Why it is difficult to solve hyperbolic problems with parareal type algorithms

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The Parareal Algorithm

$$U_{n+1}^{k+1} = F(t_{n+1}, t_n, U_n^k) + G(t_{n+1}, t_n, U_n^{k+1}) - G(t_{n+1}, t_n, U_n^k).$$

Convergence Results for Linear Problems

For the Dahlquist test equation

$$u' = \lambda u, \quad u(0) = u_0, \quad \Re(\lambda) \leq 0.$$

Theorem (Superlinear Convergence)

Let $F(t_{n+1}, t_n, U_n^k)$ denote the exact solution at t_{n+1} and $G(t_{n+1}, t_n, U_n^k) = R(\lambda \Delta T) U_n^k$ be a one step method. If the method is in its region of absolute stability, $|R(\lambda \Delta T)| \leq 1$, then at iteration k, we have

$$\max_{1 \leq n \leq N} |u(t_n) - U_n^k| \leq \frac{|e^{\lambda \Delta T} - R(\lambda \Delta T)|^k}{k!} \prod_{j=1}^{k} (N-j) \max_{1 \leq n \leq N} |u(t_n) - U_n^0|.$$

If the local truncation error is bounded by $C \Delta T^{p+1}$, then

$$\max_{1 \leq n \leq N} |u(t_n) - U_n^k| \leq \frac{(CT)^k}{k!} \Delta T^{pk} \max_{1 \leq n \leq N} |u(t_n) - U_n^0|.$$

Convergence Results for Linear Problems

Theorem (Linear Convergence)

Let $F(t_{n+1}, t_n, U_n^k)$ denote the exact solution at t_{n+1} and $G(t_{n+1}, t_n, U_n^k) = R(\lambda \Delta T) U_n^k$ be a one step method. If ΔT is such that the method is in its region of absolute stability, then at iteration k, we have

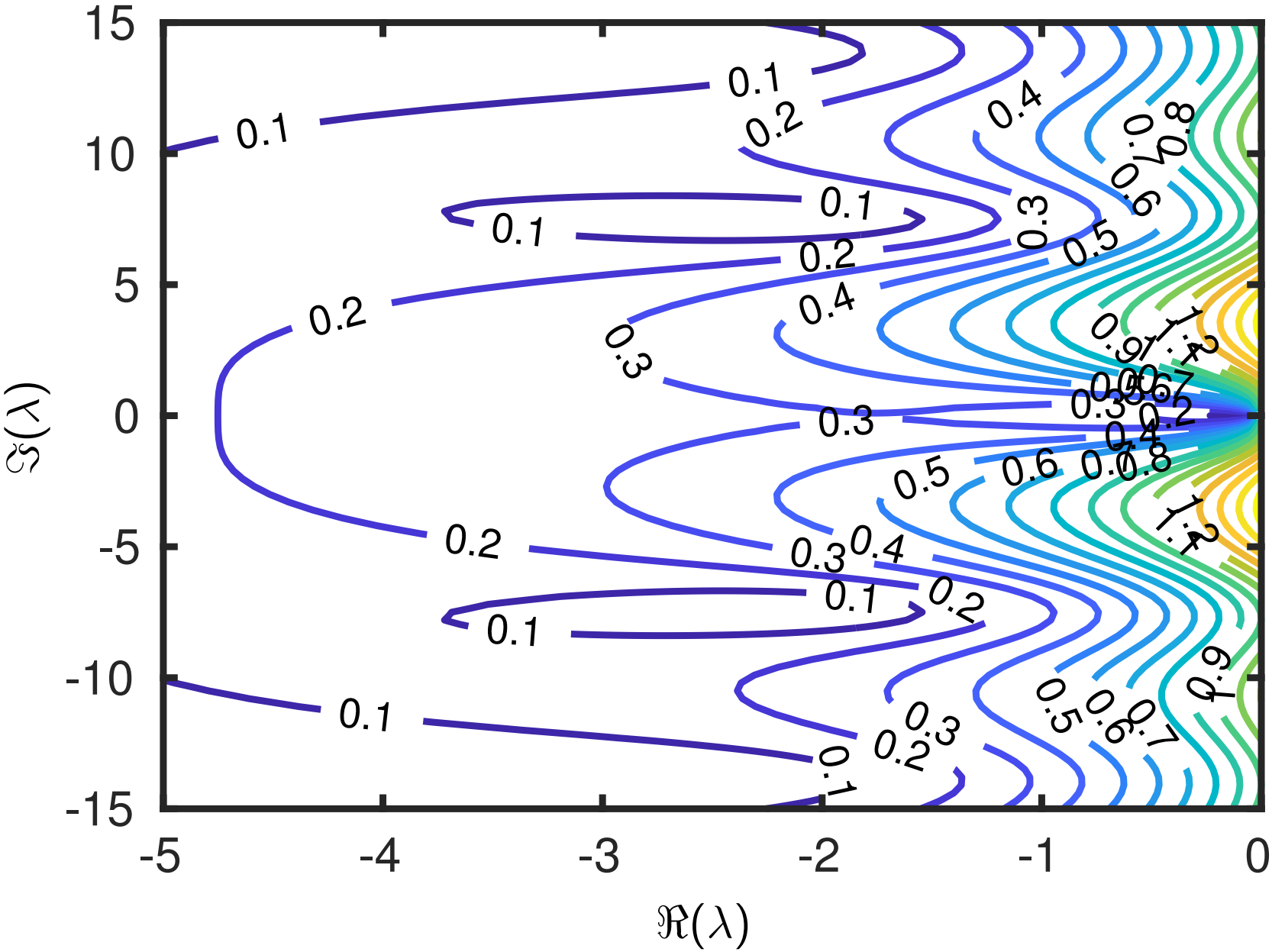
$$\sup_{n>0} |u(t_n) - U_n^k| \leq \left(\frac{|e^{\lambda \Delta T} - R(\lambda \Delta T)|}{1 - |R(\lambda \Delta T)|} \right)^k \sup_{n>0} |u(t_n) - U_n^0|.$$

If the local truncation error is bounded by $C \Delta T^{p+1}$, then for ΔT small, we have

$$\sup_{n>0} |u(t_n) - U_n^k| \leq \left(\frac{C \Delta T^p}{\Re(-\lambda) + O(\Delta T)} \right)^k \sup_{n>0} |u(t_n) - U_n^0|.$$

Note: uniform convergence bound for all time if the convergence factors are less than one !

Contraction Factor (F: exact, G: BE)



Corollary (Heat Equation)

For $u_t = \Delta u$ with L-stable method, parareal converges superlinearly on bounded time intervals,

$$\max_{1\le n\le N} ||u(t_n) - U_n^k||_2 \le \frac{\gamma_s^k}{k!}\prod_{j=1}^{k}(N-j)\max_{1\le n\le N} ||u(t_n) - U_n^0||_2,$$

and linearly on unbounded time intervals

$$\sup_{n>0} ||u(t_n) - U_n^k||_2 \le \gamma_l^k \sup_{n>0} ||u(t_n) - U_n^0||_2,$$

with universal constants for each L-stable method.

method	order	γ_s	γ_l
BE	1	0.2036321888	0.2984256075
SDIRK 3.1	3	0.1717941220	0.2338191487
SDIRK 3.2	3	0.2073822267	0.1718033767
Radau IIA	5	0.0634592650	0.0677592165

Convergence for Advection Problems

Corollary (Advection Equation)

For $u_t = u_x$ with backward Euler in time, parareal satisfies on bounded time intervals the superlinear convergence bound

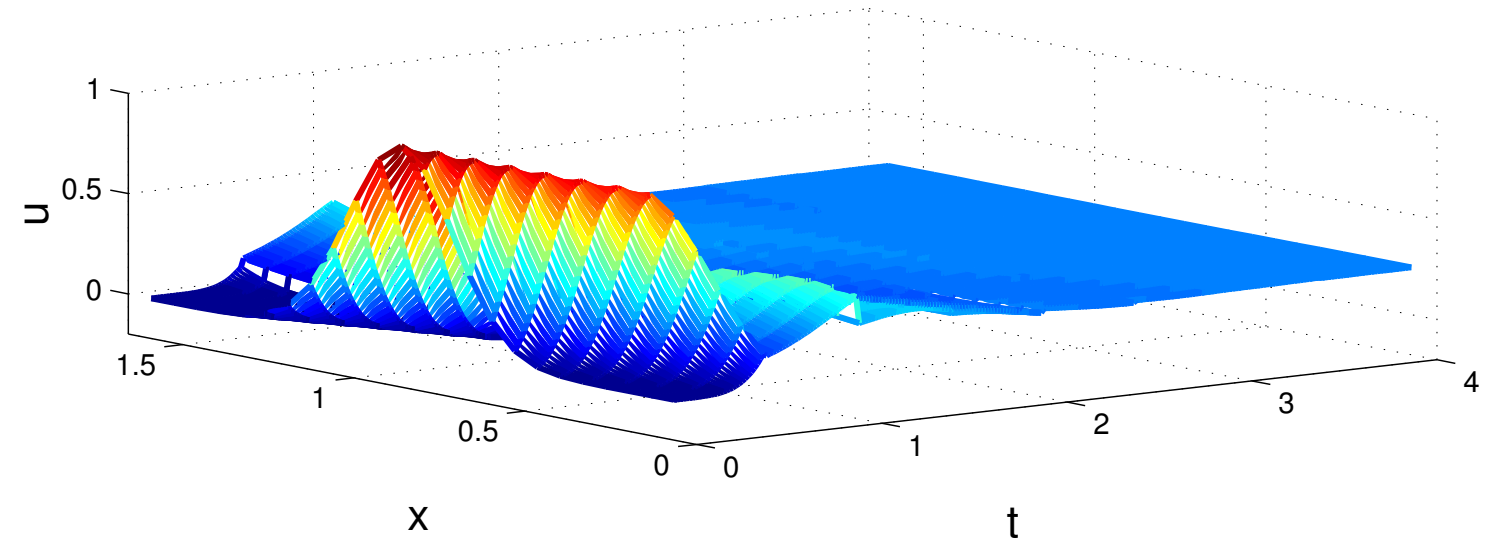
$$\max_{1\leq n\leq N} ||u(t_n) - U_n^k||_2 \leq \frac{\alpha_s^k}{k!} \prod_{j=1}^{k} (N-j) \max_{1\leq n\leq N} ||u(t_n) - U_n^0||_2,$$

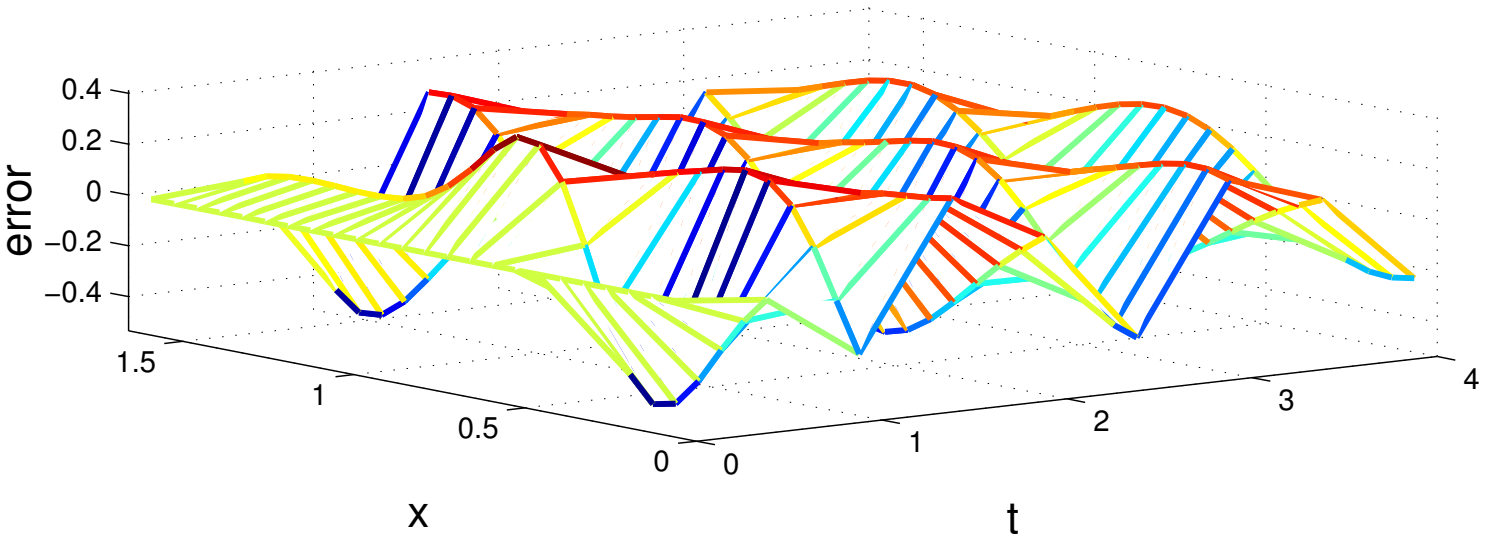
where the constant α_s is universal, $\alpha_s = 1.224353426$.

Remarks:

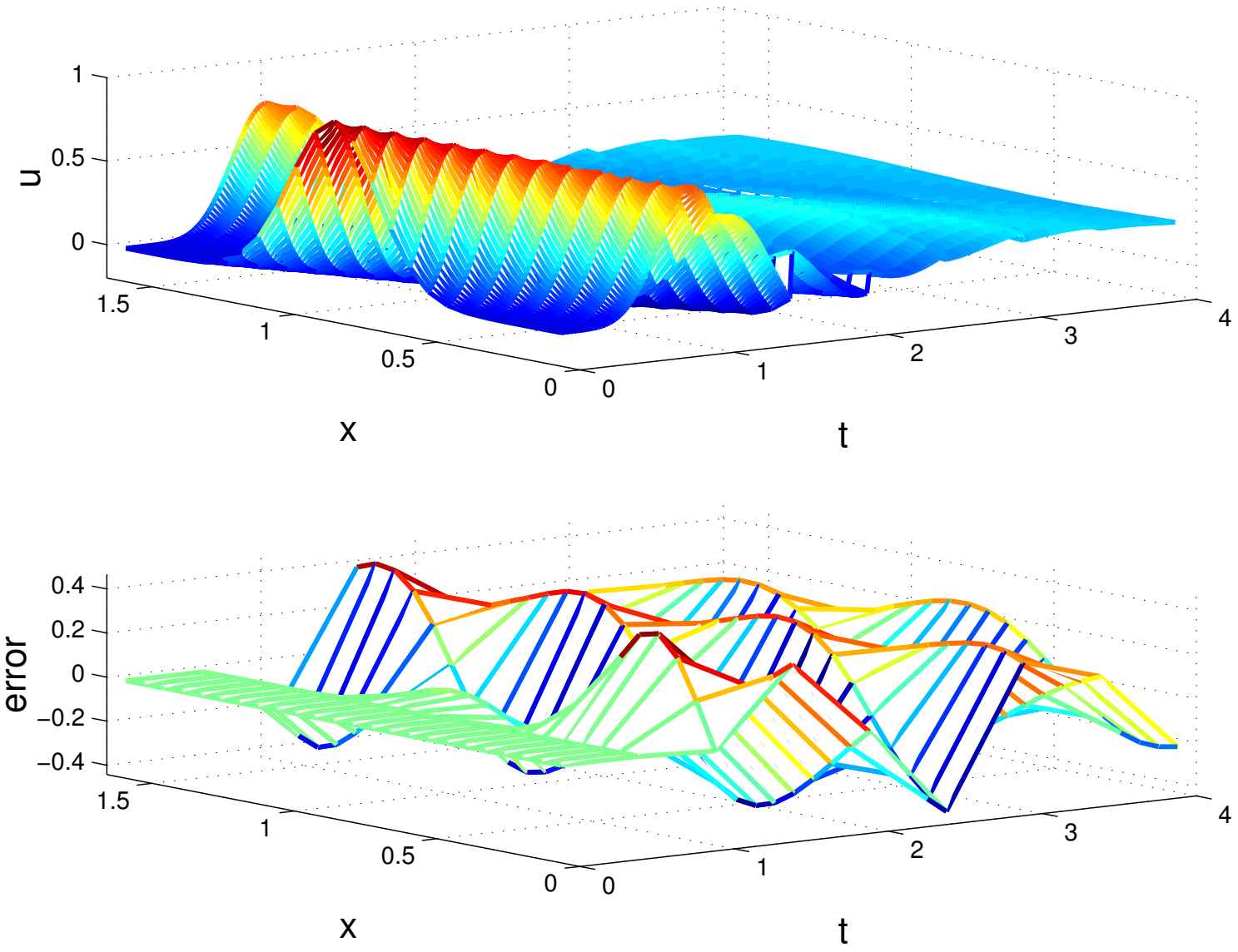
- No convergence result for unbounded time intervals.
- As soon as more than N iterations are needed, the method is not interesting any more for parallelization.

Periodic Advection Equation: $T = 4$, Iteration 1

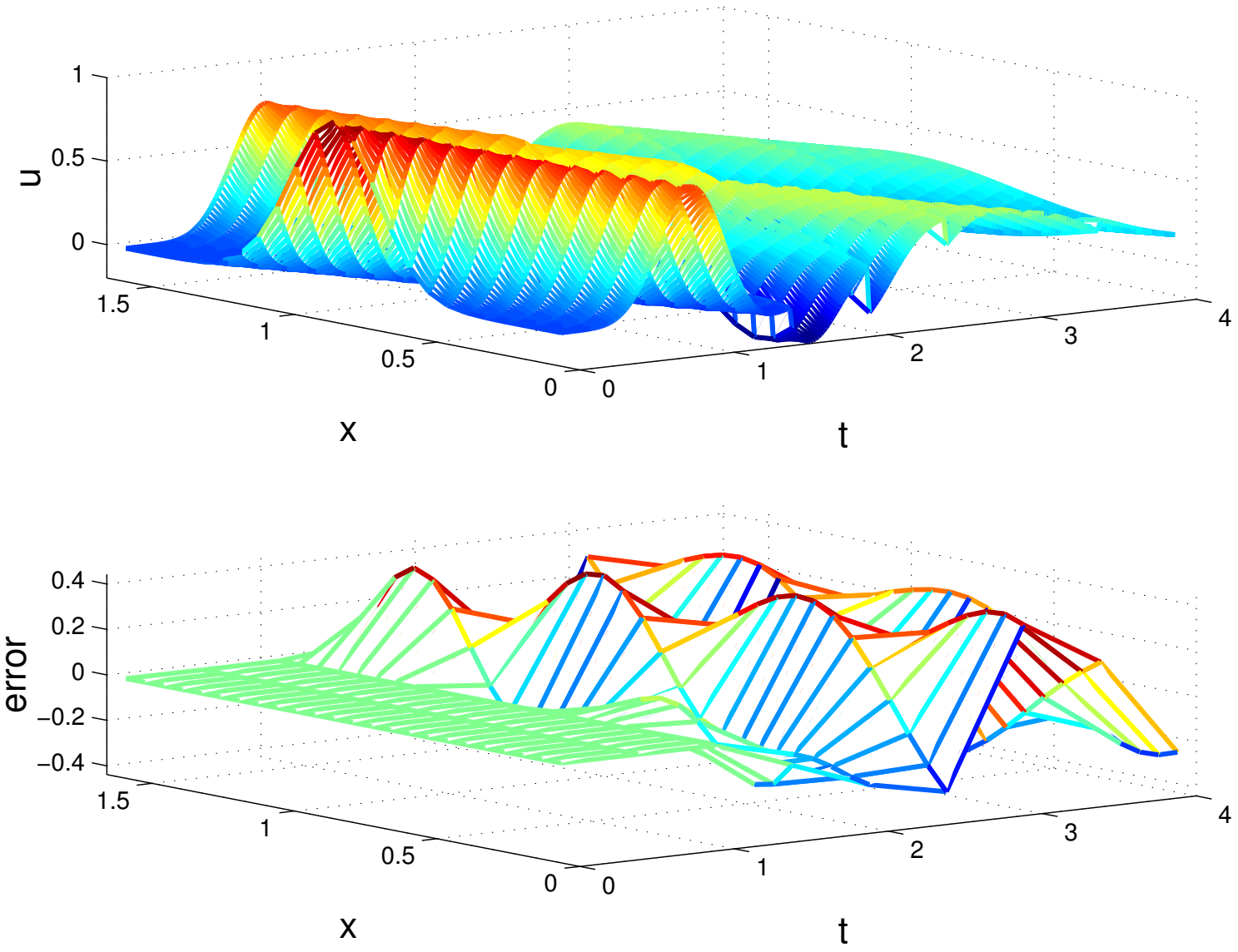




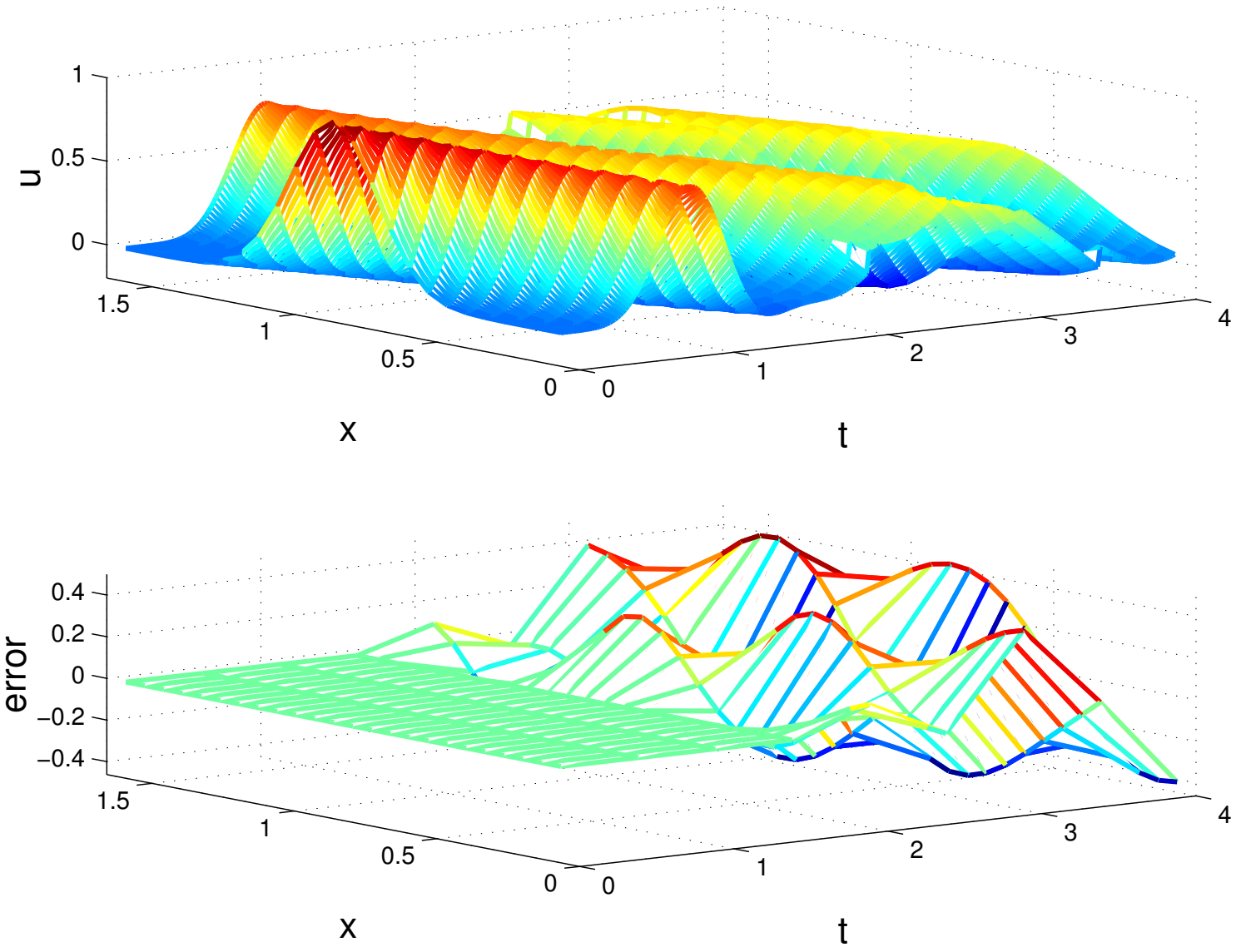
Periodic Advection Equation: $T = 4$, Iteration 2



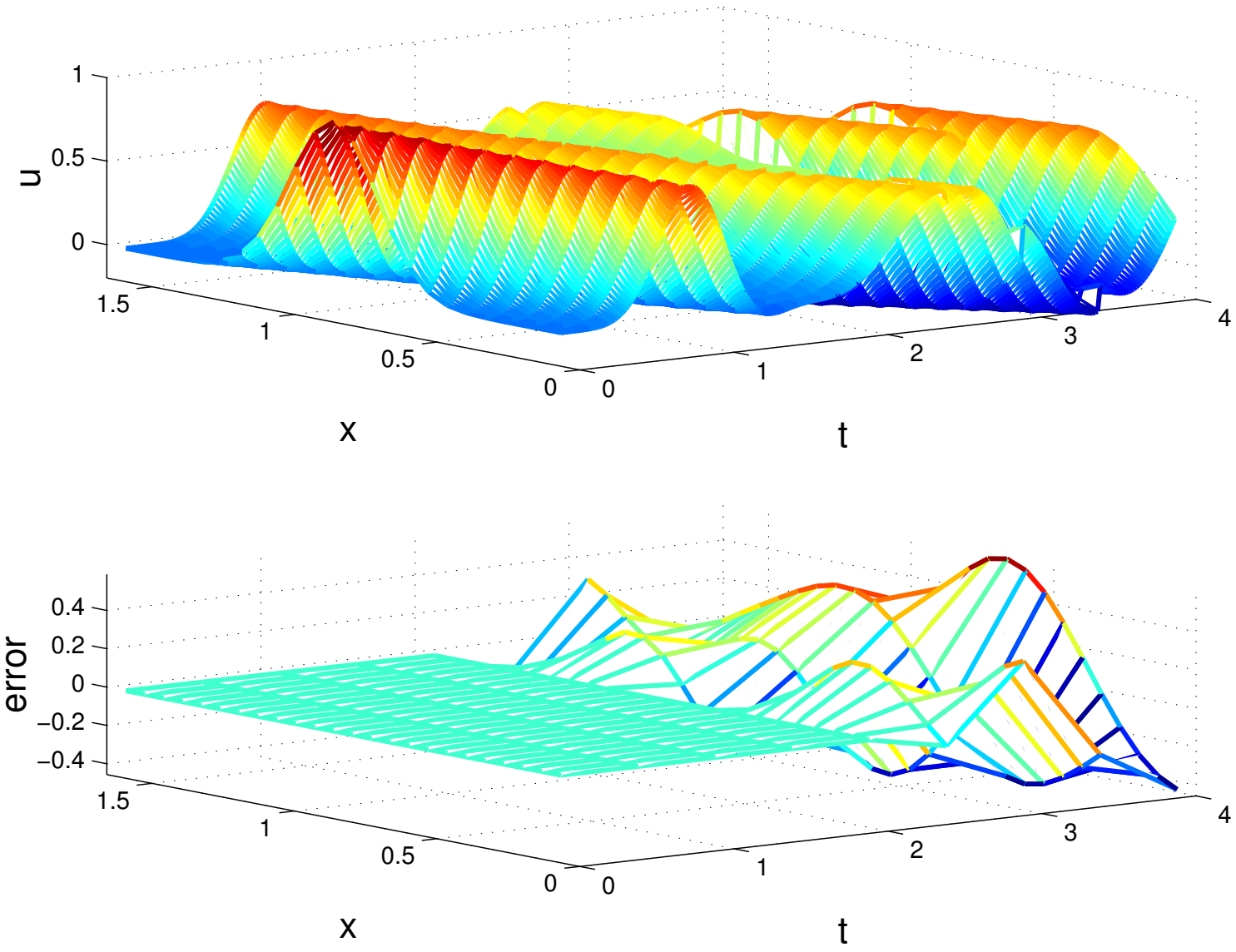
Periodic Advection Equation: $T = 4$, Iteration 3



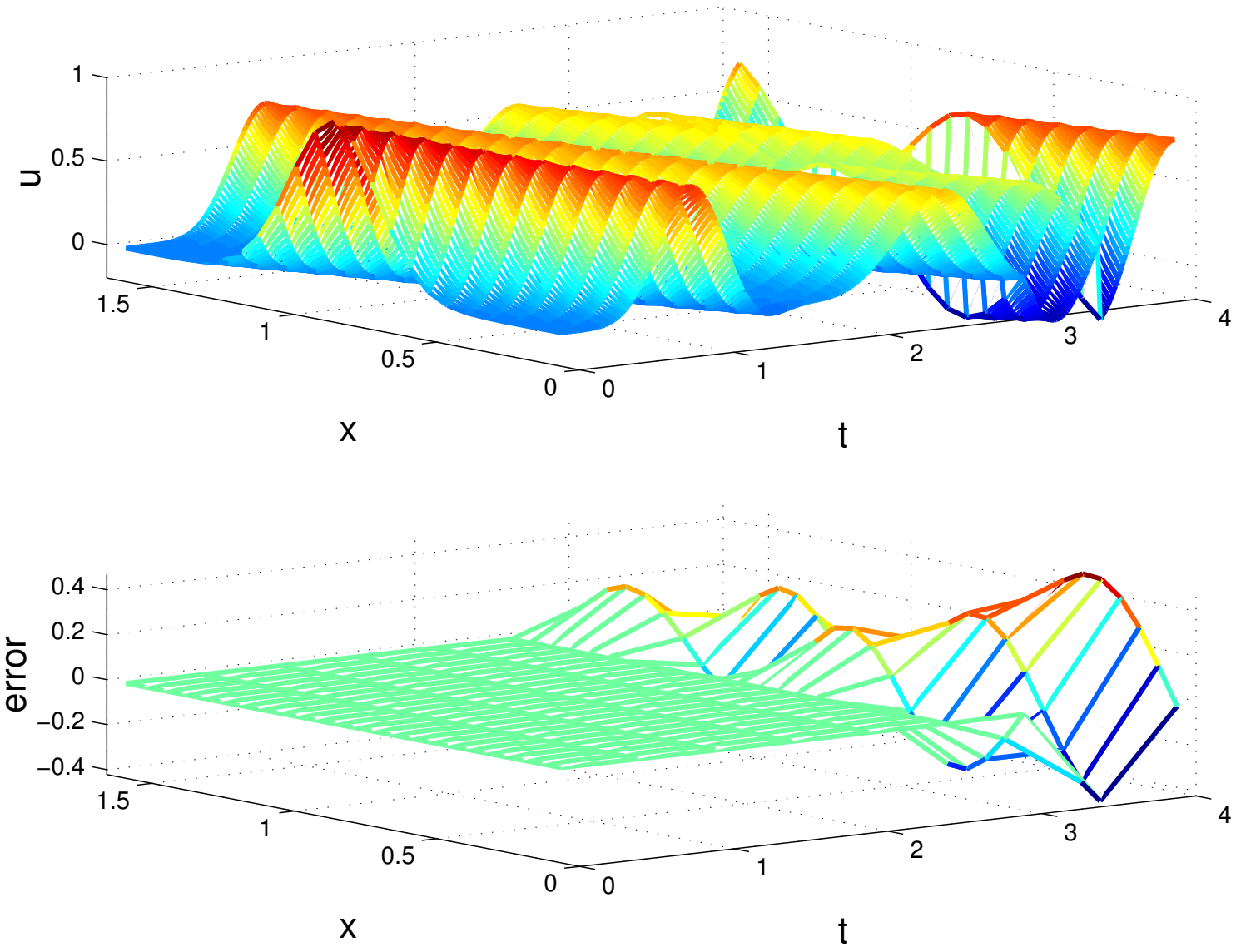
Periodic Advection Equation: $T = 4$, Iteration 4



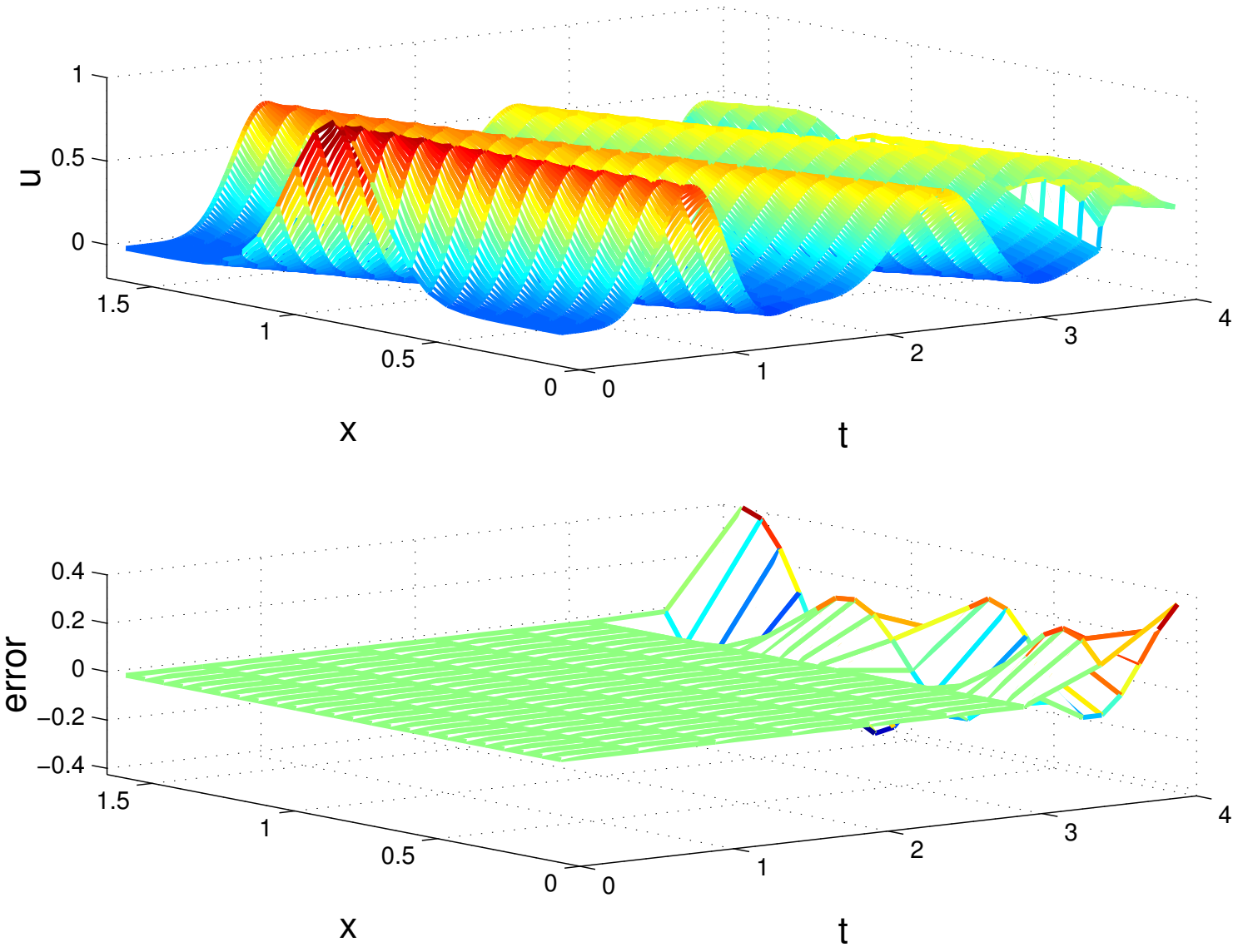
Periodic Advection Equation: $T = 4$, Iteration 5



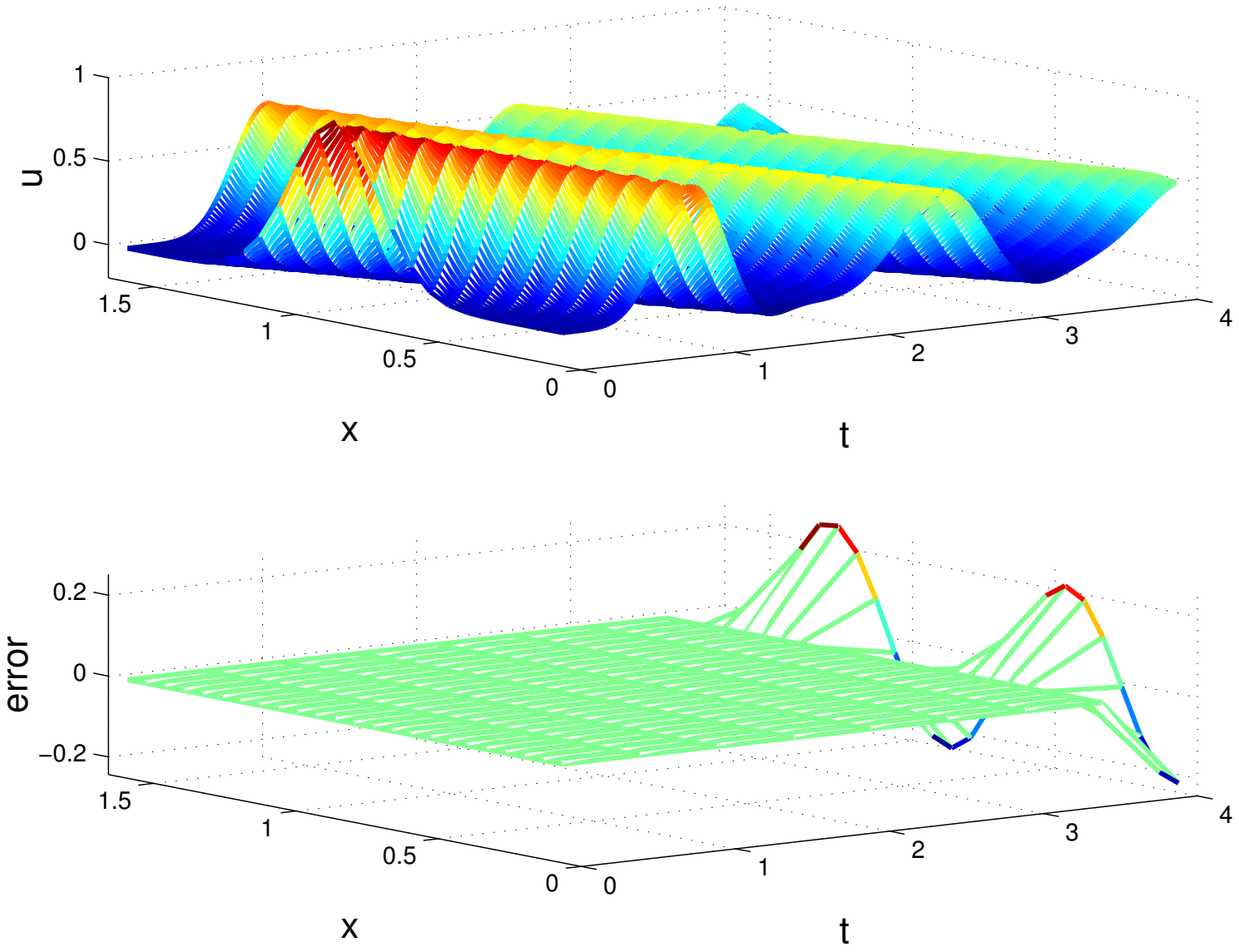
Periodic Advection Equation: $T = 4$, Iteration 6



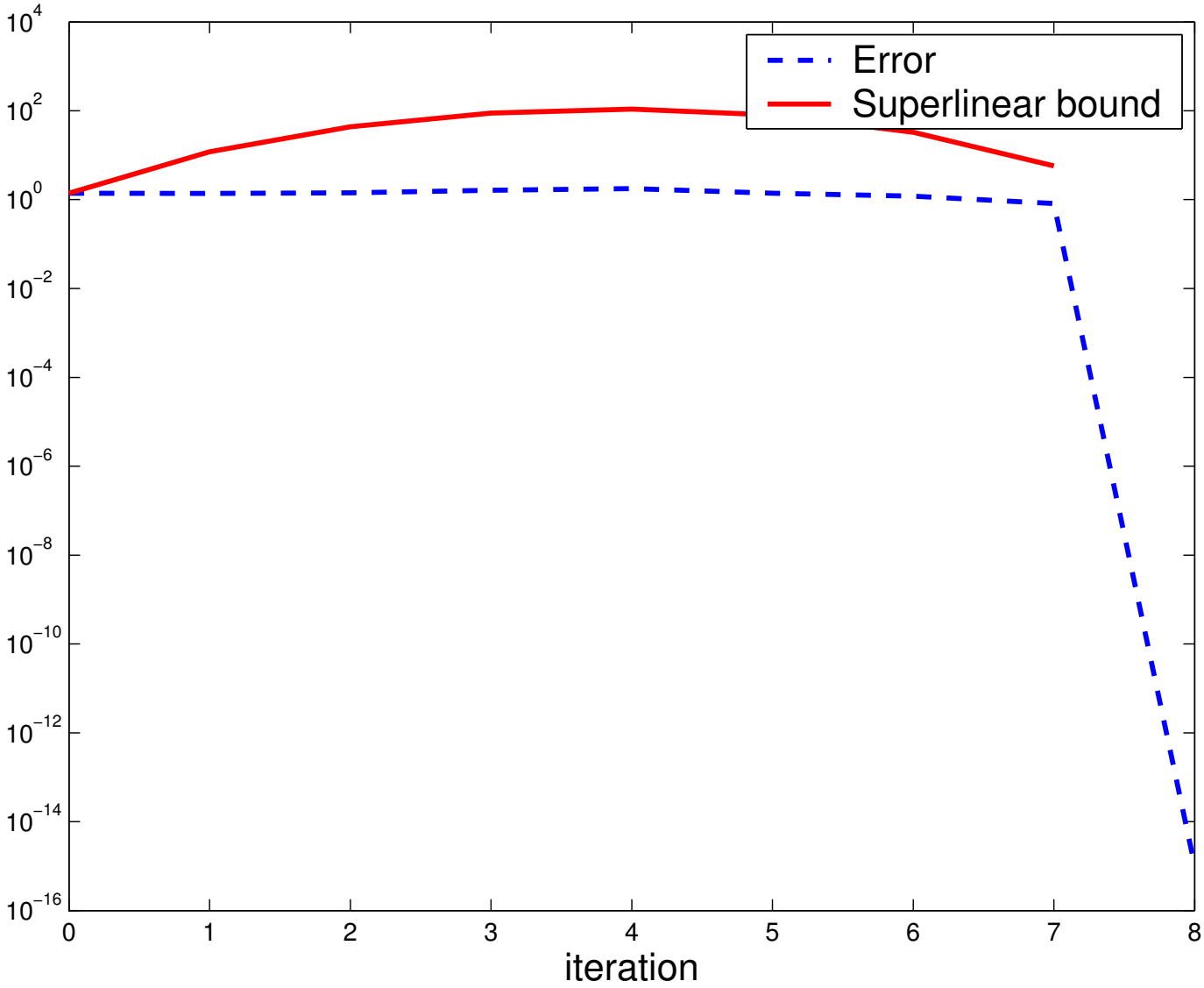
Periodic Advection Equation: $T = 4$, Iteration 7



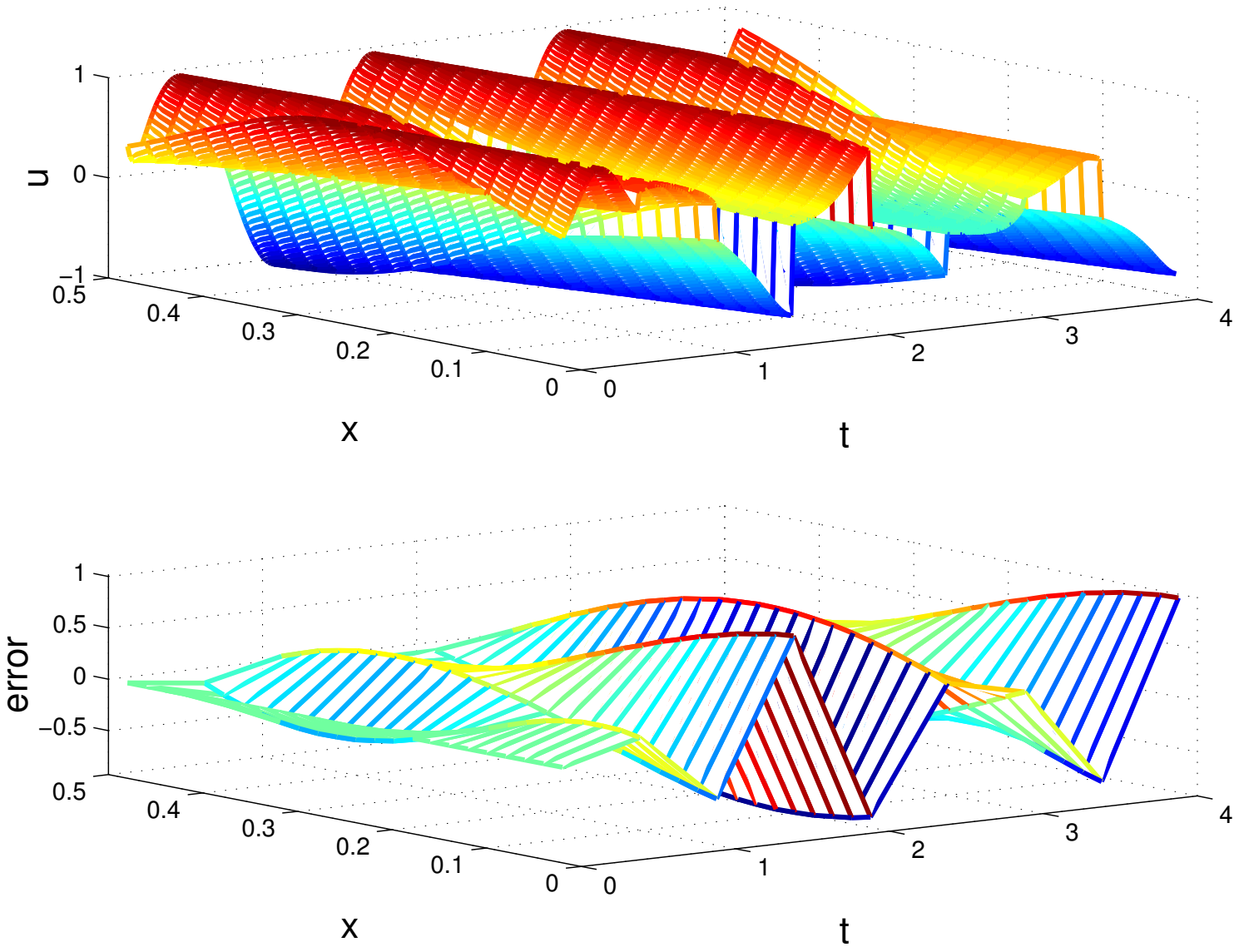
Periodic Advection Equation: $T = 4$, Iteration 8



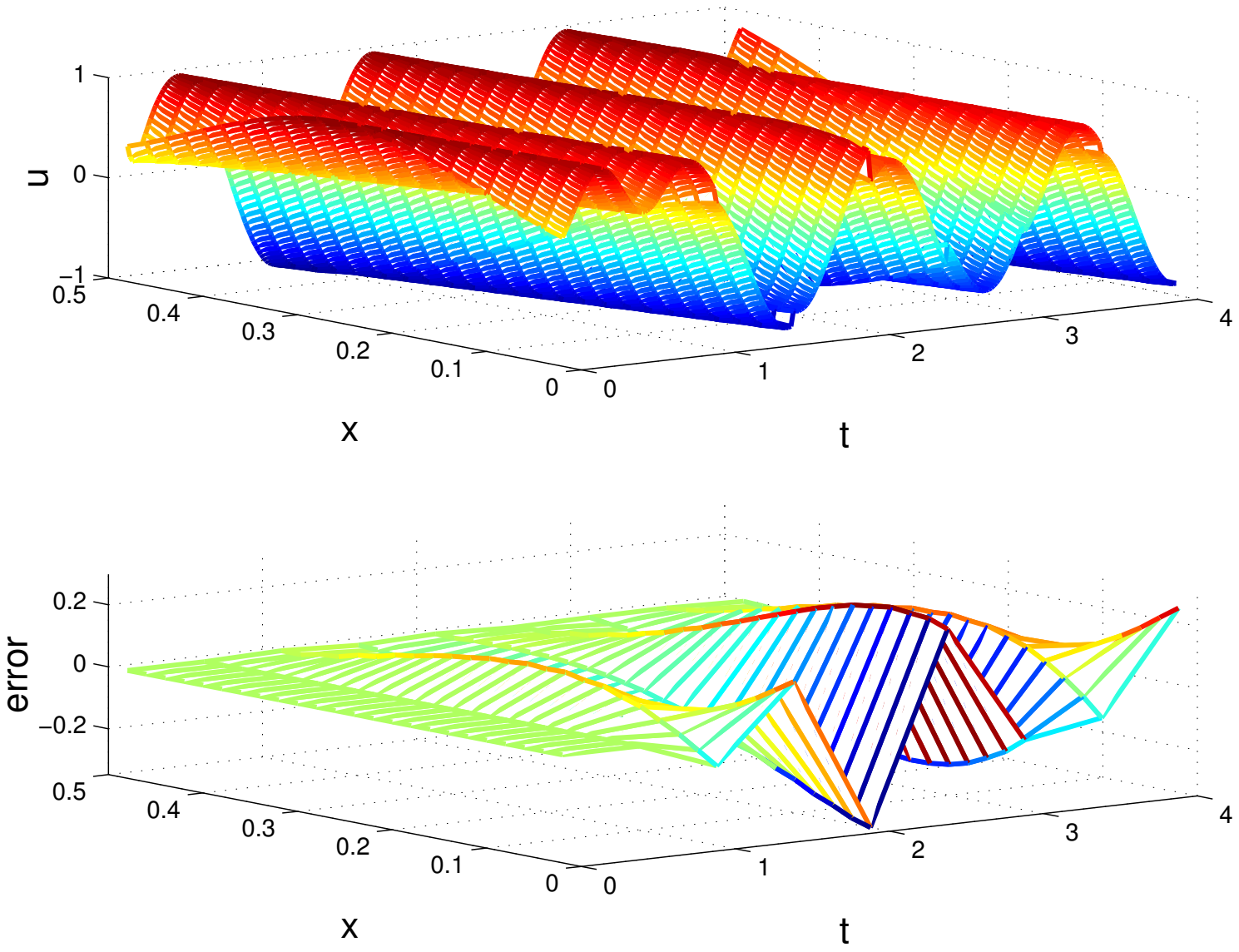
Periodic Advection Equation Error: $T = 4$



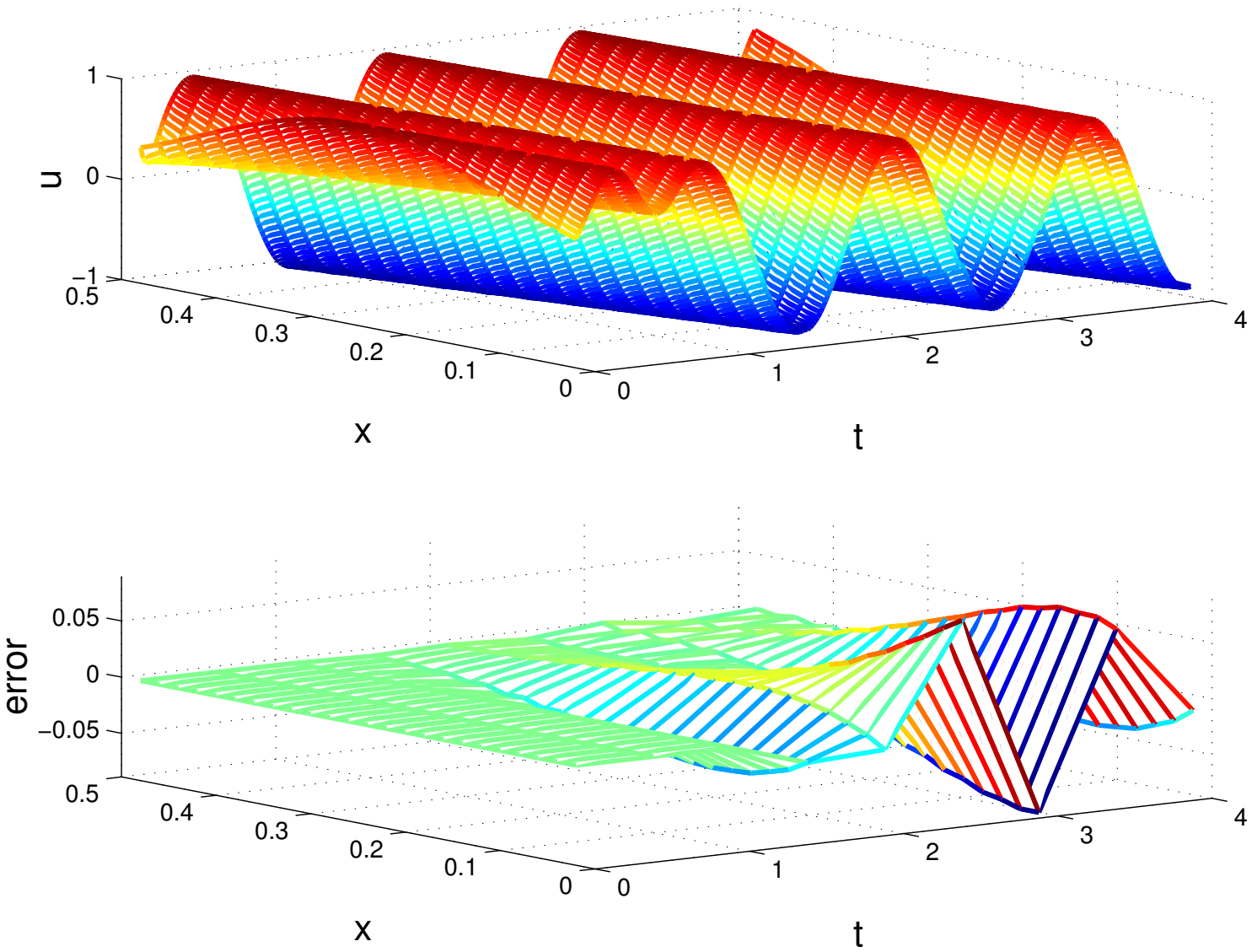
Non-Periodic Advection: $T = 4$, Iteration 1



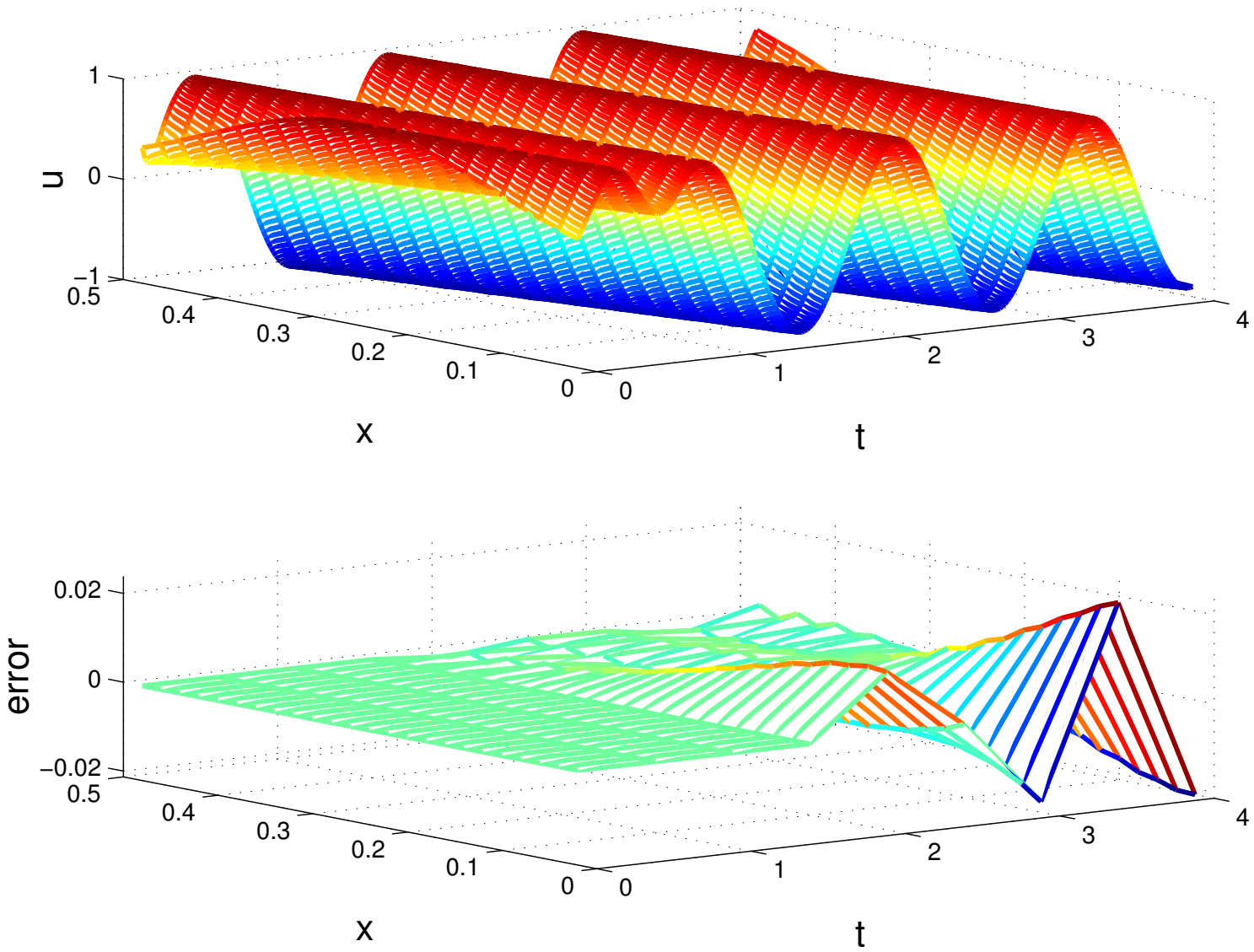
Non-Periodic Advection: $T = 4$, Iteration 2



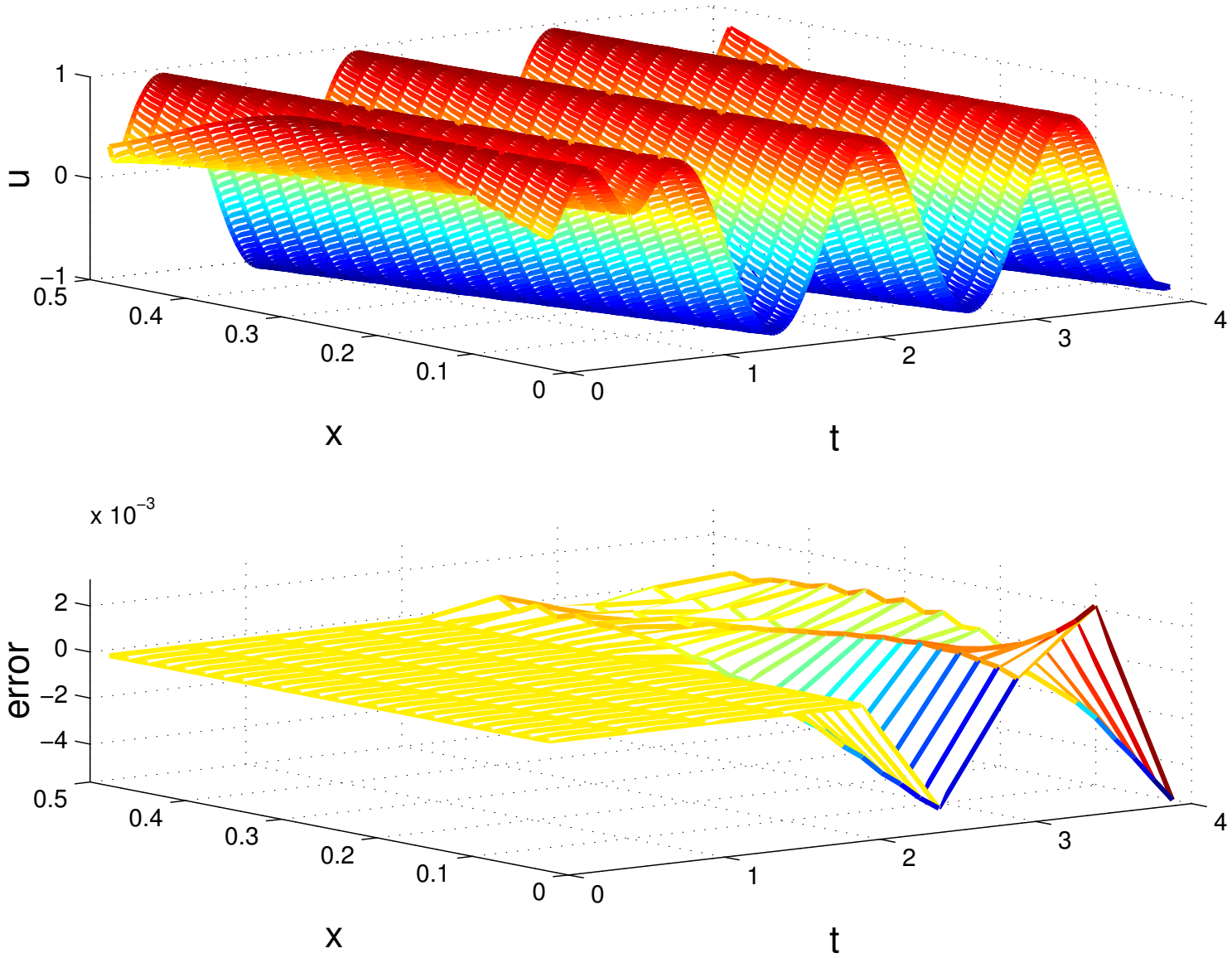
Non-Periodic Advection: $T = 4$, Iteration 3



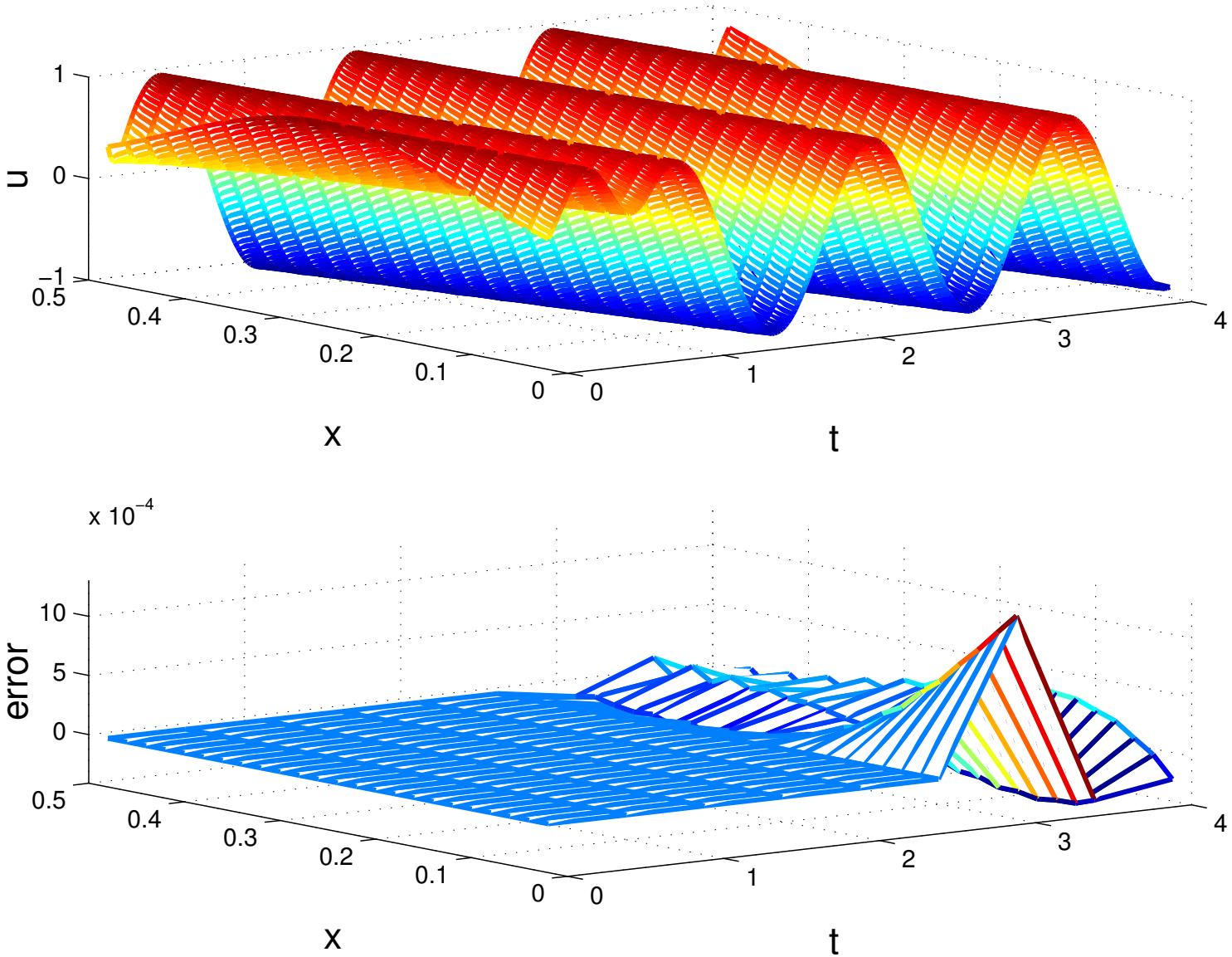
Non-Periodic Advection: $T = 4$, Iteration 4



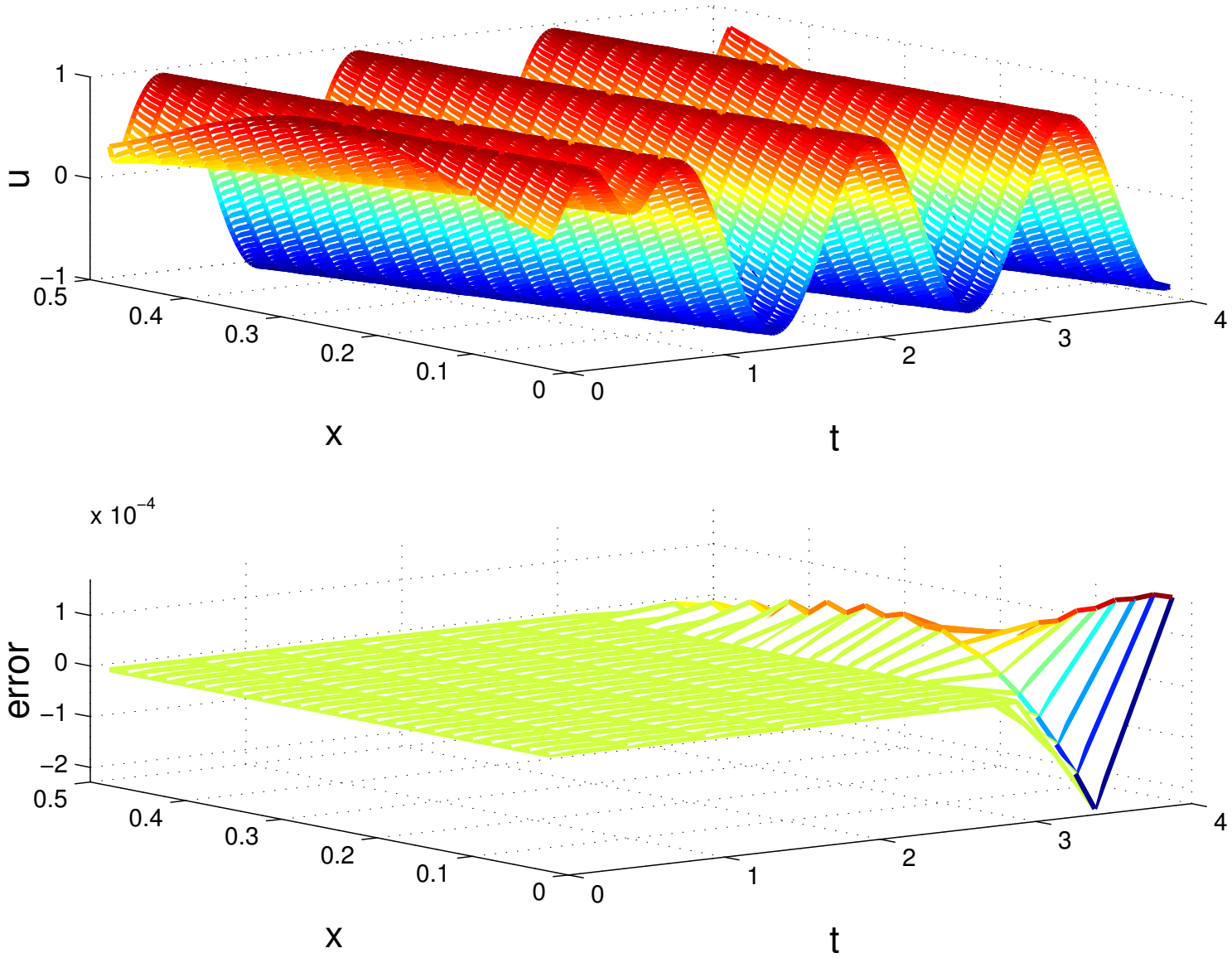
Non-Periodic Advection: $T = 4$, Iteration 5



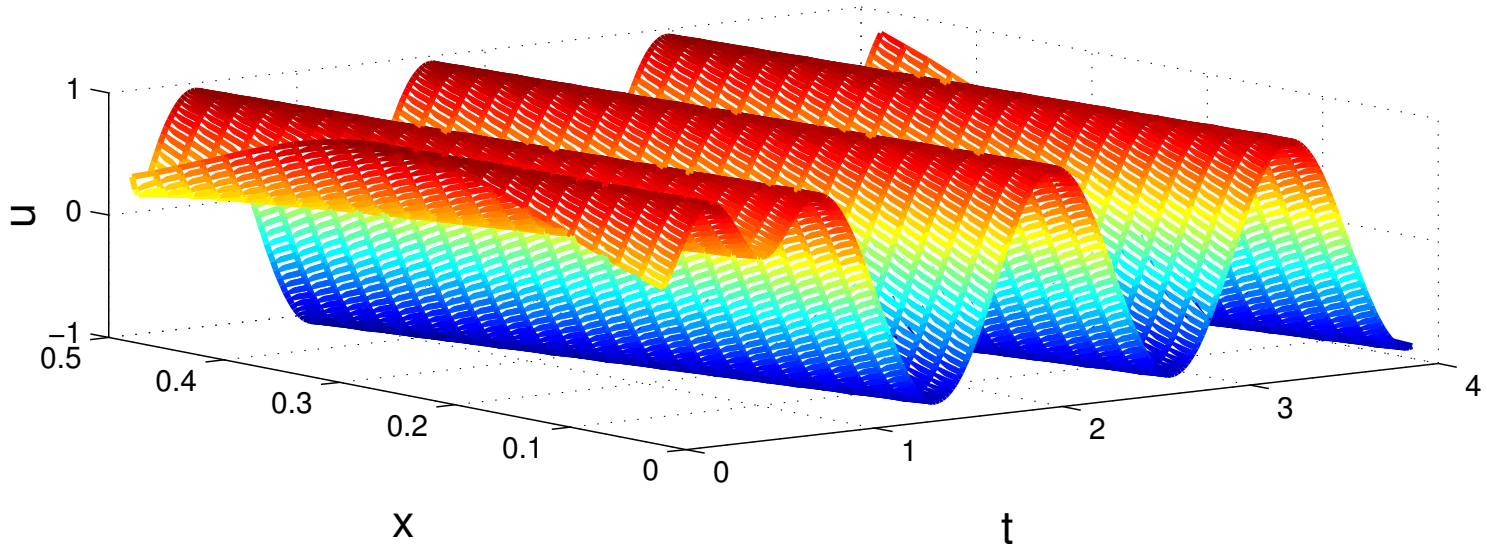
Non-Periodic Advection: $T = 4$, Iteration 6

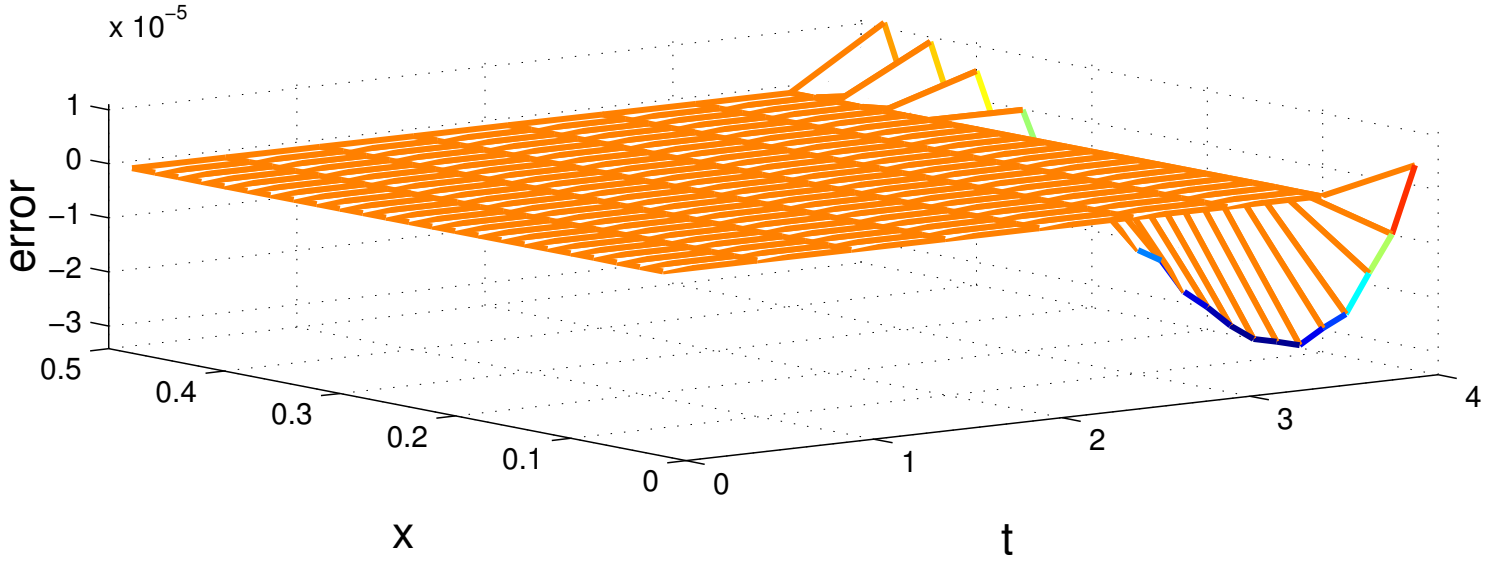


Non-Periodic Advection: $T = 4$, Iteration 6

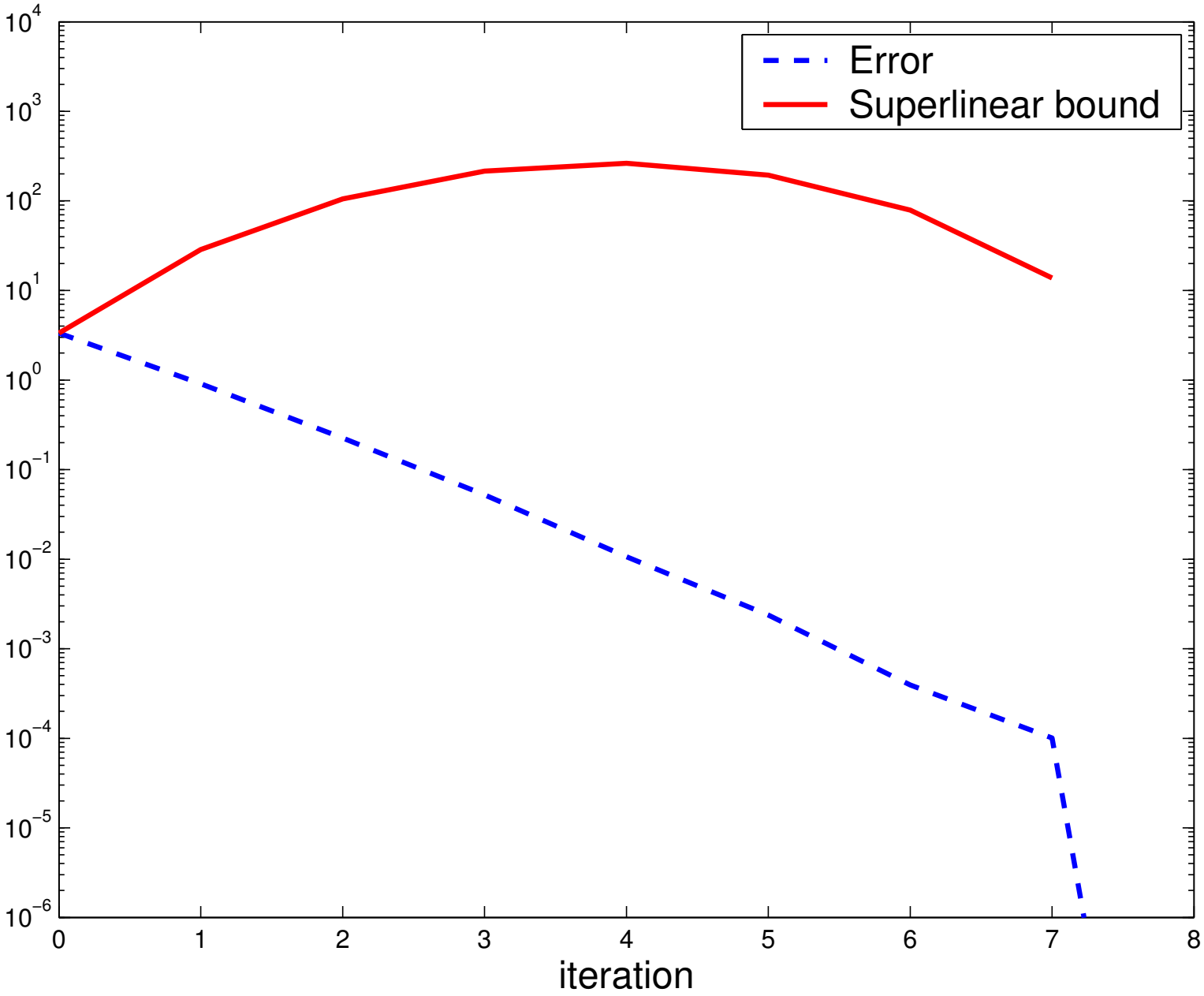


Non-Periodic Advection: $T = 4$, Iteration 7





Non-Periodic Advection Error: $T = 4$



Convergence Result for the Non-Periodic Case

Theorem (Linear Convergence in L^1)

For non-periodic advection $u_t = a u_x$ on the domain $(0, L)$, with F exact and G satisfying a technical Assumption ($\tilde{a}$), parareal satisfies the convergence estimate

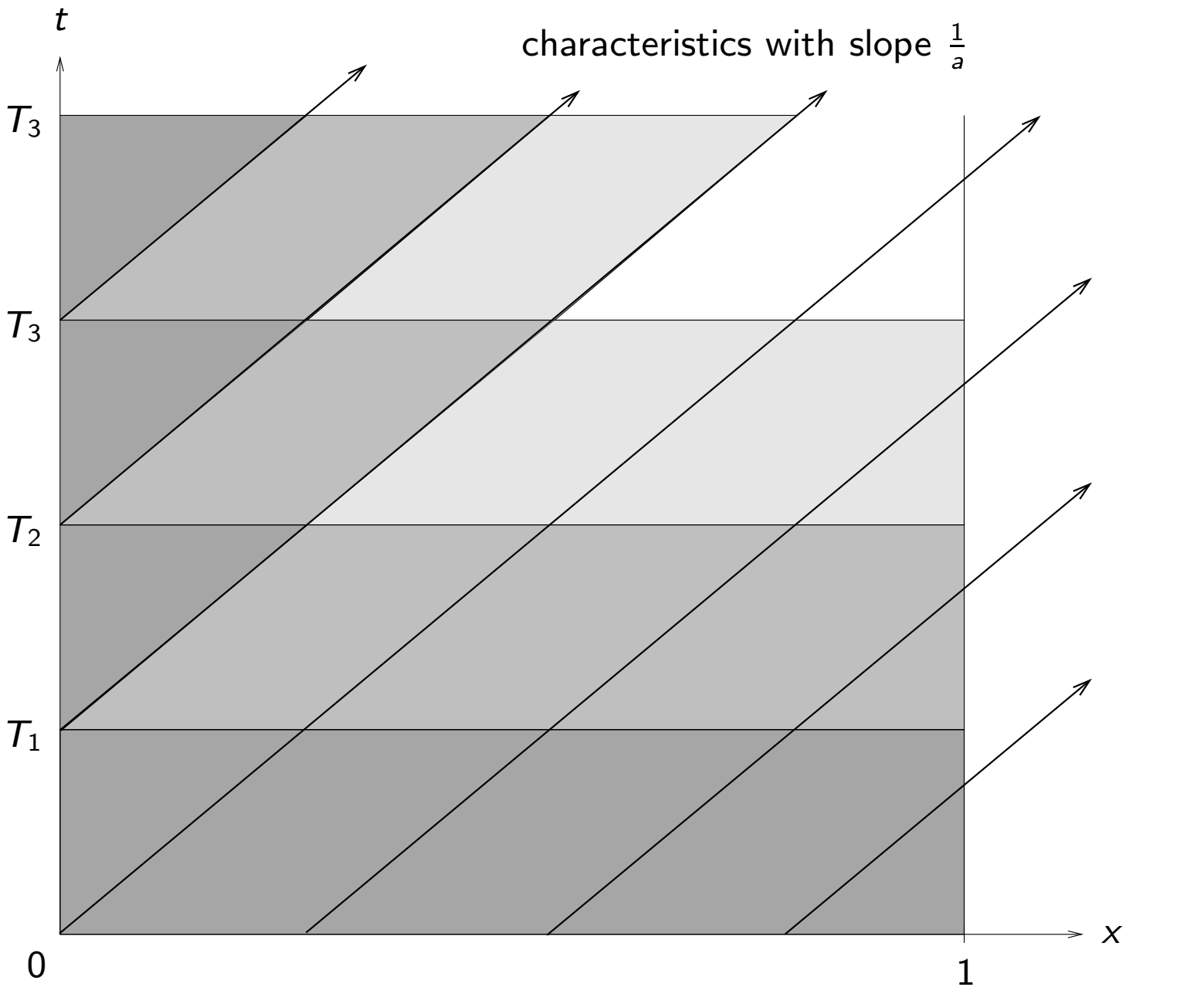
$$\sum_{n=0}^{N} ||u(\cdot, T_n) - U_n^k||_1 \leq C \max(L - k\tilde{a}\Delta T, 0) \times \max(N - k, 0),$$

where the constant C can be estimated by

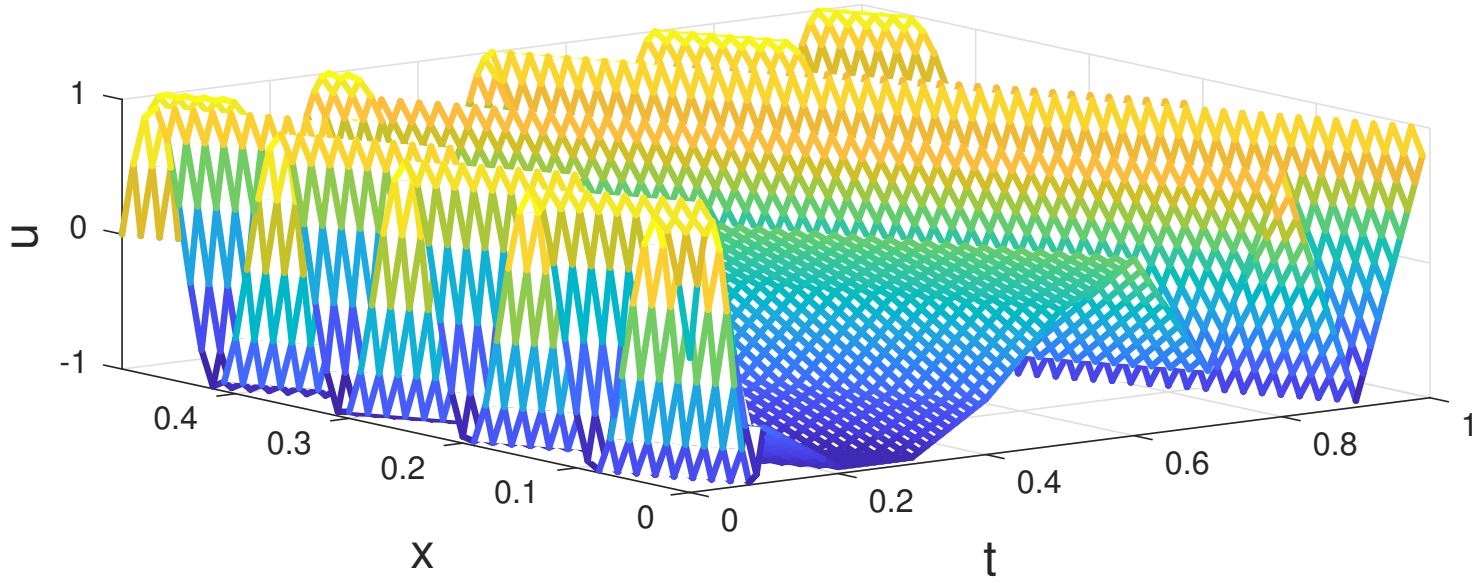
$$C = \max_{n=1,2,\ldots,N} ||u(\cdot, T_n) - U_n^0||_\infty.$$

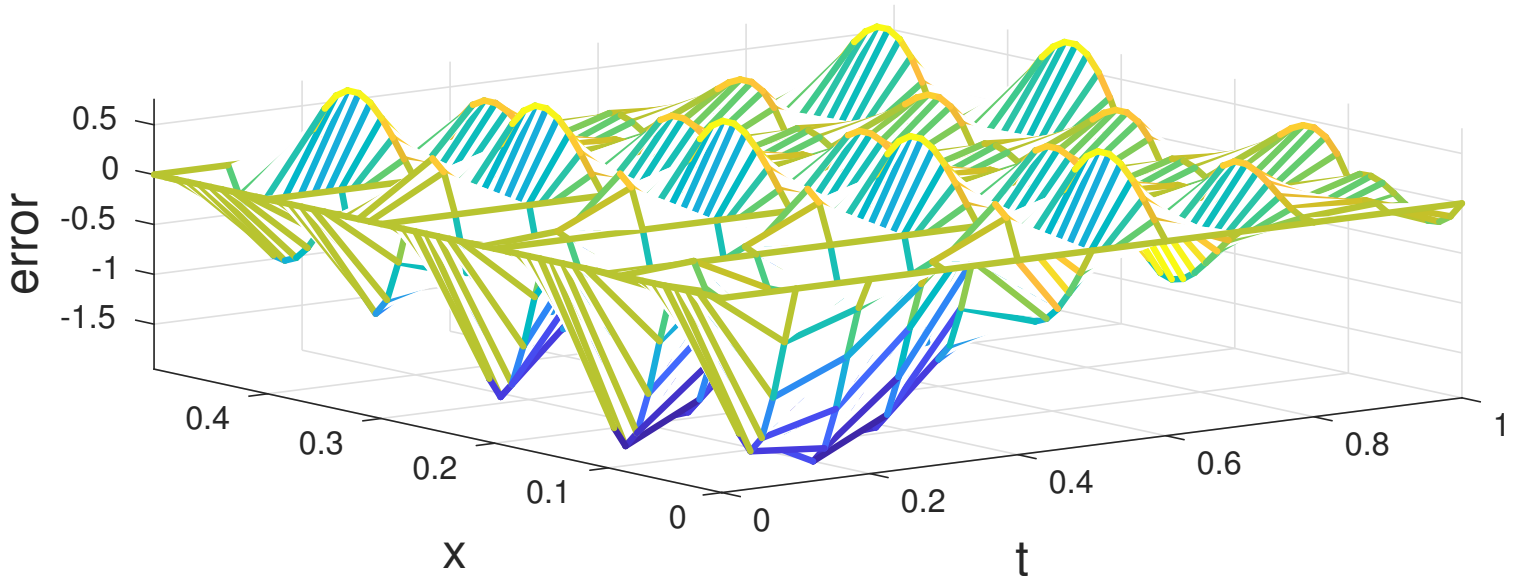
G (2008): Analysis of the Parareal Algorithm Applied to Hyperbolic Problems Using Characteristics, SeMA

Graphical Convergence Proof

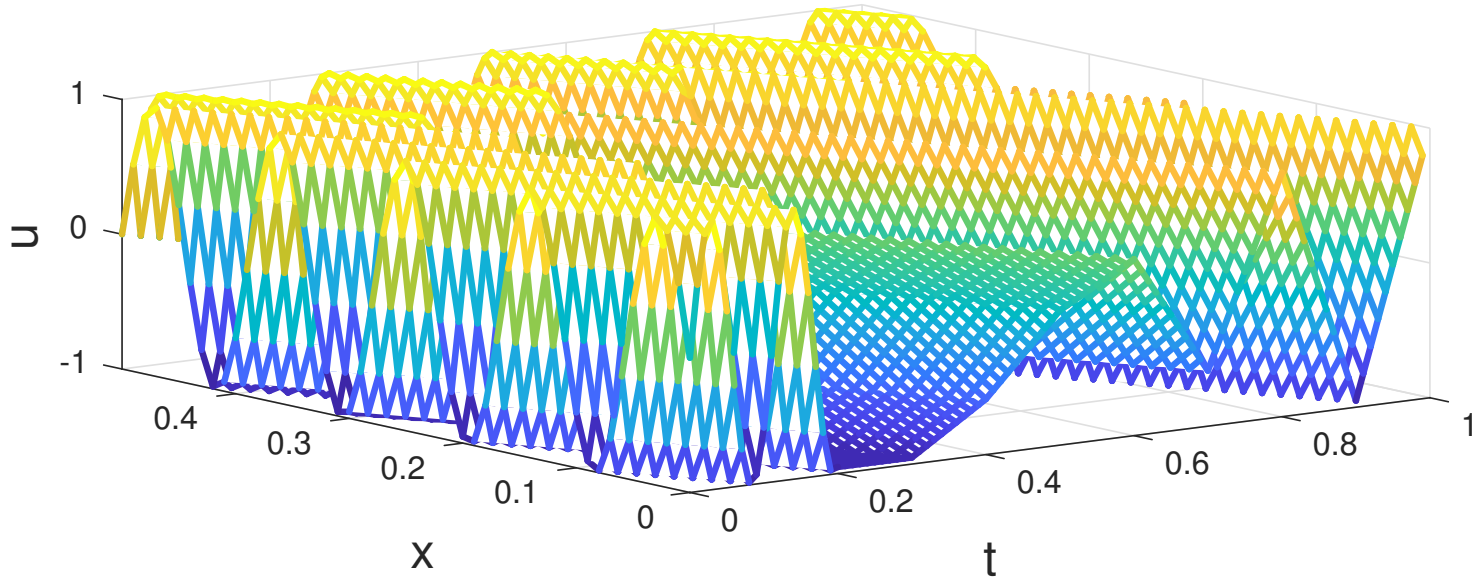


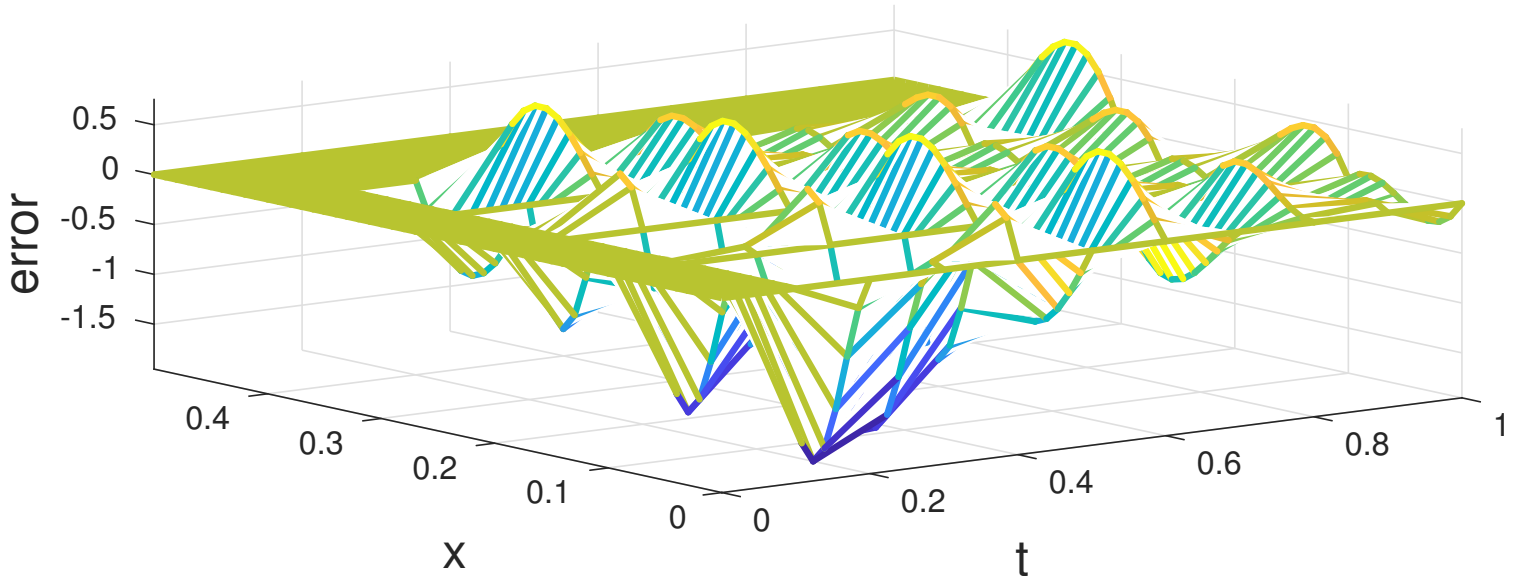
Non-Periodic Advection: $T = 1$, Iteration 1



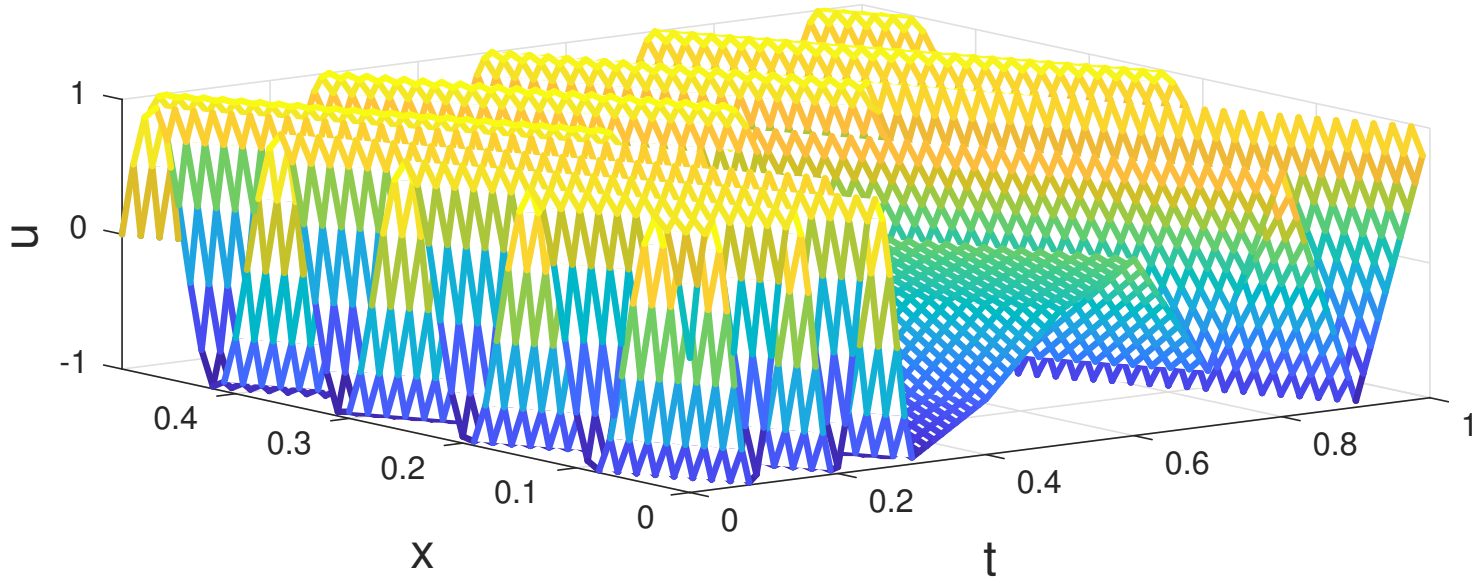


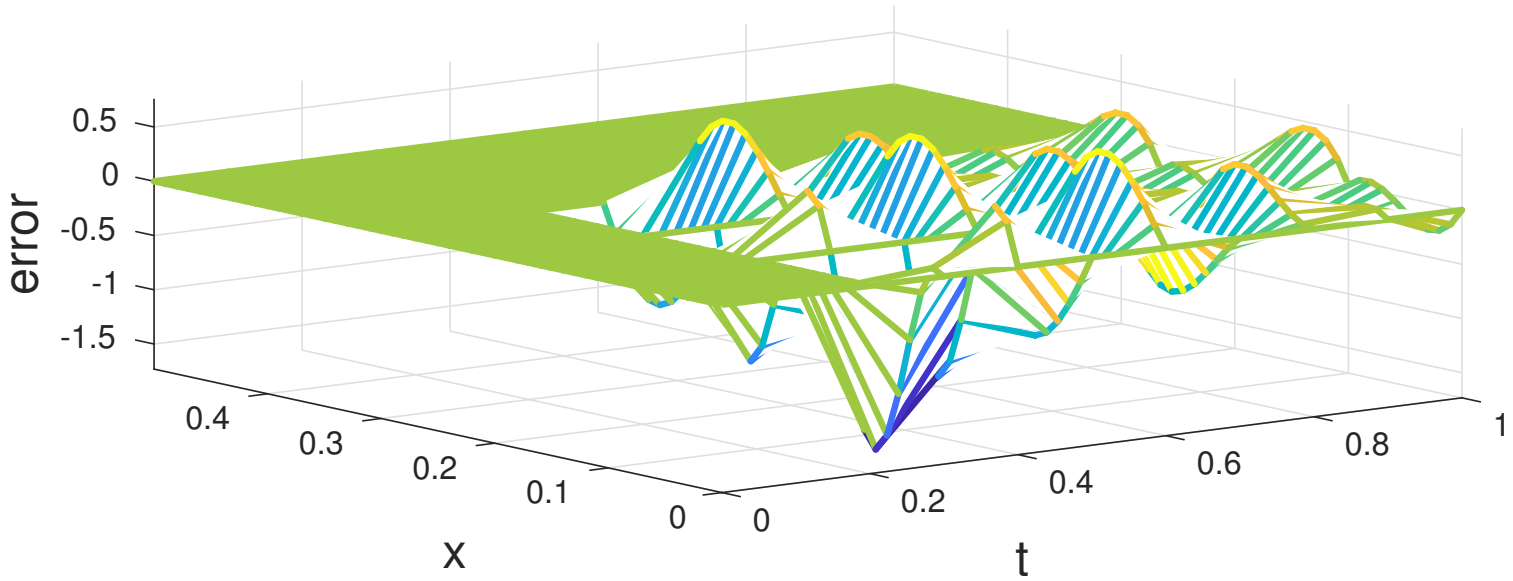
Non-Periodic Advection: $T = 1$, Iteration 2



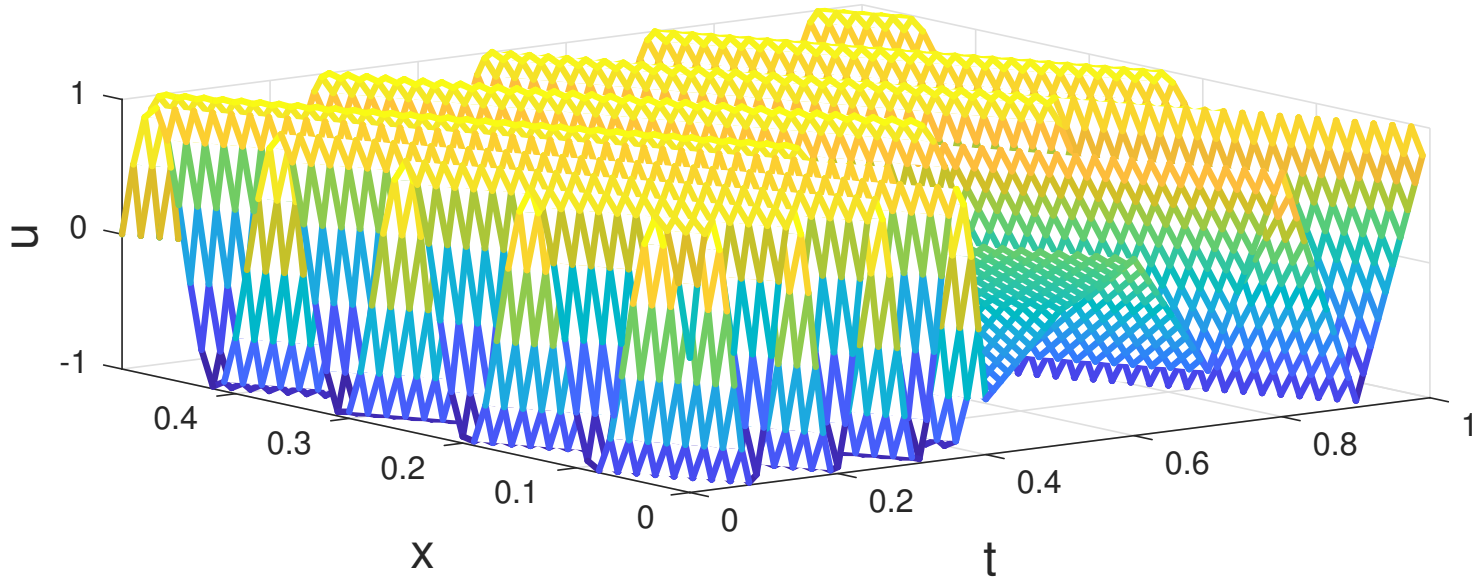


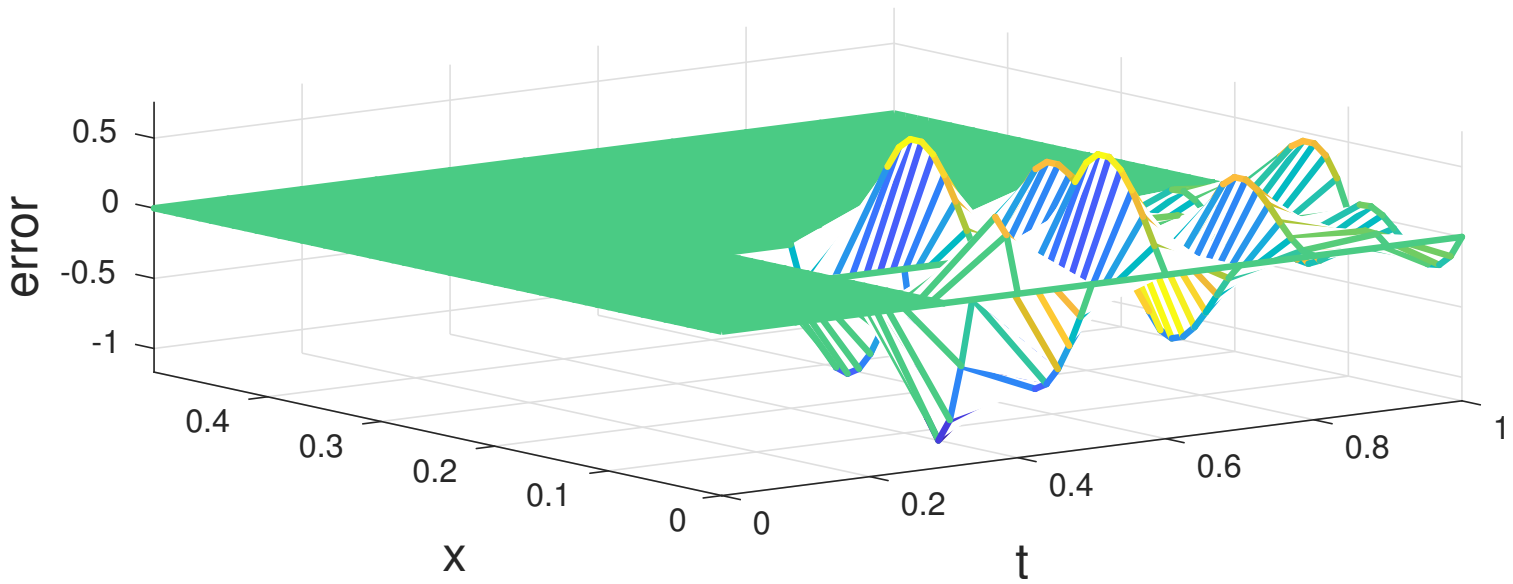
Non-Periodic Advection: $T = 1$, Iteration 3



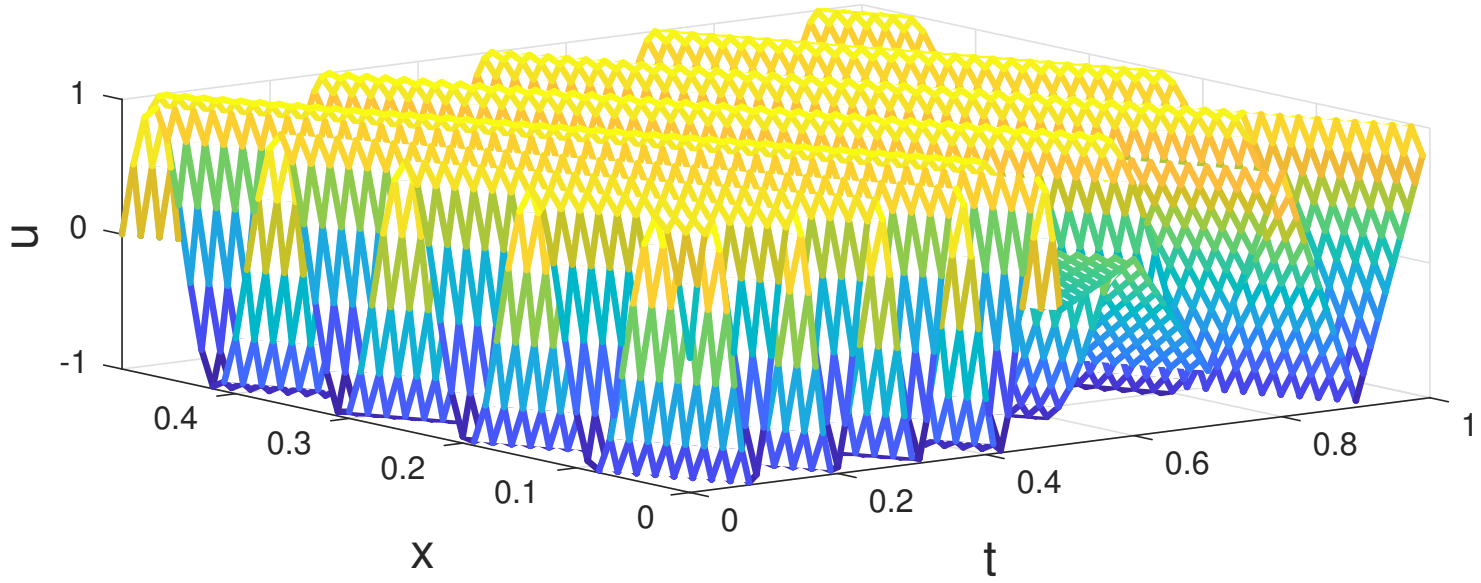


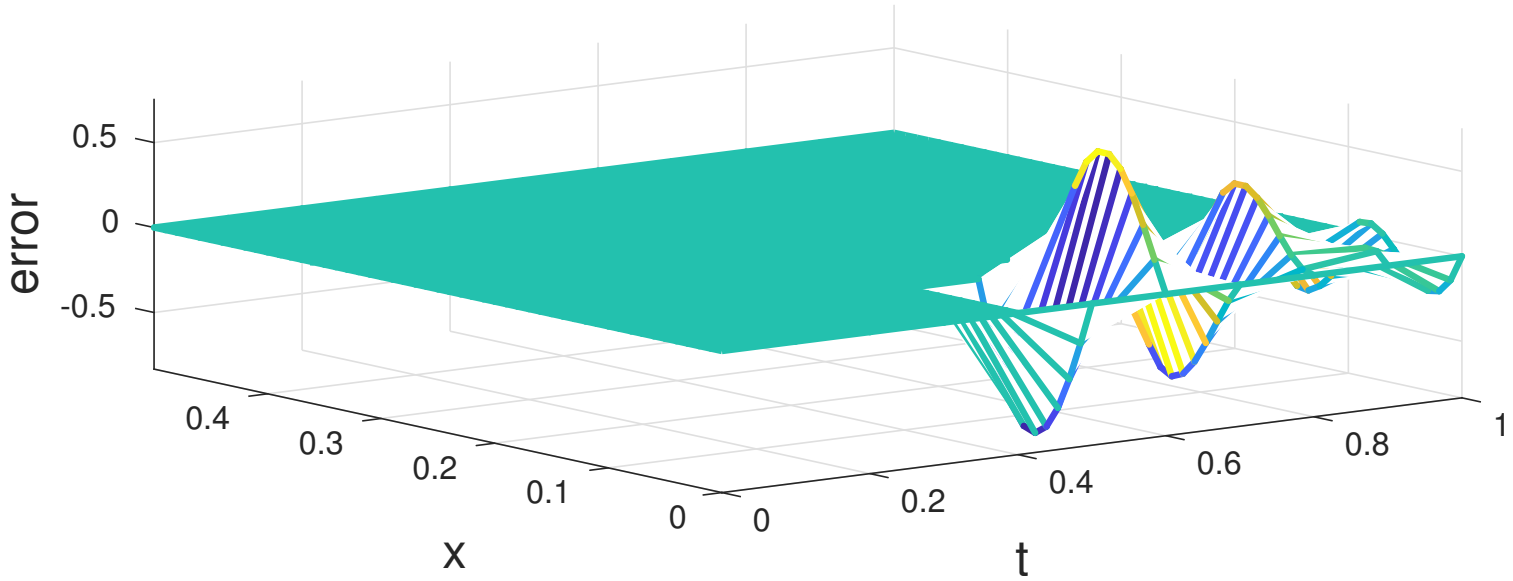
Non-Periodic Advection: $T = 1$, Iteration 4



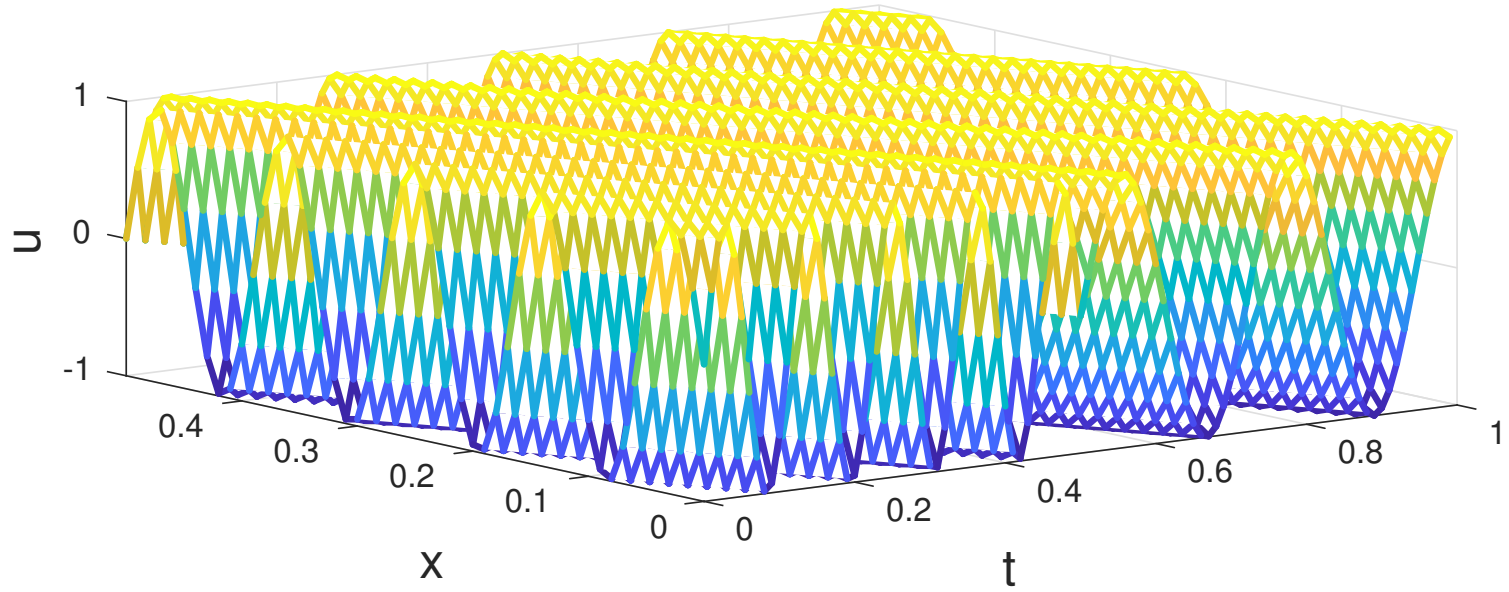


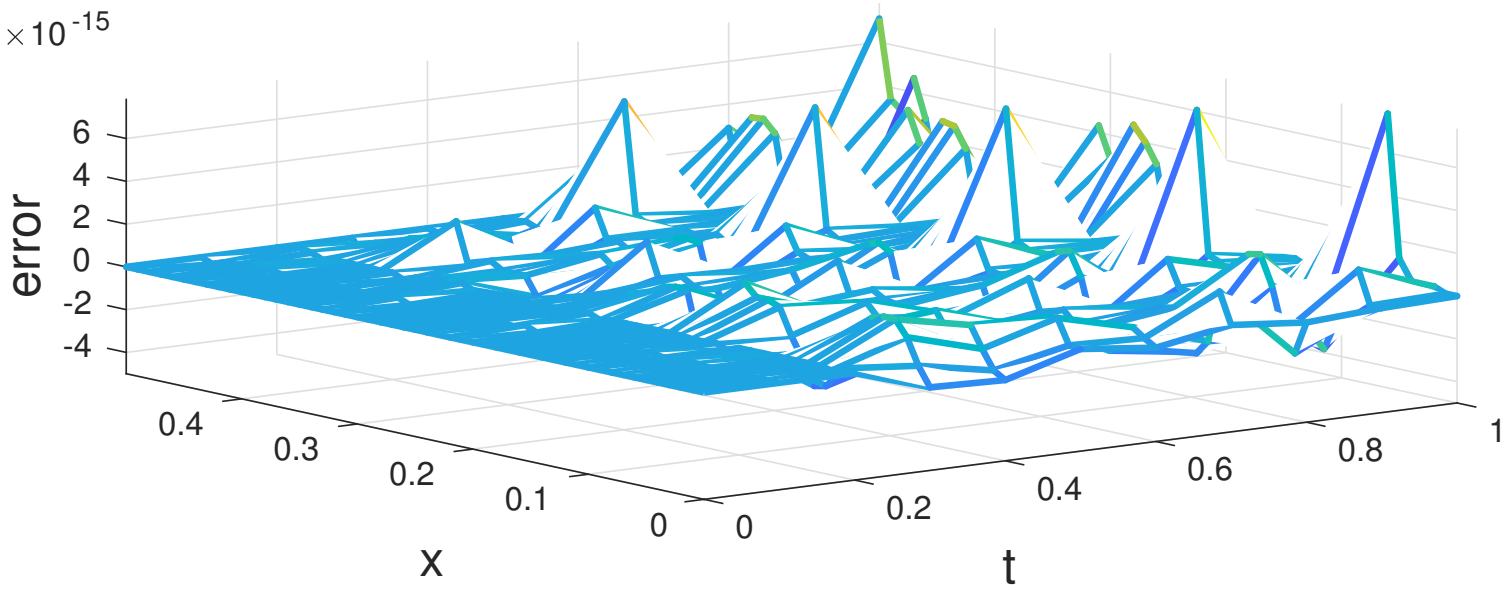
Non-Periodic Advection: $T = 1$, Iteration 5



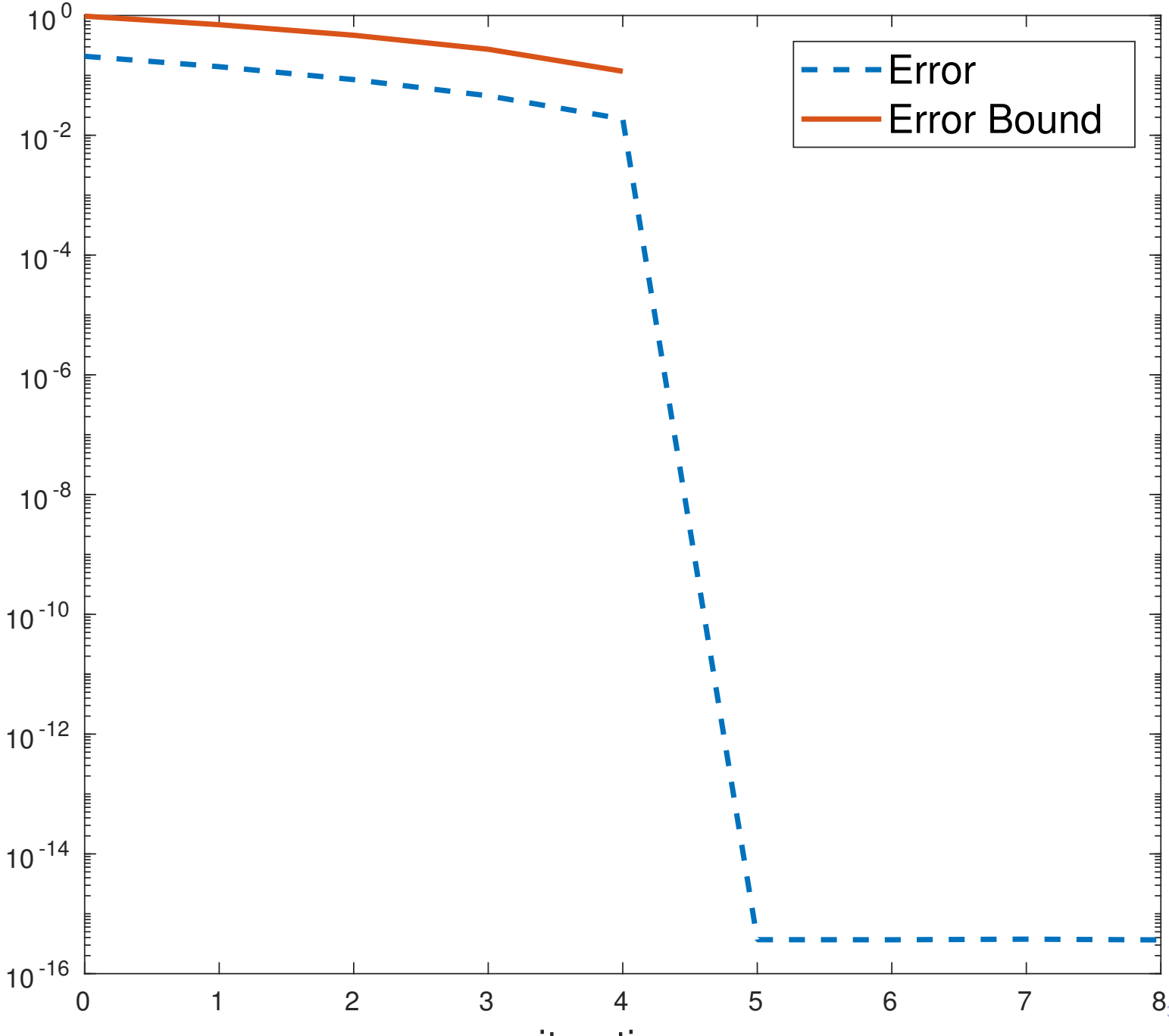


Non-Periodic Advection: $T = 1$, Iteration 6

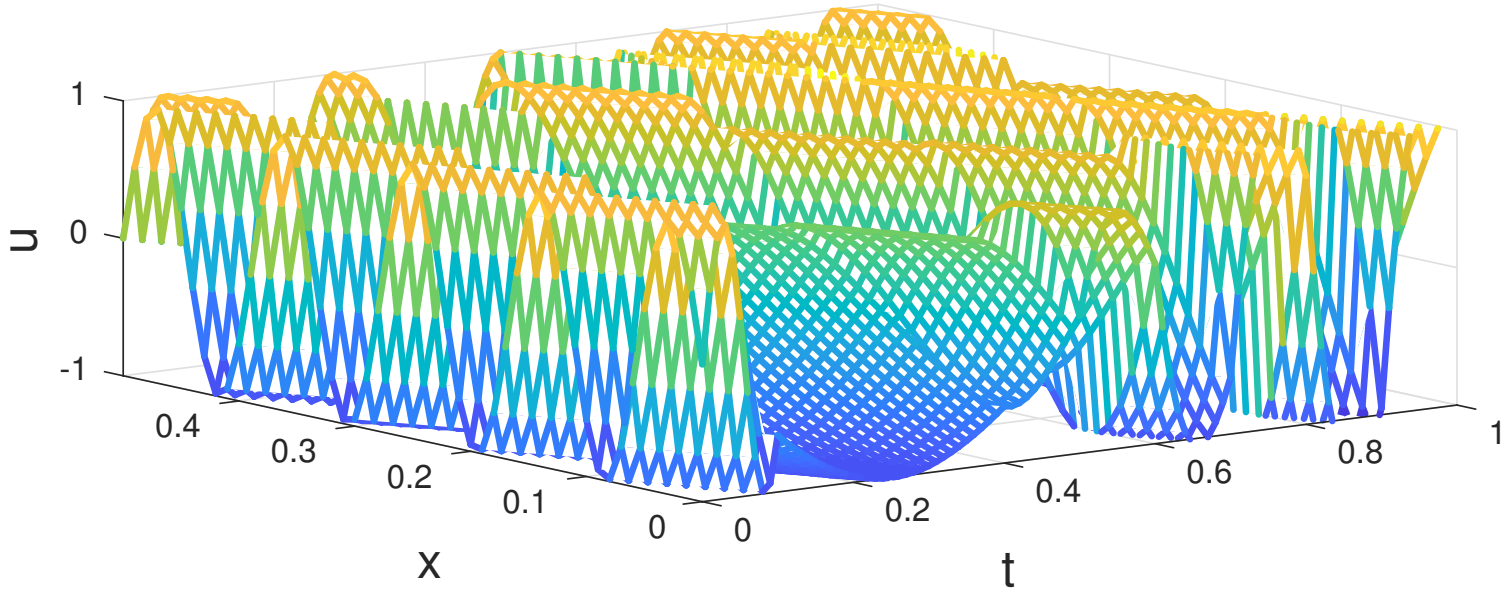


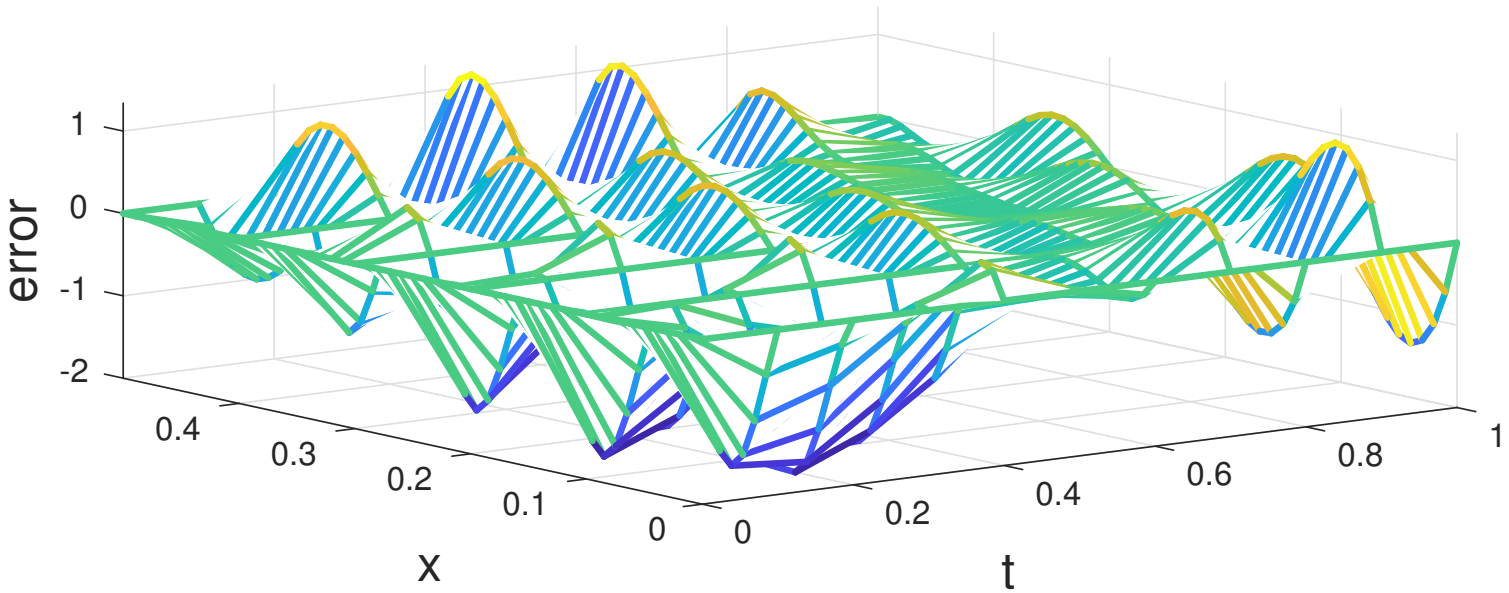


Non-Periodic Advection: Error in L^1

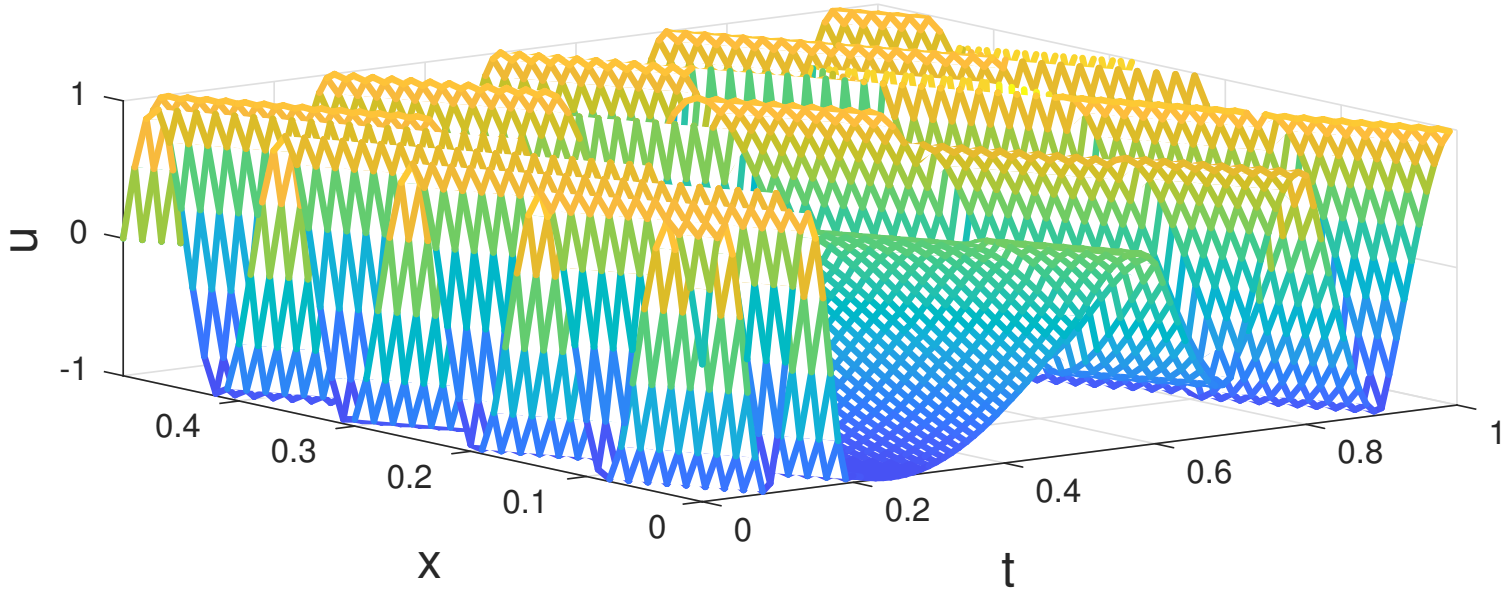


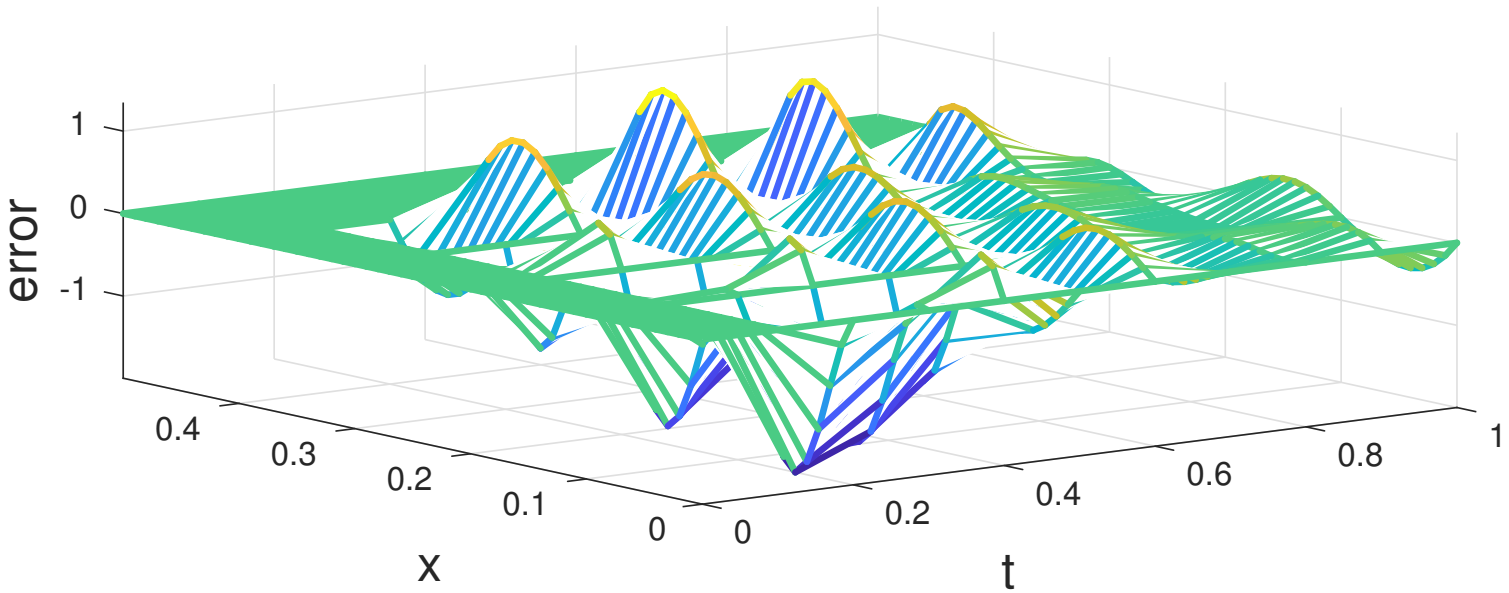
With Interpolation: $T = 1$, Iteration 1



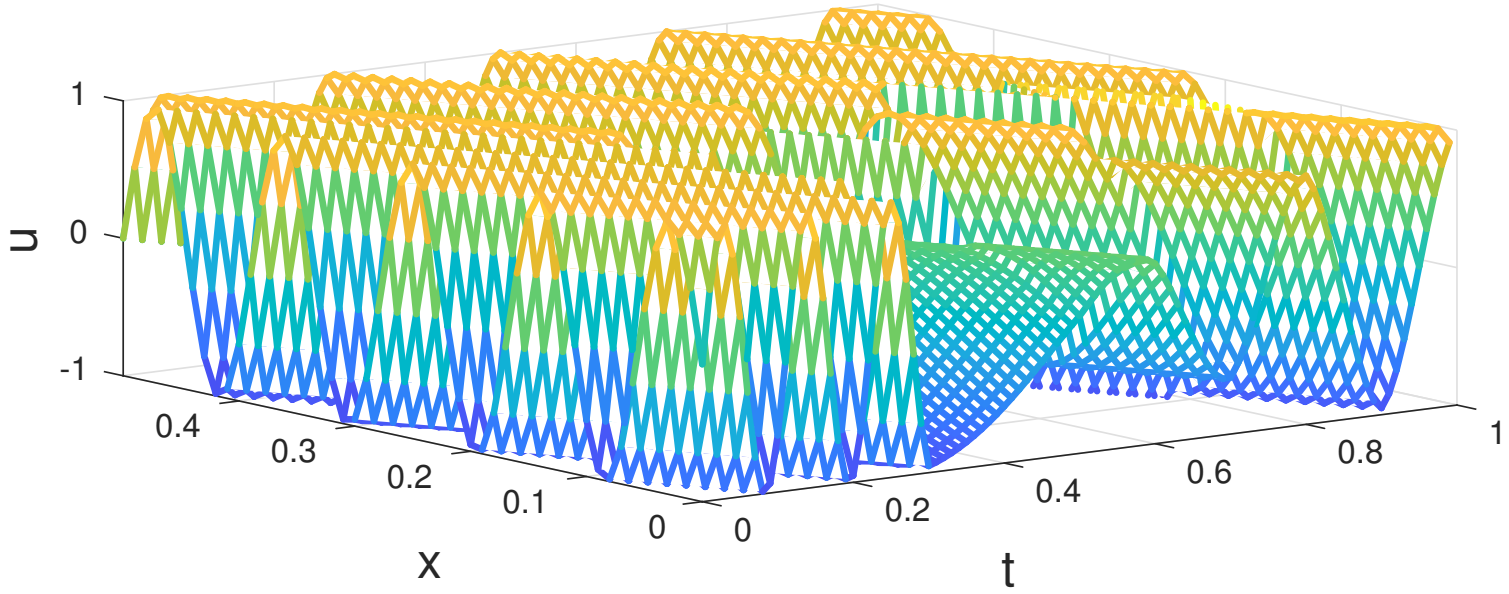


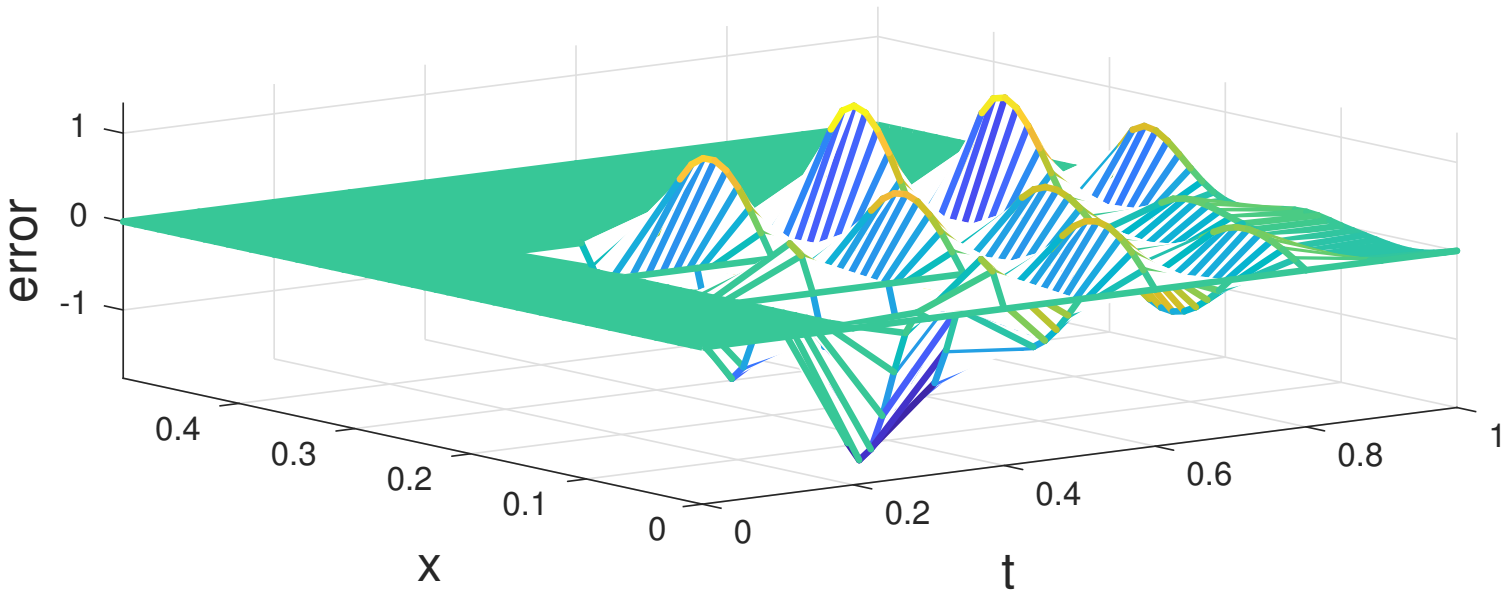
With Interpolation: $T = 1$, Iteration 2



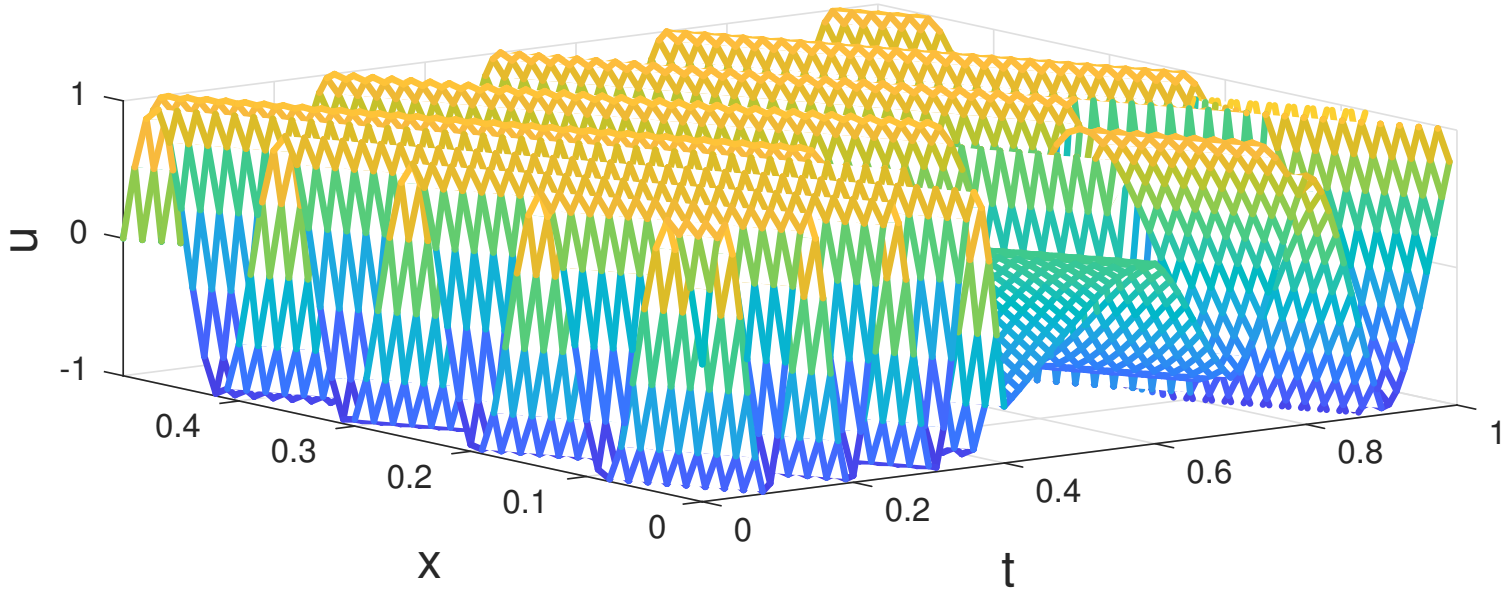


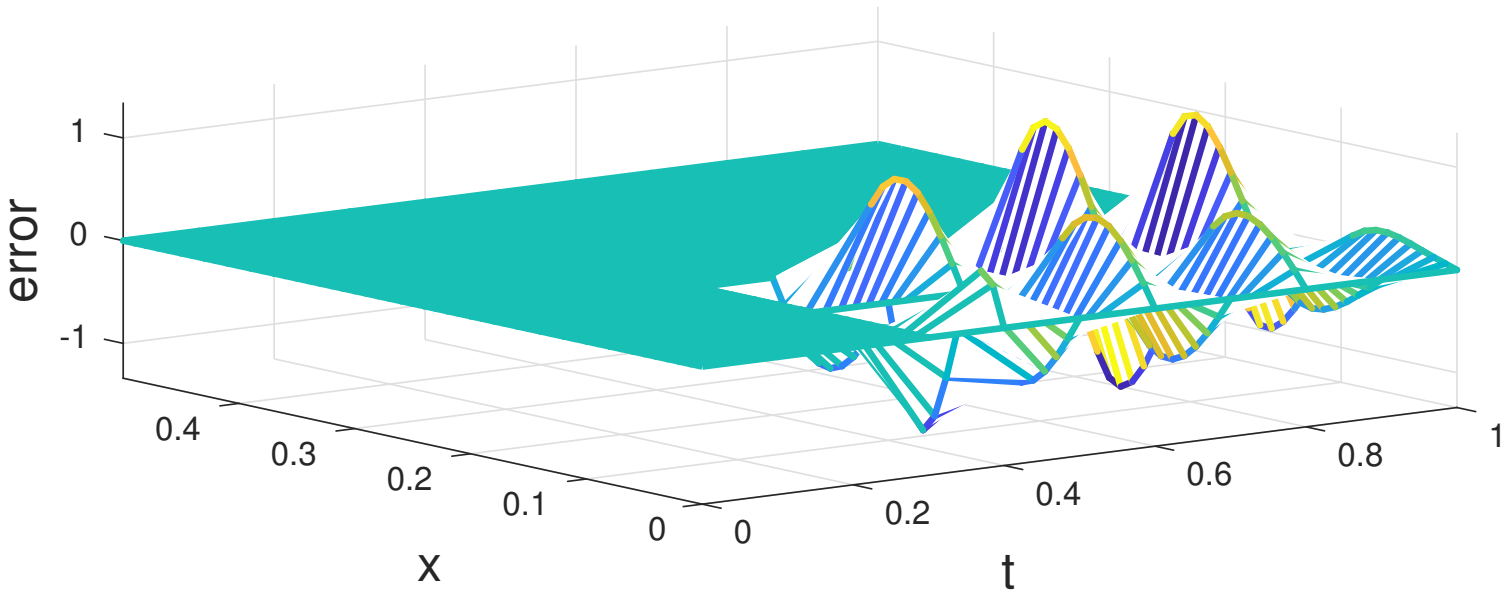
With Interpolation: $T = 1$, Iteration 3



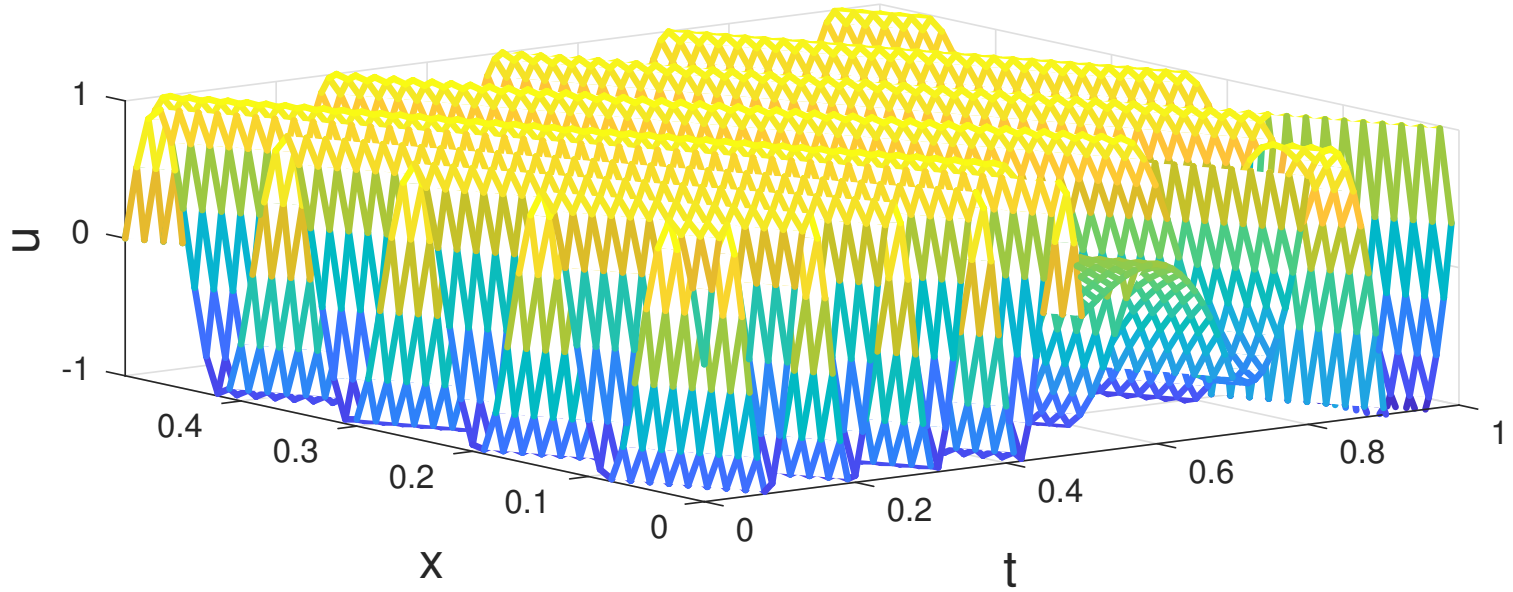


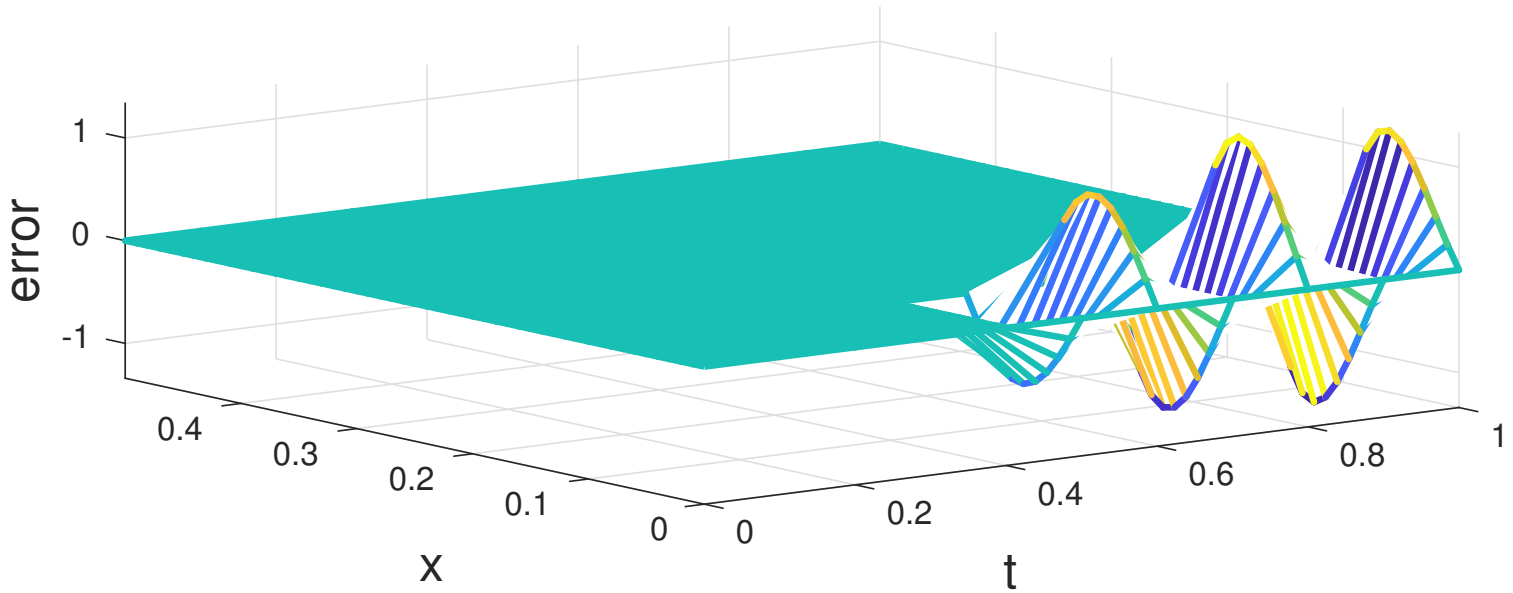
With Interpolation: $T = 1$, Iteration 4



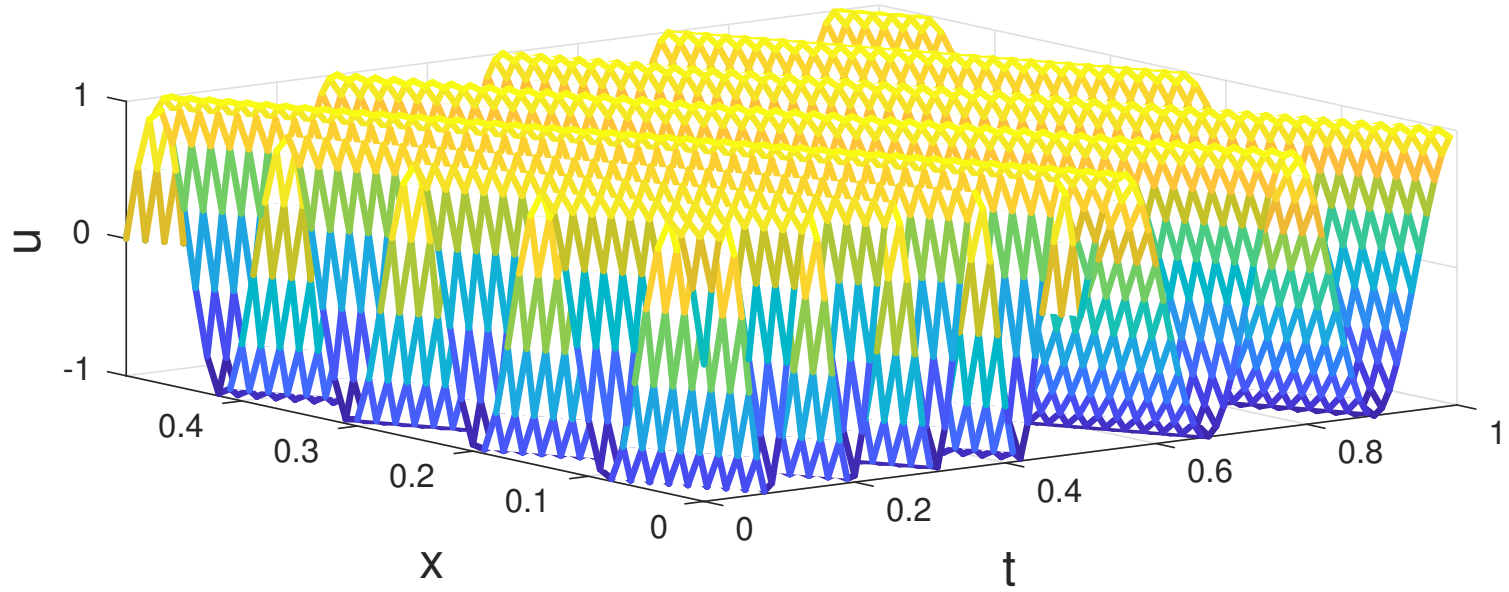


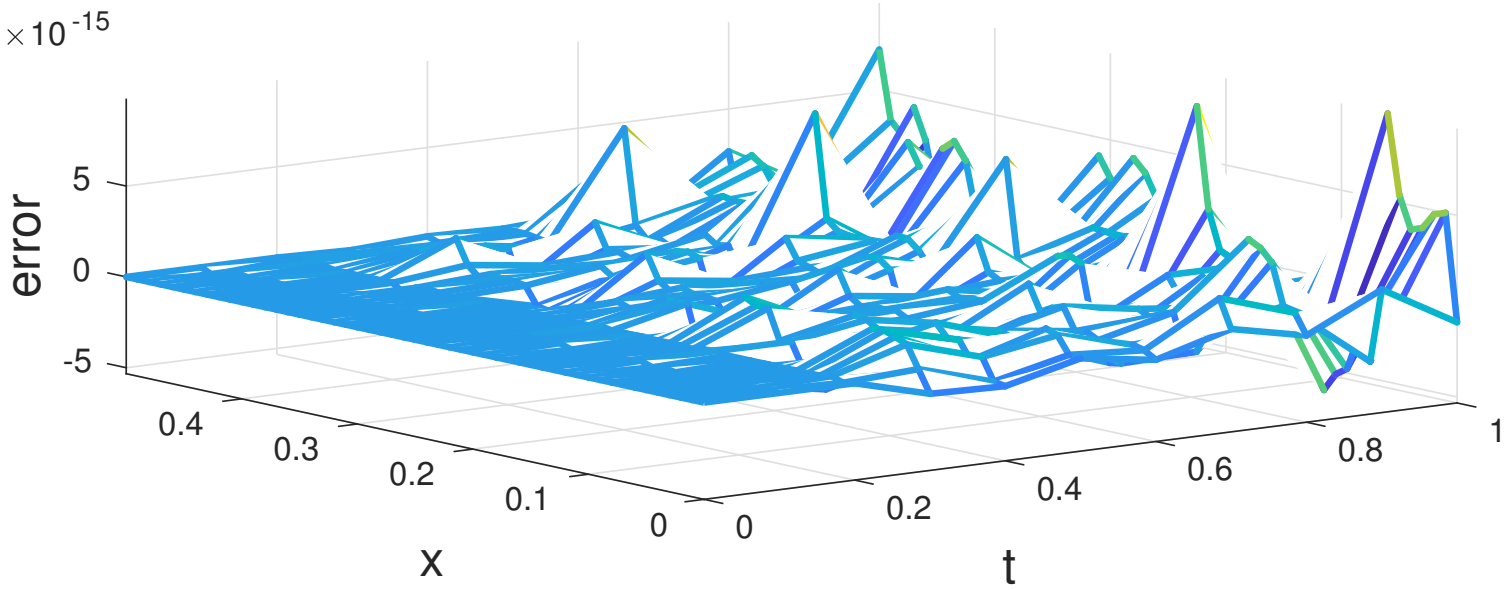
With Interpolation: $T = 1$, Iteration 5



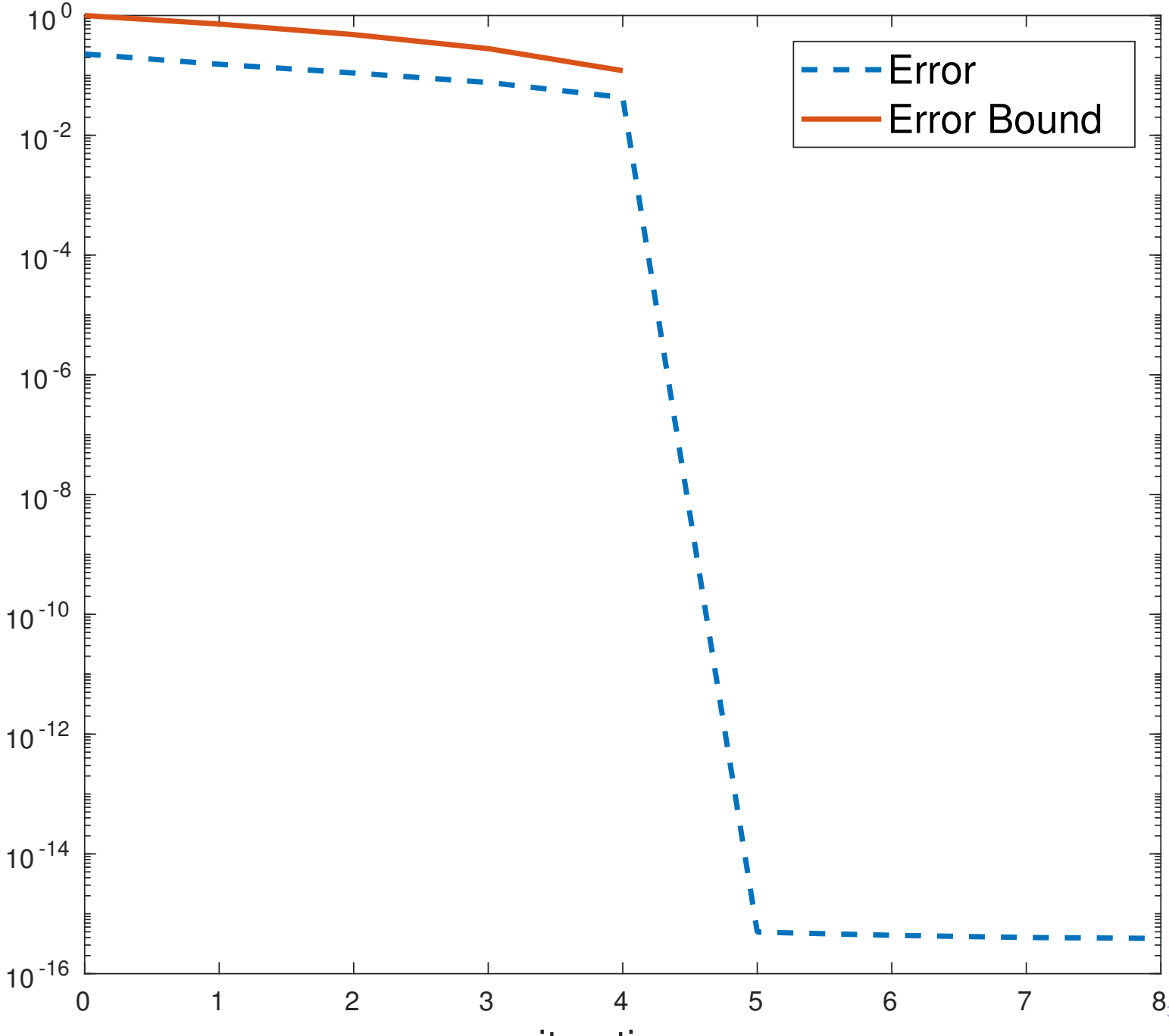


With Interpolation: $T = 1$, Iteration 6

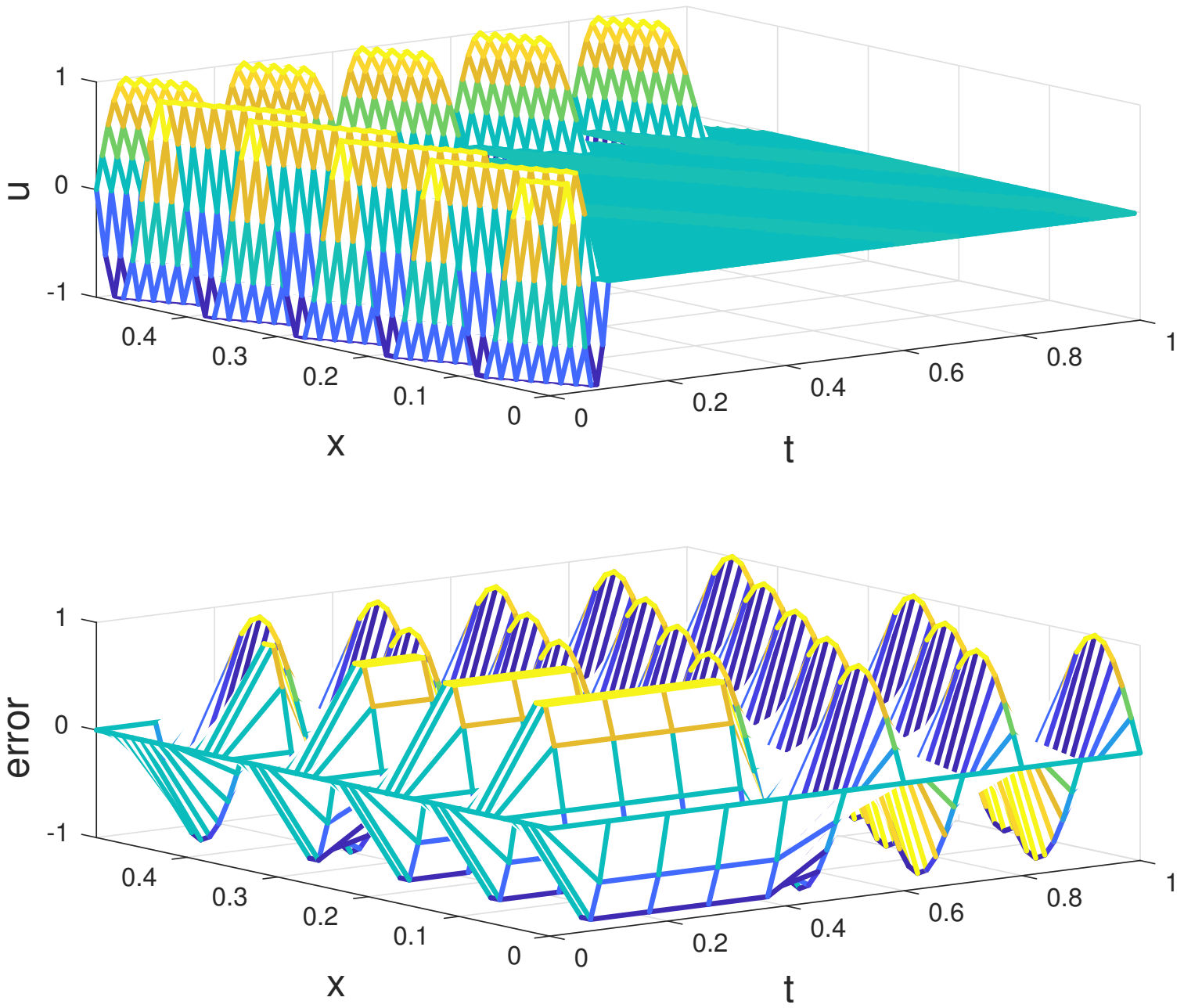




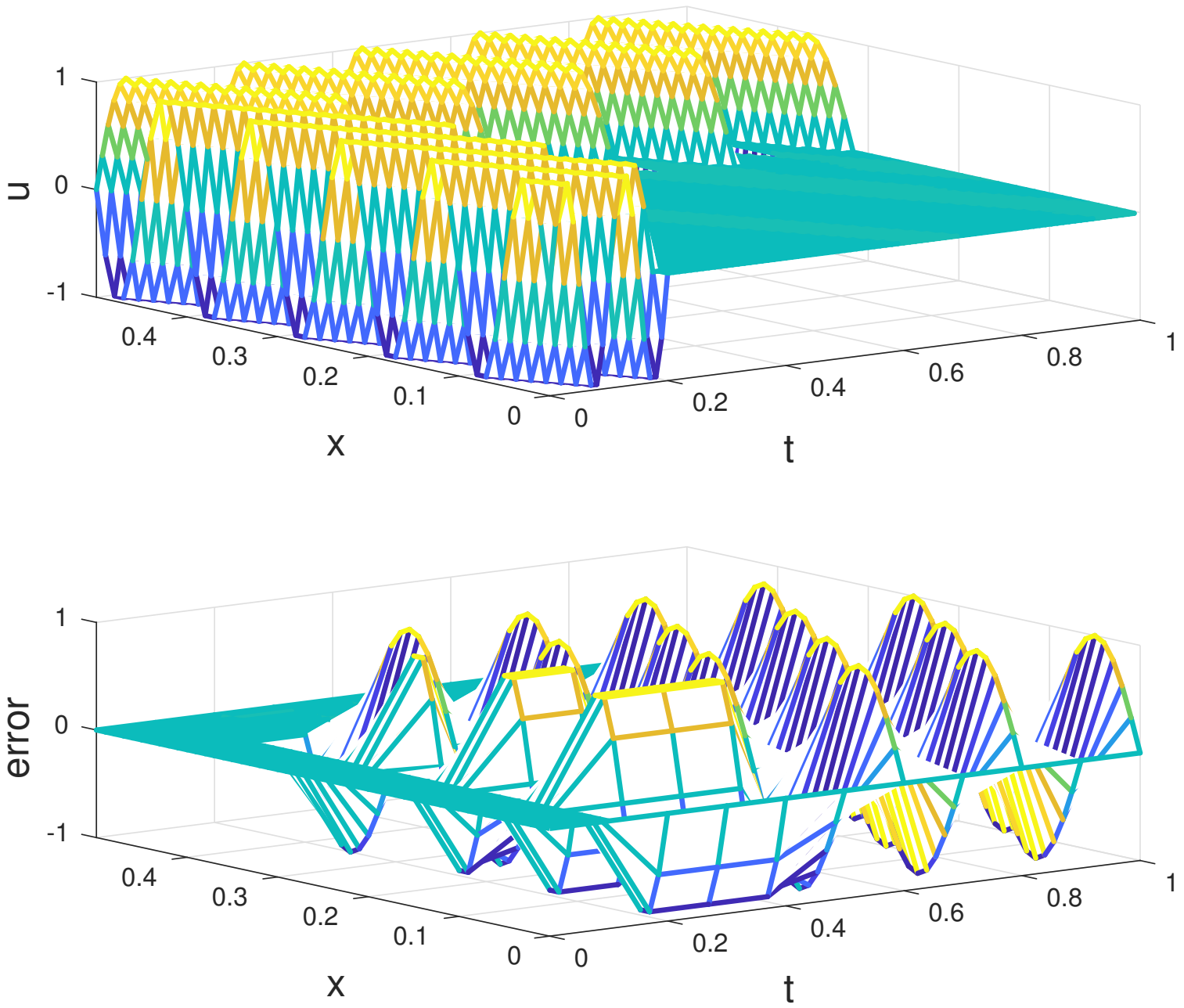
With Interpolation: Error in L^1



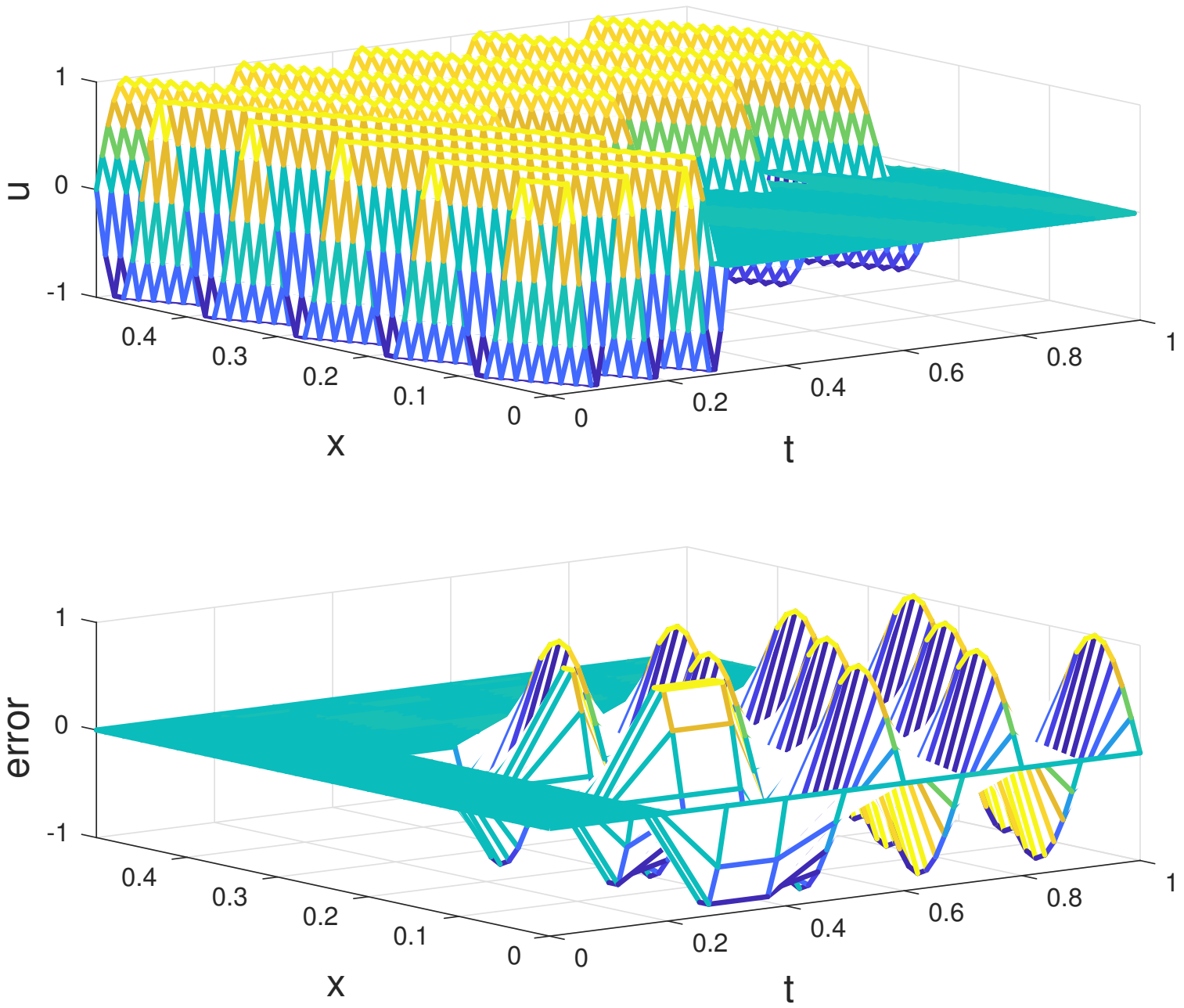
With Interplation Exact G: $T = 1$, Iteration 1



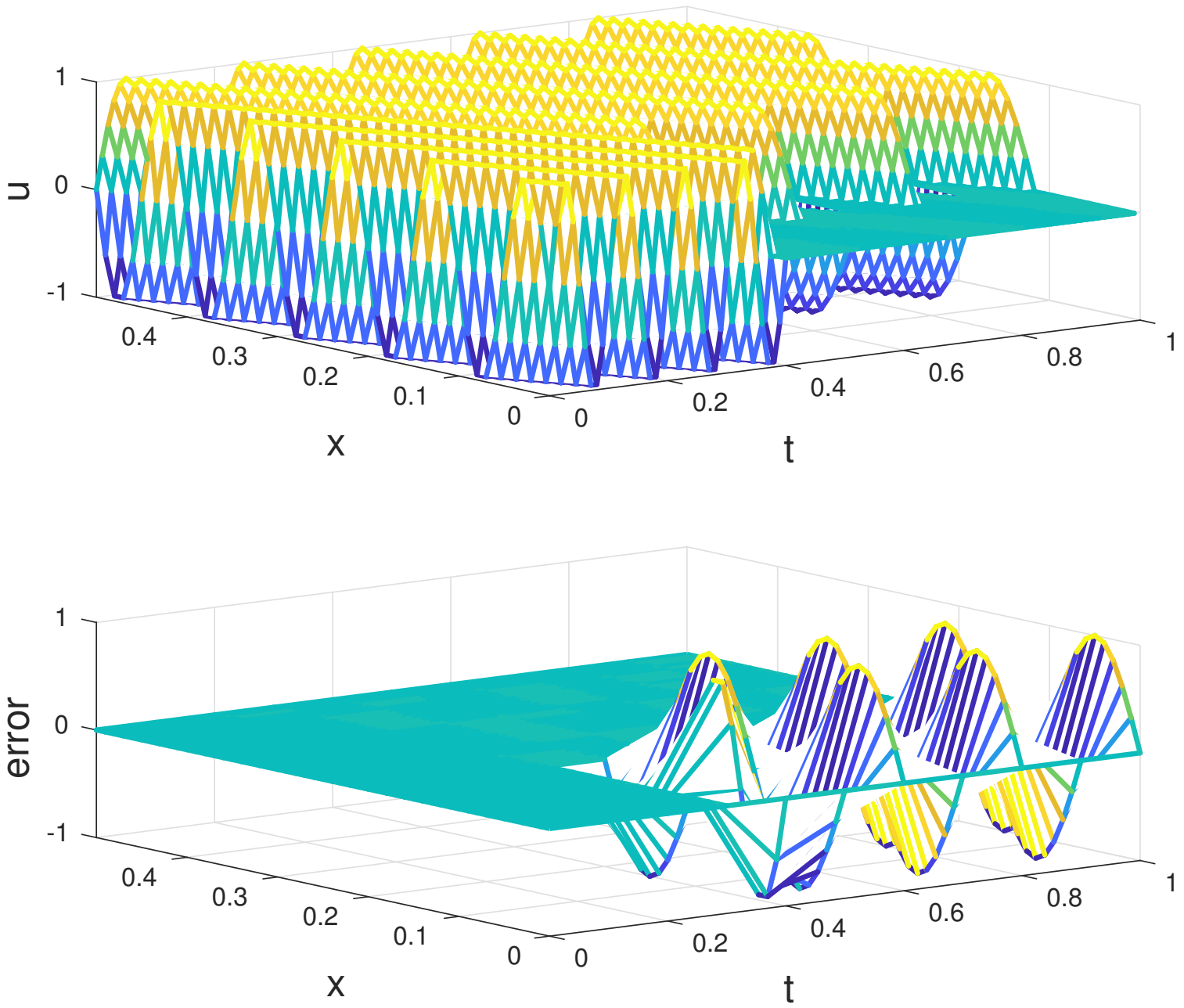
With Interplation Exact G: $T = 1$, Iteration 2



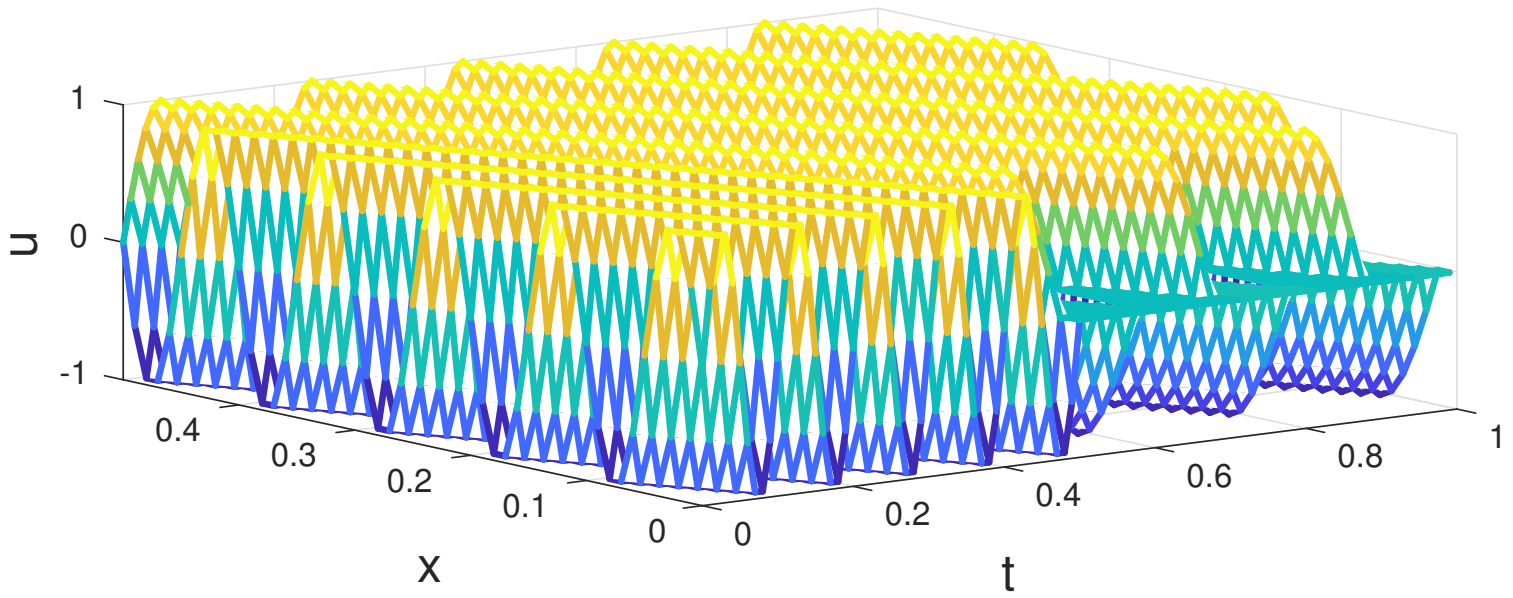
With Interplation Exact G: $T = 1$, Iteration 3

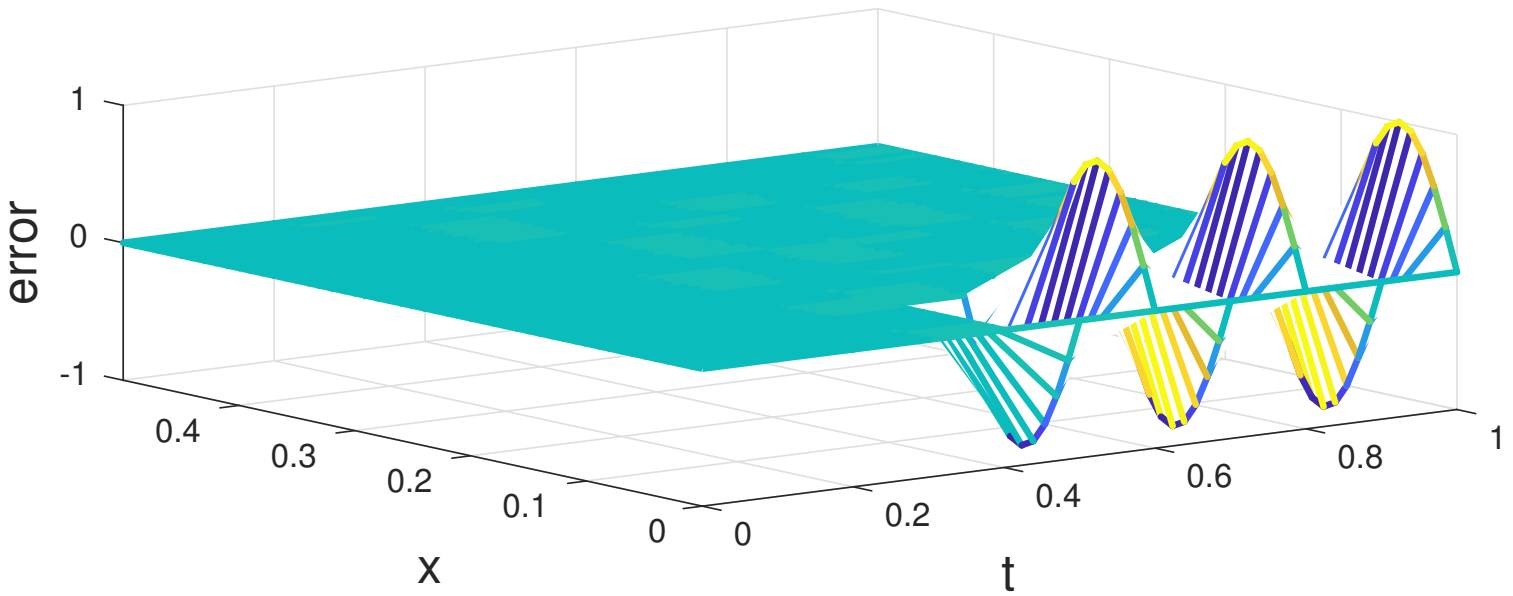


With Interplation Exact G: $T = 1$, Iteration 4

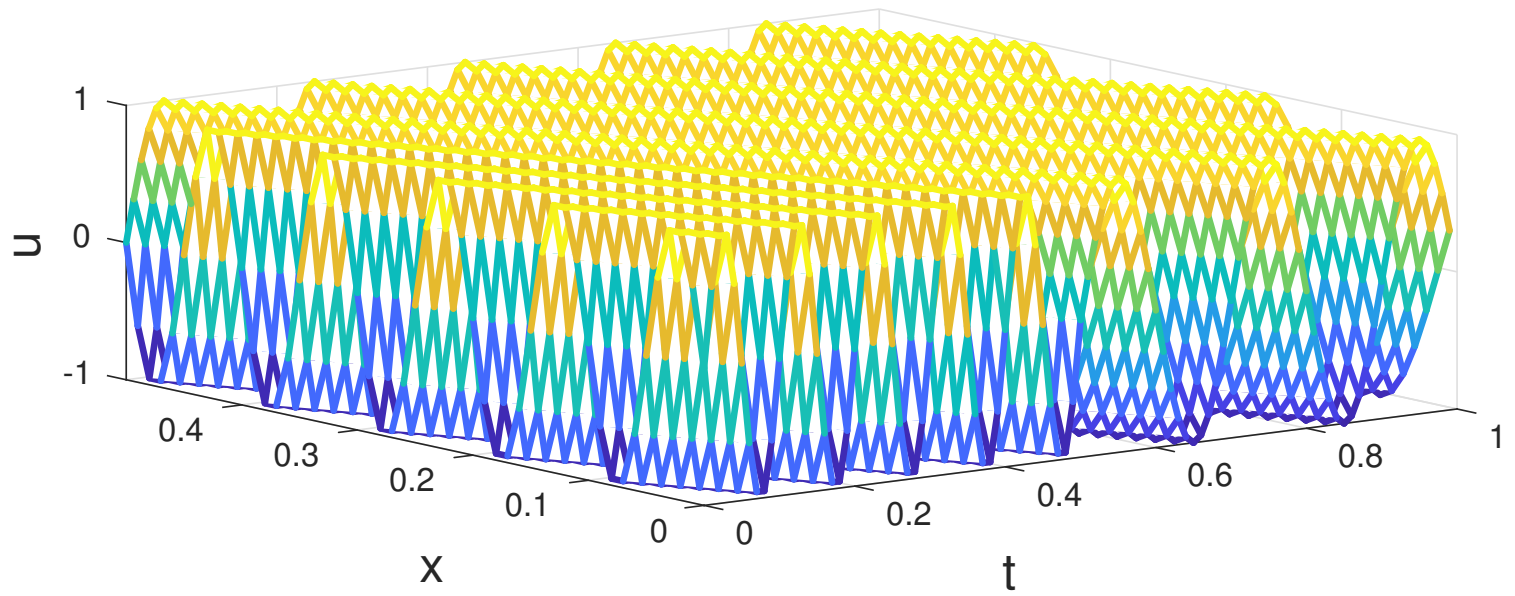


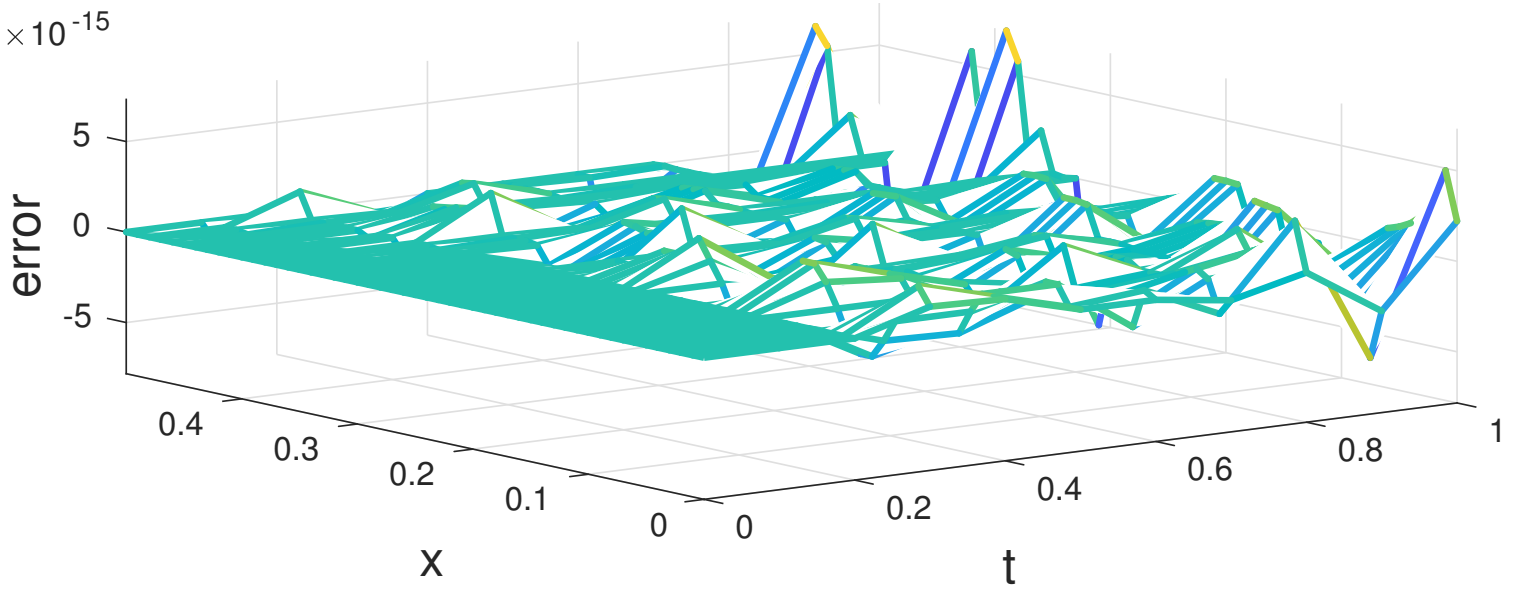
With Interplation Exact G: $T = 1$, Iteration 5



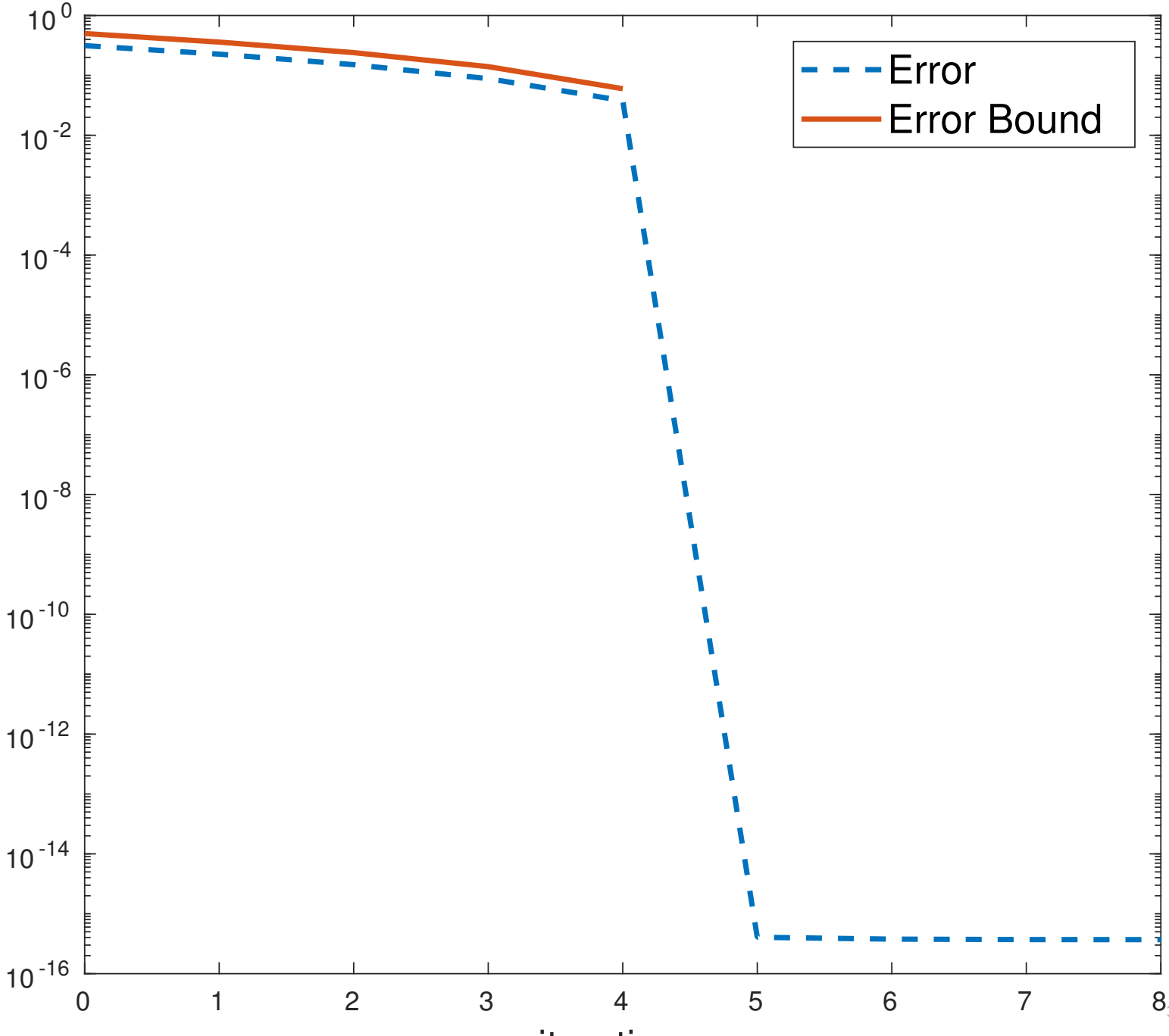


With Interplation Exact G: $T = 1$, Iteration 6

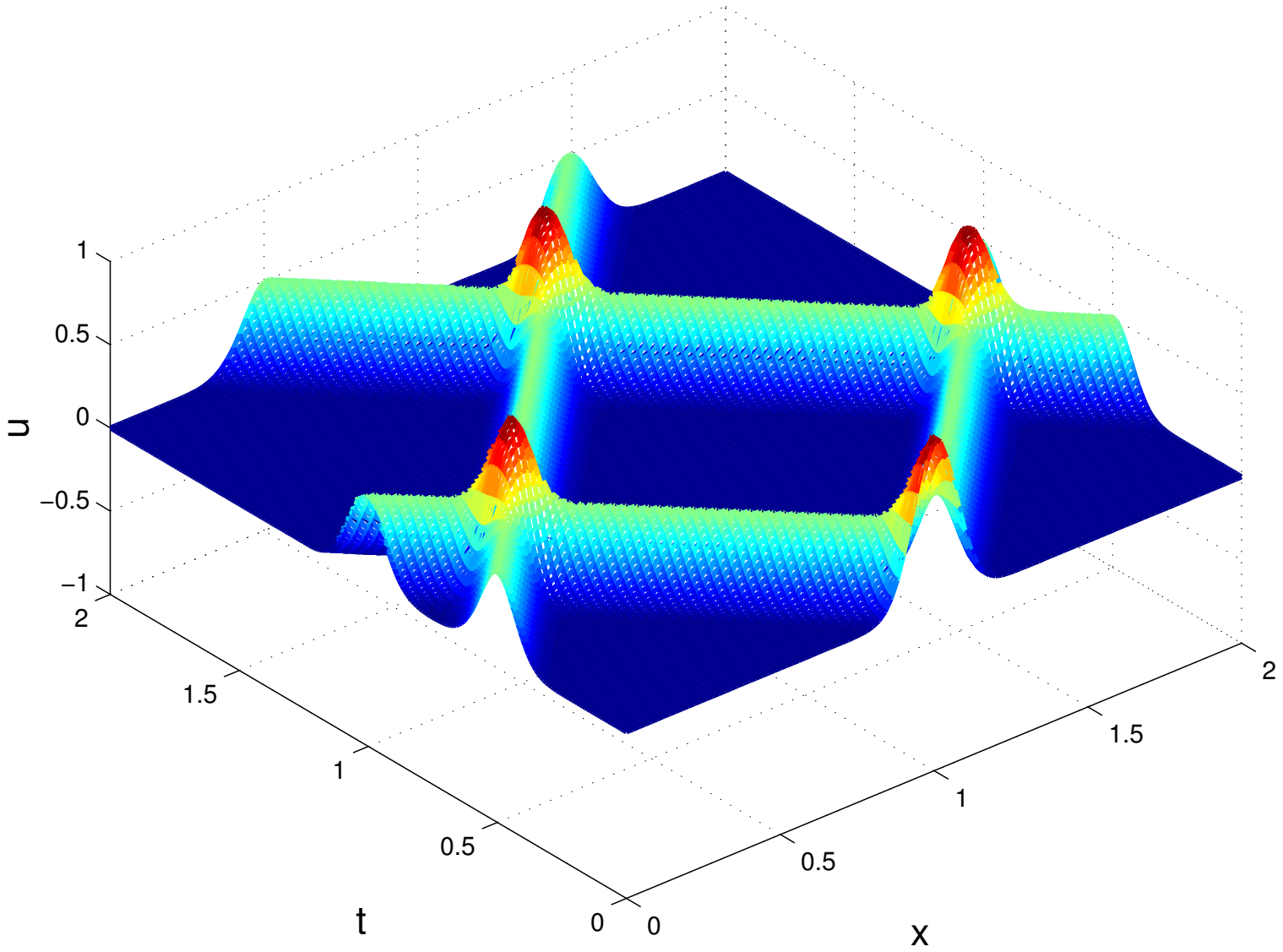




With Interplation Exact G: Error in L^1

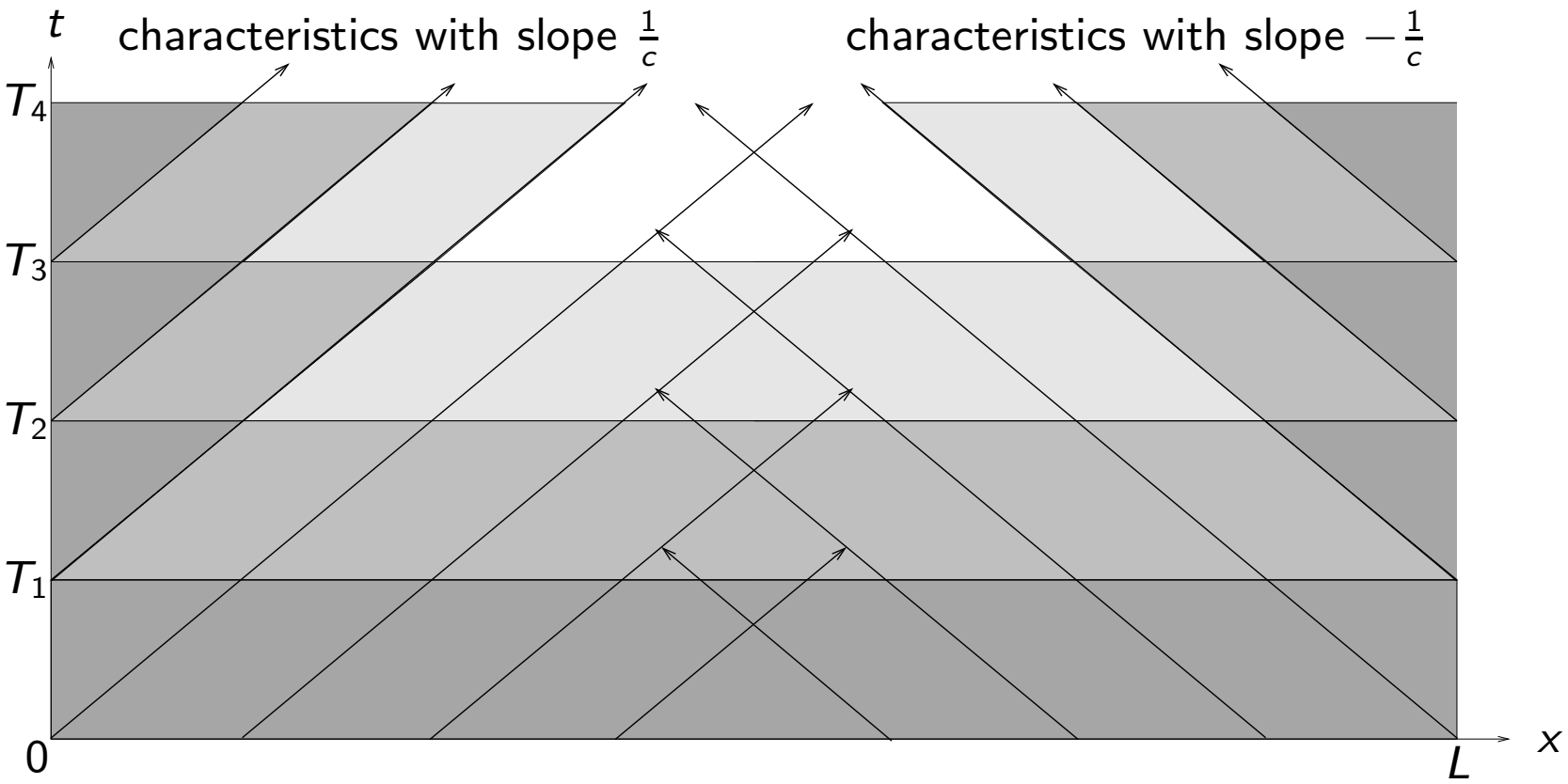


Second Order Wave Equation

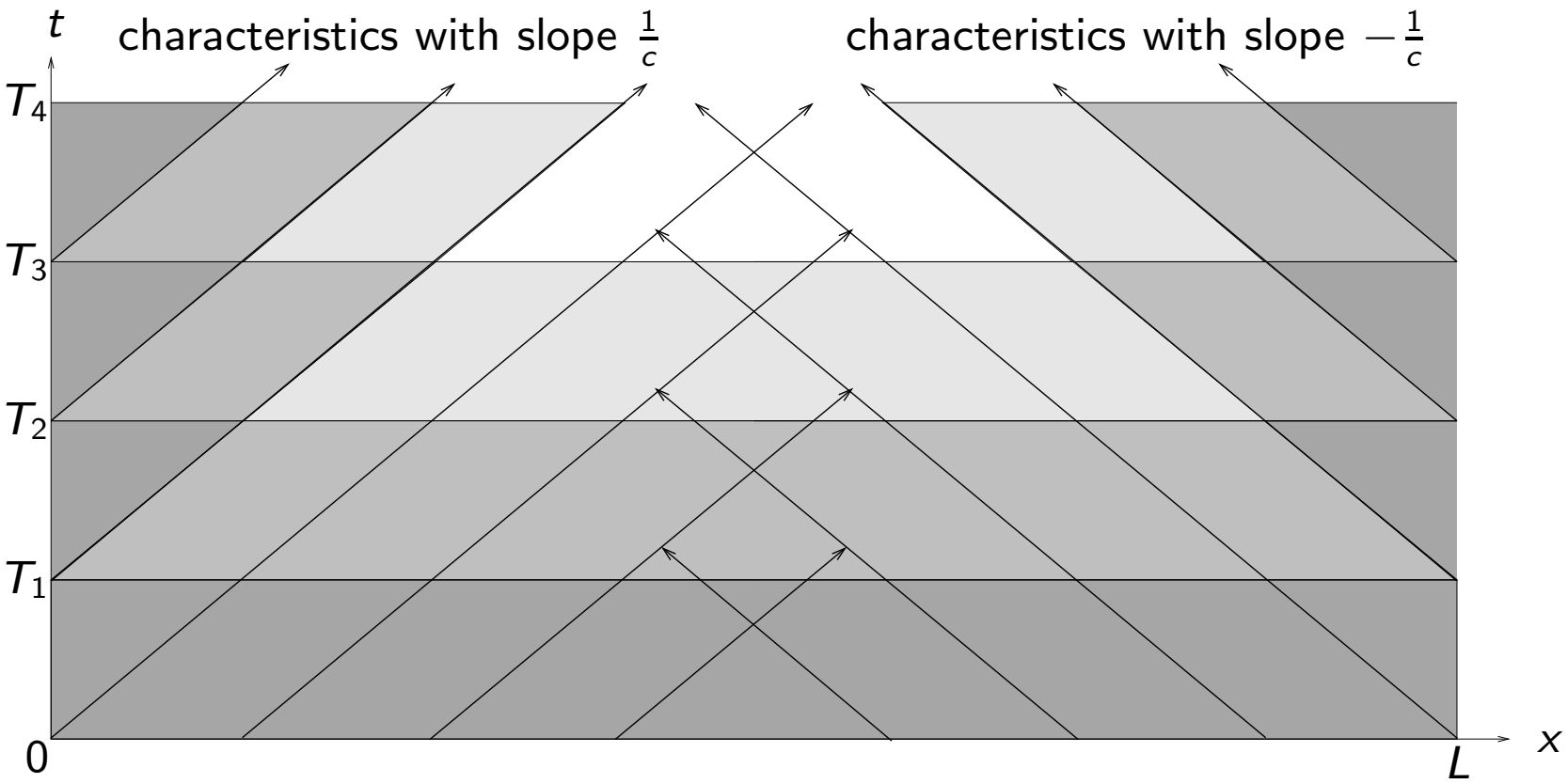


$$u_{tt} = c^2 \Delta u.$$

Same Argument as Non-Periodic Advection?

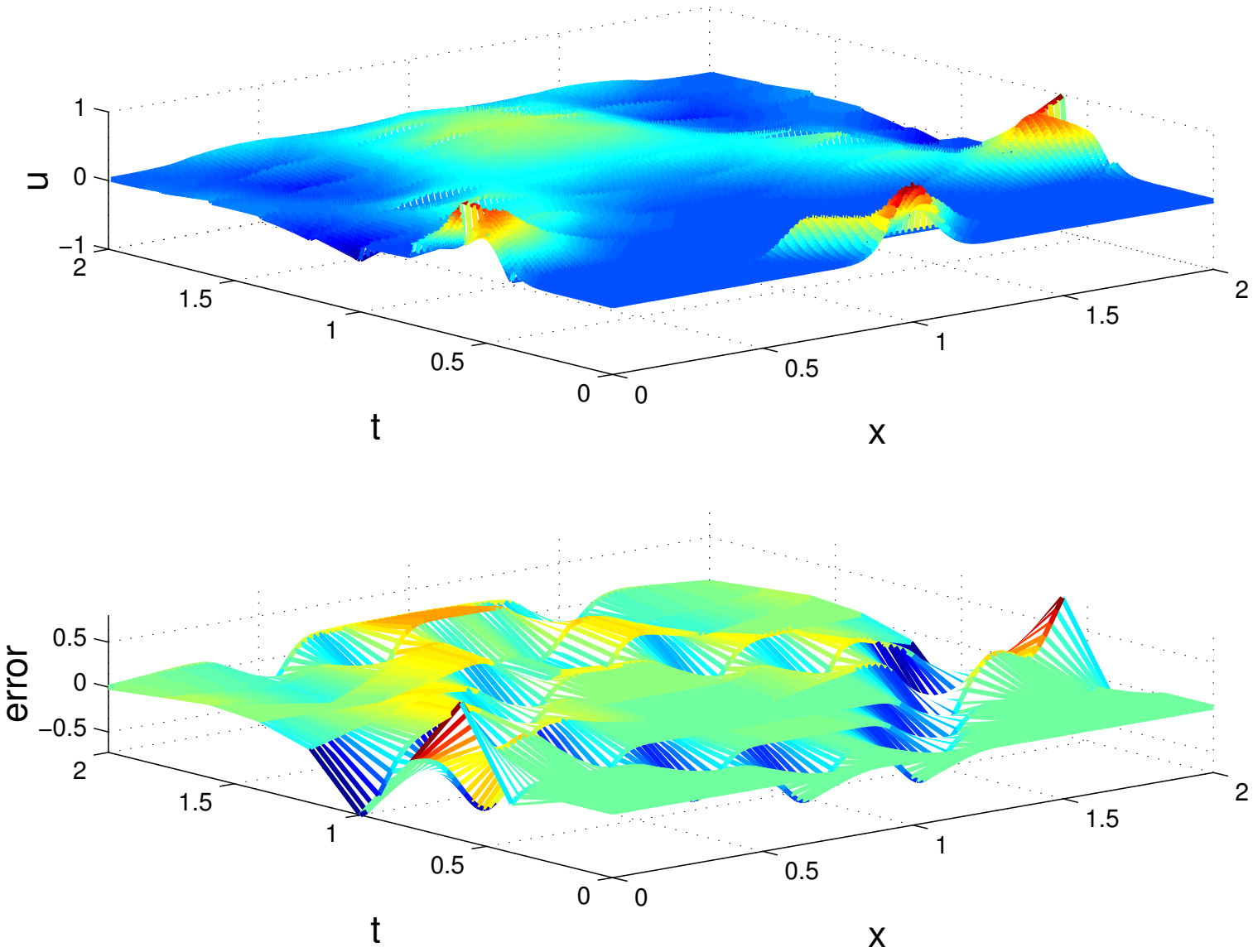


Same Argument as Non-Periodic Advection?

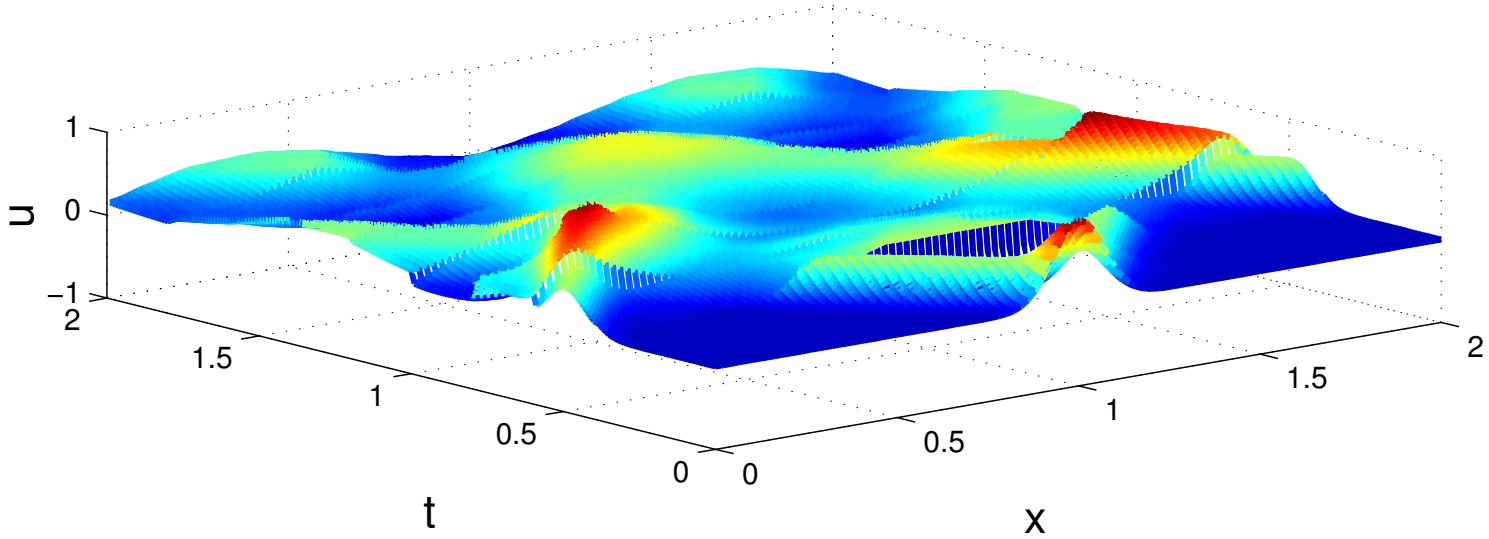


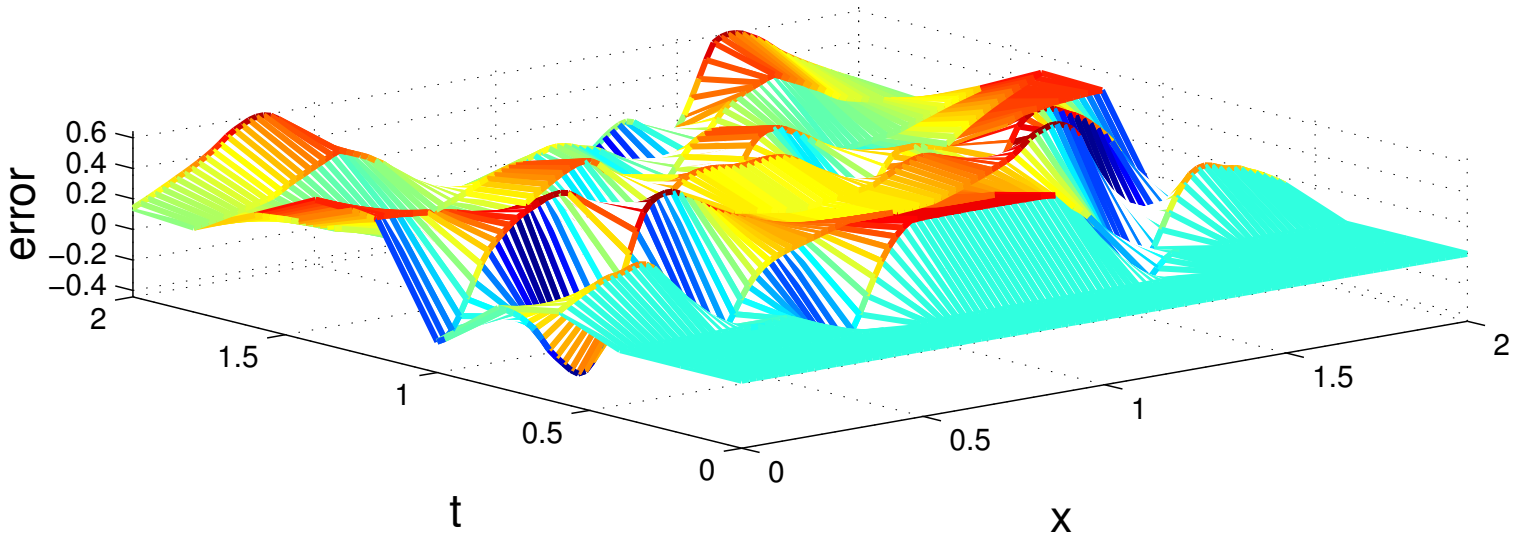
No: The two propagation directions do not allow a similar convergence argument like for the advection equation.

Wave Equation Equation: $T = 2$, Iteration 1

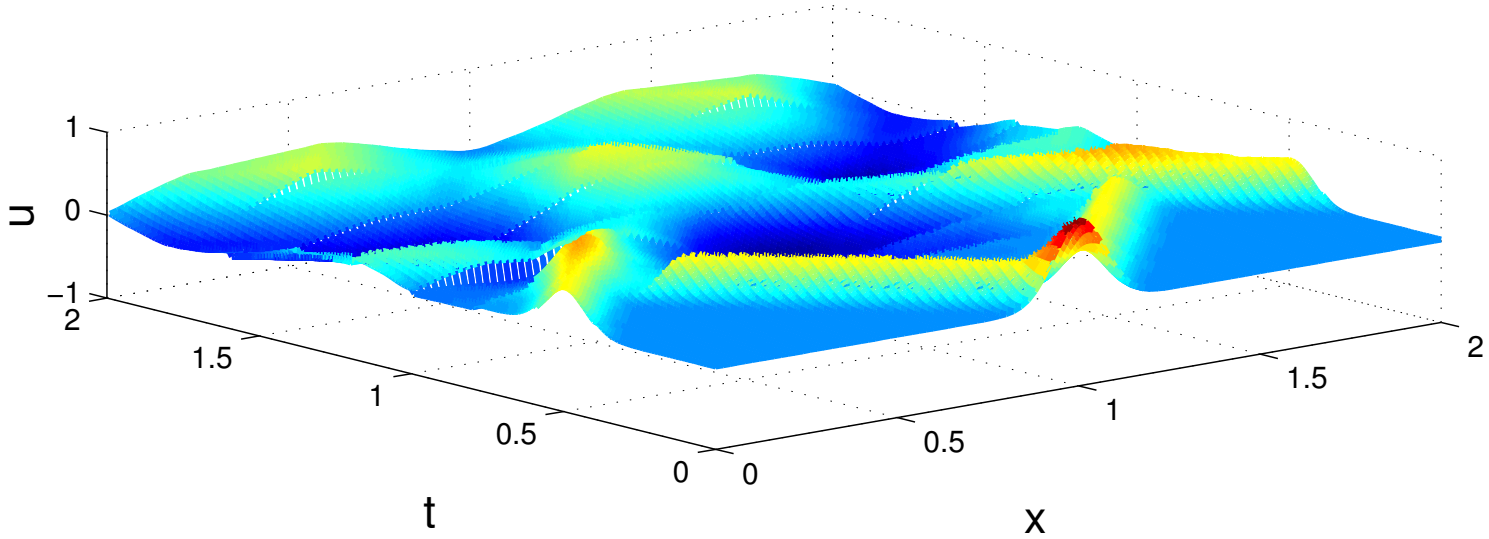


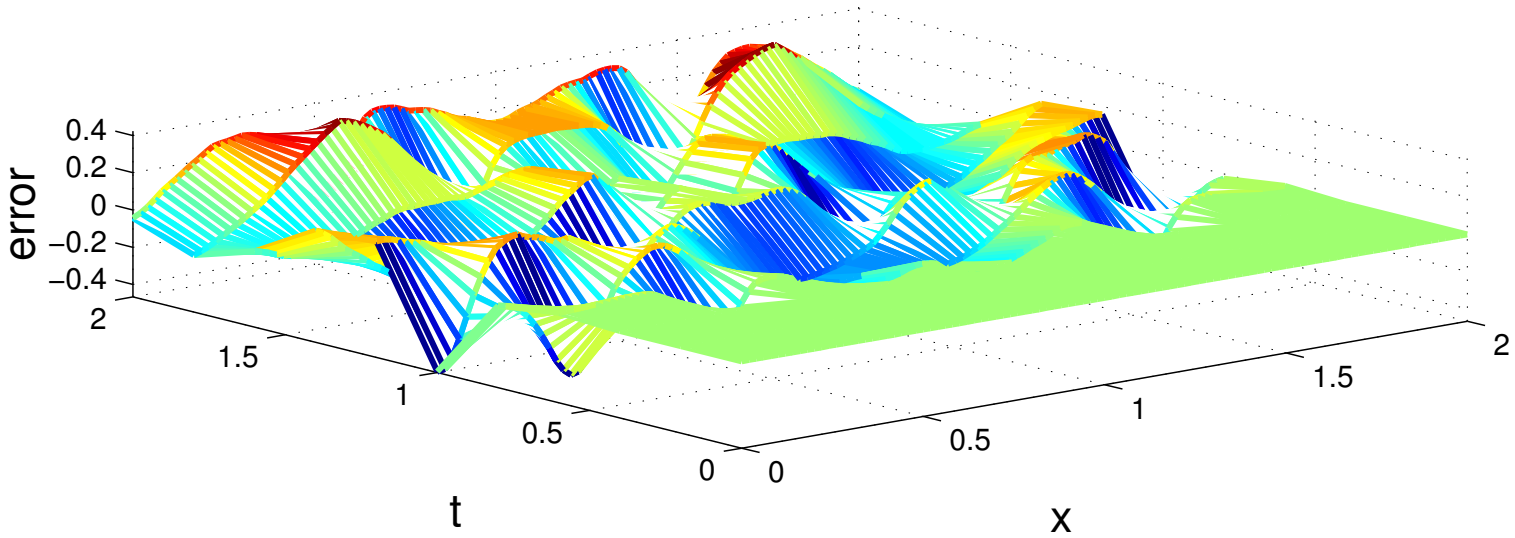
Wave Equation Equation: $T = 2$, Iteration 2



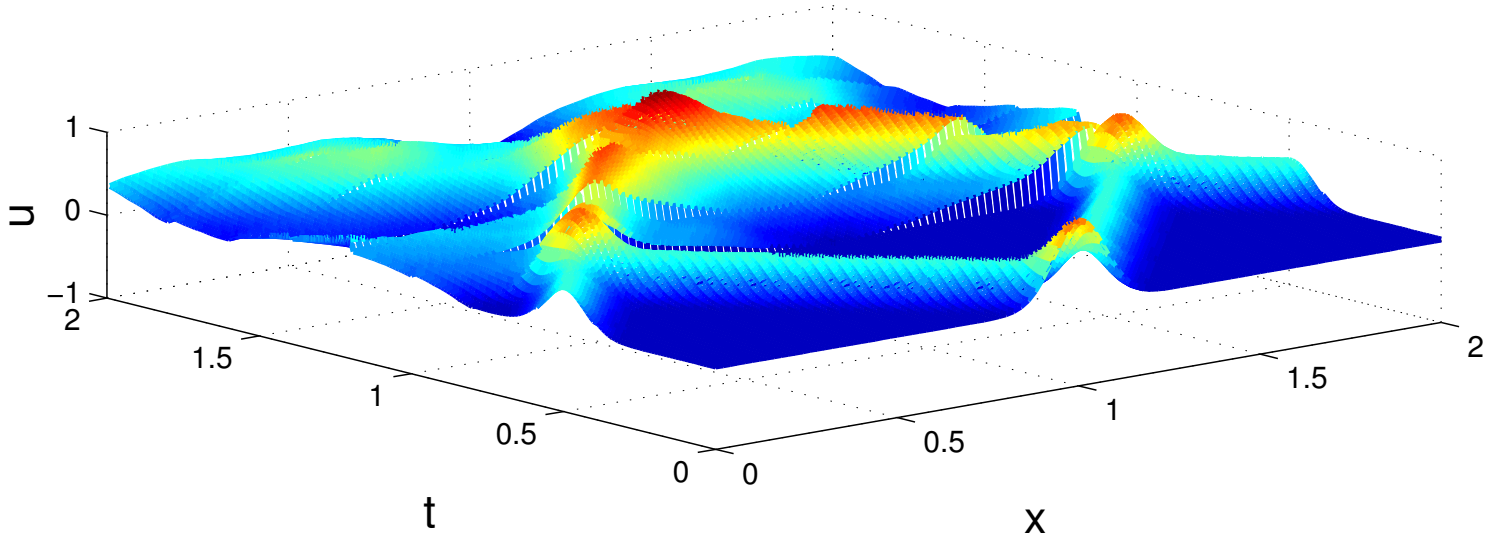


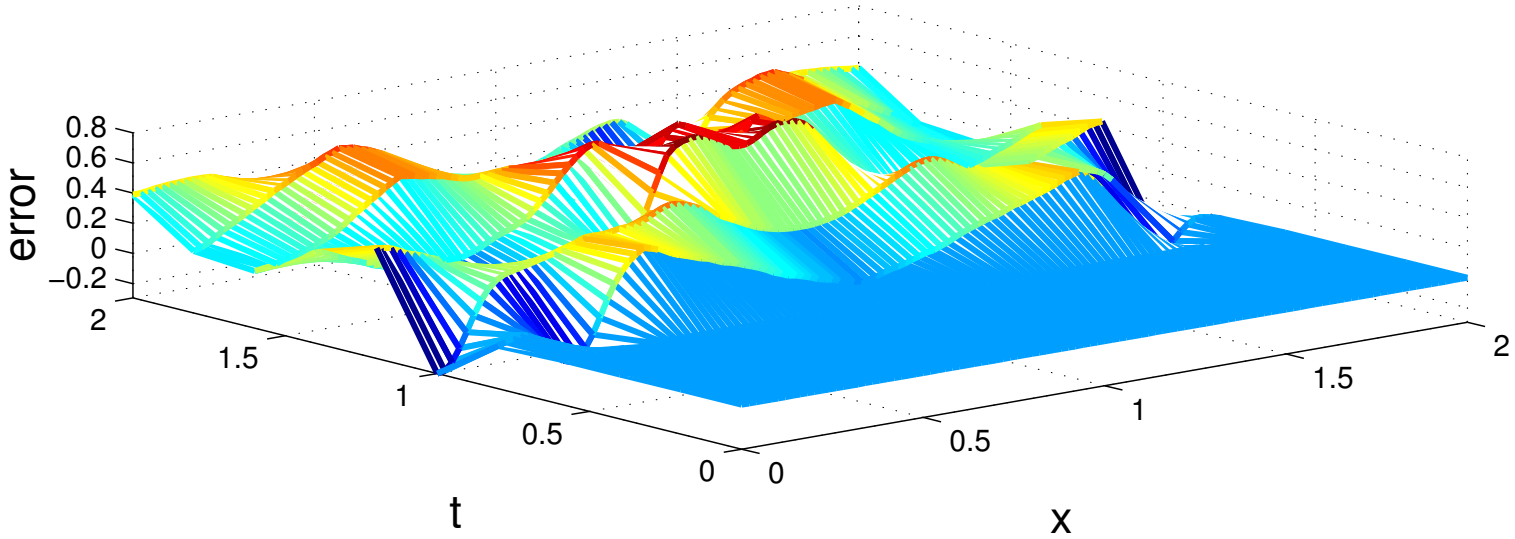
Wave Equation Equation: $T = 2$, Iteration 3



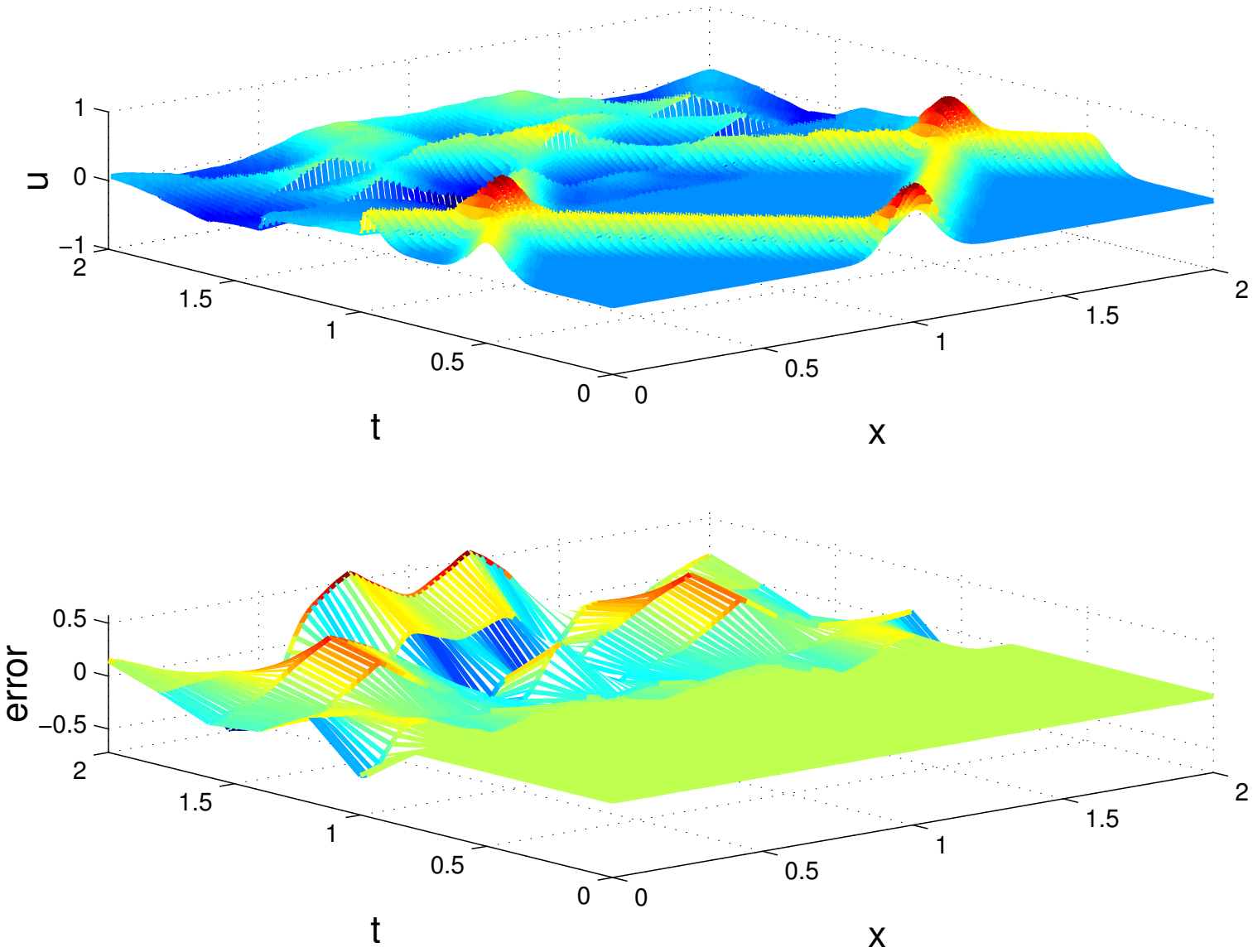


Wave Equation Equation: $T = 2$, Iteration 4

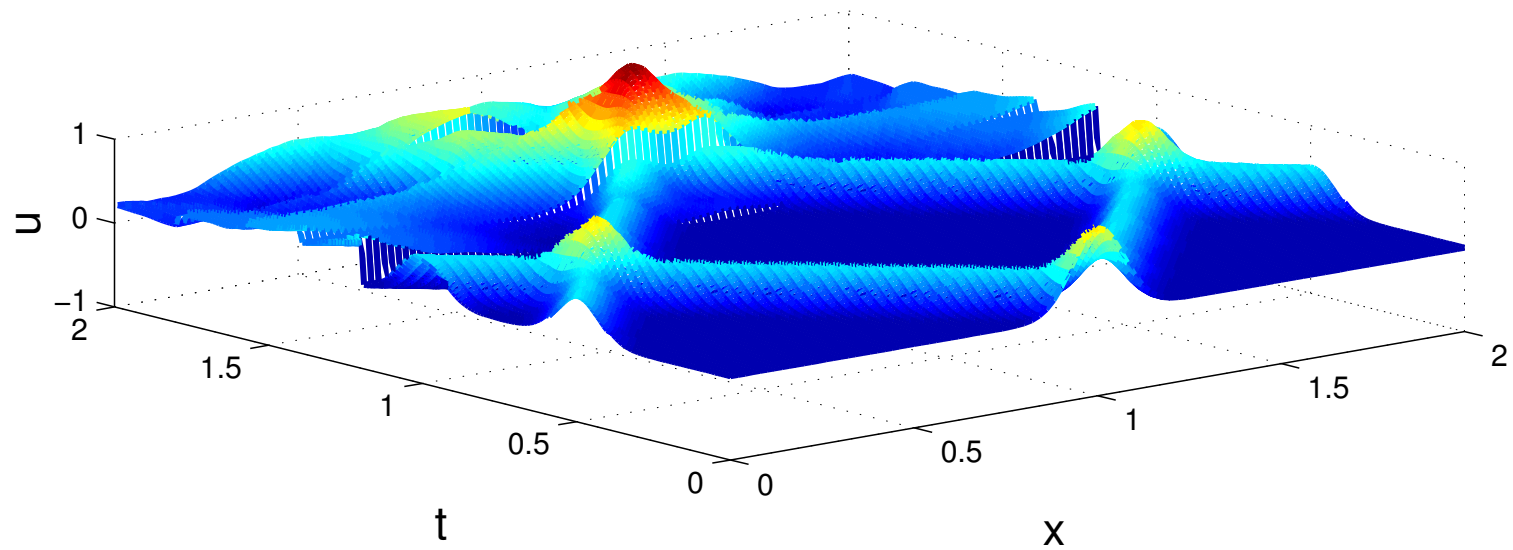


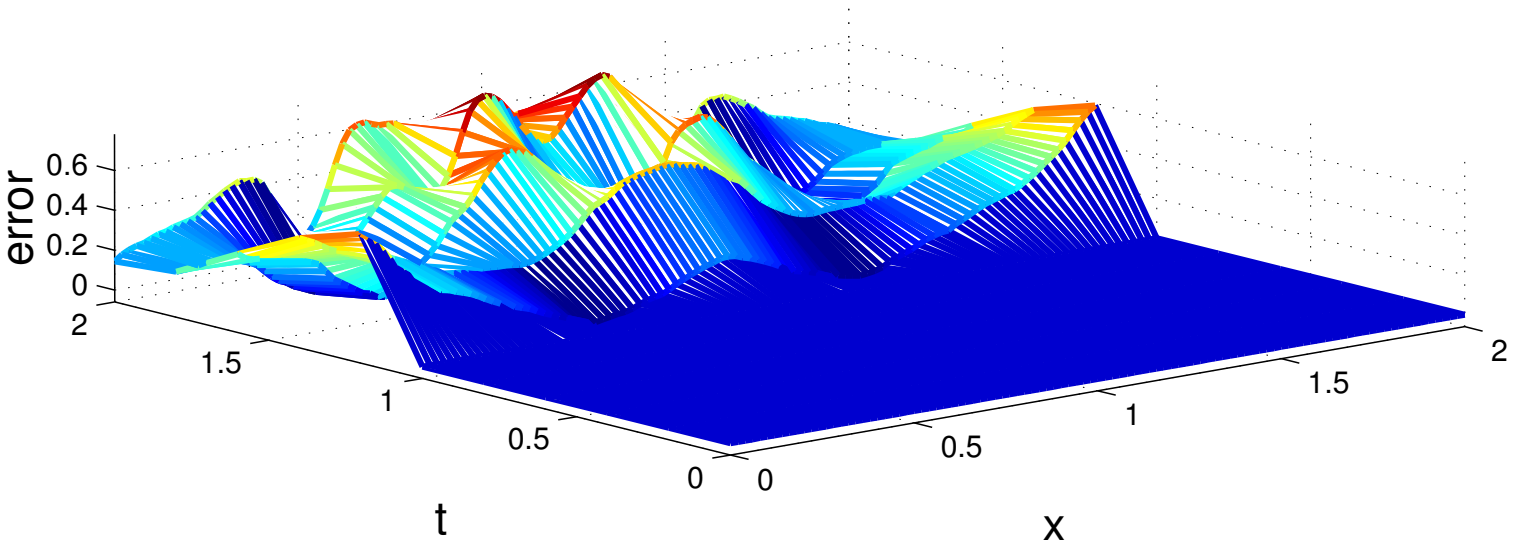


Wave Equation Equation: $T = 2$, Iteration 5

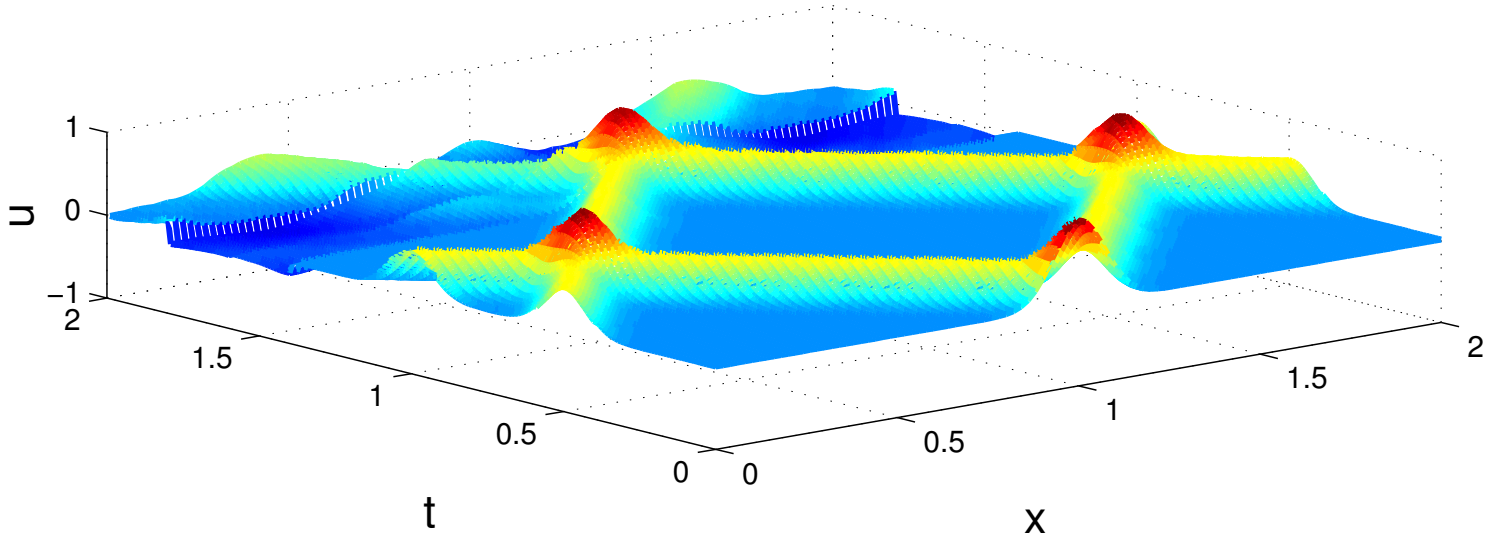


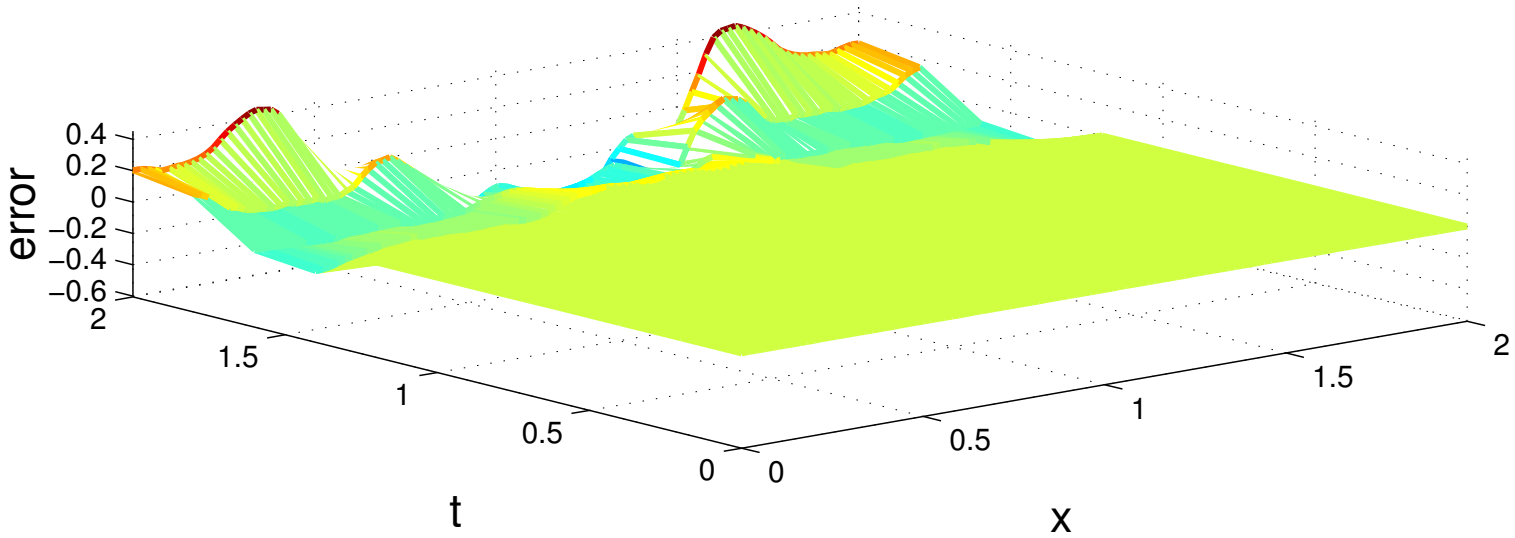
Wave Equation Equation: $T = 2$, Iteration 6



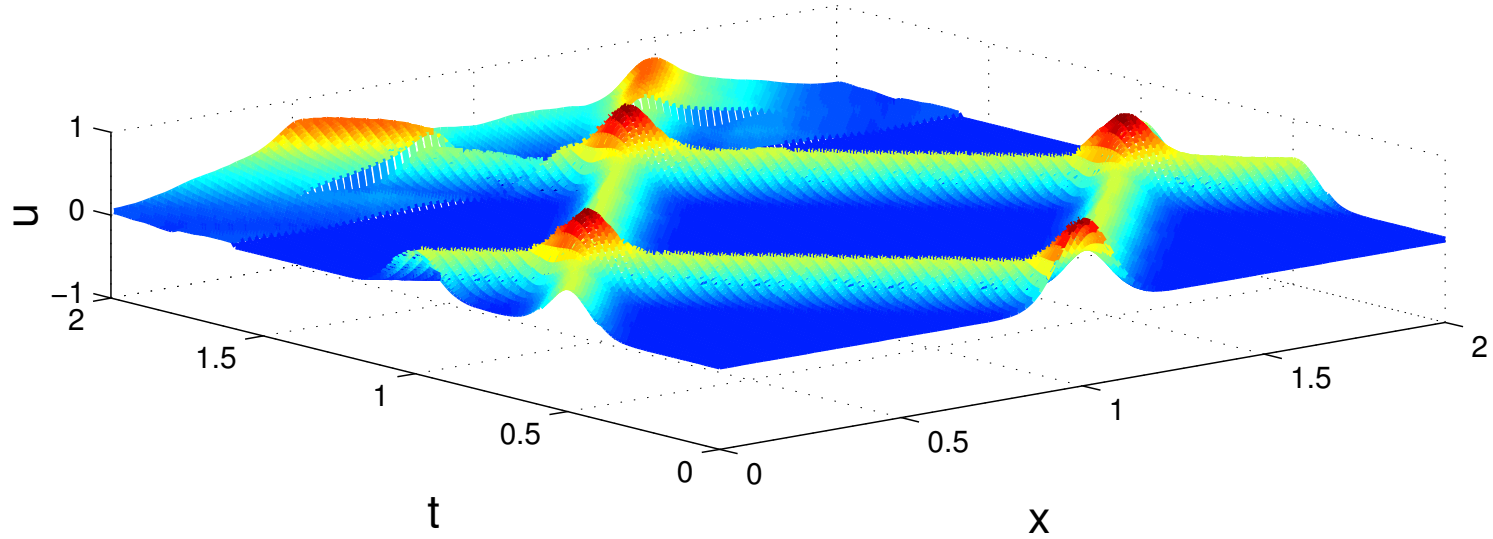


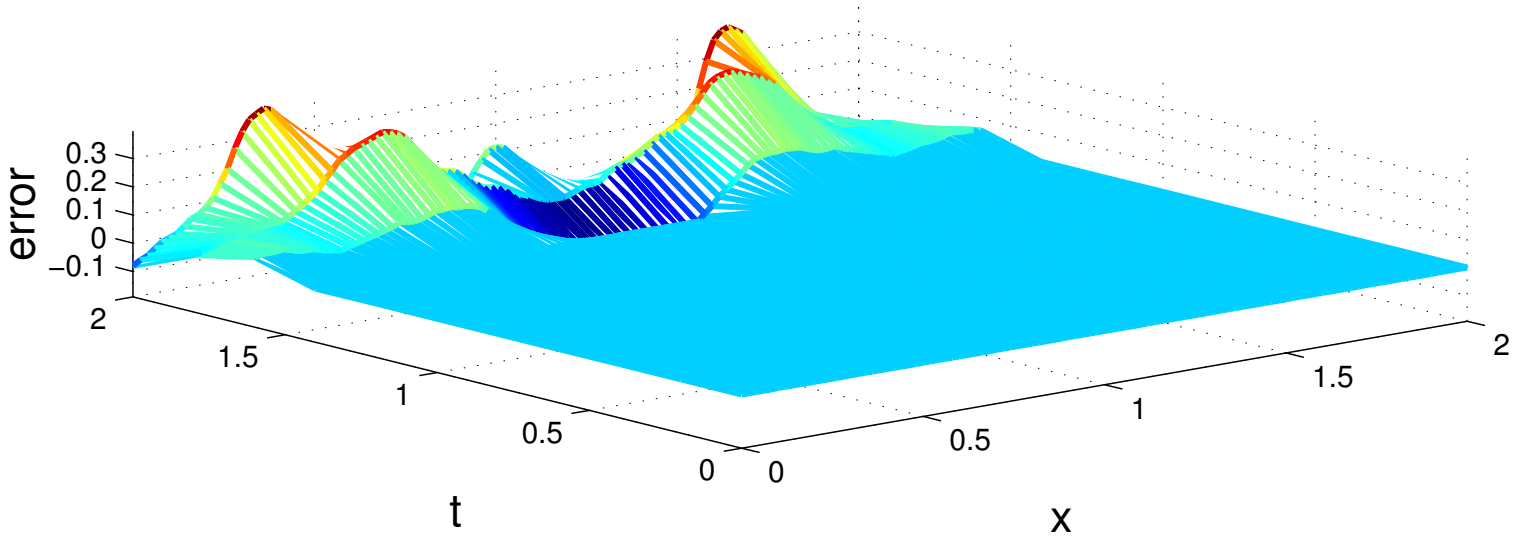
Wave Equation Equation: $T = 2$, Iteration 7





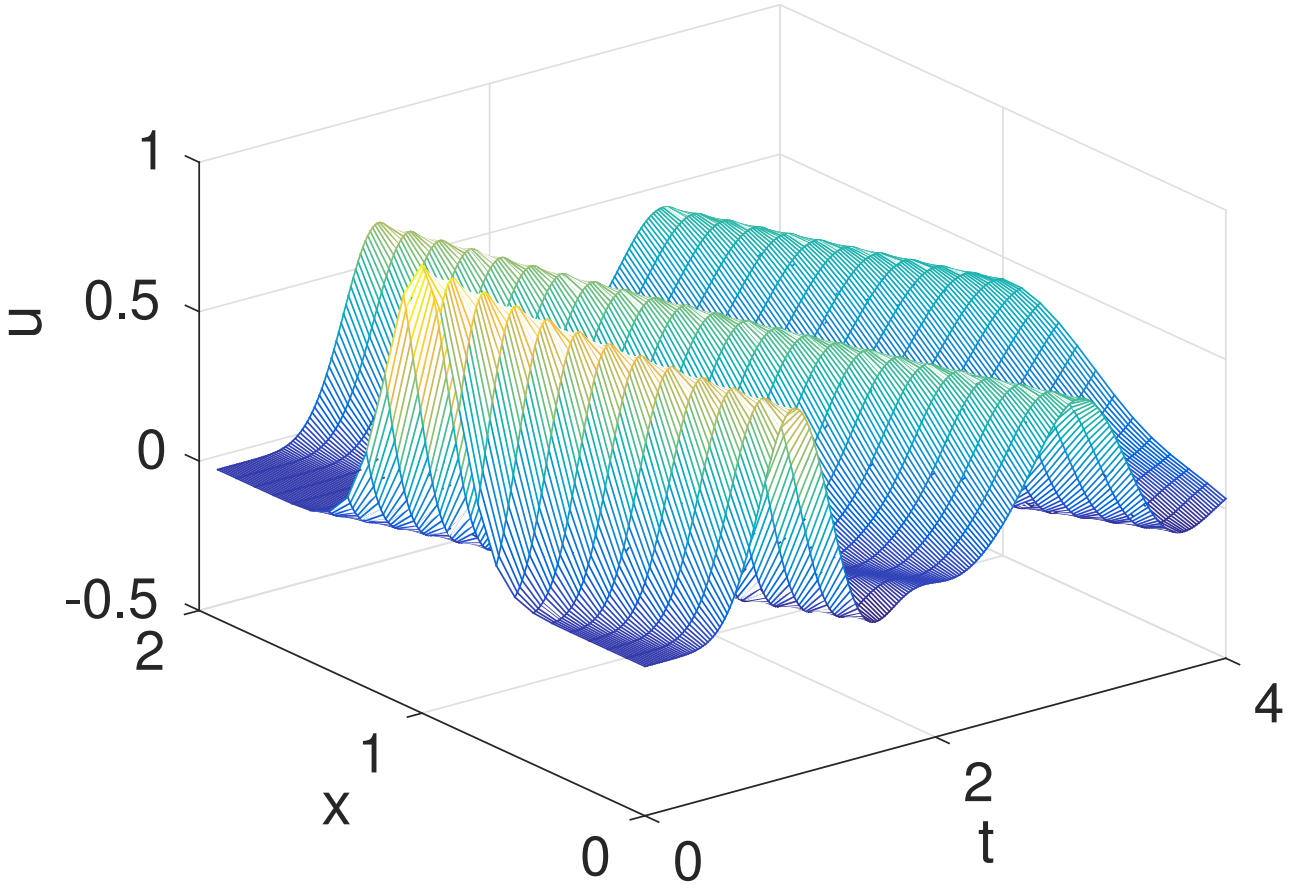
Wave Equation Equation: $T = 2$, Iteration 8





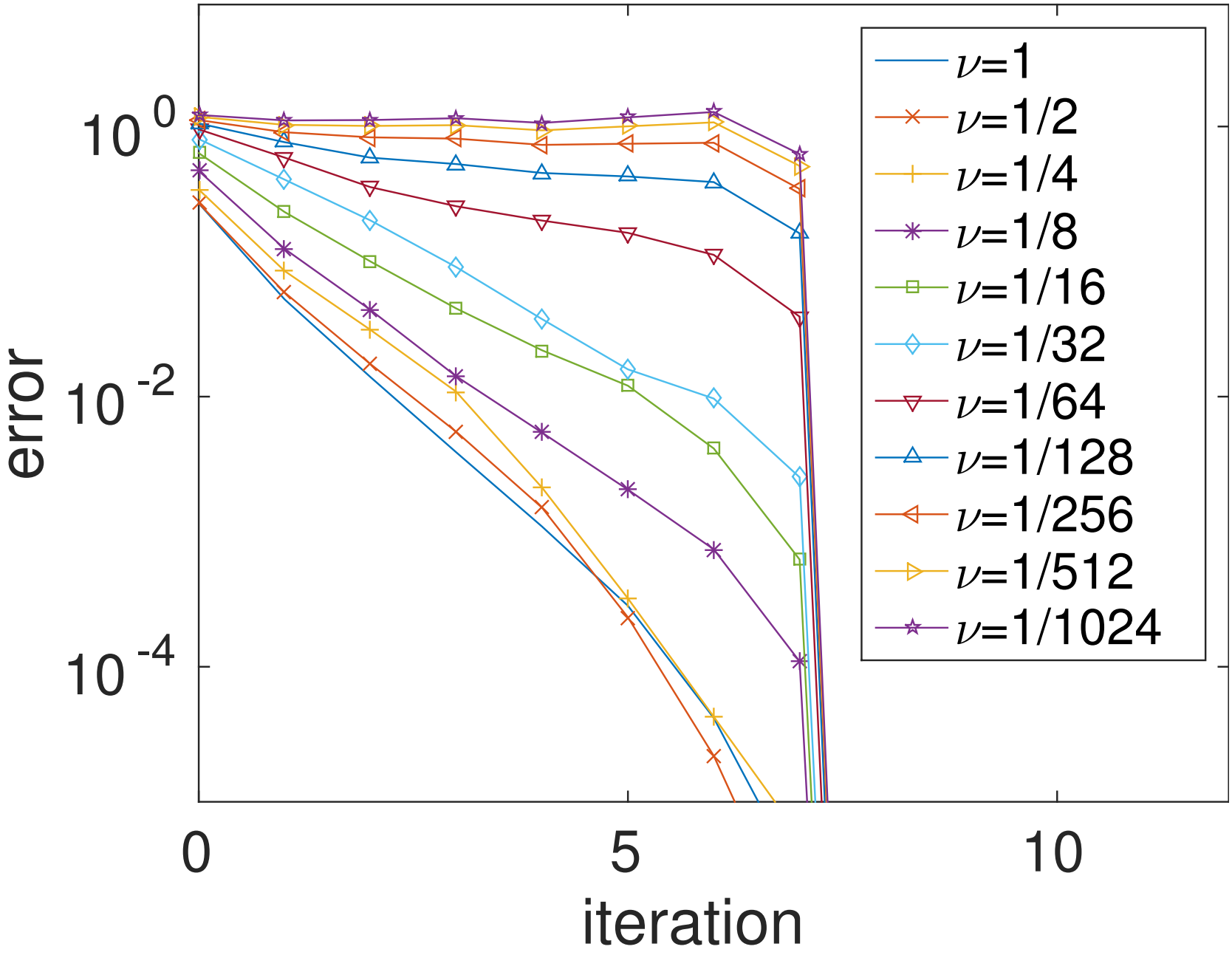
Advection Diffusion Equation

$$u_t = \nu u_{xx} + u_x \quad \text{in } (0,2) \times (0,T), \quad T = 4.$$



G (2017): Note on the Degradation of the Performance of the Parareal Algorithm as a Function of the Reynolds Number, Oberwolfach Report

Performance Dependence on Reynolds Number



Conclusions

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  - Superlinear convergence estimate for finite T
  - Linear convergence estimate for all T
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  - in the periodic case only superlinear convergence estimate (indicating convergence is too late)
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- For the wave equation (and hyperbolic problems in general) no convergence estimate except finite step convergence (again too late for speedup)
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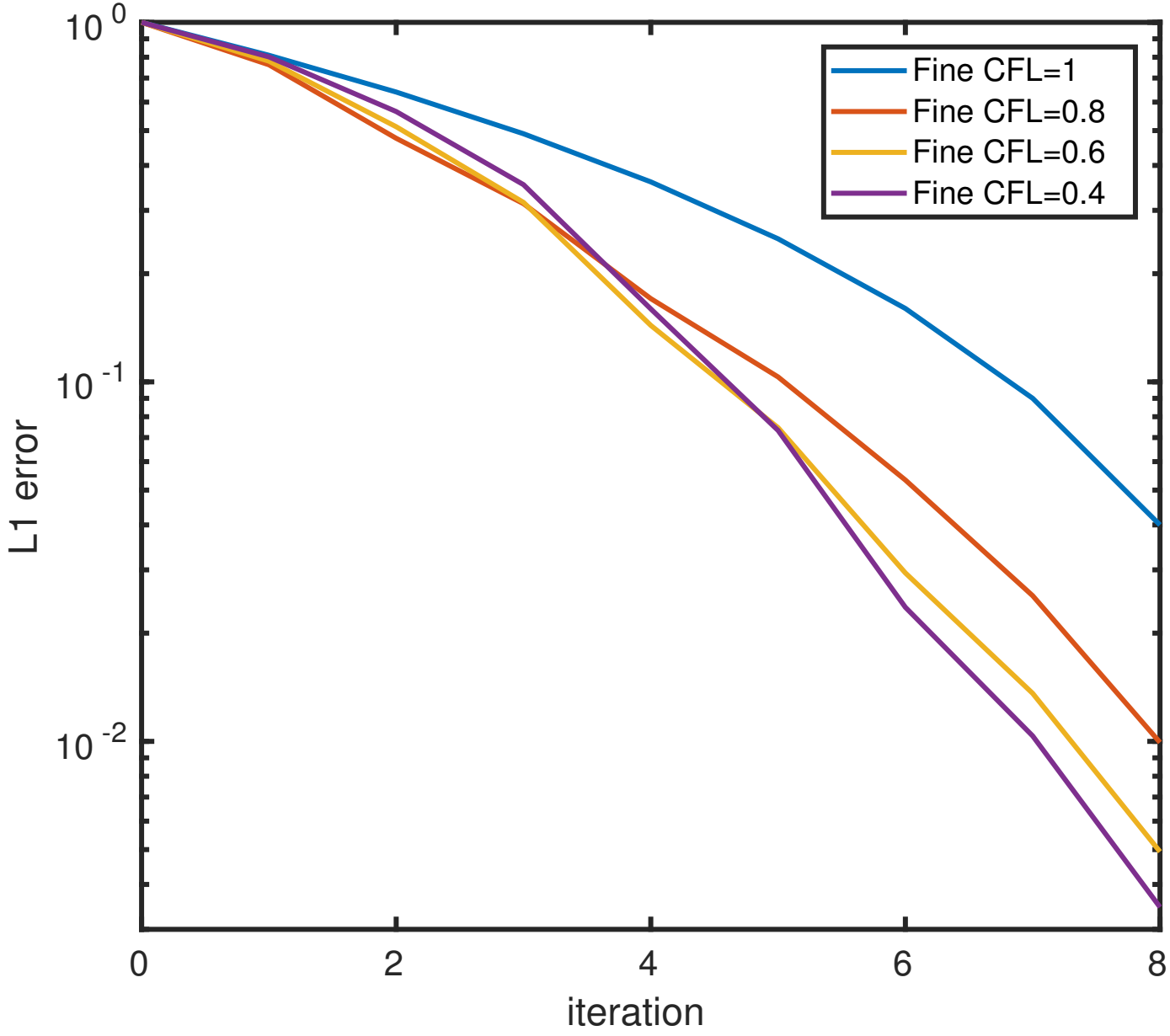
- Good PinT methods for hyperbolic problems (Optimized Schwarz Waveform Relaxation, Tent Pitching, ParaExp, Parallelization by Diagonalization, RIDC)

Influence of the CFL on Convergence



Non-Periodic Advection equation, Coarse CFL = 1