A posteriori error estimates for space-time domain decomposition method for two-phase flow problem

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Roscoff Marine Station, May 02–05, 2018

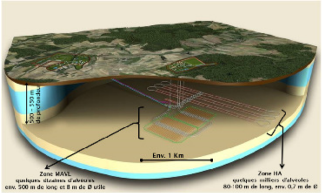
OUTLINE

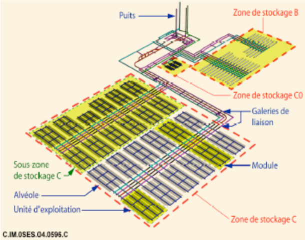
- Motivations and problem setting
1. Robin domain decomposition for a two-phase flow problem
2. Estimates and stopping criteria in a two-phase flow problem
3. Numerical experiments

OUTLINE

Geological disposal of nuclear waste

Deep underground repository
(High-level radioactive waste)





Challenges:

- Different materials → strong heterogeneity, different time scales.
- Large differences in spatial scales.
- Long-term computations.

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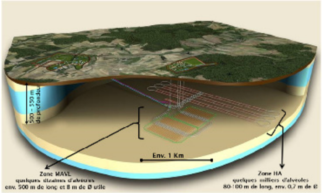

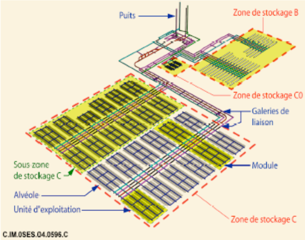

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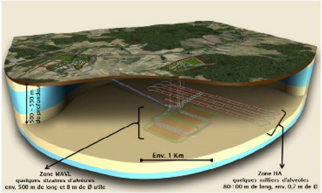

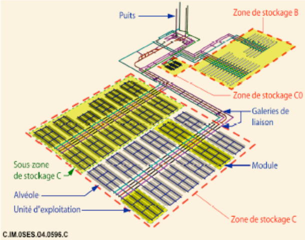

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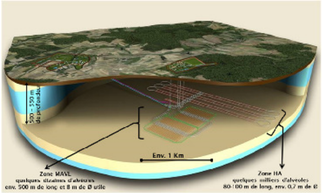

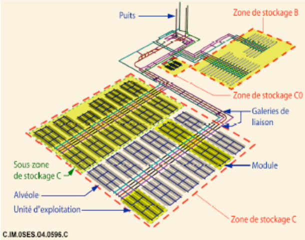

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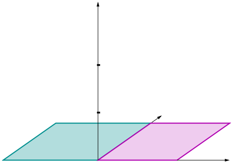
OUTLINE

Domain decomposition in space



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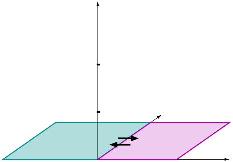

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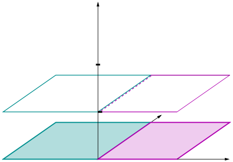

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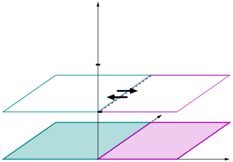

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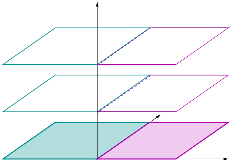

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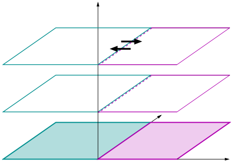

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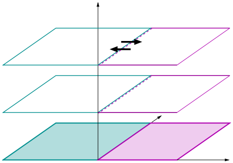

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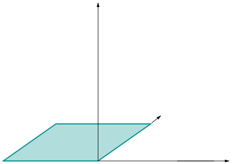

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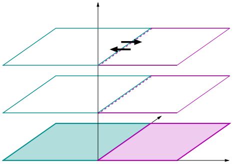

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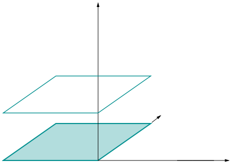

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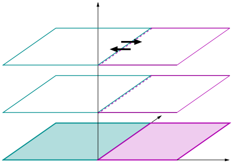

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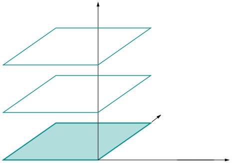

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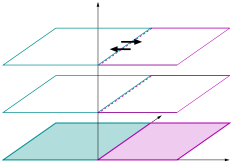

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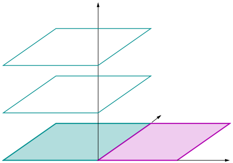

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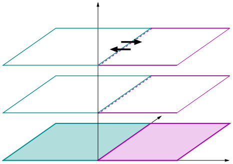

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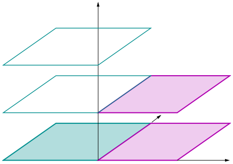

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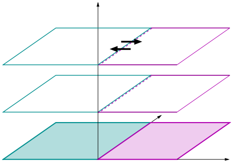

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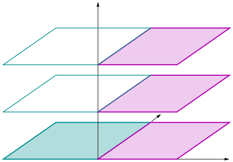

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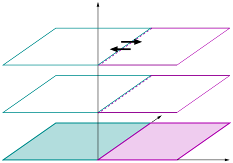

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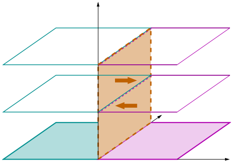

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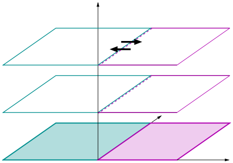

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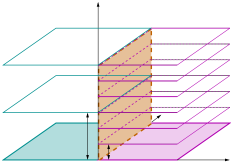

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✓ Different time steps can be used in each subdomain according to its physical properties.
· · · Following [Halpern-C.J.-Szeftel (12), Hoang-C.J.-Jaffré-Kern-Roberts (13)]

Two–phase immiscible flow with discontinuous capillary pressure curves

· · · Following [Enchery-Eymard-Michel 06]

Nonlinear (degenerate) diffusion equation in each subdomain

For $f \in L^2(\Omega \times (0,T))$ and a final time $T > 0$, find $u_i : \Omega_i \times [0,T] \to [0,1]$, $i = 1,2$, such that:

$$\begin{aligned} \partial_t u_i - \Delta \varphi_i(u_i) &= f, && \text{in } \Omega_i \times (0,T), \\ u_i(\cdot,0) &= u_0, && \text{in } \Omega_i, \\ u_i &= g_i, && \text{on } \Gamma_i^D \times (0,T). \end{aligned}$$

Kirchhoff transform φ_i

$$\varphi_i(u_i) = \int_0^{u_i} \lambda_i(a)\pi_i'(a)\mathrm{d}a$$

Capillary pressure

$\pi_i(u_i) \,:\, [0,1] \to \mathbb{R}$

Global mobility of the gas

$\lambda_i(u_i) \,:\, [0,1] \to \mathbb{R}$

- $\Omega \subset \mathbb{R}^d$, $d = 2,3$
- u scalar unknown gas saturation
- $1 - u$ is the water saturation

- u_0 initial gas saturation
- g boundary gas saturation

with the nonlinear interface conditions (**physical transmission conditions**)

$$
\begin{aligned}
\nabla\varphi_1(u_1)\cdot\mathbf{n}_1 &= -\nabla\varphi_2(u_2)\cdot\mathbf{n}_2, \quad &&\text{on } \Gamma\times(0,T),\\
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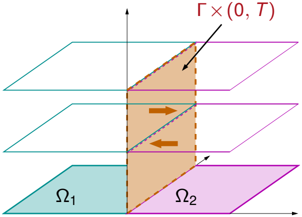

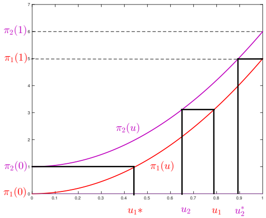

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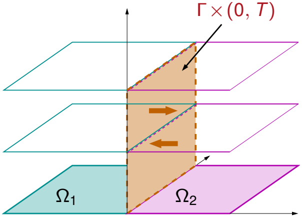

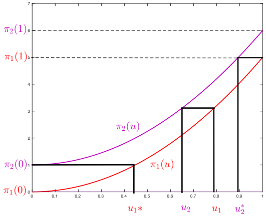

· · · Following [Chavent - Jaffré (86), Enchéry et al. (06), Cances (08), Ern et al (10), Brenner et al. (13)]

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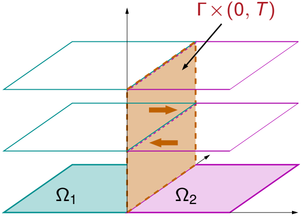

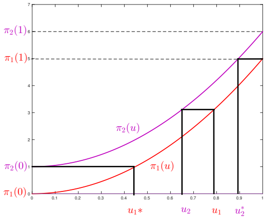

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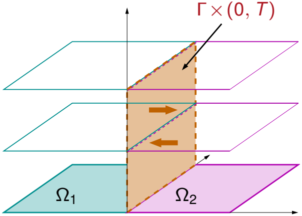

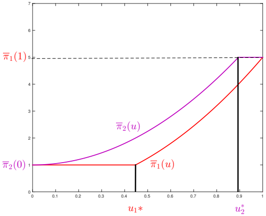

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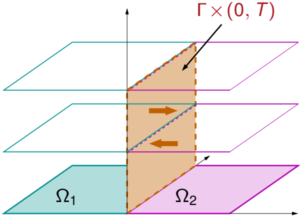

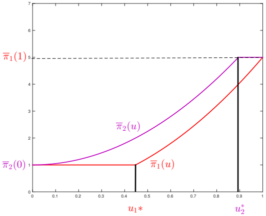

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with the nonlinear interface conditions (**Robin transmission conditions**)

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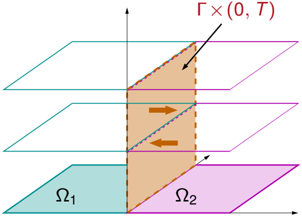
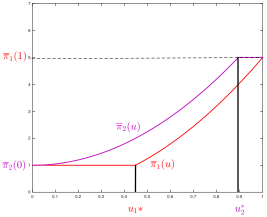
where $\alpha_{i,j}$ are free parameters which optimized convergence rates.





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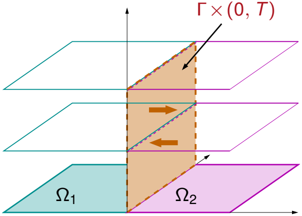

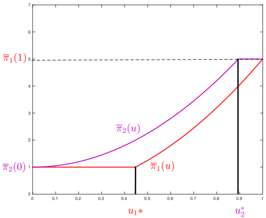

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- where $\overline{\pi}_1 : u \mapsto \max(\pi_1(u), \pi_2(0))$ and $\overline{\pi}_2 : u \mapsto \min(\pi_2(u), \pi_1(1))$
- $\Pi_i(u) := \int_{\pi_2(0)}^{\overline{\pi}_i} \min_{j\in\{1,2\}} (\lambda_j \circ \pi_j^{-1}(u))\, du$ ··· smoother than $\overline{\pi}_i$
- Extended to the Ventcell DD method in [Ahmed-S-A.H.-Japhet-Kern-Vohralík (18)]

We now define a weak solution to this problem which satisfies:

1. $u \in H^1(0,T;H^{-1}(\Omega))$;
2. $u(\cdot,0) = u_0$;
3. $\varphi_i(u_i) \in L^2(0,T;H^1_{\varphi_i(g_i)}(\Omega_i))$, where $u_i := u|_{\Omega_i}$, $i = 1,2$;

   $\cdots$ where $H^1_{\varphi_i(g_i)}(\Omega_i) := \{v \in H^1(\Omega_i), v = \varphi_i(g_i) \text{ on } \Gamma_i^D\}$
4. $\Pi(u,\cdot) \in L^2(0,T;H^1_{\Pi(g,\cdot)}(\Omega))$;

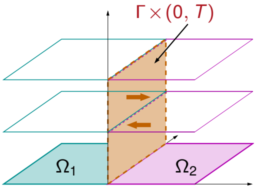
   $\cdots$ where $H^1_{\Pi(g,\cdot)}(\Omega) := \{v \in H^1(\Omega), v = \Pi(g,\cdot) \text{ on } \partial\Omega\}$
5. For all $\psi \in L^2(0,T;H^1_0(\Omega))$, the following integral equality holds:

$$\int_0^T \Big\{ \langle \partial_t u, \psi \rangle_{H^{-1}(\Omega),H^1_0(\Omega)} + \sum_{i=1}^2 (\nabla\varphi_i(u_i), \nabla\psi)_{\Omega_i} - (f,\psi) \Big\} dt = 0.$$

OSWR algorithm

For $k \geq 0$, at step k, solve **in parallel** the **space-time** Robin subdomain problems $(i = 1, 2)$:

$$
\begin{aligned}
\partial_t u_i^k - \Delta\varphi_i(u_i^k) &= f_i, && \text{in } \Omega_i \times (0, T),\\
u_i^k(\cdot, 0) &= u_0, && \text{in } \Omega_i,\\
\varphi_i(u_i^k) &= \varphi_i(g_i), && \text{on } \Gamma_i^{\mathrm{D}} \times (0, T),\\
\nabla\varphi_i(u_i^k)\cdot\mathbf{n}_i + \alpha_{i,j}\Pi_i(u_i^k) &= \Psi_i^{k-1}, && \text{on } \Gamma \times (0, T),
\end{aligned}
$$

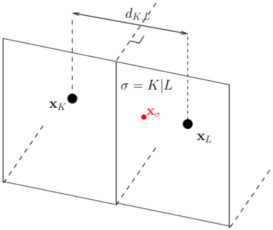


with

- $\Psi_i^{k-1} := -\nabla\varphi_j(u_j^{k-1})\cdot\mathbf{n}_j + \alpha_{i,j}\Pi_j(u_j^{k-1}), \quad j = (3 - i),\ k \geq 2,$
- Ψ_i^0 is an initial Robin guess on $\Gamma \times (0, T)$.

$\cdots$ well-posedness of Robin problem following [Ahmed-Japhet-Kern, in preparation]

☞ The discrete solution is found using the cell centered finite volume scheme in space and the backward Euler scheme in time for the subdomain problem
· · · Following [Enchéry-Eymard-Michel (2006)]
$u_{h,i}^{k,n} \in \mathbb{P}_0(\mathcal{T}_{h,i}) \times \mathbb{P}_0(\mathcal{E}_h^{\Gamma})$: unknown discrete saturation at each time step $0 \leq n \leq N$



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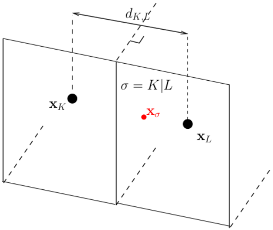

- At each OSWR DD step $k \geq 1$ and each time step $n \geq 1$, Newton–Raphson iterative linearization procedure is used to linearize the local Robin problem
  At each linearization step $m \geq 1$, find $u_{h,i}^{k,n,m} \in \mathbb{P}_0(\mathcal{T}_{h,i}) \times \mathbb{P}_0(\mathcal{E}_h^{\Gamma})$
- Define $u_{h\tau,i}^{k,m}|_{I_n} := u_{h,i}^{k,n,m}$ where I_n is a subinterval in time
- For a posteriori estimates: P_τ^1 continuous, piecewise affine in time functions

OUTLINE

- $\underbrace{\|u - \tilde{u}_{h\tau}^{k,m}\|_\sharp}_{\textbf{unknown}}$

- $\underbrace{\|u - \tilde{u}_{h\tau}^{k,m}\|_\sharp}_{\textbf{unknown}} \leq$ **Fully computable estimators**

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- Goal : $\underbrace{\|u - \tilde{u}_{h\tau}^{k,m}\|_\sharp}_{\textbf{unknown}} \leq \eta_{\mathrm{sp}}^{k,m} + \eta_{\mathrm{DD}}^{k,m} + \eta_{\mathrm{tm}}^{k,m} + \eta_{\mathrm{lin}}^{k,m}$

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- Results on a posteriori error estimates valid during the iteration of an algebraic solver
  [Becker-Johnson-Rannacher (95), Arioli (04), Arioli-Loghin(07), Patera & Rønquist (01), Meidner-Rannacher-Vihharev (09), Jiránek-Strakoš-Vohralík (10), Ern-Vohralík (13)]

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- More recent results on coupling DD and a posteriori error estimates [V.Rey-C.Rey-Gosselet (14)]
  Dirichlet & Neumann subdomain problems $\Rightarrow \mathbf{H}(\mathrm{div}, \Omega)$ flux at each DD iteration
  Following [Prager-Synge (47), Ladevèze-Pelle (05), Repin (08), Ern-Vohralík (15)]

✖ not applicable to more general (e.g. Robin, Ventcell) transmission conditions

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- In our contribution: develop a posteriori estimates for DD algorithms where on the interfaces, neither the conformity of the flux nor that of the saturation are preserved for unsteady degenerated non linear problem
  Following [Nochetto-Schmidt-Verdi (00), Cancès-Pop-Vohralík (14), Di Pietro-Vohralík-Yousef (15)]

Steady diffusion equation

$$
\begin{aligned}
\mathbf{u} &= -\boldsymbol{S}\nabla p, && \text{in} \quad \Omega \\
\nabla \cdot \mathbf{u} &= f, && \text{in} \quad \Omega \\
p &= g_{\mathrm{D}} && \text{on} \quad \Gamma_{\mathrm{D}} \cap \partial\Omega \\
-\mathbf{u} \cdot \boldsymbol{n} &= g_{\mathrm{N}} && \text{on} \quad \Gamma_{\mathrm{N}} \cap \partial\Omega
\end{aligned}
$$

S-A.H., C. Japhet, M. Kern, and M. Vohralík, A posteriori stopping criteria for optimized Schwarz domain decomposition algorithms in mixed formulations, Comput. Methods Appl. Math., (2018), Accepted.

Unsteady diffusion equation

$$
\begin{aligned}
\mathbf{u} &= -\boldsymbol{S}\nabla p, && \text{in} \quad \Omega\times(0,T) \\
\phi\frac{\partial p}{\partial t}+\nabla\cdot\mathbf{u} &= f, && \text{in} \quad \Omega\times(0,T) \\
p &= g_{\mathrm{D}} && \text{on} \quad \Gamma_{\mathrm{D}}\cap\partial\Omega\times(0,T) \\
-\mathbf{u}\cdot\boldsymbol{n} &= g_{\mathrm{N}} && \text{on} \quad \Gamma_{\mathrm{N}}\cap\partial\Omega\times(0,T) \\
p(\cdot,0) &= p_0 && \text{in} \quad \Omega
\end{aligned}
$$

S-A.H., C. Japhet, M. Kern, and M. Vohralík, A posteriori stopping criteria for optimized Schwarz domain decomposition algorithms in mixed formulations, Comput. Methods Appl. Math., (2018), Accepted.

S-A.H., C. Japhet, and M. Vohralík, A posteriori stopping criteria for space-time domain decomposition for the heat equation in mixed formulations, Electron. Trans. Numer. Anal., (2018), Accepted .
PINT 2017 by M. Kern

In this contribution: we take up the path initiated in the two papers above

- $\underbrace{\|u - \tilde{u}_{h\tau}^{k,m}\|_\sharp}_{\textbf{unknown}} \leq \underbrace{\textbf{Fully computable estimators}}_{\text{depend on } \mathbf{H}(\mathrm{div},\Omega) \text{ flux and a saturation which have good properties}}$

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- FV method gives $u_{h,i}^{k,n,m} \notin H^1(\Omega_i),\ i = 1,2 \implies \begin{cases} \varphi_i(u_{h,i}^{k,n,m}) \notin H^1(\Omega_i) \\ \Pi_i(u_{h,i}^{k,n,m}) \notin H^1(\Omega_i) \implies \Pi(u_h^{k,n,m}) \notin H^1(\Omega) \end{cases}$

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- Robin DD method gives $\mathbf{u}_h^{k,n,m} \notin \mathbf{H}(\text{div}\ , \Omega)$ and $\Pi(u_h^{k,n,m})$ jumps accros Γ

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Strategy:

Follow [Nochetto-Schmidt-Verdi (00), Cancès-Pop-Vohralík (14), Di Pietro-Vohralík-Yousef (15), S-A.H., C. Japhet, M. Kern, and M. Vohralík (18)]

Extension to Robin DD for nonlinear problem in this work

- $\underbrace{\|u - \tilde{u}_{h\tau}^{k,m}\|_\sharp}_{\text{unknown}} \leq \underbrace{\text{Fully computable estimators}}_{\text{depend on } \mathbf{H}(\mathrm{div},\Omega) \text{ flux and a saturation which have good properties}}$

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Extension to Robin DD for nonlinear problem in this work

- Postprocessing: $\tilde{u}_{h\tau}^{k,m}$ ($u_{h\tau}^{k,m}$ is piecewise constant and not suitable for the energy norm)
  where $\tilde{u}_{h\tau}^{k,m} := \varphi_i^{-1}(\tilde{\varphi}_{h\tau,i}^{k,m})$ with $\tilde{\varphi}_{h\tau,i}^{k,m} \in P_\tau^1(\mathcal{P}_2(\mathcal{T}_{h,i}))$
  $\tilde{u}_{h\tau}^{k,m}$ used for theoretical analysis and $\tilde{\varphi}_{h\tau,i}^{k,m}$ used in practice for the estimators

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- FV method gives $u_{h,i}^{k,n,m} \notin H^1(\Omega_i),\ i = 1,2 \implies \begin{cases} \varphi_i(u_{h,i}^{k,n,m}) \notin H^1(\Omega_i) \\ \Pi_i(u_{h,i}^{k,n,m}) \notin H^1(\Omega_i) \implies \Pi(u_h^{k,n,m}) \notin H^1(\Omega) \end{cases}$

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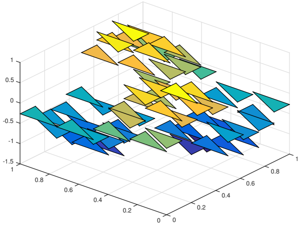
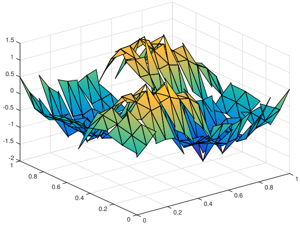
- Saturation and flux reconstructions:
  - Reconstruction saturation $s_{h,i}^{k,n,m} := \varphi_i^{-1}(\hat{\varphi}_{h,i}^{k,n,m})$
    - where $\hat{\varphi}_{h\tau,i}^{k,m} \in P_\tau^1(\mathcal{P}_2(\mathcal{T}_{h,i}) \cap H^1(\Omega_i))$-conforming in each subdoamin
    - modified to ensure the continuity across the interface: $\Pi_1(s_{h,1}^{k,n,m}) = \Pi_2(s_{h,2}^{k,n,m})$
  - $\boldsymbol{\sigma}_{h\tau}^{k,m}$: $\mathbf{H}(\mathrm{div}, \Omega)$-conforming and local conservative in each element, piecewise constant in time

Potential reconstructions (2 subdomains)

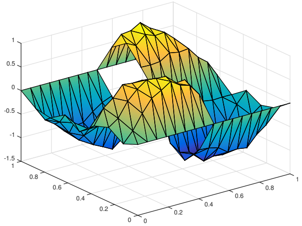


$u_h^{k,n,m}$ (from DD solver)



$\tilde{\varphi}_h^{k,n,m}$: postprocessing

then $\tilde{u}_{h\tau}^{k,m} := \varphi_i^{-1}(\tilde{\varphi}_{h\tau,i}^{k,m})$



$\hat{\varphi}_h^{k,n,m} \in H^1(\Omega_i)$

then $s_{h,i}^{k,n,m} := \varphi_i^{-1}(\hat{\varphi}_{h,i}^{k,n,m})$

Following [Di Pietro-Vohralík-Yousef (14), Cancès-Pop-Vohralík (14)]
Extension to Robin DD

$$Q_{t,i} := L^2(0,t;L^2(\Omega_i)), \qquad X_t := L^2(0,t;H_0^1(\Omega)), \qquad X_t' := L^2(0,t;H^{-1}(\Omega)).$$

$$\|u-\tilde{u}_{h\tau}^{k,m}\|_\star^2 := \sum_{i=1}^2 \|\varphi_i(u_i)-\varphi_i(\tilde{u}_{h\tau,i}^{k,m})\|_{Q_{T,i}}^2 + \frac{L_\varphi}{2}\|u-\tilde{u}_{h\tau}^{k,m}\|_{X'}^2 + \frac{L_\varphi}{2}\|(u-\tilde{u}_{h\tau}^{k,m})(\cdot,T)\|_{H^{-1}(\Omega)}^2$$

$$\|u-\tilde{u}_{h\tau}^{k,m}\|_\sharp^2 := \|u-\tilde{u}_{h\tau}^{k,m}\|_\star^2 + 2\sum_{i=1}^2\int_0^T\left(\|\varphi_i(u_i)-\varphi_i(\tilde{u}_{h\tau,i}^{k,m})\|_{Q_{t,i}}^2 + \int_0^t \|\varphi_i(u_i)-\varphi_i(\tilde{u}_{h\tau,i}^{k,m})\|_{Q_{s,i}}^2 e^{t-s}\mathrm{d}s\right)\mathrm{d}t;$$

where L_φ is the maximal Lipschitz constant of the functions φ_i

Theorem

If $\bar{\varphi} \in L^2(0,T;H_0^1(\Omega))$, where $\bar{\varphi}|_{\Omega_i} := \varphi_i(u_i) - \varphi_i(s_{h\tau,i}^{k,m})$, $\quad i = 1,2$, then

$$\|u-\tilde{u}_{h\tau}^{k,m}\|_\sharp \le \sqrt{\frac{L_\varphi}{2}}\sqrt{2e^T-1}\,\eta_{\mathrm{IC}}^{k,m} + \eta_{\mathrm{sp}}^{k,m} + \eta_{\mathrm{tm}}^{k,m} + \eta_{\mathrm{dd}}^{k,m} + \eta_{\mathrm{lin}}^{k,m}$$

which depend on $\boldsymbol{\sigma}_{h\tau}^{k,m}$, $\hat{\varphi}_{h\tau,i}^{k,m}$, $\tilde{\varphi}_{h\tau,i}^{k,m}$

0 - Postprocessing function $\tilde{\varphi}_{h,i}^{k,n,m}$ of $\varphi_i(u_{h\tau,i}^{k,m})$

$\tilde{\varphi}_{h,i}^{k,n,m} \in \mathcal{P}_2(\mathcal{T}_{h,i})$ at each iteration k, at each time step n, $n = 0, ..., N$, and at each linearization step m, is constructed as:

$$-\nabla \tilde{\varphi}_{h,i}^{k,n,m}|_K = \mathbf{u}_{h,i}^{k,n,m}|_K, \qquad \forall K \in \mathcal{T}_{h,i},$$
$$\frac{(\varphi^{-1}(\tilde{\varphi}_{h,i}^{k,n,m}), 1)_K}{|K|} = u_K^{k,n,m}|_K, \qquad \forall K \in \mathcal{T}_{h,i}.$$

- $\tilde{\varphi}_{h,i}^{k,n,m} \notin H^1(\Omega_i)$

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1 - Piecewise continuous polynomial $\hat{\varphi}_{h,i}^{k,n,m}$ in each subdomain

$$\hat{\varphi}_{h,i}^{k,n,m}(\mathbf{x}) := \mathcal{I}_{\rm av}(\tilde{\varphi}_{h,i}^{k,n,m})(\mathbf{x}) = \frac{1}{|\mathcal{T}_{\mathbf{x}}|} \sum_{K \in \mathcal{T}_{\mathbf{x}}} \tilde{\varphi}_{h,i}^{k,n,m}|_K(\mathbf{x}) \in \mathbb{P}_2(\mathcal{T}_{h,i}) \cap H^1(\Omega_i)$$
$$\hat{\varphi}_{h,i}^{k,n,m}(\mathbf{x}) := \varphi_i(g_i(\mathbf{x})) \text{ on } \Gamma_i^{\rm D}.$$

2 - Reconstruction saturation

reconstruction saturation in each subdomain: $s_h^{k,n,m}|_{\Omega_i} := \varphi_i^{-1}(\hat{\varphi}_{h,i}^{k,n,m})$

According to the weak solution u, we require that

- $s_{h\tau}^{k,n,m}|_{\Omega_i} \in H^1(0,T;H^{-1}(\Omega))$
- $\varphi_i(s_{h\tau,i}^{k,m}) \in L^2(0,T;H^1_{\varphi_i(g_i)}(\Omega_i))$

  $\cdots \quad \varphi_i(s_h^{k,n,m}|_{\Omega_i}) := \varphi_i(\varphi_i^{-1}(\hat{\varphi}_{h,i}^{k,n,m})) = \hat{\varphi}_{h,i}^{k,n,m} \in H^1_{\varphi_i(g_i)}(\Omega_i)$
- $\Pi_1(s_h^{k,n,m}|_{\Omega_1}) = \Pi_2(s_h^{k,n,m}|_{\Omega_2})$ on Γ

  $\cdots$ where Π_i, $1 \le i \le 2$, is chosen as follows:

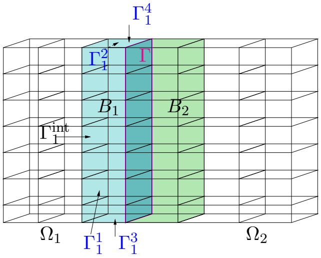
$$\Pi_i(s_h^{k,n,m}|_{\Omega_i}(\mathbf{x}_\Gamma)) = \frac{\Pi_i(\varphi_i^{-1}(\hat{\varphi}_{h,i}^{k,n,m}(\mathbf{x}_\Gamma))) + \Pi_j(\varphi_j^{-1}(\hat{\varphi}_{h,j}^{k,n,m}(\mathbf{x}_\Gamma)))}{2}.$$

- $\frac{1}{|K|}(s_h^{k,n,m},1)_K = u_K^{k,n,m}, \quad \forall K \in \mathcal{T}_h$

  $\cdots$ using suitable constants $\alpha_K^{k,n,m}$ and the b_K the bubble function on K.

3 - Equilibrated flux reconstruction $\boldsymbol{\sigma}_{h\tau}^{k,m}$

$$\boldsymbol{\sigma}_{h\tau}^{k,m} \in P_\tau^0(\mathbf{H}(\mathrm{div},\Omega)),$$
$$\left(f^n - \frac{u_K^{k,n,m} - u_K^{k,n-1}}{\tau^n} - \nabla\cdot\boldsymbol{\sigma}_h^{k,n,m}, 1\right)_K = 0, \quad \forall K \in \mathcal{T}_h.$$



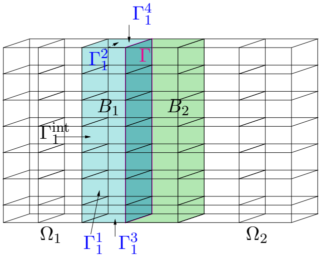
- Average of the fluxes on the interface

where B_1 and B_2 are the two bands surrounding the interface Γ in 3D

3 - Equilibrated flux reconstruction $\boldsymbol{\sigma}_{h\tau}^{k,m}$

$$\boldsymbol{\sigma}_{h\tau}^{k,m} \in P_\tau^0(\mathbf{H}(\mathrm{div},\Omega)),$$

$$\left(f^n - \frac{u_K^{k,n,m} - u_K^{k,n-1}}{\tau^n} - \nabla\cdot\boldsymbol{\sigma}_h^{k,n,m}, 1\right)_K = 0, \quad \forall K \in \mathcal{T}_h.$$



- Average of the fluxes on the interface
- Misfit of mass balance in each subdomain

where B_1 and B_2 are the two bands surrounding the interface Γ in 3D

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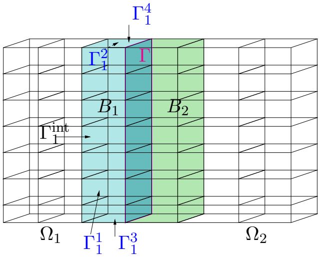

- Average of the fluxes on the interface
- Misfit of mass balance in each subdomain
- Distribute the misfit by coarse grid problem

where B_1 and B_2 are the two bands surrounding the interface Γ in 3D

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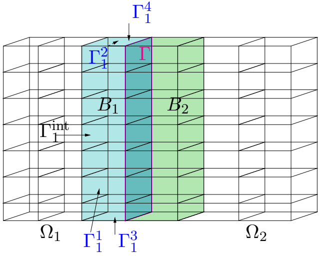

- Average of the fluxes on the interface
- Misfit of mass balance in each subdomain
- Distribute the misfit by coarse grid problem
- Add the corrections to the averages

where B_1 and B_2 are the two bands surrounding the interface Γ in 3D

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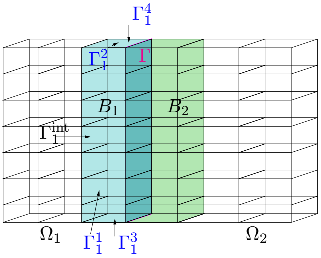


- Average of the fluxes on the interface
- Misfit of mass balance in each subdomain
- Distribute the misfit by coarse grid problem
- Add the corrections to the averages
- Solve local Neumann problem in the bands

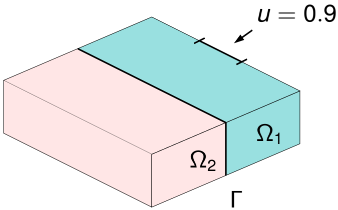
where B_1 and B_2 are the two bands surrounding the interface Γ in 3D

OUTLINE

Numerical experiment with two rock types

Let $\Omega = [0,1]^3$, $\Omega = \Omega_1 \cap \Omega_2$, where $\Gamma = \{x = 1/2\}$. We consider the capillary pressure functions and the global mobilities given respectively by

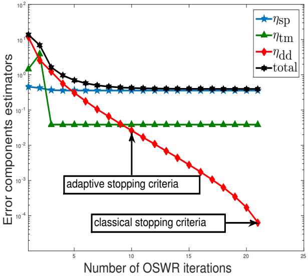
$$\pi_1(u) = 5u^2, \qquad \pi_2(u) = 5u^2 + 1, \qquad \lambda_i(u) = u(1-u),\ i \in \{1,2\}.$$



- Homogeneous Neumann boundary conditions are fixed on the remaining part of $\partial\Omega$
- $f = 0$ in Ω and $u_0 = 0$

- $\pi_2(0) = \pi_1(u_1^*) \Rightarrow u_1^* = \dfrac{1}{\sqrt{5}}$
  Here, the gas cannot enter the subdomain Ω_2 if $\pi_1(u_1)$ is lower than the entry pressure $\pi_1(u_1^*)$, with $u_1^* = \dfrac{1}{\sqrt{5}} \approx 0.44$.
- Robin transmission conditions $\alpha = \alpha_{1,2} = \alpha_{2,1}$.
- The implementation is based on the Matlab Reservoir Simulation Toolbox (MRST)

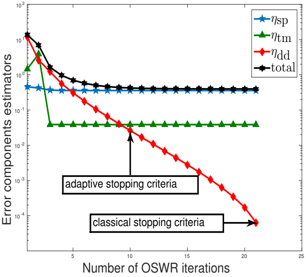
Stopping criterion



DD:

- Classical stopping criterion: Residual $\leq 10^{-6}$
- Adaptive stopping criterion: $\eta_{\text{dd}}^{k,m} \leq 0.1 \max\left\{\eta_{\text{sp}}^{k,m}, \eta_{\text{tm}}^{k,m}\right\}$.

Stopping criterion



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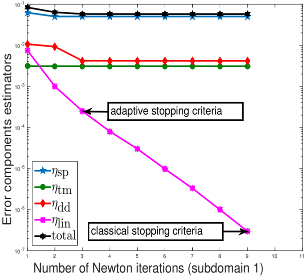
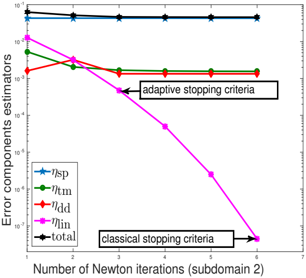
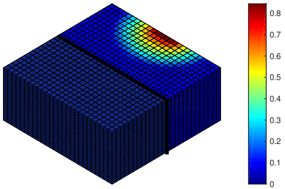
Newton at final iteration of OSWR, $t = 6.6$:

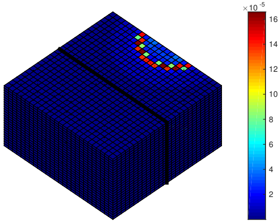
- Classical stopping criterion: Residual $\leq 10^{-8}$
- $\eta_{\mathrm{lin},i}^{k,n,m} \leq 0.1 \max\left\{\eta_{\mathrm{sp},i}^{k,n,m}, \eta_{\mathrm{tm},i}^{k,n,m}, \eta_{\mathrm{dd},i}^{k,n,m}\right\}$, $i = 1, 2$



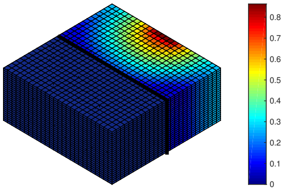




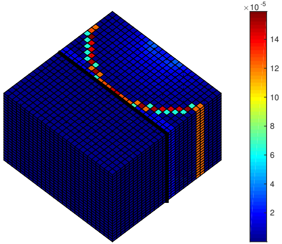
Saturation $u(t)$ for $t = 2.9$



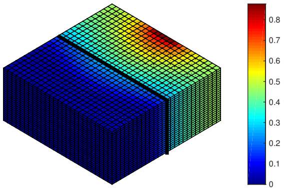
Estimated error for $t = 2.9$



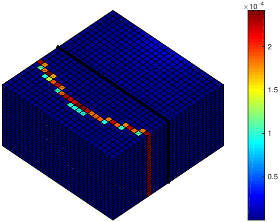
Saturation $u(t)$ for $t = 6.6$



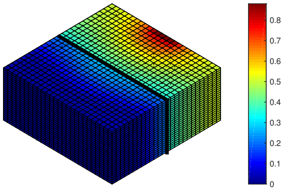
Estimated error for $t = 6.6$



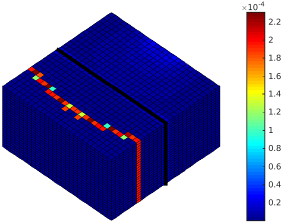
Saturation $u(t)$ for $t = 13$



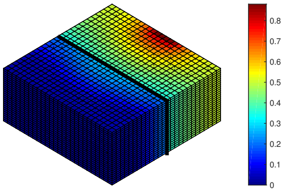
Estimated error for $t = 13$



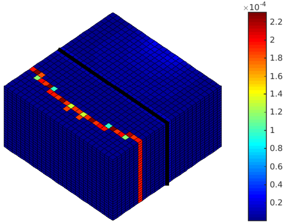
Saturation $u(t)$ for $t = 15$



Estimated error for $t = 15$



Saturation $u(t)$ for $t = 15$



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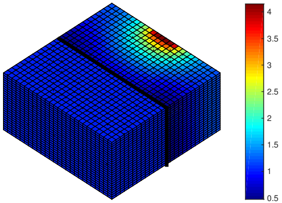
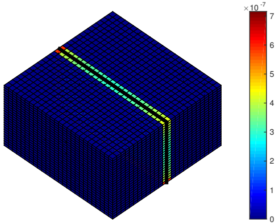
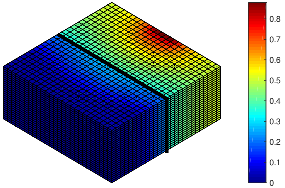


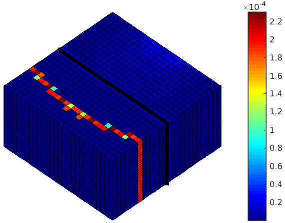
Capillary pressure $\pi(u(t), \cdot)$ for $t = 6.6$



Estimated DD error for $t = 6.6$



Saturation $u(t)$ for $t = 15$



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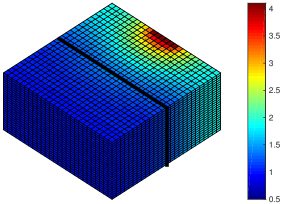
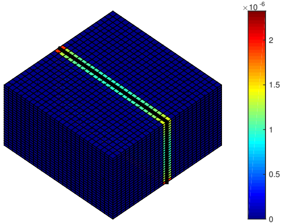


Capillary pressure $\pi(u(t), \cdot)$ for $t = 15$



Estimated DD error for $t = 15$

Conclusions

- The quality of the result is ensured by **controlling the error** between the approximate solution and the exact solution at each iteration of the DD algorithm.
- Different **components** of the error have been **distinguished**.
- An **efficient stopping criterion** for the DD iterations has been established.
- Many of the **DD** and **linearization** iterations usually performed can be **saved**.

Future work

- Assess how much computing time can be saved
- Develop an a posteriori coarse-grid corrector
- Extend to advection-diffusion

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S-A.H.-Japhet-Kern-Vohralík, accepted, 2018 (steady case)

S-A.H.-Kern-Japhet-Vohralík, EDP-Normandie Proceedings, 2018 (unsteady case - heat equation)

S-A.H.-Japhet-Vohralík, accepted, 2018 (unsteady case - heterogenous)

Ahmed-S-A.H.-Japhet-Kern-Vohralík, Preprint hal, 2018, submitted (two phase flow - nonlinear)

Thank you for your attention!