ROSCOFF

Convergence Acceleration of the PinT Integration of Advection Equation using Accurate Phase Calculation Method

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Acknowledgement

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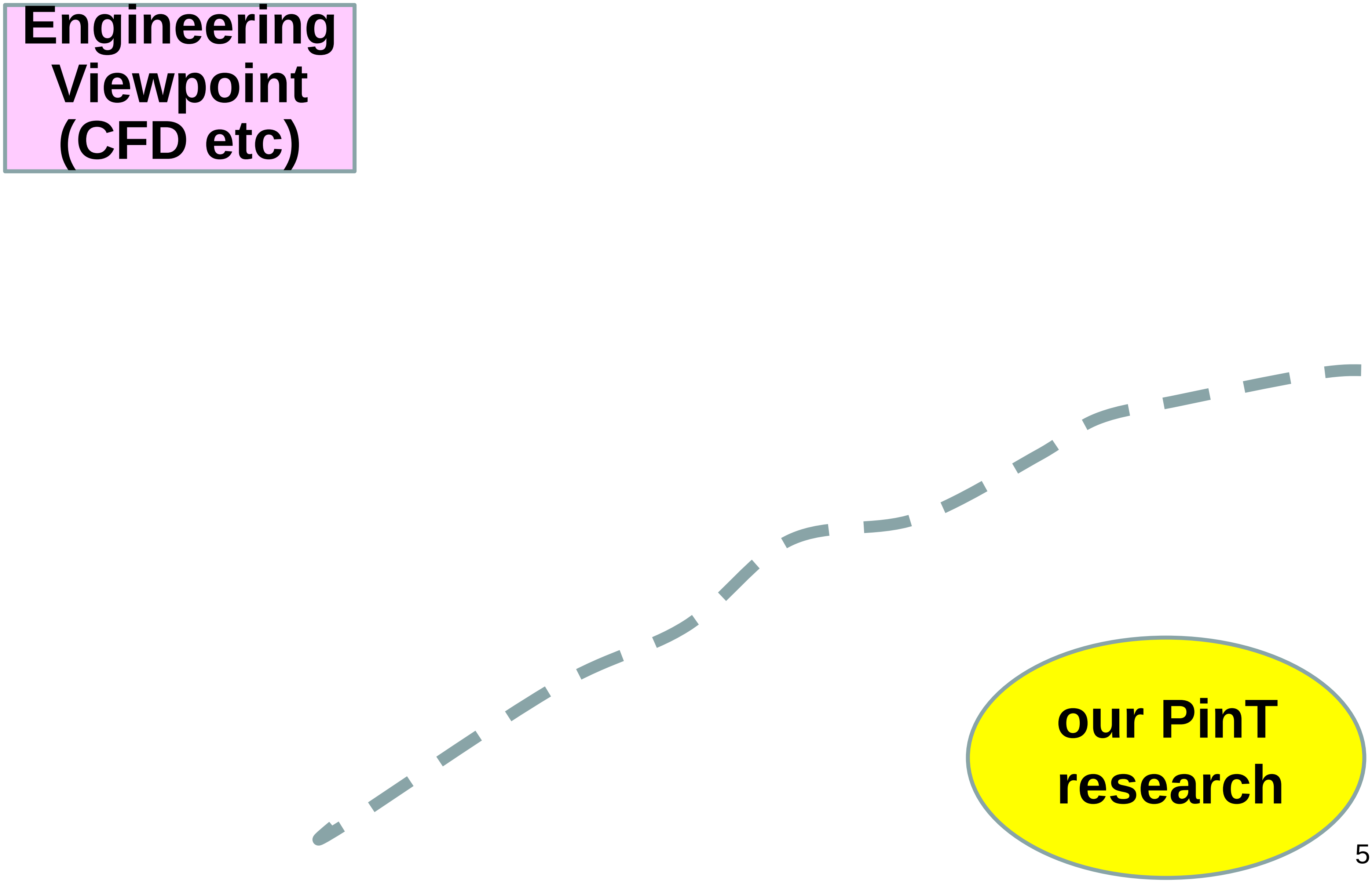
Outline

- Introduction
- How do we develop **very accurate phase calclation method** ?
    - Method **improving the calculation method** of advection
    - Check impacts of conventional methods improvement
- Parareal calculation
- Summary and Future Work

Introduction

Basis direction of our PinT research



Basis direction of our PinT research

Engineering Viewpoint (CFD etc)

Loves widely usable method, especially the method usable to hyperbolic PDEs

our PinT research

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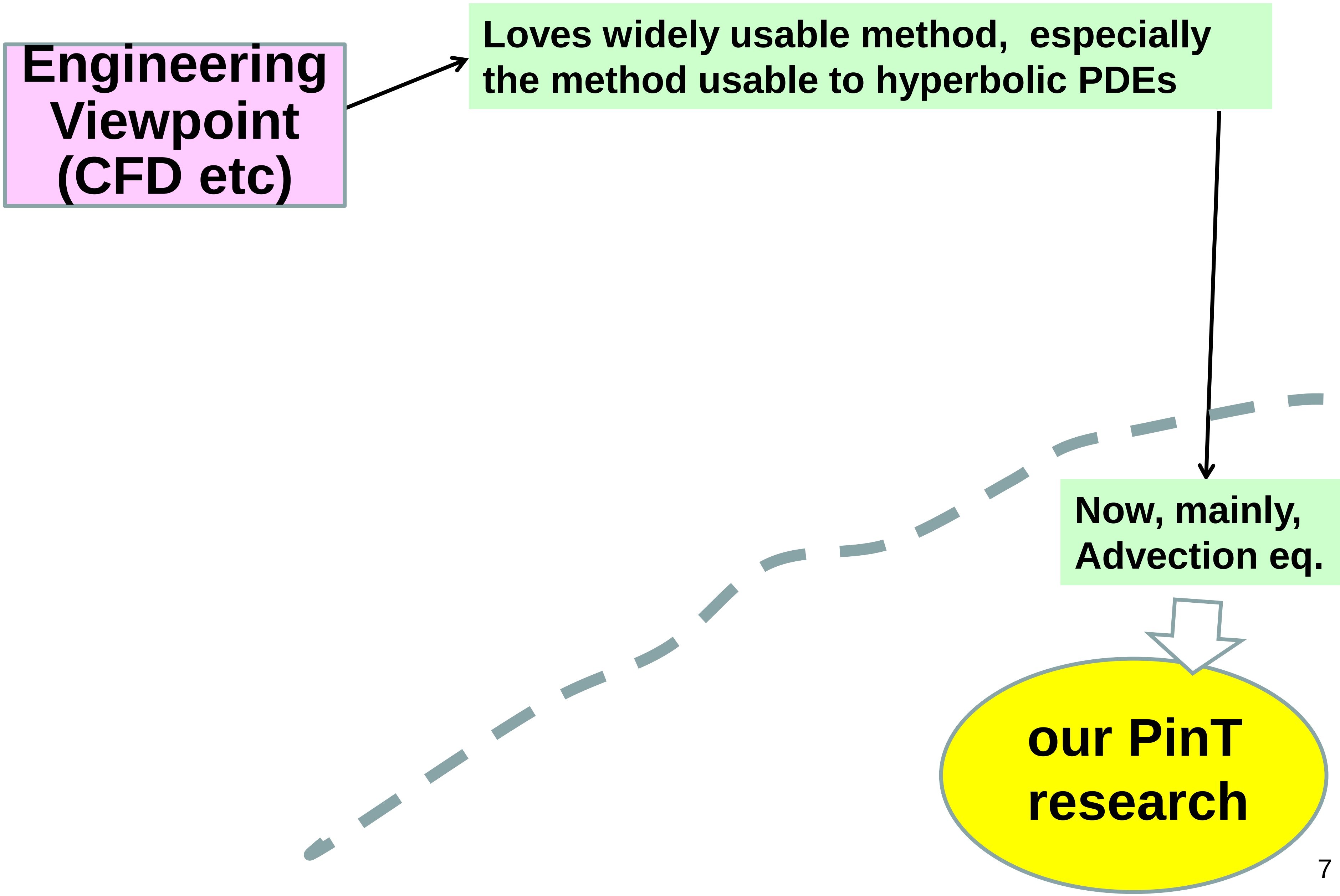

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Love simple.→ Pararael method is better. do not like complex platform or frame work.

Now, mainly Advection eq.

our PinT research

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Would you change your solver to incorporate your cord in PinT-Frame?

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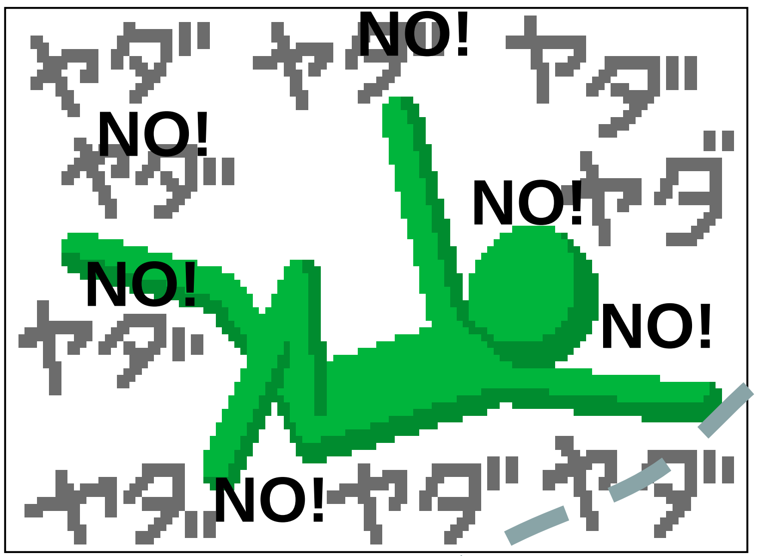

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our PinT research

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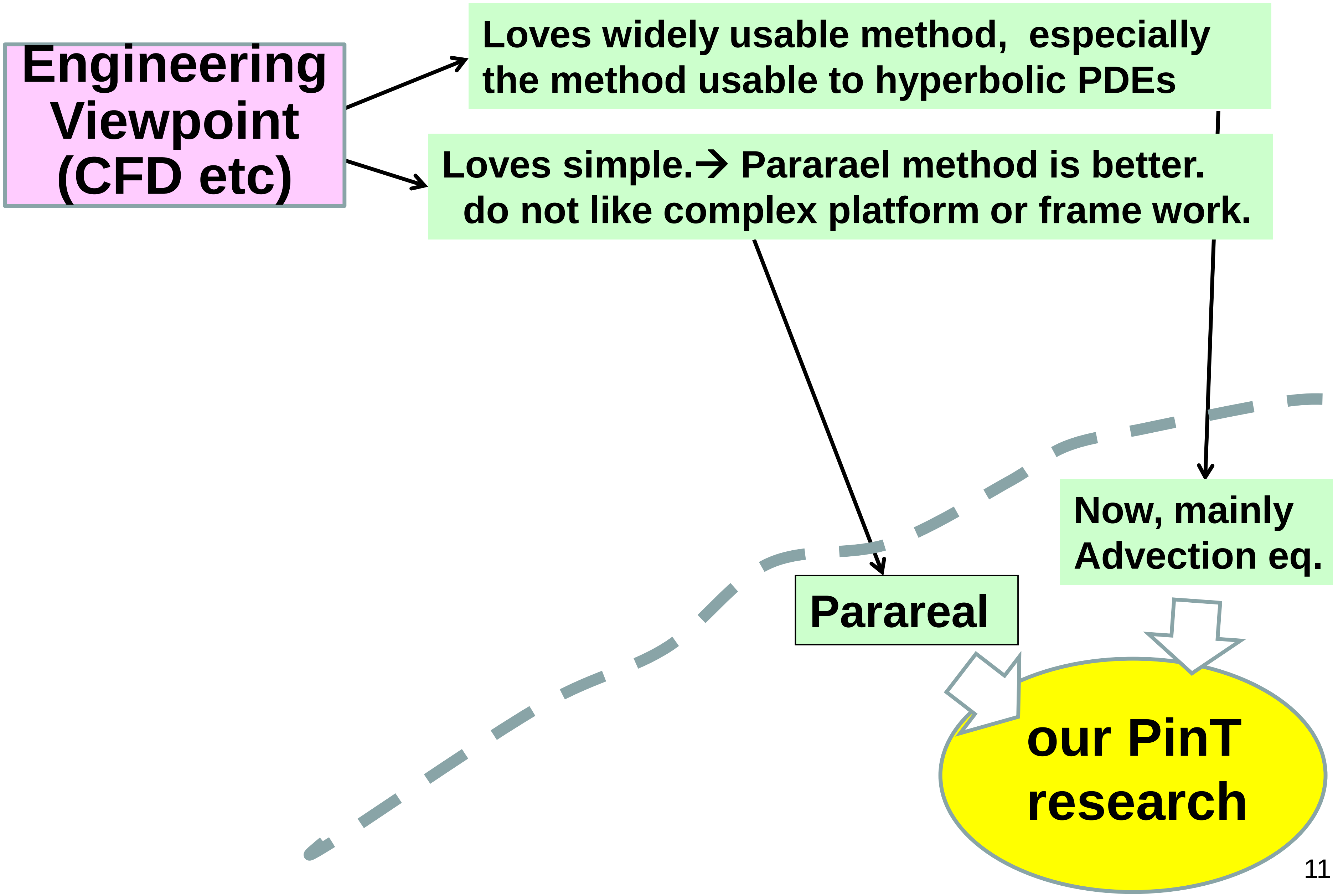

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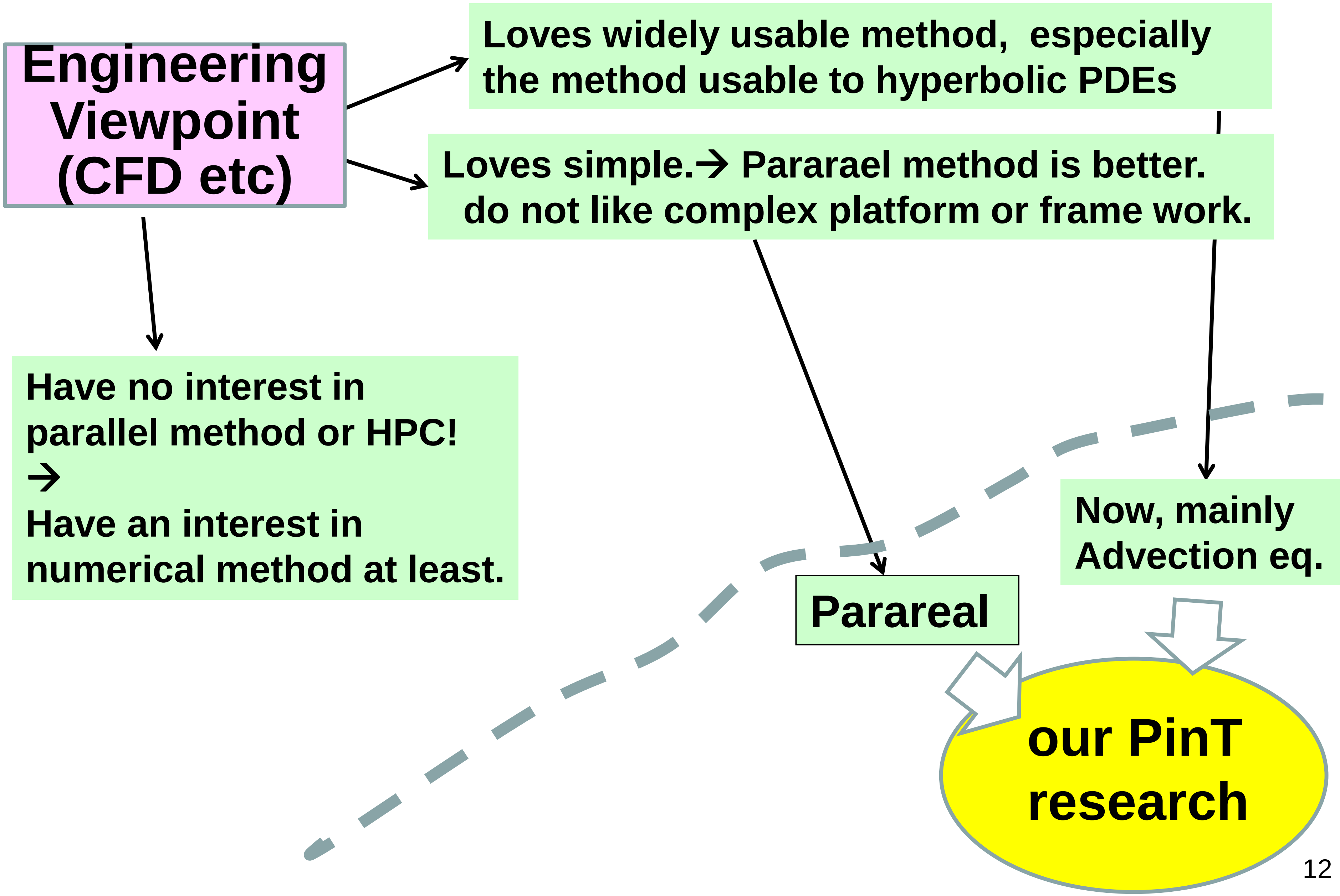

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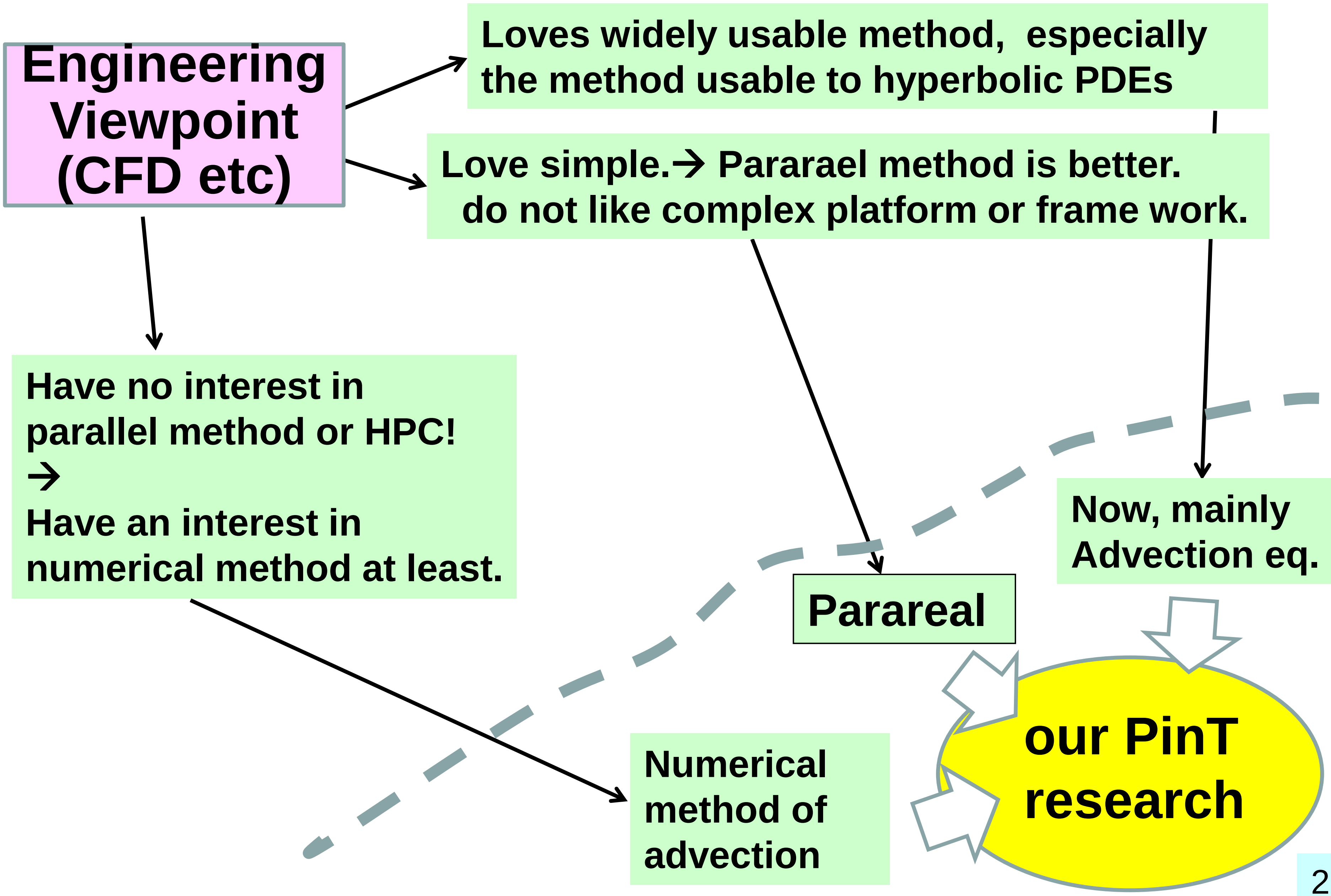
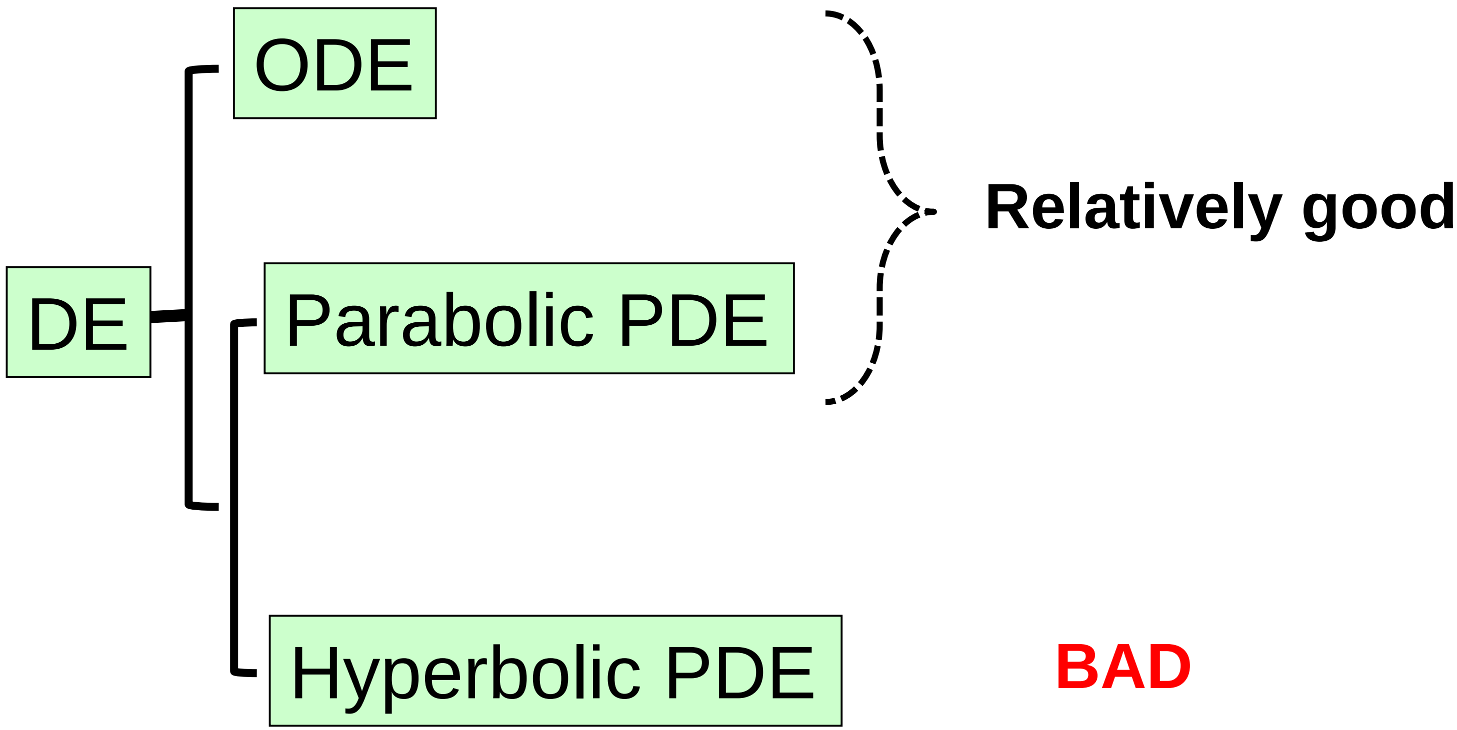
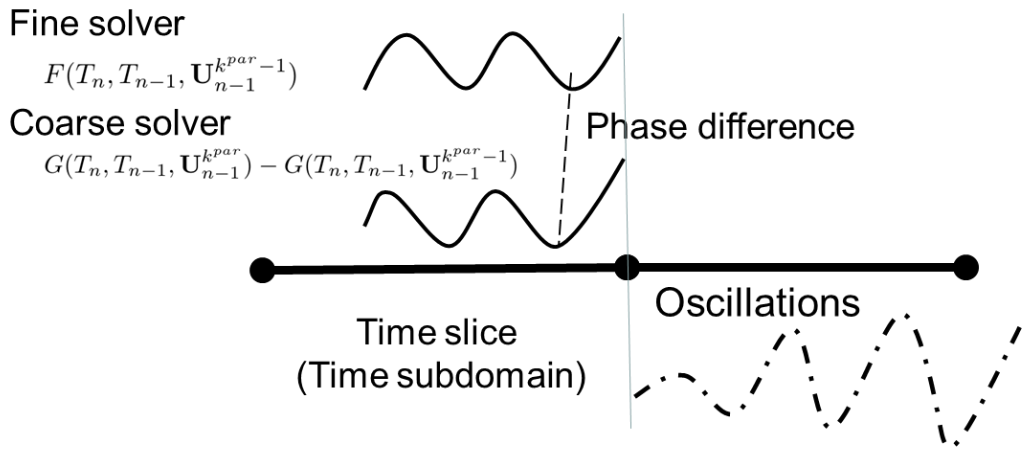


Issue of the parareal method for hyperbolic PDEs:

Parareal convergence



Why so bad?



*M. Gander and M. Petcu, Analysis of a Krylov subspace enhanced parareal algorithm for linear problems, ESAIM Proc, 25, (2008), 114-129.

- **Hyperbolic PDEs represents wave phenomena.**
- **If there is Phase Difference between fine and coarse solver's result → Oscillations appears at the edge of time slice.**
- **That gives damage the convergence of the parareal method.**

Our challenge

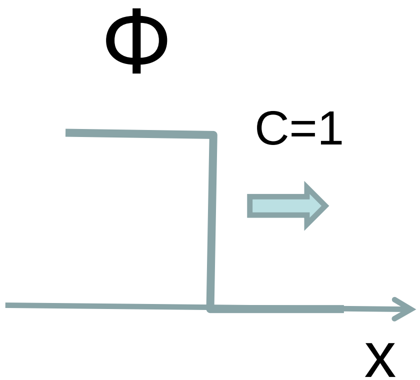
Reducing the phase difference between fine/coarse solvers

Example of oscillations in parareral iteration for advection equation

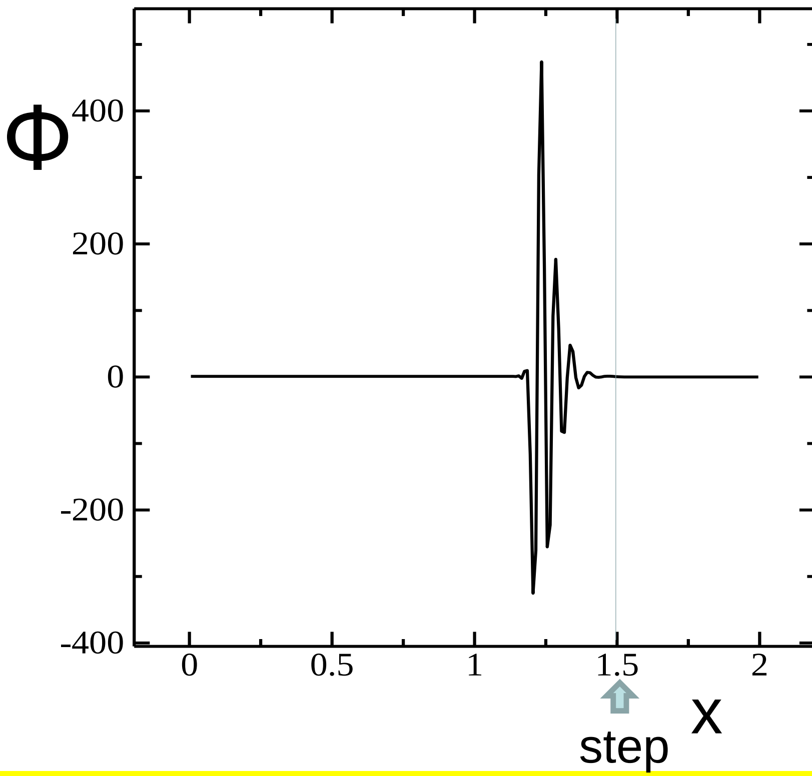
Profile of Φ at t=0.5 after 10 iteration

with time-coarsening ratio Rfc=25 without relaxation of iteration

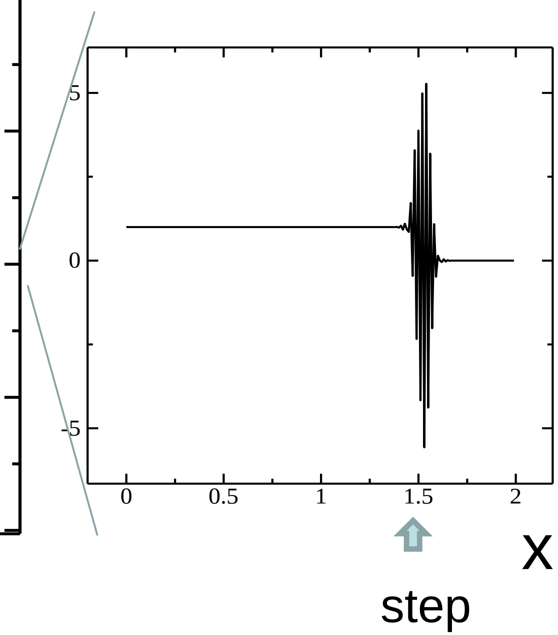
$$\partial_t \phi + c\partial_x \phi = 0$$



Fine/coarse solver :TVD/CN



Fine/coarse solver
CIP3rd method

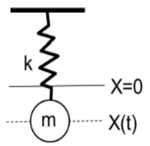


Oscillation amplitudes much depend on numerical integration methods (may be accuracy of pahse calcultion).

We studied the impact of phase difference on the parareal convergence using most simple problem.

(a) Most simple problem:

→ Simple harmonic motion
→ Simplest hyperbolic PDE



$$\frac{d^2X}{dt^2} + \left(\frac{2\pi}{Tcyc}\right)^2 X = 0$$

This gives the exact phase to fine/coarse solver.

We tried to check the effect of phase difference by adding the erro to coarse solver **by value ε.**

(b) Time integrator:

Modified Newmark-β Method

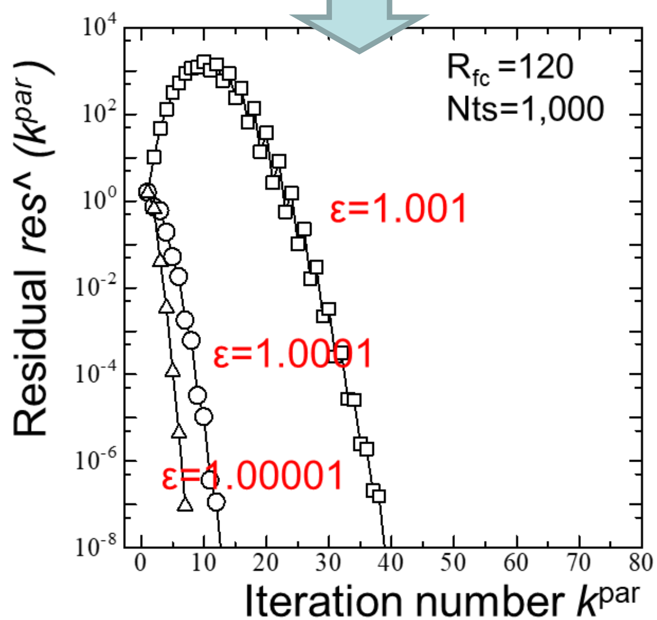
This method can give the exact phase for the simple harmonic motion independent on time step width ← by the modified δt', δT'.

Fine solver

$$\delta t' \leftarrow \frac{\delta t}{\left\{\frac{2\pi}{Tcyc}\sqrt{\left(\sin^2\pi\frac{\delta t}{Tcyc}\right)^{-1} - 1}\right\}}$$

Coarse solver

$$\delta T' \leftarrow \frac{\delta T}{\left\{\frac{2\pi}{T_{nbm}(\delta t)}\sqrt{\left(\sin^2\pi\frac{\delta t}{T_{nbm}(\delta t)}\right)^{-1} - 1}\right\}}$$



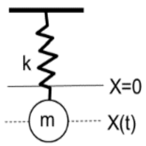
This results shows that very very small phase difference causes the convergence difficulty.

We studied the impact of phase difference on the parareal convergence using most simple problem.

(a) Most simple problem:

→ Simple harmonic motion

→ Simplest hyperbolic PDE



$$\frac{d^2X}{dt^2} + \left(\frac{2\pi}{Tcyc}\right)^2 X = 0$$

This gives the exact phase to fine/coarse solver.

We tried to check the effect of phase difference by adding the erro to coarse solver **by value ε.**

(b) Time integrator:

Therefore, we focus on development of ***the method that reduce phase difference between fine/coarse solver by apply the very accurate phase calclation method to fine/coarse solver.***

How do we develop very accurate phase calclation method ?

This research approach:1

Approach based on the engineering method

Dramatically Improving the conventional calculation method of advection equation to increase the phase accuracy

Conventional method

$$\phi_i^n = \phi_i^{n-1} + S\zeta_F \Sigma_{l=-m^-}^{+m^+} a_l \phi_{id+ls(c)}^{n-1}$$

$$\left\{ \begin{aligned} \partial_t \phi + c\partial_x \phi &= 0 \\ \partial_t g + c\partial_x g &= 0 \\ g &= \partial_x \phi \end{aligned} \right.$$

Improvement method

$$\begin{aligned} \partial_t \phi + c\partial_x \phi &= 0 \\ \partial_t g + c\partial_x g &= 0 \\ \partial_t \chi + c\partial_x \chi &= 0 \\ g &= \partial_x \phi \\ \chi &= \partial_x g \end{aligned}$$

$$\phi_{ida^+}^{n-1} + S\zeta_F \Sigma_{l=-m^-}^{+m^+} a_l \phi_{ida^+ + ls(c)}^{n-1} = \phi_{ida^-}^{n} + S\zeta_F \Sigma_{l=-m^+}^{+m^-} a_l \phi_{ida^- - ls(c)}^{n}$$

Approach based on mathematics of parareal method

$$\mathbf{U}_n^k = F(T_n, T_{n-1}, \mathbf{U}_{n-1}^{k-1}) + \gamma\left\{G(T_n, T_{n-1}, \mathbf{U}_{n-1}^{k}) - G(T_n, T_{n-1}, \mathbf{U}_{n-1}^{k-1})\right\}$$

Speedup

$$\approx \frac{N_{ts}}{K^{par} + \frac{N_{ts}}{R_{fc}}} = \frac{N_{ts}}{K^{par} + K_0^{par}} = \frac{N_{ts}}{\hat{K}^{par}}$$

Residual

$$\frac{res^{(K^{par})}}{res^{(1)}} = \frac{(C'TdT^{mp})^{K^{par}-1}}{(K^{par}-1)!} \prod_{j=1}^{K^{par}-1} (N_{ts} - j)$$

- δT ← δt
- Reduce the time span: T → Σ^{nc}_{l=1} T_{l}
- etc

Methods overview of advection term calculation

Conventional main method

Stabilization:
numerical damping
Accuracy:
Space and time higher order terms
Advecting only amplitude of variables

$$\partial_t \phi + c\partial_x \phi = 0$$

Methods that are tried in this study

1st: CIP scheme improvement: advecting the much phase information by the gradient and curveutre of value Φ.

2nd: STRS scheme: achieving the stabilization and error elimination using "space and time reversal symmetry " base on the physics.

3rd: Hybrid of CIP method and STRS scheme

$$\partial_t \phi + c\partial_x \phi = 0$$
$$\partial_t g + c\partial_x g = 0$$
$$g = \partial_x \phi$$

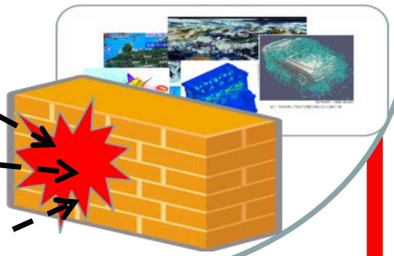
Hybrid:
STRS-CIP

$$\partial_{t'} \phi + c\partial_{x'} \phi = 0$$

not yet success

not enough

Main issue : Gap of phase accuracy between fine and coarse solver

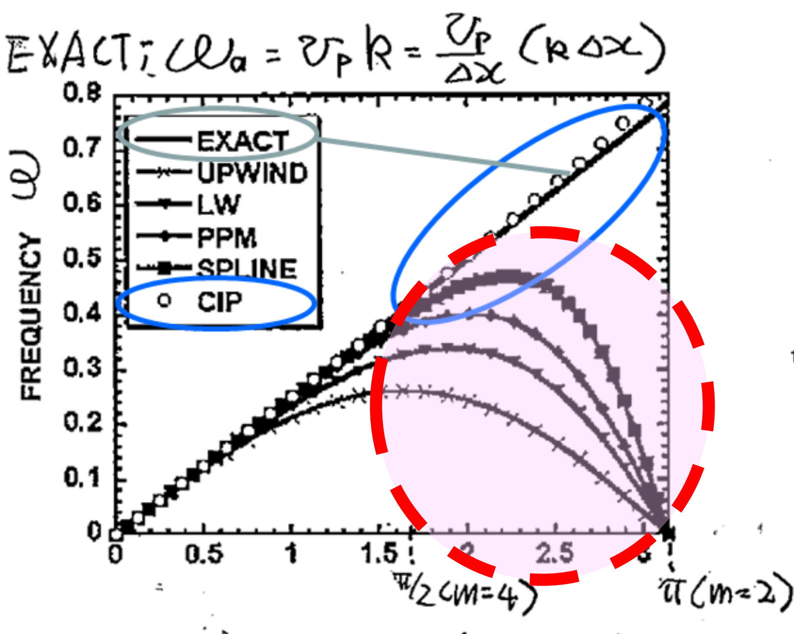


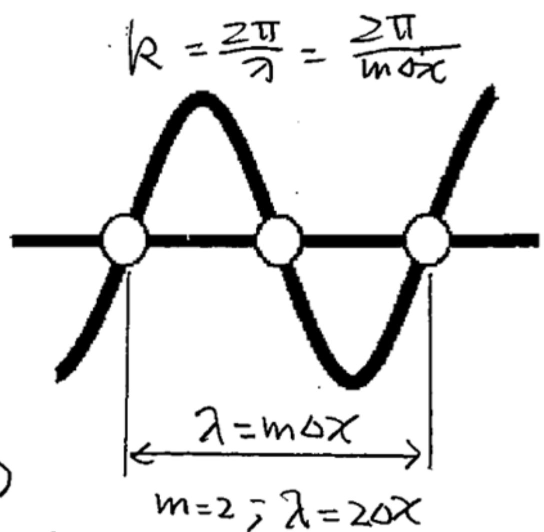
There is a limit in the use.

This research approach:2

Conventional methods of advection equation lose phase accuracy for high grid based wave number waves except CIP3rd method.

Dispersion relations numerical calculation for advection equation.



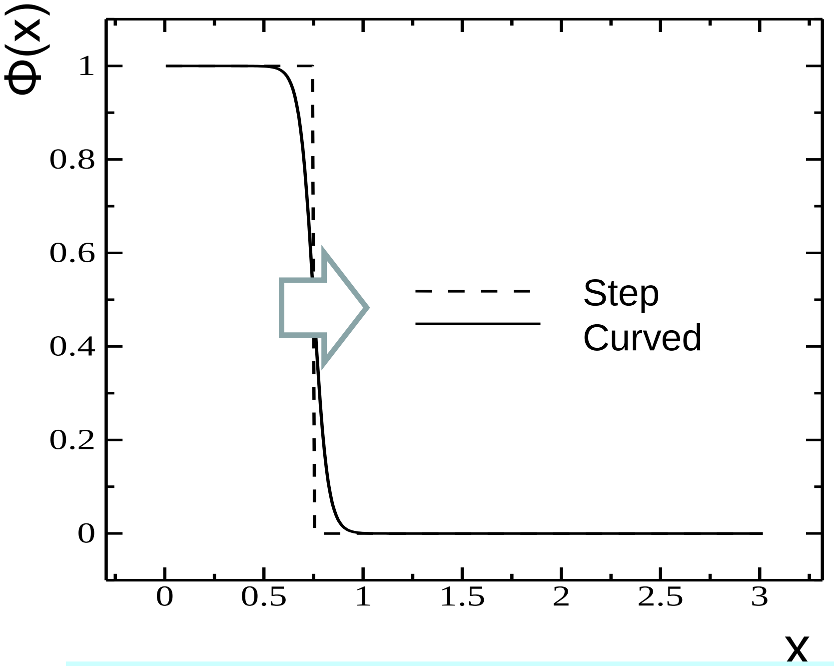


like wave number

Grid based wave number : k=2π/λ= 2π/m/Δx

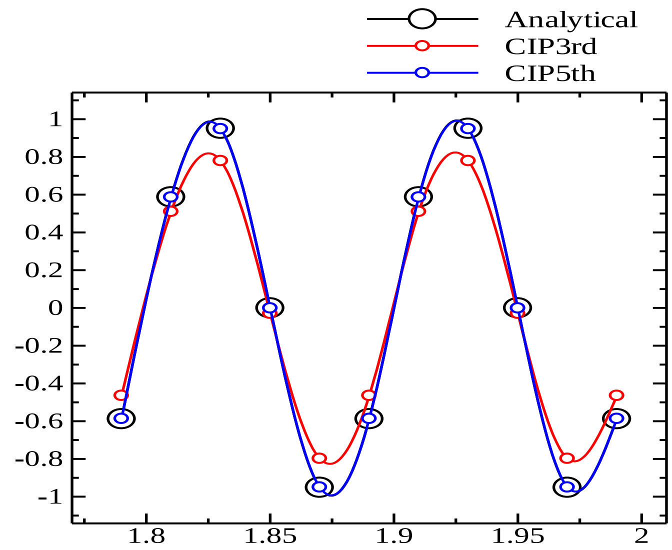
Simple and Typical Benchmark Problem including high grid based wave number waves

Step wise advection problem



Most tough problem: Including broad and high grid based wave number waves

Sin wave advection problem with very rough grids

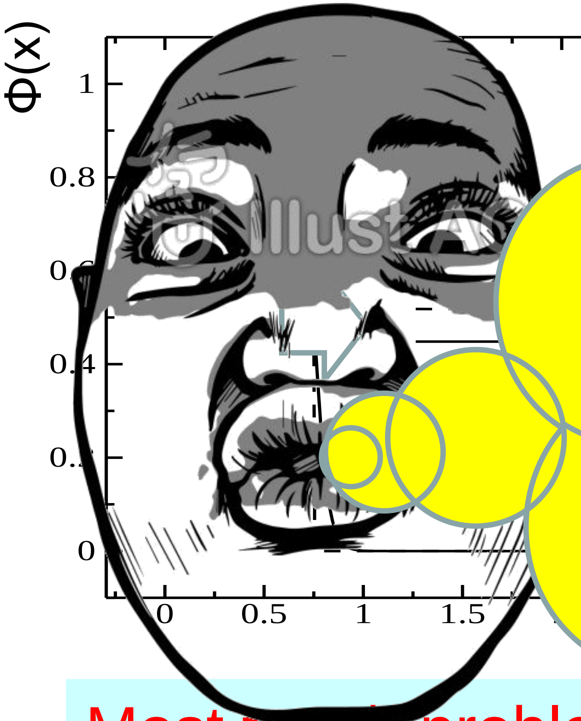


Most simple problem: Including high grid based wave number one wave.

Simple and Typical Benchmark Problem including high grid based wave n... er ... s

Step wise advection pr...

...lem



Engineering peoples using CFD every time ask me that the parareal method work well for step wise shape or rough grid sin wave advection?

Most tough problem: Including broad and high grid based wave number waves

...pic problem ...ng high grid based wave number one wave.

Method improving the calculation method of advection

Conventional calculation method of advection term

(A) Groupe using only variables amplitude

Linear type(CFL-free form used by the Semi-Lagrangian scheme)

Upwind: 1st order

$$\phi_i^n = \phi_i^{n-1} + \frac{1}{2}\zeta_F\left(\phi_{id-s(c)} - \phi_{id}\right)$$

Lax-Wndroff: 2nd order

$$\phi_i^n = \phi_i^{n-1} + \frac{1}{2}\zeta_F\left\{(\zeta_F+1)\phi_{id-s(c)}^{n-1} - 2\phi_{id}^{n-1} + (\zeta_F-1)\phi_{id+s(c)}^{n-1}\right\}$$

QUICK: 2nd order

$$\phi_i^n = \phi_i^{n-1} + \frac{1}{8}\zeta_F\left\{-\phi_{id-2s(c)}^{n-1} + 7\phi_{id-s(c)}^{n-1} - 3\phi_{id}^{n-1} - 3\phi_{id+s(c)}^{n-1}\right\}$$

QUICKEST: 3rd order

$$\phi_i^n = \phi_i^{n-1} + \frac{1}{6}\zeta_F\left\{-(\zeta_F^2-1)\phi_{id-2s(c)}^{n-1} - 3(\zeta_F^2-\zeta_F-2)\phi_{id-s(c)}^{n-1}\right.$$
$$\left.-3(-\zeta_F^2+2\zeta_F+1)\phi_{id}^{n-1} - (\zeta_F^2-3\zeta_F+2)\phi_{id+s(c)}^{n-1}\right\}$$

Upwind: 3rd order

$$\phi_i^n = \phi_i^{n-1} + \frac{1}{6}\zeta_F\left\{-\phi_{id-2s(c)}^{n-1} + 6\phi_{id-s(c)}^{n-1} - 3\phi_{id} - 2\phi_{id+s(c)}^{n-1}\right\}$$

Kawamura and Kuwahara: 3rd order

$$\phi_i^n = \phi_i^{n-1} + \frac{1}{6}\zeta_F\left\{-2\phi_{id-2s(c)}^{n-1} + 10\phi_{id-s(c)}^{n-1} - 9\phi_{id}^{n-1} + 2\phi_{id+s(c)} - \phi_{id+2s(c)}^{n-1}\right\}$$

Central : 4th order

$$\phi_i^n = \phi_i^{n-1} + \frac{1}{12}\zeta_F\left\{-\phi_{id-2s(c)}^{n-1} + 8\phi_{id-s(c)}^{n-1} - 8\phi_{id+s(c)}^{n-1} + \phi_{id+2s(c)}^{n-1}\right\}$$

Non linear type

TVD 3rd oeder

$$\partial_t\phi_i = -\Delta i\hat{f}$$
$$\hat{f}_{i+1/2} = \hat{f}_{i+1/2}^{(1)}$$
$$+\frac{1}{4}c_{i+1/2}^{+}\left\{(1-k)\Psi(r_{i-1/2}^{+})\Delta\phi_{i-1/2} + (1+k)\Psi(r_{i+1/2}^{-})\Delta\phi_{i+1/2}\right\}$$
$$-\frac{1}{4}c_{i+1/2}^{-}\left\{(1-k)\Psi(r_{i+3/2}^{-})\Delta\phi_{i+2/3} + (1+k)\Psi(r_{i+1/2}^{+})\Delta\phi_{i+1/2}\right\}$$
$$\hat{f}_{i+1/2}^{(1)} = \frac{1}{2}\left\{c_{i+1/2}^{+}\phi_i + c_{i+1/2}^{-}\phi_{i+1}\right\}$$
$$c_{i+1/2}^{+} = (c+|c|)_{i+1/2},\quad c_{i+1/2}^{-} = (c-|c|)_{i+1/2}$$

PPM : Piecewise-Parabolic Method
ENO : Essentially Non-oscillatory
WENO: weighted **ENO**

General formula

$$\phi_i^n = \phi_i^{n-1} - S\zeta_F\Sigma_{l=-m^-}^{+m^+}a_l\phi_{i+ls(c)}^{n-1}$$

(B) Groupe using variables and those gradients

CIP3rd method:

$$\partial_t\phi + c\partial_x\phi = 0$$
$$\partial_t g + c\partial_x g = 0$$
$$g = \partial_x\phi$$

$$\phi^n(x_i) = F_{id}^{n-1}(x_i - c\delta t)$$
$$g^n(x_i) = \partial_x F_{id}^{n-1}(x_i - c\delta t)$$

Improve the CIP3rd Method to CIP5th method

What is CIP scheme?

■ . **Constrained Interpolation Profile scheme?**

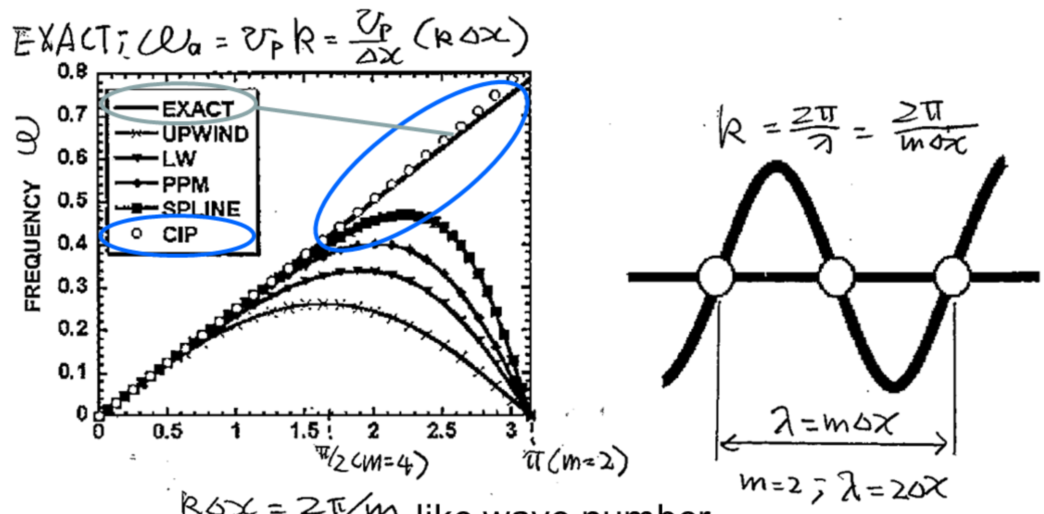
■ CIP method advects **variable's gradients as the phase information.**

$$\partial_t \phi + c\partial_x \phi = 0$$

$$\partial_t g + c\partial_x g = 0$$

$$g = \partial_x \phi$$

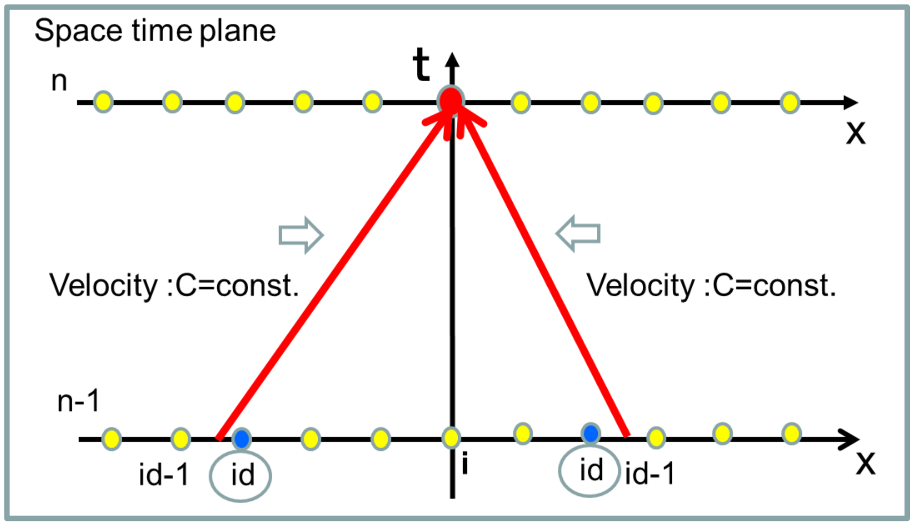
■ The phase accuracy of CIP method is higher than other conventional methods for **especially high wave number.**



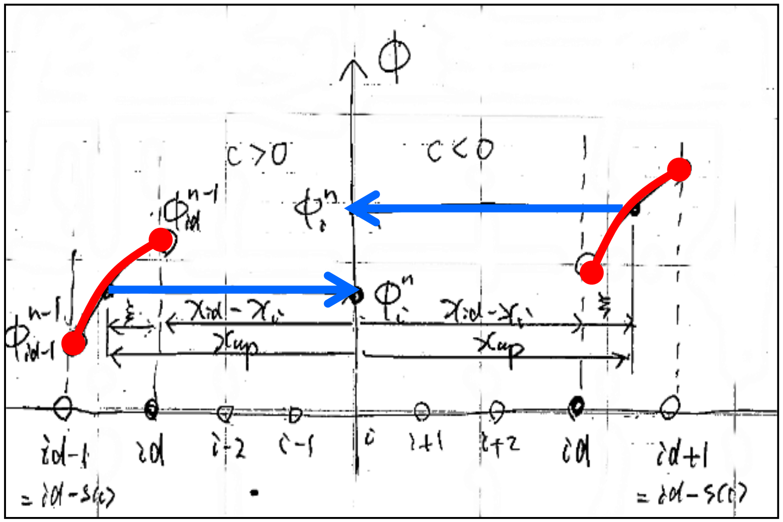
Up stream calculation and time integration

pefrormed **by back-trace and shift operation** **(CFL free formula is used here: we can easily use large δT for case solver)** ∗Considering the equation on the grid i

Back-trace :



Shit operation



id = grid that is near i grid of cell (id, id-1)

$$id = i - \mathrm{INT}\left(\frac{\mathrm{x_up}}{\Delta \mathrm{x}}\right) = \mathrm{i} - \mathrm{INT}\left(\frac{\mathrm{c}\delta\mathrm{t}}{\Delta\mathrm{x}}\right)$$

Back-trace points finding

$$\xi_F = x - x_{id} = x_i - c\delta t - x_{id} = -c\delta t - (x_i - x_{id}) = -c\delta t - \Delta x(i - id) = -c\delta t + \Delta x\mathrm{INT}\left(\frac{\mathrm{c}\delta\mathrm{t}}{\Delta\mathrm{x}}\right)$$

Upstream finding

$$D_F = -s(c)\Delta x, \quad s(c) = \mathrm{SIGN}(1.0, \mathrm{c})$$

Detail of Formula

Point

CIP 3rd method

CIP 5th method (We developed it as more accurate CIP at this time.)

$$\partial_t \phi + c\partial_x \phi = 0$$
$$\partial_t g + c\partial_x g = 0$$
$$g = \partial_x \phi$$ (gradient)

$$\partial_t \phi + c\partial_x \phi = 0$$
$$\partial_t g + c\partial_x g = 0, \quad \partial_t \chi + c\partial_x \chi = 0$$
$$g = \partial_x \phi, \quad \chi = \partial_x g$$

(gradient, curvature)

- **Space discretization: by the cubic interpolation function**

$$\phi(x) = F_{id}(x) = a_{id}(x - x_{id})^3 + b_{id}(x - x_{id})^2 + g_{id}(x - x_{id}) + \phi_{id}$$
$$g(x) = \partial_x F_{id}(x) = 3a_{id}(x - x_{id})^2 + 2b_{id}(x - x_{id}) + g_{id}(x - x_{id})$$

by the 5th interpolation function

$$\phi(x) = F_{id}(x) = a_{id}(x - x_{id})^5 + b_{id}(x - x_{id})^4 + c_{id}(x - x_{id})^3 + \chi_{id}/2(x - x_{id})^2 + g_{id}(x - x_{id}) + \phi_{id}$$
$$g(x) = \partial_x F_{id}(x) = 5a_{id}(x - x_{id})^4 + 4b_{id}(x - x_{id})^3 + 3c_{id}(x - x_{id})^2 + \chi_{id}(x - x_{id}) + g_{id}$$
$$\chi(x) = \partial_x^2 F_{id}(x) = 20a_{id}(x - x_{id})^3 + 12b_{id}(x - x_{id})^2 + 6c_{id}(x - x_{id}) + \chi_{id}$$

$$a_{id} = \frac{1}{D_F^3}\left\{-2(\phi_{id-s(c)} - \phi_{id}) + (g_{id-s(c)} + g_{id})D_F\right\}$$
$$b_{id} = \frac{1}{D_F^2}\left\{3(\phi_{id-s(c)} - \phi_{id})1(g_{id-s(c)} + 2g_{id})D_F\right\}$$

$$a_{id} = \frac{1}{D_F^5}\left\{6(\phi_{id-s(c)} - \phi_{id}) - 3(g_{id-s(c)} + g_{id})D_F + 1/2(\chi_{id-s(c)} - \chi_{id})D_F^2\right\}$$
$$b_{id} = \frac{1}{D_F^4}\left\{-15(\phi_{id-s(c)} - \phi_{id}) + 7(g_{id-s(c)} + 8/7g_{id})D_F - (\chi_{id-s(c)} - 3/2\chi_{id})D_F^2\right\}$$
$$c_{id} = \frac{1}{D_F^3}\left\{10(\phi_{id-s(c)} - \phi_{id}) - 4(g_{id-s(c)} - 3/2g_{id})D_F + 1/2(\chi_{id-s(c)} - 3\chi_{id})D_F^2\right\}$$

- **Up-date(time integration): by Semi-Lagrange scheme**

$$\phi^n(x_i) = F_{id}^{n-1}(x_i - c\delta t)$$
$$g^n(x_i) = \partial_x F_{id}^{n-1}(x_i - c\delta t)$$

$$\phi^n(x_i) = F_{id}^{n-1}(x_i - c\delta t)$$
$$g^n(x_i) = \partial_x F_{id}^{n-1}(x_i - c\delta t)$$
$$\chi^n(x_i) = \partial_x^2 F_{id}^{n-1}(x_i - c\delta t)$$

$$\phi^n(x_i) = a_{id}^{n-1}\xi_F^3 + b_{id}^{n-1}\xi_F^2 + g_{id}^{n-1}\xi_F^3 + \phi_{id}$$
$$g^n(x_i) = 3a_{id}^{n-1}\xi_F^2 + 2b_{id}^{n-1}\xi_F + g_{id}^{n-1}$$

$$\phi^n(x_i) = a_{id}^{n-1}\xi_F^5 + b_{id}^{n-1}\xi_F^4 + c_{id}^{n-1}\xi_F^3 + \chi_{id}/2\xi_F^2 + g_{id}\xi_F + \phi_{id}$$
$$g^n(x_i) = 5a_{id}^{n-1}\xi_F^4 + 4b_{id}^{n-1}\xi_F^3 + 3c_{id}^{n-1}\xi_F^2 + \chi_{id}\xi_F + g_{id}^{n-1}$$
$$\chi^n(x_i) = 20a_{id}^{n-1}\xi_F^3 + 12b_{id}^{n-1}\xi_F^2 + 6c_{id}^{n-1}\xi_F + \chi_{id}$$

Code of CIP-5th method

Set of the advection parameters

Calculation of the coefficient of spline function

Update of variables

```
do j=1,Ny; do i=1,Nx
    cx  =0.5d0*(v1(i-1,j,1)+v1(i,j,1))
    ida =i-int(cx*dt/dx)
    xgi =-cx*dt+dx*real(i-ida)
    ais =sign(1.0,cx)
    idam=ida-int(ais)
    fv  =              v1(ida,j,3)
    gfi =gf (ida,j,1) !  dfai/dx
    ggfi=ggf(ida,j,1) ! ddfai/dx/dx
    dfi =v1 (idam,j,3)-v1(ida,j,3)
    aid1=-dx5*ais*(  6.0* dfi &
   &               + 3.0*(    gf (idam,j,1)+     gfi)*dx*ais  &
   &               + 0.5*(    ggf(idam,j,1)-    ggfi)*ddx     )
    bid1= dx4*    (-15.0* dfi &
   &               -      (7.0*gf (idam,j,1)+8.0* gfi)*dx*ais  &
   &               -      (    ggf(idam,j,1)-1.5*ggfi)*ddx     )
    cid1=-dx3*ais*( 10.0* dfi &
   &               + 4.0*(    gf (idam,j,1)+1.5* gfi)*dx*ais  &
   &               + 0.5*(    ggf(idam,j,1)-3.0*ggfi)*ddx     )
    v2  (i,j,3)= &
  &((((     aid1*xgi+     bid1)*xgi+    cid1)*xgi+0.5*ggfi)*xgi+gfi)*xgi+fv
    gfn (i,j,1)= &
  & ((( 5.0*aid1*xgi+ 4.0*bid1)*xgi+3.0*cid1)*xgi+    ggfi)*xgi+gfi
    ggfn(i,j,1)= &
  &  ((20.0*aid1*xgi+12.0*bid1)*xgi+6.0*cid1)*xgi+    ggfi
end do; end do
```

CIP-5th method is very simple

and we can easily develop CIP5th code based on CIP3rd method code.

Improve the Groupe using only variables to no-dumping and accurate phase method: **STRS scheme**

*Katsuhiro Watanabe, a novel framework to construct amplitude preserving wave propagation schemes, Japan Society for Industrial and Applied Mathematics Annual meeting (2010), 123-124.(written in Japanese)

What is STRS(Space-Time Reversal Symmetry) scheme?

STRS scheme is based on a **symmetry of advection equation**. That symmetry is **the PARITY CONSERVATIVENESS**, which is expressed by following formula **in CONTINUUM SPACE**.

Parity Transformation $P : \begin{pmatrix} x \\ t \end{pmatrix} \mapsto \begin{pmatrix} -x \\ -t \end{pmatrix} = \begin{pmatrix} x' \\ t' \end{pmatrix}$

CONSERVATIVE

$$\partial_t \phi + c\partial_x \phi = 0 \quad \Longleftrightarrow \quad \partial_{t'} \phi + c\partial_{x'} \phi = 0$$

$$c = c(x,t) = \lim_{x \to 0} \frac{\Delta x}{\Delta t} = \lim_{x \to 0} \frac{-\Delta x}{-\Delta t} = \lim_{x \to 0} \frac{\Delta x'}{\Delta t'} = c' = c'(t', x')$$

Space and Time Reversal Symmetry guarantees the CONSERVATIVENESS of the amplitude Φ.

parity transformation (also called parity inversion) is the flip in the sign of coordinate
パリティ変換 (**parity transformation**) は一つの座標の符号を反転させることである。
パリティ反転 (parity inversion) とも呼ぶ。

How to construct the STRS scheme of linear type of advection differencing schemes

General formula: $$\phi_i^n = \phi_i^{n-1} + S\zeta_F \Sigma_{l=-m^-}^{+m^+} a_l \phi_{id+ls(c)}^{n-1} \qquad \text{(A)}$$

We can convert it to STRS scheme mechanically.

- 1st step: Perform the **Parity Transformation** on RHS of eq.(A)
- 2nd step: Replace LHS of eq.(A) by that.

Then we get formula (B.1). Let's check the STRS of eq.(B.1)

$$\phi_{ida^-}^n + S\zeta_F \Sigma_{l=-m^+}^{+m^-} a_l \phi_{ida^- -ls(c)}^n = \phi_{ida^+}^{n-1} + S\zeta_F \Sigma_{l=-m^-}^{+m^+} a_l \phi_{ida^+ +ls(c)}^{n-1} \qquad \text{(B.1)}$$

CONSERVATIVE

$$P : \begin{pmatrix} n-1 \\ m^{\mp} \\ ida^+ \\ l \end{pmatrix} \mapsto \begin{pmatrix} n \\ m^{\pm} \\ ida^- \\ -l \end{pmatrix}$$

Parity Transformation in the discretization space

$$\phi_{ida^+}^{n-1} + S\zeta_F \Sigma_{l=-m^-}^{+m^+} a_l \phi_{ida^+ +ls(c)}^{n-1} = \phi_{ida^-}^n + S\zeta_F \Sigma_{l=-m^+}^{+m^-} a_l \phi_{ida^- -ls(c)}^n \qquad \text{(B.2)}$$

This scheme gives
(a) **stable**, (b) **no damping of amplitudes** numerical methods.

Most simple example: STRS scheme of Upwind 1st order

$$\left(1-\frac{1}{2}\zeta_F\right)\phi^{n}_{ida-}+\frac{1}{2}\zeta_F\phi^{n}_{ida-+s(c)}=\left(1-\frac{1}{2}\zeta_F\right)\phi^{n-1}_{ida+}+\frac{1}{2}\zeta_F\phi^{n-1}_{ida+-s(c)}$$

Stability analysis using fourier transform

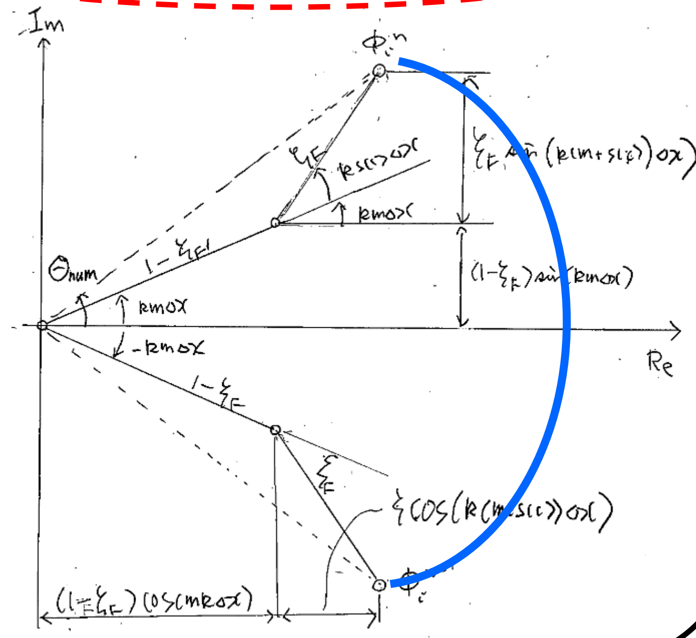
$$\phi_{ida\mp}=\phi_i e^{\mp jkm\Delta x}$$

$$\phi_{ida\mp\pm s(c)}=\phi_i e^{\mp jkm\Delta x}e^{\mp jks(c)\Delta x}$$

Time development formula of value Φ for mode l in complex plane

$$\frac{\phi_i^n}{\phi_i^{n-1}}=\frac{\left(1-\zeta_F\right)+\zeta_F e^{+jks(c)\Delta x}\right) e^{+jkm\Delta x}}{\left(1-\zeta_F\right)+\zeta_F e^{-jks(c)\Delta x}\right) e^{-jkm\Delta x}}$$

No damping of amplitude



In this case, phase correction can be done as here.

Numerical phase speed

$$\tan(2\theta_{num}) = \frac{(1-\zeta_F)\sin(mk\Delta x) + \zeta_F \sin(k(m+s(c))\Delta x}{(1-\zeta_F)\sin(mk\Delta x) + \zeta_F \sin(k(m+s(c))\Delta x} = Q(\zeta_F)$$

We can adjust **the numerical phase speed** to **correct phase speed for one mode**.

$$2\tan^{-1}\left(Q(\zeta_F^{adj})\right) = \theta_{phy} = 2\zeta_{phy} k\Delta x$$

Solve ◯ and use it!

from physical dispersion relation

Kawamura and Kuwahara-3rd, central-4th etc. schemes can be transformed to STRS scheme as same way!

However, phase adjustment is available for upwind 1st and one mode case, very special case only.

Improve the CIP3rd method by STRS scheme:

STRS-CIP scheme

STRS-CIP formula

We can easily get STRS-CIP formula from CIP3rd scheme
by the **Parity Transformation.**

$$
(\mathbf{I}+\mathbf{a})\begin{pmatrix}\phi \\ D_F g\end{pmatrix}_{id^-}^{n} + \mathbf{b}\begin{pmatrix}\phi \\ D_F g\end{pmatrix}_{id^-+s(c)}^{n}
= (\mathbf{I}+\mathbf{a})\begin{pmatrix}\phi \\ D_F g\end{pmatrix}_{id^+}^{n-1} + \mathbf{b}\begin{pmatrix}\phi \\ D_F g\end{pmatrix}_{id^+-s(c)}^{n-1}
$$

$$
\mathbf{a} = \left[\begin{array}{c|c} 2\left(\frac{\xi}{D}\right)_*^3 - 3\left(\frac{\xi}{D}\right)_*^2 & \left(\frac{\xi}{D}\right)_*^3 - 2\left(\frac{\xi}{D}\right)_*^2 + \left(\frac{\xi}{D}\right)_* \\ \hline 6\left(\frac{\xi}{D}\right)_*^2 - 6\left(\frac{\xi}{D}\right)_* & 3\left(\frac{\xi}{D}\right)_*^2 - 4\left(\frac{\xi}{D}\right)_* \end{array}\right]
$$

$$
\mathbf{b} = \left[\begin{array}{c|c} -\left\{2\left(\frac{\xi}{D}\right)_*^3 - 3\left(\frac{\xi}{D}\right)_*^2\right\} & \left(\frac{\xi}{D}\right)_*^3 - \left(\frac{\xi}{D}\right)_*^2 \\ \hline -\left\{6\left(\frac{\xi}{D}\right)_*^2 - 6\left(\frac{\xi}{D}\right)_*\right\} & 3\left(\frac{\xi}{D}\right)_*^2 - 2\left(\frac{\xi}{D}\right)_* \end{array}\right]
$$

$$
D_F = -s(c)\Delta x \quad D_R = s(c)\Delta x
$$

The formula is Parity CONSERVATIVE,
but this still dose not work.
Reason why, not yet clear.

Then, we tried an approximation version : STRS-CIP3rd_mod.

- **Method :**

1st step: get the gradient g by CIP3rd.
2nd step: get the value Φ using STRS-CIP3rd formula with given gradient g.

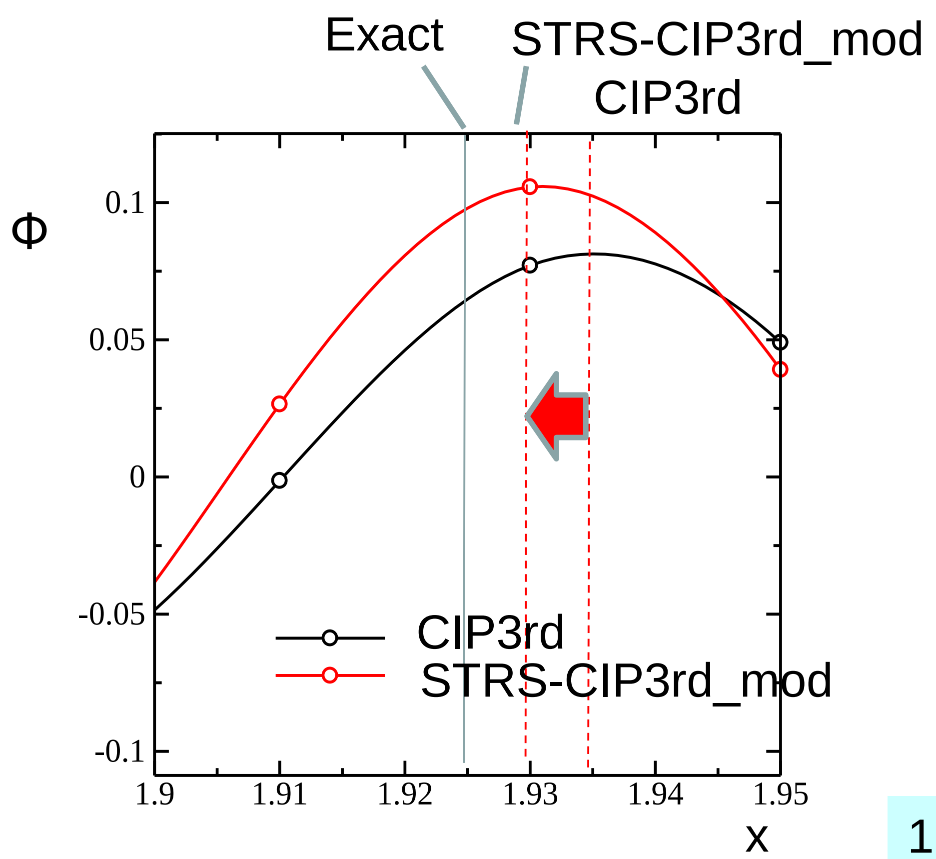
- **Check the improvement :**
  - sin wave (5grids/wave) advection
  - Space [0, 2] × Time: [0, 2]
  - CFL = 0.1

We can improve CIP3rdd by STRS approximation.

But that improvement is small.
Then, we skipped this one this study.

→ Future challenge.

Results of Φ distribution (t=2: after 20 cycles)



Check impacts of conventional methods improvement

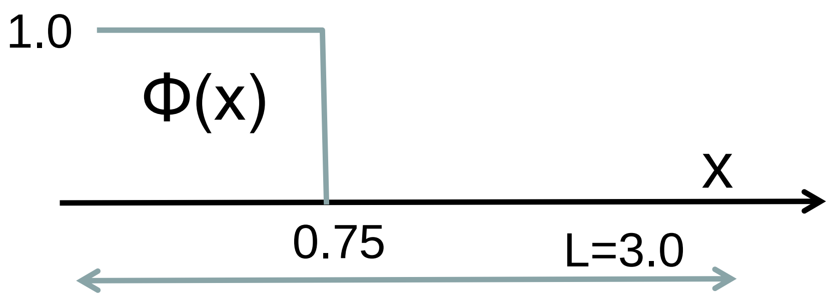
Benchmark: Step Shape Advection

Check the CIP5th method performance by CIP3rd vs CIP5th method

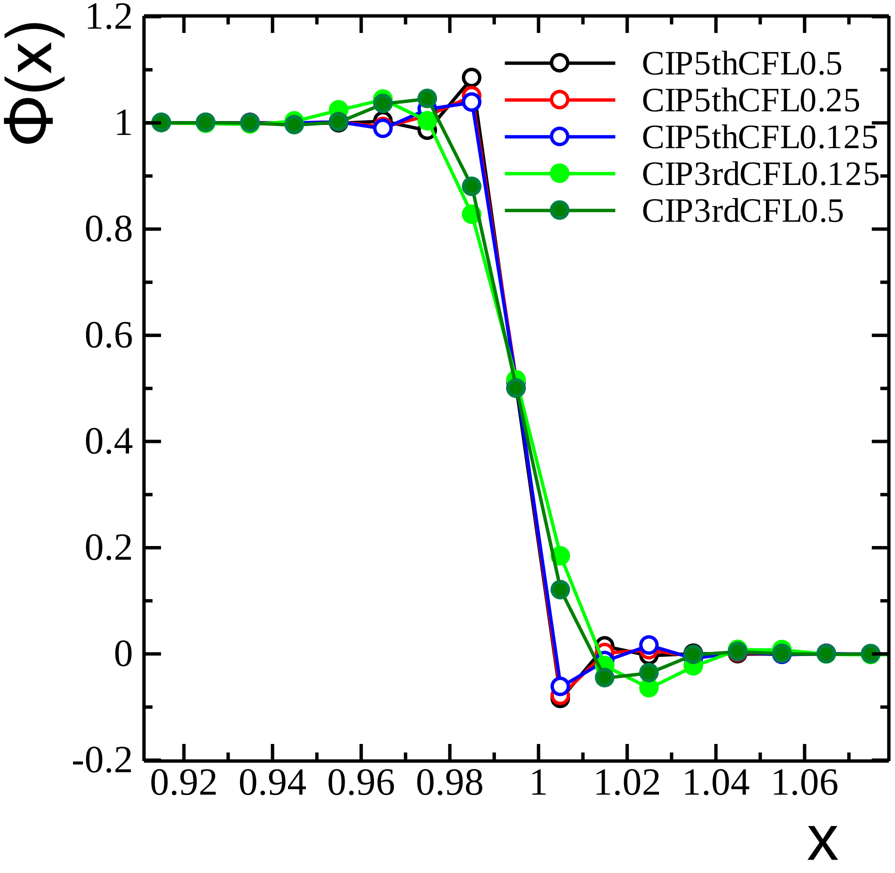
Parameters of the test

Physical condition	● 1D advection of step shape ・speed c=1.0 ● Space [0,3]×Time: [0,0.5 or 2.25]
Numerical analysis condition	● Num. of meshes: 300 → dx=0.01 ● Width of time step →dt=0.005,0.0025,0.00125 → CFL=0.5, 0.25, 0.125 ● Boundary condition：continuous ● Initial condition → x=0--0.5:Φ=1.0, x > 0.5: Φ=0.0
Advection numerical method	● CIP3rd vs CIP5th method

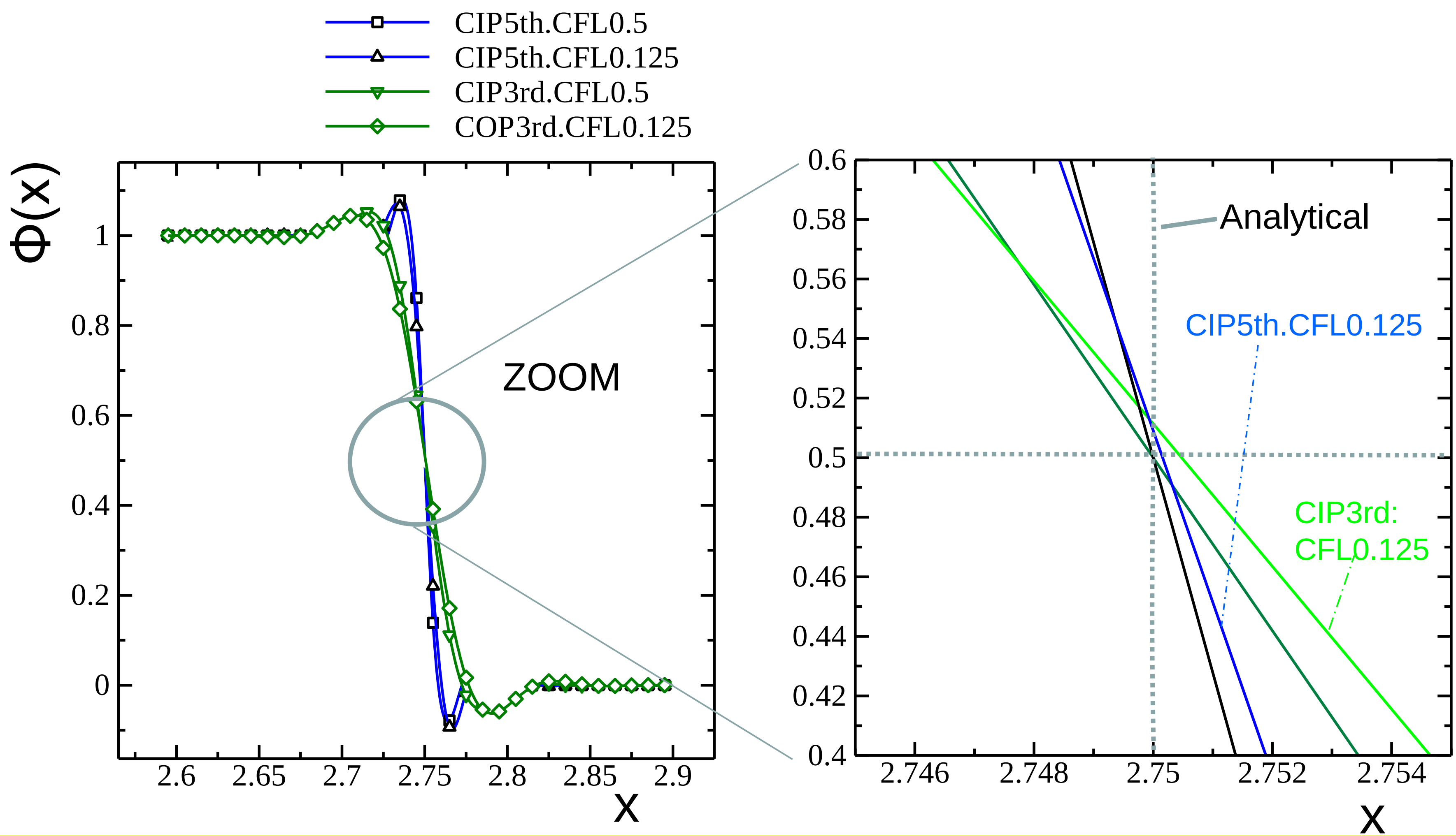
Initial condition



(a) Results of Φ distribution (t=0.5)



(b) Results of Φ distribution (t=2.25) (*) Zoomed part



CIP-5th method → Step shape is sharp.

→ Phase accuracy is better.

➔ Improvement has been achieved!

Benchmark: Sin Wave Advection

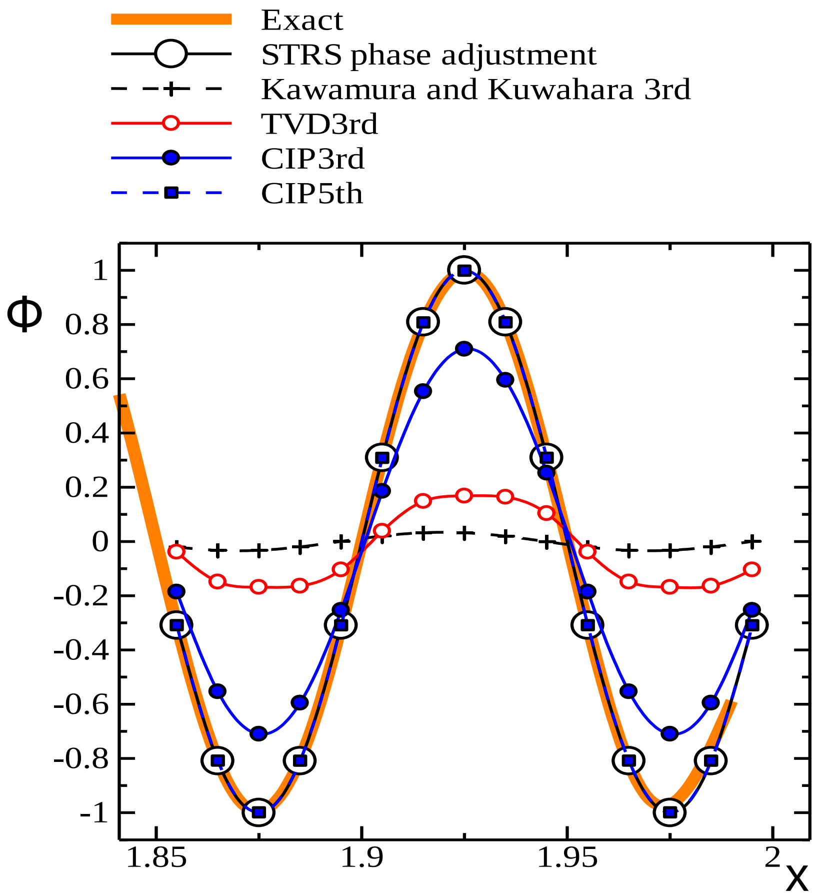
No-damping and no phase error STRS scheme using phase adjustment for one mode

Parameters of the test

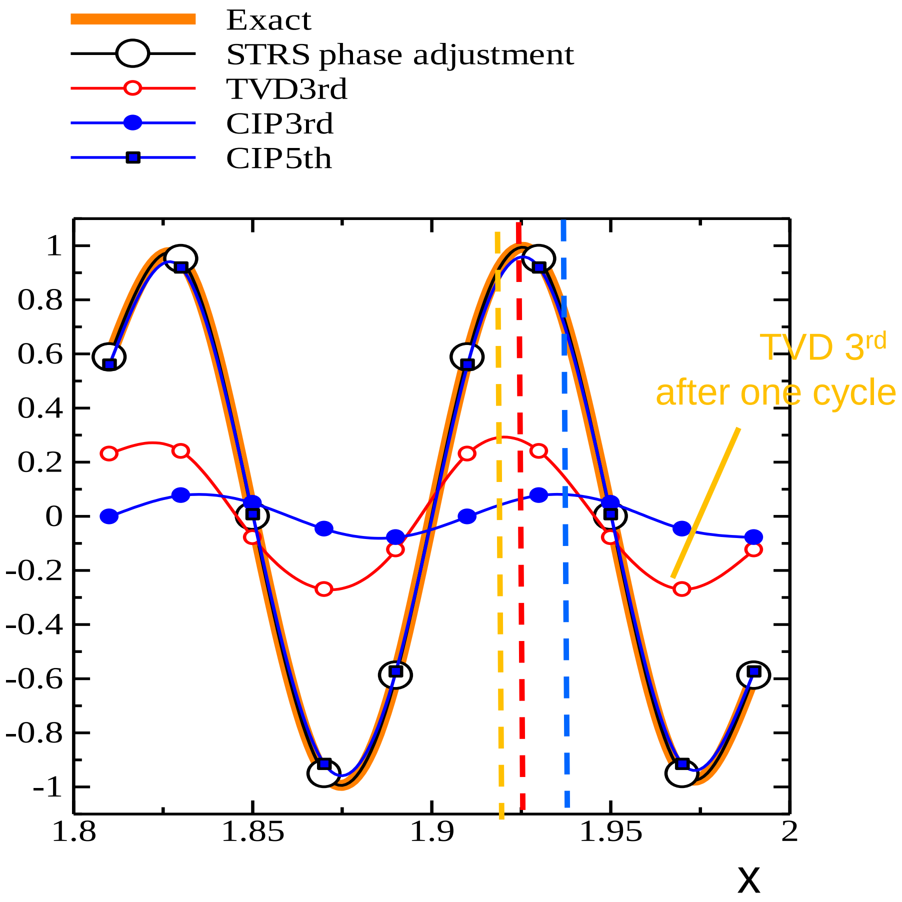
Test problem	● Advection of sin wave (one mode wave) ・$\Phi(x)=\sin(2\pi m\Delta x((i-1)/\lambda+0.5))$、$\lambda=0.1$ → $g(x)= d\Phi(x)/dx$ $=2\pi m/\lambda \cos(2\pi m\Delta x((i-1)/\lambda+0.5))$ ・10grids/wave(m=10) or 5grid/wave(m=20) ・velocity c=1.0 ● Space [0, 2]×Time: [0, 2]
Analysis condition: space and time descritaization	● dx=0.01 or 0.02、200 or 100 meshes →L=dx×200=2 ● dt =0.001 or 0.002(CFL=0.1) ● Boundary condition：cyclic
STRS scheme vs Conventional scheme	● TVD 3rd (3rd order) ● CIP scheme 3rd order ● CIP 5th (5th ordr) ● STRS-Upwind 1st order with phase adjustment (Exact for one mode wave)

Results after T=2.0 (20 cycles)

10 grids/wave



5 grids/wave



Phase improvement has been achieved by CIP5th and STRS phase adjust cases

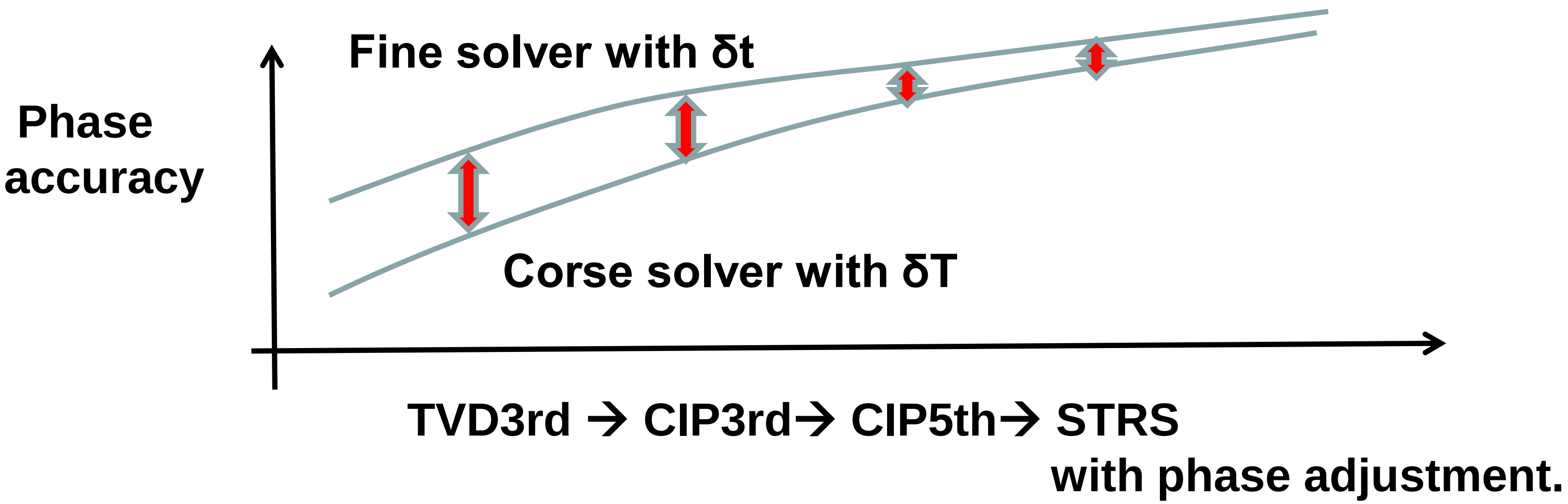
Parareal calculation

- δt: time step width of fine solver： set by the CFL condition: $\Delta x/v > \delta t$
- δT: time step width of coarse solver： $\delta T \gg \delta t$

Purpose of benchmark test

Study the impact of the *phase difference between fine/coarse solver to* the parareal convergence

- **Set the same method of advection calculation in fine/coarse solver**
- **Phase accuracy increases along TVD3rd → CIP3rd→ CIP5th→ STRS.**
  → *phase difference decrease!*



Parareal codes for each methods

Fine solver δt | **Coarse solver δT**

Parareal_CN-TVD as reference.

Fine solver δt	Coarse solver δT
Time loop: Numerical flux construction → Time integrator: Crank Nicolson	Time loop: Numerical flux construction → Time integrator: Crank Nicolson

Parareal_CIP3rd, CIP5th

Fine solver δt	Coarse solver δT
Time loop: CIP function construction → Time integrator: Semi-Lagrange	Time loop: CIP function construction → Time integrator: Semi-Lagrange

Parareal_STRS with phase adjustment

Fine solver δt	Coarse solver δT
Time loop: STRS coefficient construction → Time integrating: STRS-Euler-1st	Time loop: STRS coefficient construction → Time integrating: STRS-Euler-1st

same method in fine/coarse solver
→ δt << δT : only difference

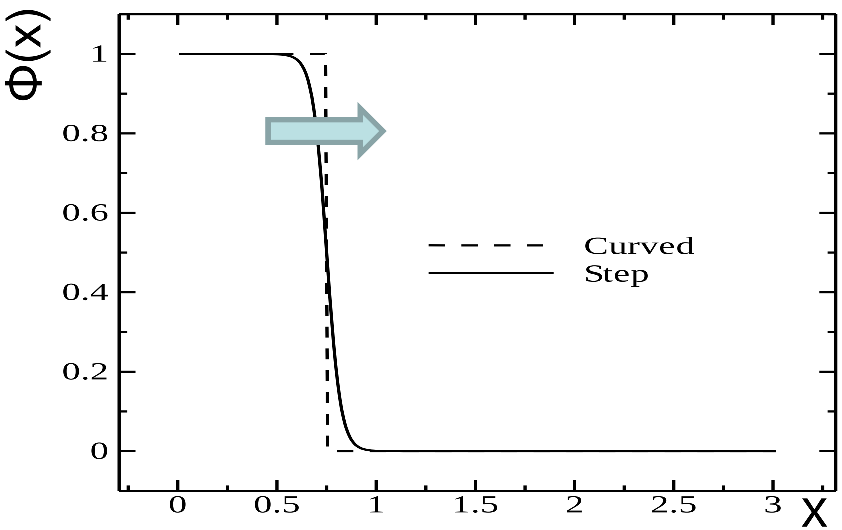
Convergence test of the parareal iteration

Benchmark: Step Advection

Numerical test: Parameters

Test problem	● **C = 1.0 and Space [0,3] × Time: [0, 2.0]** ● **(a) advection of step shape** ● **(b) advection of step like wave** f(x)=0.5(1-tanh((x-xo)/xi)、 xi : width of step → xo=1.0, → xi =SQRT(2D/k)=SQRT(2/k): 0.035(k=1600),0.07(k=400) } See the initial condition bellow.
Space and time descritaization	● dx=0.01, 200meshes (10grids/wave) →L=dx × 200=2 ● δt =0.001(CFL=0.1) ● Boundary condition ： continuous
PinT condition	● Number of time slices: 20 ● Time coarsening factor Rfc = 25 (δT= 0.025)

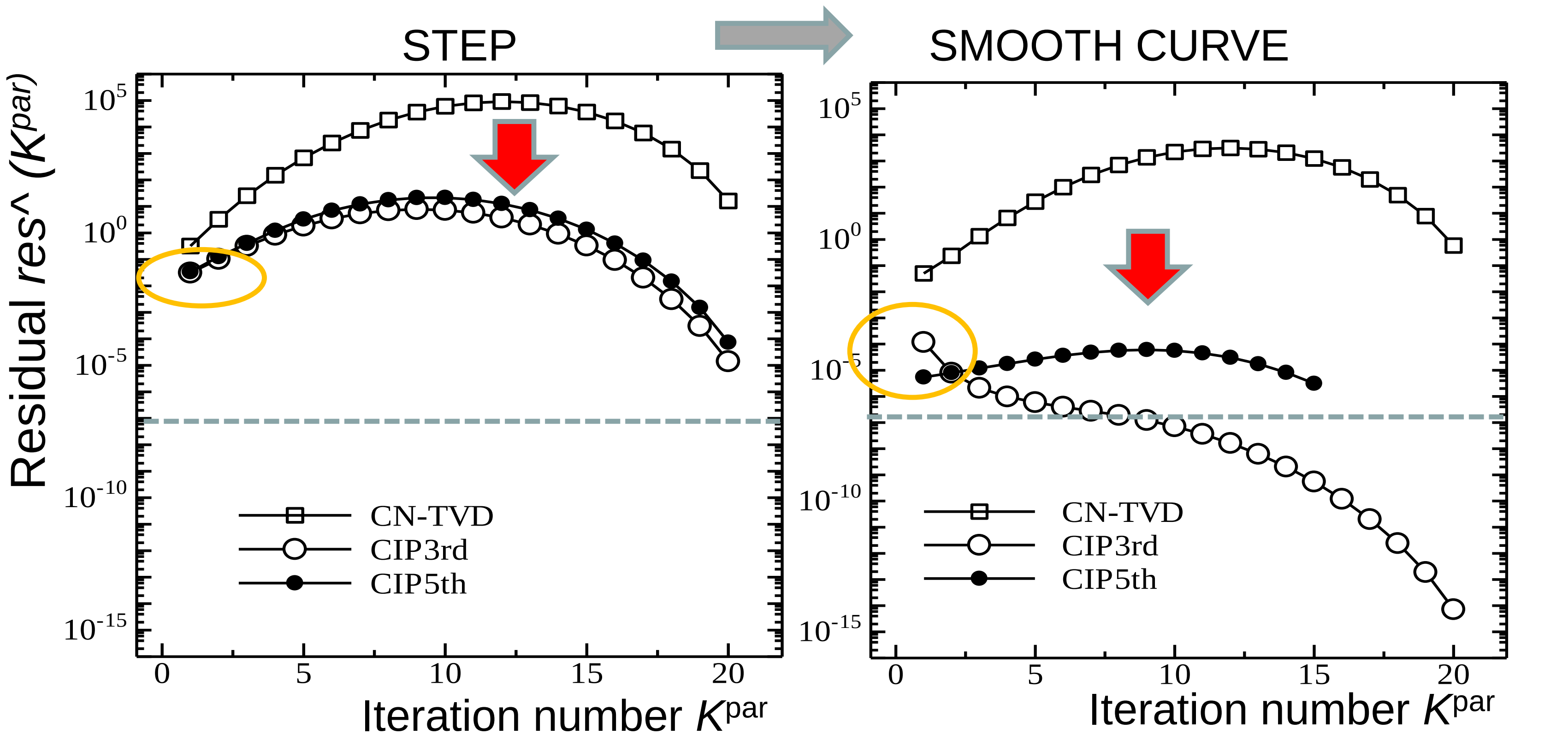
Initial condition



$$res^{(K^{par})} = max_n \left[\sqrt{ \frac{\sum_{i=1}^{N_{DOF}} |U_{i,n-1}^{K^{par}} - U_{i,n-1}^{K^{par}-1}|^2}{N_{DOF}} } \right]$$

"Step" and "Smooth curves" are used as initial condition.
→ When smoothness of curve increases, **the number of grid based high wave number waves decrease.**

Results : Residual during the parareal iteration :



Relaxation parameter α=1.0

$$\mathbf{U}_n^k = F(T_n, T_{n-1}, \mathbf{U}_{n-1}^{k-1}) + \alpha \left\{ G(T_n, T_{n-1}, \mathbf{U}_{n-1}^{k}) - G(T_n, T_{n-1}, \mathbf{U}_{n-1}^{k-1}) \right\}$$

∗ *CIP methods* and *reduce of the grid based high wave number waves* improves the convergence.

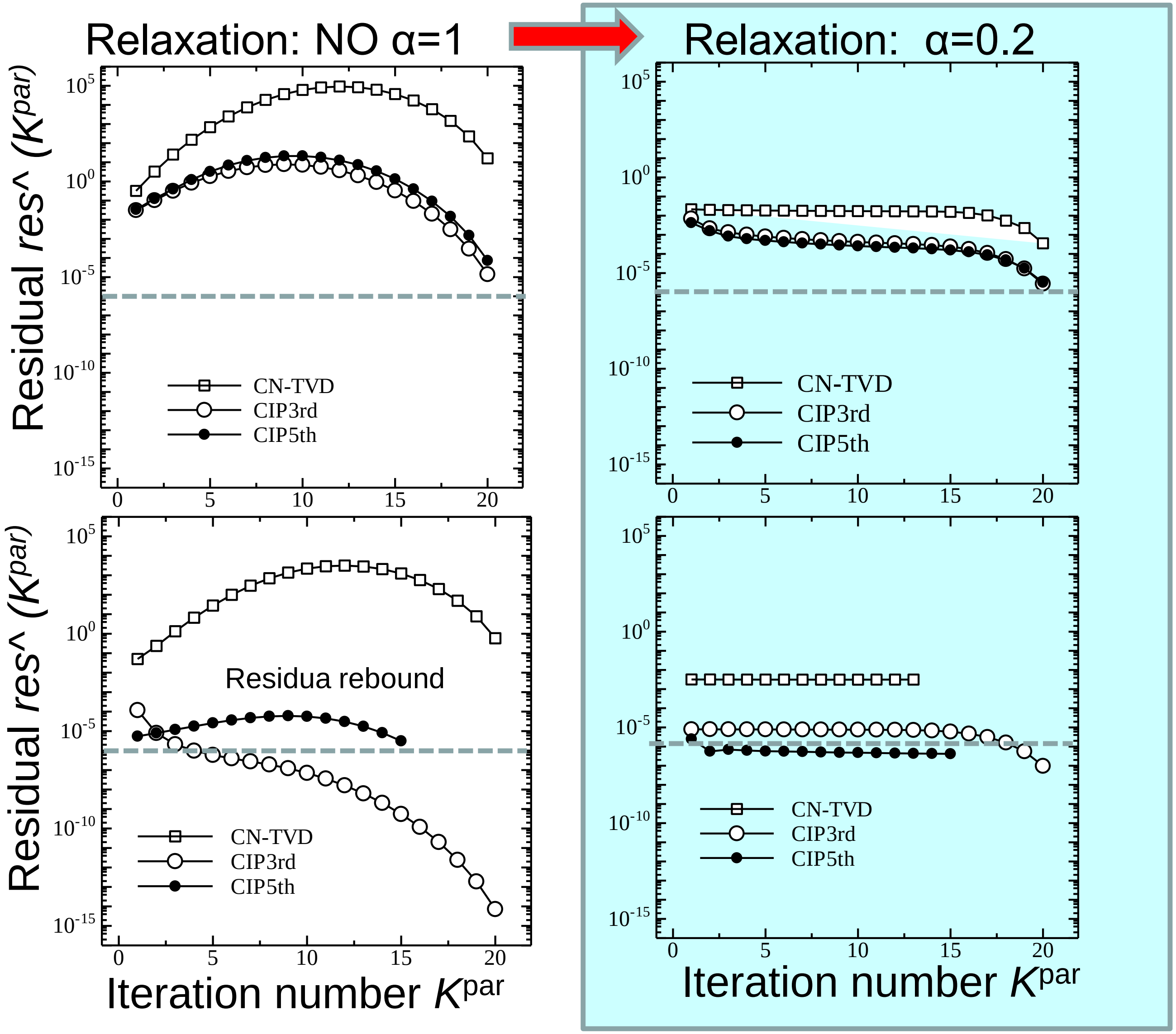
∗ CIP5th has not so much effectiveness than CIP3rd.

→ Reason why not yet clear ?

Influence of pararealiteration realxation

Step

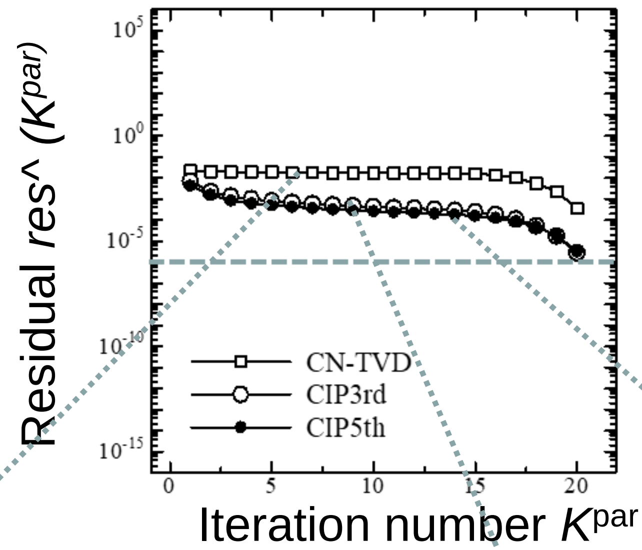
SMOOTH
CURVE



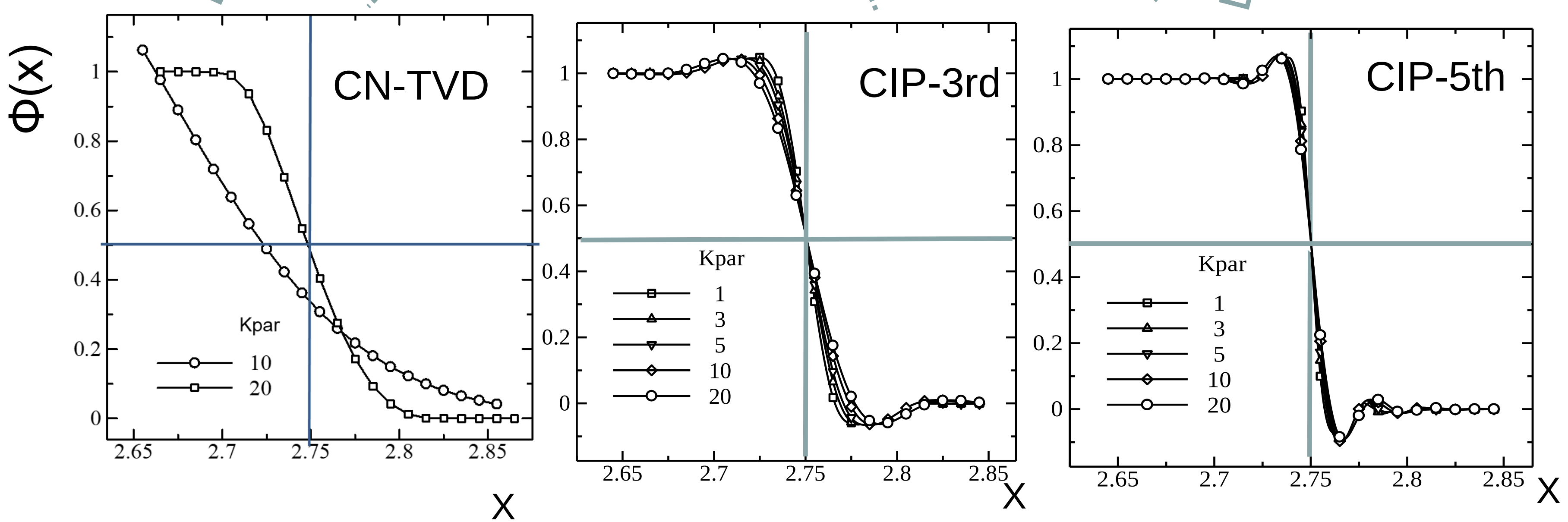
Relaxation is effective for residual rebound,
but Not so much effective ?

CIP-5th looks not so much effective than CIP-3rd, really ? Then, check the profile of variable along iteration･･･

Change of the profile Φ along Kpar.



CIP-5th is very accurate even for Kpar=1.



Profile show that CIP-5th is effective even for first sate of the iteration!

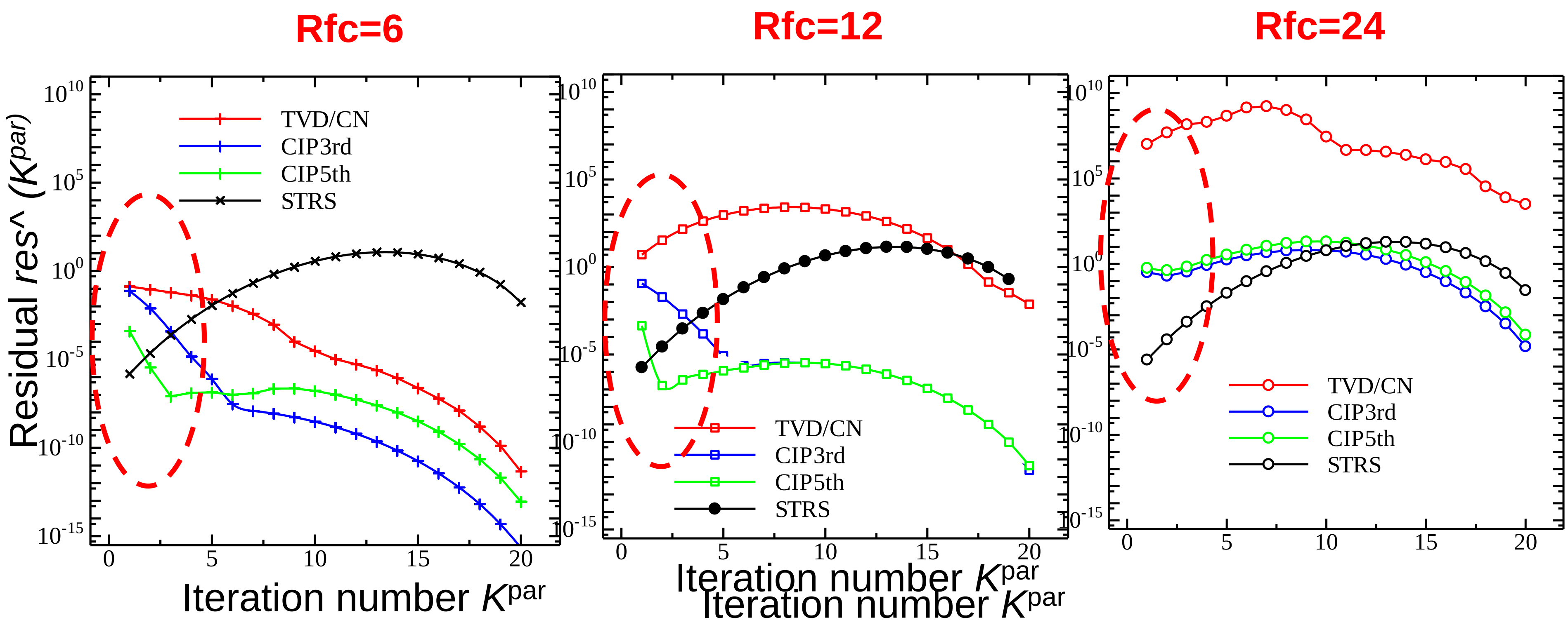
Benchmark:
sin wave with rough grids

Parameters of numerical test

Test problem	● Advection of sin wave (one mode wave) ・ Φ(x)=sin(2πmΔx((i-1)+0.5)) (m=10) → g(x)= dΦ(x)/dx=2πm cos(2πmΔx((i-1)+0.5)) ・ Velocity c=1.0 ● Space [0,2]×Time: [0,2.0]
Analysis condition: space and time descritaization	● dx=0.01、200meshes（10grids/wave） →L=dx×200=2 ● δt =0.001(CFL=0.1) ● Boundary condition：cyclic
PinT condition	● Number of time slices: 20 ● Time coarsening factor: Rfc = δt/δT = 6, 12, 24 (δT=0.006, 0.012, 0.024)

$$res^{(K^{par})} = max_n \left[\sqrt{\frac{\sum_{i=1}^{N_{DOF}} |U_{i,n-1}^{K^{par}} - U_{i,n-1}^{K^{par}-1}|^2}{N_{DOF}}} \right]$$

Results



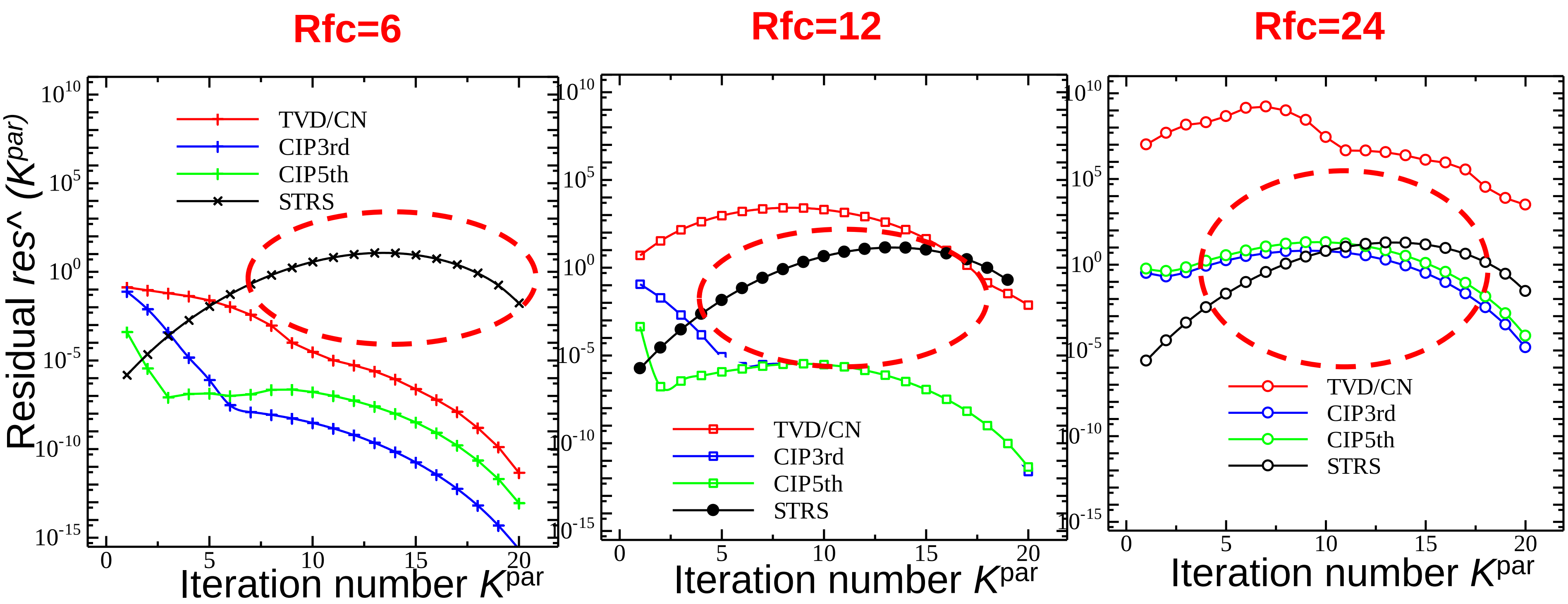
Relaxation **α=1.0**

$$\mathbf{U}_n^k = F(T_n, T_{n-1}, \mathbf{U}_{n-1}^{k-1}) + \alpha \left\{ G(T_n, T_{n-1}, \mathbf{U}_{n-1}^{k}) - G(T_n, T_{n-1}, \mathbf{U}_{n-1}^{k-1}) \right\}$$

At the stage of iteration start,

→ Residual corresponding to the phase difference

→ Small phase difference gives small residual.



Relaxation **α=1.0**

$$\mathbf{U}_n^k = F(T_n, T_{n-1}, \mathbf{U}_{n-1}^{k-1}) + \alpha \left\{ G(T_n, T_{n-1}, \mathbf{U}_{n-1}^{k}) - G(T_n, T_{n-1}, \mathbf{U}_{n-1}^{k-1}) \right\}$$

Along the iteration,

→ Smaller phase difference causes larger residual rebound!

Good and bad news

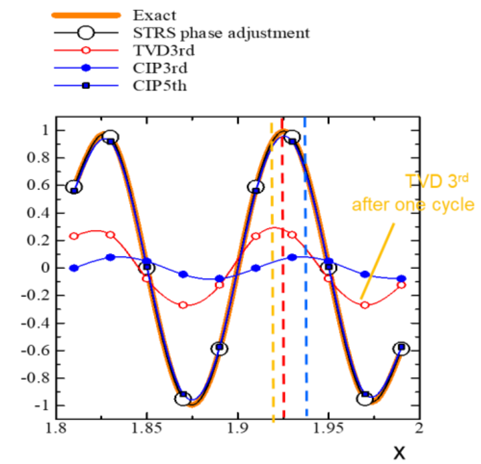
Good: **we achieved very small residual at the start stage of iteration for very tough problem.**

Bad: along the iteration, smaller phase difference causes the residual rebound, reason why not yet unclear ???

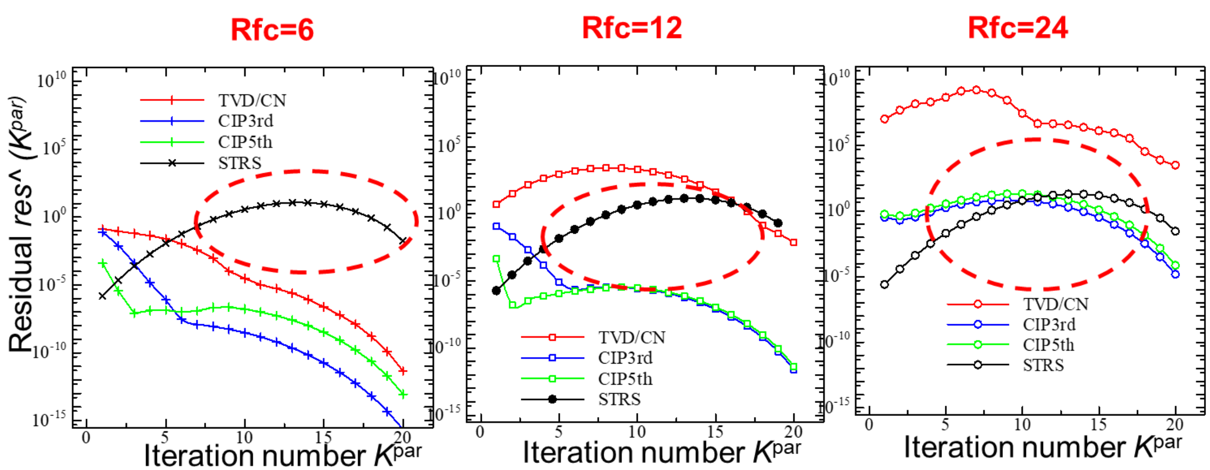
Summary and Future Work

Summary:

- We have achieved BIG STEP in the CFD method view.

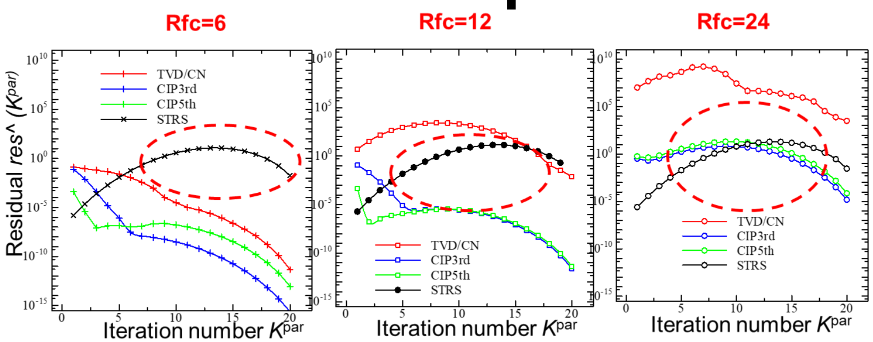


- But, that BIG STEP dose not work well for the parareal method. Still, we have the residual rebound problem.



Future work

- **Now, I have tool that help us to study the impact of the phase difference to parareal convergence. Using that tool, we continue to develop the method for PinT of advection equation.**



- **Also, development of STRS-CIP scheme is challenge. Maybe, it gives another BIG STEP.**

$$(\mathbf{I}+\mathbf{a})\begin{pmatrix}\phi\\ D_F g\end{pmatrix}^{n}_{id^-}+\mathbf{b}\begin{pmatrix}\phi\\ D_F g\end{pmatrix}^{n}_{id^-+s(c)}$$
$$=(\mathbf{I}+\mathbf{a})\begin{pmatrix}\phi\\ D_F g\end{pmatrix}^{n-1}_{id^+}+\mathbf{b}\begin{pmatrix}\phi\\ D_F g\end{pmatrix}^{n-1}_{id^+-s(c)}$$