\varepsilon(\overline{\mathcal{A}_i}). \tag{2.33}$$

Since the first time ζ at which all the mass has been reduced to dust has a finite expectation and since $\overline{\mathcal{A}} \subset [0,\zeta]$, $E\left[\mathrm{mes}_\gamma^\varepsilon(\overline{\mathcal{A}})\right]$ is finite. Moreover, $\sum_i (F_i(T)) = 1$ and $F_1(T) < 1$ a.s., which implies that $E\left[\sum_i (F_i(T))^{-\alpha\gamma}\right] < 1$ when $\gamma > 1/|\alpha|$. Combining this with (2.33) and the fact that the random variables $\overline{\mathcal{A}_i}$ are independent of $\mathcal{F}(T)$ and have the same law as $\overline{\mathcal{A}}$ implies that $E\left[\mathrm{mes}_\gamma^\varepsilon(\overline{\mathcal{A}})\right] = 0$ for all positive ε as soon as $\gamma > 1/|\alpha|$. So by definition, $m_\gamma(\overline{\mathcal{A}}) \overset{a.s.}{=} 0$ for $\gamma > 1/|\alpha|$ and then $\dim_H(\overline{\mathcal{A}}) \leq 1/|\alpha|$ a.s. ■

Proof of Proposition 2.1 (iv). We first prove that $P(M(t) = 0) = 0$ for all $t > 0$. To do so, fix $t > 0$ and take s such that $0 < s < t$. Recall that the fragmentation and scaling properties applied at time s give

$$M(t) = M(s) + \sum_i F_i(s) 1_{\{F_i(s)>0\}} M^{(i)}\left((t-s)F_i^\alpha(s)\right) \tag{2.34}$$

where the $M^{(i)}$ are mutually independent, independent of $\mathcal{F}(s)$ and with the same distribution as M. Since $\nu\left(\mathcal{S}^\downarrow\right) = \infty$, the number of splits before time s is almost surely infinite. So if $M(s) = 0$, that is no mass is lost at time s, none of the fragments with positive mass appeared before s has entirely vanished at time s, so that there is an infinite number of fragments with positive mass present at time s. In particular, if $M(t) = 0$, then $M(s) = 0$ and $F_i(s) > 0$ for all $i \geq 1$. This gives with (2.34) that when $M(t) = 0$, then $M^{(i)}\left((t-s)F_i^\alpha(s)\right) = 0$ and $F_i^\alpha(s) \underset{i\to\infty}{\nearrow} \infty$. But this event has probability 0 since $P(M(u) = 0) < 1$ for some u large enough. Therefore, $P(M(t) = 0) = 0$ and this holds for all $t > 0$.

Next, take again $0 < s < t$. The mass $M^{(1)}$ being that introduced in (2.34), remark that conditionally on $F_1(s) > 0$, we have that $1_{\{F_1(s)>0\}} M^{(1)}\left((t-s)F_1^\alpha(s)\right) > 0$ a.s. since we have just proved that $P(M(u) > 0) = 1$ for all $u > 0$. Hence, by (2.34), $M(t) > M(s)$ a.s. conditionally on $F_1(s) > 0$. In others words, $P(M(s) < M(t) \mid s < \zeta) = 1$. Since this holds for all $0 < s < t$ and since the dust's mass M is a non-decreasing function,

$$P(M(s) < M(t) \text{ for all } 0 \leq s < t \leq \zeta) = 1.$$

Hence M is a.s. strictly increasing on $[0,\zeta]$ and $\mathrm{supp}(dM) = [0,\zeta]$ ■

2.6.2 Hölder continuity of the dust's mass M

Notice that Proposition 2.1 (ii) implies that a.s. M cannot be Hölder continuous of order $\gamma > 1 \wedge (1/|\alpha|)$, since the γ-Hölder-continuity of M yields to $\dim_H(dM) \geq \gamma$ (see Section 13.7 in [33]). We have moreover:

Proposition 2.2 *Suppose that assumptions* $(A1)$ *and* $(A2)$ *hold. Then,*

(i) the mass M is a.s. Hölder-continuous of order γ for every $\gamma < (1/2) \wedge (A/2|\alpha|)$.

(ii) if $\nu(s_{N+1} > 0) = 0$ for some integer N, the mass M is a.s. Hölder-continuous of order γ for every $\gamma < 1 \wedge \left(\underline{p}/|\alpha|\right)$.

The upper bound $1 \wedge \left(\underline{p}/|\alpha|\right)$ is larger than $(1/2) \wedge (A/2|\alpha|)$ as soon as $\underline{p} \geq A/2$ or $|\alpha| \leq 2\underline{p}$. Remark also that when $\nu(s_{N+1} > 0) = 0$ for some integer N, the coefficient $A = 1$ (see (2.16)) and the coefficient $\underline{p} = 1$ if and only if ν is moreover finite.

Part (i) of Proposition 2.2 is just a consequence of Lemma 2.6:

Proof of Proposition 2.2 (i). Consider $\gamma \in]0, 1 \wedge (A/|\alpha|)[$ and remark that for all $t > s \geq 0$,

$$\begin{aligned} (M(t) - M(s))^2 &= \int_s^t \int_s^t dM(u)dM(v) \\ &\leq (t-s)^\gamma \int_s^t \int_s^t \frac{dM(u)dM(v)}{|u-v|^\gamma}. \end{aligned}$$

The integral $\int_0^\infty \int_0^\infty |u-v|^{-\gamma} dM(u)dM(v)$ is a.s. finite by Lemma 2.6, and then,

$$|M(t) - M(s)| \leq B(t-s)^{\gamma/2} \text{ for all } t > s \geq 0$$

for some a.s. finite constant B. ■

The proof of the second part of Proposition 2.2 is slightly longer. The point is to use the well-known Kolmogorov criterion (see e.g. [60], p. 26, Theorem 2.1). In that aim, we first prove the following lemma.

Lemma 2.7 *Suppose that there exists an integer N such that $\nu(s_{N+1} > 0) = 0$ and fix an integer $n \geq 2$. Suppose moreover that for all $k \in \{1, ..., n-1\}$ there exist a finite constant C_k and a positive real number $a_k < k \wedge \left(\left(k - 1 + \underline{p}\right)/|\alpha|\right)$ such that*

$$E\left[(M(t) - M(s))^k\right] \leq C_k (t-s)^{a_k} \text{ for all } t \geq s \geq 0. \tag{2.35}$$

Then, for all $a < \inf_{\substack{n_1+n_2+...+n_l=n \\ n_i \in \mathbb{N}\backslash\{0\}}} (a_{n_1} + ... + a_{n_l}) \wedge ((n-1)/|\alpha|)$, there exists a finite constant $C_{n,a}$ such that

$$E\left[(M(t) - M(s))^n\right] \leq C_{n,a} (t-s)^a \text{ for all } t \geq s \geq 0.$$

Proof. Consider n points tagged independently, as explained in Section 2.3.2, and denote by $D_1, ..., D_n$ their respective times of reduction to dust. The r.v. $D_i, 1 \leq i \leq n$, have the same distribution as D (see (2.6)). By construction, the D_i's are independent conditionally on F, and therefore, by formula (2.7) , we have that

$$E\left[\prod_{i=1}^{n} 1_{\{s<D_i\leq t\}}\right] = E\left[(M(t)-M(s))^n\right]. \tag{2.36}$$

As in the proof of Theorem 2.1 (i), the goal is now to "introduce some independence" in order to bound from above this expectation. To that end, consider T_n, the first time at which the n tagged points do not belong to the same fragment and consider the distribution of the tagged points at that time. More precisely, for each integer $l \geq 2$ and each l-tuple $(n_1, n_2, ..., n_l) \in (\mathbb{N}\backslash \{0\})^l$ satisfying $n_1 + n_2 + ... + n_l = n$, consider the event

$$A_{(n_1,...,n_l)} = \left\{ \begin{array}{l} U_1, U_2, ..., U_l \text{ belong all to different fragments at time } T_n \text{ and there} \\ \text{are } n_k \text{ tagged points in the fragment containing } U_k,\ 1 \leq k \leq l. \end{array} \right\}$$

Since the number of such events is finite and since the law of $(D_1, ..., D_n)$ is exchangeable, we just have to prove that for a fixed l-tuple $(n_1, n_2, ..., n_l)$ and all $a < (a_{n_1} + ... + a_{n_l}) \wedge (n-1) / |\alpha|$, there exists a finite constant C such that

$$E\left[\prod_{i=1}^{n} 1_{\{s<D_i\leq t\}} 1_{\left\{A_{(n_1,n_2,...,n_l)}\right\}}\right] \leq C\,(t-s)^a \text{ for all } t \geq s \geq 0. \tag{2.37}$$

Conditionally on $A_{(n_1,n_2,...,n_l)}$, there are l tagged fragments at time T_n, with respective masses, $\lambda_1(T_n), ..., \lambda_l(T_n)$ and containing each, respectively, $n_1, ..., n_l$ tagged points. Write then

$$\prod_{i=1}^{n} 1_{\{s<D_i\leq t\}} 1_{\left\{A_{(n_1,n_2,...,n_l)}\right\}} = \prod_{k=1}^{l} \prod_{\substack{i:\ U_i, U_k \in \text{same} \\ \text{fragment at time } T_n}} 1_{\{s<D_i\leq t\}} 1_{\left\{A_{(n_1,n_2,...,n_l)}\right\}}$$

and recall that the l fragments evolve independently after time T_n. Recall also the scaling property of the fragmentation and consider the identity (2.36) (that holds for every integer n, and in particular the n_k's). Then, setting $M(t) := 0$ for $t < 0$, there exists a random process $\widetilde{M}$ with the same law as M and independent of $\mathcal{F}(T_n), (\lambda_1(T_n), ..., \lambda_n(T_n))$ and $A_{(n_1,n_2,...,n_l)}$ such that

$$\begin{aligned} & E\left[\prod_{i=1}^{n} 1_{\{s<D_i\leq t\}} 1_{\left\{A_{(n_1,n_2,...,n_l)}\right\}}\right] \\ & = E\Big[\prod_{k=1}^{l} E\Big[\left(\widetilde{M}\left((t-T_n)\,\lambda_k^{\alpha}(T_n)\right) - \widetilde{M}\left((s-T_n)\,\lambda_k^{\alpha}(T_n)\right)\right)^{n_k} \\ & \qquad \mid \mathcal{F}(T_n), \lambda_1(T_n), ..., \lambda_l(T_n), A_{(n_1,n_2,...,n_l)}\Big]\, 1_{\left\{A_{(n_1,n_2,...,n_l)}\right\}}\Big]. \end{aligned}$$

Now consider the assumptions we have made in the statement. Since M is a.s. bounded by 1, the inequality (2.35) holds actually by replacing a_{n_k} by any $b_{n_k} \leq a_{n_k}$ and C_{n_k} by $C_{n_k} \vee 1$. Therefore, for each l-tuple $(b_{n_1}, ..., b_{n_l})$ such that $b_{n_k} \leq a_{n_k}$, $1 \leq k \leq l$, there exists a finite deterministic constant C such that

$$\begin{aligned} & \prod_{k=1}^{l} \left[E\left(\widetilde{M}\left((t-T_n)\,\lambda_k^{\alpha}(T_n)\right) - \widetilde{M}\left((s-T_n)\,\lambda_k^{\alpha}(T_n)\right)\right)^{n_k} \mid \mathcal{F}(T_n), \lambda_1(T_n), ..., \lambda_l(T_n), A_{(n_1,n_2,...,n_l)}\right] \\ & \underset{a.s.}{\leq} C\,(t-s)^{b_{n_1}+...+b_{n_l}} \prod_{k=1}^{l} \lambda_k^{\alpha b_{n_k}}(T_n). \end{aligned}$$

And then

$$E\left[\prod_{i=1}^{n}1_{\{s<D_i\leq t\}}1_{\left\{A_{n_1,n_2,\ldots,n_l}\right\}}\right]\leq C\,(t-s)^{b_{n_1}+\ldots+b_{n_l}}\,E\left[\prod_{k=1}^{l}\lambda_k^{\alpha b_{n_k}}(T_n)1_{\left\{A_{(n_1,n_2,\ldots,n_l)}\right\}}\right].\qquad(2.38)$$

To see when the latter expectation is finite we use Lemma 2.2. Since, by assumption, $\nu\left(s_{N+1}>0\right)=0$ and $|\alpha|\,b_{n_k}<n_k$ (recall that $\underline{p}\leq 1$) for $1\leq k\leq l$,

$$\int_{\mathcal{S}^{\downarrow}}\sum_{1\leq i_1<\ldots<i_l\leq N}\prod_{k=1}^{l}s_{i_k}^{n_k-|\alpha|b_{n_k}}1_{\left\{s_{i_k}>0\right\}}\nu(ds)\leq N^{l-1}\int_{\mathcal{S}^{\downarrow}}\sum_{2\leq i_2\leq N}s_{i_2}^{n_2+\ldots+n_l-|\alpha|b_{n_2}-\ldots-|\alpha|b_{n_l}}\nu(ds),$$

which is finite, by definition of $\underline{p}$, as soon as $n_2+\ldots+n_l-|\alpha|\,b_{n_2}-\ldots-|\alpha|\,b_{n_l}>1-\underline{p}$. This holds here since $|\alpha|\,b_{n_k}<n_k-1+\underline{p}$ for $k\geq 2$. Thus, by Lemma 2.2,

$$E\left[\prod_{k=1}^{l}\lambda_k^{\alpha b_{n_k}}(T_n)1_{\{\lambda_1(T_n)\geq\lambda_2(T_n)\geq\ldots\geq\lambda_l(T_n)\}}1_{\left\{A_{(n_1,n_2,\ldots,n_l)}\right\}}\right]<\infty$$

as soon as $\sum_{k=1}^{l}b_{n_k}<(n-1)/|\alpha|$. By exchangeability, the expectation in the right hand side of inequality (2.38) is then finite and thus the upper bound (2.37) and the required result are proved.

Proof of Proposition 2.2 (ii). For all integer $n\geq 1$, define

$$\gamma_n:=\sup\left\{a\geq 0:\exists\,C<\infty\text{ such that }E\left[(M(t)-M(s))^n\right]\leq C\,(t-s)^a\ \text{ for all }t\geq s\geq 0\right\}.$$

It is well-defined since M is a.s. bounded by 1. Our goal is to prove that the claim

$$C(k):\quad\gamma_n\geq n\left(\frac{k-1}{k}\wedge\frac{\underline{p}}{|\alpha|}\wedge\frac{k-1}{k\,|\alpha|}\right)\ \text{ for all }n\geq 1,$$

holds for all integers $k\geq 1$. If this is true, the proof is finished, since the Kolmogorov criterion then asserts that for each $k\geq 1$ and every γ such that

$$\gamma<\left(\frac{k-1}{k}\wedge\frac{\underline{p}}{|\alpha|}\wedge\frac{k-1}{k\,|\alpha|}\right)$$

there is a γ-Hölder-continuous version of M. Since M is non-decreasing, it is actually M that is a.s. Hölder-continuous with these orders γ. Letting $k\to\infty$, M is then a.s. γ-Hölder-continuous for every $\gamma<\left(\underline{p}/|\alpha|\right)\wedge 1$.

So let us prove by induction the claims $C(k)$, $k\geq 1$. That $C(1)$ holds is obvious. To prove $C(2)$, remark first that $\gamma_1=1$. This is a consequence of formula (2.7), which gives $E\left[M(t)-M(s)\right]=E\left[1_{\{s<D<t\}}\right]$ and then of assumptions $(A1)$ and $(A2)$, which, by Lemma 2.1, imply that D has a bounded density. Then, $\gamma_1=1$ and Lemma 2.7 lead to $\gamma_2\geq 2\left(1\wedge\left(\left(\underline{p}\wedge 1/2\right)/|\alpha|\right)\right)$. And next, using recursively the same lemma and the fact that $\underline{p}\leq 1$, we get that

$$\gamma_n\geq n\left(1\wedge\left(\underline{p}/|\alpha|\right)\wedge\left(1/2\,|\alpha|\right)\right)\ \text{ for all }n\geq 1.$$

Which proves the claim $C(2)$. Fix now an integer $k \geq 2$ and suppose that $C(k)$ holds. We want to prove $C(k+1)$. By Hölder's inequality,

$$E\left[(M(t)-M(s))^{k+1}\right] \leq E\left[(M(t)-M(s))^{kn}\right]^{1/n} E\left[(M(t)-M(s))^{n/(n-1)}\right]^{(n-1)/n}. \quad (2.39)$$

First, remark the existence of a finite constant C such that $E\left[(M(t)-M(s))^{n/(n-1)}\right] \leq C(t-s)$ since $0 \leq M(t)-M(s) \leq 1$ for $t \geq s$ and since D has a bounded density. Next, by claim $C(k)$,

$$\gamma_{nk} \geq n\left((k-1) \wedge \left(k\underline{p}/\left|\alpha\right|\right) \wedge \left((k-1)/\left|\alpha\right|\right)\right) \text{ for all } n \geq 1,$$

and this implies, with the previous remark and (2.39), that

$$\gamma_{k+1} \geq (k-1) \wedge \left(k\underline{p}/\left|\alpha\right|\right) \wedge \left((k-1)/\left|\alpha\right|\right) + (n-1)/n \text{ for all } n \geq 1.$$

Letting $n \to \infty$ and using that $k-1>0$, it is easy to see that

$$\begin{aligned} \gamma_{k+1} &\geq (k-1) \wedge \left(k\underline{p}/\left|\alpha\right|\right) \wedge \left((k-1)/\left|\alpha\right|\right) + 1 \\ &\geq k \wedge \left((k+1)\,\underline{p}/\left|\alpha\right|\right) \wedge \left(k/\left|\alpha\right|\right). \end{aligned}$$

When $n \leq k+1$,

$$E\left[(M(t)-M(s))^{n}\right] \leq E\left[(M(t)-M(s))^{(k+1)}\right]^{n/(k+1)}$$

and then $\gamma_n \geq n\gamma_{k+1}/(k+1)$. Hence,

$$\gamma_n \geq n\left(k \wedge \left((k+1)\underline{p}/\left|\alpha\right|\right) \wedge \left(k/\left|\alpha\right|\right)\right)/(k+1) \text{ for all } n \leq k+1.$$

Next, by applying Lemma 2.7 recursively, we get that

$$\gamma_n \geq n\left(k \wedge \left((k+1)\underline{p}/\left|\alpha\right|\right) \wedge \left(k/\left|\alpha\right|\right)\right)/(k+1) \text{ for } n > k+1$$

and so $C(k+1)$ holds. Hence the claims $C(k)$ hold for every integers $k \geq 1$. ■

2.7 Appendix: proof of Lemma 2.2

For this technical proof, it is easier to work with partition-valued fragmentations, so we first recall some background on the subject. The following recalls hold for any self-similar fragmentation. We refer to [9], [13] and [14] for details.

Define by $\mathcal{P}$ the set of partitions of $\mathbb{N}\setminus\{0\}$ and for $\pi \in \mathcal{P}$ and $i \in \mathbb{N}\setminus\{0\}$, denote by π_i the block of π having i as least element, when such a block exists, and set $\pi_i := \emptyset$ otherwise, so that $(\pi_1, \pi_2, ...)$ are the blocks of π. A random partition is called *exchangeable* if its distribution is invariant under finite permutations. Kingman [45] shows that the blocks of every exchangeable partition π have asymptotics frequencies a.s., that is ($\#$ denoting the counting measure on $\mathbb{N}\setminus\{0\}$) :

$$\lim_{n\to\infty} \frac{\#(\pi_i \cap \{1, ...n\})}{n} \text{ exists a.s. for all } i.$$

Let $|\pi|^{\downarrow}$ denote the decreasing rearrangement of these limits.

Now, let F be a $\mathcal{S}^{\downarrow}$-valued fragmentation with index of self-similarity α and consider I, one of its interval representation as explained in Section 2.2. By picking independent r.v. U_i, $i \geq 1$, uniformly distributed on $]0,1[$ and independent of I, we can construct an α-self-similar partition-valued fragmentation $(\Pi(t), t \geq 0)$ as follows: for each $t \geq 0$, $\Pi(t)$ is the random partition of $\mathbb{N} \backslash \{0\}$ such that two integers i, j belong to the same block of $\Pi(t)$ if and only if U_i and U_j belong to the same interval component of $I(t)$. If $U_i \notin I(t)$, then the block of $\Pi(t)$ containing i is $\{i\}$. This process Π is exchangeable and called *partition-valued representation* of F. By the strong law of large number, the law of F can be recovered from Π, as the law of the decreasing rearrangement of asymptotic frequencies of Π :

$$\left(|\Pi(t)|^{\downarrow}, t \geq 0\right) \overset{law}{=} F.$$

In the homogeneous case ($\alpha = 0$), the partition-valued fragmentation $(\Pi(t), t \geq 0)$ can be constructed from a Poisson point process (PPP) with an intensity measure depending on the dislocation measure ν. We explain the construction for a fragmentation with no erosion and a dislocation measure ν such that $\nu\left(\sum_i s_i < 1\right) = 0$. First, for every $s = (s_1, s_2, ...) \in \mathcal{S}^{\downarrow}$, consider the *paintbox* partition Π_s (introduced by Kingman, see e.g. [45]) defined as follows: let $(Z_i)_{i \geq 1}$ be an iid sequence of random variable such that $P\left(Z_1 = j\right) = s_j$ for $j \geq 1$ and let then Π_s be the partition such that two integers i, j are in the same block if and only if $Z_i = Z_j$. Introduce next the measure κ_ν defined by

$$\kappa_\nu(B) = \int_{\mathcal{S}^{\downarrow}} P(\Pi_s \in B)\nu(ds), \ B \in \mathcal{P}. \tag{2.40}$$

Bertoin [13] shows that κ_ν is an exchangeable measure and that the fragmentation Π is a pure jumps process whose jumps correspond to the atoms of a PPP $\left((\Delta(t), k(t)), t \geq 0\right)$ on $\mathcal{P} \times \mathbb{N} \backslash \{0\}$ with intensity $\kappa_\nu \otimes \#$. By this, we mean that Π jumps exactly at the times of occurrence of atoms of the PPP and that at such times t, $\Pi(t^-)$ jumps to $\Pi(t)$ as follows: the blocks of $\Pi(t)$ are the same as those of $\Pi(t^-)$, except $\Pi(t^-)_{k(t)}$, which is replaced by the blocks $\{n_i : i \in \Delta(t)_1\}, \{n_i : i \in \Delta(t)_2\}, ...$ where $n_1 < n_2 < ...$ are the elements of the block $\Pi(t^-)_{k(t)}$. Berestycki adapts in [9] this PPP-construction to homogeneous $\mathcal{S}^{\downarrow}$-valued fragmentations.

This partition point of view and the Poissonian construction lead to the following lemma.

Lemma 2.8 *Let I_h be a homogeneous interval fragmentation, with no erosion and with a dislocation measure ν such that $\nu\left(\sum_i s_i < 1\right) = 0$. In this fragmentation, tag independently n fragments as explained in Section 2.3.2 and let $U_{1,h}, ..., U_{n,h}$ denote the tagged points. Define $\lambda_{1,h}(t), ..., \lambda_{n,h}(t)$ to be the masses at time t of these tagged fragments and $T_{n,h}$ the first time at which the tagged points do not all belong to the same fragment. For every l-tuple $(n_1, n_2, ..., n_l) \in (\mathbb{N} \backslash \{0\})^l$ such that $n_1 + n_2 + ... + n_l = n$, define then $A_{(n_1,...,n_l),h}$ by*

$$A_{(n_1,...,n_l),h} := \left\{ \begin{array}{l} U_{1,h}, U_{2,h}, ..., U_{l,h} \text{ belong all to different fragments at time } T_{n,h} \text{ and} \\ \text{there are } n_k \text{ tagged points in the fragment containing } U_{k,h}, \ 1 \leq k \leq l. \end{array} \right\}$$

Then,

(i) $\lambda_{1,h}(T_{n,h}-) = \lambda_{2,h}(T_{n,h}-) = ... = \lambda_{n,h}(T_{n,h}-)$ *by definition of* $T_{n,h}$,

(ii) $A_{(n_1,...,n_l),h}$ *and* $\left(\frac{\lambda_{1,h}(T_{n,h})}{\lambda_{1,h}(T_{n,h}-)}, \frac{\lambda_{2,h}(T_{n,h})}{\lambda_{1,h}(T_{n,h}-)}, ..., \frac{\lambda_{n,h}(T_{n,h})}{\lambda_{1,h}(T_{n,h}-)}\right)$ *are independent of* $\lambda_{1,h}(T_{n,h}-)$,

(ii) there is a positive finite constant C *such that for every positive measurable function* f *on* $]0,1]^l$,

$$E\left[f\left(\frac{\lambda_{1,h}(T_{n,h})}{\lambda_{1,h}(T_{n,h}-)}, \frac{\lambda_{2,h}(T_{n,h})}{\lambda_{1,h}(T_{n,h}-)} ..., \frac{\lambda_{l,h}(T_{n,h})}{\lambda_{1,h}(T_{n,h}-)}\right) 1_{\left\{A_{(n_1,n_2,...,n_l),h}\right\}}\right]$$
$$= C\int_{\mathcal{S}^{\downarrow}} \sum_{i_1 \neq i_2 \neq ... \neq i_l} s_{i_1}^{n_1} s_{i_2}^{n_2} ... s_{i_l}^{n_l} f(s_{i_1}, ..., s_{i_l}) 1_{\left\{s_{i_1}>0,...,s_{i_l}>0\right\}} \nu(ds).$$

Proof. Let $(\Pi_h(t), t \geq 0)$ be the homogeneous partition-valued fragmentation constructed from I_h and the $U_{i,h}$'s, and let $((\Delta(t), k(t)), t \geq 0)$ be the PPP on $\mathcal{P} \times \mathbb{N}\backslash\{0\}$ with intensity $\kappa_\nu \otimes \#$ describing the jumps of Π_h. Define then $\mathcal{P}_n^*$ to be the set of partitions of $\mathbb{N}\backslash\{0\}$ such that integers $1, 2, ..., n$ do not belong to the same block and remark that

$$T_{n,h} = \inf\{t \geq 0 : \Pi_h(t) \in \mathcal{P}_n^*\} = \inf\{t \geq 0 : \Delta(t) \in \mathcal{P}_n^* \text{ and } k(t) = 1\}.$$

Setting Δ_i for the block of $\Delta(T_{n,h})$ containing i, $1 \leq i \leq n$, the event $A_{(n_1,n_2,...,n_l),h}$ can therefore be written as

$$A_{(n_1,n_2,...,n_l),h} = \left\{\begin{array}{l} 1, 2, ..., l \text{ belong to distinct blocks of } \Delta(T_{n,h}) \\ \text{and } \mathrm{Card}(\Delta_k \cap \{1, ..., n\}) = n_k,\ 1 \leq k \leq l. \end{array}\right\} \tag{2.41}$$

and using the exchangeability of κ_ν and the independence of $\Delta(T_{n,h})$ and $\Pi_h(T_{n,h}-)$, we get that

$$\frac{\#(\Delta_i \cap \{1, ...k\})}{k} \underset{k\to\infty}{\overset{a.s.}{\to}} \frac{\lambda_{i,h}(T_{n,h})}{\lambda_{1,h}(T_{n,h}-)},\ 1 \leq i \leq n,$$

and then assertion (ii).

Next, to prove (iii), note that formula (2.40) leads to

$$\kappa_\nu(\mathcal{P}_n^*) = \int_{\mathcal{S}^{\downarrow}} \left(1 - \sum_i s_i^n\right) \nu(ds)$$

which is positive and finite since $1 - \sum_i s_i^n \leq n(1 - s_1)$ and $(1 - s_1)$ is integrable with respect to ν. It is then a standard result of PPP's theory that $T_{n,h}$ has an exponential law with parameter $\kappa_\nu(\mathcal{P}_n^*)$ and that the distribution of $\Delta(T_{n,h})$ is given by $\kappa_\nu(\cdot \cap \mathcal{P}_n^*)/\kappa_\nu(\mathcal{P}_n^*)$. Thus, by definition of κ_ν,

$$E\left[f\left(\frac{\lambda_{1,h}(T_{n,h})}{\lambda_{1,h}(T_{n,h}-)}, \frac{\lambda_{2,h}(T_{n,h})}{\lambda_{1,h}(T_{n,h}-)}, ..., \frac{\lambda_{l,h}(T_{n,h})}{\lambda_{1,h}(T_{n,h}-)}\right) 1_{\left\{A_{(n_1,n_2,...,n_l),h}\right\}}\right]$$
$$= \frac{1}{\kappa_\nu(\mathcal{P}_n^*)} \int_{\mathcal{S}^{\downarrow}} E\left[f(\Pi_{s,1}, ..., \Pi_{s,l}) 1_{\left\{A^s_{(n_1,n_2,...,n_l),h}\right\}}\right] \nu(ds),$$

where $A^s_{(n_1,n_2,...,n_l),h}$ is defined as $A_{(n_1,n_2,...,n_l),h}$ by replacing in (2.41) $\Delta(T_{n,h})$ by Π_s. It is then easy to check with the definition of Π_s that the required formula holds.

Proof of Lemma 2.2. The first part of the proof consists in shifting the problem to a homogeneous fragmentation with the same dislocation measure ν. This can be done by using the construction of self-similar fragmentations from homogeneous ones recalled in Section 2.2. So, consider a homogeneous interval fragmentation I_h from which we construct the α-self-similar one by time-change (2.1). In this homogeneous fragmentation, tag independently n fragments as in the previous lemma. Keeping the notation introduced there, is easy to see that

$$\left(\lambda_{1,h}(T_{n,h}),...,\lambda_{n,h}(T_{n,h}),1_{\left\{A_{(n_1,n_2,...,n_l),h}\right\}}\right)\overset{law}{=}\left(\lambda_1(T_n),...,\lambda_n(T_n)1_{\left\{A_{(n_1,n_2,...,n_l)}\right\}}\right).$$

So that the aim of this proof is to find for which l-tuples $(a_1,...,a_l)$, the expectation

$$E\left[\prod_{k=1}^{l}\lambda_{k,h}^{-a_k}(T_{n,h})1_{\left\{\lambda_{1,h}(T_{n,h})\geq\lambda_{2,h}(T_{n,h})\geq...\geq\lambda_{l,h}(T_{n,h})\right\}}1_{\left\{A_{(n_1,n_2,...,n_l),h}\right\}}\right]$$

is finite.

By Lemma 2.8, we have that

$$\begin{aligned}&E\left[\prod_{k=1}^{l}\lambda_{k,h}^{-a_k}(T_{n,h})1_{\left\{\lambda_{1,h}(T_{n,h})\geq...\geq\lambda_{l,h}(T_{n,h})\right\}}1_{\left\{A_{(n_1,n_2,...,n_l),h}\right\}}\right]\\&=E\left[(\lambda_{1,h}(T_{n,h}-))^{-\sum_{k=1}^{l}a_k}\right]\\&\times E\left[\prod_{k=1}^{l}\left(\frac{\lambda_{k,h}(T_{n,h})}{\lambda_{1,h}(T_{n,h}-)}\right)^{-a_k}1_{\left\{\lambda_{1,h}(T_{n,h})\geq...\geq\lambda_{l,h}(T_{n,h})\right\}}1_{\left\{A_{(n_1,n_2,...,n_l),h}\right\}}\right]\end{aligned}$$

and that

$$\begin{aligned}&E\left[\prod_{k=1}^{l}\left(\frac{\lambda_{k,h}(T_{n,h})}{\lambda_{1,h}(T_{n,h}-)}\right)^{-a_k}1_{\left\{\lambda_{1,h}(T_{n,h})\geq...\geq\lambda_{l,h}(T_{n,h})\right\}}1_{\left\{A_{n_1,n_2,...,n_l,h}\right\}}\right]<\infty\\&\Leftrightarrow\int_{\mathcal{S}^{\downarrow}}\sum_{i_1<...<i_l}\prod_{k=1}^{l}s_{i_k}^{n_k-a_k}1_{\left\{s_{i_k}>0\right\}}\nu(ds)<\infty.\end{aligned}$$

So it just remains to specify for which $(a_1,...,a_l)$, the expectation $E\left[(\lambda_{1,h}(T_{n,h}-))^{-\sum_{k=1}^{l}a_k}\right]$ is finite. To that end, remark that given $\lambda_{1,h}$, the probability that the tagged points $U_{2,h},...,U_{n,h}$ belong to the same fragment as $U_{1,h}$ at time t is equal to $\lambda_{1,h}^{n-1}(t)$, since the $U_{i,h}$'s are independent and uniformly distributed on $]0,1[$. In other words,

$$P(T_{n,h}>t\mid\lambda_{1,h})=\lambda_{1,h}^{n-1}(t)\qquad\forall t>0.$$

As recalled in Section 2.3, the process $(\lambda_{1,h}(t),t\geq 0)$ can be expressed in the form $(\exp(-\xi_t),t\geq 0)$, for some pure jumps subordinator ξ with Laplace exponent ϕ given by (2.4). Therefore $P(T_{n,h}>t\mid\lambda_{1,h})=e^{-(n-1)\xi_t}$ and for all $a\in\mathbb{R}$:

$$\begin{aligned}E\left[\lambda_{1,h}^{-a}(T_{n,h}-)\right]&=E\left[\int_0^\infty e^{a\xi_{t^-}}P(T_{n,h}\in dt\mid\lambda_{1,h})\right]\\&=E\left[\int_0^\infty\sum_{0<s<t}\left(e^{a\xi_s}-e^{a\xi_{s^-}}\right)P(T_{n,h}\in dt\mid\lambda_{1,h})\right]+1\\&=E\left[\sum_{0<s<\infty}\left(e^{a\xi_s}-e^{a\xi_{s^-}}\right)e^{-(n-1)\xi_s}\right]+1\\&=E\left[\sum_{0<s<\infty}e^{(a-(n-1))\xi_{s^-}}\left(e^{(a-(n-1))\Delta_s}-e^{-(n-1)\Delta_s}\right)\right]+1\quad(\Delta_s=\xi_s-\xi_{s^-}).\end{aligned}$$

Finally, using the Master Formula (see [60], p.475), we get

$$E\left[\lambda_{1,h}^{-a}(T_{n,h}-)\right] = E\left[\int_0^\infty e^{(a-(n-1))\xi_s}ds\right]\int_0^\infty (e^{(a-(n-1))x} - e^{-(n-1)x})\pi(dx) + 1,$$

π being the Lévy measure of ξ. The integral $\int_0^\infty (e^{(a-(n-1))x} - e^{-(n-1)x})\pi(dx)$ is finite as soon as $a \leq n-1$ and the expectation $E\left[\int_0^\infty e^{(a-(n-1))\xi_s}ds\right]$ is finite if and only if $a < n-1$, since $E\left[e^{-q\xi_s}\right] = e^{-s\phi(q)}$ where $\phi > 0$ on $]0,\infty[$, $\phi \in [-\infty, 0]$ on $]-\infty, 0]$. This completes the proof. ■

Chapitre 3

The genealogy of self-similar fragmentations with a negative index as a continuum random tree

Abstract: We encode a certain class of stochastic fragmentation processes, namely self-similar fragmentation processes with a negative index of self-similarity, into a metric family tree which belongs to the family of Continuum Random Trees of Aldous. When the splitting times of the fragmentation are dense near 0, the tree can in turn be encoded into a continuous height function, just as the Brownian Continuum Random Tree is encoded in a normalized Brownian excursion. Under mild hypotheses, we then compute the Hausdorff dimensions of these trees, and the maximal Hölder exponents of the height functions.

3.1 Introduction

Self-similar fragmentation processes describe the evolution of an object that falls apart, so that different fragments keep on collapsing independently with a rate that depends on their sizes to a certain power, called the *index* of the self-similar fragmentation. A genealogy is naturally associated with such fragmentation processes, by saying that the common ancestor of two fragments is the block that included these fragments for the last time, before a dislocation had definitely separated them. With an appropriate coding of the fragments, one guesses that there should be a natural way to define a genealogy tree, rooted at the initial fragment, associated with any such fragmentation. It would be natural to put a metric on this tree, e.g. by letting the distance from a fragment to the root of the tree be the time at which the fragment disappears.

Conversely, it turns out that trees have played a key role in models involving self-similar fragmentations, notably, Aldous and Pitman [5] have introduced a way to log the so-called Brownian *Continuum Random Tree* (CRT) [3] that is related to the standard additive coalescent. Bertoin [14] has shown that a fragmentation that is somehow dual to the Aldous-Pitman fragmentation can be obtained as follows. Let $\mathcal{T}_B$ be the Brownian CRT, which is considered as an "infinite tree with edge-lengths" (formal definitions are given below). Let $\mathcal{T}_t^1, \mathcal{T}_t^2, \dots$ be the distinct tree components of the forest obtained by removing all the vertices of $\mathcal{T}$ that are at distance less than t from the root, and arranged by decreasing order of "size". Then the sequence $F_B(t)$ of these sizes defines as t varies a self-similar fragmentation. A moment of thought points out that the notion of genealogy defined above precisely coincides with the tree we have fragmented in this way, since a split occurs precisely at branchpoints of the tree. Fragmentations of CRT's that are different from the Brownian one and that follow the same kind of construction have been studied in [56].

The goal of this paper is to show that any self-similar fragmentation process with negative index can be obtained by a similar construction as above, for a certain instance of CRT. We are interested in negative indices, because in most interesting cases when the self-similarity index is non-negative, all fragments have an "infinite lifetime", meaning that the pieces of the fragmentation remain macroscopic at all times. In this case, the family tree defined above will be unbounded and without endpoints, hence looking completely different from the Brownian CRT. By contrast, as soon as the self-similarity index is negative, a loss of mass occurs, that makes the fragments disappear in finite time (see [15]). In this case, the metric family tree will be a bounded object, and in fact, a CRT. To state our results, we first give a rigorous definition of the involved objects. Call

$$\mathcal{S}^{\downarrow} = \left\{ \mathbf{s} = (s_1, s_2, \ldots) : s_1 \geq s_2 \geq \ldots \geq 0; \sum_{i \geq 1} s_i \leq 1 \right\},$$

and endow it with the topology of pointwise convergence.

Definition 3.1 *A Markovian $\mathcal{S}^{\downarrow}$-valued process $(F(t), t \geq 0)$ starting at $(1, 0, \ldots)$ is a ranked self-similar fragmentation with index $\alpha \in \mathbb{R}$ if it is continuous in probability and satisfies the following fragmentation property. For every $t, t' \geq 0$, given $F(t) = (x_1, x_2, \ldots)$, $F(t + t')$ has the same law as the decreasing rearrangement of the sequences $x_1 F^{(1)}(x_1^{\alpha} t'), x_2 F^{(2)}(x_2^{\alpha} t'), \ldots$, where the $F^{(i)}$'s are independent copies of F.*

By a result of Bertoin [14] and Berestycki [9], the laws of such fragmentation processes are characterized by a 3-tuple (α, c, ν), where α is the index, $c \geq 0$ is an "erosion" constant, and ν is a σ-finite measure on $\mathcal{S}^{\downarrow}$ that integrates $\mathbf{s} \mapsto 1 - s_1$ such that $\nu(\{(1, 0, 0 \ldots)\}) = 0$. Informally, c measures the rate at which fragments melt continuously (a phenomenon we will not be much interested in here), while ν measures instantaneous breaks of fragments: a piece with size x breaks into fragments with masses $x\mathbf{s}$ at rate $x^{\alpha} \nu(\mathrm{d}\mathbf{s})$. Notice that some mass can be lost within a sudden break: this happens as soon as $\nu(\sum_i s_i < 1) \neq 0$, but we will not be interested in this phenomenon here either. The loss of mass phenomenon stated above is completely different from erosion or sudden loss of mass: it is due to the fact that small fragments tend to decay faster when $\alpha < 0$.

On the other hand, let us define the notion of CRT. An $\mathbb{R}$-tree (with the terminology of Dress and Terhalle [27]; it is called a *continuum tree set* in Aldous [3]) is a complete metric space (T, d), whose elements are called *vertices*, which satisfies the following two properties:

- For $v, w \in T$, there exists a unique geodesic $[[v, w]]$ going from v to w, i.e. there exists a unique isomorphism $\varphi_{v,w} : [0, d(v, w)] \to T$ with $\varphi_{v,w}(0) = v$ and $\varphi_{v,w}(d(v, w)) = w$, and its image is called $[[v, w]]$.

- For any $v, w \in T$, the only non-self-intersecting path going from v to w is $[[v, w]]$, i.e. for any continuous injective function $s \mapsto v_s$ from $[0, 1]$ to T with $v_0 = v$ and $v_1 = w$, $\{v_s : s \in [0, 1]\} = [[v, w]]$.

We will furthermore consider $\mathbb{R}$-trees that are *rooted*, that is, one vertex is distinguished as being the root, and we call it $\varnothing$. A *leaf* is a vertex which does not belong to $[[\varnothing, w[[:= \varphi_{\varnothing,w}([0, d(\varnothing, w)))$ for any vertex w. Call $\mathcal{L}(T)$ the set of leaves of T, and $\mathcal{S}(T) = T \setminus \mathcal{L}(T)$ its skeleton. An $\mathbb{R}$ -tree is *leaf-dense* if T is the closure of $\mathcal{L}(T)$. We also call *height* of a vertex v the quantity $\mathrm{ht}(v) = d(\varnothing, v)$. Last, for T an $\mathbb{R}$-tree and $a > 0$, we let $a \otimes T$ be the $\mathbb{R}$-tree in which all distances are multiplied by a.

Definition 3.2 *A continuum tree is a pair (T, μ) where T is an $\mathbb{R}$-tree and μ is a probability measure on T, called the mass measure, which is non-atomic and satisfies $\mu(\mathcal{L}(T)) = 1$ and such that for every non-leaf vertex w, $\mu\{v \in T : [[\varnothing, v]] \cap [[\varnothing, w]] = [[\varnothing, w]]\} > 0$. The set of vertices just defined is called the fringe subtree rooted at w. A CRT is a random variable $\omega \mapsto (T(\omega), \mu(\omega))$ on a probability space $(\Omega, \mathcal{F}, P)$ whose values are continuum trees.*

Notice that the definition of a continuum tree implies that the $\mathbb{R}$-tree T satisfies certain extra properties, for example, its set of leaves must be uncountable and have no isolated point. Also, the definition of a CRT is a little inaccurate as we did not endow the space of $\mathbb{R}$-trees with a σ-field. This problem is in fact circumvented by the fact that CRTs are in fact entirely described by the sequence of their *marginals*, that is, of the subtrees spanned by the root and k leaves chosen with law μ given μ, and these subtrees, which are interpreted as finite *trees with edge-lengths*, are random variables (see Sect. 3.2.2). The reader should keep in mind that by the "law" of a CRT we mean the sequence of these marginals. Another point of view is taken in [30], where the space of $\mathbb{R}$-trees is endowed with a metric.

For (T, μ) a continuum tree, and for every $t \geq 0$, let $T_1(t), T_2(t), \ldots$ be the tree components of $\{v \in T : \mathrm{ht}(v) > t\}$, ranked by decreasing order of μ-mass. A continuum random tree (T, μ) is said to be *self-similar* with index $\alpha < 0$ if for every $t \geq 0$, conditionally on $(\mu(T_i(t)), i \geq 1)$, $(T_i(t), i \geq 1)$ has the same law as $(\mu(T_i(t))^{-\alpha} \otimes T^{(i)}, i \geq 1)$ where the $T^{(i)}$'s are independent copies of T.

Our first result is

Theorem 3.1 *Let F be a ranked self-similar fragmentation process with characteristic 3-tuple (α, c, ν), with $\alpha < 0$. Suppose also that F is not constant, that $c = 0$ and $\nu(\sum_i s_i < 1) = 0$. Then there exists an α-self-similar CRT $(\mathcal{T}_F, \mu_F)$ such that, writing $F'(t)$ for the decreasing sequence of masses of connected components of the open set $\{v \in \mathcal{T}_F : \mathrm{ht}(v) > t\}$, the process $(F'(t), t \geq 0)$ has the same law as F. The tree $\mathcal{T}_F$ is leaf-dense if and only if ν has infinite total mass.*

The next statement is a kind of converse to this theorem.

Proposition 3.1 *Let $(\mathcal{T}, \mu)$ be a self-similar CRT with index $\alpha < 0$. Then the process $F(t) = ((\mu(\mathcal{T}_i(t), i \geq 1), t \geq 0)$ is a ranked self-similar fragmentation with index α, it has no erosion and its dislocation measure ν satisfies $\nu(\sum_i s_i < 1) = 0$. Moreover, $\mathcal{T}_F$ and $\mathcal{T}$ have the same law.*

These results are proved in Sect. 3.2. There probably exists some notion of continuum random tree extending the former which would include fragmentations with erosion or with sudden loss of mass, but we do not pursue this here.

The next result, to be proved in Sect. 3.3, deals with the Hausdorff dimension of the set of leaves of the CRT $\mathcal{T}_F$.

Theorem 3.2 *Let F be a ranked self-similar fragmentation with characteristics (α, c, ν) satisfying the hypotheses of Theorem 3.1. Writing $\dim_{\mathcal{H}}$ for Hausdorff dimension, one has*

$$\dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F)) = \frac{1}{|\alpha|} \ \text{a.s.} \tag{3.1}$$

as soon as $\int_{\mathcal{S}^{\downarrow}} \left(s_1^{-1} - 1\right) \nu(\mathrm{d}\mathbf{s}) < \infty$.

Some comments about this formula. First, notice that under the extra integrability assumption on ν, the dimension of the whole tree is $\dim_{\mathcal{H}}(\mathcal{T}_F) = (1/|\alpha|) \vee 1$ because the skeleton $\mathcal{S}(\mathcal{T}_F)$ has dimension 1 as a countable union of segments. The value -1 is therefore critical for α, since the above formula shows that the dimension of $\mathcal{T}_F$ as to be 1 as soon as $\alpha \leq -1$. It was shown in a previous work by Bertoin [15] that when $\alpha < -1$, for every fixed t the number of fragments at time t is a.s. finite, so that -1 is indeed the threshold under which fragments decay extremely fast. One should then picture the CRT $\mathcal{T}_F$ as a “dead tree” looking like a handful of thin sticks connected to each other, while when $|\alpha| < 1$ the tree looks more like a dense “bush”. Last, the integrability assumption in the theorem seems to be reasonably mild; its heuristic meaning is that when a fragmentation occurs, the largest resulting fragment is not too small. In particular, it is always satisfied in the case of fragmentations for which $\nu(s_{N+1} > 0) = 0$, since then $s_1 > 1/N$ for ν-a.e. $\mathbf{s}$. Yet, we point out that when $\int_{\mathcal{S}^{\downarrow}} \left(s_1^{-1} - 1\right) \nu(\mathrm{d}\mathbf{s}) = \infty$, one anyway obtains the following bounds for the Hausdorff dimension of $\mathcal{L}(\mathcal{T}_F)$:

$$\frac{\varrho}{|\alpha|} \leq \dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F)) \leq \frac{1}{|\alpha|} \ \text{a.s.}$$

where

$$\varrho := \sup\left\{p \leq 1 : \int_{\mathcal{S}^{\downarrow}} \left(s_1^{-p} - 1\right) \nu(\mathrm{d}\mathbf{s}) < \infty\right\}. \tag{3.2}$$

We do not know whether the condition $\int_{\mathcal{S}^{\downarrow}} (s_1^{-1} - 1)\nu(\mathrm{d}\mathbf{s}) < \infty$ is necessary for (3.1), as we are not aware of any self-similar fragmentation with index α such that the associated CRT has leaf-dimension strictly less than $1/|\alpha|$.

It is worth noting that these results allow as a special case to compute the Hausdorff dimension of the so-called *stable trees* of Duquesne and Le Gall [29], which were used to construct fragmentations in the manner of Theorem 3.1 in [56]. The dimension of the stable tree (as well as finer results of Hausdorff measures on more general Lévy trees) has been obtained independently in [30]. The stable tree is a CRT whose law depends on parameter $\beta \in (1, 2]$, and it satisfies the required self-similarity property of Proposition 3.1 with index $1/\beta - 1$. We check that the associated dislocation measure satisfies the integrability condition of Theorem 3.2 in Sect. 3.3.5, so that

Corollary 3.1 *Fix $\beta \in (1, 2]$. The β-stable tree has Hausdorff dimension $\beta/(\beta - 1)$.*

An interesting process associated with a given continuum tree (T, μ) is the so-called *cumulative height profile* $\bar{W}_T(h) = \mu\{v \in T : \mathrm{ht}(v) \leq h\}$, which is non-decreasing and bounded by 1 on $\mathbb{R}_+$. It may happen that the Stieltjes measure $\mathrm{d}\bar{W}_T(h)$ is absolutely continuous with respect to Lebesgue measure, in which case its density $(W_T(h), h \geq 0)$ is called the *height profile*, or *width process* of the tree. In our setting, for any fragmentation F satisfying the hypotheses of Theorem 3.1, the cumulative height profile has the following interpretation: one has $(\bar{W}_{\mathcal{T}_F}(h), h \geq 0)$ has the same law as $(M_F(h), h \geq 0)$, where $M_F(h) = 1 - \sum_{i \geq 1} F_i(h)$ is the total mass lost by the fragmentation at time h. Detailed conditions for existence (or non-existence) of the width profile $\mathrm{d}M_F(h)/\mathrm{d}h$ have been given in [39]. It was also proved there that under some mild assumptions $\dim_{\mathcal{H}}(\mathrm{d}M_F) \geq 1 \wedge A/|\alpha|$ a.s., where A is a ν-dependent parameter introduced in (3.10) below, and

$$\dim_{\mathcal{H}}(\mathrm{d}M_F) := \inf\{\dim_{\mathcal{H}}(E) \,:\, \mathrm{d}M_F(E) = 1\}.$$

The upper bound we obtain for $\dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F))$ allows us to complete this result:

Corollary 3.2 *Let F be a ranked self-similar fragmentation with same hypotheses as in Theorem 3.1. Then* $\dim_{\mathcal{H}}(\mathrm{d}M_F) \leq 1 \wedge 1/|\alpha|$ *a.s.*

Notice that this result re-implies the fact from [39] that the height profile does not exist as soon as $|\alpha| \geq 1$.

The last motivation of this paper (Sect. 3.4) is about relations between CRTs and their so-called encoding *height processes*. The fragmentation F_B of [14], as well as the fragmentations from [56], were defined out of certain random functions $(H_s, 0 \leq s \leq 1)$. Let us describe briefly the construction of F_B. Let B^{exc} be the standard Brownian excursion with duration 1, and consider the open set $\{s \in [0,1] : 2B^{\mathrm{exc}}_s > t\}$. Write $F(t)$ for the decreasing sequence of the lengths of its interval components. Then F has the same law as the fragmentation F'_B defined out of the Brownian CRT in the same way as in Theorem 3.1. This is immediate from the description of Le Gall [51] and Aldous [3] of the Brownian tree as being encoded in the Brownian excursion. To be concise, define a pseudo-metric on $[0,1]$ by letting $\overline{d}(s,s') = 2B^{\mathrm{exc}}_s + 2B^{\mathrm{exc}}_{s'} - 4\inf_{u \in [s,s']} B^{\mathrm{exc}}_u$, with the convention that $[s,s'] = [s',s]$ if $s' < s$. We can define a true metric space by taking the quotient with respect to the equivalence relation $s \equiv s' \iff \overline{d}(s,s') = 0$. Call $(\mathcal{T}_B, d)$ this metric space. Write μ_B for the measure induced on $\mathcal{T}_B$ by Lebesgue measure on $[0,1]$. Then $(\mathcal{T}_B, \mu_B)$ is the Brownian CRT, and the equality in law of the fragmentations F_B and F'_B follows immediately from the definition of the mass measure. Our next result generalizes this construction.

Theorem 3.3 *Let F be a ranked self-similar fragmentation with same hypotheses as in Theorem 3.1, and suppose ν has infinite total mass. Then there exists a continuous random function $(H_F(s), 0 \leq s \leq 1)$, called the height function, such that $H_F(0) = H_F(1)$, $H_F(s) > 0$ for every $s \in (0,1)$, and such that F has the same law as the fragmentation F' defined by: $F'(t)$ is the decreasing rearrangement of the lengths of the interval components of the open set $I_F(t) = \{s \in (0,1) : H_F(s) > t\}$.*

An interesting point in this construction is also that it shows that a large class of self-similar fragmentation with negative index has a natural *interval representation*, given by $(I_F(t), t \geq 0)$.

Bertoin [14, Lemma 6] had already constructed such an interval representation, I'_F say, but ours is different qualitatively. We will see in the sequel that our representation is intuitively obtained by putting the intervals obtained from the dislocation of a largest interval in exchangeable random order, while Bertoin's method is to put these same intervals from left to right by size-biased random order. In particular, For example, Bertoin's interval fragmentation I'_F cannot be written in the form $I'_F(t) = \{s \in (0,1) : H(s) > t\}$ for any continuous process H.

In parallel to the computation of the Hausdorff dimension of the CRTs built above, we are able to estimate Hölder coefficients for the height processes of these CRTs. Our result is

Theorem 3.4 *Suppose $\nu(\mathcal{S}^{\downarrow}) = \infty$, and set*

$$\vartheta_{\mathrm{low}} := \quad \sup\left\{b > 0 : \lim_{x\downarrow 0} x^b \nu(s_1 < 1-x) = \infty\right\},$$
$$\vartheta_{\mathrm{up}} := \quad \inf\left\{b > 0 : \lim_{x\downarrow 0} x^b \nu(s_1 < 1-x) = 0\right\}.$$

Then the height process H_F is a.s. Hölder-continuous of order γ for every $\gamma < \vartheta_{\mathrm{low}} \wedge |\alpha|$, and, provided that $\int_{\mathcal{S}^{\downarrow}}(s_1^{-1}-1)\nu(\mathrm{d}\mathbf{s}) < \infty$, a.s. not Hölder-continuous of order γ for every $\gamma > \vartheta_{\mathrm{up}} \wedge |\alpha|$.

Again we point out that one actually obtains an upper bound for the maximal Hölder coefficient even when $\int_{\mathcal{S}^{\downarrow}}(s_1^{-1}-1)\nu(\mathrm{d}\mathbf{s}) = \infty$: with ϱ defined by (3.2), a.s. H_F cannot be Hölder-continuous of order γ for any $\gamma > \vartheta_{\mathrm{up}} \wedge |\alpha|/\varrho$.

Note that $\vartheta_{\mathrm{low}}, \vartheta_{\mathrm{up}}$ depend only on the characteristics of the fragmentation process, and more precisely, on the behavior of ν when s_1 is close to 1. By contrast, our Hausdorff dimension result for the tree depended on a hypothesis on the behavior of ν when s_1 is near 0. Remark also that ϑ_{up} may be strictly smaller than 1. Therefore, the Hausdorff dimension of $\mathcal{T}_F$ is in general not equal to the inverse of the maximal Hölder coefficient of the height process, as one could have expected. However, this turns out to be true in the case of the stable tree, as will be checked in Section 3.4.4:

Corollary 3.3 *The height process of the stable tree with index $\beta \in (1,2]$ is a.s. Hölder-continuous of any order $\gamma < 1-1/\beta$, but a.s. not of order $\gamma > 1-1/\beta$.*

When $\beta = 2$, this just states that the Brownian excursion is Hölder-continuous of any order $< 1/2$, a result that is well-known for Brownian motion and which readily transfers to the normalized Brownian excursion (e.g. by rescaling the first excursion of Brownian motion whose duration is greater than 1). The general result had been obtained in [29] by completely different methods.

Last, we mention that most of our results extend to a more general class of fragmentations in which a fragment with mass x splits to give fragments with masses $x\mathbf{s}$, $\mathbf{s} \in \mathcal{S}^{\downarrow}$, at rate $\tau(x)\nu(\mathrm{d}\mathbf{s})$ for some non-negative continuous function τ on $(0,1]$ (see [38] for a rigorous definition). The proofs of the above theorems easily adapt to give the following results: when $\liminf_{x\to 0} x^{-b}\tau(x) > 0$ for some $b < 0$, the fragmentation can be encoded as above into a

CRT and, provided that ν is infinite, into a height function. The set of leaves of the CRT then has a Hausdorff dimension smaller than $1/|b|$ and the height function is γ-Hölder continuous for every $\gamma < \vartheta_{\mathrm{low}} \wedge |b|$. If moreover $\limsup_{x\to 0} x^{-a}\tau(x) < \infty$ for some $a < 0$ and $\int_{\mathcal{S}^{\downarrow}} \left(s_1^{-1} - 1\right) \nu(\mathrm{d}\mathbf{s}) < \infty$, the Hausdorff dimension is larger than $1/|a|$ and the height function cannot have a Hölder coefficient $\gamma > \vartheta_{\mathrm{sup}} \wedge |a|$.

3.2 The CRT $\mathcal{T}_F$

Building the CRT $\mathcal{T}_F$ associated with a ranked fragmentation F will be done by determining its "marginals", i.e. the subtrees spanned by a finite but arbitrary number of randomly chosen leaves. To this purpose, it will be useful to use partition-valued fragmentations, which we first define, as well as a certain family of trees with edge-lengths.

3.2.1 Exchangeable partitions and partition-valued self-similar fragmentations

Let $\mathcal{P}_\infty$ be the set of (unordered) partitions of $\mathbb{N} = \{1, 2, \ldots\}$ and $[n] = \{1, 2, \ldots, n\}$. For $i, j \in \mathbb{N}$, we write $i \overset{\pi}{\sim} j$ if i and j are in the same block of π. We adopt the following ordering convention: for $\pi \in \mathcal{P}_\infty$, we let $(\pi_1, \pi_2, \ldots)$ be the blocks of π, so that π_i is the block containing i provided that i is the smallest integer of the block and $\pi_i = \varnothing$ otherwise. We let $\mathbb{O} = \{\{1\}, \{2\}, \ldots\}$ be the partition of $\mathbb{N}$ into singletons. If $B \subset \mathbb{N}$ and $\pi \in \mathcal{P}_\infty$ we let $\pi \cap B$ (or $\pi|_B$) be the restriction of π to B, i.e. the partition of B whose collection of blocks is $\{\pi_i \cap B, i \geq 1\}$. If $\pi \in \mathcal{P}_\infty$ and $B \in \pi$ is a block of π, we let

$$|B| = \lim_{n\to\infty} \frac{\#(B \cap [n])}{n}$$

be the asymptotic frequency of the block B, whenever it exists. A random variable π with values in $\mathcal{P}_\infty$ is called *exchangeable* if its law is invariant under the natural action of permutations of $\mathbb{N}$ on $\mathcal{P}_\infty$. By a theorem of Kingman [44, 1], all the blocks of such random partitions admit asymptotic frequencies a.s. For π whose blocks have asymptotic frequencies, we let $|\pi| \in \mathcal{S}^{\downarrow}$ be the decreasing sequence of these frequencies. Kingman's theorem more precisely says that the law of any exchangeable random partition π is a (random) "paintbox process", a term we now explain. Take $\mathbf{s} \in \mathcal{S}^{\downarrow}$ (the paintbox) and consider a sequence $U_1, U_2, \ldots$ of i.i.d. variables in $\mathbb{N} \cup \{0\}$ (the colors) with $P(U_1 = j) = s_j$ for $j \geq 1$ and $P(U_1 = 0) = 1 - \sum_k s_k$. Define a partition π on $\mathbb{N}$ by saying that $i \neq j$ are in the same block if and only if $U_i = U_j \neq 0$ (i.e. i and j have the same color, where 0 is considered as colorless). Call $\rho_{\mathbf{s}}(\mathrm{d}\pi)$ its law, the $\mathbf{s}$-*paintbox* law. Kingman's theorem says that the law of any random partition is a mixing of paintboxes, i.e. it has the form $\int_{\mathbf{s}\in\mathcal{S}^{\downarrow}} m(\mathrm{d}\mathbf{s})\rho_{\mathbf{s}}(\mathrm{d}\pi)$ for some probability measure m on $\mathcal{S}^{\downarrow}$. A useful consequence is that the block of an exchangeable partition π containing 1, or some prescribed integer i, is a *size-biased pick* from the blocks of π, i.e. the probability it equals a non-singleton block π_j conditionally on $(|\pi_j|, j \geq 1)$ equals $|\pi_j|$. Similarly,

Lemma 3.1 *Let π be an exchangeable random partition which is a.s. different from the trivial partition $\mathbb{O}$, and B an infinite subset of $\mathbb{N}$. For any $i \in \mathbb{N}$, let*

$$\widetilde{i} = \inf\{j \geq i : j \in B \text{ and } \{j\} \notin \pi\},$$

then $\widetilde{i} < \infty$ a.s. and the block $\widetilde{\pi}$ of π containing $\widetilde{i}$ is a size-biased pick among the non-singleton blocks of π, i.e. if we denote these by $\pi'_1, \pi'_2, \ldots$,

$$P(\widetilde{\pi} = \pi'_k | (|\pi'_j|, j \geq 1)) = |\pi'_k| / \sum_j |\pi'_j|.$$

For any sequence of partitions $(\pi^{(i)}, i \geq 1)$, define $\pi = \bigcap_{i \geq 1} \pi^{(i)}$ by

$$k \overset{\pi}{\sim} j \iff k \overset{\pi^{(i)}}{\sim} j \quad \forall i \geq 1.$$

Lemma 3.2 *Let $(\pi^{(i)}, i \geq 1)$ be a sequence of independent exchangeable partitions and set $\pi := \bigcap_{i \geq 1} \pi^{(i)}$. Then, a.s. for every $j \in \mathbb{N}$,*

$$|\pi_j| = \prod_{i \geq 1} \left| \pi^{(i)}_{k(i,j)} \right|,$$

where $(k(i,j), j \geq 1)$ is defined so that $\pi_j = \bigcap_{i \geq 1} \pi^{(i)}_{k(i,j)}$.

Proof. First notice that $k(i,j) \leq j$ for all $i \geq 1$ a.s. This is clear when $\pi_j \neq \varnothing$, since $j \in \pi_j$ and then $j \in \pi^{(i)}_{k(i,j)}$. When $\pi_j = \varnothing$, $j \in \pi_m$ for some $m < j$ and then m and j belong to the same block of $\pi^{(i)}$ for all $i \geq 1$. Thus $k(i,j) \leq m < j$. Using then the paintbox construction of exchangeable partitions explained above and the independence of the $\pi^{(i)}$'s, we see that the r.v. $\prod_{i \geq 1} \mathbf{1}_{\{m \in \pi^{(i)}_{k(i,j)}\}}$, $m \geq j+1$, are iid conditionally on $(|\pi^{(i)}_{k(i,j)}|, i \geq 1)$ with a mean equal to $\prod_{i \geq 1} |\pi^{(i)}_{k(i,j)}|$. The law of large numbers therefore gives

$$\prod_{i \geq 1} \left| \pi^{(i)}_{k(i,j)} \right| = \lim_{n \to \infty} \frac{1}{n} \sum_{j+1 \leq m \leq n} \prod_{i \geq 1} \mathbf{1}_{\left\{m \in \pi^{(i)}_{k(i,j)}\right\}} \text{ a.s.}$$

On the other hand, the random variables $\prod_{i \geq 1} \mathbf{1}_{\{m \in \pi^{(i)}_{k(i,j)}\}} = \mathbf{1}_{\{m \in \pi_j\}}$, $m \geq j+1$, are i.i.d. conditionally on $|\pi_j|$ with mean $|\pi_j|$ and then the limit above converges a.s. to $|\pi_j|$, again by the law of large numbers. $\blacksquare$

We now turn our attention to partition-valued fragmentations.

Definition 3.3 *Let $(\Pi(t), t \geq 0)$ be a Markovian $\mathcal{P}_\infty$-valued process with $\Pi(0) = \{\mathbb{N}, \varnothing, \varnothing, \ldots\}$ that is continuous in probability and exchangeable as a process (meaning that the law of Π is invariant under the action of permutations). Call it a partition-valued self-similar fragmentation with index $\alpha \in \mathbb{R}$ if moreover $\Pi(t)$ admits asymptotic frequencies for all t, a.s., if the process $(|\Pi(t)|, t \geq 0)$ is continuous in probability, and if the following fragmentation property is satisfied. For $t, t' \geq 0$, given $\Pi(t) = (\pi_1, \pi_2, \ldots)$, the sequence $\Pi(t+t')$ has the same law as the partition with blocks $\pi_1 \cap \Pi^{(1)}(|\pi_1|^\alpha t'), \pi_2 \cap \Pi^{(2)}(|\pi_2|^\alpha t'), \ldots$, where $(\Pi^{(i)}, i \geq 1)$ are independent copies of Π.*

Bertoin [14] has shown that any such fragmentation is also characterized by the same 3-tuple (α, c, ν) as above, meaning that the laws of partition-valued and ranked self-similar fragmentations are in a one-to-one correspondence. In fact, for every (α, c, ν), one can construct a version of the partition-valued fragmentation Π with parameters (α, c, ν), and then $(|\Pi(t)|, t \geq 0)$ is a ranked fragmentation with parameters (α, c, ν). Let us build this version now. It is done following [13, 14] by a Poissonian construction. Recall the notation $\rho_{\mathbf{s}}(\mathrm{d}\pi)$, and define $\kappa_\nu(\mathrm{d}\pi) = \int_{\mathcal{S}^\downarrow} \nu(\mathrm{d}\mathbf{s})\rho_{\mathbf{s}}(\mathrm{d}\pi)$. Let $\#$ be the counting measure on $\mathbb{N}$ and let (Δ_t, k_t) be a $\mathcal{P}_\infty \times \mathbb{N}$-valued Poisson point process with intensity $\kappa_\nu \otimes \#$. We may construct a process $(\Pi^0(t), t \geq 0)$ by letting $\Pi^0(0)$ be the trivial partition $(\mathbb{N}, \varnothing, \varnothing, \ldots)$, and saying that Π^0 jumps only at times t when an atom (Δ_t, k_t) occurs. When this is the case, Π^0 jumps from the state $\Pi^0(t-)$ to the following partition $\Pi^0(t)$: replace the block $\Pi^0_{k_t}(t-)$ by $\Pi^0_{k_t}(t-) \cap \Delta_t$, and leave the other blocks unchanged. Such a construction can be made rigorous by considering restrictions of partitions to the first n integers and by a consistency argument. Then Π^0 has the law of the fragmentation with parameters $(0, 0, \nu)$.

Out of this "homogeneous" fragmentation, we construct the $(\alpha, 0, \nu)$-fragmentation by introducing a time-change. Call $\lambda_i(t)$ the asymptotic frequency of the block of $\Pi^0(t)$ that contains i, and write

$$T_i(t) = \inf\left\{ u \geq 0 : \int_0^u \lambda_i(r)^{-\alpha}\mathrm{d}r > t \right\}. \tag{3.3}$$

Last, for every $t \geq 0$ we let $\Pi(t)$ be the random partition such that i, j are in the same block of $\Pi(t)$ if and only if they are in the same block of $\Pi^0(T_i(t))$, or equivalently of $\Pi^0(T_j(t))$. Then $(\Pi(t), t \geq 0)$ is the wanted version. Let $(\mathcal{G}(t), t \geq 0)$ be the natural filtration generated by Π completed up to P-null sets. According to [14], the fragmentation property holds actually for $\mathcal{G}$-stopping times and we shall refer to it as the *strong fragmentation property.* In the homogeneous case, we will rather call $\mathcal{G}^0$ the natural filtration.

When $\alpha < 0$, the loss of mass in the ranked fragmentations shows up at the level of partitions by the fact that a positive fraction of the blocks of $\Pi(t)$ are singletons for some $t > 0$. This last property of self-similar fragmentations with negative index allows us to build a collection of trees with edge-lengths.

3.2.2 Trees with edge-lengths

A tree is a finite connected graph with no cycles. It is *rooted* when a particular vertex (the root) is distinguished from the others, in this case the edges are by convention oriented, pointing from the root, and we define the out-degree of a vertex v as being the number of edges that point outward from v. A *leaf* in a rooted tree is a vertex with out-degree 0. For $k \geq 1$, let $\mathbf{T}_k$ be the set of rooted trees with exactly k labeled leaves (the names of the labels may change according to what we see fit), the other vertices (except the root) begin unlabeled , and such that the root is the only vertex that has out-degree 1. If $\mathbf{t} \in \mathbf{T}_k$, we let $E(\mathbf{t})$ be the set of its edges.

A *tree with edge-lengths* is a pair $\vartheta = (\mathbf{t}, \mathbf{e})$ for $\mathbf{t} \in \bigcup_{k\geq 1} \mathbf{T}_k$ and $\mathbf{e} = (e_i, i \in E(t)) \in (\mathbb{R}_+ \setminus \{0\})^{E(\mathbf{t})}$. Call $\mathbf{t}$ the *skeleton* of ϑ. Such a tree is naturally equipped with a distance $d(v, w)$ on the set of its vertices, by adding the lengths of edges that appear in the unique path connecting v and w in the skeleton (which we still denote by $[[v, w]]$). The height of a vertex is its distance to the root. We let $\mathbb{T}_k$ be the set of trees with edge-lengths whose skeleton is in $\mathbf{T}_k$. For $\vartheta \in \mathbb{T}_k$, let e_{root} be the length of the unique edge connected to the root, and for $e < e_{\mathrm{root}}$ write $\vartheta - e$ for the tree with edge-lengths that has same skeleton and same edge-lengths as ϑ, but for the edge pointing outward from the root which is assigned length $e_{\mathrm{root}} - e$.

We also define an operation MERGE as follows. Let $n \geq 2$ and take $\vartheta_1, \vartheta_2, \dots, \vartheta_n$ respectively in $\mathbb{T}_{k_1}, \mathbb{T}_{k_2}, \dots, \mathbb{T}_{k_n}$, with leaves $(L_i^1, 1 \leq i \leq k_1), (L_i^2, 1 \leq i \leq k_2), \dots, (L_i^n, 1 \leq i \leq k_n)$ respectively. Let also $e > 0$. The tree with edge-lengths $\mathtt{MERGE}((\vartheta_1, \dots, \vartheta_n); e) \in \mathbb{T}_{\sum_i k_i}$ is defined by merging together the roots of $\vartheta_1, \dots, \vartheta_n$ into a single vertex $\bullet$, and by drawing a new edge root $\to \bullet$ with length e.

Last, for $\vartheta \in \mathbb{T}_k$ and i vertices $v_1, \dots, v_i$, define the subtree spanned by the root and $v_1, \dots, v_i$ as follows. For every $p \neq q$, let $b(v_p, v_q)$ be the branchpoint of v_p and v_q, that is, the highest point in the tree that belongs to $[[\mathrm{root}, v_p]] \cap [[\mathrm{root}, v_q]]$. The spanned tree is the tree with edge-lengths whose vertices are the root, the vertices $v_1, \dots, v_i$ and the branchpoints $b(v_p, v_q)$, $1 \leq p \neq q \leq i$, and whose edge-lengths are given by the respective distances between this subset of vertices of the original tree.

3.2.3 Building the CRT

Now for $B \subset \mathbb{N}$ finite, define $\mathcal{R}(B)$, a random variable with values in $\mathbb{T}_{\#B}$, whose leaf-labels are of the form L_i for $i \in \mathbb{N}$, as follows. Let $D_i = \inf\{t \geq 0 : \{i\} \in \Pi(t)\}$ be the first time when $\{i\}$ "disappears", i.e. is isolated in a singleton of $\Pi(t)$. For B a finite subset of $\mathbb{N}$ with at least two elements, let $D_B = \inf\{t \geq 0 : \#(B \cap \Pi(t)) \neq 1\}$ be the first time when the restriction of $\Pi(t)$ to B is non-trivial, i.e. has more than one block. By convention, $D_{\{i\}} = D_i$. For every $i \geq 1$, define $\mathcal{R}(\{i\})$ as a single edge root $\to L_i$, and assign this edge the length D_i. For B with $\#B \geq 2$, let $B_1, \dots, B_i$ be the non-empty blocks of $B \cap \Pi(D_B)$, arranged in increasing order of least element, and define a tree $\mathcal{R}(B)$ recursively by

$$\mathcal{R}(B) = \mathtt{MERGE}((\mathcal{R}(B_1) - D_B, \dots, \mathcal{R}(B_i) - D_B); D_B).$$

Last, define $\mathcal{R}(k) = \mathcal{R}([k])$. Notice that by definition of the distance, the distance between L_i and L_j in $\mathcal{R}(k)$ for any $k \geq i \vee j$ equals $D_i + D_j - 2D_{\{i,j\}}$.

We now state the key lemma that allows us to describe the CRT out of the family $(\mathcal{R}(k), k \geq 1)$ which is the candidate for the marginals of $\mathcal{T}_F$. By Aldous [3], it suffices to check two properties, called *consistency* and *leaf-tightness*. Notice that in [3], only binary trees (in which branchpoint have out-degree 2) are considered, but as noticed therein, this translates to our setting with minor changes.

Lemma 3.3 *(i) The family $(\mathcal{R}(k), k \geq 1)$ is consistent in the sense that for every k and $j \leq k$, $\mathcal{R}(j)$ has the same law as the subtree of $\mathcal{R}(k)$ spanned by the root and j distinct leaves $L_1^k, \dots, L_j^k$ taken uniformly at random from the leaves $L_1, \dots, L_k$ of $\mathcal{R}(k)$, independently of $\mathcal{R}(k)$.*

(ii) The family $(\mathcal{R}(k), k \geq 1)$ is leaf-tight, that is, with the above notations,

$$\min_{2 \leq j \leq k} d(L_1^k, L_j^k) \xrightarrow{p} 0.$$

Proof. The consistency property is an immediate consequence of the fact that the process Π is exchangeable. Taking j leaves uniformly out of the k ones of $\mathcal{R}(k)$ is just the same as if we had chosen exactly the leaves $L_1, L_2, \dots, L_j$, which give rise to the tree $\mathcal{R}(j)$, and this is (i).

For (ii), first notice that we may suppose by exchangeability that $L_1^k = L_1$. The only point is then to show that the minimal distance of this leaf to the leaves $L_2, \dots, L_k$ tends to 0 in probability as $k \to \infty$. Fix $\eta > 0$ and for $\varepsilon > 0$ write $t_\varepsilon^1 = \inf\{t \geq 0 : |\Pi_1(t)| < \varepsilon\}$, where $\Pi_1(t)$ is the block of $\Pi(t)$ containing 1. Then t_ε^1 is a stopping time with respect to the natural filtration $(\mathcal{F}_t, t \geq 0)$ associated with Π and $t_\varepsilon^1 \uparrow D_1$ as $\varepsilon \downarrow 0$. By the strong Markov property and exchangeability, one has that if $K(\varepsilon) = \inf\{k > 1 : k \in \Pi_1(t_\varepsilon^1)\}$, then $P(D_1 + D_{K(\varepsilon)} - 2t_\varepsilon^1 < \eta) = E[P_{\Pi(t_\varepsilon^1)}(D_1 + D_{K(\varepsilon)} < \eta)]$ where P_π is the law of the fragmentation Π started at π (the law of Π under P_π is the same as that of the family of partitions $\left(\left\{\text{blocks of } \pi_1 \cap \Pi^{(1)}(|\pi_1|^\alpha t), \pi_2 \cap \Pi^{(2)}(|\pi_2|^\alpha t), \dots\right\}, t \geq 0\right)$ where the $\Pi^{(i)}$'s, $i \geq 1$, are independent copies of Π under $P_{\{\mathbb{N}, \varnothing, \varnothing, \dots\}}$). By the self-similar fragmentation property and exchangeability this is greater than $P(D_1 + D_2 < \varepsilon^\alpha \eta)$, which in turn is greater than $P(2\zeta < \varepsilon^\alpha \eta)$ where ζ is the first time where $\Pi(t)$ becomes the partition into singletons, which by [15] is finite a.s. This last probability thus goes to 1 as $\varepsilon \downarrow 0$. Taking $\varepsilon = \varepsilon(n) \downarrow 0$ quickly enough as $n \to \infty$ and applying the Borel-Cantelli lemma, we a.s. obtain a sequence $K(\varepsilon(n))$ such that $d(L_1, L_{K(n)}) \leq D_1 + D_{K(\varepsilon(n))} - 2t_{\varepsilon(n)} < \eta$. Hence the result. ■

For a rooted $\mathbb{R}$-tree T and k vertices $v_1, \dots, v_k$, we define exactly as for marked trees the subtree spanned by the root and $v_1, \dots, v_k$, as an element of $\mathbb{T}_k$. A consequence of [3, Theorem 3] is then:

Lemma 3.4 *There exists a CRT $(\mathcal{T}_\Pi, \mu_\Pi)$ such that if $Z_1, \dots, Z_k$ is a sample of k leaves picked independently according to μ_Π conditionally on μ_Π, the subtree of $\mathcal{T}_\Pi$ spanned by the root and $Z_1, \dots, Z_k$ has the same law as $\mathcal{R}(k)$.*

In the sequel, sequences like $(Z_1, Z_2, \dots)$ will be called *exchangeable sequences with directing measure* μ_Π.

Proof of Theorem 3.1. We have to check that the tree $\mathcal{T}_\Pi$ of the preceding lemma gives rise to a fragmentation process with the same law as $F = |\Pi|$. By construction, we have that for every $t \geq 0$ the partition $\Pi(t)$ is such that i and j are in the same block of $\Pi(t)$ if and only if L_i and L_j are in the same connected component of $\{v \in \mathcal{T}_\Pi : \mathrm{ht}(v) > t\}$. Hence, the law of large numbers implies that if $F'(t)$ is the decreasing sequence of the μ-masses of these connected components, then $F'(t) = F(t)$ a.s. for every t. Hence, F' is a version of F, so we can set $\mathcal{T}_F = \mathcal{T}_\Pi$. That $\mathcal{T}_F$ is α-self-similar is an immediate consequence of the fragmentation and self-similar properties of F.

We now turn to the last statement of Theorem 3.1. With the notation of Lemma 3.4 we will show that the path $[[\varnothing, Z_1]]$ is almost-surely in the closure of the set of leaves of $\mathcal{T}_F$ if and only if $\nu(\mathcal{S}^\downarrow) = \infty$. Then it must hold by exchangeability that so do the paths $[[\varnothing, Z_i]]$ for every $i \geq 1$, and this is sufficient because the definition of the CRTs implies that $\mathcal{S}(\mathcal{T}_F) = \bigcup_{i \geq 1} [[\varnothing, Z_i[[$, see [3, Lemma 6] (the fact that $\mathcal{T}_F$ is a.s. compact will be proved below). To this end, it suffices to show that for any $a \in (0,1)$, the point aZ_1 of $[[\varnothing, Z_1]]$ that is at a proportion a from $\varnothing$ (the point $\varphi_{\varnothing, Z_1}(ad(\varnothing, Z_1))$ with the above notations) can be approached closely by leaves, that is, for $\eta > 0$ there exists $j > 1$ such that $d(aZ_1, Z_j) < \eta$. It thus suffices to check that for any

$\delta > 0$

$$P(\exists 2 \leq j \leq k : |D_{\{1,j\}} - aD_1| < \delta \text{ and } D_j - D_{\{1,j\}} < \delta) \underset{k\to\infty}{\to} 1, \tag{3.4}$$

with the above notations derived from Π (this is a slight variation of [3, (iii) a). Theorem 15]).

Suppose that $\nu(\mathcal{S}^\downarrow) = \infty$. Then for every rational $r > 0$ such that $|\Pi_1(r)| \neq 0$ and for every $\delta > 0$, the block containing 1 undergoes a fragmentation in the time-interval $(r, r+\delta/2)$. This is obvious from the Poisson construction of the self-similar fragmentation Π given above, because ν is an infinite measure so there is an infinite number of atoms of (Δ_t, k_t) with $k_t = 1$ in any time-interval with positive length. Therefore, there exists an infinite number of elements of $\Pi_1(r)$ that are isolated in singletons of $\Pi(r+\delta)$, e.g. because of Lemma 3.5 below which asserts that only a finite number of the blocks of $\Pi(r+\delta/2)$ "survive" at time $r+\delta$, i.e. is not completely reduced to singletons. Thus, an infinite number of elements of $\Pi_1(r)$ correspond to leaves of some $\mathcal{R}(k)$ for k large enough. By taking r close to aD_1 we thus have the result.

On the other hand, if $\nu(\mathcal{S}^\downarrow) < \infty$, it follows from the Poisson construction that the state $(1, 0, \ldots)$ is a holding state, so the first fragmentation occurs at a positive time, so the root cannot be approached by leaves. ■

Remark. We have seen that we may actually build simultaneously the trees $(\mathcal{R}(k), k \geq 1)$ on the same probability space as a measurable functional of the process $(\Pi(t), t \geq 0)$. This yields, by redoing the "special construction" of Aldous [3], a *stick-breaking construction* of the tree $\mathcal{T}_F$, by now considering the trees $\mathcal{R}(k)$ as $\mathbb{R}$-trees obtained as finite unions of segments rather than trees with edge-lengths (one can check that it is possible to switch between the two notions). The mass measure is then defined as the limit of the empirical measure on the leaves $L_1, \ldots, L_n$. The special CRT thus constructed is a subset of ℓ^1 in [3], but we consider it as universal, i.e. up to isomorphism. The tree $\mathcal{R}(k+1)$ is then obtained from $\mathcal{R}(k)$ by branching a new segment with length $D_{k+1} - \max_{B \subset [k], B \neq \varnothing} D_{B \cup \{k+1\}}$, and $\mathcal{T}_F$ can be reinterpreted as the completion of the metric space $\bigcup_{k\geq 1} \mathcal{R}(k)$. On the other hand, call $L_1, L_2, \ldots$ as before the leaves of $\bigcup_{k\geq 1} \mathcal{R}(k)$, L_k being the leaf corresponding to the k-th branch. One of the subtleties of the special construction of [3] is that $L_1, L_2, \ldots$ is not itself an exchangeable sample with the mass measure as directing law. However, considering such a sample $Z_1, Z_2, \ldots$, we may construct a random partition $\Pi'(t)$ for every t by letting $i \sim^{\Pi'(t)} j$ if and only if Z_i and Z_j are in the same connected component of the forest $\{v \in \mathcal{T}_F : \mathrm{ht}(v) > t\}$. Then easily $\Pi'(t)$ is again a partition-valued self-similar fragmentation, and in fact $|\Pi'(t)| = F(t)$ a.s. for every t so Π' has same law as Π (Π' can be interpreted as a "relabeling" of the blocks of Π). As a conclusion, up to this relabeling, we may and will assimilate $\mathcal{T}_F$ as the completion of the increasing union of the trees $\mathcal{R}(k)$, while $L_1, L_2, \ldots$ will be considered as an exchangeable sequence with directing law μ_F.

Proof of Proposition 3.1. The fact that the process F defined out of a CRT $(\mathcal{T}, \mu)$ with the stated properties is a $\mathcal{S}^\downarrow$-valued self-similar fragmentation with index α is straightforward and left to the reader. The treatment of the erosion and sudden loss of mass is a little more subtle. Let $Z_1, Z_2, \ldots$ be an exchangeable sample directed by the measure μ, and for every $t \geq 0$ define a random partition $\Pi(t)$ by saying that i and j are in the same block of $\Pi(t)$ if Z_i and Z_j fall in the same tree component of $\{v \in \mathcal{T} : \mathrm{ht}(v) > t\}$. By the arguments above, Π defines a self-similar partition-valued fragmentation such that $|\Pi(t)| = F(t)$ a.s. for every t. Notice that if we show that the erosion coefficient $c = 0$ and that no sudden loss of mass occur, it will immediately follow that $\mathcal{T}$ has the same law as $\mathcal{T}_F$.

Now suppose that $\nu(\sum_i s_i < 1) \neq 0$. Then (e.g. by the Poisson construction of fragmentations described above) there exist a.s. two distinct integers i and j and a time D such that i and j are in the same block of $\Pi(D-)$ but $\{i\} \in \Pi(D)$ and $\{j\} \in \Pi(D)$. This implies that $Z_i = Z_j$, so μ has a.s. an atom and $(\mathcal{T}, \mu)$ cannot be a CRT. On the other hand, suppose that the erosion coefficient $c > 0$. Again from the Poisson construction, we see that there a.s. exists a time D such that $\{1\} \notin \Pi(D-)$ but $\{1\} \in \Pi(D)$, and nevertheless $\Pi(D) \cap \Pi_1(D-)$ is not the trivial partition $\mathbb{O}$. Taking j in a non-trivial block of this last partition and denoting its death time by D', we obtain that the distance from Z_1 to Z_j is $D' - D$, while the height of Z_1 is D and that of Z_j is D'. This implies that Z_1 is a.s. not in the set of leaves of $\mathcal{T}$, again contradicting the definition of a CRT. ■

3.3 Hausdorff dimension of $\mathcal{T}_F$

Let (M, d) be a compact metric space. For $\mathcal{E} \subseteq M$, the Hausdorff dimension of $\mathcal{E}$ is the real number

$$\dim_{\mathcal{H}}(\mathcal{E}) := \inf \{\gamma > 0 : m_\gamma(\mathcal{E}) = 0\} = \sup \{\gamma > 0 : m_\gamma(\mathcal{E}) = \infty\}, \tag{3.5}$$

where

$$m_\gamma(\mathcal{E}) := \sup_{\varepsilon > 0} \inf \sum_i \Delta(E_i)^\gamma, \tag{3.6}$$

the infimum being taken over all collections $(E_i, i \geq 1)$ of subsets of $\mathcal{E}$ with diameter $\Delta(E_i) \leq \varepsilon$, whose union covers $\mathcal{E}$. This dimension is meant to measure the "fractal size" of the considered set. For background on this subject, we mention [33] (in the case $M = \mathbb{R}^n$, but the generalization to general metric spaces of the results we will need is straightforward).

The goal of this section is to prove Theorem 3.2 and more generally that

$$\frac{\varrho}{|\alpha|} \leq \dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F)) \leq \frac{1}{|\alpha|} \text{ a.s.}$$

where ϱ is the ν-dependent parameter defined by (3.2). The proof is divided in the two usual upper and lower bound parts. In Section 3.3.1, we first prove that $\mathcal{T}_F$ is indeed compact and that $\dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F)) \leq 1/|\alpha|$ a.s., which is true without any extra integrability assumption on ν. We then show that this upper bound yields $\dim_{\mathcal{H}}(\mathrm{d}M_F) \leq 1 \wedge 1/|\alpha|$ a.s. (Corollary 3.2). Sections 3.3.2 to 3.3.4 are devoted to the lower bound $\dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F)) \geq \varrho/|\alpha|$ a.s. This is obtained by using appropriate subtrees of $\mathcal{T}_F$ (we will see that the most naive way to apply Frostman's energy method with the mass measure μ_F fails in general). That Theorem 3.2 applies to stable trees is proved in Sect. 3.3.5.

3.3.1 Upper bound

We begin by stating the expected

Lemma 3.5 *The tree $\mathcal{T}_F$ is a.s. compact.*

Proof. For $t \geq 0$ and $\varepsilon > 0$, denote by N_t^ε the number of blocks of $\Pi(t)$ not reduced to singletons that are not entirely reduced to dust at time $t+\varepsilon$. We first prove that N_t^ε is a.s. finite. Let $(\Pi_i(t), i \geq 1)$ be the blocks of $\Pi(t)$, and $(|\Pi_i(t)|, i \geq 1)$, their respective asymptotic frequencies. For integers i such that $|\Pi_i(t)| > 0$, that is $\Pi_i(t) \neq \varnothing$ and $\Pi_i(t)$ is not reduced to a singleton, let $\zeta_i := \inf\{s > t : \Pi_i(t) \cap \Pi(s) = \mathbb{O}\}$ be the first time at which the block $\Pi_i(t)$ is entirely reduced to dust. Applying the fragmentation property at time t, we may write ζ_i as $\zeta_i = t + |\Pi_i(t)|^{|\alpha|} \widetilde{\zeta}_i$ where $\widetilde{\zeta}_i$ is a r.v. independent of $\mathcal{G}(t)$ that has same distribution as $\zeta = \inf\{t \geq 0 : \Pi(t) = \mathbb{O}\}$, the first time at which the fragmentation is entirely reduced to dust. Now, fix $\varepsilon > 0$. The number of blocks of $\Pi(t)$ that are not entirely reduced to dust at time $t + \varepsilon$, which could be *a priori* infinite, is then given by

$$N_t^\varepsilon = \sum_{i:|\Pi_i(t)|>0} \mathbf{1}_{\left\{|\Pi_i(t)|^{|\alpha|}\widetilde{\zeta}_i > \varepsilon\right\}}.$$

From Proposition 15 in [38], we know that there exist two constants C_1, C_2 such that $P(\zeta > t) \leq C_1 e^{-C_2 t}$ for all $t \geq 0$. Consequently, for all $\delta > 0$,

$$\begin{aligned} E\left[N_t^\varepsilon \mid \mathcal{G}(t)\right] &\leq C_1 \sum_{i:|\Pi_i(t)|>0} e^{-C_2\varepsilon|\Pi_i(t)|^\alpha} \\ &\leq C(\delta)\varepsilon^{-\delta} \sum_i |\Pi_i(t)|^{|\alpha|\delta}, \end{aligned} \tag{3.7}$$

where $C(\delta) = \sup_{x\in\mathbb{R}^+}\left\{C_1 x^\delta e^{-C_2 x}\right\} < \infty$. Since $\sum_i |\Pi_i(t)| \leq 1$ a.s, this shows by taking $\delta = 1/|\alpha|$ that $N_t^\varepsilon < \infty$ a.s.

Let us now construct a covering of supp (μ) with balls of radius 5ε. Recall that we may suppose that the tree $\mathcal{T}_F$ is constructed together with an exchangeable leaf sample $(L_1, L_2, \ldots)$ directed by μ_F. For each $l \in \mathbb{N} \cup \{0\}$, we introduce the set

$$B_l^\varepsilon = \{k \in \mathbb{N} : \{k\} \notin \Pi(l\varepsilon), \{k\} \in \Pi((l+1)\varepsilon)\},$$

some of which may be empty when $\nu(\mathcal{S}^\downarrow) < \infty$, since the tree is not leaf-dense. For $l \geq 1$, the number of blocks of the partition $B_l^\varepsilon \cap \Pi((l-1)\varepsilon)$ of B_l^ε is less than or equal to $N_{(l-1)\varepsilon}^\varepsilon$ and so is a.s. finite. Since the fragmentation is entirely reduced to dust at time $\zeta < \infty$ a.s., $N_{l\varepsilon}^\varepsilon$ is equal to zero for $l \geq \zeta/\varepsilon$ and then, defining

$$N_\varepsilon := \sum_{l=0}^{[\zeta/\varepsilon]} N_{l\varepsilon}^\varepsilon$$

we have $N_\varepsilon < \infty$ a.s. ($[\zeta/\varepsilon]$ denotes here the largest integer smaller than ζ/ε). Now, consider a finite random sequence of pairwise distinct integers $\sigma(1), ..., \sigma(N_\varepsilon)$ such that for each $1 \leq l \leq [\zeta/\varepsilon]$ and each non-empty block of $B_l^\varepsilon \cap \Pi((l-1)\varepsilon)$, there is a $\sigma(i), 1 \leq i \leq N_\varepsilon$, in this block. Then each leaf L_j belongs then to a ball of center $L_{\sigma(i)}$, for an integer $1 \leq i \leq N_\varepsilon$, and of radius 4ε. Indeed, fix $j \geq 1$. It is clear that the sequence $(B_l^\varepsilon)_{l\in\mathbb{N}\cup\{0\}}$ forms a partition of $\mathbb{N}$. Thus, there exists a unique block B_l^ε containing j and in this block we consider the integer $\sigma(i)$ that belongs to the same block as j in the partition $B_l^\varepsilon \cap \Pi(((l-1)\vee 0)\varepsilon)$. By definition (see Section 3.2.3), the distance between the leaves L_j and $L_{\sigma(i)}$ is $d(L_j, L_{\sigma(i)}) = D_j + D_{\sigma(i)} - 2D_{\{j,\sigma(i)\}}$.

By construction, j and $\sigma(i)$ belong to the same block of $\Pi(((l-1)\vee 0)\,\varepsilon)$ and both die before $(l+1)\,\varepsilon$. In other words, $\max(D_j, D_{\sigma(i)}) \leq (l+1)\,\varepsilon$ and $D_{\{j,\sigma(i)\}} \geq ((l-1)\vee 0)\,\varepsilon$, which implies that $d(L_j, L_{\sigma(i)}) \leq 4\varepsilon$. Therefore, we have covered the set of leaves $\{L_j, j\geq 1\}$ by at most N_ε balls of radius 4ε. Since the sequence $(L_j)_{j\geq 1}$ is dense in supp (μ), this induces by taking balls with radius 5ε instead of 4ε a covering of supp (μ) by N_ε balls of radius 5ε. This holds for all $\varepsilon > 0$ so supp (μ) is a.s. compact. The compactness of $\mathcal{T}_F$ follows. ■

Let us now prove the upper bound for $\dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F))$. The difficulty for finding a "good" covering of the set $\mathcal{L}(\mathcal{T}_F)$ is that as soon as ν is infinite, this set is dense in $\mathcal{T}_F$, and thus one cannot hope to find its dimension by the plain box-counting method, because the skeleton $\mathcal{S}(\mathcal{T}_F)$ has a.s. Hausdorff dimension 1 as a countable union of segments. However, we stress that the covering with balls of radius 5ε of the previous lemma is a good covering of the *whole tree*, because the box-counting method leads to the right bound $\dim_{\mathcal{H}}(\mathcal{T}_F) \leq (1/|\alpha|)\vee 1$, and this is sufficient when $|\alpha| < 1$. When $|\alpha| \geq 1$ though, we may lose the details of the structure of $\mathcal{L}(\mathcal{T}_F)$. We will thus try to find a sharp "cutset" for the tree, motivated by the computation of the dimension of leaves of discrete infinite trees.

Proof of Theorem 3.2: upper bound. For every $i\in\mathbb{N}$ and $t\geq 0$ let $\Pi_{(i)}(t)$ be the block of $\Pi(t)$ containing i and for $\varepsilon > 0$ let

$$t_i^\varepsilon = \inf\{t\geq 0 : |\Pi_{(i)}(t)| < \varepsilon\}.$$

Define a partition Π^ε by $i \sim^{\Pi^\varepsilon} j$ if and only if $\Pi_{(i)}(t_i^\varepsilon) = \Pi_{(j)}(t_j^\varepsilon)$. One easily checks that this random partition is exchangeable, moreover it has a.s. no singleton. Indeed, notice that for any i, $\Pi_{(i)}(t_i^\varepsilon)$ is the block of $\Pi(t_i^\varepsilon)$ that contains i, and this block cannot be a singleton because the process $(|\Pi_{(i)}(t)|, t\geq 0)$ reaches 0 continuously. Therefore, $\Pi^{(\varepsilon)}$ admits asymptotic frequencies a.s., and these frequencies sum to 1. Then let

$$\zeta_{(i)}^\varepsilon = \sup_{j\in\Pi_{(i)}(t_i^\varepsilon)} \inf\{t\geq t_i^\varepsilon : |\Pi_{(j)}(t)| = 0\} - t_i^\varepsilon$$

be the time after t_i^ε when the fragment containing i vanishes entirely (notice that $\zeta_{(i)}^\varepsilon = \zeta_{(j)}^\varepsilon$ whenever $i \sim^{\Pi^\varepsilon} j$). We also let b_i^ε be the unique vertex of $[[\varnothing, L_i]]$ at distance t_i^ε from the root, notice that again $b_i^\varepsilon = b_j^\varepsilon$ whenever $i \sim^{\Pi^\varepsilon} j$.

We claim that

$$\mathcal{L}(\mathcal{T}_F) \subseteq \bigcup_{i\in\mathbb{N}} \overline{B}(b_i^\varepsilon, \zeta_{(i)}^\varepsilon),$$

where $\overline{B}(v,r)$ is the closed ball centered at v with radius r in $\mathcal{T}_F$. Indeed, for $L\in\mathcal{L}(\mathcal{T}_F)$, let b_L be the vertex of $[[\varnothing, L]]$ with minimal height such that $\mu_F(\mathcal{T}_{b_L}) < \varepsilon$, where $\mathcal{T}_{b_L}$ is the fringe subtree of $\mathcal{T}_F$ rooted at b_L. Since $b_L\in\mathcal{S}(\mathcal{T}_F)$, $\mu_F(\mathcal{T}_{b_L}) > 0$ and there exist infinitely many i's with $L_i\in\mathcal{T}_{b_L}$. But then, it is immediate that for any such i, $t_i^\varepsilon = \mathrm{ht}(b_L) = \mathrm{ht}(b_i^\varepsilon)$. Since $(L_i, i\geq 1)$ is dense in $\mathcal{L}(\mathcal{T}_F)$, and since for every j with $L_j\in\mathcal{T}_{b_i^\varepsilon}$ one has $d(b_i^\varepsilon, L_j)\leq \zeta_{(i)}^\varepsilon$ by definition, it follows that $L\in\overline{B}(b_i^\varepsilon, \zeta_{(i)}^\varepsilon)$. Therefore, $(\overline{B}(b_i^\varepsilon, \zeta_{(i)}^\varepsilon), i\geq 1)$ is a covering of $\mathcal{L}(\mathcal{T}_F)$.

The next claim is that this covering is fine as $\varepsilon\downarrow 0$, namely

$$\sup_{i\in\mathbb{N}} \zeta_{(i)}^\varepsilon \underset{\varepsilon\downarrow 0}{\to} 0 \qquad \text{a.s.}$$

Indeed, if it were not the case, we would find $\eta > 0$ and $i_n, n \geq 0$, such that $\zeta_{(i_n)}^{1/2^n} \geq \eta$ and $d(b_{i_n}^{1/2^n}, L_{i_n}) \geq \eta/2$ for every n. Since $\mathcal{T}_F$ is compact, we may extract a subsequence such that $L_{i_n} \to v$ for some $v \in \mathcal{T}_F$. Now, since $\mu_F(\mathcal{T}_{b_{i_n}^{1/2^n}}) \leq 2^{-n}$, it follows that we may find a vertex $b \in [[\varnothing, v]]$ at distance at least $\eta/4$ from v, such that $\mu_F(\mathcal{T}_b) = 0$, and this does not happen a.s.

To conclude, let $\zeta_i^\varepsilon = \zeta_{(i)}^\varepsilon \mathbf{1}_{\{\Pi_{(i)}(t_i^\varepsilon) = \Pi_i(t_i^\varepsilon)\}}$ (we just choose one i representing each class of Π^ε above). By the self-similarity property applied at the $(\mathcal{G}(t), t \geq 0)$-stopping time t_i^ε, ζ_i^ε has the same law as $|\Pi_i(t_i^\varepsilon)|^{|\alpha|}\zeta$, where ζ has same law as $\inf\{t \geq 0 : |\Pi(t)| = (0, 0, \ldots)\}$ and is taken independent of $|\Pi_i(t_i^\varepsilon)|$. Therefore,

$$E\left[\sum_{i \geq 1} (\zeta_i^\varepsilon)^{1/|\alpha|}\right] = E[\zeta^{1/|\alpha|}] E\left[\sum_{i \geq 1} |\Pi_i(t_i^\varepsilon)|\right] = E[\zeta^{1/|\alpha|}] < \infty. \tag{3.8}$$

The fact that $E[\zeta^{1/|\alpha|}]$ is finite comes from the fact that ζ has exponential moments. Because our covering is a fine covering as $\varepsilon \downarrow 0$, it finally follows that (with the above notations)

$$m_{1/|\alpha|}(\mathcal{L}(\mathcal{T}_F)) \leq \liminf_{\varepsilon \downarrow 0} \sum_{i : \Pi_{(i)}(t_i^\varepsilon) = \Pi_i(t_i^\varepsilon)} (\zeta_i^\varepsilon)^{1/|\alpha|} \qquad \text{a.s.},$$

which is a.s. finite by (3.8) and Fatou's Lemma. ■

Proof of Corollary 3.2. By Theorem 3.1, the measure $\mathrm{d}M_F$ has same law as $\mathrm{d}\overline{W}_{\mathcal{T}_F}$, the Stieltjes measure associated with the cumulative height profile $\overline{W}_{\mathcal{T}_F}(t) = \mu_F\{v \in \mathcal{T}_F : \mathrm{ht}(v) \leq t\}$, $t \geq 0$. To bound from above the Hausdorff dimension of $\mathrm{d}\overline{W}_{\mathcal{T}_F}$, note that

$$\mathrm{d}\overline{W}_{\mathcal{T}_F}(\mathrm{ht}(\mathcal{L}(\mathcal{T}_F))) = \int_{\mathcal{T}_F} \mathbf{1}_{\{\mathrm{ht}(v) \in \mathrm{ht}(\mathcal{L}(\mathcal{T}_F))\}} \mu_F(\mathrm{d}v) = 1$$

since $\mu_F(\mathcal{L}(\mathcal{T}_F)) = 1$. By definition of $\dim_{\mathcal{H}}(\mathrm{d}\overline{W}_{\mathcal{T}_F})$, it is thus sufficient to show that $\dim_{\mathcal{H}}(\mathrm{ht}(\mathcal{L}(\mathcal{T}_F))) \leq 1/|\alpha|$ a.s. To do so, remark that ht is Lipschitz and that this property easily leads to

$$\dim_{\mathcal{H}}(\mathrm{ht}(\mathcal{L}(\mathcal{T}_F))) \leq \dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F)).$$

The conclusion hence follows from the majoration $\dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F))) \leq 1/|\alpha|$ proved above. ■

3.3.2 A first lower bound

Recall that Frostman's energy method to prove that $\dim_{\mathcal{H}}(\mathcal{E}) \geq \gamma$ where $\mathcal{E}$ is a subset of a metric space (M, d) is to find a nonzero positive measure $\eta(\mathrm{d}x)$ on $\mathcal{E}$ such that $\int_{\mathcal{E}}\int_{\mathcal{E}} \frac{\eta(\mathrm{d}x)\eta(\mathrm{d}y)}{d(x,y)^\gamma} < \infty$. A naive approach for finding a lower bound of the Hausdorff dimension of $\mathcal{T}_F$ is thus to apply this method by taking $\eta = \mu_F$ and $\mathcal{E} = \mathcal{L}(\mathcal{T}_F)$. The result states as follows.

Lemma 3.6 *For any fragmentation process F satisfying the hypotheses of Theorem 3.1, one has*

$$\dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F)) \geq \frac{A}{|\alpha|} \wedge \left(1 + \frac{p}{|\alpha|}\right),$$

where

$$\underline{p} := -\inf\left\{q : \int_{\mathcal{S}^\downarrow}\left(1-\sum_{i\geq 1} s_i^{q+1}\right)\nu(\mathrm{d}\mathbf{s}) > -\infty\right\} \in [0,1], \tag{3.9}$$

and

$$A := \sup\left\{a \leq 1 : \int_{\mathcal{S}^\downarrow}\sum_{1\leq i<j} s_i^{1-a}s_j\nu(\mathrm{d}\mathbf{s}) < \infty\right\} \in [0,1]. \tag{3.10}$$

Proof. By Lemma 3.4 (recall that $(\mathcal{T}_\Pi, \mu_\Pi) = (\mathcal{T}_F, \mu_F)$ by Theorem 3.1) we have

$$\int_{\mathcal{T}_F}\int_{\mathcal{T}_F}\frac{\mu_F(\mathrm{d}x)\mu_F(\mathrm{d}y)}{d(x,y)^\gamma} \overset{a.s.}{=} E\left[\frac{1}{d(L_1,L_2)^\gamma}\,|\mathcal{T}_F,\mu_F\right]$$

so that

$$E\left[\int_{\mathcal{T}_F}\int_{\mathcal{T}_F}\frac{\mu_F(\mathrm{d}x)\mu_F(\mathrm{d}y)}{d(x,y)^\gamma}\right] = E\left[\frac{1}{d(L_1,L_2)^\gamma}\right]$$

and by definition, $d(L_1,L_2) = D_1 + D_2 - 2D_{\{1,2\}}$. Applying the strong fragmentation property at the stopping time $D_{\{1,2\}}$, we can rewrite D_1 and D_2 as

$$D_1 = D_{\{1,2\}} + \lambda_1^{|\alpha|}(D_{\{1,2\}})\widetilde{D}_1 \qquad D_2 = D_{\{1,2\}} + \lambda_2^{|\alpha|}(D_{\{1,2\}})\widetilde{D}_2$$

where $\lambda_1(D_{\{1,2\}})$ (resp. $\lambda_2(D_{\{1,2\}})$) is the asymptotic frequency of the block containing 1 (resp. 2) at time $D_{\{1,2\}}$ and $\widetilde{D}_1$ and $\widetilde{D}_2$ are independent with the same law as D_1 and independent of $\mathcal{G}\left(D_{\{1,2\}}\right)$. Therefore,

$$d(L_1,L_2) = \lambda_1^{|\alpha|}(D_{\{1,2\}})\widetilde{D}_1 + \lambda_2^{|\alpha|}(D_{\{1,2\}})\widetilde{D}_2,$$

and

$$E\left[\frac{1}{d(L_1,L_2)^\gamma}\right] \leq 2E\left[\lambda_1^{\alpha\gamma}(D_{\{1,2\}});\lambda_1(D_{\{1,2\}}) \geq \lambda_2(D_{\{1,2\}})\right]E\left[D_1^{-\gamma}\right]. \tag{3.11}$$

By [39, Lemma 2] the first expectation in the right-hand side of inequality (3.11) is finite as soon as $|\alpha|\,\gamma < A$, while by [38, Sect. 4.2.1] the second expectation is finite as soon as $\gamma < 1+\underline{p}/\,|\alpha|$. That $\dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F)) \overset{a.s.}{\geq} \left((A/\,|\alpha|)\wedge\left(1+\underline{p}/\,|\alpha|\right)\right)$ follows. ■

Let us now make a comment about this bound. For dislocation measures such that $\nu(s_{N+1}>0)=0$ for some $N\geq 1$, the constant A equals 1 since for all $a<1$,

$$\int_{\mathcal{S}^\downarrow}\sum_{i<j}s_i^{1-a}s_j\nu(\mathrm{d}\mathbf{s}) \leq \int_{\mathcal{S}^\downarrow}(N-1)\sum_{2\leq j\leq N}s_j\nu(\mathrm{d}\mathbf{s}) \leq (N-1)\int_{\mathcal{S}^\downarrow}(1-s_1)\,\nu(\mathrm{d}\mathbf{s}) < \infty.$$

In such cases, if moreover $\underline{p}=1$, the “naive” lower bound of Lemma 3.6 is thus equal to $1/\,|\alpha|$. A typical setting in which this holds is when $\nu(\mathcal{S}^\downarrow)<\infty$ and $\nu(s_{N+1}>0)=0$ and therefore, for such dislocation measures the “naive” lower bound is also the best possible.

3.3.3 A subtree of $\mathcal{T}_F$ and a reduced fragmentation

In the general case, in order to improve this lower bound, we will thus try to transform the problem on F into a problem on an auxiliary fragmentation that satisfies the hypotheses above. The idea is as follows: fix an integer N and $0 < \varepsilon < 1$. Consider the subtree $\mathcal{T}_F^{N,\varepsilon} \subset \mathcal{T}_F$ constructed from $\mathcal{T}_F$ by keeping, at each branchpoint, the N largest fringe subtrees rooted at this branchpoint (that is the subtrees with the largest masses) and discarding the others in order to yield a tree in which branchpoints have out-degree at most N. Also, we remove the accumulation of fragmentation times by discarding all the fringe subtrees rooted at the branchpoints but the largest one, as soon as the proportion of its mass compared to the others is larger than $1-\varepsilon$. Then there exists a probability $\mu_F^{N,\varepsilon}$ such that $(\mathcal{T}_F^{N,\varepsilon}, \mu_F^{N,\varepsilon})$ is a CRT, to which we will apply the energy method.

Let us make the definition precise. Define $\mathcal{L}^{N,\varepsilon} \subset \mathcal{L}(\mathcal{T}_F)$ to be the set of leaves L such that for every branchpoint $b \in [[\varnothing, L]]$, $L \in \mathcal{F}_b^{N,\varepsilon}$ with $\mathcal{F}_b^{N,\varepsilon}$ defined by

$$\begin{cases} \mathcal{F}_b^{N,\varepsilon} = \mathcal{T}_b^1 \cup \ldots \cup \mathcal{T}_b^N & \text{if } \mu_F(\mathcal{T}_b^1)/\mu_F\left(\bigcup_{i\geq 1} \mathcal{T}_b^i\right) \leq 1-\varepsilon \\ \mathcal{F}_b^{N,\varepsilon} = \mathcal{T}_b^1 & \text{if } \mu_F(\mathcal{T}_b^1)/\mu_F\left(\bigcup_{i\geq 1} \mathcal{T}_b^i\right) > 1-\varepsilon \end{cases}, \tag{3.12}$$

where $\mathcal{T}_b^1, \mathcal{T}_b^2 \ldots$ are the connected components of the fringe subtree of $\mathcal{T}_F$ rooted at b, from whom b has been removed (the connected components of $\{v \in \mathcal{T}_F : \mathrm{ht}(v) > b\}$) and ranked in decreasing order of μ_F-mass. Then let $\mathcal{T}_F^{N,\varepsilon} \subset \mathcal{T}_F$ be the subtree of $\mathcal{T}_F$ spanned by the root and the leaves of $\mathcal{L}^{N,\varepsilon}$, i.e.

$$\mathcal{T}_F^{N,\varepsilon} = \{v \in \mathcal{T}_F : \exists L \in \mathcal{L}^{N,\varepsilon}, v \in [[\varnothing, L]]\}.$$

The set $\mathcal{T}_F^{N,\varepsilon} \subset \mathcal{T}_F$ is plainly connected and closed in $\mathcal{T}_F$, thus an $\mathbb{R}$-tree.

Now let us try to give a sense to "taking at random a leaf in $\mathcal{T}_F^{N,\varepsilon}$". In the case of $\mathcal{T}_F$, it was easy because, from the partition-valued fragmentation Π, it sufficed to look at the fragment containing 1 (or some prescribed integer). Here, it is not difficult to show (as we will see later) that the corresponding leaf L_1 a.s. never belongs to $\mathcal{T}_F^{N,\varepsilon}$ when the dislocation measure ν charges the set $\{s_1 > 1-\varepsilon\} \cup \{s_{N+1} > 0\}$. Therefore, we will have to use several random leaves of $\mathcal{T}_F$. For any leaf $L \in \mathcal{L}(\mathcal{T}_F) \setminus \mathcal{L}(\mathcal{T}_F^{N,\varepsilon})$ let $b(L)$ be the highest vertex v of $[[\varnothing, L]]$ such that $v \in \mathcal{T}_F^{N,\varepsilon}$. Call it the branchpoint of L and $\mathcal{T}_F^{N,\varepsilon}$.

Now take at random a leaf Z_1 of $\mathcal{T}_F$ with law μ_F conditionally on μ_F, and define recursively a sequence $(Z_n, n \geq 1)$ with values in $\mathcal{T}_F$ as follows. Let Z_{n+1} be independent of $Z_1, \ldots, Z_n$ conditionally on $(\mathcal{T}_F, \mu_F, b(Z_n))$, and take it with conditional law

$$P(Z_{n+1} \in \cdot | \mathcal{T}_F, \mu_F, b(Z_n)) = \mu_F(\cdot \cap \mathcal{F}_{b(Z_n)}^{N,\varepsilon})/\mu_F(\mathcal{F}_{b(Z_n)}^{N,\varepsilon}).$$

Lemma 3.7 *Almost surely, the sequence $(Z_n, n \geq 1)$ converges to a random leaf $Z^{N,\varepsilon} \in \mathcal{L}(\mathcal{T}_F^{N,\varepsilon})$. If $\mu_F^{N,\varepsilon}$ denotes the conditional law of $Z^{N,\varepsilon}$ given $(\mathcal{T}_F, \mu_F)$, then $(\mathcal{T}_F^{N,\varepsilon}, \mu_F^{N,\varepsilon})$ is a CRT, provided ε is small enough.*

To prove this and for later use we first reconnect this discussion to partition-valued fragmentations. Recall from Sect. 3.2.1 the construction of the homogeneous fragmentation Π^0

with characteristics $(0,0,\nu)$ out of a $\mathcal{P}_\infty \times \mathbb{N}$-valued Poisson point process $((\Delta_t, k_t), t \geq 0)$ with intensity $\kappa_\nu \otimes \#$. For any partition $\pi \in \mathcal{P}_\infty$ that admits asymptotic frequencies whose ranked sequence is $\mathbf{s}$, write $\pi_i^\downarrow$ for the block of π with asymptotic frequency s_i (with some convention for ties, e.g. taking the order of least element). We define a function $\texttt{GRIND}^{N,\varepsilon} : \mathcal{P}_\infty \to \mathcal{P}_\infty$ that reduces the smallest blocks of the partition to singletons as follows. If π does not admit asymptotic frequencies, let $\texttt{GRIND}^{N,\varepsilon}(\pi) = \pi$, else let

$$\texttt{GRIND}^{N,\varepsilon}(\pi) = \begin{cases} \left(\pi_1^\downarrow, ..., \pi_N^\downarrow, \text{singletons}\right) & \text{if } s_1 \leq 1-\varepsilon \\ \left(\pi_1^\downarrow, \text{singletons}\right) & \text{if } s_1 > 1-\varepsilon. \end{cases}$$

Now for each $t \geq 0$ write $\Delta_t^{N,\varepsilon} = \texttt{GRIND}^{N,\varepsilon}(\Delta_t)$, so $((\Delta_t^{N,\varepsilon}, k_t), t \geq 0)$ is a $\mathcal{P}_\infty \times \mathbb{N}$-valued Poisson point process with intensity measure $\kappa_{\nu^{N,\varepsilon}} \otimes \#$, where $\nu^{N,\varepsilon}$ is the image of ν by the function

$$\mathbf{s} \in \mathcal{S}^\downarrow \mapsto \begin{cases} (s_1, ..., s_N, 0, ...) \text{ if } s_1 \leq 1-\varepsilon \\ (s_1, 0, ...) \text{ if } s_1 > 1-\varepsilon. \end{cases}$$

From this Poisson point process we construct first a version $\Pi^{0,N,\varepsilon}$ of the $\left(0,0,\nu^{N,\varepsilon}\right)$ fragmentation, as explained in Section 3.2.1. For every time $t \geq 0$, the partition $\Pi^{0,N,\varepsilon}(t)$ is finer than $\Pi^0(t)$ and the blocks of $\Pi^{0,N,\varepsilon}(t)$ non-reduced to singleton are blocks of $\Pi^0(t)$. Next, using the time change (3.3) , we construct from $\Pi^{0,N,\varepsilon}$ a version of the $\left(\alpha,0,\nu^{N,\varepsilon}\right)$ fragmentation, that we denote by $\Pi^{N,\varepsilon}$.

Note that for dislocation measures ν such that $\nu^{N,\varepsilon}\left(\sum s_i < 1\right) = 0$, Theorem 3.2 is already proved, by the previous subsection. For the rest of this subsection and next subsection, we shall thus focus on dislocation measures ν such that $\nu^{N,\varepsilon}\left(\sum s_i < 1\right) > 0$. In that case, in $\Pi^{0,N,\varepsilon}$ (unlike for Π^0) each integer i is eventually isolated in a singleton a.s. within a sudden break and this is why a μ_F-sampled leaf on $\mathcal{T}_F$ cannot be in $\mathcal{T}_F^{N,\varepsilon}$, in other words, μ_F and $\mu_F^{N,\varepsilon}$ are a.s. singular. Recall that we may build $\mathcal{T}_F$ together with an exchangeable μ_F-sample of leaves $L_1, L_2, \ldots$ on the same probability space as Π (or Π^0). We are going to use a subfamily of $(L_1, L_2, \ldots)$ to build a sequence with the same law as $(Z_n, n \geq 1)$ built above. Let $i_1 = 1$ and

$$i_{n+1} = \inf\{i > i_n : L_{i_{n+1}} \in \mathcal{F}^{N,\varepsilon}_{b(L_{i_n})}\}.$$

It is easy to see that $(L_{i_n}, n \geq 1)$ has the same law as $(Z_n, n \geq 1)$. From this, we build a decreasing family of blocks $B^{0,N,\varepsilon}(t) \in \Pi^0(t)$, $t \geq 0$, by letting $B^{0,N,\varepsilon}(t)$ be the unique block of $\Pi^0(t)$ that contains all but a finite number of elements of $\{i_1, i_2, \ldots\}$.

Here is a useful alternative description of $B^{0,N,\varepsilon}(t)$. Let $D_i^{0,N,\varepsilon}$ be the death time of i for the fragmentation $\Pi^{0,N,\varepsilon}$ that is

$$D_i^{0,N,\varepsilon} = \inf\{t \geq 0 : \{i\} \in \Pi^{0,N,\varepsilon}(t)\}.$$

By exchangeability the $D_i^{0,N,\varepsilon}$'s are identically distributed and

$$D_1^{0,N,\varepsilon} = \inf\{t \geq 0 : k_t = 1 \text{ and } \{1\} \in \Delta_t^{N,\varepsilon}\}$$

so it has an exponential law with parameter $\int_{\mathcal{S}^\downarrow}(1 - \sum_i s_i)\nu^{N,\varepsilon}(\mathrm{d}\mathbf{s})$. Then notice that $B^{0,N,\varepsilon}(t)$ is the block admitting i_n as least element when $D_{i_n}^{0,N,\varepsilon} \leq t < D_{i_{n+1}}^{0,N,\varepsilon}$. Indeed, by construction we have

$$i_{n+1} = \inf\{i \in B^{0,N,\varepsilon}(D_{i_n}^{0,N,\varepsilon}-) : \{i\} \notin \Pi^{0,N,\varepsilon}(D_{i_n}^{0,N,\varepsilon})\}.$$

Moreover, the asymptotic frequency $\lambda_1^{0,N,\varepsilon}(t)$ of $B^{0,N,\varepsilon}(t)$ exists for every t and equals the μ_F-mass of the tree component of $\{v \in \mathcal{T}_F : \mathrm{ht}(v) > t\}$ containing L_{i_n} for $D_{i_n}^{0,N,\varepsilon} \le t < D_{i_{n+1}}^{0,N,\varepsilon}$.

Notice that at time $D_{i_n}^{0,N,\varepsilon}$, either one non-singleton block coming from $B^{0,N,\varepsilon}(D_{i_n}^{0,N,\varepsilon}-)$, or up to N non-singleton blocks may appear; by Lemma 3.1, $B^{0,N,\varepsilon}(D_{i_n}^{0,N,\varepsilon})$ is then obtained by taking at random one of these blocks with probability proportional to its size.

Proof of Lemma 3.7. For $t \ge 0$ let $\lambda^{0,N,\varepsilon}(t) = |B^{0,N,\varepsilon}(t)|$ and

$$T^{0,N,\varepsilon}(t) := \inf\left\{u \ge 0 : \int_0^u \left(\lambda^{0,N,\varepsilon}(r)\right)^{-\alpha} \mathrm{d}r > t\right\} \tag{3.13}$$

and write $B^{N,\varepsilon}(t) := B^{0,N,\varepsilon}(T^{0,N,\varepsilon}(t))$, for $T^{0,N,\varepsilon}(t) < \infty$ and $B^{N,\varepsilon}(t) = \varnothing$ otherwise, so for all $t \ge 0$, $B^{N,\varepsilon}(t) \in \Pi^{N,\varepsilon}(t)$. Let also $D_{i_n}^{N,\varepsilon} := T^{0,N,\varepsilon}(D_{i_n}^{0,N,\varepsilon})$ be the death time of i_n in the fragmentation $\Pi^{N,\varepsilon}$. It is easy to see that $b_n = b(L_{i_n})$ is the branchpoint of the paths $[[\varnothing, L_{i_n}]]$ and $[[\varnothing, L_{i_{n+1}}]]$, so the path $[[\varnothing, b_n]]$ has length $D_{i_n}^{N,\varepsilon}$. The "edges" $[[b_n, b_{n+1}]]$, $n \in \mathbb{N}$, have respective lengths $D_{i_{n+1}}^{N,\varepsilon} - D_{i_n}^{N,\varepsilon}$, $n \in \mathbb{N}$. Since the sequence of death times $(D_{i_n}^{N,\varepsilon}, n \ge 1)$ is increasing and bounded by ζ (the first time at which Π is entirely reduced to singletons), the sequence $(b_n, n \ge 1)$ is Cauchy, so it converges by completeness of $\mathcal{T}_F$. Now it is easy to show that $D_{i_n}^{0,N,\varepsilon} \to \infty$ as $n \to \infty$ a.s., so $\lambda^{0,N,\varepsilon}(t) \to 0$ as $t \to \infty$ a.s. (see also the next lemma). Therefore, the fragmentation property implies $d(L_{i_n}, b_n) \to 0$ a.s. so L_{i_n} is also Cauchy, with the same limit, and the limit has to be a leaf which we denote $L^{N,\varepsilon}$ (of course it has same distribution as the $Z^{N,\varepsilon}$ of the lemma's statement). The fact that $L^{N,\varepsilon} \in \mathcal{T}_F^{N,\varepsilon}$ a.s. is obtained by checking (3.12), which is true since it is verified for each branchpoint $b \in [[\varnothing, b_n]]$ for every $n \ge 1$ by construction.

We now sketch the proof that $(\mathcal{T}_F^{N,\varepsilon}, \mu_F^{N,\varepsilon})$ is indeed a CRT, leaving details to the reader. We need to show non-atomicity of $\mu_F^{N,\varepsilon}$, but it is clear that when performing the recursive construction of $Z^{N,\varepsilon}$ twice with independent variables, $(Z_n, n \ge 1)$ and $(Z'_n, n \ge 1)$ say, there exists a.s. some n such that Z_n and Z'_n end up in two different fringe subtrees rooted at some of the branchpoints b_n, provided that ε is small enough so that $\nu(1 - s_1 \ge \varepsilon) \ne 0$ (see also below the explicit construction of two independently $\mu_F^{N,\varepsilon}$-sampled leaves). On the other hand, all of the subtrees of $\mathcal{T}_F$ rooted at the branchpoints of $\mathcal{T}_F^{N,\varepsilon}$ have positive μ_F-mass, so they will end up being visited by the intermediate leaves used to construct a $\mu_F^{N,\varepsilon}$-i.i.d. sample, so the condition $\mu_F^{N,\varepsilon}(\{v \in \mathcal{T}_F^{N,\varepsilon} : [[\varnothing, v]] \cap [[\varnothing, w]] = [[\varnothing, w]]\}) > 0$ for every $w \in \mathcal{S}(\mathcal{T}_F^{N,\varepsilon})$ is satisfied.
■

It will also be useful to sample two leaves $(L_1^{N,\varepsilon}, L_2^{N,\varepsilon})$ that are independent with same distribution $\mu_F^{N,\varepsilon}$ conditionally on $\mu_F^{N,\varepsilon}$ out of the exchangeable family $L_1, L_2, \ldots$. A natural way to do this is to use the family $(L_1, L_3, L_5, \ldots)$ to sample the first leaf in the same way as above, and to use the family $(L_2, L_4, \ldots)$ to sample the other one. That is, let $j_1^1 = 1, j_1^2 = 2$ and define recursively $(j_n^1, j_n^2, n \ge 1)$ by letting

$$\begin{cases} j_{n+1}^1 = \inf\{j \in 2\mathbb{N}+1, j > j_n^1 : L_j \in \mathcal{F}_{b(L_{j_n^1})}^{N,\varepsilon}\} \\ j_{n+1}^2 = \inf\{j \in 2\mathbb{N}, j > j_{n+1}^1 : L_j \in \mathcal{F}_{b(L_{j_n^2})}^{N,\varepsilon}\} \end{cases}.$$

It is easy to check that $(L_{j_n^1}, n \ge 1)$ and $(L_{j_n^2}, n \ge 1)$ are two independent sequences distributed as $(Z_1, Z_2, \ldots)$ of Lemma 3.7. Therefore, these sequences a.s. converge to limits $L_1^{N,\varepsilon}, L_2^{N,\varepsilon}$, and these are independent with law $\mu_F^{N,\varepsilon}$ conditionally on $\mu_F^{N,\varepsilon}$. We let $\mathcal{D}_k = \mathrm{ht}(L_k^{N,\varepsilon})$, $k = 1, 2$.

Similarly as above, for every $t \geq 0$ we let $B_k^{0,N,\varepsilon}(t)$, $k = 1, 2$ (resp. $B_k^{N,\varepsilon}(t)$) be the block of $\Pi^0(t)$ (resp. $\Pi(t)$) that contains all but the first few elements of $\{j_1^k, j_2^k, \ldots\}$, and we call $\lambda_k^{0,N,\varepsilon}(t)$ (resp. $\lambda_k^{N,\varepsilon}(t)$) its asymptotic frequency. Last, let $\mathcal{D}^0_{\{1,2\}} = \inf\{t \geq 0 : B_1^{0,N,\varepsilon}(t) \cap B_2^{0,N,\varepsilon}(t) = \varnothing\}$ (and define similarly $\mathcal{D}_{\{1,2\}}$). Notice that for $t < \mathcal{D}^0_{\{1,2\}}$, we have $B_1^{0,N,\varepsilon}(t) = B_2^{0,N,\varepsilon}(t)$, and by construction the two least elements of the blocks $(2\mathbb{N}+1) \cap B_1^{0,N,\varepsilon}(t)$ and $(2\mathbb{N}) \cap B_1^{0,N,\varepsilon}(t)$ are of the form j_n^1, j_m^2 for some n, m. On the other hand, for $t \geq \mathcal{D}^0_{\{1,2\}}$, we have $B_1^{0,N,\varepsilon}(t) \cap B_2^{0,N\varepsilon}(t) = \varnothing$, and again the least elements of $(2\mathbb{N}+1) \cap B_1^{0,N,\varepsilon}(t)$ and $(2\mathbb{N}) \cap B_2^{0,N\varepsilon}(t)$ are of the form j_n^1, j_m^2 for some n, m. In any case, we let $j^1(t) = j_n^1, j^2(t) = j_m^2$ for these n, m.

3.3.4 Lower bound

Since $\mu_F^{N,\varepsilon}$ is a measure on $\mathcal{L}(\mathcal{T}_F)$, we want to show that for every $a < \varrho$, the integral $\int_{\mathcal{T}_F^{N,\varepsilon}} \int_{\mathcal{T}_F^{N,\varepsilon}} \frac{\mu_F^{N,\varepsilon}(\mathrm{d}x)\mu_F^{N,\varepsilon}(\mathrm{d}y)}{d(x,y)^{a/|\alpha|}}$ is a.s. finite for suitable N and ε. So consider $a < \varrho$, and note that

$$E\left[\int_{\mathcal{T}_F^{N,\varepsilon}} \int_{\mathcal{T}_F^{N,\varepsilon}} \frac{\mu_F^{N,\varepsilon}(\mathrm{d}x)\mu_F^{N,\varepsilon}(\mathrm{d}y)}{d(x,y)^{a/|\alpha|}}\right] = E\left[\frac{1}{d(L_1^{N,\varepsilon}, L_2^{N,\varepsilon})^{a/|\alpha|}}\right],$$

where $d(L_1^{N,\varepsilon}, L_2^{N,\varepsilon}) = \mathcal{D}_1 + \mathcal{D}_2 - 2\mathcal{D}_{\{1,2\}}$, with notations above. The fragmentation property at the stopping time $\mathcal{D}_{\{1,2\}}$ leads to

$$\mathcal{D}_k = \mathcal{D}_{\{1,2\}} + \lambda_k^{N,\varepsilon}(\mathcal{D}_{\{1,2\}})^{|\alpha|}\widetilde{\mathcal{D}}_k, \ k = 1, 2,$$

where $\widetilde{\mathcal{D}}_1, \widetilde{\mathcal{D}}_2$ are independent with the same distribution as $\mathcal{D}$, the height of the leaf $L^{N,\varepsilon}$ constructed above, and independent of $\mathcal{G}(\mathcal{D}_{\{1,2\}})$. Therefore, the distance $d(L_1^{N,\varepsilon}, L_2^{N,\varepsilon})$ can be rewritten as

$$d(L_1^{N,\varepsilon}, L_2^{N,\varepsilon}) = \left(\lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}})\right)^{|\alpha|} \widetilde{\mathcal{D}}_1 + \left(\lambda_2^{N,\varepsilon}(\mathcal{D}_{\{1,2\}})\right)^{|\alpha|} \widetilde{\mathcal{D}}_2$$

and

$$E\left[d(L_1^{N,\varepsilon}, L_2^{N,\varepsilon})^{-a/|\alpha|}\right] \leq 2E\left[\left(\lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}})\right)^{-a}; \lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}}) \geq \lambda_2^{N,\varepsilon}(\mathcal{D}_{\{1,2\}})\right] E\left[\mathcal{D}^{-a/|\alpha|}\right].$$

Therefore, that $\dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F)) \geq \varrho/|\alpha|$ is directly implied by the following Lemmas 3.8 and 3.10.

Lemma 3.8 *The quantity $E[\mathcal{D}^{-\gamma}]$ is finite for every $0 \leq \gamma \leq \varrho/|\alpha|$.*

The proof uses the following technical lemma. Recall that $\lambda^{N,\varepsilon}(t) = |B^{N,\varepsilon}(t)|$.

Lemma 3.9 *One can write $\lambda^{N,\varepsilon} = \exp\left(-\xi_{\rho(\cdot)}\right)$, where ξ (tacitly depending on N, ε) is a subordinator with Laplace exponent*

$$\Phi_\xi(q) = \int_{\mathcal{S}^\downarrow} \left((1 - s_1^q)\, \mathbf{1}_{\{s_1 > 1-\varepsilon\}} + \sum_{i=1}^N (1 - s_i^q) \frac{s_i \mathbf{1}_{\{s_1 \leq 1-\varepsilon\}}}{s_1 + \ldots + s_N}\right)\nu(\mathrm{d}\mathbf{s}), \ q \geq 0, \tag{3.14}$$

and ρ is the time-change

$$\rho(t) = \inf\left\{u \geq 0 : \int_0^u \exp(\alpha\xi_r)\mathrm{d}r > t\right\},\ t \geq 0.$$

Proof. Recall the construction of the process $B^{0,N,\varepsilon}$ from Π^0, which itself was constructed from a Poisson process $(\Delta_t, k_t, t \geq 0)$. From the definition of $B^{0,N,\varepsilon}(t)$, we have

$$B^{0,N,\varepsilon}(t) = \bigcap_{0\leq s\leq t} \bar{\Delta}_s^{N,\varepsilon},$$

where the sets $\bar{\Delta}_s^{N,\varepsilon}$ are defined as follows. For each $s \geq 0$, let $i(s)$ be the least element of the block $B^{0,N,\varepsilon}(s-)$ (so that $B^{0,N,\varepsilon}(s-) = \Pi^0_{i(s)}(s-)$), so $(i(s), s \geq 0)$ is an $(\mathcal{F}(s-), s \geq 0)$-adapted jump-hold process, and the process $\{\Delta_s : k_s = i(s), s \geq 0\}$ is a Poisson point process with intensity κ_ν. Then for each s such that $k_s = i(s)$, $\bar{\Delta}_s^{N,\varepsilon}$ consists in a certain block of Δ_s, and precisely, $\bar{\Delta}_s^{N,\varepsilon}$ is the block of Δ_s containing

$$\inf\left\{i \in B^{0,N,\varepsilon}(s-) : \{i\} \notin \Delta_s^{N,\varepsilon}\right\},$$

the least element of $B^{0,N,\varepsilon}(s-)$ which is not isolated in a singleton of $\Delta_s^{N,\varepsilon}$ (such an integer must be of the form i_n for some n by definition). Now $B^{0,N,\varepsilon}(s-)$ is $\mathcal{F}(s-)$-measurable, hence independent of Δ_s. By Lemma 3.1, $\bar{\Delta}_s^{N,\varepsilon}$ is thus a size-biased pick among the non-void blocks of $\Delta_s^{N,\varepsilon}$, and by definition of the function $\texttt{GRIND}^{N,\varepsilon}$, the process $(|\bar{\Delta}_s^{N,\varepsilon}|, s \geq 0)$ is a $[0,1]$-valued Poisson point process with intensity $\omega(s)$ characterized by

$$\int_{[0,1]} f(s)\omega(\mathrm{d}s) = \int_{\mathcal{S}^\downarrow}\left(\mathbf{1}_{\{s_1>1-\varepsilon\}}f(s_1) + \mathbf{1}_{\{s_1\leq 1-\varepsilon\}}\sum_{i=1}^N f(s_i)\frac{s_i}{s_1+\ldots+s_N}\right)\nu(\mathrm{d}\mathbf{s}),$$

for every positive measurable function f. Then $|B^{0,N,\varepsilon}(t)| = \prod_{0\leq s\leq t}|\bar{\Delta}_s^{N,\varepsilon}|$ a.s. for every $t \geq 0$. To see this, denote for every $k \geq 1$ by $\Delta_{s_1}^{N,\varepsilon,k}, \Delta_{s_2}^{N,\varepsilon,k},\ldots$ the atoms $\Delta_s^{N,\varepsilon}$, $s \leq t$, such that $|\Delta_s^{N,\varepsilon}|_1 \in [1-k^{-1}, 1-(k+1)^{-1})$. Complete this a.s. finite sequence of partitions by partitions $\mathbf{1}$ and call $\Gamma^{(k)}$ their intersection, i.e. $\Gamma^{(k)} := \bigcap_{i\geq 1}(\Delta_{s_i}^{N,\varepsilon,k})$. By Lemma 3.2, $|\Gamma^{(k)}_{n_k}| \overset{a.s.}{=} \prod_{i\geq 1}|\overline{\Delta}_{s_i}^{N,\varepsilon,k}|$, where n_k is the index of the block $\bigcap_{i\geq 1}\overline{\Delta}_{s_i}^{N,\varepsilon,k}$ in the partition $\Gamma^{(k)}$. These partitions $\Gamma^{(k)}$, $k \geq 1$, are exchangeable and clearly independent. Applying again Lemma 3.2 gives $|\bigcap_{k\geq 1}\Gamma^{(k)}_{n_k}| \overset{a.s.}{=} \prod_{k\geq 1}\prod_{i\geq 1}|\overline{\Delta}_{s_i}^{N,\varepsilon,k}|$, which is exactly the equality mentioned above. The exponential formula for Poisson processes then shows that $(\xi_t, t \geq 0) = (-\log(\lambda^{0,N,\varepsilon}(t)), t \geq 0)$ is a subordinator with Laplace exponent Φ_ξ. The result is now obtained by noticing that (3.3) rewrites $\lambda^{N,\varepsilon}(t) = \lambda^{0,N,\varepsilon}(\rho(t))$ in our setting. ■

Proof of Lemma 3.8. By the previous lemma, $\mathcal{D} = \inf\{t \geq 0 : \lambda^{N,\varepsilon}(t) = 0\}$, which equals $\int_0^\infty \exp(\alpha\xi_t)\mathrm{d}t$ by the definition of ρ. According to Theorem 25.17 in [64], if for some positive γ the quantity

$$\Phi_\xi(-\gamma) := \int_{\mathcal{S}^\downarrow}\left(\left(1-s_1^{-\gamma}\right)\mathbf{1}_{\{s_1>1-\varepsilon\}} + \sum_{i=1}^N\left(1-s_i^{-\gamma}\right)\frac{s_i\mathbf{1}_{\{s_i>0\}}\mathbf{1}_{\{s_1\leq 1-\varepsilon\}}}{s_1+\ldots+s_N}\right)\nu(\mathrm{d}\mathbf{s})$$

is finite, then $E[\exp(\gamma\xi_t)] < \infty$ for all $t \geq 0$ and it equals $\exp(-t\Phi_\xi(-\gamma))$. Notice that $\Phi_\xi(-\gamma) > -\infty$ for $\gamma < \varrho \leq 1$. Indeed for such γ's, $\int_{\mathcal{S}^\downarrow} \left(s_1^{-\gamma} - 1\right) \mathbf{1}_{\{s_1 > 1-\varepsilon\}} \nu(\mathrm{d}\mathbf{s}) < \infty$ by definition and

$$\int_{\mathcal{S}^\downarrow} \left(\sum_{i=1}^N \left(s_i^{1-\gamma} - s_i\right) \frac{\mathbf{1}_{\{s_1 \leq 1-\varepsilon\}}}{s_1 + ... + s_N} \right) \nu(\mathrm{d}\mathbf{s}) \leq N \int_{\mathcal{S}^\downarrow} \frac{s_1^{1-\gamma} \mathbf{1}_{\{s_1 \leq 1-\varepsilon\}}}{s_1} \nu(\mathrm{d}\mathbf{s}),$$

which is finite by the definition of ϱ and since ν integrates $(1 - s_1)$. This implies in particular that ξ_t has finite expectation for every t, and it follows by [25] that $E[\mathcal{D}^{-1}] < \infty$. Then, following the proof of Proposition 2 in [19] and using again that $\Phi_\xi(-\gamma) > -\infty$ for $\gamma < \varrho$,

$$E\left[\left(\int_0^\infty \exp(\alpha\xi_t)\mathrm{d}t\right)^{-k-1}\right] = \frac{-\Phi_\xi(-|\alpha|\,k)}{k} E\left[\left(\int_0^\infty \exp(\alpha\xi_t)\mathrm{d}t\right)^{-k}\right]$$

for every integer $k < \varrho / |\alpha|$. Hence, using induction, $E[(\int_0^\infty \exp(\alpha\xi_t))^{-k-1}]$ is finite for $k = [\varrho/|\alpha|]$ if $\varrho/|\alpha| \notin \mathbb{N}$ and for $k = \varrho/|\alpha| - 1$ otherwise. In both cases, we see that $E[\mathcal{D}^{-\gamma}] < \infty$ for every $\gamma \leq \varrho/|\alpha|$. $\blacksquare$

Lemma 3.10 *For any $a < \varrho$, there exists N, ε such that*

$$E\left[\left(\lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}})\right)^{-a}; \lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}}) \geq \lambda_2^{N,\varepsilon}(\mathcal{D}_{\{1,2\}})\right] < \infty.$$

The ingredient for proving Lemma 3.10 is the following lemma, which uses the notations around the construction of the leaves $(L_1^{N,\varepsilon}, L_2^{N,\varepsilon})$.

Lemma 3.11 *With the convention* $\log(0) = -\infty$, *the process*

$$\sigma(t) = -\log\left|B_1^{0,N,\varepsilon}(t) \cap B_2^{0,N,\varepsilon}(t)\right| \quad , \quad t \geq 0$$

is a killed subordinator (its death time is $\mathcal{D}^0_{\{1,2\}}$) with Laplace exponent

$$\Phi_\sigma(q) = \mathtt{k}^{N,\varepsilon} + \int_{\mathcal{S}^\downarrow} \left((1 - s_1^q)\,\mathbf{1}_{\{s_1 > 1-\varepsilon\}} + \sum_{i=1}^N (1 - s_i^q) \frac{s_i^2 \mathbf{1}_{\{s_1 \leq 1-\varepsilon\}}}{(s_1 + ... + s_N)^2} \right) \nu(\mathrm{d}\mathbf{s}), \ q \geq 0, \quad (3.15)$$

where the killing rate $\mathtt{k}^{N,\varepsilon} := \int_{\mathcal{S}^\downarrow} \sum_{i \neq j} s_i s_j \frac{\mathbf{1}_{\{s_1 \leq 1-\varepsilon\}}}{(s_1 + ... + s_N)^2} \nu(\mathrm{d}\mathbf{s}) \in (0, \infty)$. *Moreover, the pair*

$$(l_1^{N,\varepsilon}, l_2^{N,\varepsilon}) = \exp(\sigma(\mathcal{D}^0_{\{1,2\}}-))(\lambda_1^{0,N,\varepsilon}(\mathcal{D}^0_{\{1,2\}}), \lambda_2^{0,N,\varepsilon}(\mathcal{D}^0_{\{1,2\}}))$$

is independent of $\sigma(\mathcal{D}^0_{\{1,2\}}-)$ with law characterized by

$$E\left[f\left(l_1^{N,\varepsilon}, l_2^{N,\varepsilon}\right)\right] = \frac{1}{\mathtt{k}^{N,\varepsilon}} \int_{\mathcal{S}^\downarrow} \sum_{1 \leq i \neq j \leq N} f(s_i, s_j) \frac{s_i s_j \mathbf{1}_{\{s_1 \leq 1-\varepsilon\}} \mathbf{1}_{\{s_i > 0\}} \mathbf{1}_{\{s_j > 0\}}}{(s_1 + ... + s_N)^2} \nu(\mathrm{d}\mathbf{s})$$

for any positive measurable function f.

Proof. We again use the Poisson construction of Π^0 out of $(\Delta_t, k_t, t \geq 0)$ and follow closely the proof of Lemma 3.9. For every $t \geq 0$ we have

$$B_k^{0,N,\varepsilon}(t) = \bigcap_{0 \leq s \leq t} \bar{\Delta}_s^k \quad , \quad k = 1, 2,$$

where $\bar{\Delta}_s^k$ is defined as follows. Let $J^k(s), k = 1, 2$ be the integers such that $B_k^{0,N,\varepsilon}(s-) = \Pi^0_{J^k(s)}(s-)$, so $\{\Delta_s : k_s = J^k(s), s \geq 0\}$, $k = 1, 2$ are two Poisson processes with same intensity κ_ν, which are equal for s in the interval $[0, \mathcal{D}^0_{\{1,2\}})$. Then for s with $k_s = J^k(s)$, let $\bar{\Delta}_s^k$ be the block of Δ_s containing $j^k(s)$. If $B_1^{0,N,\varepsilon}(s-) = B_2^{0,N,\varepsilon}(s-)$ notice that $j^1(s), j^2(s)$ are the two least integers of $(2\mathbb{N}+1) \cap B_1^{0,N,\varepsilon}(s-)$ and $(2\mathbb{N}) \cap B_2^{0,N,\varepsilon}(s-)$ respectively that are not isolated as singletons of $\Delta_s^{N,\varepsilon}$, so $\bar{\Delta}_s^1 = \bar{\Delta}_s^2$ if these two integers fall in the same block of $\Delta_s^{N,\varepsilon}$. Hence by a variation of Lemma 3.1, $(|\bar{\Delta}_s^1 \cap \bar{\Delta}_s^2|, s \geq 0)$ is a Poisson process whose intensity is the image measure of $\kappa_{\nu^{N,\varepsilon}}(\pi \mathbf{1}_{\{1 \sim 2\}})$ by the map $\pi \mapsto |\pi|$, and killed at an independent exponential time (namely $\mathcal{D}^0_{\{1,2\}}$) with parameter $\kappa_{\nu^{N,\varepsilon}}(1 \nsim 2)$ (here $1 \sim 2$ means that 1 and 2 are in the same block of π). This implies (3.15).

The time $\mathcal{D}^0_{\{1,2\}}$ is the first time when the two considered integers fall into two distinct blocks of $\Delta_s^{N,\varepsilon}$. It is then easy by the Poissonian construction and the paintbox representation to check that these blocks have asymptotic frequencies $(l_1^{N,\varepsilon}, l_2^{N,\varepsilon})$ which are independent of $\sigma(\mathcal{D}^0_{\{1,2\}}-)$, and have the claimed law. ■

Proof of Lemma 3.10. First notice, from the fact that self-similar fragmentations are time-changed homogeneous fragmentations, that

$$(\lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}}), \lambda_2^{N,\varepsilon}(\mathcal{D}_{\{1,2\}})) \stackrel{d}{=} (\lambda_1^{0,N,\varepsilon}(\mathcal{D}^0_{\{1,2\}}), \lambda_2^{0,N,\varepsilon}(\mathcal{D}^0_{\{1,2\}})).$$

Thus, with the notations of Lemma 3.11,

$$\begin{aligned} & E\left[\left(\lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}})\right)^{-a} ; \lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}}) \geq \lambda_2^{N,\varepsilon}(\mathcal{D}_{\{1,2\}})\right] \\ & \quad = E\left[\exp(a\sigma(\mathcal{D}^0_{\{1,2\}}-))\right] E\left[\left(l_1^{N,\varepsilon}\right)^{-a} ; l_1^{N,\varepsilon} \geq l_2^{N,\varepsilon}\right]. \end{aligned}$$

First, define for every $a > 0$ $\Phi_\sigma(-a)$ by replacing q by $-a$ in (3.15) and then remark that $\Phi_\sigma(-a) > -\infty$ when $a < \varrho$. Indeed, $\int_{\mathcal{S}^\downarrow} \left(s_1^{-a} - 1\right) \mathbf{1}_{\{s_1 > 1-\varepsilon\}} \nu(\mathrm{d}\mathbf{s})$ is then finite and, since $\sum_{1 \leq i \leq N} s_i^{2-a} \leq \left(\sum_{1 \leq i \leq N} s_i\right)^{2-a} \quad (2 - a \geq 1)$,

$$\sum_{1 \leq i \leq N} \left(s_i^{2-a} - s_i^2\right) \frac{\mathbf{1}_{\{s_1 \leq 1-\varepsilon\}}}{(s_1 + ... + s_N)^2} \leq \frac{\mathbf{1}_{\{s_1 \leq 1-\varepsilon\}}}{s_1^a}$$

which, by assumption, is integrable with respect to ν. Then, consider a subordinator $\widetilde{\sigma}$ with Laplace transform $\Phi_\sigma - \mathtt{k}^{N,\varepsilon}$ and independent of $\mathcal{D}^0_{\{1,2\}}$, such that $\sigma = \widetilde{\sigma}$ on $(0, \mathcal{D}^0_{\{1,2\}})$. As in the proof of Lemma 3.8, we use Theorem 25.17 of [64], which gives $E\left[\exp(a\widetilde{\sigma}(t))\right] = \exp\left(-t\left(\Phi_\sigma(-a) - \mathtt{k}^{N,\varepsilon}\right)\right)$ for all $t \geq 0$. Hence, by independence of $\widetilde{\sigma}$ and $\mathcal{D}^0_{\{1,2\}}$,

$$\begin{aligned} E\left[\exp(a\sigma(\mathcal{D}^0_{\{1,2\}}-))\right] &= E\left[\exp(a\widetilde{\sigma}(\mathcal{D}^0_{\{1,2\}}))\right] \\ &= \mathtt{k}^{N,\varepsilon} \int_0^\infty \exp(-t\mathtt{k}^{N,\varepsilon}) \exp\left(-t(\Phi_\sigma(-a) - \mathtt{k}^{N,\varepsilon})\right) \mathrm{d}t, \end{aligned}$$

which is finite if and only if $\Phi_\sigma(-a) > 0$. Recall that $\Phi_\sigma(-a)$ is equal to

$$\int_{\mathcal{S}^\downarrow} \left(1 - s_1^{-a}\right) \mathbf{1}_{\{s_1 > 1-\varepsilon\}} \nu(\mathrm{d}\mathbf{s}) + \int_{\mathcal{S}^\downarrow} \left(\sum_{1 \le i \ne j \le N} s_i s_j + \sum_{1 \le i \le N} \left(s_i^2 - s_i^{2-a}\right) \right) \frac{\mathbf{1}_{\{s_1 \le 1-\varepsilon\}}}{(s_1 + ... + s_N)^2} \nu(\mathrm{d}\mathbf{s}). \tag{3.16}$$

Since

$$\sum_{1 \le i \ne j \le N} s_i s_j + \sum_{1 \le i \le N} (s_i^2 - s_i^{2-a}) = (\sum_{1 \le i \le N} s_i)^2 - \sum_{1 \le i \le N} s_i^{2-a},$$

the integrand in the second term converges to $\left(1 - \sum_i s_i^{2-a}\right) \mathbf{1}_{\{s_1 \le 1-\varepsilon\}}$ as $N \to \infty$ and is dominated by $\left(1 + s_1^{-a}\right) \mathbf{1}_{\{s_1 \le 1-\varepsilon\}}$. So, by dominated convergence, the second term of (3.16) converges to $\int_{\mathcal{S}^\downarrow} (1 - \sum_i s_i^{2-a}) \mathbf{1}_{\{s_1 \le 1-\varepsilon\}} \nu(\mathrm{d}\mathbf{s})$ as $N \to \infty$. This last integral converges to a strictly positive quantity as $\varepsilon \downarrow 0$, and since $\int_{\mathcal{S}^\downarrow} \left(1 - s_1^{-a}\right) \mathbf{1}_{\{s_1 > 1-\varepsilon\}} \nu(\mathrm{d}\mathbf{s}) \to 0$ as $\varepsilon \to 0$, $\Phi_\sigma(-a)$ is strictly positive for N and $1/\varepsilon$ large enough. Hence $E[\exp(a\sigma(\mathcal{D}^0_{\{1,2\}}-))] < \infty$ for N and $1/\varepsilon$ large enough.

On the other hand, Lemma 3.11 implies that the finiteness of $E[(l_1^{N,\varepsilon})^{-a} \mathbf{1}_{\{l_1^{N,\varepsilon} \ge l_2^{N,\varepsilon}\}}]$ is equivalent to that of $\int_{\mathcal{S}^\downarrow} \sum_{1 \le i \ne j \le N} s_i^{1-a} s_j \frac{\mathbf{1}_{\{s_1 \le 1-\varepsilon\}}}{(s_1 + ... + s_N)^2} \nu(\mathrm{d}\mathbf{s})$. But this integral is finite for all integers N and every $0 < \varepsilon < 1$, since $\sum_{1 \le i \ne j \le N} s_i^{1-a} s_j \le N^2 s_1^{2-a}$ and ν integrates $s_1^{-a} \mathbf{1}_{\{s_1 \le 1-\varepsilon\}}$. Hence the result. ■

3.3.5 Dimension of the stable tree

This section is devoted to the proof of Corollary 3.1. Recall from [56] that the fragmentation F_- associated with the β-stable tree has index $1/\beta - 1$ (where $\beta \in (1, 2]$). In the case $\beta = 2$, the tree is the Brownian CRT and the fragmentation is binary (it is the fragmentation F_B of the Introduction), so that the integrability assumption of Theorem 3.2 is satisfied and then the dimension is 2. So suppose $\beta < 2$. The main result of [56] is that the dislocation measure $\nu_-(\mathrm{d}\mathbf{s})$ of F_- has the form

$$\nu_-(\mathrm{d}\mathbf{s}) = C(\beta) E\left[T_1; \frac{\Delta T_{[0,1]}}{T_1} \in \mathrm{d}\mathbf{s}\right]$$

for some constant $C(\beta)$, where $(T_x, x \ge 0)$ is a stable subordinator with index $1/\beta$ and $\Delta T_{[0,1]} = (\Delta_1, \Delta_2, \ldots)$ is the decreasing rearrangement of the sequence of jumps of T accomplished within the time-interval $[0, 1]$ (so that $\sum_i \Delta_i = T_1$). By Theorem 3.2, to prove Corollary 3.1 it thus suffices to check that $E[T_1(T_1/\Delta_1 - 1)]$ is finite. The problem is that computations involving jumps of subordinators are often quite involved; they are sometimes eased by using size-biased picked jumps, whose laws are more tractable. However, one can check that if Δ_* is a size-biased picked jump among $(\Delta_1, \Delta_2, \ldots)$, the quantity $E[T_1(T_1/\Delta_* - 1)]$ is infinite, therefore we really have to study the joint law of (T_1, Δ_1). This has been done in Perman [59], but we will re-explain all the details we need here.

Recall that the process $(T_x, x \ge 0)$ can be put in the Lévy-Itô form $T_x = \sum_{0 \le y \le x} \Delta(y)$, where $(\Delta(y), y \ge 0)$ is a Poisson point process with intensity $cu^{-1-1/\beta}\mathrm{d}u$ (the Lévy measure of T) for some constant $c > 0$. Therefore, the law of the largest jump of T before time 1 is characterized by

$$P(\Delta_1 < v) = P\left(\sup_{0\leq y\leq 1} \Delta(y) < v\right) = \exp\left(-c\beta v^{-1/\beta}\right) \qquad v > 0,$$

and by the restriction property of Poisson processes, conditionally on $\Delta_1 = v$, one can write $T_1 = v + T_1^{(v)}$, where $(T_x^{(v)}, x \geq 0)$ is a subordinator with Lévy measure $cu^{-1-1/\beta}\mathbf{1}_{\{0\leq u\leq v\}}\mathrm{d}u$. The Laplace transform of $T_x^{(v)}$ is given by the Lévy-Khintchine formula

$$E[\exp(-\lambda T_x^{(v)})] = \exp\left(-x\int_0^v \frac{c(1-e^{-\lambda u})}{u^{1+1/\beta}}\mathrm{d}u\right) \quad \lambda, x \geq 0,$$

in particular, $T_1^{(v)}$ admits moments of all order (by differentiating in λ) and $v^{-1}T_1^{(v)}$ has the same law as $T^{(1)}_{v^{-1/\beta}}$ (by changing variables). We then obtain

$$\begin{aligned} E[T_1(T_1/\Delta_1 - 1)] &= E\left[\Delta_1\left(1 + \frac{T_1^{(\Delta_1)}}{\Delta_1}\right)\frac{T_1^{(\Delta_1)}}{\Delta_1}\right] \\ &= K_1\int_{\mathbb{R}_+} \mathrm{d}v\, v^{-1/\beta}e^{-\beta c v^{-1/\beta}} E\left[\left(1 + \frac{T_1^{(v)}}{v}\right)\frac{T_1^{(v)}}{v}\right] \\ &= K_1\int_{\mathbb{R}_+} \mathrm{d}v\, v^{-1/\beta}e^{-\beta c v^{-1/\beta}} E\left[\left(1 + T^{(1)}_{v^{-1/\beta}}\right)T^{(1)}_{v^{-1/\beta}}\right] \end{aligned}$$

where $K_1 = K(\beta) > 0$. Since $T_1^{(1)}$ has a moment of orders 1 and 2, the expectation in the integrand is dominated by some $K_2 v^{-1/\beta} + K_3 v^{-2/\beta}$. It is then easy to see that the integrand is integrable both near 0 and ∞ since $\beta < 2$. Hence $\int_{\mathcal{S}^\downarrow}\left(s_1^{-1} - 1\right)\nu_-(\mathrm{d}\mathbf{s}) < \infty$.

3.4 The height function

We now turn to the proof of the results related to the height function, starting with Theorem 3.3. The height function we are going to build will in fact satisfy more than stated there: we will show that under the hypotheses of Theorem 3.3, there exists a process H_F that encodes $\mathcal{T}_F$ in the sense given in the introduction, that is, $\mathcal{T}_F$ is isometric to the quotient $((0,1), \overline{d})/\equiv$, where $\overline{d}(u,v) = H_F(u) + H_F(v) - 2\inf_{s\in[u,v]} H_F(s)$ and $u \equiv v \iff \overline{d}(u,v) = 0$. Once we have proved this, the result is obvious since $I_F(t)/\equiv$ is the set of vertices of $\mathcal{T}_F$ that are above level t.

3.4.1 Construction of the height function

Recall from [3] that to encode a CRT, defined as a projective limit of consistent random $\mathbb{R}-$trees $(\mathcal{R}(k), k \geq 1)$, in a continuous height process, one first needs to enrich the structure of the $\mathbb{R}$-trees with consistent *orders* on each set of children of some node. The sons of a given node of $\mathcal{R}(k)$ are thus labelled as first, second, etc... This induces a *planar* representation of the tree.

This representation also induces a total order on the vertices of $\mathcal{R}(k)$, which we call $\preceq_k$, by the rule $v \preceq w$ if either v is an ancestor of w, or the branchpoint $b(v,w)$ of v and w is such that the edge leading toward v is earlier than the edge leading toward w (for the ordering on children of $b(v,w)$). In turn, the knowledge of $\mathcal{R}(k), \preceq_k$, or even of $\mathcal{R}(k)$ and the restriction of $\preceq_k$ to the leaves $L_1, \ldots, L_k$ of $\mathcal{R}(k)$, allows us to recover the planar structure of $\mathcal{R}(k)$. The family of planar trees $(\mathcal{R}(k), \preceq_k, k \geq 1)$ is said to be *consistent* if furthermore for every $1 \leq j < k$ the planar tree $(\mathcal{R}(j), \preceq_j)$ has the same law as the planar subtree of $(\mathcal{R}(k), \preceq_k)$ spanned by j leaves $L_1^1, \ldots, L_j^k$ taken independently uniformly at random among the leaves of $\mathcal{R}(k)$.

We build such a consistent family out of the consistent family of unordered trees $(\mathcal{R}(k), k \geq 1)$ as follows. Starting from the tree $\mathcal{R}(1)$, which we endow with the trivial order on its only leaf, we build recursively the total order on $\mathcal{R}(k+1)$ from the order $\preceq_k$ on $\mathcal{R}(k)$, so that the restriction of $\preceq_{k+1}$ to the leaves $L_1, \ldots, L_k$ of $\mathcal{R}(k)$ equals $\preceq_k$. Given $\mathcal{R}(k+1), \preceq_k$, let $b(L_{k+1})$ be the father of L_{k+1}. We distinguish two cases:

1. if $b(L_{k+1})$ is a vertex of $\mathcal{R}(k)$, which has r children $c_1, c_2, \ldots, c_r$ in $\mathcal{R}(k)$, choose J uniformly in $\{1, 2, \ldots, r+1\}$ and let $c_{J-1} \preceq_{k+1} L_{k+1} \preceq_{k+1} c_J$, that is, turn L_{k+1} into the j-th son of $b(L_{k+1})$ in $\mathcal{R}(k+1)$ with probability $1/(r+1)$ (here c_0 (resp. c_{r+1}) is the predecessor (resp. successor) of c_1 (resp. c_r) for $\preceq_k$; we simply ignore them if they do not exist)

2. else, $b(L_{k+1})$ must have a unique son s besides L_{k+1}. Let s' be the predecessor of s for $\preceq_k$ and s'' its successor (if any), and we let $s' \preceq_{k+1} L_{k+1} \preceq_{k+1} s$ with probability $1/2$ and $s \preceq_{k+1} L_{k+1} \preceq_{k+1} s''$ with probability $1/2$.

It is easy to see that this procedure uniquely determines the law of the total order $\preceq_{k+1}$ on $\mathcal{R}(k+1)$ given $\mathcal{R}(k+1), \preceq_k$, and hence the law of $(\mathcal{R}(k), \preceq_k, k \geq 1)$ (the important thing being that the order is total).

Lemma 3.12 *The family of planar trees $(\mathcal{R}(k), \preceq_k, k \geq 1)$ is consistent. Moreover, given $\mathcal{R}(k)$, the law of $\preceq_k$ can be obtained as follows: for each vertex v of $\mathcal{R}(k)$, endow the (possibly empty) set $\{c_1(v), \ldots, c_i(v)\}$ of children of v in uniform random order, this independently over different vertices.*

Proof. The second statement is obvious by induction. The first statement follows, since we already know that the family of unordered trees $(\mathcal{R}(k), k \geq 1)$ is consistent. ■

As a consequence, there exists a.s. a unique total order $\preceq$ on the set of leaves $\{L_1, L_2 \ldots\}$ such that the restriction $\preceq_{|[k]} = \preceq_k$. One can check that this order extends to a total order on the set $\mathcal{L}(\mathcal{T}_F)$: if L, L' are distinct leaves, we say that $L \preceq L'$ if and only if there exist two sequences $L_{\phi(k)} \preceq L_{\varphi(k)}, k \geq 1$, the first one decreasing and converging to L and the second increasing and converging to L'. In turn, this extends to a total order (which we still call $\preceq$) on the whole tree $\mathcal{T}_F$. Theorem 3.3 is now a direct application of [3, Theorem 15 (iii)], the only thing to check being the conditions a) and b) therein (since we already know that $\mathcal{T}_F$ is compact). Precisely, condition (iii) a) rewritten to fit our setting spells:

$$\lim_{k\to\infty} P(\exists 2 \leq j \leq k : |D_{\{1,j\}} - aD_1| \leq \delta \text{ and } D_j - D_{\{1,j\}} < \delta \text{ and } L_j \preceq L_1) = 1.$$

This is thus a slight modification of (3.4), and the proof goes similarly, the difference being that we need to keep track of the order on the leaves. Precisely, consider again some rational $r < aD_1$ close to aD_1, so that $|\Pi_1(r)| \neq 0$. The proof of (3.4) shows that within the time-interval $[r, r+\delta]$, infinitely many integers of $\Pi_1(r)$ have been isolated into singletons. Now, by definition of $\preceq$, the probability that any of these integers j satisfies $L_j \preceq_j L_1$ is $1/2$. Therefore, infinitely many integers of $\Pi_1(r)$ give birth to a leaf L_j that satisfy the required conditions, a.s. The proof of [3, Condition (iii) b)] is exactly similar, hence proving Theorem 3.3.

It is worth recalling the detailed construction of the process H_F, which is taken from the proof of [3, Theorem 15] with a slight modification (we use the leaves L_i rather than a new sample $Z_i, i \geq 1$, but one checks that the proof remains valid). Given the continuum ordered tree $(\mathcal{T}_F, \mu_F, \preceq, (L_i, i \geq 1))$,

$$U_i = \lim_{n\to\infty} \frac{\#\{j \leq n : L_j \preceq L_i\}}{n},$$

a limit that exists a.s. Then the family $(U_i, i \geq 1)$ is distributed as a sequence of independent sequence of uniformly distributed random variables on $(0,1)$, and since $\preceq$ is a total order, one has $U_i \leq U_j$ if and only if $L_i \preceq L_j$. Next, define $H_F(U_i)$ to be the height of L_i in $\mathcal{T}_F$, and extend it by continuity on $[0,1]$ (which is a.s. possible according to [3, Theorem 15]) to obtain H_F. In fact, one can define $\tilde{H}_F(U_i) = L_i$ and extend it by continuity on $\mathcal{T}_F$, in which case $\tilde{H}_F$ is an isometry between $\mathcal{T}_F$ and $((0,1), \overline{d})/\equiv$ that maps (the equivalence class of) U_i to L_i for $i \geq 1$, and which preserves order.

Writing $I_F(t) = \{s \in (0,1) : H_F(s) > t\}$, and $|I_F(t)|$ for the decreasing sequence of the lengths of the interval components of $I_F(t)$, we know from the above that $(|I_F(t)|, t \geq 0)$ has the same law as F. More precisely,

Lemma 3.13 *The processes $(|I_F(t)|, t \geq 0)$ and $(F(t), t \geq 0)$ are equal.*

Proof. Let $\Pi'(t)$ be the partition of $\mathbb{N}$ such that $i \sim^{\Pi'(t)} j$ if and only if U_i and U_j fall in the same interval component of $I_F(t)$. The isometry $\tilde{H}_F$ allows us to assimilate L_i to U_i, then the interval component of $I_F(t)$ containing U_i corresponds to the tree component of $\{v \in \mathcal{T}_F : \mathrm{ht}(v) > t\}$ containing L_i, therefore U_j falls in this interval if and only if $i \sim^{\Pi(t)} j$, and $\Pi'(t) = \Pi(t)$. By the law of large numbers and the fact that $(U_j, j \geq 1)$ is distributed as a uniform i.i.d. sample, it follows that the length of the interval equals the asymptotic frequency of the block of $\Pi(t)$ containing i, a.s. for every t. One inverts the assertions "a.s." and "for every t" by a simple monotony argument, showing that if $(U_i, i \geq 1)$ is a uniform i.i.d. sample, then a.s. for every sub-interval (a,b) of $(0,1)$, the asymptotic frequency $\lim_{n\to\infty} n^{-1}\#\{i \leq n : U_i \in (a,b)\} = b - a$ (use distribution functions). ■

We will also need the following result, which is slightly more accurate than just saying, as in the introduction, that $(I_F(t), t \geq 0)$ is an "interval representation" of F:

Lemma 3.14 *The process $(I_F(t), t \geq 0)$ is a self-similar interval fragmentation, meaning that it is nested ($I_F(t') \subseteq I_F(t)$ for every $0 \leq t \leq t'$), continuous in probability, and for every $t, t' \geq 0$, given $I_F(t) = \bigcup_{i\geq 1} I_i$ where I_i are pairwise disjoint intervals, $I_F(t+t')$ has the same law as $\bigcup_{i\geq 1} g_i(I_F^{(i)}(t'|I_i|^{\alpha}))$, where the $I_F^{(i)}, i \geq 1$ are independent copies of I_F, and g_i is the orientation-preserving affine function that maps $(0,1)$ to I_i.*

Here, the "continuity in probability" is with respect to the Hausdorff metric D on compact subsets of $[0,1]$, and it just means that $P(D(I_F^c(t_n), I_F^c(t)) > \varepsilon) \to 0$ as $n \to \infty$ for any sequence $t_n \to t$ and $\varepsilon > 0$ (here $A^c = [0,1] \setminus A$).

Proof. The fact that $I_F(t)$ is nested is trivial. Now recall that the different interval components of $I_F(t)$ encode the tree components of $\{v \in \mathcal{T}_F : \mathrm{ht}(v) > t\}$, call them $\mathcal{T}_1(t), \mathcal{T}_2(t), \ldots$. We already know that these trees are rescaled independent copies of $\mathcal{T}_F$, that is, they have the same law as $\mu_F(\mathcal{T}_i(t))^{-\alpha} \otimes \mathcal{T}^{(i)}, i \geq 1$, where $\mathcal{T}^{(i)}, i \geq 1$ are independent copies of $\mathcal{T}_F$. So let $\mathcal{T}^{(i)} = \mu_F(\mathcal{T}_i(t))^{\alpha} \otimes \mathcal{T}_i(t)$. Now, the orders induced by $\preceq$ on the different $\mathcal{T}^{(i)}$'s have the same law as $\preceq$ and are independent, because they only depend on the L_j's that fall in each of them. Therefore, the trees $(\mathcal{T}^{(i)}, \mu^{(i)}, \preceq^{(i)})$ are independent copies of $(\mathcal{T}_F, \mu_F, \preceq)$, where $\mu^{(i)}(\cdot) = \mu_F((\mu_F(\mathcal{T}_i(t))^{-\alpha} \otimes \cdot) \cap \mathcal{T}_i(t))/\mu_F(\mathcal{T}_i(t))$ and $\preceq^{(i)}$ is the order on $\mathcal{T}^{(i)}$ induced by the restriction of $\preceq$ to $\mathcal{T}_i(t)$. It follows by our previous considerations that their respective height processes $H^{(i)}$ are independent copies of H_F, and it is easy to check that given $I_F(t) = \bigcup_{i\geq 1} I_i$ (where I_i is the interval corresponding to $\mathcal{T}_i(t)$), the excursions of H_F above t are precisely the processes $\mu(\mathcal{T}_i(t))^{-\alpha} H^{(i)} = |I_i|^{-\alpha} H^{(i)}$. The self-similar fragmentation property follows at once, as the fact that I_F is Markov. Thanks to these properties, we may just check the continuity in probability at time 0, and it is trivial because H_F is a.s. continuous and positive on $(0,1)$. ■

It appears that besides these elementary properties, the process H_F is quite hard to study. In order to move one step further, we will try to give a "Poissonian construction" of H_F, in the same way as we used properties of the Poisson process construction of Π^0 to study $\mathcal{T}_F$. To begin with, we move "back to the homogeneous case" by time-changing. For every $x \in (0,1)$, let $I_x(t)$ be the interval component of $I_F(t)$ containing x, and $|I_x(t)|$ be its length ($= 0$ if $I_x(t) = \varnothing$). Then set

$$T_t^{-1}(x) = \inf\left\{u \geq 0 : \int_0^u |I_x(r)|^{\alpha} \mathrm{d}r > t\right\},$$

and let $I_F^0(t)$ be the open set constituted of the union of the intervals $I_x(T_t^{-1}(x)), x \in (0,1)$ (it suffices in fact to take the union of the $I_{U_i}(T_t^{-1}(U_i)), i \geq 1$). From [14] and Lemma 3.14, $(I_F^0(t), t \geq 0)$ is a self-similar homogeneous interval fragmentation.

3.4.2 A Poissonian construction

Recall that the process $(\Pi(t), t \geq 0)$ is constructed out of a homogeneous fragmentation $(\Pi^0(t), t \geq 0)$, which has been appropriately time-changed, and where $(\Pi^0(t), t \geq 0)$ has itself been constructed out of a Poisson point process $(\Delta_t, k_t, t \geq 0)$ with intensity $\kappa_\nu \otimes \#$. Further, we mark this Poisson process by considering, for each jump time t of this Poisson process, a sequence $(U_i(t), i \geq 1)$ of i.i.d. random variables that are uniform on $(0,1)$, so that these sequences are independent over different such t's. We are going to use the marks to build an order on the non-void blocks of Π^0. It is convenient first to formalize what we precisely call an *order* on a set A: it is a subset $\mathcal{O}$ of $A \times A$ satisfying:

1. $(i,i) \in \mathcal{O}$ for every $i \in A$
2. $(i,j) \in \mathcal{O}$ and $(j,i) \in \mathcal{O}$ imply $i = j$
3. $(i,j) \in \mathcal{O}$ and $(j,k) \in \mathcal{O}$ imply $(i,k) \in \mathcal{O}$.

If $B \subseteq A$, the restriction to B of the order $\mathcal{O}$ is $\mathcal{O}_{|B} = \mathcal{O} \cap (B \times B)$. We now construct a process $(\mathcal{O}(t), t \geq 0)$, with values in the set of orders of $\mathbb{N}$, as follows. Let $\mathcal{O}(0) = \{(i,i), i \in \mathbb{N}\}$ be the trivial order, and let $n \in \mathbb{N}$. Let $0 < t_1 < t_2 < \ldots < t_K$ be the times of occurrence of jumps of the Poisson process $(\Delta_t, k_t, t \geq 0)$ such that both $k_t \leq n$ and $(\Delta_t)_{|[n]}$ (the restriction of Δ_t to $[n]$) is non-trivial. Let $\mathcal{O}^n(0) = \mathcal{O}_{|[n]}(0)$, and define a process $\mathcal{O}^n(t)$ to be constant on the time-intervals $[t_{i-1}, t_i)$ (where $t_0 = 0$), where inductively, given $\mathcal{O}^n(t_{i-1}) = \mathcal{O}^n(t_i-)$, $\mathcal{O}^n(t_i)$ is defined as follows. Let $J_n(t_i) = \{j \in \Pi^0_{k_{t_i}}(t_i-) : j \leq n \text{ and } \Pi^0_j(t_i) \neq \varnothing\}$ so that $k_{t_i} \in J_n(t_i)$ as soon as $\Pi^0_{k_{t_i}}(t_i-) \neq \varnothing$. Let then

$$\mathcal{O}^n(t_i) = \mathcal{O}^n(t_i-) \cup \bigcup_{\substack{j,k \in J_n(t_i): \\ U_j(t_i) < U_k(t_i)}} \{(j,k)\} \cup \bigcup_{\substack{j:(j,k_{t_i}) \in \mathcal{O}^n(t_i-) \\ j \neq k_{t_i} \\ k \in J_n(t_i)}} \{(j,k)\} \cup \bigcup_{\substack{j:(k_{t_i},j) \in \mathcal{O}^n(t_i-) \\ j \neq k_{t_i} \\ k \in J_n(t_i)}} \{(k,j)\}.$$

In words, we order each set of new blocks in random order in accordance with the variables $U_m(t_i), 1 \leq m \leq n$, and these new blocks have the same relative position with other blocks as had their father, namely the block $\Pi^0_{k_{t_i}}(t_i-)$.

It is easy to see that the orders thus defined are consistent as n varies, i.e. $(\mathcal{O}^{n+1}(t))_{|[n]} = \mathcal{O}^n(t)$ for every n, t, and it easily follows that there exists a unique process $(\mathcal{O}(t), t \geq 0)$ such that $\mathcal{O}_{|[n]}(t) = \mathcal{O}^n(t)$ for every n, t (for existence, take the union over $n \in \mathbb{N}$, and unicity is trivial). The process $\mathcal{O}$ thus obtained allows us to build an interval-valued version of the fragmentation $\Pi^0(t)$, namely, for every $t \geq 0$ and $j \geq 0$ let

$$I^0_j(t) = \left(\sum_{k \neq j:(k,j) \in \mathcal{O}(t)} |\Pi^0_k(t)|, \sum_{k:(k,j) \in \mathcal{O}(t)} |\Pi^0_k(t)| \right)$$

(notice that $I^0_j(t) = \varnothing$ if $\Pi^0_j(t) = \varnothing$). Write $I^0(t) = \bigcup_{j \geq 1} I^0_j(t)$, and notice that the length $|I^0_j(t)|$ of $I^0_j(t)$ equals the asymptotic frequency of $\Pi^0_j(t)$ for every $j \geq 1, t \geq 0$.

Proposition 3.2 *The processes $(I^0_F(t), t \geq 0)$ and $(I^0(t), t \geq 0)$ have the same law.*

As a consequence, we have obtained a construction of an object with the same law as I^0_F with the help of a marked Poisson process in $\mathcal{P}_\infty$, and this is the one we are going to work with.

Proof. Let $I^0_F(i,t)$ be the interval component of $I^0_F(t)$ containing U_i if i is the least j such that U_j falls in this component, and $I^0_F(i,t) = \varnothing$ otherwise. Let $\mathcal{O}_F(t) = \{(i,i), i \in \mathbb{N}\} \cup \{(j,k) : I^0_F(j,t) \text{ is located to the left of } I^0_F(k,t) \text{ and both are nonempty}\}$. Since the lengths of the interval components of I^0_F and I^0 are the same, the only thing we need to check is that the processes $\mathcal{O}$ and $\mathcal{O}_F$ have the same law. But then, for $j \neq k$, $(j,k) \in \mathcal{O}_F(t)$ means that the branchpoint $b(L_j, L_k)$ of L_j and L_k has height less than t, and the subtree rooted at $b(L_j, L_k)$ containing L_j has been placed before that containing L_k. Using Lemma 3.12, we see that given $\mathcal{T}_F, L_1, L_2, \ldots$, the subtrees rooted at any branchpoint b of $\mathcal{T}_F$ are placed in exchangeable random order independently over branchpoints. Precisely, letting $\mathcal{T}^b_1$ be the subtree containing the leaf with least label, $\mathcal{T}^b_2$ the subtree different from $\mathcal{T}^b_1$ containing the leaf with least label, and so on, the first subtrees $\mathcal{T}^b_1, \ldots, \mathcal{T}^b_k$ are placed in any of the $k!$ possible linear orders, consistently as k varies. Therefore (see e.g. [3, Lemma 10]), there exist independent uniform$(0,1)$ random variables $U_1^b, U_2^b, \dots$ independent over b's such that $\mathcal{T}_i^b$ is on the "left" of $\mathcal{T}_j^b$ (for the order $\mathcal{O}_F$) if and only if $U_i^b \le U_j^b$. This is exactly how we defined the order $\mathcal{O}(t)$. ■

Remark. As the reader may have noticed, this construction of an interval-valued fragmentation has in fact little to do with pure manipulation of intervals, and it is actually almost entirely performed in the world of partitions. We stress that it is in fact quite hard to construct directly such an interval fragmentation out of the plain idea: "start from the interval $(0,1)$, take a Poisson process $(s(t), k_t, t \ge 0)$ with intensity $\nu(\mathrm{d}\mathbf{s}) \otimes \#$, and at a jump time of the Poisson process turn the k_t-th interval component $I_{k_t}(t-)$ of $I(t-)$ (for some labelling convention) into the open subset of $I_{k_t}(t-)$ whose components sizes are $|I_{k_t}(t-)|s_i(t), i \ge 1$, and placed in exchangeable order". Using partitions helps much more than plainly giving a natural "labelling convention" for the intervals. In the same vein, we refer to the work of Gnedin [37], which shows that exchangeable interval (composition) structures are in fact equivalent to "exchangeable partitions+order on blocks".

For every $x \in (0,1)$, write $I_x^0(t)$ for the interval component of $I_F^0(t)$ containing x, and notice that $I_x^0(t-) = \bigcap_{s\uparrow t} I_x^0(s)$ is well-defined as a decreasing intersection. For $t \ge 0$ such that $I_x^0(t) \ne I_x^0(t-)$, let $s^x(t)$ be the sequence $|I_F^0(t) \cap I_x^0(t-)|/|I_x^0(t-)|$, where $|I_F^0(t) \cap I_x^0(t-)|$ is the decreasing sequence of lengths of the interval components of $I_F^0(t) \cap I_x^0(t-)$. The useful result on the Poissonian construction is given in the following

Lemma 3.15 *The process $(s^x(t), t \ge 0)$ is a Poisson point process with intensity $\nu(\mathrm{d}\mathbf{s})$, and more precisely, the order of the interval components of $I_F^0(t) \cap I_x^0(t-)$ is exchangeable: there exists a sequence of i.i.d. uniform random variables $(U_i^x(t), i \ge 1)$, independent of $(\mathcal{G}^0(t-), s^x(t))$ such that the interval with length $s_i^x(t)|I_x^0(t-)|$ is located on the left of the interval with length $s_j^x(t)|I_x^0(t-)|$ if and only if $U_i^x(t) \le U_j^x(t)$.*

Proof. Let $i(t,x) = \inf\{i \in \mathbb{N} : U_i \in I_x^0(t)\}$. Then $i(t,x)$ is an increasing jump-hold process in $\mathbb{N}$. If now $I_x^0(t) \ne I_x^0(t-)$, it means that there has been a jump of the Poisson process Δ_t, k_t at time t, so that $k_t = i(t,x)$, and then $s^x(t)$ is equal to the decreasing sequence $|\Delta_t|$ of asymptotic frequencies of Δ_t, therefore $s^x(t) = |\Delta_t|$ when $k_t = i(t-,x)$, and since $i(t-,x)$ is progressive, its jump times are stopping times so the process $(s^x(t), t \ge 0)$ is in turn a Poisson process with intensity $\nu(\mathrm{d}\mathbf{s})$. Moreover, by Proposition 3.2 and the construction of I^0, each time an interval splits, the corresponding blocks are put in exchangeable order, which gives the second half of the lemma. ■

3.4.3 Proof of Theorem 3.4

3.4.3.1 Hölder-continuity of H_F

We prove here that the height process is a.s. Hölder-continuous of order γ for every $\gamma < \vartheta_{\mathrm{low}} \wedge |\alpha|$. The proof will proceed in three steps.

First step: Reduction to the behavior of H_F near 0. By a theorem of Garsia, Rodemich and Rumsey (see e.g. [26]), the finiteness of $\int_0^1\int_0^1 \frac{|H_F(x)-H_F(y)|^{n+n_0}}{|x-y|^{\gamma n}}\mathrm{d}x\mathrm{d}y$ leads to the $\left(\frac{\gamma n-2}{n+n_0}\right)$-Hölder-continuity of H_F, so that when the previous integral is finite for every n, the height process H_F is Hölder-continuous of order δ for every $\delta < \gamma$, whatever is n_0. To prove Theorem 3.4 it is thus sufficient to show that for every $\gamma < \vartheta_{\text{low}} \wedge |\alpha|$ there exists a $n_0(\gamma)$ such that

$$E\left[\int_0^1 \int_0^1 \frac{|H_F(x) - H_F(y)|^{n+n_0(\gamma)}}{|x-y|^{\gamma n}} \mathrm{d}x\mathrm{d}y\right] < \infty \text{ for every positive integer } n.$$

Now take V_1, V_2 uniform independent on $(0,1)$, independently of H_F. The expectation above then becomes $E\left[\frac{|H_F(V_1)-H_F(V_2)|^{n+n_0(\gamma)}}{|V_1-V_2|^{\gamma n}}\right]$.

Consider next I_F the interval fragmentation constructed from H_F (see Section 3.4.1). By Lemma 3.14, $H_F(V_1)$ and $H_F(V_2)$ may be rewritten as

$$H_F(V_i) = D_{\{1,2\}} + \lambda_i^{|\alpha|}(D_{\{1,2\}})\widetilde{D}_i,\ i = 1,2,$$

where $D_{\{1,2\}}$ is the first time at which V_1 and V_2 belong to different intervals of I_F and $\widetilde{D}_1, \widetilde{D}_2$ have the same law as $H_F(V_1)$ and are independent of $\mathcal{H}(D_{\{1,2\}})$, where $\mathcal{H}(t), t \geq 0$ is the natural completed filtration associated with I_F. The r.v. $\widetilde{D}_1$ and $\widetilde{D}_2$ can actually be described more precisely. Say that at time $D_{\{1,2\}}$, V_1 belongs to an interval $\left(a_1, a_1 + \lambda_1(D_{\{1,2\}})\right)$ and V_2 to $\left(a_2, a_2 + \lambda_2(D_{\{1,2\}})\right)$. Then there exist two iid processes independent of $\mathcal{H}(D_{\{1,2\}})$ and with the same law as H_F, let us denote them $H_F^{(1)}$ and $H_F^{(2)}$, such that $\widetilde{D}_i = H_F^{(i)}\left(\frac{V_i - a_i}{\lambda_i(D_{\{1,2\}})}\right)$, $i = 1,2$. Since $V_i \in \left(a_i, a_i + \lambda_i(D_{\{1,2\}})\right)$, the random variables $\widetilde{V}_i = (V_i - a_1)\,\lambda_i^{-1}(D_{\{1,2\}})$ are iid, with the uniform law on $(0,1)$ and independent of $H_F^{(1)}, H_F^{(2)}$ and $\mathcal{H}(D_{\{1,2\}})$. And when $V_1 > V_2$,

$$V_1 - V_2 \geq \lambda_1(D_{\{1,2\}})\widetilde{V}_1 + \lambda_2(D_{\{1,2\}})\left(1 - \widetilde{V}_2\right)$$

since a_1 is then larger than $a_2 + \lambda_2(D_{\{1,2\}})$. This gives

$$\begin{aligned} E\left[\frac{|H_F(V_1) - H_F(V_2)|^{n+n_0(\gamma)}}{|V_1 - V_2|^{\gamma n}}\right] &= 2E\left[\frac{|H_F(V_1) - H_F(V_2)|^{n+n_0(\gamma)}}{(V_1 - V_2)^{\gamma n}}\mathbf{1}_{\{V_1 > V_2\}}\right] \\ &\leq 2E\left[\frac{\left(\lambda_1^{|\alpha|}(D_{\{1,2\}})\widetilde{D}_1 + \lambda_2^{|\alpha|}(D_{\{1,2\}})\widetilde{D}_2\right)^{n+n_0(\gamma)}}{\left(\lambda_1(D_{\{1,2\}})\widetilde{V}_1 + \lambda_2(D_{\{1,2\}})\left(1-\widetilde{V}_2\right)\right)^{\gamma n}}\right] \end{aligned}$$

and this last expectation is bounded from above by

$$2^{n+n_0(\gamma)}E\left[\left(\lambda_1(D_{\{1,2\}})\right)^{(n+n_0(\gamma))|\alpha|-\gamma n}\right]\left(E\left[\frac{H_F^{n+n_0(\gamma)}(V_1)}{V_1^{\gamma n}}\right] + E\left[\frac{H_F^{n+n_0(\gamma)}(V_1)}{(1-V_1)^{\gamma n}}\right]\right).$$

The expectation involving λ_1 is bounded by 1 since $\gamma < |\alpha|$. And since V_1 is independent of H_F, the two expectations in the parenthesis are equal (reversing the order $\preceq$ and performing the construction of H_F gives a process with the same law and shows that $H_F(x) \overset{law}{=} H_F(1-x)$ for every $x \in (0,1)$) and finite as soon as

$$\sup_{x \in (0,1)} E\left[H_F(x)^{n+n_0(\gamma)}\right] x^{-\gamma n} < \infty. \tag{3.17}$$

The rest of the proof thus consists of finding an integer $n_0(\gamma)$ such that (3.17) holds for every n. To do so, we will have to observe the interval fragmentation I_F at nice stopping times depending on x, say $\mathbb{T}_x^{(\gamma)}$, and then use the strong fragmentation property (which also holds for interval fragmentations, see [14]) at time $\mathbb{T}_x^{(\gamma)}$. This gives

$$H_F(x) = \mathbb{T}_x^{(\gamma)} + \left(S_x(\mathbb{T}_x^{(\gamma)})\right)^{|\alpha|} \overline{H}_F(P_x(\mathbb{T}_x^{(\gamma)})) \tag{3.18}$$

where $S_x(\mathbb{T}_x^{(\gamma)})$ is the length of the interval containing x at time $\mathbb{T}_x^{(\gamma)}$, $P_x(\mathbb{T}_x^{(\gamma)})$ the relative position of x in that interval and $\overline{H}_F$ a process with the same law as H_F and independent of $\mathcal{H}(\mathbb{T}_x^{(\gamma)})$.

Second step: Choice and properties of $\mathbb{T}_x^{(\gamma)}$. Let us first introduce some notation in order to prove the forthcoming Lemma 3.16. Recall that we have called I_F^0 the homogeneous interval fragmentation related to I_F by the time changes $T_t^{-1}(x)$ introduced in Section 3.4.1. In this homogeneous fragmentation, let

$I_x^0(t) = (a_x(t), b_x(t))$ be the interval containing x at time t

$S_x^0(t)$ the length of this interval

$P_x^0(t) = (x - a_x(t))/S_x^0(t)$ the relative position of x in $I_x(t)$.

Similarly, we define $P_x^0(t-)$ to be the relative position of x in the interval $I_x^0(t-)$, which is well-defined as an intersection of nested intervals. $S_x^0(t-)$ is the size of this interval. We will need the following inequalities in the sequel:

$$P_x^0(t) \leq x/S_x^0(t) \qquad P_x^0(t-) \leq x/S_x^0(t-).$$

Next recall the Poisson point process construction of the interval fragmentation I_F^0, and the Poisson point process $(s^x(t))_{t\geq 0}$ of Lemma 3.15. Set

$$\sigma(t) := -\ln\left(\prod_{s\leq t} s_1^x(t)\right) \qquad t \geq 0,$$

with the convention $s_1^x(t) = 1$ when t is not a time of occurrence of the point process. By Lemma 3.15, the process σ is a subordinator with intensity measure $\nu(-\ln s_1 \in x)$, which is infinite. Consider then T_x^{exit}, the first time at which x is not in the largest sub-interval of I_x^0 when I_x^0 splits, that is

$$T_x^{\mathrm{exit}} := \inf\left\{t : S_x^0(t) < \exp(-\sigma(t))\right\}.$$

By definition, the size of the interval containing x at time $t < T_x^{\mathrm{exit}}$ is given by $S_x^0(t) = \exp(-\sigma(t))$. We will need to consider the first time at which this size is smaller than a, for a in $(0,1)$, and so we introduce

$$T_a^\sigma := \inf\left\{t : \exp(-\sigma(t)) < a\right\}.$$

Note that $P_x^0(t) \leq x\exp(\sigma(t))$ when $t < T_x^{\mathrm{exit}}$ and that $P_x^0(T_x^{\mathrm{exit}}-) \leq x\exp(\sigma(T_x^{\mathrm{exit}}-))$.

Finally, to obtain a nice $\mathbb{T}_x^{(\gamma)}$ as required in the preceding step, we stop the homogeneous fragmentation at time

$$T_x^{\mathrm{exit}} \wedge T_{x^\varepsilon}^\sigma$$

for some ε to be determined (and depending on γ) and then take for $\mathbb{T}_x^{(\gamma)}$ the self-similar counterpart of this stopping time, that is $\mathbb{T}_x^{(\gamma)} = T^{-1}_{T_x^{\text{exit}} \wedge T^{\sigma}_{x^{\varepsilon}}}(x)$. More precisely, we have

Lemma 3.16 *For every $\gamma < \vartheta_{\text{low}} \wedge |\alpha|$, there exists a family of random stopping times $\mathbb{T}_x^{(\gamma)}$, $x \in (0,1)$, and an integer $N(\gamma)$ such that*

(i) for every $n \geq 0$, $\exists C_1(n) : E\left[\left(\mathbb{T}_x^{(\gamma)}\right)^n\right] \leq C_1(n) x^{\gamma n} \quad \forall\, x \in (0,1)$,

(ii) $\exists C_2$ such that $E\left[\left(S_x(\mathbb{T}_x^{(\gamma)})\right)^n\right] \leq C_2 x^{\gamma}$ for every x in $(0,1)$ and $n \geq N(\gamma)$.

Proof. Fix $\gamma < \vartheta_{\text{low}} \wedge |\alpha|$ and then $\varepsilon < 1$ such that $\gamma/(1-\varepsilon) < \vartheta_{\text{low}}$. The times $\mathbb{T}_x^{(\gamma)}, x \in (0,1)$, are constructed from this ε by

$$\mathbb{T}_x^{(\gamma)} = T^{-1}_{T_x^{\text{exit}} \wedge T^{\sigma}_{x^{\varepsilon}}}(x),$$

and it may be clear that these times are stopping times with respect to $\mathcal{H}$. A first remark is that the function $x \in (0,1) \mapsto S_x(\mathbb{T}_x^{(\gamma)})$ is bounded from above by 1 and that $x \in (0,1) \mapsto \mathbb{T}_x^{(\gamma)}$ is bounded from above by ζ, the first time at which the fragmentation is entirely reduced to dust, that is, in others words, the supremum of H_F on $[0,1]$. Since ζ has moments of all orders, it is thus sufficient to prove statements (i) and (ii) for $x \in (0, x_0)$ for some well chosen $x_0 > 0$. Another remark, using the definition of $T_t^{-1}(x)$, is that $\mathbb{T}_x^{(\gamma)} \leq T_x^{\text{exit}} \wedge T^{\sigma}_{x^{\varepsilon}}$ and $S_x(\mathbb{T}_x^{(\gamma)}) = S_x^0\left(T_x^{\text{exit}} \wedge T^{\sigma}_{x^{\varepsilon}}\right)$, so that we just have to prove (i) and (ii) by replacing in the statement $\mathbb{T}_x^{(\gamma)}$ by $T_x^{\text{exit}} \wedge T^{\sigma}_{x^{\varepsilon}}$ and $S_x(\mathbb{T}_x^{(\gamma)})$ by $S_x^0\left(T_x^{\text{exit}} \wedge T^{\sigma}_{x^{\varepsilon}}\right)$.

We shall thus work with the homogeneous fragmentation. When I_x^0 splits to give smaller intervals, we divide these sub-intervals into three groups: the largest sub-interval, the group of sub-intervals on its left and the group of sub-intervals on its right. With the notations of Lemma 3.15, the lengths of the intervals belonging to the group on the left are the $s_i^x(t) S_x^0(t-)$ with i such that $U_i^x(t) < U_1^x(t)$ and similarly, the lengths of the intervals on the right are the $s_i^x(t) S_x^0(t-)$ with i such that $U_i^x(t) > U_1^x(t)$. An important point is that when $T_x^{\text{exit}} < T^{\sigma}_{x^{\varepsilon}}$, then at time T_x^{exit}, the point x belongs to the group of sub-intervals on the left resulting from the fragmentation of $I_x^0(T_x^{\text{exit}}-)$. Indeed, when $T_x^{\text{exit}} < T^{\sigma}_{x^{\varepsilon}}$, then $\exp(-\sigma(T_x^{\text{exit}})) \geq x^{\varepsilon} \geq x$, which becomes $s_1^x(T_x^{\text{exit}}) \exp(-\sigma(T_x^{\text{exit}}-)) \geq x$. Then using that $P_x^0(T_x^{\text{exit}}-) \leq x \exp(\sigma(T_x^{\text{exit}}-))$, we obtain $s_1^x(T_x^{\text{exit}}) \geq P_x^0(T_x^{\text{exit}}-)$ and thus that x does not belong to the group on the right at time T_x^{exit} (x belongs to the group on the right at a time t if and only if $P_x^0(t-) > \sum_{i:U_i^x(t) \leq U_1^x(t)} s_i^x(t)$). Hence x belongs to the union of intervals on the left at time T_x^{exit} when $T_x^{\text{exit}} < T^{\sigma}_{x^{\varepsilon}}$. In other words,

$$T_x^{\text{exit}} = \inf\left\{t : \sum\nolimits_{i:U_i^x(t) < U_1^x(t)} s_i^x(t) > P_x^0(t-)\right\} \text{ when } T_x^{\text{exit}} < T^{\sigma}_{x^{\varepsilon}}.$$

The key-point, consequence of Lemma 3.15, is that the process $\left(\sum_{i:U_i^x(t) < U_1^x(t)} s_i^x(t)\right)_{t \geq 0}$ is a marked Poisson point process with an intensity measure on $[0,1]$ given by

$$\mu(\mathrm{d}u) := \int_{\mathcal{S}^{\downarrow}} p(\mathbf{s}, \mathrm{d}u) \nu(\mathrm{d}\mathbf{s}), \ u \in [0,1],$$

where for a fixed $\mathbf{s}$ in $\mathcal{S}^{\downarrow}$, $p(\mathbf{s}, \mathrm{d}u)$ is the law of $\sum_{i:U_i<U_1} s_i$, the U_i's being uniform and independent random variables. We refer to Kingman [46] for details on marked Poisson point processes. Observing then that for any a in $(0,1/2)$ and for a fixed $\mathbf{s}$ in $\mathcal{S}^{\downarrow}$

$$\mathbf{1}_{\{1-s_1>2a\}} \leq \mathbf{1}_{\left\{\sum_{i:U_i<U_1} s_i>a\right\}} + \mathbf{1}_{\left\{\sum_{i:U_i>U_1} s_i>a\right\}},$$

we obtain that $\mathbf{1}_{\{1-s_1>2a\}} \leq 2P\left(\sum_{i:U_i<U_1} s_i > a\right)$ and then the following inequality

$$\mu\left((a,1]\right) \geq \frac{1}{2}\nu\left(s_1 < 1-2a\right).$$

This, recalling the definition of ϑ_{low} and that $\gamma/(1-\varepsilon) < \vartheta_{\mathrm{low}}$, leads to the existence of a positive x_0 and a positive constant C such that

$$\mu\left(\left(x^{1-\varepsilon},1\right]\right) \geq C\left(x^{-(1-\varepsilon)}\right)^{\gamma/(1-\varepsilon)} = Cx^{-\gamma} \text{ for all } x \text{ in } (0,x_0). \tag{3.19}$$

Proof of (i). We again have to introduce a hitting time, that is the first time at which the Poisson point process $\left(\sum_{i:U_i^x(t)<U_1^x(t)} s_i^x(t), t \geq 0\right)$ belongs to $(x^{1-\varepsilon},1)$:

$$H_{x^{1-\varepsilon}} := \inf\left\{t : \sum_{i:U_i^x(t)<U_1^x(t)} s_i^x(t) > x^{1-\varepsilon}\right\}.$$

By the theory of Poisson point processes, this time has an exponential law with parameter $\mu\left((x^{1-\varepsilon},1]\right)$. Hence, given inequality (3.19), it is sufficient to show that $T_x^{\mathrm{exit}} \wedge T_{x^\varepsilon}^{\sigma} \leq H_{x^{1-\varepsilon}}$ to obtain (i) for x in $(0,x_0)$ and then (i) (we recall that it is already known that $\sup_{x\in[x_0,1)} x^{-\gamma n} E\left[\left(\mathbb{T}_x^{(\gamma)}\right)^n\right]$ is finite). On the one hand, since $P_x^0(t) \leq x\exp(\sigma(t))$ when $t < T_x^{\mathrm{exit}}$,

$$P_x^0(H_{x^{1-\varepsilon}}-) \leq x\exp(\sigma(H_{x^{1-\varepsilon}}-)) < x\exp(\sigma(H_{x^{1-\varepsilon}})) \text{ when } H_{x^{1-\varepsilon}} < T_x^{\mathrm{exit}}.$$

On the other hand, $H_{x^{1-\varepsilon}} < T_{x^\varepsilon}^{\sigma}$ yields

$$x\exp(\sigma(H_{x^{1-\varepsilon}})) \leq x^{1-\varepsilon} < \sum_{i:U_i^x(H_{x^{1-\varepsilon}})<U_1^x(H_{x^{1-\varepsilon}})} s_i^x(H_{x^{1-\varepsilon}}),$$

and combining these two remarks, we get that $H_{x^{1-\varepsilon}} < T_x^{\mathrm{exit}} \wedge T_{x^\varepsilon}^{\sigma}$ implies

$$P_x^0(H_{x^{1-\varepsilon}}-) < \sum_{i:U_i^x(H_{x^{1-\varepsilon}})<U_1^x(H_{x^{1-\varepsilon}})} s_i^x(H_{x^{1-\varepsilon}}).$$

Yet this is not possible, because this last relation on $H_{x^{1-\varepsilon}}$ means that, at time $H_{x^{1-\varepsilon}}$, x is not in the largest sub-interval resulting from the splitting of $I_x^0(H_{x^{1-\varepsilon}}-)$, which implies $H_{x^{1-\varepsilon}} \geq T_x^{\mathrm{exit}}$ and this does not match with $H_{x^{1-\varepsilon}} < T_x^{\mathrm{exit}} \wedge T_{x^\varepsilon}^{\sigma}$. Hence $T_x^{\mathrm{exit}} \wedge T_{x^\varepsilon}^{\sigma} \leq H_{x^{1-\varepsilon}}$ and (i) is proved.

Proof of (ii). Take $N(\gamma) \geq \gamma/\varepsilon \vee 1$. When $T_{x^\varepsilon}^{\sigma} \leq T_x^{\mathrm{exit}}$, using the definition of $T_{x^\varepsilon}^{\sigma}$ and the right continuity of σ, we have

$$S_x^0(T_x^{\mathrm{exit}} \wedge T_{x^\varepsilon}^{\sigma}) \leq \exp(-\sigma(T_{x^\varepsilon}^{\sigma})) \leq x^{\varepsilon}$$

and consequently $\left(S_x^0(T_x^{\mathrm{exit}}\wedge T_{x^\varepsilon}^\sigma)\right)^{N(\gamma)}\leq x^\gamma$. Thus it just remains to show that

$$E\left[\left(S_x^0(T_x^{\mathrm{exit}}\wedge T_{x^\varepsilon}^\sigma)\right)^{N(\gamma)}\mathbf{1}_{\left\{T_x^{\mathrm{exit}}<T_{x^\varepsilon}^\sigma\right\}}\right]\leq x^\gamma \text{ for } x<x_0.$$

When $T_x^{\mathrm{exit}}<T_{x^\varepsilon}^\sigma$, we know - as explained at the beginning of the proof - that x belongs at time T_x^{exit} to the group of sub-intervals on the left resulting from the fragmentation of $I_x^0(T_x^{\mathrm{exit}}-)$ and hence that $S_x^0(T_x^{\mathrm{exit}}\wedge T_{x^\varepsilon}^\sigma)^{N(\gamma)}\leq s_i^x(T_x^{\mathrm{exit}})$ for some i such that $U_i^x(T_x^{\mathrm{exit}})<U_1^x(T_x^{\mathrm{exit}})$. More roughly,

$$S_x^0(T_x^{\mathrm{exit}}\wedge T_{x^\varepsilon}^\sigma)^{N(\gamma)}\mathbf{1}_{\left\{T_x^{\mathrm{exit}}<T_{x^\varepsilon}^\sigma\right\}}\leq\sum\nolimits_{i:U_i^x(T_x^{\mathrm{exit}})<U_1^x(T_x^{\mathrm{exit}})}s_i^x(T_x^{\mathrm{exit}})\mathbf{1}_{\left\{T_x^{\mathrm{exit}}<T_{x^\varepsilon}^\sigma\right\}}.$$

To evaluate the expectation of this random sum, recall from the proof of (i) that $T_x^{\mathrm{exit}}\leq H_{x^{1-\varepsilon}}$ when $T_x^{\mathrm{exit}}<T_{x^\varepsilon}^\sigma$ and remark that either $T_x^{\mathrm{exit}}<H_{x^{1-\varepsilon}}$ and then

$$\sum\nolimits_{i:U_i^x(T_x^{\mathrm{exit}})<U_1^x(T_x^{\mathrm{exit}})}s_i^x(T_x^{\mathrm{exit}})\leq x^{1-\varepsilon}\leq x^\gamma\quad(\gamma<\vartheta_{\mathrm{low}}(1-\varepsilon)\leq 1-\varepsilon)$$

or $T_x^{\mathrm{exit}}=H_{x^{1-\varepsilon}}$ and then

$$\sum\nolimits_{i:U_i^x(T_x^{\mathrm{exit}})<U_1^x(T_x^{\mathrm{exit}})}s_i^x(T_x^{\mathrm{exit}})=\sum\nolimits_{i:U_i^x(H_{x^{1-\varepsilon}})<U_1^x(H_{x^{1-\varepsilon}})}s_i^x(H_{x^{1-\varepsilon}}).$$

There we conclude with the following inequality

$$\begin{aligned}E\left[\sum\nolimits_{i:U_i^x(H_{x^{1-\varepsilon}})<U_1^x(H_{x^{1-\varepsilon}})}s_i^x(H_{x^{1-\varepsilon}})\right] &= \frac{\int_{\mathcal{S}^\downarrow}E\left[\sum_{i:U_i<U_1}s_i\mathbf{1}_{\left\{\sum_{i:U_i<U_1}s_i>x^{1-\varepsilon}\right\}}\right]\nu(\mathrm{d}s)}{\mu\left((x^{1-\varepsilon},1]\right)}\\ &\leq C^{-1}x^\gamma\int_{\mathcal{S}^\downarrow}(1-s_1)\,\nu(\mathrm{d}s),\ x\in(0,x_0).\end{aligned}$$

■

Third step: Proof of (3.17). Fix $\gamma<\vartheta_{\mathrm{low}}\wedge|\alpha|$ and take $\mathbb{T}_x^{(\gamma)}$ and $N(\gamma)$ as introduced in Lemma 3.16. Let then $n_0(\gamma)$ be an integer larger than $N(\gamma)/|\alpha|$. According to the first step, Theorem 3.4 is proved if (3.17) holds for this $n_0(\gamma)$ and every integer $n\geq 1$. To show this, it is obviously sufficient to prove that for all integers $n\geq 1$ and $m\geq 0$, there exists a finite constant $C(n,m)$ such that

$$E\left[H_F(x)^{m+n+n_0(\gamma)}\right]\leq C(n,m)x^{\gamma n}\quad\forall x\in(0,1).$$

This can be proved by induction: for $n=1$ and every $m\geq 0$, using (3.18), we have

$$\begin{aligned}E\left[H_F(x)^{m+1+n_0(\gamma)}\right] &\leq 2^{m+1+n_0(\gamma)}\\ &\times\left(E\left[\left(\mathbb{T}_x^{(\gamma)}\right)^{m+1+n_0(\gamma)}\right]+E\left[\left(S_x(\mathbb{T}_x^{(\gamma)})\right)^{|\alpha|(m+1+n_0(\gamma))}\widetilde{\zeta}^{m+1+n_0(\gamma)}\right]\right)\end{aligned}$$

where $\widetilde{\zeta}$ is the maximum of $\overline{H}_F$ on $(0,1)$. Recall that this maximum is independent of $S_x(\mathbb{T}_x^{(\gamma)})$ and has moments of all orders. Since moreover $|\alpha|(m+1+n_0(\gamma))\geq N(\gamma)$, we can apply Lemma 3.16 to deduce the existence of a constant $C(1,m)$ such that

$$E\left[H_F(x)^{m+1+n_0(\gamma)}\right]\leq C(1,m)x^\gamma\ \text{ for } x \text{ in } (0,1).$$

Now suppose that for some fixed n and every $m \geq 0$,

$$E\left[H_F(x)^{m+n+n_0(\gamma)}\right] \leq C(n,m)x^{\gamma n} \quad \forall x \in (0,1).$$

Then,

$$\begin{aligned} E\left[\left(\overline{H}_F\left(P_x(\mathbb{T}_x^{(\gamma)})\right)\right)^{m+n+1+n_0(\gamma)} \mid \mathcal{H}\left(\mathbb{T}_x^{(\gamma)}\right)\right] &\leq C(n,m+1)\left(P_x(\mathbb{T}_x^{(\gamma)})\right)^{\gamma n} \\ &\leq C(n,m+1)\left(S_x(\mathbb{T}_x^{(\gamma)})\right)^{-\gamma n} x^{\gamma n} \end{aligned}$$

since $P_x(\mathbb{T}_x^{(\gamma)}) \leq x/S_x(\mathbb{T}_x^{(\gamma)})$. Next, by (3.18),

$$\begin{aligned} E\left[H_F(x)^{m+n+1+n_0(\gamma)}\right] &\leq 2^{m+n+1+n_0(\gamma)} E\left[\left(\mathbb{T}_x^{(\gamma)}\right)^{m+n+1+n_0(\gamma)}\right] \\ &+2^{m+n+1+n_0(\gamma)} C(n,m+1) E\left[\left(S_x(\mathbb{T}_x^{(\gamma)})\right)^{|\alpha|(m+n+1+n_0(\gamma))-\gamma n}\right] x^{\gamma n}. \end{aligned}$$

Since $\gamma < |\alpha|$, the exponent $|\alpha|(m+n+1+n_0(\gamma)) - \gamma n \geq N(\gamma)$, and hence Lemma 3.16 applies to give, together with the previous inequality, the existence of a finite constant $C(n+1,m)$ such that

$$E\left[H_F(x)^{m+n+1+n_0(\gamma)}\right] \leq C(n+1,m)x^{\gamma(n+1)}$$

for every x in $(0,1)$. This holds for every m and hence the induction, formula (3.17) and Theorem 3.4 are proved.

3.4.3.2 Maximal Hölder exponent of the height process

The aim of this subsection is to prove that a.s. H_F cannot be Hölder-continuous of order γ for any $\gamma > \vartheta_{\mathrm{up}} \wedge |\alpha|/\varrho$.

We first prove that H_F cannot be Hölder-continuous with an exponent γ larger than ϑ_{up}. To see this, consider the interval fragmentation I_F and let U be a r.v. independent of I_F and with the uniform law on $(0,1)$. By Corollary 2 in [14], there is a subordinator $(\theta(t), t \geq 0)$ with no drift and a Lévy measure given by

$$\pi_\theta(dx) = e^{-x}\sum_{i=1}^{\infty} \nu(-\log s_i \in dx), x \in (0,\infty),$$

such that the length of the interval component of I_F containing U at time t is equal to $\exp(-\theta(\rho_\theta(t)))$, $t \geq 0$, ρ_θ being the time-change

$$\rho_\theta(t) = \inf\left\{u \geq 0 : \int_0^u \exp\left(\alpha\theta(r)\right) dr > t\right\}, t \geq 0.$$

Denoting by Leb the Lebesgue measure on $(0,1)$, we then have that

$$\mathrm{Leb}\left\{x \in (0,1) : H_F(x) > t\right\} \geq \exp(-\theta(\rho_\theta(t))). \tag{3.20}$$

On the other hand, recall that H_F is anyway a.s. continuous and introduce for every $t > 0$

$$x_t := \inf \{x : H_F(x) = t\},$$

so that $x < x_t \Rightarrow H_F(x) < t$. Hence $x_t \leq$Leb$\{x \in (0,1) : H_F(x) < t\}$ and this yields, together with (3.20),

$$x_t \leq 1 - \exp(-\theta(\rho_\theta(t)) \text{ a.s. for every } t \geq 0.$$

Now suppose that H_F is a.s. Hölder-continuous of order γ. The previous inequality then gives

$$t = H_F(x_t) \leq C x_t^\gamma \leq C\left(\theta(\rho_\theta(t))\right)^\gamma \tag{3.21}$$

so that it is sufficient to study the behavior of $\theta(\rho_\theta(t))$ as $t \to 0$ to obtain an upper bound for γ. It is easily seen that $\rho_\theta(t) \sim t$ as $t \downarrow 0$, so we just have to focus on the behavior of $\theta(t)$ as $t \to 0$. By [10, Theorem III.4.9], for every $\delta > 1$, $\lim_{t\to 0}\left(\theta(t)/t^\delta\right) = 0$ as soon as $\int_0^1 \overline{\pi_\theta}(t^\delta)dt < \infty$, where $\overline{\pi_\theta}(t^\delta) = \int_{t^\delta}^\infty \pi_\theta(dx)$. To see when this quantity is integrable near 0, remark first that

$$\overline{\pi_\theta}(u) = \overline{\pi_\theta}(1) + \int_u^1 e^{-x}\nu(-\log s_1 \in dx) \text{ when } u < 1,$$

(since $s_i \leq 1/2$ for $i \geq 2$) and second that

$$\int_u^1 e^{-x}\nu(-\log s_1 \in dx) \leq \nu(s_1 < e^{-u}).$$

Hence,

$$\int_0^1 \overline{\pi_\theta}(t^\delta)dt \leq \overline{\pi_\theta}(1) + \int_0^1 \nu(s_1 < e^{-t^\delta})dt$$

and by the definition of ϑ_{up} this integral is finite as soon as $1/\delta > \vartheta_{\text{up}}$. Thus $\lim_{t\to 0}\left(\theta(t)/t^\delta\right) = 0$ for every $\delta < 1/\vartheta_{\text{up}}$ and this implies, recalling (3.21), that $\gamma\delta < 1$ for every $\delta < 1/\vartheta_{\text{up}}$. Which gives $\gamma \leq \vartheta_{\text{up}}$.

It remains to prove that H_F cannot be Hölder-continuous with an exponent γ larger than $|\alpha|/\varrho$. This is actually a consequence of the results we have on the minoration of $\dim_{\mathcal{H}}(\mathcal{T}_F)$. Indeed, recall the definition of the function $\tilde{H}_F : (0,1) \to \mathcal{T}_F$ introduced Section 3.4.1 and in particular that for $0 < x < y < 1$

$$d\left(\tilde{H}_F(x), \tilde{H}_F(y)\right) = H_F(x) + H_F(y) - 2\inf_{z\in[x,y]} H_F(z),$$

which shows that the γ-Hölder continuity of H_F implies that of $\tilde{H}_F$. It is now well known that, since $\tilde{H}_F : (0,1) \to \mathcal{T}_F$, the γ-Hölder continuity of $\tilde{H}_F$ leads to $\dim_{\mathcal{H}}(\mathcal{T}_F) \leq \dim_{\mathcal{H}}((0,1))/\gamma = 1/\gamma$. Hence H_F cannot be Hölder-continuous with an order $\gamma > 1/\dim_{\mathcal{H}}(\mathcal{T}_F)$. Recall then that $\dim_{\mathcal{H}}(\mathcal{T}_F) \geq \varrho/|\alpha|$. Hence H_F cannot be Hölder-continuous with an order $\gamma > |\alpha|/\varrho$.

3.4.4 Height process of the stable tree

To prove Corollary 3.3, we will check that $\nu_-(1-s_1>x)\sim Cx^{1/\beta-1}$ for some $C>0$ as $x\downarrow 0$, where ν_- is the dislocation measure of the fragmentation F_- associated with the stable (β) tree. In view of Theorem 3.4 this is sufficient, since the index of self-similarity is $1/\beta-1$ and $\int_{\mathcal{S}^\downarrow}\left(s_1^{-1}-1\right)\nu(\mathrm{d}\mathbf{s})<\infty$, as proved in Sect. 3.3.5. Recalling the definition of ν_- in Sect. 3.3.5 and the notations therein, we want to prove

$$E\left[T_1\mathbf{1}_{\{1-\Delta_1/T_1>x\}}\right]\sim Cx^{1/\beta-1}\qquad \text{as } x\downarrow 0$$

Using the above notations, the quantity on the left can be rewritten as

$$E\left[(\Delta_1+T_1^{(\Delta_1)})\mathbf{1}_{\left\{T_1^{(\Delta_1)}/(\Delta_1+T_1^{(\Delta_1)})>x\right\}}\right]=E\left[\Delta_1(1+\Delta_1^{-1}T_1^{(\Delta_1)})\mathbf{1}_{\left\{\Delta_1^{-1}T_1^{(\Delta_1)}>x(1-x)^{-1}\right\}}\right].$$

Recalling the law of Δ_1 and the fact that $v^{-1}T_1^{(v)}$ has same law as $T^{(1)}_{v^{-1/\beta}}$, this is

$$c\int_0^\infty \mathrm{d}v\,v^{-1/\beta}e^{-c\beta v^{-1/\beta}}E\left[(1+T^{(1)}_{v^{-1/\beta}})\mathbf{1}_{\left\{T^{(1)}_{v^{-1/\beta}}>x(1-x)^{-1}\right\}}\right].$$

By [64, Proposition 28.3], since $T^{(1)}$ and T share the same Lévy measure on a neighborhood of 0, $T^{(1)}_v$ admits a continuous density $q^{(1)}_v(x),x\geq 0$ for every $v>0$. We thus can rewrite the preceding quantity as

$$c\int_0^\infty\frac{\mathrm{d}v}{v^{1/\beta}}e^{-c\beta v^{-1/\beta}}\int_{x/(1-x)}^\infty(1+u)q^{(1)}_{v^{-1/\beta}}(u)\mathrm{d}u=c\beta\int_{x/(1-x)}^\infty\mathrm{d}u(1+u)\int_0^\infty\frac{\mathrm{d}w}{w^\beta}e^{-c\beta w}q^{(1)}_w(u)$$

by Fubini's theorem and the change of variables $w=v^{-1/\beta}$. The behavior of this as $x\downarrow 0$ is the same as that of $c\beta J(x)$ where $J(x)=\int_x^\infty\mathrm{d}u j(u)$, and where $j(u)=\int_0^\infty\mathrm{d}w\,w^{-\beta}e^{-c\beta w}q^{(1)}_w(u)$. Write $\mathcal{J}(x)=\int_0^x J(u)\mathrm{d}u$ for $x\geq 0$, and consider the Stieltjes-Laplace transform $\hat{\mathcal{J}}$ of $\mathcal{J}$ evaluated at $\lambda\geq 0$:

$$\begin{aligned}\hat{\mathcal{J}}(\lambda)=\int_0^\infty e^{-\lambda u}J(u)\mathrm{d}u &= \lambda^{-1}\int_0^\infty(1-e^{-\lambda u})j(u)\mathrm{d}u\\ &= \lambda^{-1}\int_0^\infty\frac{\mathrm{d}w}{w^\beta}e^{-c\beta w}\int_0^\infty\mathrm{d}uq^{(1)}_w(u)(1-e^{-\lambda u})\\ &= \lambda^{-1}\int_0^\infty\frac{\mathrm{d}w}{w^\beta}e^{-c\beta w}(1-e^{-w\Phi^{(1)}(\lambda)})\end{aligned}$$

where as above $\Phi^{(1)}(\lambda)=c\int_0^1u^{-1-1/\beta}(1-e^{-\lambda u})\mathrm{d}u$. Integrating by parts yields

$$\begin{aligned}\hat{\mathcal{J}}(\lambda) &= \frac{\lambda^{-1}}{\beta-1}\int_0^\infty\frac{\mathrm{d}w}{w^{\beta-1}}e^{-c\beta w}((c\beta+\Phi^{(1)}(\lambda))e^{-w\Phi^{(1)}(\lambda)}-c\beta)\\ &= \lambda^{-1}\frac{\Gamma(2-\beta)}{\beta-1}((c\beta+\Phi^{(1)}(\lambda))^{\beta-1}-(c\beta)^{\beta-1})\end{aligned}$$

Changing variables in the definition of $\Phi^{(1)}$, we easily obtain that $\Phi^{(1)}(\lambda)\sim C\lambda^{1/\beta}$ as $\lambda\to\infty$ for some $C>0$, so finally we obtain that $\hat{\mathcal{J}}(\lambda)\sim C\lambda^{-1/\beta}$ as $\lambda\to\infty$ for some other $C>0$. Since $\mathcal{J}$ is non-decreasing, Feller's version of Karamata's Tauberian theorem [21, Theorem 1.7.1'] gives $\mathcal{J}(x)\sim Cx^{1/\beta}$ as $x\downarrow 0$, and since J is monotone, the monotone convergence theorem [21, Theorem 1.7.2b] gives $J(x)\sim\beta^{-1}Cx^{1/\beta-1}$ as $x\downarrow 0$, as wanted.

Chapitre 4

Equilibrium for fragmentation with immigration

Abstract: This paper introduces stochastic processes that describe the evolution of systems of particles in which particles immigrate according to a Poisson measure and split according to a self-similar fragmentation. Criteria for existence and absence of stationary distributions are established and uniqueness is proved. Also, convergence rates to the stationary distribution are given. Linear equations which are the deterministic counterparts of fragmentation with immigration processes are next considered. As in the stochastic case, existence and uniqueness of solutions, as well as existence and uniqueness of stationary solutions, are investigated.

4.1 Introduction

The aim of this paper is to study random and deterministic models that describe the evolution of systems of particles in which two independent phenomena take place: immigration and fragmentation of particles. Particles immigrate and split into smaller particles, which in turn continue splitting, at rates that depend on their mass. Such situation occurs for example in grinding lines ([7], [53]) where macroscopic blocks are continuously placed in tumbling ball mills that reduce them to microscopic fragments. These microscopic fragments then undergo a chemical process to extract the minerals. In such systems, one may expect to attain an equilibrium, as the immigration may compensate for the fragmentation of particles. The investigation of existence and uniqueness of such stationary state, as well as convergence to the stationary state, is one of the main points of interest of this paper. It will be undertaken both in random and deterministic settings.

We first introduce continuous times *fragmentation with immigration Markov processes.* Roughly, their dynamics are described as follows. The immigration is coded by a Poisson measure with intensity $I(\mathrm{d}\mathbf{s})\mathrm{d}t$, $t \geq 0$, where I is a measure supported on $\mathcal{D}$, the set of decreasing sequences $\mathbf{s} = (s_j, j \geq 1)$ that converge to 0. That is, if $(\mathbf{s}(t_i), t_i)$ denotes the atoms of this Poisson measure, a group of particles with masses $(s_1(t_i), s_2(t_i), ...)$ immigrates at time t_i

for each $t_i \geq 0$. We further impose that I integrates $\sum_{j\geq 1}(s_j \wedge 1)$, which means that the total mass of immigrants on a finite time interval is finite a.s. The particles fragment independently of the immigration, according to a "self-similar fragmentation with index $\alpha \in \mathbb{R}$" as introduced by Bertoin in [13],[14]. This means that each particle split independently of others with a rate proportional to its mass to the power α and that the resulting particles continue splitting with the same rules. Rigorous definitions are given in Subsections 4.1.1 and 4.1.2 below. Some examples of such processes arise from classical stochastic processes, as Brownian motions with positive drift. This is detailed in Section 4.4.

Let FI denote a fragmentation with immigration process. Our first purpose is to know whether it is possible to find a *stationary distribution* for FI. Under some conditions that depend both on the dynamics of the fragmentation and on the immigration, we construct a random variable $\mathbf{U}_{\text{stat}}$ in $\mathcal{D}$ whose distribution is stationary for FI. Let α_I be the I-dependent parameter defined by

$$\alpha_I := -\sup\left\{a > 0 : \int_{\mathcal{D}} s_1^a \mathbf{1}_{\{s_1 \geq 1\}} I(\mathrm{d}\mathbf{s}) < \infty\right\}.$$

When $\alpha_I < 0$, we obtain that the stationary state $\mathbf{U}_{\text{stat}}$ exists as soon as the index of self-similarity α is larger than α_I and that there is no stationary distribution when α is smaller than α_I. In this latter case, too many large particles are brought in the ball mill which is not able to grind them fast enough. These results are made precise in Theorems 4.1, 4.2 and 4.3, Section 4.2, where we also study whether $\mathbf{U}_{\text{stat}}$ is in l^p, $p \geq 0$. In addition, the stationary solution is proved unique.

It is easily checked from the construction of $\mathbf{U}_{\text{stat}}$ that

$$FI(t) \stackrel{\text{law}}{\to} \mathbf{U}_{\text{stat}}$$

as soon as the stationary distribution exists and that this convergence holds independently of the initial distribution. One standard problem is to investigate the rate of convergence to this stationary state. Our approach is based on a coupling method. This provides rates of convergence that differ significantly according as $\alpha < 0$, $\alpha = 0$ or $\alpha > 0$: one obtains that the convergence takes place at a geometric rate when $\alpha = 0$, at rate $t^{-1/\alpha}$ when $\alpha > 0$, whereas the rate of convergence depends both on I and α when $\alpha < 0$.

We next turn to deterministic models, namely *fragmentation with immigration equations*. Roughly, these equations are obtained by adding an immigration term to a family of well-known fragmentation equations with mass loss ([31],[55],[38]): we consider that particles with mass in the interval $(x, x + \mathrm{d}x)$ arrive at rate $\mu_I(\mathrm{d}x)$ which is defined from I by

$$\int_0^\infty f(x)\mu_I(\mathrm{d}x) := \int_{\mathcal{D}} \sum_{j\geq 1} f(s_j) I(\mathrm{d}\mathbf{s}),$$

for all positive measurable functions f. Solutions to the fragmentation with immigration equation do not always exist. We give conditions for existence and then show uniqueness. The obtained solution is closely related to the stochastic model $(FI(t), t \geq 0)$: it is - in a sense to be specified - related to the expectations of the random measures $\sum_{k\geq 1} \delta_{FI_k(t)}$, $t \geq 0$. In this deterministic setting, one may also expect the existence of stationary solutions. Provided the average mass immigrated by unit time is finite, we construct explicitly a stationary solution which is proved unique. Note that here the hypothesis for existence only involves I, not α, contrary to the stochastic case.

This paper is organized as follows. In the remainder of this section we first review the definition and some properties of self-similar fragmentations (Subsection 4.1.1), then we set down the definition of fragmentation with immigration processes (Subsection 4.1.2). The study of existence and uniqueness of a stationary distribution is undertaken in Section 4.2, where we also give criteria for existence of a stationary distribution for more general Markov processes with immigration. In Section 4.3, we investigate the rate of convergence to the stationary distribution. Section 4.4 is devoted to examples of fragmentation with immigration processes constructed from Brownian motions with positive drift or from height functions coding continuous state branching processes with immigration, as introduced in [49]. Section 4.5 concerns the fragmentation with immigration equation.

4.1.1 Self-similar fragmentations

State space. We endow the state space

$$\mathcal{D} = \{\mathbf{s} = (s_j)_{j\geq 1} : s_1 \geq s_2 \geq ... \geq 0, \lim_{j\to\infty} s_j = 0\}$$

with the uniform distance

$$d(\mathbf{s}, \mathbf{s}') := \sup_{j\geq 1} \left| s_j - s'_j \right|.$$

Clearly, as $n \to \infty$, $d(\mathbf{s}, \mathbf{s}^n) \to 0$ is equivalent to $s_j^n \to s_j$ for all $j \geq 1$ which in turn is equivalent to $\sum_{j\geq 1} f(s_j^n) \to \sum_{j\geq 1} f(s_j)$ for all continuous functions f with compact support in $(0, \infty)$. Hence $\mathcal{D}$ identifies with the set of Radon counting measures on $(0, \infty)$ with bounded support endowed with the topology of vague convergence through the homeomorphism

$$\mathbf{s} \in \mathcal{D} \mapsto \sum\nolimits_{j\geq 1} \delta_{s_j} \mathbf{1}_{\{s_j>0\}}.$$

With a slight abuse of notations, we also call $\mathbf{s}$ the measure $\sum_{j\geq 1} \delta_{s_j} \mathbf{1}_{\{s_j>0\}}$. It is then natural to denote by "$\mathbf{s} + \mathbf{s}'$" the decreasing rearrangement of the concatenation of sequences $\mathbf{s}, \mathbf{s}'$ and by $\langle \mathbf{s}, f \rangle$ the sum $\sum_{j\geq 1} f(s_j)\mathbf{1}_{\{s_j>0\}}$. More generally, we denote by "$\sum_{i\geq 1} \mathbf{s}^i$" the measure $\sum_{i\geq 1} \sum_{j\geq 1} \delta_{s_j^i} \mathbf{1}_{\{s_j^i>0\}}$. This point measure does not necessarily corresponds to a sequence in $\mathcal{D}$, but when it does, it represents the decreasing rearrangement of the concatenation of sequences $\mathbf{s}^1, \mathbf{s}^2, ...$.

For all $p \geq 0$, let l^p be the subset of $\mathcal{D}$ of sequences $s_1 \geq s_2 \geq ... \geq 0$ such that $\sum_{j\geq 1} s_j^p < \infty$. When $p = 0$, we use the convention $0^0 = 0$, which means that l^0 is the space of sequences with at most a finite number of non-zero terms. Let also $\mathcal{D}_1$ be the subset of $\mathcal{D}$ of sequences such that $\sum_{j\geq 1} s_j \leq 1$. Clearly $l^p \subset l^{p'}$ when $p \leq p'$ and $\mathcal{D}_1 \subset l^1$. At last, set $\mathbf{0} := (0, 0, ...)$.

Self-similar fragmentations.

Definition 4.1 *A standard self-similar fragmentation* $(F(t), t \geq 0)$ *with index* $\alpha \in \mathbb{R}$ *is a* $\mathcal{D}_1$*-valued Markov process continuous in probability such that:*

- $F(0) = (1, 0, ...)$

- *for each* $t_0 \geq 0$*, conditionally on* $F(t_0) = (s_1, s_2, ...)$*, the process* $(F(t + t_0), t \geq 0)$ *has the same law as the process obtained for each* $t \geq 0$ *by ranking in the decreasing order the components of sequences* $s_1 F^{(1)}(s_1^\alpha t)$*,* $s_2 F^{(2)}(s_2^\alpha t), ...$*, where the* $F^{(j)}$*'s are independent copies of* F*.*

This means that the particles present at a time t_0 evolve independently and that the evolution process of a particle with mass m has the same distribution as m times the process starting from a particle with mass 1, up to the time change $t \mapsto tm^\alpha$. According to [9] and [14], a self-similar fragmentation is Feller - hence possesses a càdlàg version which we shall always consider - and its distribution is characterized by a 3-tuple (α, c, ν): α is the index of self-similarity, $c \geq 0$ an *erosion* coefficient and ν a *dislocation measure*, which is a sigma-finite non-negative measure on $\mathcal{D}$ that does not charge $(1, 0, ...)$ and satisfies

$$\int_{\mathcal{D}_1} (1 - s_1)\nu(\mathrm{d}\mathbf{s}) < \infty.$$

Roughly speaking, the erosion is a deterministic continuous phenomenon and the dislocation measure describes the rates of sudden dislocations: a fragment with mass m splits into fragments with masses $m\mathbf{s}, \mathbf{s} \in \mathcal{D}_1$, at rate $m^\alpha \nu(\mathrm{d}\mathbf{s})$. In case $\nu(\mathcal{D}_1) < \infty$, this means that a particle with mass m splits after a time T with an exponential law with parameter $m^\alpha \nu(\mathcal{D}_1)$ into particles with masses $m\mathbf{s}$, where $\mathbf{s}$ is distributed according to $\nu(\cdot)/\nu(\mathcal{D}_1)$ and is independent of T. For more details on these fundamental properties of self-similar fragmentations, we refer to [9],[13] and [14].

Definition 4.2 *For any random* $\mathbf{u} \in \mathcal{D}$*, a fragmentation process* (α, c, ν) *starting from* $\mathbf{u}$ *is defined by*

$$F^{(\mathbf{u})}(t) := \sum\nolimits_{j \geq 1} (u_j F^{(j)}(u_j^\alpha t)), \ t \geq 0, \tag{4.1}$$

where the $F^{(j)}$*'s are i.i.d copies of a standard* (α, c, ν)*-fragmentation* F*, independent of* $\mathbf{u}$.

Clearly, $F^{(\mathbf{u})}(t) \in \mathcal{D}$ for all $t \geq 0$ and, according to the branching property of F, $F^{(\mathbf{u})}$ is Markov. It is plain that such fragmentation process converges a.s. to $\mathbf{0}$ as $t \to \infty$, provided $\nu(\mathcal{D}_1) \neq 0$.

We now review some facts about standard (α, c, ν)-fragmentations that we will need. In the remainder of this subsection, F denotes a standard (α, c, ν)-fragmentation.

Tagged particle. We are interested in the evolution process of the mass of a particle tagged at random in the fragmentation. So, consider a point tagged at random at time 0 according to the mass distribution of the particle, independently of the fragmentation, and let $\lambda(t)$ denotes the mass at time t of the particle containing this tagged point. Conditionally on F, $\lambda(t) = F_k(t)$ with probability $F_k(t)$, $k \geq 1$, and $\lambda(t) = 0$ with probability $1 - \sum_{k\geq 1} F_k(t)$.

Suppose first that $\alpha = 0$. Bertoin [13] shows that $\lambda \overset{\text{law}}{=} \exp(-\xi(.))$, where ξ is a subordinator (i.e. a right-continuous increasing process with values in $[0,\infty]$ and with stationary and independent increments on the interval $\{t : \xi(t) < \infty\}$), with Laplace exponent ϕ given by

$$\phi(q) := c(q+1) + \int_{\mathcal{D}_1} \left(1 - \sum\nolimits_{j\geq 1} s_j^{1+q}\right) \nu(\mathrm{d}\mathbf{s}),\ q \geq 0. \tag{4.2}$$

We recall that ϕ characterizes ξ, since $E\left[\exp(-q\xi(t))\right] = \exp(-t\phi(q))$ for all $t, q \geq 0$ (for background on subordinators, we refer to [10], chapter III). When $c > 0$ or $\nu(\sum_{j\geq 1} s_j < 1) > 0$, one sees that the subordinator ξ is killed at rate $k = \phi(0) > 0$: that is there exists a subordinator $\overline{\xi}$ with Laplace exponent $\overline{\phi} = \phi - k$ and an exponential r.v. $\mathbf{e}\,(k)$ with parameter k, independent of $\overline{\xi}$, such that

$$\xi(t) = \overline{\xi}(t)\mathbf{1}_{\{t<\mathbf{e}(k)\}} + \infty\mathbf{1}_{\{t\geq\mathbf{e}(k)\}}$$

for all $t \geq 0$.

Now when $\alpha \in \mathbb{R}$, Bertoin [14] shows that $\lambda \overset{\text{law}}{=} \exp(-\xi(\rho(.)))$ where ξ is the same subordinator as above and ρ is the time-change

$$\rho(t) := \inf\left\{u \geq 0 : \int_0^u \exp(\alpha\xi(r))\mathrm{d}r > t\right\},\ t \geq 0. \tag{4.3}$$

This implies that

$$\sum\nolimits_{k\geq 1} f\left(F_k(t)\right) = E\left[f(\exp(-\xi(\rho(t))))\exp(\xi(\rho(t))) \mid F\right] \tag{4.4}$$

for every positive measurable function f supported on a compact of $(0,\infty)$ (with the convention $0 \times \infty = 0$), and in particular that

$$E\left[\sum\nolimits_{k\geq 1} f\left(F_k(t)\right)\right] = E\left[f(\exp(-\xi(\rho(t))))\exp(\xi(\rho(t)))\right]. \tag{4.5}$$

Formation of dust when $\alpha < 0$. When the index of self-similarity α is negative, for all dislocation measures ν, the total mass $\sum_{k\geq 1} F_k(t)$ of the fragmentation F decreases as time passes to reach 0 in finite time even if there is no erosion ($c = 0$) and no mass is lost within sudden dislocations ($\nu(\sum_{j\geq 1} s_j < 1) = 0$). This is due to an intensive fragmentation of small particles which reduces macroscopic particles to an infinite number of zero-mass particles or *dust*. To say this precisely, introduce

$$\zeta := \inf\left\{t \geq 0 : \sum\nolimits_{k\geq 1} F_k(t) = 0\right\} \tag{4.6}$$

the first time at which the total mass reaches 0. According to Proposition 14 in [38], there exist C, C' some positive finite constants such that for any $t \geq 0$,

$$P(\zeta > t) \leq C\exp(-C't^{\Gamma}) \tag{4.7}$$

where Γ is a (c,ν)-dependent parameter defined by

$$\Gamma := \begin{cases} (1-\lambda)^{-1} \text{ when } \phi(q)-cq \text{ varies regularly with index } 0<\lambda<1 \text{ as } q\to\infty \\ 1 \text{ otherwise.} \end{cases} \tag{4.8}$$

Note that $E[\zeta] < \infty$.

4.1.2 Fragmentation with immigration processes

As said previously, the immigration and fragmentation phenomena occur independently. The immigration is coded by a Poisson measure on $l^1 \times [0,\infty)$ with an intensity $I(\mathrm{d}\mathbf{s})\mathrm{d}t$ such that

$$\int_{l^1} \sum\nolimits_{j\geq 1} (s_j \wedge 1)\, I(\mathrm{d}\mathbf{s}) < \infty \tag{H1}$$

and we call such measure I an *immigration measure.* The hypothesis (H1) implies that the total mass of particles that have immigrated during a time t is almost surely finite (for an introduction to Poisson measures, we refer to [46]). On the other hand, the particles fragment according to a self-similar fragmentation (α, c, ν).

Definition 4.3 *Let* $\mathbf{u}$ *be a random sequence of* $\mathcal{D}$ *and let* $((\mathbf{s}(t_i), t_i), i \geq 1)$ *be the atoms of a Poisson measure with intensity* $I(\mathrm{d}\mathbf{s})\mathrm{d}t$ *independent of* $\mathbf{u}$. *Then, conditionally on* $\mathbf{u}$ *and* $((\mathbf{s}(t_i), t_i), i \geq 1)$, *let* $F^{(\mathbf{u})}, F^{(\mathbf{s}(t_i))}, i \geq 1$, *be independent fragmentation processes* (α, c, ν) *starting respectively from* $\mathbf{u}$, $\mathbf{s}(t_1)$, $\mathbf{s}(t_2)$, ... *With probability one, the sum*

$$FI^{(\mathbf{u})}(t) := F^{(\mathbf{u})}(t) + \sum\nolimits_{t_i \leq t} F^{(\mathbf{s}(t_i))}(t-t_i)$$

belongs to $\mathcal{D}$ *for all* $t \geq 0$, *and the process* $FI^{(\mathbf{u})}$ *is called a fragmentation with immigration process with parameters* (α, c, ν, I) *starting from* $\mathbf{u}$.

The reason why $\sum_{t_i \leq t} F^{(\mathbf{s}(t_i))}(t-t_i) \in \mathcal{D}$ a.s. is that $\sum_{t_i \leq t} \sum_{j \geq 1} s_j(t_i) < \infty$ (by hypothesis (H1)) and then that $\sum_{t_i \leq t} F^{(\mathbf{s}(t_i))}(t-t_i) \in l^1$, since $\sum_{k\geq 1} F_k^{(\mathbf{s}(t_i))}(t-t_i) \leq \sum_{j\geq 1} s_j(t_i)$. Note also that when $p \geq 1$, $FI^{(\mathbf{u})} \in l^p$ as soon as $\mathbf{u} \in l^p$.

In this definition, the sequence $\mathbf{u}$ represents the masses of particles present at time 0 and at each time $t_i \geq 0$, some particles of masses $\mathbf{s}(t_i)$ immigrate. At time t, two families of particles are then present: those resulting from the fragmentation of $\mathbf{u}$ during a time t and those resulting from the fragmentation of $\mathbf{s}(t_i)$ during a time $t - t_i$, $t_i \leq t$.

It is easy to see that the process $FI^{(\mathbf{u})}$ is Markov and even Feller (cf. the proof of Proposition 1.1, [9]). Hence we may and will always consider càdlàg versions of $FI^{(\mathbf{u})}$.

In the rest of this paper, we denote by FI a fragmentation with immigration (α, c, ν, I) (without any specified starting point) and we always exclude the trivial cases $\nu = 0$ or $I = 0$.

Remark. One may wonder why we do not more generally consider some fragmentation with immigration processes with values in $\mathcal{R}$, the set of Radon point measures on $(0,\infty)$. Indeed, for all (random) $\mathbf{u} \in \mathcal{R}$ and all $t \geq 0$, it is always possible to define the point measure

$$FI^{(\mathbf{u})}(t) := F^{(\mathbf{u})}(t) + \sum\nolimits_{t_i \leq t} F^{(\mathbf{s}(t_i))}(t-t_i),\ t \geq 0, \tag{4.9}$$

where $F^{(\mathbf{u})}(t)$ is defined similarly as (4.1) and is independent of $F^{(\mathbf{s}(t_i))}, i \geq 1$, some independent fragmentations (α, c, ν) starting respectively from $\mathbf{s}(t_1)$, $\mathbf{s}(t_2)$, ... The sum involving the terms $F^{(\mathbf{s}(t_i))}(t-t_i)$, $t_i \leq t$, is in $\mathcal{D}$, as noticed in the definition 4.3 above. The issue is that in general, starting from some $\mathbf{u} \in \mathcal{R}\backslash\mathcal{D}$, the measures $F^{(\mathbf{u})}(t)$ do not necessarily belong to $\mathcal{R}$, as the masses of the initial particles may accumulate in some bounded interval (a,b) after fragmentation. As an example, one can check that for most of dislocation measures ν, $F^{(\mathbf{u})}(t) \notin \mathcal{R}$ a.s. as soon as $\alpha > 0$, $\mathbf{u} \in \mathcal{R}\backslash\mathcal{D}$ and $t > 0$. That is why we study fragmentation with immigration processes on $\mathcal{D}$. However, in Section 4.5, we shall use some of these measures $FI^{(\mathbf{u})}(t)$, $\mathbf{u} \in \mathcal{R}$, and we give (Proposition 4.4) some sufficient conditions on $\mathbf{u}$ and α for $F^{(\mathbf{u})}(t)$ (equivalently $FI^{(\mathbf{u})}(t)$) to be a.s. Radon. These conditions do not ensure that the process $FI^{(\mathbf{u})}$ is $\mathcal{R}$-valued, as we do not know if a.s. for *all* t, $FI^{(\mathbf{u})}(t) \in \mathcal{R}$.

4.2 Existence and uniqueness of the stationary distribution

This section is devoted to the existence and uniqueness of a stationary distribution for FI and to properties of the stationary state, when it exists. We begin by establishing some criteria for existence and uniqueness of a stationary distribution, which are available for a class of Markov processes with immigration including fragmentation with immigration processes. This is undertaken in Subsection 4.2.1 where we more specifically obtain an explicit construction of a stationary state. We then apply these results to fragmentation with immigration processes (Subsection 4.2.2).

From now on, for any r.v. X, $\mathcal{L}(X)$ denotes the distribution of X.

4.2.1 The candidate for a stationary distribution for Markov processes with immigration

Recall that $\mathcal{R}$ denotes the set of Radon point measures on $(0,\infty)$ and equip it with the topology of vague convergence. We first study $\mathcal{R}$-valued branching processes with immigration and then extend the results to a larger class of Markov processes.

Let X be a $\mathcal{R}$-valued Markov process that satisfies the following *branching property*: for all $\mathbf{u}, \mathbf{v} \in \mathcal{R}$, the sum of two independent processes $X^{(\mathbf{u})}$ and $X^{(\mathbf{v})}$ starting respectively from $\mathbf{u}$ and $\mathbf{v}$ is distributed as $X^{(\mathbf{u}+\mathbf{v})}$. A moment of thought shows that this is equivalent to $\sum_{i\geq 1} X^{(\mathbf{u}_i)} \overset{\text{law}}{=} X^{(\sum_{i\geq 1}\mathbf{u}_i)}$ for all sequences $(\mathbf{u}_i, i \geq 1)$ such that $\sum_{i\geq 1} \mathbf{u}_i \in \mathcal{R}$ a.s., where $X^{(\mathbf{u}_1)}, X^{(\mathbf{u}_2)}, ...$ are independent processes, starting respectively from $\mathbf{u}_1, \mathbf{u}_2, ...$. Consider then I, a non-negative σ-finite measure on $\mathcal{R}$, and let $((\mathbf{s}(t_i), t_i), i \geq 1)$ be the atoms of a Poisson measure with intensity

$I(\mathrm{d}\mathbf{s})\mathrm{d}t, t \geq 0$. Conditionally on this Poisson measure, let $X^{(\mathbf{s}(t_i))}$ be independent versions of X, starting respectively from $\mathbf{s}(t_1)$, $\mathbf{s}(t_2)$, ... In order to define an *X-process with immigration,* we need and will suppose in this section that a.s.

$$\sum\nolimits_{t_i \leq t} X^{(\mathbf{s}(t_i))}(t - t_i) \in \mathcal{R} \text{ for all } t \geq 0.$$

In particular, this holds when I is an immigration measure and X a fragmentation process, as explained just after Definition 4.3.

Definition 4.4 *For every random $\mathbf{u} \in \mathcal{R}$, let $X^{(\mathbf{u})}$ be a version of X starting from $\mathbf{u}$ and consider $((X^{(\mathbf{r}(v_i))}, v_i), i \geq 1)$ a version of $((X^{(\mathbf{s}(t_i))}, t_i), i \geq 1)$ independent of $X^{(\mathbf{u})}$. Then, the process defined by*

$$XI^{(\mathbf{u})}(t) := X^{(\mathbf{u})}(t) + \sum\nolimits_{v_i \leq t} X^{(\mathbf{r}(v_i))}(t - v_i),\ t \geq 0, \tag{4.10}$$

is a $\mathcal{R}$-valued Markov process and is called X-process with immigration starting from $\mathbf{u}$.

We point out that the Markov property of XI results both from the Markov property and from the branching property of X. A moment of reflection shows that the law of the point measure

$$\mathbf{U}_{\text{stat}} := \sum\nolimits_{t_i \geq 0} X^{(\mathbf{s}(t_i))}(t_i) \tag{4.11}$$

is a natural candidate for a stationary distribution for XI (in some sense, it is the limit as $t \to \infty$ of $XI^{(\mathbf{0})}(t)$), provided that it belongs to $\mathcal{R}$. The problem is that it does not necessarily belong to $\mathcal{R}$, as the components of $\mathbf{U}_{\text{stat}}$ may accumulate in some bounded interval (a, b).

Lemma 4.1 (i) *If $\mathbf{U}_{\text{stat}} \in \mathcal{R}$ a.s., then the distribution $\mathcal{L}(\mathbf{U}_{\text{stat}})$ is a stationary distribution for XI and for any random $\mathbf{u} \in \mathcal{R}$ such that $X^{(\mathbf{u})}(t) \overset{\mathrm{P}}{\to} \mathbf{0}$ as $t \to \infty$,*

$$XI^{(\mathbf{u})}(t) \overset{\text{law}}{\to} \mathbf{U}_{\text{stat}} \text{ as } t \to \infty.$$

(ii) *If $P(\mathbf{U}_{\text{stat}} \notin \mathcal{R}) > 0$, then there exists no stationary distribution for XI and if $P(\mathbf{U}_{\text{stat}} \notin \mathcal{D}) > 0$, then there exists no stationary distribution on $\mathcal{D}$ for XI.*

Proof. **(i)** Assume $\mathbf{U}_{\text{stat}} \in \mathcal{R}$ a.s. and consider a version $XI^{(\mathbf{U}_{\text{stat}})}$ of the X-process with immigration starting from $\mathbf{U}_{\text{stat}}$. We want to prove that $XI^{(\mathbf{U}_{\text{stat}})}(t) \overset{\text{law}}{=} \mathbf{U}_{\text{stat}}$ for every $t \geq 0$. So fix $t > 0$. By definition of XI and using the Markov and branching properties of X, we see that there exists $((X^{(\mathbf{r}(v_i))}, v_i), i \geq 1)$ an independent copy of $((X^{(\mathbf{s}(t_i))}, t_i), i \geq 1)$ such that

$$XI^{(\mathbf{U}_{\text{stat}})}(t) \overset{\text{law}}{=} \sum\nolimits_{t_i \geq 0} X^{(\mathbf{s}(t_i))}(t_i + t) + \sum\nolimits_{v_i \leq t} X^{(\mathbf{r}(v_i))}(t - v_i).$$

By independence of $((\mathbf{r}(v_i), v_i), i \geq 1)$ and $((\mathbf{s}(t_i), t_i), i \geq 1)$, the concatenation of

$$((\mathbf{r}(v_i), t - v_i), v_i \leq t) \text{ and } ((\mathbf{s}(t_i), t_i + t), i \geq 1)$$

has same law as $((\mathbf{s}(t_i), t_i), i \geq 1)$. Hence

$$XI^{(\mathbf{U}_{\mathrm{stat}})}(t) \overset{\mathrm{law}}{=} \sum_{t_i \geq 0} X^{(\mathbf{s}(t_i))}(t_i) = \mathbf{U}_{\mathrm{stat}}.$$

Similarly, one obtains that for all $t \geq 0$,

$$XI^{(\mathbf{u})}(t) \overset{\mathrm{law}}{=} X^{(\mathbf{u})}(t) + \sum_{v_i \leq t} X^{(\mathbf{r}(v_i))}(v_i) \tag{4.12}$$

where $((X^{(\mathbf{r}(v_i))}, v_i), i \geq 1)$ is distributed as $((X^{(\mathbf{s}(t_i))}, t_i), i \geq 1)$ and is independent of $X^{(\mathbf{u})}$. Suppose now that $X^{(\mathbf{u})}(t) \overset{\mathrm{P}}{\to} \mathbf{0}$ as $t \to \infty$. Clearly, $\sum_{v_i \leq t} X^{(\mathbf{r}(v_i))}(v_i) \underset{t \to \infty}{\overset{\mathrm{a.s.}}{\to}} \sum_{v_i \geq 0} X^{(\mathbf{r}(v_i))}(v_i)$ and therefore

$$X^{(\mathbf{u})}(t) + \sum_{v_i \leq t} X^{(\mathbf{r}(v_i))}(v_i) \overset{\mathrm{P}}{\to} \sum_{v_i \geq 0} X^{(\mathbf{r}(v_i))}(v_i) \text{ as } t \to \infty.$$

Since the limit here is distributed as $\mathbf{U}_{\mathrm{stat}}$ and since (4.12) holds, one has $XI^{(\mathbf{u})}(t) \overset{\mathrm{law}}{\to} \mathbf{U}_{\mathrm{stat}}$.

(ii) Suppose that there exists a stationary distribution $\mathcal{L}_{\mathrm{stat}}$. Our aim is to show that $P(\mathbf{U}_{\mathrm{stat}} \notin \mathcal{R}) = 0$. To do so, let $XI^{(\mathcal{L}_{\mathrm{stat}})}$ be an X-process with immigration starting from an initial sequence distributed according to $\mathcal{L}_{\mathrm{stat}}$. Replacing $\mathbf{u}$ by $XI^{(\mathcal{L}_{\mathrm{stat}})}(0)$ in (4.12), we get

$$XI^{(\mathcal{L}_{\mathrm{stat}})}(0) \overset{\mathrm{law}}{=} X^{(XI^{(\mathcal{L}_{\mathrm{stat}})}(0))}(t) + \sum_{t_i \leq t} X^{(\mathbf{s}(t_i))}(t_i).$$

Introduce then for any $0 < a < b < \infty$ the event

$$E_{a,b} := \left\{ \sum_{t_i \geq 0} \langle X^{(\mathbf{s}(t_i))}(t_i), \mathbf{1}_{(a,b)} \rangle = \infty \right\}$$

and fix some $N > 0$. The identity in law obtained above yields

$$\begin{aligned} P\left(\langle XI^{(\mathcal{L}_{\mathrm{stat}})}(0), \mathbf{1}_{(a,b)} \rangle < N\right) &\leq P(\textstyle\sum_{t_i \leq t} \langle X^{(\mathbf{s}(t_i))}(t_i), \mathbf{1}_{(a,b)} \rangle < N) \\ &\leq P(\textstyle\sum_{t_i \leq t} \langle X^{(\mathbf{s}(t_i))}(t_i), \mathbf{1}_{(a,b)} \rangle < N, E_{a,b}) + P\left(\Omega \backslash E_{a,b}\right). \end{aligned}$$

The first probability in this latter sum converges to 0 as $t \to \infty$ by definition of $E_{a,b}$ and therefore

$$P(\langle XI^{(\mathcal{L}_{\mathrm{stat}})}(0), \mathbf{1}_{(a,b)} \rangle < N) \leq P\left(\Omega \backslash E_{a,b}\right) \quad \forall N > 0.$$

Letting $N \to \infty$, we get $P\left(\Omega \backslash E_{a,b}\right) = 1$ (because $\mathcal{L}_{\mathrm{stat}}$ is supported on $\mathcal{R}$) and then $P(E_{a,b}) = 0$. This implies that $P(\mathbf{U}_{\mathrm{stat}} \notin \mathcal{R}) = 0$.

Now, replacing $\mathcal{R}$ by $\mathcal{D}$ and $E_{a,b}$ by $E_{a,\infty}$, we obtain similarly that $P(\mathbf{U}_{\mathrm{stat}} \notin \mathcal{D}) = 0$ as soon as there exists a stationary distribution $\mathcal{L}_{\mathrm{stat}}$ such that $\mathcal{L}_{\mathrm{stat}}(\mathcal{D}) = 1$. ■

Let us now extend these results to Markov processes that take values in some σ-compact space E and that do not necessarily possess a branching property. In order to introduce some immigration and some branching property, we will work on $\mathfrak{M}_E$, the set of point measures on E: if $\mathfrak{m} \in \mathfrak{M}_E$, either $\mathfrak{m} = \sum_{i \geq 1} \delta_{x^{(i)}}$ for some sequence $(x^{(i)}, i \geq 1)$ of points of E, or $\mathfrak{m} = \mathfrak{o}$, where $\mathfrak{o}$ is the trivial measure: $\mathfrak{o}(E) = 0$. The subset of measures of $\mathfrak{M}_E$ that are Radon is denoted by $\mathfrak{M}_E^{\mathrm{Radon}}$ and is equipped with the topology of vague convergence. Consider then I, a non-negative σ-finite measure on E, and $(X(t), t \geq 0)$, a Markov process with values in E. For any $\mathfrak{m} = \sum_{i \geq 1} \delta_{x^{(i)}} \in \mathfrak{M}_E$, set

$$\mathcal{X}^{(\mathfrak{m})}(t) := \sum\nolimits_{i \geq 1} \delta_{X^{(x^{(i)})}(t)}, \ t \geq 0,$$

where $X^{(x^{(1)})}, X^{(x^{(2)})}$, ... are independent versions of X, starting respectively from $x^{(1)}, x^{(2)}$,... If $\mathfrak{m} = \mathfrak{o}$, $\mathcal{X}^{(\mathfrak{m})}(t) := \mathfrak{o}$, $\forall t \geq 0$.

We now construct some $\mathcal{X}$-*process with immigration.* Let $\mathfrak{m}$ be a random element of $\mathfrak{M}_E^{\mathrm{Radon}}$ and $((x(t_i), t_i), i \geq 1)$ be the atoms of a Poisson measure with intensity $I(\mathrm{d}\mathbf{s})\mathrm{d}t$, $t \geq 0$, independent of $\mathfrak{m}$. Conditionally on this Poisson measure and on $\mathfrak{m}$, let $\mathcal{X}^{(\mathfrak{m})}$ and $\mathcal{X}^{(\delta_{x(t_i)})}, i \geq 1$, be independent versions of $\mathcal{X}$ starting respectively from $\mathfrak{m}$,$\delta_{x(t_1)}, \delta_{x(t_2)}, \ldots$. Define then

$$\mathcal{XI}^{(\mathfrak{m})}(t) := \mathcal{X}^{(\mathfrak{m})}(t) + \sum\nolimits_{t_i \leq t} \mathcal{X}^{(\delta_{x(t_i)})}(t - t_i), \ t \geq 0,$$

and suppose that a.s. for all $t \geq 0$, $\mathcal{XI}^{(\mathfrak{m})} \in \mathfrak{M}_E^{\mathrm{Radon}}$. Then $\mathcal{XI}^{(\mathfrak{m})}$ is Markovian and called $\mathcal{X}$-process with immigration starting from $\mathfrak{m}$. Introduce next the point measure

$$\mathcal{U}_{\mathrm{stat}} := \sum\nolimits_{t_i \geq 0} \mathcal{X}^{(\delta_{x(t_i)})}(t_i) = \sum\nolimits_{i \geq 1} \delta_{X^{(x(t_i))}(t_i)}.$$

We the same kind of arguments as above, one obtains the following result.

Lemma 4.2 (i) *Assume $\mathcal{U}_{\mathrm{stat}} \in \mathfrak{M}_E^{\mathrm{Radon}}$ a.s. Then the distribution $\mathcal{L}(\mathcal{U}_{\mathrm{stat}})$ is a stationary distribution for $\mathcal{XI}$ and $\mathcal{XI}^{(\mathfrak{m})}(t) \overset{\mathrm{law}}{\to} \mathcal{U}_{\mathrm{stat}}$ as soon as $\mathcal{X}^{(\mathfrak{m})}(t) \overset{\mathrm{P}}{\to} \mathfrak{o}$ as $t \to \infty$.*

(ii) *If $P(\mathcal{U}_{\mathrm{stat}} \notin \mathfrak{M}_E^{\mathrm{Radon}}) > 0$, there exists no stationary distribution for $\mathcal{XI}$.*

4.2.2 Conditions for existence and properties of FI's stationary distribution

Up to now, I is an immigration measure as defined in Section 4.1.2, that is I satisfies hypothesis (H1). Let FI denote a fragmentation with immigration (α, c, ν, I). By definition, the fragmentation process satisfies the branching property and for every $\mathbf{u} \in \mathcal{D}$, $F^{(\mathbf{u})}(t) \overset{\mathrm{a.s.}}{\to} \mathbf{0}$ as $t \to \infty$. Then the results of Lemma 4.1 can be rephrased as follows: if $((\mathbf{s}(t_i), t_i), i \geq 1)$ are the atoms of a Poisson measure with intensity $I(\mathrm{d}\mathbf{s})\mathrm{d}t$ and if conditionally on this Poisson measure, $F^{(\mathbf{s}(t_1))}, F^{(\mathbf{s}(t_2))}, \ldots$ are independent (α, c, ν)-fragmentations starting respectively from $\mathbf{s}(t_1), \mathbf{s}(t_2), \ldots$ then there is a stationary distribution for the fragmentation with immigration (α, c, ν, I) if and only if

$$\mathbf{U}_{\mathrm{stat}} = \sum\nolimits_{t_i \geq 0} F^{(\mathbf{s}(t_i))}(t_i) \in \mathcal{D} \text{ a.s.}$$

In this case,

$$FI^{(\mathbf{u})}(t) \overset{\mathrm{law}}{\to} \mathbf{U}_{\mathrm{stat}} \text{ as } t \to \infty$$

for all $\mathbf{u} \in \mathcal{D}$ and therefore $\mathcal{L}(\mathbf{U}_{\mathrm{stat}})$ is the *unique* stationary distribution for FI. The point is then to see when $\mathbf{U}_{\mathrm{stat}}$ belongs to $\mathcal{D}$ and when it does not. The results are given in Subsection

4.2.2.1 where we further investigate whether $\mathbf{U}_{\mathrm{stat}}$ is in l^p or not, $p \geq 0$. This is particularly interesting when $\mathbf{U}_{\mathrm{stat}} \in l^1$ a.s.: then the total mass of the system converges to an equilibrium, which means that the immigration compensates the mass lost by formation of dust (when $\alpha < 0$), by erosion or within sudden dislocations. When $\mathbf{U}_{\mathrm{stat}} \in \mathcal{D}$ a.s., we also investigate the behavior of its small components. The proofs are detailed in Subsection 4.2.2.2.

4.2.2.1 Statement of results

Let F denote an (α, c, ν)-fragmentation. In the statements below, we shall sometimes suppose that

$$c = 0,\ \nu\left(\sum_{j\geq 1} s_j < 1\right) = 0 \text{ and } \int_{\mathcal{D}_1} \sum_{j\geq 1} |\log(s_j)|\, s_j \nu(\mathrm{d}\mathbf{s}) < \infty \tag{H2}$$

or

$$\nexists\, 0 < r < 1 : F_i(t) \in \{r^n, n \in \mathbb{N}\}\ \ \forall t \geq 0, i \geq 1, \text{ and (H2) holds.} \tag{H3}$$

In term of ξ, the subordinator driving a tagged fragment of F, the hypothesis (H2) means that $E[\xi(1)] < \infty$. We shall also use the convention $l^p = l^0$ when $p \leq 0$.

We now state our results on the existence of a stationary distribution; they depend heavily on the value of the index α.

Theorem 4.1 *Suppose $\alpha < 0$.*

(i) *If either $\int_{l^1} \sum_{j\geq 1} s_j^{-\alpha} \mathbf{1}_{\{s_j \geq 1\}} I(\mathrm{d}\mathbf{s}) < \infty$ or $\int_{l^1} s_1^{-\alpha} \ln s_1 \mathbf{1}_{\{s_1 \geq 1\}} I(\mathrm{d}\mathbf{s}) < \infty$, then the stationary state $\mathbf{U}_{\mathrm{stat}} \in l^p$ a.s. for all $p > 1 + \alpha$.*

(ii) *There exists no stationary distribution when $\int_{l^1} s_1^{-\alpha} \mathbf{1}_{\{s_1 \geq 1\}} I(\mathrm{d}\mathbf{s}) = \infty$.*

Theorem 4.2 *Suppose $\alpha = 0$.*

(i) *If $\int_{l^1} \ln s_1 \mathbf{1}_{\{s_1 \geq 1\}} I(\mathrm{d}\mathbf{s}) < \infty$, then with probability one, $\mathbf{U}_{\mathrm{stat}} \in l^p$ for all $p > 1$ and does not belong to l^1 when $c = 0$ and $\nu(\sum_{j\geq 1} s_j < 1) = 0$.*

(ii) *There exists no stationary distribution when $\int_{l^1} \ln s_1 \mathbf{1}_{\{s_1 \geq 1\}} I(\mathrm{d}\mathbf{s}) = \infty$ and (H2) holds.*

Theorem 4.3 *Suppose $\alpha > 0$. If $\int_{l^1} s_1^{\varepsilon} \mathbf{1}_{\{s_1 \geq 1\}} I(\mathrm{d}\mathbf{s}) < \infty$ for some $\varepsilon > 0$, then $\mathbf{U}_{\mathrm{stat}} \in l^p$ a.s. for p large enough and if (H3) holds, then $\mathbf{U}_{\mathrm{stat}} \notin l^{1+\alpha}$ a.s. More precisely, for every $\gamma > 0$,*

(i) *if $\int_{l^1} \sum_{j\geq 1} s_j^{\gamma} \mathbf{1}_{\{s_j \geq 1\}} I(\mathrm{d}\mathbf{s}) < \infty$, then $\mathbf{U}_{\mathrm{stat}} \in l^p$ a.s. for all $p > 1 + \alpha / (\gamma \wedge 1)$,*

(ii) *if $\int_{l^1} s_1^{\gamma} \mathbf{1}_{\{s_1 \geq 1\}} I(\mathrm{d}\mathbf{s}) = \infty$ and (H3) holds, then $\mathbf{U}_{\mathrm{stat}} \notin l^{1+\alpha/(\gamma\wedge 1)}$ a.s.*

When $-1 < \alpha < 0$, the result of Theorem 4.1 (i) can be completed (see the remark following Proposition 4.1 below): in most cases, either $\mathbf{U}_{\mathrm{stat}} \notin l^{1+\alpha}$ a.s. or both events $\{\mathbf{U}_{\mathrm{stat}} = \mathbf{0}\}$ and $\{\mathbf{U}_{\mathrm{stat}} \notin l^{1+\alpha}\}$ have positive probabilities.

It is interesting to notice that the above conditions for existence or absence of a stationary distribution depend only on α and I, provided hypothesis (H3) holds. For a fixed immigration measure I, let

$$\alpha_I = \inf\left\{\alpha < 0 : \int_{l^1} s_1^{-\alpha}\mathbf{1}_{\{s_1\geq 1\}}I(\mathrm{d}\mathbf{s}) < \infty\right\} \tag{4.13}$$

and let then α vary. According to the above theorems, the values $\alpha = \alpha_I$ and $\alpha = -1$ are critical. Indeed, provided $\alpha_I < 0$, the stationary distribution exists when $\alpha > \alpha_I$ and does not exist when $\alpha < \alpha_I$. Moreover, the stationary state $\mathbf{U}_{\mathrm{stat}}$ is a.s. composed by a finite number of particles as soon as $\alpha_I < \alpha < -1$, whereas when $\alpha > -1$, $\mathbf{U}_{\mathrm{stat}} \notin l^{1+\alpha}$ with a positive probability (which equals 1 when $\alpha \geq 0$ and depends on further hypothesis on I and α when $-1 < \alpha < 0$)

Let us try to explain these results. By the scaling property of fragmentation processes, particles with mass ≥ 1 split faster when α is larger. This explains that when α is too small some particles may accumulate in intervals of type (a,∞), $a > 0$, which implies that $\mathbf{U}_{\mathrm{stat}} \notin \mathcal{D}$. For α large enough, particles with mass ≥ 1 become rapidly smaller, but particles with mass ≤ 1 split more slowly when α is larger. Therefore, small particles accumulate and $\mathbf{U}_{\mathrm{stat}} \notin l^p$ when p is too small. Moreover the smallest p such that $\mathbf{U}_{\mathrm{stat}} \in l^p$ increases as α increases. When $\alpha < -1$, it is known that small particles are very quickly reduced to dust (see e.g. Proposition 2, [15]). This implies that $\mathbf{U}_{\mathrm{stat}} \in l^0$ provided it belongs to $\mathcal{D}$.

Small particles behavior. Suppose that $-1 < \alpha < 0$ and $\int_{l^1}\sum_{j\geq 1} s_j^{-\alpha}\mathbf{1}_{\{s_j\geq 1\}}I(\mathrm{d}\mathbf{s}) < \infty$, so that $\mathbf{U}_{\mathrm{stat}} \in \mathcal{D}$ a.s., according to Theorem 4.1 (i). Consider then the random function

$$\varepsilon \mapsto \overline{\mathbf{U}}_{\mathrm{stat}}(\varepsilon) := \mathbf{U}_{\mathrm{stat}}([\varepsilon,\infty))$$

which counts the number of components of $\mathbf{U}_{\mathrm{stat}}$ larger than ε. We want to investigate the limiting behavior of $\overline{\mathbf{U}}_{\mathrm{stat}}(\varepsilon)$ as $\varepsilon \to 0$. In that aim, we make the following technical hypothesis

$$\int_{\mathcal{D}_1}\sum\nolimits_{j>i\geq 1} s_i^{1+\alpha}s_j\nu(\mathrm{d}\mathbf{s}) < \infty \text{ and } \int_{\mathcal{D}_1}(1-s_1)^{\theta}\,\nu(\mathrm{d}\mathbf{s}) < \infty \text{ for some } \theta < 1 \tag{H4}$$

as well as hypothesis (H3).

Proposition 4.1 *Under the previous hypotheses,*

(i) *if* $\int_{l^1}\sum_{j\geq 1} s_j^{-\alpha}\mathbf{1}_{\{s_j\leq 1\}}I(\mathrm{d}\mathbf{s}) < \infty$, *there exists a finite r.v.* X, $0 < P(X=0) < 1$, *such that*

$$\overline{\mathbf{U}}_{\mathrm{stat}}(\varepsilon)\varepsilon^{1+\alpha} \underset{\varepsilon\to 0}{\to} X \ \textit{a.s.}$$

(ii) *if* $\int_{l^1} s_1^{-\alpha}1_{\{s_j\leq 1\}}I(\mathrm{d}\mathbf{s}) = \infty$, *one has* $\liminf_{\varepsilon\to 0}\varepsilon^{1+\alpha}\overline{\mathbf{U}}_{\mathrm{stat}}(\varepsilon) > 0$ *a.s.*

In particular, this implies that $P(\mathbf{U}_{\mathrm{stat}} \notin l^{1+\alpha}) = 1$ when the assumption of the second statement is satisfied. This is not true when the assumption of the first statement holds: in such case, $0 < P(\mathbf{U}_{\mathrm{stat}} = \mathbf{0}) \leq P(\mathbf{U}_{\mathrm{stat}} \in l^{1+\alpha}) < 1$ (see the proof of (i) for the first inequality).

When $\alpha \geq 0$ or $\alpha < -1$, some information on the behavior of $\overline{\mathbf{U}}_{\mathrm{stat}}(\varepsilon)$ as $\varepsilon \to 0$ can be deduced from Theorems 4.1, 4.2 and 4.3. As an example, $\overline{\mathbf{U}}_{\mathrm{stat}}(0) < \infty$ a.s. when $\alpha_I < \alpha < -1$.

Remark. It is possible to show that $\mathbf{U}_{\text{stat}} \in \mathcal{R}$ a.s. as soon as $\int_{l^1} \sum_{j\geq 1} s_j \mathbf{1}_{\{s_j \geq 1\}} I(\mathrm{d}\mathbf{s}) < \infty$ and that $P(\mathbf{U}_{\text{stat}} \notin \mathcal{R}) > 0$ as soon as $\alpha > -1$, $\int_{l^1} s_1^{-\alpha} \mathbf{1}_{\{s_1 \geq 1\}} I(\mathrm{d}\mathbf{s}) = \infty$ and hypotheses (H3) and (H4) hold. The first claim can be proved by using some arguments of the proof of the forthcoming Proposition 4.5 and the second claim is a consequence of Theorems 4 (i) and 7 of [39], which are also used below to prove Proposition 4.1.

4.2.2.2 Proofs

Let F be a standard (α, c, ν)-fragmentation and for every $p \in \mathbb{R}$ and $t \geq 0$, define

$$M(p,t) := \sum\nolimits_{k\geq 1} (F_k(t))^p \mathbf{1}_{\{F_k(t)>0\}},$$

which is a.s. finite at least when $p \geq 1$ (since it is bounded from above by 1). That $\mathbf{U}_{\text{stat}}$ belongs to some l^p-space is closely related to the behavior of the function $t \mapsto M(p,t)$. Indeed,

$$\mathbf{U}_{\text{stat}} = \sum\nolimits_{i\geq 1} \sum\nolimits_{j\geq 1} s_j(t_i) F^{(i,j)}(s_j^{\alpha}(t_i) t_i)$$

where the $F^{(i,j)}$'s, $i,j \geq 1$, are i.i.d copies of F, independent of $((\mathbf{s}(t_i), t_i), i \geq 1)$. Then, $\mathbf{U}_{\text{stat}} \in l^p \Leftrightarrow M(p) < \infty$ with

$$\begin{aligned} M(p) &= \int_{(0,\infty)} x^p \mathbf{U}_{\text{stat}}(\mathrm{d}x) \\ &= \sum\nolimits_{i\geq 1} \sum\nolimits_{j\geq 1} s_j^p(t_i) M^{(i,j)}(p, s_j^{\alpha}(t_i) t_i) \mathbf{1}_{\{s_j(t_i)>0\}} \end{aligned}$$

where the $M^{(i,j)}(p,\cdot)$'s, $i,j \geq 1$, are i.i.d copies of $M(p,\cdot)$, independent of $((\mathbf{s}(t_i), t_i), i \geq 1)$. Using the tagged particle approach as explained in Section 4.1.1, one obtains the following results on $M(p,\cdot)$.

Lemma 4.3 (i) *Suppose $\alpha \leq 0$. Then $\int_0^\infty \exp(\lambda t) E[M(p,t)]\, \mathrm{d}t < \infty$ as soon as $p \geq 1+\alpha$ and $\lambda < \phi(p-1-\alpha)$. In particular, $E[M(p,t)] < \infty$ for a.e. $t \geq 0$ as soon as $p \geq 1+\alpha$.*

(ii) *Suppose $\alpha > 0$. Then for every $\eta > 0$ and every $p \geq 1$, there exists a random variable $I_{(\eta,p)}$ with positive moments of all orders such that*

$$M(p,t) \leq I_{(\eta,p)} t^{-\frac{p-1}{\alpha+\eta}} \text{ a.s. for every } t > 0.$$

Consequently $\int_0^\infty E[M(p,t)]\, \mathrm{d}t < \infty$ when $p > 1+\alpha$.

Bertoin (Corollary 3, [15]) shows that when $\alpha > 0$ and $p \geq 1$, the process $t^{\frac{p-1}{\alpha}} M(p,t)$ converges in probability to some deterministic limit as $t \to \infty$, provided the fragmentation satisfies hypothesis (H3). See also Brennan and Durrett [23],[24] who prove the almost sure convergence for binary fragmentations ($\nu(s_1 + s_2 < 1) = 0$) with a finite dislocation measure.

Proof. We use the notations introduced in Section 4.1.1.

(i) According to (4.5),

$$E\left[M(p,t)\right]=E\left[\exp((1-p)\overline{\xi}(\rho(t)))\mathbf{1}_{\{t<D\}}\right]$$

where $D=\inf\{t:\rho(t)\geq \mathbf{e}(k)\}$. Therefore

$$\begin{aligned}\int_0^\infty \exp(\lambda t)E\left[M(p,t)\right]\mathrm{d}t &= E\left[\int_0^D \exp(\lambda t)\exp((1-p)\overline{\xi}(\rho(t)))\mathrm{d}t\right]\\ &= E\left[\int_0^{\mathbf{e}(k)} \exp(\lambda\rho^{-1}(t))\exp((1-p+\alpha)\overline{\xi}(t))\mathrm{d}t\right].\end{aligned} \tag{4.14}$$

using for the last equality the change of variables $t\mapsto\rho(t)$ and that, by definition of ρ, $\exp(\alpha\overline{\xi}(\rho(t)))\mathrm{d}\rho(t)=\mathrm{d}t$ on $[0,D)$. The function ρ^{-1} denotes the right inverse of ρ and clearly $\rho^{-1}(t)\leq t$ since $\alpha\leq 0$. When $p\geq 1+\alpha$, this leads to

$$\int_0^\infty \exp(\lambda t)E\left[M(p,t)\right]\mathrm{d}t\leq\begin{cases} E\left[\int_0^{\mathbf{e}(k)}\exp(-\overline{\phi}(p-1-\alpha)t)\mathrm{d}t\right] & \text{if } \lambda<0\\ E\left[\int_0^{\mathbf{e}(k)}\exp((\lambda-\overline{\phi}(p-1-\alpha))t)\mathrm{d}t\right] & \text{if } \lambda\geq 0\end{cases}$$

and in both cases, the integral is finite as soon as $\lambda<\phi(p-1-\alpha)=\overline{\phi}(p-1-\alpha)+k$.

(ii) Fix $\alpha>0$, $p\geq 1$ and $\eta>0$ and recall that, according to (4.4),

$$M(p,t)=E\left[\exp(-(p-1)\xi(\rho(t)))\mathbf{1}_{\{t<D\}}\mid F\right].$$

On the one hand, one has

$$\rho(t)\exp(-\eta\xi(\rho(t)))\leq\int_0^{\rho(t)}\exp(-\eta\xi(r))\mathrm{d}r\leq\int_0^\infty\exp(-\eta\xi(r))\mathrm{d}r:=I_{(\eta)}.$$

And on the other hand, for $t<D$,

$$t=\int_0^{\rho(t)}\exp(\alpha\xi(r))\mathrm{d}r\leq\rho(t)\exp(\alpha\xi(\rho(t))).$$

Combining these inequalities, we obtain $\exp(-\left(\alpha+\eta\right)\xi(\rho(t)))\leq t^{-1}I_{(\eta)}$ for all $t<D$. Hence $M(p,t)\leq t^{-\frac{p-1}{\alpha+\eta}}I_{(\eta,p)}$ where $I_{(\eta,p)}:=E\left[I_{(\eta)}^{(p-1)/(\alpha+\eta)}\mid F\right]$. Carmona, Petit and Yor [25] have shown that $I_{(\eta)}$ has moments of all positive orders, which, by Hölder inequality, is also true for $I_{(\eta,p)}$. ■

We now turn to the proofs of Theorems 4.1, 4.2 and 4.3.

Proof of Theorem 4.1. **(i)** Fix $p>1+\alpha$ and split $M(p)$ into two sub-sums:

$$M_{\mathrm{inf}}(p)=\sum\nolimits_{i\geq 1}\sum\nolimits_{j\geq 1}s_j^p(t_i)\mathbf{1}_{\{0<s_j(t_i)<1\}}M^{(i,j)}(p,s_j^\alpha(t_i)t_i)$$

and $M_{\mathrm{sup}}(p)=M(p)-M_{\mathrm{inf}}(p)$. One has

$$E\left[M_{\mathrm{inf}}(p)\right]=\int_{l^1}(\sum\nolimits_{j\geq 1}s_j^{p-\alpha}\mathbf{1}_{\{s_j<1\}})I(\mathrm{d}\mathbf{s})\times\int_0^\infty E\left[M(p,t)\right]\mathrm{d}t$$

and both of these integrals are finite according to hypothesis (H1) and Lemma 4.3, since $p > 1+\alpha$. It remains to show that $M_{\sup}(p) < \infty$ when I integrates $\sum_{j\geq 1} s_j^{-\alpha}\mathbf{1}_{\{s_j\geq 1\}}$ or $s_1^{-\alpha}\ln s_1 \mathbf{1}_{\{s_1\geq 1\}}$.

Suppose first that $\int_{l^1}\sum_{j\geq 1} s_j^{-\alpha}\mathbf{1}_{\{s_j\geq 1\}}I(\mathrm{d}\mathbf{s}) < \infty$ and let $\zeta^{(i,j)}$ be the first time at which the fragmentation $F^{(i,j)}$ is entirely reduced to dust. Equivalently, $\zeta^{(i,j)}$ is the first time at which $M^{(i,j)}$ reaches 0. If the number of pairs (i,j) such that $s_j^{\alpha}(t_i)t_i \leq \zeta^{(i,j)}$ and $s_j(t_i)\geq 1$ is finite, then the sum $M_{\sup}(p)$ is finite because it involves at most a finite number of non-zero $M^{(i,j)}(p, s_j^{\alpha}(t_i)t_i)$ (which are a.s. all finite according to Lemma 4.3 (i)). To prove that this is the case, we use Poisson measures theory. Since the v.a. $\zeta^{(i,j)}$, $i,j\geq 1$, are i.i.d, the measure

$$\sum_{i\geq 1}\delta_{t_i^{-1}\sup_{j:s_j(t_i)\geq 1}(\zeta^{(i,j)}s_j^{-\alpha}(t_i))}$$

is a Poisson measure with intensity m defined for any positive measurable function f by

$$\int_0^\infty f(x)m(\mathrm{d}x) = \int_0^\infty\int_{l^1} E\left[f(t^{-1}\sup_{j:s_j\geq 1}(\zeta^{(1,j)}s_j^{-\alpha}))\right] I(\mathrm{d}\mathbf{s})\mathrm{d}t.$$

The integral $\int_1^\infty m(\mathrm{d}x)$ is bounded from above by $E\left[\zeta^{(1,1)}\right]\int_{l^1}\sum_{j\geq 1}s_j^{-\alpha}\mathbf{1}_{\{s_j\geq 1\}}I(\mathrm{d}\mathbf{s})$ which is finite by assumption on I and since $E\left[\zeta^{(1,1)}\right] < \infty$ (by (4.7)). This implies that a.s. there is only a finite number of integers $i\geq 1$ such that $t_i^{-1}\sup_{j:s_j(t_i)\geq 1}(\zeta^{(i,j)}s_j^{-\alpha}(t_i))\geq 1$. For each of these i, there is at most a finite number of integers $j\geq 1$ such that $s_j(t_i)\geq 1$. Hence the number of pairs (i,j) such that $s_j^{\alpha}(t_i)t_i\leq\zeta^{(i,j)}$ and $s_j(t_i)\geq 1$ is indeed a.s. finite.

Assume now that $\int_{l^1} s_1^{-\alpha}\ln s_1\mathbf{1}_{\{s_1\geq 1\}}I(\mathrm{d}\mathbf{s}) < \infty$. For any $a>0$, the number of integers $i\geq 1$ such that $at_i\leq s_1^{-\alpha}(t_i)\ln(s_1(t_i))$ and $s_1(t_i)\geq 1$ is then a.s. finite. The sum $M_{\sup}(p)$ is therefore finite if

$$\sum_{i\geq 1}\sum_{j\geq 1} s_j^{p}(t_i)\mathbf{1}_{\{at_i > s_1^{-\alpha}(t_i)\ln(s_1(t_i))\}}\mathbf{1}_{\{s_j(t_i)\geq 1\}}M^{(i,j)}(p, s_j^{\alpha}(t_i)t_i)$$

is finite for some (and then all) $a>0$. The expectation of this latter sum is bounded from above by

$$\begin{aligned}&\int_0^\infty\int_{l^1}(\sum_{j\geq 1}s_j^{p}\mathbf{1}_{\{at>s_j^{-\alpha}\ln s_j\}}\mathbf{1}_{\{s_j\geq 1\}})E\left[M(p,s_j^{\alpha}t)\right]I(\mathrm{d}\mathbf{s})\mathrm{d}t \quad (\text{as } s_j\leq s_1)\\ \leq\ &\int_{l^1}\sum_{j\geq 1}\mathbf{1}_{\{s_j\geq 1\}}I(\mathrm{d}\mathbf{s})\int_0^\infty \exp(at(p-\alpha))E\left[M(p,t)\right]\mathrm{d}t\end{aligned}$$

which is finite for a sufficiently small, according to Lemma 4.3 (i). Hence $M_{\sup}(p) < \infty$ a.s.

(ii) Suppose $\int_{l^1}s_1^{-\alpha}\mathbf{1}_{\{s_1\geq 1\}}I(\mathrm{d}\mathbf{s}) = \infty$ and let $\zeta_{1/2}^{(i,1)} := \inf\{t\geq 0 : F_1^{(i,1)}(t) < 1/2\}$ be the first time at which all components of $F^{(i,1)}$ are smaller than $1/2$, $i\geq 1$. Note that $E[\zeta_{1/2}^{(i,1)}] > 0$ since $F_1^{(i,1)}$ is càdlàg. The measure

$$\sum_{i\geq 1:s_1(t_i)\geq 1}\delta_{s_1^{-\alpha}(t_i)t_i^{-1}\zeta_{1/2}^{(i,1)}}$$

is a Poisson measure with intensity m' given by

$$\int_0^\infty f(x)m'(\mathrm{d}x) = \int_0^\infty\int_{l^1}E\left[f(s_1^{-\alpha}t^{-1}\zeta_{1/2}^{(1,1)})\right]\mathbf{1}_{\{s_1\geq 1\}}I(\mathrm{d}\mathbf{s})\mathrm{d}t.$$

By assumption on I and since $E[\zeta_{1/2}^{(1,1)}]>0$, the integral $\int_1^{\infty} m'(\mathrm{d}x)$ is infinite and consequently the number of integers i such that $\zeta_{1/2}^{(i,1)} > s_1^{\alpha}(t_i)t_i$ and $s_1(t_i)\geq 1$ is a.s. infinite. For those i, $s_1(t_i)F_1^{(i,1)}(s_1^{\alpha}(t_i)t_i)\geq 1/2$ and therefore $\mathbf{U}_{\mathrm{stat}}$ contains a sequence of terms all larger than 1/2, which implies that it is not in $\mathcal{D}$ a.s. ■

Proof of Theorem 4.2. **(i)** The second part of the proof of Theorem 4.1 (i) (replacing there α by 0) shows that $\mathbf{U}_{\mathrm{stat}}\in\cap_{p>1}l^p$ when $\int_{l^1}\ln(s_1)\mathbf{1}_{\{s_1\geq 1\}}I(\mathrm{d}\mathbf{s})<\infty$. Now, if $c=0$ and $\nu(\sum_{k\geq 1}s_k<1)=0$, the sum $M(1)$ equals $\sum_{i\geq 1}\sum_{j\geq 1}s_j(t_i)$, which is clearly a.s. infinite since $I\neq 0$.

(ii) Assume that $\int_{l^1}\ln(s_1)\mathbf{1}_{\{s_1\geq 1\}}I(\mathrm{d}\mathbf{s})=\infty$ and $E[\xi(1)]<\infty$. For each $i\geq 1$, let $\exp(-\xi^{(i,1)}(\cdot))$ denote the process of masses of the tagged particle in the fragmentation $F^{(i,1)}$. To prove that $\mathbf{U}_{\mathrm{stat}}\notin\mathcal{D}$, it suffices to show that its subsequence $\left\{s_1(t_i)\exp(-\xi^{(i,1)}(t_i)), i\geq 1\right\}^{\downarrow}\notin\mathcal{D}$. The components of this sequence are the atoms of a Poisson measure with intensity m'' given by

$$\int_0^{\infty} f(x)m''(\mathrm{d}x)=\int_0^{\infty}\int_{l^1}E\left[f\left(s_1\exp(-\xi(t)\right)\right]I(\mathrm{d}\mathbf{s})\mathrm{d}t.$$

Take then $a>E[\xi(1)]$. Since $\xi(t)/t\overset{\text{a.s.}}{\to}E[\xi(1)]$ as $t\to\infty$, there exists some t_0 such that $P(\xi(t)\leq at)\geq 1/2$ for $t\geq t_0$. Then

$$\begin{aligned}\int_1^{\infty} m''(\mathrm{d}x) &= \int_0^{\infty}\int_{l^1}P(\xi(t)\leq \ln s_1)I(\mathrm{d}\mathbf{s})\mathrm{d}t\\ &\geq \int_{l^1}\int_{t_0}^{a^{-1}\ln s_1}P(\xi(t)\leq at)\mathrm{d}tI(\mathrm{d}\mathbf{s})\\ &\geq \frac{1}{2}\int_{l^1}\left(a^{-1}\ln s_1-t_0\right)\mathbf{1}_{\{a^{-1}\ln s_1\geq t_0\}}I(\mathrm{d}\mathbf{s})\end{aligned}$$

and this last integral is infinite by assumption. Hence $\sum_{i\geq 1}\delta_{s_1(t_i)\exp(-\xi^{(i,1)}(t_i))}\notin\mathcal{D}$ a.s. and a fortiori $\mathbf{U}_{\mathrm{stat}}\notin\mathcal{D}$ a.s. ■

Proof of Theorem 4.3. Fix $p\geq 1+\alpha$. According to the Campbell formula for Poisson measures (see [46]), the sum $M(p)$ is finite if and only if

$$\int_0^{\infty}\int_{l^1}E\left[1-\exp(-\sum\nolimits_{j\geq 1}s_j^pM^{(1,j)}(p,s_j^{\alpha}t))\right]I(\mathrm{d}\mathbf{s})\mathrm{d}t<\infty. \tag{4.15}$$

(i) We first prove assertion (i) and that $\mathbf{U}_{\mathrm{stat}}\in l^p$ a.s. for p large enough when I integrates $s_1^{\varepsilon}\mathbf{1}_{\{s_1\geq 1\}}$. Suppose $p>1+\alpha$ and note that the integral (4.15) is bounded from above by

$$\begin{aligned}&\int_{l^1}\sum\nolimits_{j\geq 1}s_j^{p-\alpha}\mathbf{1}_{\{s_j<1\}}I(\mathrm{d}\mathbf{s})\int_0^{\infty}E\left[M(p,t)\right]\mathrm{d}t\\ &+\int_0^{\infty}\int_{l^1}E\left[1-\exp(-\sum\nolimits_{j\geq 1}s_j^p\mathbf{1}_{\{s_j\geq 1\}}M^{(1,j)}(p,s_j^{\alpha}t))\right]I(\mathrm{d}\mathbf{s})\mathrm{d}t.\end{aligned}$$

According to Lemma 4.3 (ii), the first component of this sum is finite and for all $\eta > 0$ there exists some i.i.d r.v. $I^{(j)}_{(\eta,p)}$ having finite moments of all positive orders and independent of $(\mathbf{s}(t_i), i \geq 1)$ such that the second component is bounded from above by

$$\begin{aligned} &\int_0^{\infty} \int_{l^1} E\left[1 - \exp(-\sum\nolimits_{j\geq 1} s_j^{p-\alpha\frac{p-1}{\alpha+\eta}} \mathbf{1}_{\{s_j \geq 1\}} I^{(j)}_{(\eta,p)} t^{-\frac{p-1}{\alpha+\eta}})\right] I(\mathrm{d}\mathbf{s})\mathrm{d}t \\ = &\int_0^{\infty} (1 - \exp(-t^{-\frac{p-1}{\alpha+\eta}}))\mathrm{d}t \times \int_{l^1} (\sum\nolimits_{j\geq 1} s_j^{\frac{p\eta+\alpha}{\alpha+\eta}} \mathbf{1}_{\{s_j \geq 1\}})^{\frac{\alpha+\eta}{p-1}} I(\mathrm{d}\mathbf{s}) E\left[I^{(1)}_{(\eta,p)}{}^{\frac{\alpha+\eta}{p-1}}\right]. \end{aligned}$$

If $p > 1+\alpha+\eta$, the first integral in this latter product is finite. So, take η small enough so that $p > 1+\alpha+\eta$ and notice then that

$$\int_{l^1} (\sum\nolimits_{j\geq 1} s_j^{\frac{p\eta+\alpha}{\alpha+\eta}} \mathbf{1}_{\{s_j \geq 1\}})^{\frac{\alpha+\eta}{p-1}} I(\mathrm{d}\mathbf{s}) \leq \int_{l^1} \sum\nolimits_{j\geq 1} s_j^{\frac{p\eta+\alpha}{p-1}} \mathbf{1}_{\{s_j \geq 1\}} I(\mathrm{d}\mathbf{s}). \quad (4.16)$$

The integral (4.15) is therefore finite as soon as the integral in the right hand side of (4.16) is finite for some $\eta > 0$ small enough. Hence we get (i).

The same argument holds to show that $\mathbf{U}_{\mathrm{stat}} \in l^p$ for p sufficiently large when there exists some $\varepsilon > 0$ such that $\int_{l^1} s_1^{\varepsilon} \mathbf{1}_{\{s_1 \geq 1\}} I(\mathrm{d}\mathbf{s}) < \infty$. Indeed, let $p > 1+\alpha+\eta$. It suffices then to show that the integral on the left hand of inequality (4.16) is finite and to do so we replace the upper bound there by

$$\int_{l^1} (\sum\nolimits_{j\geq 1} s_j^{\frac{p\eta+\alpha}{\alpha+\eta}} \mathbf{1}_{\{s_j \geq 1\}})^{\frac{\alpha+\eta}{p-1}} I(\mathrm{d}\mathbf{s}) \leq \int_{l^1} s_1^{\frac{p\eta+\alpha}{p-1}} (\sum\nolimits_{j\geq 1} \mathbf{1}_{\{s_j \geq 1\}})^{\frac{\alpha+\eta}{p-1}} I(\mathrm{d}\mathbf{s}),$$

which, by Hölder inequality, is finite as soon as p is large enough and η small enough.

(ii) We now turn to the proof of assertion (ii) and that $\mathbf{U}_{\mathrm{stat}} \notin l^{1+\alpha}$ when (H3) holds. The integral (4.15) is bounded from below by

$$\begin{aligned} &\int_0^{\infty} \int_{l^1} s_1^{-\alpha} E\left[(1 - \exp(-s_1^p M(p,t)))\mathbf{1}_{\{M(p,t) \geq rt^{-(p-1)/\alpha}\}}\right] I(\mathrm{d}\mathbf{s})\mathrm{d}t \\ \geq &\int_{l^1} s_1^{-\alpha} \int_0^{\infty} (1 - \exp(-s_1^p rt^{-(p-1)/\alpha}) P(M(p,t) \geq rt^{-(p-1)/\alpha})\mathrm{d}t I(\mathrm{d}\mathbf{s}). \end{aligned}$$

According to Corollary 3, [15], the hypothesis (H3) ensures that $t^{(p-1)/\alpha} M(p,t)$ converges in probability to some finite deterministic constant as $t \to \infty$. Hence, taking $r > 0$ small enough and then t_0 large enough, one has $P(M(p,t) \geq rt^{-(p-1)/\alpha}) \geq 1/2$ for $t \geq t_0$ and therefore the integral (4.15) is bounded from below by

$$\frac{1}{2} \int_{l^1} s_1^{-\alpha} s_1^{p\alpha/(p-1)} \mathbf{1}_{\{s_1^{p\alpha/(p-1)} \geq (t_0/t)\}} I(\mathrm{d}\mathbf{s}) \int_0^{\infty} (1 - \exp(-rt^{-(p-1)/\alpha})\mathrm{d}t$$

which is infinite as soon as $p \leq 1+\alpha$ or $\int_{l^1} s_1^{\alpha/(p-1)} \mathbf{1}_{\{s_1 \geq t_0\}} I(\mathrm{d}\mathbf{s}) = \infty$. ■

Proof of Proposition 4.1. For the standard fragmentation F, let $N_{(\varepsilon,\infty)}(t) := \sum_{k\geq 1} \mathbf{1}_{\{F_k(t) > \varepsilon\}}$ denote the number of terms larger than ε present at time t. Under the hypotheses (H3), (H4) and $\alpha > -1$, Theorems 4 (i) and 7 of [39] describe the behavior of $N_{(\varepsilon,\infty)}(t)$ as $\varepsilon \to 0$. Theorem 4 (i) states the existence of a random function L such that $\sum_{k\geq 1} F_k(t) = \int_t^\infty L(u)\mathrm{d}u$ a.s. for all t. Then Theorem 7 says that

$$\varepsilon^{1+\alpha} N_{(\varepsilon,\infty)}(t) \to KL(t) \text{ as } \varepsilon \to 0 \tag{4.17}$$

a.s. for almost every t, where $K = (1+\alpha)/\alpha^2 E[\xi(1)]$. Note that the sum $\overline{\mathbf{U}}_{\mathrm{stat}}(\varepsilon)$ rewrites

$$\overline{\mathbf{U}}_{\mathrm{stat}}(\varepsilon) = \sum\nolimits_{i,j\geq 1} N^{(i,j)}_{(\varepsilon/s_j(t_i),\infty)}(s_j^\alpha(t_i)t_i) \tag{4.18}$$

where the $N^{(i,j)}_{(\cdot,\infty)}(\cdot)$'s are i.i.d copies of $N_{(\cdot,\infty)}(\cdot)$, independent of $((\mathbf{s}(t_i), t_i), i \geq 1)$.

(i) Let $\zeta^{(i,j)}$ be the first time at which $F^{(i,j)}$ reaches $\mathbf{0}$, $i, j \geq 1$. With the same arguments as in the proof of Theorem 4.1 (i), one sees that with probability one there is at most a finite number of $t_i < \sup_{j\geq 1}(\zeta^{(i,j)} s_j^{-\alpha}(t_i))$ if and only if $\int_{l^1} E\left[\sup_{j\geq 1} \zeta^{(1,j)} s_j^{-\alpha}\right] I(\mathrm{d}\mathbf{s}) < \infty$. This integral is finite by assumption. A moment of thought then show that there is at most a finite number of integers $i, j \geq 1$ - independent of ε - such that $N^{(i,j)}_{(\varepsilon/s_j(t_i),\infty)}(s_j^\alpha(t_i)t_i) > 0$. Consequently, the sum (4.18) involves a finite number of non-zero terms and

$$\varepsilon^{1+\alpha}\overline{\mathbf{U}}_{\mathrm{stat}}(\varepsilon) \underset{\varepsilon\to 0}{\to} K \sum\nolimits_{i,j\geq 1} L^{(i,j)}(s_j^\alpha(t_i)t_i) s_j^{1+\alpha}(t_i) \text{ a.s.}$$

where the functions $L^{(i,j)}$'s are i.i.d and distributed as L. This limit, which we denote by X, is null as soon as $\mathbf{U}_{\mathrm{stat}} = \mathbf{0}$, that is as soon as there is no integer $i \geq 1$ such that $t_i < \sup_{j\geq 1}(\zeta^{(i,j)} s_j^{-\alpha}(t_i))$. This occurs, according to the Poissonian construction, with a positive probability. On the other hand, the Lebesgue measure of $\mathcal{B}_L := \{x \geq 0 : L(x) > 0\}$ (denoted by $\mathrm{Leb}(\mathcal{B}_L)$) is a.s. non-zero and then $P(X > 0) > 0$.

(ii) Suppose $\int_{l^1} s_1^{-\alpha} 1_{\{s_1\leq 1\}} I(\mathrm{d}\mathbf{s}) = \infty$ and let $\mathcal{B}_{L^{(i,j)}} := \{x \geq 0 : L^{(i,j)}(x) > 0\}$, which are i.i.d copies of $\mathcal{B}_L$. One checks that there a.s. exists a time $t_i \in \cup_{j\geq 1} s_j^{-\alpha}(t_i)\mathcal{B}_{L^{(i,j)}}$ if and only if the integral $\int_{l^1} E\left[\mathrm{Leb}(\cup_{j\geq 1} s_j^{-\alpha} \mathcal{B}_{L^{(1,j)}})\right] I(\mathrm{d}\mathbf{s})$ is infinite and that this integral is indeed infinite here, according to the assumption on I and since $\mathrm{Leb}(\mathcal{B}_L) > 0$ a.s. From this we deduce that

$$\sum\nolimits_{1\leq i,j\leq N} L^{(i,j)}(s_j^\alpha(t_i)t_i) s_j^{1+\alpha}(t_i) > 0 \text{ a.s. for } N \text{ large enough}$$

and then, by (4.17) and (4.18), that $\liminf_{\varepsilon\to 0} \varepsilon^{1+\alpha}\overline{\mathbf{U}}_{\mathrm{stat}}(\varepsilon) > 0$. ■

4.3 Rate of convergence to the stationary distribution

We are interested in the convergence in law to the stationary regime $\mathbf{U}_{\mathrm{stat}}$. It is already known, according to Lemma 4.1, that for every random $\mathbf{u} \in \mathcal{D}$ the process $FI^{(\mathbf{u})}(t)$ converges in law as $t \to \infty$ to the stationary state $\mathbf{U}_{\mathrm{stat}}$, provided it belongs to $\mathcal{D}$ a.s. The aim of this section is to strengthen this result by providing upper bounds for the rate at which this convergence takes place. The norm considered on the set of signed finite measures on $\mathcal{D}$ is

$$\|\mu\| := \sup_{\substack{f \text{ 1-Lipschitz,} \\ \sup_{\mathbf{s}\in\mathcal{D}} |f(\mathbf{s})|\leq 1}} \left| \int_{\mathcal{D}} f(\mathbf{s})\mu(\mathrm{d}\mathbf{s}) \right|.$$

By f is 1-Lipschitz, we mean that $|f(\mathbf{s}) - f(\mathbf{s}')| \leq d(\mathbf{s}, \mathbf{s}')$ for all $\mathbf{s}, \mathbf{s}' \in \mathcal{D}$. It is well-known that this norm induces the topology of weak convergence.

The main results are stated in the following Theorem 4.4. In case $\alpha < 0$, the rate of convergence depends on I and it is worthwhile making the result a little more explicit. This is done, under some regular variation type hypotheses on I, in Corollary 4.1.

Theorem 4.4 *The starting points* $\mathbf{u}$ *considered here are all deterministic.*

(i) *Suppose that* $\alpha < 0$ *and* $\int_{l^1} \sum_{j \geq 1} s_j^{-\alpha} \mathbf{1}_{\{s_j \geq 1\}} I(\mathrm{d}\mathbf{s}) < \infty$. *Then, for every* $\gamma \in [1, \Gamma]$ *(*Γ *is defined by formula (4.8)), there exists a positive finite constant* A *such that for every* $\mathbf{u}$ *satisfying* $\sum_{j \geq 1} \exp(-u_j^{\alpha}) < \infty$,

$$\left\| \mathcal{L}(FI^{(\mathbf{u})}(t)) - \mathcal{L}(\mathbf{U}_{\mathrm{stat}}) \right\| = O(t^{-(\gamma-1)} \int_{l^1} \sum_{j \geq 1} s_j^{-\alpha\gamma} \exp(-At^{\gamma} s_j^{\alpha\gamma}) I(\mathrm{d}\mathbf{s}) + \exp(-At^{\gamma} u_1^{\alpha\gamma}))$$

as $t \to \infty$.

(ii) *Suppose that* $\alpha = 0$ *and* $\int_{l^1} \sum_{j \geq 1} s_j^{1+\varepsilon} I(\mathrm{d}\mathbf{s}) < \infty$ *for some* $\varepsilon > 0$. *Then for every* $\mathbf{u} \in l^{1+\varepsilon}$ *and* $a < \phi(\varepsilon)/(2+\varepsilon)$,

$$\left\| \mathcal{L}(FI^{(\mathbf{u})}(t)) - \mathcal{L}(\mathbf{U}_{\mathrm{stat}}) \right\| = o(\exp(-at)) \text{ as } t \to \infty.$$

(iii) *Suppose that* $\alpha > 0$ *and* $\int_{l^1} \sum_{j \geq 1} s_j^{p} I(\mathrm{d}\mathbf{s}) < \infty$ *for some* $p > 0$. *Then, for every* $\mathbf{u} \in l^p$ *and every* $a < 1/\alpha$,

$$\left\| \mathcal{L}(FI^{(\mathbf{u})}(t)) - \mathcal{L}(\mathbf{U}_{\mathrm{stat}}) \right\| = o(t^{-a}) \text{ as } t \to \infty.$$

Note first that, by Theorems 4.1, 4.2 and 4.3, the assumptions we make on I imply in each case that $\mathbf{U}_{\mathrm{stat}} \in \mathcal{D}$ a.s. In case $\alpha < 0$, the given upper bound may be infinite for some γ's. The point is then to find the γ's in $[1, \Gamma]$ that give the best rate of convergence. This is possible, for example, when $\int_{l^1} \sum_{j \geq 1} \mathbf{1}_{\{s_j \geq x\}} I(\mathrm{d}\mathbf{s})$ behaves regularly as $x \to \infty$. In such case the statement (i) turns to:

Corollary 4.1 *Suppose* $\alpha < 0$ *and fix* $\mathbf{u}$ *such that* $\sum_{j \geq 1} \exp(-u_j^{\alpha}) < \infty$.

(i) *If* $\int_{l^1} \sum_{j \geq 1} \mathbf{1}_{\{s_j \geq x\}} I(\mathrm{d}\mathbf{s}) \sim l(x) x^{-\varrho}$ *as* $x \to \infty$ *for some slowly varying function* l *and some* $\varrho > 0$, *then, provided* $-\alpha < \varrho$,

$$\left\| \mathcal{L}(FI^{(\mathbf{u})}(t)) - \mathcal{L}(\mathbf{U}_{\mathrm{stat}}) \right\| = O\left(l(t^{1/|\alpha|}) t^{-(\varrho/|\alpha|-1)} \right) \text{ as } t \to \infty.$$

(ii) *If* $-\log\left(\int_{l^1} \sum_{j \geq 1} \mathbf{1}_{\{s_j \geq x\}} I(\mathrm{d}\mathbf{s}) \right) \sim l(x) x^{\varrho}$ *as* $x \to \infty$ *for some slowly varying function* l *and some* $\varrho > 0$, *then there exists a slowly varying function* l' *(which is constant when* l *is constant) such that*

$$\left\| \mathcal{L}(FI^{(\mathbf{u})}(t)) - \mathcal{L}(\mathbf{U}_{\mathrm{stat}}) \right\| = O\left(t^{-(\Gamma-1)} \exp(-l'(t) t^{\varrho\Gamma/(|\alpha|\Gamma+\varrho)}) \right) \text{ as } t \to \infty.$$

In the special case when $I(s_1 > a) = 0$ *for some* $a > 0$,

$$\left\| \mathcal{L}(FI^{(\mathbf{u})}(t)) - \mathcal{L}(\mathbf{U}_{\mathrm{stat}}) \right\| = O(\exp(-Bt^{\Gamma}))$$

for some constant $B > 0$.

Proof. **(i)** First, by integrating by parts and then using e.g. Prop. 1.5.10 of [21], one obtains that for $\gamma \in [1, \varrho / -\alpha)$

$$\int_{l^1} \sum_{j \geq 1} s_j^{-\alpha\gamma} \mathbf{1}_{\{x \geq s_j^{\alpha\gamma}\}} I(\mathrm{d}\mathbf{s}) \approx l(x^{1/\alpha\gamma}) x^{-1-\varrho/\alpha\gamma} \text{ as } x \to 0$$

(the notation $\approx$ means that the functions are equivalent up to a multiplicative constant). Then, using Karamata's Abelian-Tauberian Theorem (Th. 1.7.1' of [21]), one deduces that

$$\int_{l^1} \sum_{j \geq 1} s_j^{-\alpha\gamma} \exp(-t s_j^{\alpha\gamma}) I(\mathrm{d}\mathbf{s}) \approx l(t^{-1/\alpha\gamma}) t^{1+\varrho/\alpha\gamma} \text{ as } t \to \infty.$$

Now if $-\alpha < \varrho$, statement (i) of Theorem 4.4 applies and one can plug the above equivalence into the upper bound obtained there. Hence the conclusion.

(ii) Let $1 \leq \gamma \leq \Gamma$. By integrating by parts and then by using Theorem 4.12.10 in [21], one sees that $-\log(\int_{l^1} \sum_{j \geq 1} s_j^{-\alpha\gamma} \mathbf{1}_{\{s_j \geq x\}} I(\mathrm{d}\mathbf{s})) \sim l(x) x^{\varrho}$ as $x \to \infty$. According to de Bruijn's Abelian-Tauberian Theorem 4.12.9 in [21], this implies that

$$-\log \left(\int_{l^1} \sum_{j \geq 1} s_j^{-\alpha\gamma} \exp(-t s_j^{\alpha\gamma}) I(\mathrm{d}\mathbf{s}) \right) \approx f(t) \text{ as } t \to \infty \tag{4.19}$$

where $f(t) = 1/\Psi^{\leftarrow}(t)$ with $\Psi(t) = \Phi(t)/t$ and $\Phi^{\leftarrow}(t) = t^{\varrho/(\alpha\gamma)} / l(t^{1/(-\alpha\gamma)})$. Here $\Phi^{\leftarrow}(t) = \sup\{u \geq 0 : \phi(u) > t\}$ and similarly for Ψ. Therefore $f(t) \sim \widetilde{l}(t) t^{\varrho/(\varrho+|\alpha|\gamma)}$ for some slowly varying function $\widetilde{l}$ (to inverse regularly varying functions, we refer to chapter 1.5.7 of [21]) which is constant when l is constant. The assumption we have on I allows us to apply Theorem 4.4 (i) and the conclusion then follows by taking there $\gamma = \Gamma$ and using the equivalence (4.19). The special case when $I(s_1 > a) = 0$ is obvious. ■

Hence, our bounds for the rate of convergence depend significantly on I when $\alpha < 0$, whereas they are essentially independent of I when $\alpha \geq 0$. Also, in any case they are essentially independent of the starting point $\mathbf{u}$.

We now turn to the proof of Theorem 4.4, which relies on a coupling method that holds for $\mathcal{D}$-valued X-processes with immigration, as defined in Section 4.2.1. We first explain the method in this general context and then make precise calculus for fragmentation with immigration processes. In this latter case, if c, ν and I are fixed so that $I(s_1 > 1) = 0$ and if α varies, one sees (without any calculations, just using that particles with mass ≤ 1 split faster when α is smaller) that the employed method provides a better rate of convergence when α is smaller. When $I(s_1 > 1) > 0$ the comparison of rates of convergence as α varies is no longer possible because particles with mass larger than 1 split more slowly when α is smaller.

Proof of Theorem 4.4. Let X be a $\mathcal{D}$-valued branching process and I an immigration measure such that the processes $XI^{(\mathbf{u})}$, $\mathbf{u} \in \mathcal{D}$, defined by formula (4.10), are $\mathcal{D}$-valued X-processes with immigration. Let then $((\mathbf{s}(t_i), t_i), i \geq 1)$ be the atoms of a Poisson measure with intensity $I(\mathrm{d}\mathbf{s})\mathrm{d}t$, $t \geq 0$, and suppose that the stationary sum $\mathbf{U}_{\mathrm{stat}}$ constructed from $((\mathbf{s}(t_i), t_i), i \geq 1)$ as explained in (4.11) belongs a.s. to $\mathcal{D}$. Suppose moreover that $X^{(\mathbf{u})}(t) \overset{\text{a.s.}}{\to} \mathbf{0}$ for all $\mathbf{u} \in \mathcal{D}$.

Then, fix $\mathbf{u} \in \mathcal{D}$ and consider $X^{(\mathbf{u})}$ and $X^{(\mathbf{U}_{\text{stat}})}$ some versions of X starting respectively from $\mathbf{u}$ and $\mathbf{U}_{\text{stat}}$. Consider next $XI^{(\mathbf{0})}$ an X-process with immigration starting from $\mathbf{0}$, independent of $X^{(\mathbf{u})}$ and $X^{(\mathbf{U}_{\text{stat}})}$. Then, the processes $XI^{(\mathbf{u})}$ and $XI^{(\mathbf{U}_{\text{stat}})}$, defined respectively by $XI^{(\mathbf{u})}(t) := X^{(\mathbf{u})}(t) + XI^{(\mathbf{0})}(t)$ and $XI^{(\mathbf{U}_{\text{stat}})}(t) := X^{(\mathbf{U}_{\text{stat}})}(t) + XI^{(\mathbf{0})}(t)$, $t \geq 0$, are X-processes with immigration starting respectively from $\mathbf{u}$ and $\mathbf{U}_{\text{stat}}$.

Let now r be a deterministic function and call $\zeta_r^{(\mathbf{u})}$ the first time t at which $X_1^{(\mathbf{u})}(s) \leq r(s)$ for all $s \geq t$ and similarly $\zeta_r^{(\text{stat})}$ the first time t at which $X_1^{(\mathbf{U}_{\text{stat}})}(s) \leq r(s)$ for all $s \geq t$. Of course the interesting cases are $\zeta_r^{(\mathbf{u})} < \infty$ and $\zeta_r^{(\text{stat})} < \infty$ a.s. Such cases exist, take e.g. $r \equiv 1$.

Our goal is to evaluate the behavior of the norm $\left\|\mathcal{L}(XI^{(\mathbf{u})}(t)) - \mathcal{L}(\mathbf{U}_{\text{stat}})\right\|$ as $t \to \infty$. To do so, let $f : \mathcal{D} \to \mathbb{R}$ denote a 1-Lipschitz function on $\mathcal{D}$ such that $\sup_{\mathbf{s}\in\mathcal{D}} |f(\mathbf{s})| \leq 1$. For all $t \geq 0$, we construct a function $f_{r(t)}$ from f and $r(t)$ by setting

$$f_{r(t)}(\mathbf{s}) := \begin{cases} f(\mathbf{0}) \text{ when } s_1 \leq r(t) \\ f(s_1, ..., s_{i(r(t))}, 0, 0, ...) \text{ when } s_1 > r(t) \end{cases}$$

where $i(r(t))$ is the unique integer such that $s_{i(r(t))} > r(t)$ and $s_{i(r(t))+1} \leq r(t)$. Clearly, as f is 1-Lipschitz and $d(\mathbf{s}, \mathbf{s}') = \sup_{j\geq 1} \left|s_j - s'_j\right|$ for $\mathbf{s}, \mathbf{s}' \in \mathcal{D}$, $\left|f(\mathbf{s}) - f_{r(t)}(\mathbf{s})\right| \leq r(t)$ for every $\mathbf{s} \in \mathcal{D}$ and therefore

$$\begin{aligned} \left|E\left[f(XI^{(\mathbf{u})}(t)) - f(\mathbf{U}_{\text{stat}})\right]\right| &= \left|E\left[f(XI^{(\mathbf{u})}(t)) - f(XI^{(\mathbf{U}_{\text{stat}})}(t))\right]\right| \\ &\leq 2r(t) + \left|E\left[f_{r(t)}(XI^{(\mathbf{u})}(t)) - f_{r(t)}(XI^{(\mathbf{U}_{\text{stat}})}(t))\right]\right|. \end{aligned} \tag{4.20}$$

The time $\zeta_r^{(\mathbf{u})}$ and the function $f_{r(t)}$ are defined so that for times $t \geq \zeta_r^{(\mathbf{u})}$, $f_{r(t)}(XI^{(\mathbf{u})}(t))$ takes only into account the masses of particles that are descended from immigrated particles, not from $\mathbf{u}$. Therefore, one has

$$E\left[f_{r(t)}(XI^{(\mathbf{u})}(t))\right] = E\left[f_{r(t)}(XI^{(\mathbf{u})}(t))\mathbf{1}_{\{\zeta_r^{(\mathbf{u})} \vee \zeta_r^{(\text{stat})} > t\}}\right] + E\left[f_{r(t)}(XI^{(\mathbf{0})}(t))\mathbf{1}_{\{t \geq \zeta_r^{(\mathbf{u})} \vee \zeta_r^{(\text{stat})}\}}\right]$$

and similarly

$$E\left[f_{r(t)}(XI^{(\mathbf{U}_{\text{stat}})}(t))\right] = E\left[f_{r(t)}(XI^{(\mathbf{U}_{\text{stat}})}(t))\mathbf{1}_{\{\zeta_r^{(\mathbf{u})} \vee \zeta_r^{(\text{stat})} > t\}}\right] + E\left[f_{r(t)}(XI^{(\mathbf{0})}(t))\mathbf{1}_{\{t \geq \zeta_r^{(\mathbf{u})} \vee \zeta_r^{(\text{stat})}\}}\right].$$

Combined with (4.20) this gives

$$\begin{aligned} \left|E\left[f(XI^{(\mathbf{u})}(t)) - f(\mathbf{U}_{\text{stat}})\right]\right| &\leq 2r(t) + \left|E\left[(f_{r(t)}(XI^{(\mathbf{u})}(t)) - f_{r(t)}(XI^{(\mathbf{U}_{\text{stat}})}(t)))\mathbf{1}_{\{\zeta_r^{(\mathbf{u})} \vee \zeta_r^{(\text{stat})} > t\}}\right]\right| \\ &\leq 2r(t) + 2P(\zeta_r^{(\mathbf{u})} \vee \zeta_r^{(\text{stat})} > t) \end{aligned}$$

since $\sup_{\mathbf{s}\in\mathcal{D}} |f(\mathbf{s})| \leq 1$. This holds for all 1-Lipschitz functions f such that $\sup_{\mathbf{s}\in\mathcal{D}} |f(\mathbf{s})| \leq 1$ and therefore

$$\left\|\mathcal{L}(XI^{(\mathbf{u})}(t)) - \mathcal{L}(\mathbf{U}_{\text{stat}})\right\| \leq 2(r(t) + P(\zeta_r^{(\mathbf{u})} > t) + P(\zeta_r^{(\text{stat})} > t)). \tag{4.21}$$

The point is thus to find a function r such that the above upper bound gives the best possible rate of convergence.

In the rest of this proof, we replace X by an (α, c, ν) fragmentation process F, in order to make precise calculus. We recall that $F^{(\mathbf{u})}(t) \overset{\text{a.s.}}{\to} \mathbf{0}$ and that the assumptions of Theorem 4.4 involving I ensure that $\mathbf{U}_{\text{stat}} \in \mathcal{D}$ a.s. for all $\alpha \in \mathbb{R}$, so that inequality (4.21) holds for $FI^{(\mathbf{u})}$. The choice of the function r then differs according as $\alpha < 0$, $\alpha = 0$ and $\alpha > 0$.

Proof of (i). Here we take $r \equiv 0$. According to the definitions above, $\zeta_r^{(\mathbf{u})}$ is the first time at which $F^{(\mathbf{u})}$ reaches $\mathbf{0}$ (it may be a priori infinite) and $\zeta_r^{(\text{stat})}$ the first time at which $F^{(\mathbf{U}_{\text{stat}})}$ reaches $\mathbf{0}$. As recalled in Section 4.1.1, the first time ζ at which a 1-mass particle splitting according to the (α, c, ν)-fragmentation reaches $\mathbf{0}$ is a.s. finite since $\alpha < 0$. By self-similarity, the first time at which a particle with mass m is reduced to $\mathbf{0}$ is distributed as $m^{-\alpha}\zeta$. Hence, by definitions of $F^{(\mathbf{u})}$ and $F^{(\mathbf{U}_{\text{stat}})}$,

$$\zeta_r^{(\mathbf{u})} = \sup_{j \geq 1} u_j^{-\alpha} \zeta^{(j)} \quad \text{and} \quad \zeta_r^{(\text{stat})} = \sup_{i \geq 1, j \geq 1} (s_j^{-\alpha}(t_i)\zeta^{(i,j)} - t_i)^+$$

where $(\zeta^{(j)}, j \geq 1)$ and $(\zeta^{(i,j)}, i, j \geq 1)$ denote families of i.i.d copies of ζ such that $(\zeta^{(i,j)}, i, j \geq 1)$ is independent of $((\mathbf{s}(t_i), t_i), i \geq 1)$.

Now fix $\gamma \in [1, \Gamma]$. On the one hand, one has

$$P(\zeta_r^{(\mathbf{u})} > t) \leq \sum\nolimits_{j \geq 1} P(\zeta^{(j)} > t u_j^{\alpha})$$

which by (4.7) is bounded from above by $C_\gamma \sum_{j \geq 1} \exp(-C'_\gamma t^\gamma u_j^{\alpha\gamma})$ for some constant $C_\gamma, C'_\gamma > 0$. Let $0 < \varepsilon < C'_\gamma$. It is easy that this sum is in turn bounded for all $t \geq 1$ by $B \exp(-(C'_\gamma - \varepsilon) t^\gamma u_1^{\alpha\gamma})$, where B is a constant (depending on γ, ε and $\mathbf{u}$, not on $t \geq 1$) which is finite as soon as $\sum_{j \geq 1} \exp(-u_j^{\alpha}) < \infty$. On the other hand,

$$P(\zeta_r^{(\text{stat})} > t) \leq \int_0^\infty \int_{l^1} \sum\nolimits_{j \geq 1} P(\zeta > (t+v) s_j^{\alpha}) I(\mathrm{d}\mathbf{s}) \mathrm{d}v$$

which, again by (4.7), is bounded from above by

$$\frac{C_\gamma}{C'_\gamma \gamma t^{\gamma - 1}} \int_{l^1} \sum\nolimits_{j \geq 1} s_j^{-\alpha\gamma} \exp(-C'_\gamma t^\gamma s_j^{\alpha\gamma}) I(\mathrm{d}\mathbf{s})$$

for $t > 0$. Hence the result.

Proof of (ii). When $\alpha = 0$, the fragmentation does not reach $\mathbf{0}$ in general. We thus have to choose some function $r \neq 0$. By assumption, $\int_{l^1} \sum_{j \geq 1} s_j^{1+\varepsilon} I(\mathrm{d}\mathbf{s}) < \infty$ for some $\varepsilon > 0$. So, fix such ε, fix $\eta > 1$ and set $a := \phi(\varepsilon) / (1 + \eta(1+\varepsilon))$. Then take $r(t) := \exp(-at)$, $t \geq 0$.

In order to bound from above $P(\zeta_r^{(\mathbf{u})} > t)$ and $P(\zeta_r^{(\text{stat})} > t)$, introduce for all $x > 0$

$$\zeta_{a,x} = \sup \{t \geq 0 : F_1(t) > x \exp(-at)\}$$

the last time t at which the largest fragment of a standard fragmentation process F starting from $(1, 0, ...)$ has a mass largest than $x \exp(-at)$. Here we use the convention $\sup(\emptyset) = 0$. This time $\zeta_{a,x}$ is a.s. finite because $\exp(at)F_1(t) \overset{\text{a.s.}}{\to} 0$ when $0 \leq a < \sup_{p\geq 0} \frac{\phi(p)}{p+1}$ as explained in [15]. More precisely, one can show the existence of a positive constant $C(a)$ such that

$$P(\zeta_{a,x} > t) \leq C(a)x^{-(1+\varepsilon)} \exp(-at) \text{ for all } x > 0, t \geq 1. \tag{4.22}$$

Indeed, let $t \geq 1$ and note that

$$\begin{aligned} P(\eta t \geq \zeta_{a,x} > t) &= P\left(\exists u \in [t, \eta t[\, : F_1(u) \exp(au) > x\right) \\ &\leq P\left(F_1(t) \exp(a\eta t) > x\right) \qquad \text{(as } F_1 \searrow \text{)} \\ &\leq x^{-(1+\varepsilon)} \exp(a\eta(1+\varepsilon)t) E\left[(F_1(t))^{1+\varepsilon}\right]. \end{aligned}$$

This last expectation is bounded from above by $E\left[\sum_{k\geq 1} (F_k(t))^{1+\varepsilon}\right] = \exp(-\phi(\varepsilon)t)$, which yields $P(\eta t \geq \zeta_{a,x} > t) \leq x^{-(1+\varepsilon)} \exp(-at)$, since $a = \phi(\varepsilon) - a\eta\,(1+\varepsilon)$. Then, setting $C(a) := \sum_{n\geq 1} \exp(-a\,(\eta^{n-1} - 1))$, one obtains (4.22).

By definition, $\zeta_r^{(\mathbf{u})}$ is the supremum of times t such that $F_1^{(\mathbf{u})}(t) > \exp(-at)$. Hence there exist some independent random variables $\zeta_{a,1/u_j}^{(j)}$, $j \geq 1$, where $\zeta_{a,1/u_j}^{(j)}$ has the same distribution as $\zeta_{a,1/u_j}$, such that

$$\zeta_r^{(\mathbf{u})} = \sup_{j\geq 1} \zeta_{a,1/u_j}^{(j)}.$$

Then, by inequality (4.22) ,

$$P(\zeta_r^{(\mathbf{u})} > t) \leq C(a) \exp(-at) \sum_{j\geq 1} u_j^{1+\varepsilon}. \tag{4.23}$$

Next, by definition of $\zeta_r^{(\text{stat})}$, there exists a family of r.v. $\zeta_{a,\exp(at_i)/s_j(t_i)}^{(i,j)}$, $i, j \geq 1$, such that

$$\zeta_r^{(\text{stat})} = \sup_{i\geq 1, j\geq 1} (\zeta_{a,\exp(at_i)/s_j(t_i)}^{(i,j)} - t_i)^+$$

and, conditionally on $((\mathbf{s}(t_i), t_i), i \geq 1)$, $\zeta_{a,\exp(at_i)/s_j(t_i)}^{(i,j)} \overset{\text{law}}{=} \zeta_{a,\exp(at_i)/s_j(t_i)}$, $i, j \geq 1$, and the $\zeta_{a,\exp(at_i)/s_j(t_i)}^{(i,j)}$'s are independent. This implies that

$$P(\zeta_r^{(\text{stat})} > t) \leq \sum_{i\geq 1} \sum_{j\geq 1} P(\zeta_{a,\exp(at_i)/s_j(t_i)}^{(i,j)} > t_i + t)$$

and then, by (4.22), that

$$P(\zeta_r^{(\text{stat})} > t) \leq \frac{C(a)}{2a+\varepsilon} \exp(-at) \int_{l^1} \sum_{j\geq 1} s_j^{1+\varepsilon} I(\mathrm{d}\mathbf{s}).$$

Combining this last inequality with (4.21) and (4.23) , one obtains

$$\left\|\mathcal{L}(FI^{(\mathbf{u})}(t)) - \mathcal{L}(\mathbf{U}_{\text{stat}})\right\| \leq 2\exp(-at)(1 + C(a) \sum_{j\geq 1} u_j^{1+\varepsilon} + (2a)^{-1} C(a) \int_{l^1} \sum_{j\geq 1} s_j^{1+\varepsilon} I(\mathrm{d}\mathbf{s})).$$

This holds for every $\eta > 1$ and therefore $\left\|\mathcal{L}(FI^{(\mathbf{u})}(t)) - \mathcal{L}(\mathbf{U}_{\text{stat}})\right\| = O(\exp(-at))$ for every $a < \phi(\varepsilon)/\,(2+\varepsilon)$, provided $\mathbf{u} \in l^{1+\varepsilon}$.

Proof of (iii). Fix $0 < a < 1/\alpha$ and set $r(t) := t^{-a}$, $t > 0$. By assumption, there exists some $p > 0$ such that $\int_{l^1} \sum_{j\geq 1} s_j^p I(\mathrm{d}\mathbf{s}) < \infty$ and we call z the real number such that $z\alpha^2(a+1) = p(1-\alpha a - \alpha z)$. Note that $0 < z < \alpha^{-1} - a$. Define then for $x > 0$

$$\zeta_{a,x} := \sup\left\{t \geq 0 : F_1(t) > xt^{-a}\right\}.$$

The fact that $z \in (0, \alpha^{-1})$ allows us to choose some $\eta > 0$ and $q > 1$ such that $\frac{q-1}{\alpha+\eta} - aq = q(\alpha^{-1} - a - z)$, which, by definition of z, is also equal to $qz\alpha(a+1)/p$. According to Lemma 4.3 (ii), there exists a r.v. $I_{(\eta,q)}$ with positive moments of all orders such that

$$t^{qa} F_1^q(t) \leq I_{(\eta,q)} t^{qa - \frac{q-1}{\alpha+\eta}} = I_{(\eta,q)} t^{-qz\alpha(a+1)/p}$$

a.s. for every $t > 0$. This implies that

$$\begin{aligned} P(\zeta_{a,x} &> t) \leq P\left(\exists u \geq t : u^{qa} F_1^q(u) > x^q\right) \\ &\leq P\left(\exists u \geq t : I_{(\eta,q)} u^{-qz\alpha(a+1)/p} > x^q\right) \\ &\leq Bx^{-p/(z\alpha)} t^{-(a+1)}, \end{aligned}$$

where $B := E\left[I_{(\eta,q)}^{p/(qz\alpha)}\right] < \infty$.

A moment of thought shows that the times $\zeta_r^{(\mathbf{u})} = \sup\{t \geq 0 : F_1^{(\mathbf{u})}(t) > t^{-a}\}$ and $\zeta_r^{(\mathrm{stat})} = \sup\{t \geq 0 : F_1^{(\mathbf{U}_{\mathrm{stat}})}(t) > t^{-a}\}$ satisfy

$$\zeta_r^{(\mathbf{u})} = \sup_{j\geq 1}(u_j^{-\alpha} \zeta^{(j)}_{a,u_j^{\alpha a-1}}) \quad \text{and} \quad \zeta_r^{(\mathrm{stat})} \leq \sup_{i\geq 1, j\geq 1}(s_j^{-\alpha} \zeta^{(i,j)}_{a,s_j^{\alpha a-1}} - t_i)^+$$

where the r.v. $\zeta^{(j)}_{a,u_j^{\alpha a-1}}$, $j \geq 1$, are independent such that $\zeta^{(j)}_{a,u_j^{\alpha a-1}} \overset{\text{law}}{=} \zeta_{a,u_j^{\alpha a-1}}$ and, conditionally on $((\mathbf{s}(t_i), t_i), i \geq 1)$, the r.v. $\zeta^{(i,j)}_{a,s_j^{\alpha a-1}}$, $i, j \geq 1$, are independent such that $\zeta^{(i,j)}_{a,s_j^{\alpha a-1}} \overset{\text{law}}{=} \zeta_{a,s_j^{\alpha a-1}}$. Using then the upper bound $P(\zeta_{a,x} > t) \leq Bx^{-p/(z\alpha)} t^{-(a+1)}$, one obtains

$$P(\zeta_r^{(\mathbf{u})} > t) \leq Bt^{-(a+1)} \sum_{j\geq 1} u_j^{-\alpha(a+1)+p(1-\alpha a)/z\alpha}$$

which is equal to $Bt^{-(a+1)} \sum_{j\geq 1} u_j^p$ by definition of z. Similarly, one obtains

$$P(\zeta_r^{(\mathrm{stat})} > t) \leq a^{-1} B t^{-a} \int_{l^1} \sum_{j\geq 1} s_j^p I(\mathrm{d}\mathbf{s}).$$

Hence by (4.21),

$$\left\| \mathcal{L}(FI^{(\mathbf{u})}(t)) - \mathcal{L}(\mathbf{U}_{\mathrm{stat}}) \right\| \leq Rt^{-a}(1 + \sum_{j\geq 1} u_j^p + \int_{l^1} \sum_{j\geq 1} s_j^p I(\mathrm{d}\mathbf{s}))$$

where R is a finite real number depending on the parameters of the fragmentation and on a, but not on t and f. ■

4.4 Some examples

Here we turn our attention to examples of fragmentation with immigration processes constructed from two families of continuous processes. First, Brownian motions with positive drift provide examples of stationary fragmentation with immigration processes where particles immigrate one by one. The stationary distribution is explicit and constructed from a Poisson measure depending on the drift. Second, height functions of continuous state branching processes with immigration (as introduced in [49]) code fragmentation with immigration processes where some particles immigrate in groups and others on their own. We will see that those processes do not all have a stationary distribution.

4.4.1 Construction from Brownian motions with positive drift

Let B be a standard linear Brownian motion and for every $d > 0$, consider the Brownian motion with drift d

$$B_{(d)}(x) := B(x) + dx, \ x \geq 0.$$

For any $t > 0$, define

$$L_{(d)}(t) := \inf\left\{x \geq 0 : B_{(d)}(x) = t\right\} \qquad R_{(d)}(t) := \sup\left\{x \geq 0 : B_{(d)}(x) = t\right\}$$

the first and the last hitting times of t by $B_{(d)}$. Clearly $0 < L_{(d)}(t) < R_{(d)}(t) < \infty$ a.s., since $d > 0$. It is thus possible to consider the decreasing rearrangement of lengths of the connected components of

$$\mathcal{E}_{(d)}(t) := \left\{x \in \left[L_{(d)}(t), R_{(d)}(t)\right] : B_{(d)}(x) > t\right\}$$

which we denote by $FI_{(d)}(t)$.

Proposition 4.2 (i) *The process* $\left(FI_{(d)}(t), t \geq 0\right)$ *is a fragmentation immigration process with parameters*

- $\alpha_B = -1/2$
- $c_B = 0$
- $\nu_B(s_1 + s_2 < 1) = 0$ *and* $\nu_B(s_1 \in \mathrm{d}x) = \sqrt{2\pi^{-1}}x^{-3/2}(1-x)^{-3/2}\mathrm{d}x, \quad x \in [1/2, 1)$,
- $I_{(d)}(s_2 > 0) = 0$ *and* $I_{(d)}(s_1 \in \mathrm{d}x) = \sqrt{(2\pi)^{-1}}x^{-3/2}\exp(-xd^2/2)\mathrm{d}x, \quad x > 0.$

(ii) *The process is stationary. The stationary law is that of a Cox measure (that is a Poisson measure with random intensity) with intensity* $T(d)\sqrt{(8\pi)^{-1}}x^{-3/2}\exp(-xd^2/2)\mathrm{d}x$, $x > 0$, *where* $T(d)$ *is an exponential r.v. with parameter* d.

(iii) *There exists a constant* $L \in (0, \infty)$ *such that for every* $\mathbf{u} \in \mathcal{D}$ *satisfying* $\sum_{j \geq 1}\exp(-u_j^{-1/2}) < \infty$, *an* $\left(\alpha_B, c_B, \nu_B, I_{(d)}\right)$ *fragmentation immigration* $FI^{(\mathbf{u})}$ *starting from* $\mathbf{u}$ *converges in law to the stationary distribution* $\mathcal{L}(\mathbf{U}_{\mathrm{stat}})$ *at rate*

$$\left\|\mathcal{L}(FI^{(\mathbf{u})}(t)) - \mathcal{L}(\mathbf{U}_{\mathrm{stat}})\right\| = O(t^{-1}\exp(-Lt)).$$

Note that the immigrating particles arrive one by one.

The fragmentation part of these processes, that does not depend on d, is a well-known fragmentation process that was first constructed by Bertoin in [14]. Let $F_B^{(\mathbf{l})}$ denote this fragmentation starting from $\mathbf{l} = (l, 0, ...)$. It is a binary fragmentation, that is each particle splits exactly into two pieces, which is constructed from a Brownian excursion $e_B^{(l)}$ conditioned to have length l as follows :

$$F_B^{(\mathbf{l})}(t) := \left\{\text{lengths of connected components of } \left\{x \in [0, l] : e_B^{(l)}(x) > t\right\}\right\}^{\downarrow} \tag{4.24}$$

for all $t \geq 0$. In [14] it is proved that this process is indeed a fragmentation process with index $\alpha_B = -1/2$, no erosion and a dislocation measure ν_B as given above.

Proof. **(i)** According to Corollaries 1 and 2 in [62], the process defined by

$$Y_{(d)}(x) := B_{(d)}(x + R_{(d)}(0)), \quad x \geq 0,$$

is a $\mathrm{BES}^0(3, d)$ (which means that it is identical in law to the norm of a three dimensional Brownian motion with drift d) and is independent of $\left(B_{(d)}(x), 0 \leq x \leq R_{(d)}(0)\right)$. This last process codes the fragmentation of particles present at time 0, whereas the process $Y_{(d)}$ codes the immigration and fragmentation of immigrated particles. More precisely,

- let $e_B^{(l_1)}, ..., e_B^{(l_i)}, ...$ denote the finite excursions of $B_{(d)}$ above 0, with respective lengths $l_1, l_2, ...$. The Cameron-Martin-Girsanov theorem implies that the $(l_i, i \geq 1)$ are the finite jumps of a subordinator with Lévy measure $\sqrt{(8\pi)^{-1}} x^{-3/2} e^{-xd^2/2} \mathrm{d}x$, killed at an exponential time with parameter d, and that conditionally on $(l_i, i \geq 1)$ the excursions $e_B^{(l_1)}, e_B^{(l_2)}, ...$ are independent Brownian excursions with respective lengths $l_1, ... l_i, ...$. This gives the distribution of $FI_{(d)}(0) = (l_1, l_2, ...)^{\downarrow}$ and implies that the process $(FI_{(d)}^{[0, R_{(d)}(0)]}(t), t \geq 0)$ defined by

$$FI_{(d)}^{[0, R_{(d)}(0)]}(t) := \left\{\text{lengths of connected comp. of } \left\{x \in \left[L_{(d)}(t), R_{(d)}(0)\right] : B_{(d)}(x) > t\right\}\right\}^{\downarrow}$$

is an $(-1/2, 0, \nu_B)$ fragmentation starting from $FI_{(d)}(0)$.

- let $J_{(Y_{(d)})}(x) := \inf_{y \geq x} Y_{(d)}(y)$, $x \geq 0$ be the future infimum of $Y_{(d)}$. One has to see $J_{(Y_{(d)})}$ as the process coding the arrival of immigrating particles and $Y_{(d)} - J_{(Y_{(d)})}$ as the process coding their fragmentation. According to a generalization of Pitman's theorem (Corollary 1, [62]), $(J_{(Y_{(d)})}, Y_{(d)} - J_{(Y_{(d)})})$ is distributed as $(M_{(d)}, M_{(d)} - B_{(d)})$ where $M_{(d)}(x) := \sup_{[0,x]} B_{(d)}(y)$, $x \geq 0$. Moreover according to the Cameron-Martin-Girsanov theorem, $M_{(d)}$ is distributed as the inverse of a subordinator with Lévy measure

$$I_{(d)}(s_1 \in \mathrm{d}x) = \sqrt{(2\pi)^{-1}} x^{-3/2} \exp(-xd^2/2) \mathrm{d}x, \ x > 0,$$

and conditionally on their lengths the excursions above 0 of $M_{(d)} - B_{(d)}$ are Brownian excursions. Let $(\left(\Delta_{(d)}(t_i), t_i\right), i \geq 1)$ denote the family of jump sizes and times of the subordinator inverse of $M_{(d)}$. The sequence

$$FI_{(d)}^{[R_{(d)}(0), \infty)}(t) := \left\{\text{lengths of connected comp. of } \left\{x \in \left[R_{(d)}(0), R_{(d)}(t)\right] : B_{(d)}(x) > t\right\}\right\}^{\downarrow}$$

is the decreasing rearrangement of masses of particles that have immigrated at time $t_i \leq t$ with mass $\Delta_{(d)}(t_i)$ and that have split independently (conditionally on their masses) until time $t-t_i$ according to the fragmentation $(-1/2, 0, \nu_B)$.

• $FI_{(d)}(t)$ is the concatenation of $FI_{(d)}^{[0,R_{(d)}(0)]}(t)$ and $FI_{(d)}^{[R_{(d)}(0),\infty)}(t)$, which leads to the result. Note that $I_{(d)}$ satisfies the hypothesis (H1).

(ii) That $FI_{(d)}(t) \overset{\text{law}}{=} FI_{(d)}(0)$ is a simple consequence of the strong Markov property of B applied at time $L_{(d)}(t)$. The stationary distribution $\mathcal{L}(FI_{(d)}(0))$ is calculated in the first part of this proof.

(iii) It is easy to check that the ν_B-dependent parameter Γ_B (defined in (4.8)) is here equal to 2 and that

$$-\log\left(\int_x^\infty I_{(d)}(s_1 \in \mathrm{d}y)\right) \sim \frac{d^2 x}{2} \text{ as } x \to \infty.$$

Then we conclude with Corollary 4.1 (ii). ■

Remark. Let $Y_{(d)}$ be a $\mathrm{BES}^0(3,d)$, $d \geq 0$, and set

$$FI_{Y_{(d)}}(t) := \left\{\text{lengths of connected comp. of } \left\{x \in [L_{Y_{(d)}}(t), R_{Y_{(d)}}(t)] : Y_{(d)}(x) > t\right\}\right\}^{\downarrow}$$

where $L_{Y_{(d)}}(t) := \inf\left\{x \geq 0 : Y_{(d)}(x) = t\right\}$ and $R_{Y_{(d)}}(t) := \sup\left\{x \geq 0 : Y_{(d)}(x) = t\right\}$. According to the proof above, $FI_{Y_{(d)}}$ is an $\left(-1/2, 0, \nu_B, I_{(d)}\right)$ fragmentation with immigration starting from $\mathbf{0}$ (clearly, this is also valid for $d = 0$). Recall then the construction of the stationary state $\mathbf{U}_{\text{stat}}$ as explained in (4.11). It is easy to see that $\mathbf{U}_{\text{stat}}$ has the same law as the point measure whose atoms are the lengths of the excursions below 0 of the process obtained by reflecting $Y_{(d)}$ at the level of its future infimum. By Corollary 1, [62], this reflected process is a Brownian motion with drift d. Therefore, if $d > 0$, $\mathbf{U}_{\text{stat}} \in \mathcal{D}$ a.s. and the stationary distribution is that of the reordering of the lengths of the excursions below 0 of a Brownian motion with drift d, which is indeed the distribution of $FI_{(d)}(0)$ (by Girsanov's theorem). On the other hand, if $d = 0$, $\mathbf{U}_{\text{stat}}$ is clearly not in $\mathcal{D}$ a.s. and then there is no stationary distribution (which was already known, according to Theorem 4.1 (ii)).

This latter example of fragmentation with immigration constructed from a $\mathrm{BES}^0(3,0)$ belongs to a class of fragmentation with immigration processes which are constructed from height functions coding continuous state branching processes with immigration, that we now study.

4.4.2 Construction from height processes

The height processes we are interested in are those introduced by Lambert [49] to code continuous state branching processes with immigration. Roughly speaking, such height process is a positive continuous process whose total time spent at a level t corresponds to the amount of population belonging to the generation t. Here we are interested in height processes constructed from stable Lévy processes. Let us first remind their construction: fix $\beta \in (1,2]$ and consider $X_{(\beta)}$ a stable Lévy process with no negative jumps and with Laplace exponent

$E\left[\exp(-\lambda X_{(\beta)}(t))\right] = \exp(t\lambda^{\beta})$, $\lambda \geq 0$; consider next a subordinator Y which is not a compound Poisson process and which is independent of $X_{(\beta)}$. We denote by d_Y its drift and by π_Y its Lévy measure.

Definition 4.5 *The height process $(H_{(\beta,Y)}(x), x \geq 0)$ is defined by*

$$H_{(\beta,Y)}(x) := Y^{-1}(-J_{(\beta)}(x)) + \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{g_x}^{x} \mathbf{1}_{\{X^*_{(\beta)}(y) - \inf_{y \leq r \leq x} X^*_{(\beta)}(r) \leq \varepsilon\}} \mathrm{d}y$$

where $J_{(\beta)}(x) := \inf_{0 \leq y \leq x} X_{(\beta)}(y)$; $X^*_{(\beta)}(x) := X_{(\beta)}(x) + Y \circ Y^{-1}(-J_{(\beta)}(x))$ *and* $g_x = \sup\{0 \leq y \leq x : X^*_{(\beta)}(y-) = 0\}$ $(\sup(\emptyset) = 0)$.

In the special case when $\beta = 2$ ($X_{(2)} = \sqrt{2}B$ for some standard Brownian motion B) and $Y = \mathrm{id}$, one has ([29]) $H_{(2,\mathrm{id})} = X_{(2)} - 2J_{(2)}$, which, according to Pitman's theorem, is distributed as a $\sqrt{2}\mathrm{BES}^0(3,0)$.

By Theorem VII.1.1 in [10], the right-continuous inverse of $(-J_{(\beta)})$, which we denote by $T_{(\beta)}$ and which is defined as

$$T_{(\beta)}(x) := \inf\left\{u \geq 0 : -J_{(\beta)}(u) > x\right\}, x \geq 0,$$

is a stable subordinator with Laplace exponent $q^{1/\beta}$. In others words, $T_{(\beta)}$ has no drift and a Lévy measure given by $C_\beta x^{-1-1/\beta}\mathrm{d}x$, $x > 0$, where $C_\beta := (\beta\Gamma(1-1/\beta))^{-1}$. In the sequel, $\Delta_{T_{(\beta)}}([0,x])$ denotes the decreasing rearrangement of jumps of $T_{(\beta)}$ before time x.

According to [49], the process $H_{(\beta,Y)}$ is continuous and converges to ∞ as $x \to \infty$. Let then $L_{(\beta,Y)}(t)$ and $R_{(\beta,Y)}(t)$ be respectively the first and the last time at which $H_{(\beta,Y)}$ reaches t, $t \geq 0$, and introduce

$$\mathcal{E}_{(\beta,Y)}(t) := \left\{L_{(\beta,Y)}(t) \leq x \leq R_{(\beta,Y)}(t) : H_{(\beta,Y)}(x) > t\right\}.$$

The decreasing rearrangement of lengths of connected components of $\mathcal{E}_{(\beta,Y)}(t)$ is denoted by $FI_{(\beta,Y)}(t)$.

Proposition 4.3 *Suppose $E[Y(1)] = d_Y + \int_0^\infty x\pi_Y(\mathrm{d}x) < \infty$. Then, the process $FI_{(\beta,Y)}$ is a fragmentation with immigration process starting from $\mathbf{0}$ and with values in l^1. Its parameters are*

- $\alpha_\beta = 1/\beta - 1$
- $c_\beta = 0$
- $\int_{\mathcal{D}_1} f(\mathbf{s})\nu_\beta(\mathrm{ds}) = \beta^2 \frac{\Gamma(2-\beta^{-1})}{\Gamma(2-\beta)} E\left[T_{(\beta)}(1) f((T_{(\beta)}(1)^{-1})\Delta_{T_{(\beta)}}([0,1]))\right]$ *when* $\beta < 2$

  $= \pi^{-1/2} \int_{1/2}^{1} f(x, 1-x, 0, ...) \left(x(1-x)\right)^{-3/2} \mathrm{d}x$ *when* $\beta = 2$
- $\int_{l^1} f(\mathbf{s}) I_{\beta,Y}(\mathrm{ds}) = \int_0^\infty E\left[f(x^\beta \Delta_{T_{(\beta)}}([0,1]))\right] \pi_Y(\mathrm{d}x) + d_Y C_\beta \int_0^\infty f(x,0,0...) x^{-1-1/\beta}\mathrm{d}x$

f denoting here any positive measurable function on $\mathcal{D}$.

Note that $\nu_2 = \nu_B/\sqrt{2}$, ν_B being the measure introduced in Proposition 4.2. As we shall see below, this is directly related to the fact that $X_{(2)} = \sqrt{2}B$.

Note also that there are two distinct and independent kinds of immigration: the first integral in the definition of $I_{\beta,Y}$ codes the immigration of grouped particles (each immigrating group contains an infinite number of particles) whereas the second integral codes the immigration of particles arriving one by one.

One may show that $FI_{(\beta,Y)}$ is in some sense a fragmentation with immigration process even when the extra condition $E[Y(1)] < \infty$ is not satisfied, the only difference being then that the immigration intensity does not satisfy the hypothesis (H1). One may also prove the existence of a fragmentation immigration process $FI_{(\beta,Y)}$ when Y is a compound Poisson process: it suffices to extend the definition 4.5 of $H_{(\beta,Y)}$ to compound Poisson processes Y. In such case, $Y^{-1} \circ (-J_{(\beta)})$ (and then $H_{(\beta,Y)}$) is not continuous (there are some positive jumps) and so $L_{(\beta,Y)}(t)$ and $R_{(\beta,Y)}(t)$ may not exist for some t. Setting $FI_{(\beta,Y)}(t) := \mathbf{0}$ for those t and keeping the previous definitions for the others, we obtain a fragmentation with immigration process.

Proof. Informally, in the definition of $H_{(\beta,Y)}$ the piecewise constant process $Y^{-1} \circ (-J_{(\beta)})$ contributes to the immigration and the continuous process

$$x \mapsto \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{g_x}^{x} \mathbf{1}_{\left\{X^*_{(\beta)}(y) - \inf_{y \le r \le x} X^*_{(\beta)}(r) \le \varepsilon\right\}} \mathrm{d}y$$

to the fragmentation. The process $Y^{-1} \circ (-J_{(\beta)})$ is the future infimum of $H_{(\beta,Y)}$. We claim (details are left to the reader, see e.g. [29]) that the excursions of $H_{(\beta,Y)}$ above this future infimum are independent conditionally on their lengths and distributed as excursions of $H_{(\beta)}$ above 0 where

$$H_{(\beta)}(x) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{0}^{x} \mathbf{1}_{\left\{X_{(\beta)}(y) - \inf_{y \le r \le x} X_{(\beta)}(r) \le \varepsilon\right\}} \mathrm{d}y.$$

This process is the height process that codes a continuous state branching process with branching mechanism function $\lambda \mapsto \lambda^\beta$ (see [52]). Let $e^{(l)}_{H_{(\beta)}}$ be an excursions of $H_{(\beta)}$ conditioned to have length $l > 0$ (this makes sense, although $H_{(\beta)}$ is not Markovian for $\beta < 2$; see [29]) and for $t \ge 0$ let

$$F^l_{(\beta)}(t) := \left\{\text{lengths of the connected comp. of } \left\{x \in [0,l] : e^{(l)}_{H_{(\beta)}}(x) > t\right\}\right\}^{\downarrow}.$$

When $\beta = 2$, it is known ([29]) that $e^{(l)}_{H_{(2)}} = \sqrt{2}e^{(l)}_B$ where $e^{(l)}_B$ is a Brownian excursion with length l and therefore $F^l_{(\beta)}$ is an $(-1/2, 0, \nu_B/\sqrt{2})$ fragmentation starting from $(l, 0, ...)$ as explained in (4.24). When $\beta < 2$, Miermont [56] shows that $F^l_{(\beta)}$ is a self-similar fragmentation with parameters $(1/\beta - 1, 0, \nu_\beta)$ starting from $(l, 0, ...)$.

Note that $Y^{-1} \circ (-J_{(\beta)})$ is the inverse of the pure jump subordinator $T_{(\beta)} \circ Y$, and let $(\Delta_{T_{(\beta)} \circ Y}(t_i), t_i)$ denote the jumps and jump times of this subordinator. Then the process $FI_{(\beta,Y)}$ is a fragmentation with immigration process where

• particles arrive at times t_i in the following manner: either t_i is a jump time of Y and a group of particles with masses $\Delta_{T_{(\beta)}}(s)$, $s \in [Y(t_i-), Y(t_i))$, immigrates at time t_i (the

$\Delta_{T_{(\beta)}}(s), s \geq 0$, being the jumps of $T_{(\beta)}$); or t_i is not a jump time of Y and a unique particle with mass $T_{(\beta)}(Y(t_i)) - T_{(\beta)}(Y(t_i)-)$ immigrates at time t_i;

- each particle splits according to the $(1/\beta - 1, 0, \nu_\beta)$-fragmentation.

It remains to compute the immigration intensity $I_{\beta,Y}$. For each jump time t_i of $T_{(\beta)} \circ Y$, set

$$\mathbf{s}(t_i) := \begin{cases} \{\Delta_{T_{(\beta)}}(s), s \in [Y(t_i-), Y(t_i))\}^{\downarrow} \text{ when } t_i \text{ is a jump time of } Y \\ (\Delta_{T_{(\beta)}}(Y(t_i)), 0, ...) \text{ otherwise.} \end{cases}$$

The points $(\mathbf{s}(t_i), t_i)$'s are the atoms of a Poisson measure with intensity $I_{\beta,Y}(\mathrm{d}\mathbf{s})\mathrm{d}t$, $t \geq 0$. Call $\mathcal{J}_Y$ the set of jump times of Y and $\mathcal{J}_{T_{(\beta)}}$ that of $T_{(\beta)}$. Then fix f a positive measurable function on $\mathcal{D}$ and $t > 0$. Using the independence of Y and $T_{(\beta)}$, Fubini's theorem and that $(T_{(\beta)}(x), x \geq 0) \overset{\text{law}}{=} (x^\beta T_{(\beta)}(1), x \geq 0)$, we get

$$\begin{aligned} E\left[\sum\nolimits_{t_i \in \mathcal{J}_Y \cap [0,t]} f\left(\mathbf{s}(t_i)\right)\right] &= E\left[\sum\nolimits_{t_i \in \mathcal{J}_Y \cap [0,t]} f\left(\Delta_{T_{(\beta)}}([0, \Delta_Y(t_i)])\right)\right] \\ &= t\int_0^\infty E\left[f(x^\beta \Delta_{T_{(\beta)}}([0,1]))\right] \pi_Y(\mathrm{d}x). \end{aligned}$$

Next, set $\mathrm{Im}Y := \{Y(x), x \geq 0\}$. Again by independence of Y and $T_{(\beta)}$,

$$\begin{aligned} E\left[\sum\nolimits_{s_i \in \mathcal{J}_{T_{(\beta)}} \cap [0, Y(t)] \cap \mathrm{Im}Y} f((\Delta_{T_{(\beta)}}(s_i), 0, ..))\right] &= C_\beta \int_0^\infty f(x, 0, ..) x^{-1-1/\beta} \mathrm{d}x \\ &\quad \times E\left[\int_0^{Y(t)} \mathbf{1}_{\{s \in \mathrm{Im}Y\}} \mathrm{d}s\right]. \end{aligned}$$

Since $\int_0^{Y(t)} \mathbf{1}_{\{s \in \mathrm{Im}Y\}} \mathrm{d}s \overset{\text{a.s.}}{=} t d_Y$, the combination of computations above yields

$$E\left[\sum\nolimits_{0 \leq t_i \leq t} f(\mathbf{s}(t_i))\right] = t\int_0^\infty E\left[f(x^\beta \Delta_{T_{(\beta)}}([0,1]))\right] \pi_Y(\mathrm{d}x) + t d_Y C_\beta \int_0^\infty f(x, 0, ...) x^{-1-1/\beta} \mathrm{d}x$$

and $I_{\beta,Y}$ has the required form. It is easy to check that $I_{\beta,Y}$ satisfies the hypothesis (H1). ■

Let us now apply the results of Sections 4.2 and 4.3 to the fragmentation with immigration $FI_{(\beta,Y)}$. First, we want to apply Theorem 4.1. To do so, note that when $\gamma < 1/\beta$,

$$\int_{l^1} \sum\nolimits_{j \geq 1} s_j^\gamma \mathbf{1}_{\{s_j \geq 1\}} I_{\beta,Y}(\mathrm{d}\mathbf{s}) = \frac{C_\beta E\left[Y(1)\right]}{\gamma - 1/\beta}$$

which is finite (provided that $E\left[Y(1)\right] < \infty$). When $\beta < 2$, this holds in particular for $\gamma = -\alpha_\beta = 1 - 1/\beta$. On the other hand, the jumps of $T_{(\beta)}$ before time 1 being the atoms of a Poisson measure with intensity $C_\beta x^{-1-1/\beta}\mathrm{d}x$, $x > 0$, the distribution of the largest jump $\Delta_{T_{(\beta)}}^{\text{large}}$ is given by $P(\Delta_{T_{(\beta)}}^{\text{large}} \leq x) = \exp(-C_\beta \beta x^{-1/\beta})$ and consequently $\int_{l^1} s_1^\gamma \mathbf{1}_{\{s_1 \geq 1\}} I_{\beta,Y}(\mathrm{d}\mathbf{s}) = \infty$ when $\gamma \geq 1/\beta$, for any subordinator $Y \neq 0$. In particular $\int_{l^1} s_1^{-\alpha_2} \mathbf{1}_{\{s_1 \geq 1\}} I_{2,Y}(\mathrm{d}\mathbf{s}) = \infty$. Hence, by Theorem 4.1, $\mathbf{U}_{\text{stat}} \in \mathcal{D}$ a.s. $\Leftrightarrow \beta < 2$.

Second, in order to apply Corollary 4.1 (i) to obtain the rate of convergence to the stationary distribution, note that

$$\int_{l^1} \sum_{j\geq 1} \mathbf{1}_{\{s_j \geq x\}} I_{\beta,Y}(\mathrm{d}\mathbf{s}) = x^{-1/\beta}\beta C_\beta E\left[Y(1)\right].$$

This yields the following result.

Corollary 4.2 *Suppose* $E\left[Y(1)\right] < \infty$ *and* $Y \neq 0$. *Then the fragmentation with immigration* $(1/\beta - 1, 0, \nu_\beta, I_{\beta,Y})$ *has a stationary distribution if and only if* $\beta < 2$. *When* $\beta < 2$, *the stationary state* $\mathbf{U}_{\mathrm{stat}}$ *belongs to* l^p *for every* $p > 1/\beta$ *a.s. Moreover, a fragmentation with immigration* $FI^{(\mathbf{u})}$ *with parameters* $(1/\beta - 1, 0, \nu_\beta, I_{\beta,Y})$ *starting from* $\mathbf{u} \in \mathcal{D}$ *such that* $\sum_{j\geq 1} \exp(-u_j^{1/\beta-1}) < \infty$, *converges in law to* $\mathcal{L}(\mathbf{U}_{\mathrm{stat}})$ *at rate*

$$\left\| \mathcal{L}(FI^{(\mathbf{u})}(t)) - \mathcal{L}(\mathbf{U}_{\mathrm{stat}}) \right\| = O(t^{-\frac{2-\beta}{\beta-1}}) \text{ as } t \to \infty.$$

Moreover one checks that $\mathbf{U}_{\mathrm{stat}} \notin l^{1/\beta}$ a.s. as soon as $d_Y > 0$ or $\int_0 x^{\beta-1}\pi_Y(\mathrm{d}x) = \infty$; see Proposition 4.1 (ii).

Remark. Let $FI^{(\mathbf{0})}$ be an (α, c, ν, I) fragmentation starting from $\mathbf{0}$. As in the above examples, it is always possible to find a positive function $h_{FI^{(\mathbf{0})}}$ on $[0,\infty)$ such that, writing $\overline{FI}^{(\mathbf{0})}(t)$ for the decreasing rearrangement of lengths of connected components of $\{0 \leq x \leq R(t) : h_{FI^{(\mathbf{0})}}(x) > t\}$, $R(t) := \sup\{x \geq 0 : h_{FI^{(\mathbf{0})}}(x) \leq t\}$, then the process $\overline{FI}^{(\mathbf{0})}$ has same law as $FI^{(\mathbf{0})}$. Indeed, let $((\mathbf{s}(t_i), t_i), i \geq 1)$ be the atoms of a Poisson measure with intensity $I(\mathrm{d}\mathbf{s})\mathrm{d}t, t \geq 0$, and define

$$h_I(x) := \inf\left\{t \geq 0 : \sum_{t_i \leq t} \sum_{j \geq 1} s_j(t_i) > x\right\}.$$

This function h_I is continuous if and only if $I(l^1) = \infty$. Next, conditionally on $((\mathbf{s}(t_i), t_i), i \geq 1)$, let $F^{(s_j(t_i))}$, $i, j \geq 1$, be independent fragmentation processes starting respectively from $(s_j(t_i), 0, ...)$, $i, j \geq 1$. It is known ([14],[9]) that there exist some functions $h_{i,j}$ such that $F^{(s_j(t_i))}$ has same law as $\overline{F}^{(s_j(t_i))}$, where $\overline{F}^{(s_j(t_i))}(t)$, $t \geq 0$, is the decreasing rearrangement of lengths of connected components of $\{0 \leq x \leq s_j(t_i) : h_{i,j}(x) > t\}$. The idea, then, is to "put" the functions $h_{i,j}$, $i, j \geq 1$, on h_I, and a natural way to do this is to put them in exchangeable random order as follows: let $(U_{i,j}, i, j \geq 1)$ be a sequence of i.i.d uniform random variables, independent of the $h_{i,j}$'s, $i, j \geq 1$, and h_I. For a fixed i, say that $j \prec_i j'$ if $U_{i,j} < U_{i,j'}$. Then, for $x \in [0, \sum_{j\geq 1} s_j(t_i))$, there exists a unique integer, let us denote it by j_x, such that $\sum_{j \prec_i j_x} s_j(t_i) \leq x < \sum_{j \prec_i j_x} s_j(t_i) + s_{j_x}(t_i)$. Now, call $h_{F,i} : x \mapsto h_{i,j_x}(x)$, $0 \leq x < \sum_{j\geq 1} s_j(t_i)$, and introduce

$$h_{FI^{(\mathbf{0})}}(x) := \sum_{i\geq 1} \mathbf{1}_{\{h_I(x) = t_i\}}(t_i + h_{F,i}(x - h_I^{-1}(t_i-))), \ x \geq 0.$$

This function codes the fragmentation with immigration $FI^{(\mathbf{0})}$ in the sense required above. Moreover, when $c = \nu(\sum_{j\geq 1} s_j < 1) = 0$, one knows (see Theorem 3, [40]) that it is possible to choose some *continuous* functions $h_{i,j}$ to code the fragmentations $F^{(s_j(t_i))}$, $i, j \geq 1$, if and only if $\nu(\mathcal{D}_1) = \infty$. Consequently, it is possible to construct a continuous function $h_{FI^{(\mathbf{0})}}$ to code the process $FI^{(\mathbf{0})}$ if and only if $I(l^1) = \infty$ and $\nu(\mathcal{D}_1) = \infty$.

As in the examples of fragmentations with immigration constructed from $\mathrm{BES}^0(3,d)$ processes, it is easy to see that the law of stationary state $\mathbf{U}_{\mathrm{stat}}$ of a fragmentation with immigration FI is obtained by reflecting the function $h_{FI^{(\mathbf{0})}}$ at the level of its future infimum h_I and by considering the family of lengths of the excursions below 0 of the process obtained by this reflection.

4.5 The fragmentation with immigration equation

The deterministic counterpart of the fragmentation with immigration process (α, c, ν, I) is the following equation, namely the *fragmentation with immigration equation* (α, c, ν, I)

$$\begin{aligned} \partial_t \langle \mu_t, f \rangle &= \int_0^\infty x^\alpha \left(-cxf'(x) + \int_{\mathcal{D}_1} \left[\sum\nolimits_{j \geq 1} f(xs_j) - f(x) \right] \nu(\mathrm{d}\mathbf{s}) \right) \mu_t(\mathrm{d}x) \\ &\quad + \int_{l^1} \sum\nolimits_{j\geq 1} f(s_j) I(\mathrm{d}\mathbf{s}) \end{aligned} \tag{E}$$

where $(\mu_t, t \geq 0)$ is a family of non-negative Radon measures on $(0,\infty)$. The measure $\mu_t(\mathrm{d}x)$ corresponds to the average number per unit volume of particles with mass in the interval $(x, x+\mathrm{d}x)$ at time t. The test-functions f belong to $\mathcal{C}_c^1(0,\infty)$, the set of continuously differentiable functions with compact support in $(0,\infty)$. Note that the hypothesis (H1) implies the finiteness of the integral $\int_{l^1} \sum_{j\geq 1} f(s_j) I(\mathrm{d}\mathbf{s})$ for every $f \in \mathcal{C}_c^1(0,\infty)$. In [8], the stationary solution to this equation is studied in the special case when $\alpha = 1$, $c = 0$, $\nu(s_1 \in \mathrm{d}x) = 2\mathbf{1}_{\{x \in [1/2,1]\}}\mathrm{d}x$ and $\nu(s_1 + s_2 < 1) = 0$, $I(s_2 > 0) = 0$ and $I(s_1 \in \mathrm{d}x) = i(x)\mathrm{d}x$ for some measurable function i. Here we investigate solutions and stationary solutions to (E) in the general case.

4.5.1 Solutions to (E)

When $I = 0$, existence and uniqueness of a solution to equation (E) starting from $\delta_1(\mathrm{d}x)$ are established in Theorem 3, [38]. More precisely, the unique solution to the equation starting from $\delta_1(\mathrm{d}x)$ is given for all $t \geq 0$ by

$$\langle \eta_t, f \rangle := E\left[\sum\nolimits_{k\geq 1} f(F_k(t)) \right], \ f \in \mathcal{C}_c^1(0,\infty), \tag{4.25}$$

where F is a standard fragmentation process (α, c, ν). Now, we generalize this to the case when $I \neq 0$. In that aim, we recall that some fragmentation with immigration processes starting from $\mathbf{u} \in \mathcal{R}$ were introduced in (4.9). Recall also that ϕ is the Laplace exponent given by (4.2) and that $\overline{\phi} = \phi - \phi(0)$.

Proposition 4.4 *Let μ_0 be a non-negative Radon measure on $(0,\infty)$ and let $\mathbf{u}$ be a Poisson measure with intensity μ_0. Consider then an (α, c, ν, I) fragmentation with immigration*

$(FI^{(\mathbf{u})}(t), t \geq 0)$ *as introduced in* (4.9) *and define a family of non-negative measures* $(\mu_t, t \geq 0)$ *by*

$$\langle \mu_t, f \rangle := E\left[\sum\nolimits_{k \geq 1} f(FI_k^{(\mathbf{u})}(t))\right], \quad f \in \mathcal{C}_c^1(0,\infty), \ f \geq 0. \tag{4.26}$$

If one of the three following assertions is satisfied

(A1) $\alpha > 0$, $\int_{l^1} \sum_{j \geq 1} s_j I(\mathrm{d}\mathbf{s}) < \infty$ *and* $\int_1^\infty x \mu_0(\mathrm{d}x) < \infty$

(A2) $\alpha = 0$, $\int_{l^1} \sum_{j \geq 1} s_j \overline{\phi}(\frac{1}{\ln s_j}) \mathbf{1}_{\{s_j \geq 1\}} I(\mathrm{d}\mathbf{s}) < \infty$ *and* $\int_1^\infty x \overline{\phi}(\frac{1}{\ln x}) \mu_0(\mathrm{d}x) < \infty$

(A3) $\alpha < 0$, $\int_{l^1} \sum_{j \geq 1} s_j^{1+\alpha} \mathbf{1}_{\{s_j \geq 1\}} I(\mathrm{d}\mathbf{s}) < \infty$ *and* $\int_1^\infty x^{1+\alpha} \mu_0(\mathrm{d}x) < \infty$,

then the measures μ_t, $t \geq 0$, *are Radon and the family* $(\mu_t, t \geq 0)$ *is the unique solution to the fragmentation with immigration equation* (E) *starting from* μ_0.

Of course, $FI^{(\mathbf{u})}$ is a "usual" $\mathcal{D}$-valued fragmentation with immigration process as soon as $\mu_0 [1, \infty) < \infty$.

Remarks. 1) Notice that for all $f \in \mathcal{C}_c^1(0,\infty)$, $f \geq 0$,

$$\langle \mu_t, f \rangle = E\left[\sum\nolimits_{i \geq 1} \sum\nolimits_{k \geq 1} f\left(u_i F_k\left(u_i^\alpha t\right)\right)\right] + E\left[\sum\nolimits_{t_i \leq t} \sum\nolimits_{j \geq 1} \sum\nolimits_{k \geq 1} f(s_j(t_i) F_k(s_j^\alpha(t_i)(t - t_i)))\right],$$

where $((\mathbf{s}(t_i), t_i), i \geq 1)$ (resp. $(u_i, i \geq 1)$) are the atoms of a Poisson measure with intensity $I(\mathrm{d}\mathbf{s})\mathrm{d}t$ (resp. μ_0) and F is an (α, c, ν)-fragmentation, independent of these Poisson measures. By formula (4.5), this rewrites

$$\begin{aligned} \langle \mu_t, f \rangle &= \int_0^\infty E\left[f(x \exp(-\xi(\rho(x^\alpha t)))) \exp(\xi(\rho(x^\alpha t)))\right] \mu_0(\mathrm{d}x) \\ &\quad + \int_0^t \int_{l^1} \sum\nolimits_{j \geq 1} E\left[f(s_j \exp(-\xi(\rho(s_j^\alpha u)))) \exp(\xi(\rho(s_j^\alpha u)))\right] I(\mathrm{d}\mathbf{s})\mathrm{d}u \end{aligned} \tag{4.27}$$

where ξ is a subordinator with Laplace exponent ϕ. It is not hard to see that there exists some dislocation measures $\nu_1 \neq \nu_2$ that lead to the same ϕ. In this case, the previous formula shows that the (α, c, ν_1, I) and (α, c, ν_2, I) fragmentation with immigration equations have identical solutions.

2) Assume that one of the assertions (A1), (A2) and (A3) is satisfied, so that the measures $\mu_t, t \geq 0$, are Radon. Then, these measures are hydrodynamic limits of fragmentation with immigration processes. Indeed, let $\mathbf{u}^{(n)}$ be a Poisson measure with intensity $n\mu_0$ and call $FI^{(n)}$ a fragmentation with immigration process with parameters (α, c, ν, nI) starting from $\mathbf{u}^{(n)}$. Then, for every $t \geq 0$,

$$\frac{1}{n} FI^{(n)}(t) \overset{\text{vaguely}}{\to} \mu_t(\mathrm{d}x) \text{ a.s.}$$

This holds because $FI^{(n)}(t)$ is the sum of n i.i.d point measures distributed as $FI^{(\mathbf{u}^{(1)})}(t)$ for some (α, c, ν, I) fragmentation with immigration $FI^{(\mathbf{u}^{(1)})}$. The strong law of large numbers then implies that for every $f \in \mathcal{C}_c^1(0,\infty)$

$$\frac{1}{n} \sum\nolimits_{k \geq 1} f(FI_k^{(n)}(t)) \overset{\text{a.s.}}{\to} E\left[\sum\nolimits_{k \geq 1} f(FI_k^{(\mathbf{u}^{(1)})}(t))\right] = \langle \mu_t, f \rangle$$

and the conclusion follows by inverting the order of "for every $f \in \mathcal{C}_c^1(0,\infty)$" and "a.s.", which can be done e.g. as in the proof of Corollary 5 of [38].

Proof of Proposition 4.4. Let $\mu_t, t \geq 0$, be defined by (4.27) (equivalently (4.26)).

• It is easily seen that these measures are Radon if (A1) holds. To prove this is also valid for assertions (A2) or (A3), we need to evaluate the rate of convergence to 0 of $P(a \leq x\exp(-\xi(\rho(x^\alpha t))) \leq b)$ as $x \to \infty$, $0 < a < b < \infty$, when $\alpha \leq 0$. First, note that this probability is bounded from above by $P(x\exp(-\overline{\xi}(\rho(x^\alpha t))) \leq b)$ where $\overline{\xi} = \xi \mathbf{1}_{\{\xi<\infty\}}$ is a subordinator with Laplace exponent $\overline{\phi} = \phi - \phi(0)$. Then for $u \geq 0$ and $v > 0$,

$$\begin{aligned} P(\overline{\xi}(u) > v) &\leq (1-e^{-1})^{-1} E\left[1-\exp(-v^{-1}\overline{\xi}(u))\right] \\ &= (1-e^{-1})^{-1}\left(1-\exp(-u\overline{\phi}(v^{-1})\right). \end{aligned} \tag{4.28}$$

When $\alpha = 0$, this implies that

$$P(a \leq x\exp(-\xi(t)) \leq b) = O(\overline{\phi}((\ln x)^{-1})) \text{ as } x \to \infty. \tag{4.29}$$

When $\alpha < 0$, by definition of ρ and conditionally on $2x^\alpha t \leq \rho(x^\alpha t) < \infty$,

$$2x^\alpha t \exp(\alpha\overline{\xi}(2x^\alpha t)) \leq \int_0^{2x^\alpha t} \exp(\alpha\overline{\xi}(r))\mathrm{d}r \leq \int_0^{\rho(x^\alpha t)} \exp(\alpha\overline{\xi}(r))\mathrm{d}r = x^\alpha t$$

and consequently, $P(2x^\alpha t \leq \rho(x^\alpha t) < \infty) \leq P(\exp(\alpha\overline{\xi}(2x^\alpha t)) \leq 1/2)$ which, by (4.28), is a $O(x^\alpha)$ as $x \to \infty$. Moreover, again by (4.28), $P(x\exp(-\overline{\xi}(2x^\alpha t)) \leq b) = O(x^\alpha)$ and therefore,

$$P(a \leq x\exp(-\xi(\rho(x^\alpha t)) \leq b) = O(x^\alpha) \text{ as } x \to \infty \tag{4.30}$$

since

$$P(a \leq x\exp(-\xi(\rho(x^\alpha t)) \leq b) \leq P(2x^\alpha t \leq \rho(x^\alpha t) < \infty) + P(x\exp(-\overline{\xi}(2x^\alpha t)) \leq b).$$

Now, suppose that (A2) or (A3) holds and take $f(x) = x\mathbf{1}_{\{x\in(a,b)\}}$, $0 < a < b < \infty$. Using the results (4.29) and (4.30), one sees that $\langle \mu_t, f\rangle$ is finite. Hence μ_t is Radon.

• Suppose that (A1), (A2) or (A3) holds, so that the measures $\mu_t, t \geq 0$, are Radon. Consider then the measures η_t, $t \geq 0$, introduced in (4.25). One checks that

$$\langle \mu_t, f\rangle = \int_0^\infty \langle \eta_{x^\alpha t}, f_x\rangle\, \mu_0(\mathrm{d}x) + \int_0^t \int_{l^1} \sum\nolimits_{j\geq 1} \langle \eta_{s_j^\alpha u}, f_{s_j}\rangle I(\mathrm{d}\mathbf{s})\mathrm{d}u$$

where $f_x : y \mapsto f(xy)$, $x \in (0,\infty)$, $f \in \mathcal{C}_c^1(0,\infty)$. Theorem 3 in [38] states that $(\eta_t, t \geq 0)$ is a solution to (E) when $I = 0$, i.e.

$$\langle \eta_t, f\rangle = f(1) + \int_0^t \langle \eta_v, Af\rangle\, \mathrm{d}v$$

where

$$Af(x) = x^\alpha\left(-cxf'(x) + \int_{\mathcal{D}_1}\left[\sum\nolimits_{j\geq 1} f(xs_j) - f(x)\right]\nu(\mathrm{d}\mathbf{s})\right). \tag{4.31}$$

This equation relies on the fact that for $f \in \mathcal{C}_c^1(0,\infty)$, $A(\mathrm{id} \times f)(x) = x^{1+\alpha} G(f)(x)$ where G is the infinitesimal generator of the process $\exp(-\xi)$ (see the proof of Th.3, [38] for details).

Using then that $x^\alpha A f_x = (Af)_x$, one obtains

$$\langle \eta_{x^\alpha t}, f_x \rangle = f(x) + \int_0^t \langle \eta_{x^\alpha v}, (Af)_x \rangle \,\mathrm{d}v \tag{4.32}$$

and therefore, by Fubini's Theorem[1],

$$\begin{aligned} \langle \mu_t, f \rangle &= \langle \mu_0, f \rangle + \textstyle\int_0^t \int_0^\infty \langle \eta_{x^\alpha u}, (Af)_x \rangle \,\mu_0(\mathrm{d}x)\mathrm{d}u \\ &\quad + \textstyle\int_0^t \Big(\int_0^u \int_{l^1} \sum_{j \geq 1} \langle \eta_{s_j^\alpha v}, (Af)_{s_j} \rangle I(\mathrm{d}\mathbf{s})\mathrm{d}v + \int_{l^1} \sum_{j\geq 1} f(s_j) I(\mathrm{d}\mathbf{s}) \Big) \\ &= \langle \mu_0, f \rangle + \textstyle\int_0^t \langle \mu_u, Af \rangle \,\mathrm{d}u + t \int_{l^1} \sum_{j \geq 1} f(s_j) I(\mathrm{d}\mathbf{s}). \end{aligned}$$

Hence $(\mu_t, t \geq 0)$ is indeed a solution to (E). It remains to prove the uniqueness. This can be done with some minor changes by adapting the proof of uniqueness of a solution to the equation (E) when $I = 0$ (see the third part of the proof of Theorem 3, [38]). ■

4.5.2 Stationary solutions to (E)

As in the stochastic case, we are interested in the existence of a stationary regime. We say that a Radon measure μ_{stat} is a stationary solution to (E) if the family $(\mu_t = \mu_{\mathrm{stat}}, t \geq 0)$ is a solution to (E).

Proposition 4.5 (i) *There is a stationary solution to (E) as soon as* $\int_{l^1} \sum_{j \geq 1} s_j I(\mathrm{d}\mathbf{s}) < \infty$ *and conversely, provided that hypothesis (H2) holds, there is no stationary solution to (E) when* $\int_{l^1} \sum_{j\geq 1} s_j I(\mathrm{d}\mathbf{s}) = \infty$. *In case* $\int_{l^1} \sum_{j \geq 1} s_j I(\mathrm{d}\mathbf{s}) < \infty$, *the stationary solution* μ_{stat} *is unique and given by*

$$\mu_{\mathrm{stat}}(\mathrm{d}x) := x^{-\alpha} \mu_{\mathrm{stat}}^{(\mathrm{hom})}(\mathrm{d}x), \quad x \geq 0,$$

where the measure $\mu_{\mathrm{stat}}^{(\mathrm{hom})}$ *is independent of* α *and is constructed from* c, ν *and* I *by*

$$\langle \mu_{\mathrm{stat}}^{(\mathrm{hom})}, f \rangle := \int_0^\infty \int_{l^1} \sum_{j \geq 1} E\left[f(s_j \exp(-\xi(t))) \exp(\xi(t)) \right] I(\mathrm{d}\mathbf{s})\mathrm{d}t, \quad f \in \mathcal{C}_c^1(0,\infty). \tag{4.33}$$

(ii) *Suppose* $\int_{l^1} \sum_{j\geq 1} s_j I(\mathrm{d}\mathbf{s}) < \infty$ *and* $\int_1^\infty x \mu_0(\mathrm{d}x) < \infty$ *and let* $(\mu_t, t \geq 0)$ *be the solution to (E) starting from* μ_0. *Then,*

$$\mu_t \overset{\text{vaguely}}{\to} \mu_{\mathrm{stat}} \text{ as } t \to \infty.$$

[1] Call $[a,b]$ the support of f and suppose $f \geq 0$. Write $(Af)_x(y) = Af(xy)\mathbf{1}_{\{xy>b\}} + Af(xy)\mathbf{1}_{\{a \leq xy \leq b\}}$. On the one hand, $Af(xy)\mathbf{1}_{\{xy>b\}} \geq 0$ and Fubini's Theorem holds for this function. On the other hand, $\left|Af(xy)\mathbf{1}_{\{a\leq xy\leq b\}}\right| \leq C\mathbf{1}_{\{a \leq xy \leq b\}}$ for some constant C since Af is continuous and $\int_0^\infty \int_0^t \langle \eta_{x^\alpha u}, \mathbf{1}_{\{a \leq xy \leq b\}} \rangle \mathrm{d}u \mu_0(\mathrm{d}x) < \infty$ according to the assumptions made on μ_0. Hence Fubini's Theorem applies to $Af(xy)\mathbf{1}_{\{a\leq xy\leq b\}}$ and then to $(Af)_x$. The same argument holds for the integral involving I.

Remarks. 1) It μ_{stat} exists, then $\mathbf{U}_{\text{stat}} \in \mathcal{R}$ a.s. and the distribution $\mathcal{L}(\mathbf{U}_{\text{stat}})$ is linked to μ_{stat} by

$$\langle \mu_{\text{stat}}, f \rangle = \int_{\mathcal{R}} \sum\nolimits_{j\geq 1} f(s_j)\mathcal{L}(\mathbf{U}_{\text{stat}})(\mathrm{d}\mathbf{s}),\ f \in \mathcal{C}_c^1(0,\infty).$$

2) Call $\Lambda := \sup\{\lambda : \int_{l^1} \sum_{j\geq 1} s_j^{\lambda} I(\mathrm{d}\mathbf{s}) < \infty\}$ and suppose $\Lambda > 1$. Then the statement (i) and the relations $E\left[e^{-q\xi(t)}\right] = e^{-t\phi(q)}$, $t, q \geq 0$, imply that for all $1+\alpha < \lambda < \Lambda + \alpha$,

$$\int_0^{\infty} x^{\lambda} \mu_{\text{stat}}(\mathrm{d}x) = \phi(\lambda - \alpha - 1)^{-1} \int_{l^1} \sum\nolimits_{j\geq 1} s_j^{\lambda-\alpha} I(\mathrm{d}\mathbf{s}), \tag{4.34}$$

and that this integral is infinite as soon as $\lambda > \Lambda + \alpha$ or $\lambda \leq 1 + \alpha$, provided $\phi(0) = 0$ (which is equivalent to $c = \nu(\sum_{j\geq 1} s_j < 1) = 0$). This characterizes μ_{stat} and is more explicit than (4.33).

As an example, it allows us to obtain the more convenient expression

$$\mu_{\text{stat}}(\mathrm{d}x) = \left(x^{-\alpha} i(x) + 2x^{-\alpha-2} \int_x^{\infty} y i(y) \mathrm{d}y \right) \mathrm{d}x$$

in case ν is binary, $\nu(s_1 \in \mathrm{d}x) = 2\mathbf{1}_{\{x\in[1/2,1]\}}\mathrm{d}x$, $c = 0$, and $I(s_1 \in \mathrm{d}x) = i(x)\mathrm{d}x$, $I(s_2 > 0) = 0$ ($\alpha \in \mathbb{R}$). This latter result is proved in a different way in [8].

Others examples are given by the equations corresponding to the fragmentation with immigration processes constructed from Brownian motions with drift $d > 0$ (Section 4.4.1). The immigration measure $I_{(d)}$ satisfies $\int_{l^1} \sum_{j\geq 1} s_j^{\lambda} I_{(d)}(\mathrm{d}\mathbf{s}) < \infty$ for all $\lambda > 1/2$ and therefore there exists a stationary solution to the equation. One can use formula (4.34) to obtain

$$\mu_{\text{stat}}(\mathrm{d}x) = \frac{1}{d\sqrt{8\pi x^3}} \exp(-xd^2/2)\mathrm{d}x,\ x \geq 0.$$

This can also be shown by using remark 1) above and the stationary law $\mathcal{L}(\mathbf{U}_{\text{stat}})$ given in Proposition 4.2 (ii).

3) For fragmentations with immigration $(1/\beta - 1, 0, \nu_\beta, I_{\beta,Y})$ constructed from height processes (Section 4.4.2), the immigration term satisfies

$$\int_{l^1} \sum\nolimits_{j\geq 1} f(s_j) I_{\beta,Y}(\mathrm{d}\mathbf{s}) = E\left[Y(1)\right] \int_0^{\infty} f(x) C_\beta x^{-1-1/\beta} \mathrm{d}x$$

which shows the small influence of Y on the equation. Moreover, the latter integral is infinite when $f = \mathrm{id}$ and one checks that the hypothesis (H2) holds, which implies that for all $1 < \beta \leq 2$, the equation does not have a stationary solution.

Proof of Proposition 4.5. **(i)** We first suppose that there exists a stationary solution $\mu_t = \mu_{\text{stat}}$, $t \geq 0$, to the equation (E). Of course then $\partial_t \langle \mu_t, f \rangle = 0$ for every $t \geq 0$ and $f \in \mathcal{C}_c^1(0,\infty)$, and consequently

$$\langle \mu_{\text{stat}}, Af \rangle = -\int_{l^1} \sum\nolimits_{j\geq 1} f(s_j) I(\mathrm{d}\mathbf{s})$$

where Af is given by (4.31). Letting $t \to \infty$ in (4.32), we get by dominated convergence that $\langle \eta_{x^\alpha t}, f_x \rangle \to 0$ and then that $f(x) = -\int_0^\infty \langle \eta_{x^\alpha v}, (Af)_x \rangle \,\mathrm{d}v$, $x \in (0, \infty)$. Hence

$$\langle \mu_{\text{stat}}, Af \rangle = \int_{l^1} \sum\nolimits_{j \geq 1} \int_0^\infty \langle \eta_{s_j^\alpha v}, (Af)_{s_j} \rangle \mathrm{d}v I(\mathrm{d}\mathbf{s}).$$

We point out that this formula characterizes μ_{stat}, since $A(\mathrm{id} \times f)(x) = x^{1+\alpha} G(f)(x)$ where G is the infinitesimal generator of $\exp(-\xi)$ and since $G\left(\mathcal{C}_c^1(0,\infty)\right)$ is dense in the set of continuous functions on $(0,\infty)$ that vanish at 0 and ∞. Using then the definition of η_t and formula (4.5), one sees that for every measurable function g with compact support in $(0, \infty)$

$$\begin{aligned} \langle \mu_{\text{stat}}, g \rangle &= \int_{l^1} \sum\nolimits_{j \geq 1} \int_0^\infty E\left[g(s_j \exp(-\xi(\rho(s_j^\alpha v)))) \exp(\xi(\rho(s_j^\alpha v))) \right] \mathrm{d}v I(\mathrm{d}\mathbf{s}) \qquad (4.35) \\ &= \int_{l^1} \sum\nolimits_{j \geq 1} s_j^{-\alpha} \int_0^\infty E\left[g(s_j \exp(-\xi(v))) \exp((1+\alpha)\xi(v)) \right] \mathrm{d}v I(\mathrm{d}\mathbf{s}) \end{aligned}$$

using for the last equality the change of variables $v \mapsto \rho(s_j^\alpha v)$ and that $\exp(\alpha \xi_{\rho(v)}) \mathrm{d}\rho(v) = \mathrm{d}v$ on $[0, D)$, $D = \inf\left\{ v : \xi_{\rho(v)} = \infty \right\}$. This gives the required expression for μ_{stat}.

Note now that the previous argument implies that a stationary solution exists if and only if

$$\int_{l^1} \sum\nolimits_{j \geq 1} \int_0^\infty E\left[g(s_j \exp(-\xi(v))) \exp(\xi(v)) \right] \mathrm{d}v I(\mathrm{d}\mathbf{s}) < \infty$$

for all functions g of type $g(x) = x \mathbf{1}_{\{a \leq x \leq b\}}$, $0 < a < b$. For such function g, the previous integral is equal to

$$\int_{l^1} \sum\nolimits_{j \geq 1} s_j \mathbf{1}_{\{s_j \geq a\}} E\left[T^\xi_{\ln(s_j/a)} - T^\xi_{\ln^+(s_j/b)} \right] I(\mathrm{d}\mathbf{s}) \qquad (4.36)$$

where $T_t^\xi := \inf\{u : \xi(u) > t\}$, $t \geq 0$. If hypothesis (H2) holds and ξ is arithmetic (that is if (H3) holds), the renewal theorem applies (see e.g. Theorem I.21, [10]) and $E[T^\xi_{\ln(t/a)} - T^\xi_{\ln^+(t/b)}]$ converges as $t \to \infty$ to some finite non-zero limit. In such case, the integral (4.36) is finite if and only if $\int_{l^1} \sum_{j \geq 1} s_j \mathbf{1}_{\{s_j \geq 1\}} I(\mathrm{d}\mathbf{s}) < \infty$, $\forall\, b > a > 0$, and therefore, there exists a stationary solution if and only if $\int_{l^1} \sum_{j \geq 1} s_j \mathbf{1}_{\{s_j \geq 1\}} I(\mathrm{d}\mathbf{s}) < \infty$. This conclusion remains valid if (H2) holds and ξ is not arithmetic, since the renewal theory then implies that $\limsup_{t \to \infty} E[T^\xi_{\ln(t/a)} - T^\xi_{\ln^+(t/b)}] < \infty$, and that $\liminf_{t \to \infty} E[T^\xi_{\ln(t/a)} - T^\xi_{\ln^+(t/b)}] > 0$ as soon as $\ln b - \ln a$ is large enough. Last, to conclude when (H2) does not hold, remark first that $T_t^\xi = T_t^{\overline{\xi}} \wedge \mathbf{e}(k)$ (the subordinator $\overline{\xi}$ and the exponential r.v. $\mathbf{e}(k)$ are those defined in Section 4.1.1) and then that

$$E\left[T^\xi_{\ln(s_j/a)} - T^\xi_{\ln^+(s_j/b)} \right] \leq E\left[T^{\overline{\xi}}_{\ln(s_j/a)} - T^{\overline{\xi}}_{\ln^+(s_j/b)} \right] \leq E\left[T^{\overline{\xi}}_{\ln(b/a)} \right] < \infty.$$

In this case, the integral (4.36) is finite as soon as $\int_{l^1} \sum_{j \geq 1} s_j \mathbf{1}_{\{s_j \geq 1\}} I(\mathrm{d}\mathbf{s}) < \infty$, $\forall\, b > a > 0$.

(ii) Under the assumptions of the statement, the measures μ_t, $t \geq 0$, are Radon and therefore satisfy (4.27) for all continuous function f with compact support in $(0, \infty)$. The integral involving μ_0 converges to 0 as $t \to \infty$, since, with the assumption $\int_1^\infty x \mu_0(\mathrm{d}x) < \infty$, the dominated convergence theorem applies. Hence $\langle \mu_t, f \rangle \underset{t \to \infty}{\to} \langle \mu_{\text{stat}}, f \rangle$, using the definition (4.35) of μ_{stat}. ■

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