Time Domain Finite Element Methods for Maxwell's Equations

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Outline

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- Formulation in linear case,
- Spatial discretization,
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Definitions

- The electric displacement field, denoted by $\mathbf{D}(t) = \vec{D}(\vec{r}, t)$, in units of coulomb per metre squared C/m^2.
- The magnetic induction $\mathbf{B}(t) = \vec{B}(\vec{r}, t)$ is measured in teslas or newtons per meter per ampere.
- $\mathbf{E}(t) = \vec{E}(\vec{r}, t)$ is the electric field in newtons per coulomb N/C or volts per meter V/m.
- $\mathbf{H}(t) = \vec{H}(\vec{r}, t)$ is the magnetic field in A/m.
- $\mathbf{J}(t) = \vec{J}(\vec{r}, t)$ is a surface electric current density measured in A/m.
- ρ is a surface electric charge density measured in C/m^2
  - The continuity equation, $\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$
- ε is a material permittivity in F/m (farad per meter). $\varepsilon = \varepsilon_0$ in vacuum,
- μ is a material permeabilty in H/m. $\mu = \mu_0$ in vacuum.
- Speed of light c (m/s) and characteristic impedance $\eta(\Omega)$.
  - $c = \frac{1}{\sqrt{\varepsilon\mu}}$ and $\eta = \sqrt{\frac{\mu}{\varepsilon}}$.

Notations

Furthermore, we need the following Hilbert spaces that are related to the rotation and divergence operators:

$$\begin{aligned}
\mathbf{H}(\mathrm{curl}\,;\Omega) &:= \{\mathbf{u} \in \mathbf{L}^2(\Omega);\quad \nabla \times \mathbf{u} \in \mathbf{L}^2(\Omega)\},\\
\mathbf{H}_0(\mathrm{curl}\,;\Omega) &:= \{\mathbf{u} \in \mathbf{H}(\mathrm{curl}\,;\Omega);\quad \mathbf{u} \times \mathbf{n}|_\Gamma = 0\},\\
\mathbf{H}(\mathrm{div};\Omega) &:= \{\mathbf{u} \in \mathbf{L}^2(\Omega);\quad \nabla \cdot \mathbf{u} \in \mathbf{L}^2(\Omega)\},\\
\mathbf{H}_0(\mathrm{div};\Omega) &:= \{\mathbf{u} \in \mathbf{L}^2(\Omega);\quad \mathbf{u} \cdot \mathbf{n}|_\Gamma = 0\}.
\end{aligned}$$

Maxwell's equations in nonlinear Optics

Ω be a volume in $\mathbb{R}^3$, with boundary Γ and unit outward normal $\mathbf{n} = \vec{n}$. $\mathbf{D}(t)$, $\mathbf{B}(t)$, $\mathbf{E}(t)$ and $\mathbf{H}(t)$ represent the displacement field, magnetic induction, electric and magnetic field intensities respectively, where the time variable t belongs to some interval $(0, T)$, $T > 0$. Given a current density function $\mathbf{J}(t)$, specifying the applied current.The time-dependent Maxwell equations for nonlinear medium as

$$\frac{\partial}{\partial t}\mathbf{D}(t) + \sigma\mathbf{E}(t) - \nabla \times \mathbf{H}(t) := \mathbf{J}(t) \text{ in } \Omega \times (0, T), \quad (1)$$

$$\frac{\partial}{\partial t}\mathbf{B}(t) + \nabla \times \mathbf{E}(t) := 0 \text{ in } \Omega \times (0, T), \quad (2)$$

the following constitutive relations shall hold:

$$\mathbf{B}(t) := \mu_0\mathbf{H}(t), \quad (3)$$

$$\mathbf{D}(t) := \varepsilon_0\mathbf{E}(t) + \mathbf{P}(\mathbf{E}(t)). \quad (4)$$

ε_0 and μ_0 are vacuum permittivity and permeability respectively. Generally, the constitutive relation $\mathbf{P}(\mathbf{E})$ is approximated by a Taylor series for nonlinear optics

Introduction

$$\mathbf{P}(\mathbf{E}(t)) := \varepsilon_0\Big(\chi^1(\vec{r})\mathbf{E}(t) + \chi^2(\vec{r})|\mathbf{E}(t)|^2 + \chi^3(\vec{r})\mathbf{E}(t)^3\Big). \tag{5}$$

- Restrict the model to isotropic materials so that the second term is eliminated due to inversion symmetry, and third term is to simplified as $\chi^3(\vec{r})\Big(\mathbf{E}(t)\cdot\mathbf{E}(t)\Big)\mathbf{E}(t)$. Rewrite $\mathbf{D}(t)$ as,

$$\frac{\partial}{\partial t}\mathbf{D}(t) = \Big(\mathbf{a}\mathbf{I}_3 + 2\chi^3(\vec{r})\mathbf{C}\Big)\frac{\partial}{\partial t}\mathbf{E}(t), \tag{6}$$

where $\mathbf{I}_3$ is a (3×3) unit matrix and $\mathbf{a} = \varepsilon_0 + \chi^3(\vec{r})|\mathbf{E}(t)|^2$. Furthermore introducing $\mathbf{C}$,

$$\mathbf{C} = \begin{pmatrix} \mathbf{E}_1^2 & \mathbf{E}_1\mathbf{E}_2 & \mathbf{E}_1\mathbf{E}_3 \\ \mathbf{E}_1\mathbf{E}_2 & \mathbf{E}_2^2 & \mathbf{E}_2\mathbf{E}_3 \\ \mathbf{E}_1\mathbf{E}_3 & \mathbf{E}_2\mathbf{E}_3 & \mathbf{E}_3^2 \end{pmatrix}.$$

Introduction

For simplicity we could rewrite displacement field,

$$\frac{\partial}{\partial t}\mathbf{D}(t) = \Big(\varepsilon(\mathbf{E})\Big)\frac{\partial}{\partial t}\mathbf{E}(t). \tag{7}$$

For the spatial case electric fields $\mathbf{E}(t)$ is also satisfied:

$$\Big(\mathbf{E}(t)\cdot\frac{\partial \mathbf{E}(t)}{\partial t}\Big)\mathbf{E}(t) = \Big(\mathbf{E}(t)\cdot\mathbf{E}(t)\Big)\frac{\partial \mathbf{E}(t)}{\partial t}. \tag{8}$$

We could also rewrite $\mathbf{D}(t)$ in case of (8):

$$\frac{\partial}{\partial t}\mathbf{D}(t) = \Big(\varepsilon_0 + 3\chi^3(\vec{r})\mathbf{E}(t)\cdot\mathbf{E}(t)\Big)\frac{\partial}{\partial t}\mathbf{E}(t). \tag{9}$$

We suppose $\chi(\vec{r}) > 0$ and $\varepsilon_0 + 3\chi^3(\vec{r})|\mathbf{E}(t)|^2 \neq 0$. This condition is fulfilled for $|3\chi^3(\vec{r})|\mathbf{E}(t)|^2| < |\varepsilon_0|$. For $\chi(\vec{r}) = 0$, we obtain the linear Maxwell's equations. Thus the nonlinear Maxwell's equations (1)-(4) can be written as:

Weak Formulations

$$\varepsilon(\mathbf{E})\frac{\partial}{\partial t}\mathbf{E}(t) + \sigma\mathbf{E}(t) - \nabla\times\mathbf{H}(t) = \mathbf{J}(t) \text{ in } \Omega\times(0,T), \tag{10}$$

$$\mu_0\frac{\partial}{\partial t}\mathbf{H}(t) + \nabla\times\mathbf{E}(t) = 0 \text{ in } \Omega\times(0,T), \tag{11}$$

$$\mathbf{n}\times\mathbf{E}(t) = 0. \tag{12}$$

Multiplying equation (10) by a test function $\Phi(t) \in \mathbf{U} = \mathbf{H}_0(\text{curl}\,;\Omega)$ and integrate over Ω. Similarly multiplying (11) by $\Psi(t) \in \mathbf{V} = \mathbf{H}(\text{div};\Omega)$ and integrate over Ω. Now, we can see that the solution $(\mathbf{E}(t),\mathbf{H}(t)) \in \left[C^1(0,T;\mathbf{U}) \cap C^1(0,T;\mathbf{V})\right]^2$ of $(1)-(2)$ satisfies:

$$(\varepsilon(\mathbf{E})\partial_t\mathbf{E}(t),\Phi(t)) + (\sigma\mathbf{E}(t),\Phi(t)) - (\mathbf{H}(t),\nabla\times\Phi(t)) = (\mathbf{J}(t),\Phi(t)) \quad \forall\,\Phi(t)\in\mathbf{U}, \tag{13}$$

$$(\mu_0\partial_t\mathbf{H}(t),\Psi(t)) + (\nabla\times\mathbf{E}(t),\Psi(t)) = 0 \quad \forall\,\Psi(t)\in\mathbf{V}, \tag{14}$$

for $0 < t < T$ with initial conditions:

$$\mathbf{E}(0) = \mathbf{E}_0 \quad \text{and} \quad \mathbf{H}(0) = \mathbf{H}_0, \tag{15}$$

Notations

Let $\mathcal{P}_K$ be the space of scalar real-valued polynomials in the three variables of maximum degree of k, and $\tilde{\mathcal{P}}_k$ be the space of scalar real-valued homogeneous polynomials of degree exactly k. For any integer $k \geq 1$ and we define the following subspaces of $\mathbf{P}_k := [\mathcal{P}_k]^3$.

$$\begin{aligned} \mathbf{D}_k &= \mathbf{P}_{k-1} \oplus \tilde{\mathcal{P}}_{k-1} \cdot r, \;\; r =< x_1, x_2, x_3 > \\ \mathbf{R}_k &= \mathbf{P}_{k-1} \oplus \mathbf{S}_k, \;\; \text{where} \\ \mathbf{S}_k &= \{\mathbf{p} \in (\tilde{P}_k)^3; \quad \mathbf{p}(\mathbf{x}) \cdot \mathbf{x} = 0\}. \end{aligned}$$

i.e $\mathbf{S}_k \subset \mathbf{P}_k$ and $\mathbf{R}_k \subset \mathbf{P}_k$.

$$\mathbf{U}_h = \{\mathbf{v}^h \in \mathbf{H}(\text{curl}\,; \Omega); \quad \mathbf{v}^h|_K \in \mathbf{R}_k \; \forall K \in \mathcal{T}_h\}, \tag{16}$$

$$\mathbf{V}_h = \{\mathbf{u}^h \in \mathbf{U}; \quad \mathbf{u}^h|_K \in \mathbf{D}_k \; \forall K \in \mathcal{T}_h\}. \tag{17}$$

Spatial discretization for nonlinear case

Let $\mathbf{U}_h \subset \mathbf{U}$ and $\mathbf{V}_h \subset \mathbf{V}$ be finite dimensional subspaces of given spaces. We may find $(\mathbf{E}_h(t), \mathbf{H}_h(t)) \in C^1(0, T; \mathbf{U}_h) \times C^1(0, T; \mathbf{V}_h)$ such that:

$$(\varepsilon(\mathbf{E}_h)\partial_t \mathbf{E}_h(t), \Phi_h(t)) + (\sigma \mathbf{E}_h(t), \Phi_h(t)) - (\mathbf{H}_h(t), \nabla \times \Phi_h(t)) = (\mathbf{J}_h(t), \Phi_h(t)) \quad \forall \Phi_h(t) \in \mathbf{U}_h, \tag{18}$$

$$(\mu_0 \partial_t \mathbf{H}_h(t), \Psi_h(t)) + (\nabla \times \mathbf{E}_h(t), \Psi_h(t)) = 0 \quad \forall\, \Psi_h(t) \in \mathbf{V}_h, \tag{19}$$

for $0 < t < T$, subject to the initial conditions:

$$\mathbf{E}_h(0) = \mathbf{E}_0 \text{ and } \mathbf{H}_h(0) = \mathbf{H}_0. \tag{20}$$

Weak Formulation and Spatial discretization for Linear case

For $\chi(\vec{r}) = 0$ in (10). Then, we obtain a linear Maxwell's equations, and time-independent dielectric permittivity ε, magnetic permeability μ and electric conductivity σ. The weak solution $(\mathbf{E}(t), \mathbf{H}(t))$ of the system (1)-(2) for linear Maxwell's equations satisfies,

$$(\varepsilon \mathbf{E}_t(t), \Phi(t)) + (\sigma \mathbf{E}(t), \Phi(t)) - (\mathbf{H}(t), \nabla \times \Phi(t)) = (\mathbf{J}(t), \Phi(t)) \quad \forall\, \Phi(t) \in \mathbf{H}_0(\mathrm{curl}\,; \Omega), \tag{21}$$

$$(\mu \mathbf{H}_t(t), \Psi(t) + (\nabla \times \mathbf{E}(t), \Psi(t)) = 0 \quad \forall\, \Psi(t) \in \mathbf{H}(\mathrm{div}; \Omega). \tag{22}$$

$\mathbf{U}_h \subset \mathbf{U}$ and $\mathbf{V}_h \subset \mathbf{V}$ be finite dimensional subspaces of given spaces. We may find $(\mathbf{E}_h(t), \mathbf{H}_h(t)) \in C^1(0, T; \mathbf{U}_h) \times C^1(0, T; \mathbf{V}_h)$ such that:

$$(\varepsilon \partial_t \mathbf{E}_h(t), \Phi_h(t)) + (\sigma \mathbf{E}_h(t), \Phi_h(t)) - (\mathbf{H}_h(t), \nabla \times \Phi_h(t)) = (\mathbf{J}(t), \Phi_h(t)) \quad , \forall\, \Phi_h(t) \in \mathbf{U}_h, \tag{23}$$

$$(\mu \partial_t \mathbf{H}_h(t), \Psi_h(t)) + (\nabla \times \mathbf{E}_h(t), \Psi_h(t)) = 0, \ \forall\, \Psi_h(t) \in \mathbf{V}_h, \tag{24}$$

for $0 < t < T$, subject to the initial conditions,

$$\mathbf{E}_h(0) = \mathbf{E}_0 \ \text{ and } \ \mathbf{H}_h(0) = \mathbf{H}_0. \tag{25}$$

Conserve Energy for linear case

The method (23)-(25) is conserve energy. Take $\mathbf{J}(t) = 0$, $\sigma = 0$ and choose $\Phi_h(t) = \mathbf{E}_h(t)$, and $\Psi_h(t) = \mathbf{H}_h(t)$ in (23)-(25) and adding (23)-(25), we obtain:

$$\frac{1}{2}\left(\frac{\partial}{\partial t}\|\mathbf{E}_h(t)\|_\varepsilon^2 + \frac{\partial}{\partial t}\|\mathbf{H}_h(t)\|_\mu^2\right) = 0. \tag{26}$$

Furthermore,

$$\|\mathbf{E}_h(t)\|_\varepsilon^2 + \|\mathbf{H}_h(t)\|_\mu^2 = \|\mathbf{E}_h(0)\|_\varepsilon^2 + \|\mathbf{H}_h(0)\|_\mu^2. \tag{27}$$

which states that energy in the discrete system is independent of time.

Matrices

The method uses edge finite elements as a basis for the electric field and face finite elements for the magnetic flux density. The edge elements have tangential continuity whereas the face elements have normal continuity across interfaces. The Matrices in these equations have the following form:

$$\{\mathbf{M}^1_\alpha\}_{ij} = \int_\Omega \alpha \Phi^1_i \cdot \Phi^1_j d\Omega,$$

$$\{\mathbf{M}^2_\alpha\}_{ij} = \int_\Omega \alpha \Phi^2_i \cdot \Phi^2_j d\Omega,$$

$$\{\mathbf{G}^{12}_\alpha\}_{ij} = \int_\Omega \alpha (\nabla \times \Phi^1_i) \cdot \Phi^2_j d\Omega.$$

Spatial Discretization for nonlinear case

This leads to the following semi-discrete matrix equations for liner case:

$$\mathbf{M}^{(1)}_{\textbf{Update}} \frac{\partial \mathbf{e}}{\partial t} + \mathbf{M}^{(1)}_{\sigma} \mathbf{e} = \left(\mathbf{G}^{(12)}\right)^{\top} \mathbf{h} + \mathbf{J}^{(1)}, \tag{28}$$

$$\mathbf{M}^{(2)}_{\mu} \frac{\partial \mathbf{h}}{\partial t} = -\mathbf{G}^{(12)} \mathbf{e}, \tag{29}$$

Here $\mathbf{e}$ and $\mathbf{h}$ are the electric fields and magnetic fields degrees of freedom with size $\dim \mathbf{U}_h$ and $\dim \mathbf{V}_h$ respectively. $\mathbf{M}^1_{\textbf{Update}}$ is the positive definite mass matrix with size $\dim \mathbf{U}_h \times \dim \mathbf{U}_h$. The matrix $\mathbf{M}^1_{\textbf{Update}}$ will have to update or calculate at each time step, e.g mass matrix $\mathbf{M}^1_{\textbf{Update}}$ is obtained at the time step n by the approximated value of electric field $\mathbf{e}$ at the time step $n-1$. $\mathbf{M}^1_{\sigma}$ is also positive definite matrix with size $\dim \mathbf{U}_h \times \dim \mathbf{U}_h$. $\mathbf{M}^2_{\mu}$ is also symmetric and positive definite with size $\dim \mathbf{V}_h \times \dim \mathbf{V}_h$. Vectors $\mathbf{e}$ and $\mathbf{h}$ have different dimensions. The $\mathbf{G}^{12}$ matrix is a discrete representation of curl with size $\dim \mathbf{V}_h \times \dim \mathbf{U}_h$. However, $\mathbf{G}^{12}$ matrix is rectangular. $\mathbf{J}^{(1)}$ is a discrete current source.

Spatial Discretization for linear case

This linear Maxwell's equations leads to the following semi-discrete matrix equations for liner case:

$$\mathbf{M}_{\varepsilon}^{(1)} \frac{\partial \mathbf{e}}{\partial t} + \mathbf{M}_{\sigma}^{(1)} \mathbf{e} = \left(\mathbf{G}^{(12)}\right)^{\top} \mathbf{h} + \mathbf{J}^{(1)}, \tag{30}$$

$$\mathbf{M}_{\mu}^{(2)} \frac{\partial \mathbf{h}}{\partial t} = -\mathbf{G}^{(12)} \mathbf{e}, \tag{31}$$

where $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ are the first 1-form and 2-form mass matrices respectively. The matrix $\mathbf{G}^{(12)}$ is a discrete representation of the curl operator. $\mathbf{e}$ and $\mathbf{h}$ are the vectors of electric fields and magnetic fields degrees of freedom. $\mathbf{J}^{(1)}$ is a discrete current source. Since the vectors $\mathbf{e}$ and $\mathbf{h}$ have different dimensions, $\mathbf{G}^{(12)}$ is rectangular.

Time discretization using symplectic method in nonlinear case

Compute the number of time steps. **nstep** $= \dfrac{t_{final} - t_0}{\Delta t}$

Set the initial conditions

$e_1 \leftarrow e_{Initial}$

$h_1 \leftarrow h_{Initial}$

loop over time steps.

for i=1 to nstep do

begin integration method update

$e_{in} \leftarrow e_i$

$h_{in} \leftarrow h_i$

update the field values

for j $= 1$ **to k do**

$$e_{out} \leftarrow e_{in} + \alpha_j \Delta t (\mathbf{M}^{(1)}_{\mathbf{Update}})^{-1} \left(\left(\mathbf{G}^{(12)} \right)^T h_{in} - \mathbf{M}^{(1)}_{\sigma} e_{in} + \mathbf{J}^{(1)} \right)$$

$$h_{out} \leftarrow h_{in} + \beta_j \Delta t (\mathbf{M}^{(2)}_{\mu})^{-1} \left(\mathbf{G}^{(12)} \right) e_{out}$$

Time discretization using symplectic method in nonlinear case

$e_{in} = e_{out}$

$h_{in} = h_{out}$

end for

Update field value for this time step

$e_{i+1} \leftarrow e_{out}$

$h_{i+1} \leftarrow h_{out}$

end for

$e_{final} \leftarrow e_{nstep+1}$

$h_{final} \leftarrow h_{nstep+1}$.

The value of β and α corresponding to the order of integration. **Order=1**

$$\beta_1 = 1, \qquad \alpha_1 = 1.$$

Order=2,

Time discretization using symplectic method in nonlinear case

$$\beta_1 = \frac{1}{2}, \qquad \alpha_1 = 0,$$
$$\beta_2 = \frac{1}{2}, \qquad \alpha_2 = 1.$$

Order=3,

$$\beta_1 = \frac{2}{3}, \qquad \alpha_1 = \frac{7}{24},$$
$$\beta_2 = -\frac{2}{3}, \qquad \alpha_2 = \frac{3}{4},$$
$$\beta_3 = 1, \qquad \alpha_3 = -\frac{1}{24}.$$

Order=4,

$$\beta_1 = \frac{2 + 2^{\frac{1}{3}} + 2^{\frac{-1}{3}}}{6}, \qquad \alpha_1 = 0,$$
$$\beta_2 = \frac{1 - 2^{\frac{1}{3}} - 2^{\frac{-1}{3}}}{6}, \qquad \alpha_2 = \frac{1}{2 - 2^{\frac{1}{3}}},$$

CFL stability condition

$$\beta_3 = \frac{1 - 2^{\frac{1}{3}} - 2^{\frac{-1}{3}}}{6}, \qquad \alpha_3 = \frac{1}{1 - 2^{\frac{2}{3}}}.$$

$$\beta_4 = \frac{2 + 2^{\frac{1}{3}} + 2^{\frac{-1}{3}}}{6}, \qquad \alpha_4 = \frac{1}{2 - 2^{\frac{1}{3}}}.$$

Here we should have to describe the stable time step Δt. The CFL stability condition for symplectic integration method is,

$$\Delta t \leq \frac{2}{\sqrt{\rho((\mathbf{M}_\varepsilon^{(1)})^{-1}(\left(\mathbf{G}^{(12)}\right)^T \mathbf{M}_\mu^{(2)} \mathbf{G}^{(12)})}},$$

where ρ is spectral radius function.

Validation of Simulation

Numerical experiment are performed for the fully discretized Maxwell's equations to check the stability and convergence properties, where the frequency $f = \frac{\sqrt{3}}{2} c_0$ Hz and c_0 is the speed of light in vacuum. $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$ are the vacuum permittivity and permeability respectively. The angular frequency is $\omega = 2\pi f$ (rad$\cdot$s^{-1}). The exact electric and magnetic fields are given that

$$
\begin{aligned}
\mathbf{E}_1(t) &= -\cos(\pi x)\sin(\pi y)\sin(\pi z)\cos(\omega t),\\
\mathbf{E}_2(t) &= 0.0,\\
\mathbf{E}_3(t) &= \sin(\pi x)\sin(\pi y)\cos(\pi z)\cos(\omega t),\\
\mathbf{H}_1(t) &= -\frac{\pi}{\omega}\sin(\pi x)\cos(\pi y)\cos(\pi z)\sin(\omega t),\\
\mathbf{H}_2(t) &= \frac{2\pi}{\omega}\cos(\pi x)\sin(\pi y)\cos(\pi z)\sin(\omega t),\\
\mathbf{H}_3(t) &= -\frac{\pi}{\omega}\cos(\pi x)\cos(\pi y)\sin(\pi z)\sin(\omega t).
\end{aligned}
$$

Absolute Error for sympletic method

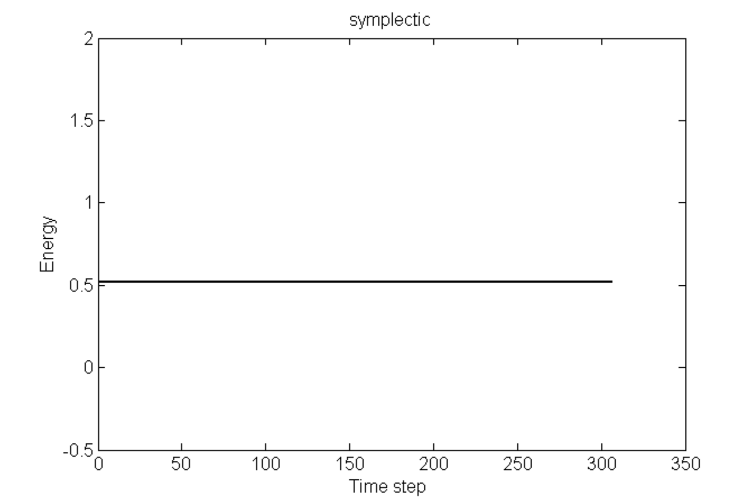
We measure the $\mathbf{L}_2$-norm of the error for a sequence of successively refined meshes starting from a uniform coarse mesh. Here define notation $Err(\mathbf{E}) = \|\mathbf{E}(t^n) - \mathbf{E}_h^n\|_{\mathbf{L}^2_\varepsilon(\Omega)}$ and $Err(\mathbf{H}) = \|\mathbf{H}(t^n) - \mathbf{H}_h^n\|_{\mathbf{L}^2_\mu(\Omega)}$ that are used in Table 1.

Table: Absolute Error

Refined Level	**Electric and Magnetic Fields absolute error**			
	$Err(\mathbf{E})$	$Err(\mathbf{B})$	***stable time=*** Δt	***steps***
l=2	2.786666	1.17524e-08	0.282302ns	300
l=3	0.733713	2.2434e-09	0.140619ns	600

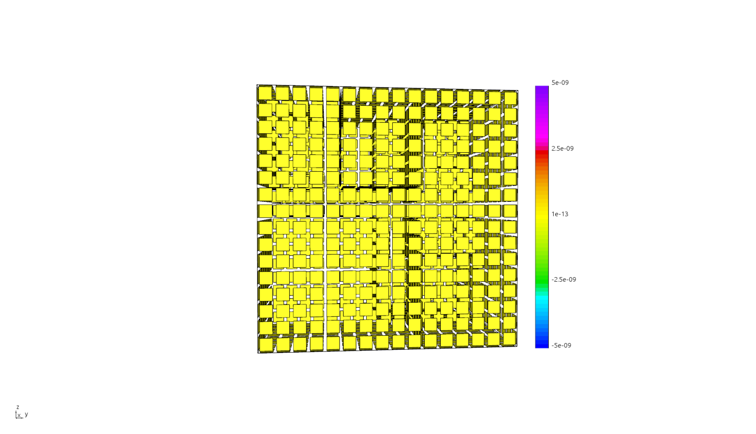
- Using **MFEM** and **Hypre**.
- HperPCG solver.
- Set tolerance = 1.0e-12.
- Processors in space up to 60.

Conservation of numerical energy



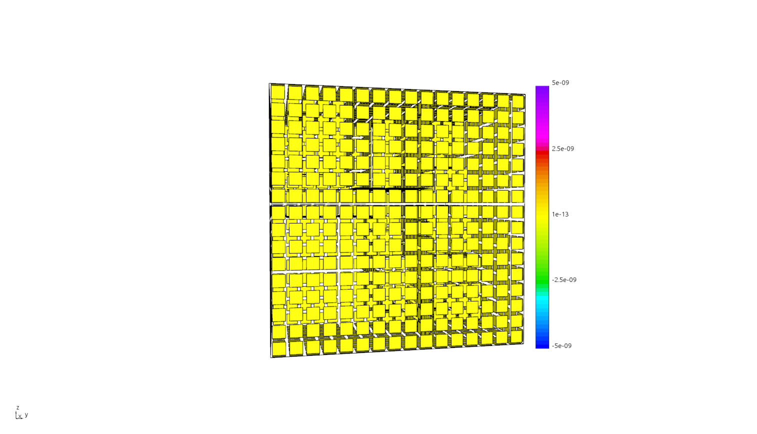
Energy of the system.

Trivial electric field at final time step



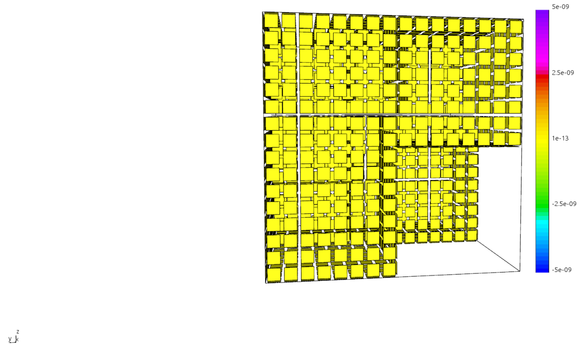
The scale shows value of electric field at final time step ($n = 1200$), by employing the Sympletic time integration method, when electric and magnetic fields, and current source are initialized zero.

Trivial magnetic field at final time step



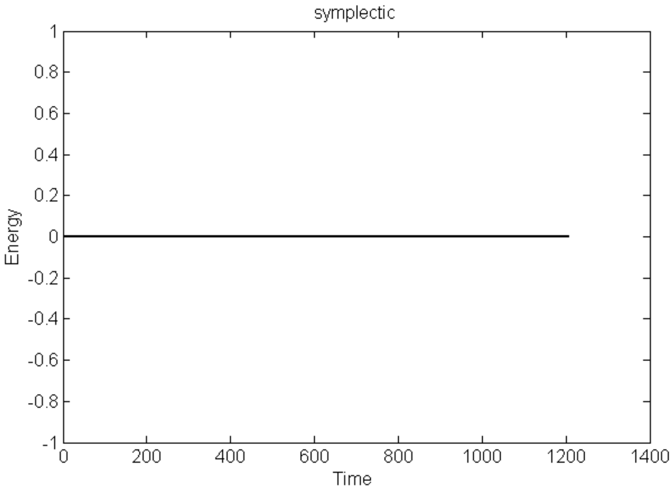
The scale shows value of electric field at final time step ($n = 1200$), by employing the Sympletic time integration method, when electric and magnetic fields, and current source are initialized zero.

Trivial current source at final time step



The scale shows value of current source at final time step ($n = 1200$), by employing the Sympletic time integration method, when electric and magnetic fields, and current source are initialized zero.

Zero energy



Energy of the system is zero, in case of trivial solution.

Time discretization employing Backward Euler for linear case

For brevity, the backward Euler method for the system is given here as an example:

Compute the number of time steps.

$$\textbf{nstep} = \frac{t_{final} - t_0}{\Delta t}$$

Set the initial conditions

$e_1 \leftarrow e_{Initial}$

$h_1 \leftarrow h_{Initial}$

Loop over time steps

for i=1 to nstep do

Begin integration method update

$e_{in} \leftarrow e_i$

$h_{in} \leftarrow h_i$

Update the field values

Time discretization Backward Euler for linear case

$e_{out} \leftarrow e_{in} + \Delta t(\mathbf{M}_{\varepsilon}^{(1)})^{-1}\left(\left(\mathbf{G}^{(12)}\right)^T h_{out} - \mathbf{M}_{\sigma}^{(1)} e_{out} + \mathbf{J}^{(1)}\right)$

$h_{out} \leftarrow h_{in} + \Delta t(\mathbf{M}_{\mu}^{(2)})^{-1}\left(\mathbf{G}^{(12)}\right) e_{out}$

Update the field values for this time step

$e_{i+1} \leftarrow e_{out}$

$h_{i+1} \leftarrow h_{out}$

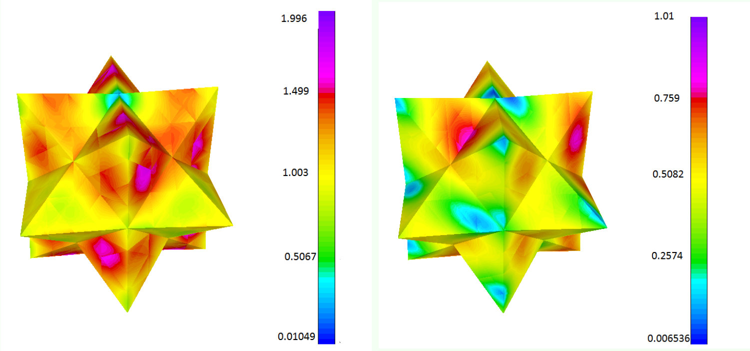
end for

$$e_{final} \leftarrow e_{nstep+1}$$

$$h_{final} \leftarrow h_{nstep+1}$$

Plane wave with the wavelengths varying from 1.0 μm to 2.0 μm and the $\mathbf{E}(t)$ field magnitude varying from 1 V/m to 1.5×10^8 were injected into the material with ε_0 and μ_0 and $\chi^3 = 2 \times 10^{-18}$, . The propagation of these plane waves was simulated through 2000 times steps of $\frac{1}{4}\Delta t$ and snapshots are taken to measure the wave velocities.

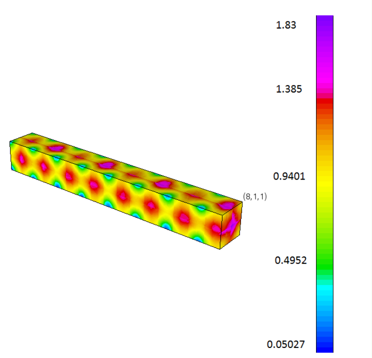
Escher meshes employing Backward Euler



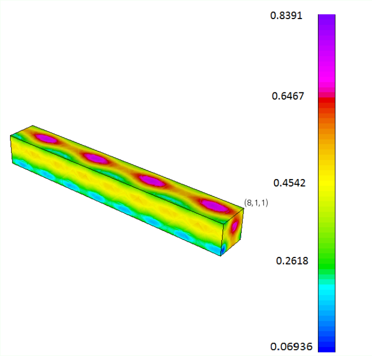
(a) Electric field (b) Magnetic field

The scale shows value of electric and magnetic fields at final time step ($n = 1000$), by employing the backward Euler method. Time step $\Delta t = 0.001$

Beam tetrahedron meshes employing Backward Euler



(a) Electric field



(b) Magnetic field

The scale shows value of electric and magnetic fields at final time step ($n = 1000$), by employing the backward Euler method. Time step $\Delta t = 0.001$.

Conclusion and future work

- Higher order in space and time, in case of complex material ε and μ,
- parallel in space,
- Energy conserving,
- Perform also number of experiments for A-Stable and L-Stable method (SDIRK23Solver, SDIRK34Solver and Backward Euler solver),
- nonlinear Optics
- Convergence and error estimation theoretically.
- Future work
  - validation in nonlinear Optics,
  - parallel in space and time parallelism in XBraid.
  - when material parameters ε and μ are complex $\mathbf{R}^{3x3}$ Matrix, space-dependent parameters functions.
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