Ladder times, dual subordinators and the exponential law

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Abstract

Let S be an oscillating, real-valued random walk. Denote by L_n the number of ladder times of $(S_k, k \leq n)$ and $\widehat{L}_n$ the number of ladder times of $(S_{n-k}, k \leq n)$. We prove that $L_n \widehat{L}_n / n$ converges to the exponential law as $n \to \infty$. A similar result holds for Lévy processes.

Keywords: random walk, Lévy process, ladder time, exponential random variable.

1 Introduction

Let S be a real-valued random walk. Suppose that S is oscillating, that is, almost surely, $\liminf_{n\to\infty} S_n = -\infty$ and $\limsup_{n\to\infty} S_n = \infty$. Recall that $k \geq 0$ is a strong ladder time for S if $S_k > S_i$ for every $i < k$, and a weak ladder time if $S_k \geq S_i$ for every $i < k$ (in particular, 0 is a strong and weak ladder time with this definition).

Consider the path of $Q_n = (S_0, S_1, \ldots, S_n)$ and let L_n be the number of strong ladder times of Q_n. Similarly, consider the time reversed-path $\widehat{Q}_n = (S_n - S_n, S_{n-1} - S_n, \ldots, S_1 - S_n, -S_n)$ and let $\widehat{L}_n$ be the number of weak ladder times of $\widehat{Q}_n$. See for instance Figure 1, where the ladder times and dual ladder times have been circled, and $n = 19$, $L_n = 6$, $\widehat{L}_n = 5$.

It was shown in [5] that for every random walk S, even without the oscillating condition:

$$\mathbb{E}(L_n \widehat{L}_n) = n + 1$$

It turns out that in fact, the product $L_n \widehat{L}_n$ has a limit distribution:

Theorem 1 *Suppose that the random walk S is oscillating. Then*

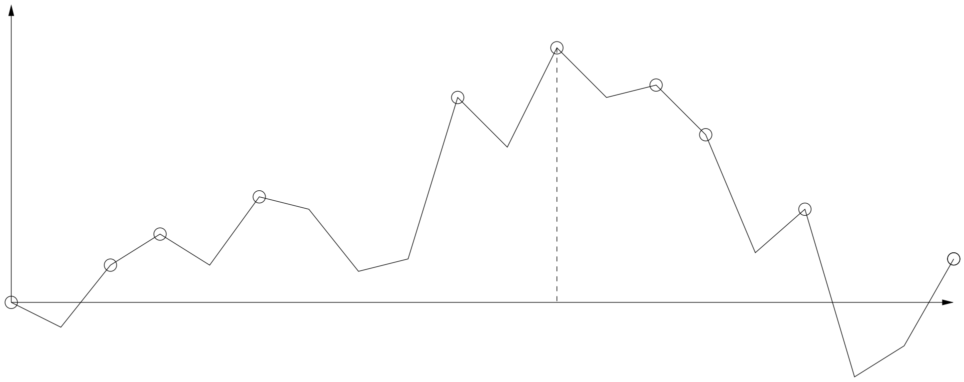

Figure 1: Ladder times and dual ladder times

(i) For every positive integer k, as $n \to \infty$

$$\mathbb{E}\left[\left(\frac{L_n \widehat{L}_n}{n+1}\right)^k\right] \to k!$$

(ii) As a consequence, the random variable $L_n \widehat{L}_n/(n+1)$ converges in distribution to an exponential random variable with mean 1 as $n \to \infty$.

Remark 1 When a random walk is not oscillating and, say, tends to $+\infty$ almost surely, the expectation of T_1, the first strong ladder time, is finite. Hence denoting $m = \mathbb{E}(T_1)$, the law of large numbers tells us that almost surely, $L_n \sim n/m$. On the other hand, as $n \to \infty$, $\widehat{L}_n$ converges in law to a geometric random variable with mean m, see for instance [4], chap. XII.7. Therefore , in the non-oscillating case, $L_n \widehat{L}_n/(n+1)$ converges in law to G/m, where G is a geometric random variable with mean m. So there is a formal analogy with Theorem 1, in the sense that geometric laws are analogues of exponential laws.

Theorem 1 has an analogue in continuous time. Let X be a Lévy process and let I be its infimum process, $I_t = \inf\{X_s, s \leq t\}$. Suppose that 0 is regular for $(0,\infty)$ and $(-\infty, 0)$, that is, starting at 0, X visits immediately $(0,\infty)$ and $(-\infty, 0)$. In that case, one can define (L_t), the local time at 0 of the reflected process $(S_t - I_t)$, see [2], chap. VI.1.

As in the case of random walks, consider the time reversal $\widehat{X}_s^{(t)} = (X_s - X_t, 0 \leq s \leq t)$, its infimum process $\widehat{I}_s^{(t)} = \inf\{\widehat{X}_s^{(t)}, s \leq t\}$, and $\widehat{L}_s^{(t)}$ the local time at 0 of the reflected process $\widehat{X}_s^{(t)} - \widehat{I}_s^{(t)}$. Then we have:

Theorem 2 *Let X be a Lévy process such that 0 is regular for $(0,\infty)$ and $(-\infty,0)$. Then for every t, the random variable $L_t\widehat{L}_t^{(t)}$ is exponentially distributed with mean t.*

Remark 2 It is well-known [2], chap. VI.1, that the inverse of L, namely the process Y defined by

$$Y_t = \inf\{s, L_s > t\} \tag{1}$$

is a subordinator. Similarly, there is a subordinator $\widehat{Y}$ related to the local time $\widehat{L}$ of the time reversal. Then Y and $\widehat{Y}$ are dual in the sense that they are related by the Wiener-Hopf formula. Let $\phi, \widehat{\phi}$ be their Laplace exponents:

$$\mathbb{E}(\exp(-\lambda Y_t)) = \exp(-t\phi(\lambda)) \tag{2}$$

$$\mathbb{E}(\exp(-\lambda \widehat{Y}_t)) = \exp(-t\widehat{\phi}(\lambda)) \tag{3}$$

Then

$$\phi(\lambda)\widehat{\phi}(\lambda) = \lambda \tag{4}$$

Bertoin-Yor noticed in [3] that when two independent subordinators $Y, \widehat{Y}$ satisfy (4), defining the exponential functionals

$$\xi = \int_0^\infty \exp(-Y_t)dt$$

$$\widehat{\xi} = \int_0^\infty \exp(-\widehat{Y}_t)dt$$

then the product is exponentially distributed:

$$\xi\widehat{\xi} \sim \exp(1)$$

It is not clear whether there is a relation between this result and Theorem 2.

Remark 3 The result of [3] yields several factorizations of the exponential distribution as a product of two independent random variables. It turns out that Theorem 2 also induces a factorization :

Corollary 1 *Let $\mathbf{e}$ be a standard exponential random variable. Then*

$$\mathbf{e} \overset{d}{=} XX'G^\alpha(1-G)^{1-\alpha}$$

where $\alpha \in (0,1)$ and X, X', G are independent random variables given by

$$\mathbb{P}(G \in dt) = \frac{t^{\alpha-1}(1-t)^{-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)} dt \tag{5}$$

$$\mathbb{P}(X \in dt) = c\mathbb{P}(Y_t = 1)dt \tag{6}$$

$$\mathbb{P}(X' \in dt) = c'\mathbb{P}(Y'_t = 1)dt \tag{7}$$

where (Y_t) and (Y'_t) are stable subordinators of index α, $1-\alpha$, respectively.

We also have the following factorization as an infinite product:

Corollary 2 *(i) Let R be random variable distributed according to the Rayleigh law:*

$$\mathbb{P}(R \in dt) = t\exp(-t^2/2)dt$$

Then R has the representation as an infinite product

$$R \stackrel{d}{=} \prod_{n \geq 0} X_n$$

where the X_n are independent random variables given by

$$\mathbb{P}(X_n \in dx) = c_n x^{2^{n+1}-1} \exp\left(-x^{2^{n+2}}\right) dx$$

(ii) Let $\mathbf{e}$ be a standard exponential random variable. Then

$$\mathbf{e} \stackrel{d}{=} \prod_{n \geq 0} X_n^2$$

where the X_n are as above.

Remark 4 As in [5], we shall prove Therorem 1 by combinatorial arguments. It should be clear that Theorem 2 could be deduced from Theorem 1 by approximating the Lévy process by random walks. However, since the local time of the reflected process is not a continuous functional, this would involve some tedious discussions, and it is more convenient to use the Wiener-Hopf formula in the context of Lévy processes.

The advantage of the combinatorial method is that it is potentially adaptable to show a similar result for bridges. We shall investigate this question in a subsequent work. However, it is easy to check that, in the Brownian case, the extension of Theorem 2 to bridges holds. For a Brownian motion B, the quantitities $L_t, \widehat{L}_t$ correspond, up to a multiplicative constant, to S_t, $S_t - B_t$ respectively, where $S_t = \sup\{B_s, s \leq t\}$. Then we have:

Proposition 1 *Let $(B_t, t \geq 0)$, be a linear Brownian motion and let*

$$S_t = \sup\{B_s, s \leq t\}$$

Then for every $a \in \mathbb{R}$, conditionally on $B_1 = a$, the product $S_1(S_1 - B_1)$ is exponentially distributed.

We now proceed to the proof of our results.

2 Proof of the results

2.1 Outline of the proof

To prove (i) of Theorem 1, we establish an "equivalence" between:

- Choosing k points on a path of length n.
- Choosing k ladder times and k dual ladder times on a path of length n.

In the first case we have $\sim n^k/k!$ choices (because the choice is up to the ordering of the k points). In the second case we have $\sim L_n^k \widehat{L}_n^k / k!^2$ choices (here the choice is up to the ordering of the k ladder times and of the k dual ladder times). This yields

$$\frac{n^k}{k!} \sim \frac{\mathbb{E}(L_n^k \widehat{L}_n^k)}{k!^2}$$

The "equivalence" is shown by using a path transform. Sections 2.2 and 2.3 give a formal proof using this idea.

2.2 A deterministic path transform

Given a path $P = (P_1, P_2, \ldots P_n)$, we denote by $\widehat{P}$ the path $(P_n - P_n, P_{n-1} - P_n, \ldots, P_1 - P_n, -P_n)$. We say that a time k is a dual weak ladder time for P if $n - k$ is a weak ladder time for $\widehat{P}$. We denote by $Lad(P)$ the set of ladder times (including time 0) and by $\widehat{Lad}(P)$ the set of dual weak ladder times (including time n).

Given two integers $k \leq n$, we first describe a path transformation $F_{n,k}$ between

- Triples (P, f, σ) where P is a path of length n, f is an increasing function
$$f : \{1, 2, \ldots k\} \to \{1, 2, \ldots n - 1\}$$
and σ is a permutation of $\{1, 2, \ldots k\}$.

- Triples (Q, g, h) where Q is a path of length n such that $Lad(Q)$ has cardinality at least k, g is an increasing function

  $$g : \{1, 2, \dots k\} \to Lad(Q)$$

  and h is a function

  $$h : \{1, 2, \dots k\} \to \widehat{Lad}(Q)$$

Here is the definition of $F_{n,k}$. Consider a triple of the first kind. Put $a_i = P_{f(i)}$ for $1 \le i \le k$ and $a_0 = 0$, $a_{k+1} = n$. For $i \in \{0, 1, 2, \dots k\}$ define

$$b_i = \min\{j, S_j \ge S_h \text{ for all } h \in [a_j, a_{j+1}]\}$$

For each $i \in \{0, 1, \dots k\}$, let A_i be the part of the path P between time a_i and time b_i and let B_i be the part of the path P between time b_i and time a_{i+1}. Now concatenate successively

$$A_0, A_1, \dots A_k, B_0, B_{\sigma(1)}, \dots B_{\sigma(k)}$$

This way one obtains a path Q of length n. If $|A_i|$ denotes the length of the path A_i, then the function g is defined by

$$g(i) = |A_0| + \dots + |A_i| \tag{8}$$

for $i \in \{1, 2 \dots k\}$ and h is defined by

$$h(j) = |A_0| + \dots + |A_k| + |B_0| + \dots + |B_{\sigma(j-1)}| \tag{9}$$

for $j \in \{1, 2 \dots k\}$. Then one obtains a triple (Q, g, h) and it is easily verified that is a triple of the second kind. See Figure 2 for a path of the first kind, where the a_i are circled black and the points b_i are circled white, and Figure 3 for its transform, where the permutation σ is the identity.

Conversely, given a triple (Q, g, h) of the second kind, using (8) and (9), one can identify $A_0, \dots, A_k$ and $B_0, \dots B_k$. This way one obtains a path P, together with the times $a_0, a_1, \dots a_{k+1}$ and $b_0, \dots, b_k$, which yields the function f. Moreover, if $h(i) \ne h(j)$ when $i \ne j$, one can define the permutation σ by

$$\sigma(i) = Card\{j \in \{1, 2, \dots k\}, h(j) \le h(i)\} \tag{10}$$

and one checks that $(Q, g, h) = F_{n,k}(P, f, \sigma)$. On the other hand, if $h(i) = h(j)$ for two integers $i \ne j$, then the permutation σ is not uniquely determined, although for every permutation σ satisfying (10), one still has $(Q, g, h) = F_{n,k}(P, f, \sigma)$.

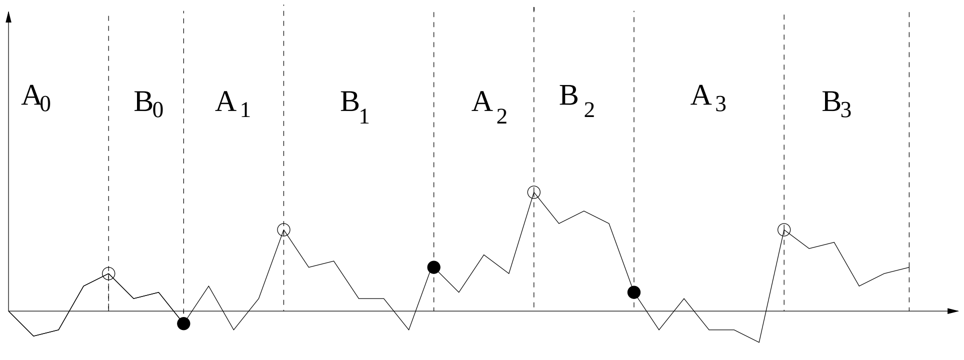

Figure 2: A path of the first kind...

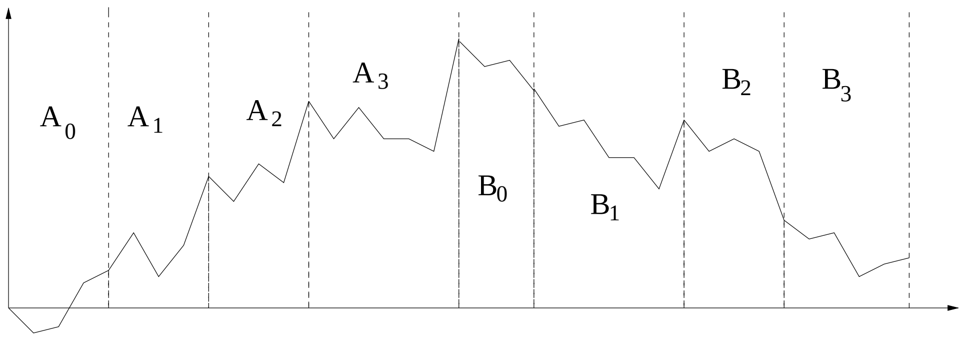

Figure 3: ...and its transform, a path of the second kind

Remark that the situation when $h(i) = h(j)$ for two integers $i \neq j$ occurs if and only if for some h, $|B_h| = 0$ or, equivalently, $a_h = b_h$. This is the case if and only for some h,

$$P_{f(h)} = \max\{P_{f(h)}, P_{f(h)+1}, \dots, P_{f(h+1)}\}$$

So $F_{n,k}$ is a bijection

- The set of triples of the first kind such that for every $h \leq k$,

  $$P_{f(h)} \neq \max\{P_{f(h)}, P_{f(h)+1}, \dots, P_{f(h+1)}\}$$

  Denote this set $S(n,k)$

- The set of triples of the second kind such that $\widehat{Lad}(Q) \geq k$ and h is injective. Denote this set $T(n,k)$.

2.3 Proof of Theorem 1

Let S be the random walk of Theorem 1. The law of S induces a probability measure on the set of paths of length n. Moreover, consider the counting measures on the set of increasing functions $f : \{1,2,\dots k\} \to \{0,1,\dots n\}$, and the counting measure on the set of permutations of $\{1,2\dots k\}$. Then by taking the product measure, one obtains a measure $\mathbb{P}_1$ on the set of triples of the first kind, with total mass $(n+1)n(n-1)\dots(n-k+2)$, since the number of increasing functions is $(n+1)n(n-1)\dots(n-k+2)/k!$.

Similarly, one obtains a measure $\mathbb{P}_2$ on the set of triples of the second kind by choosing the counting measure on the set of possible functions g, h. Then the total mass of $\mathbb{P}_2$ is

$$\frac{1}{k!}\mathbb{E}[L_n(L_n-1)\dots(L_n-k+1)\widehat{L}_n^k]$$

(Note that the quantity inside the expectation equals 0 when $L_n < k$.)

Moreover, $F_{n,k}$ is a bijection between $S(n,k)$ and $T(n,k)$ and it preserves the set of increments of the paths, and therefore the measure, since the probability of a path is determined by its increments. Hence

$$\mathbb{P}_1(S(n,k)) = \mathbb{P}_2(T(n,k))$$

We have

$$\mathbb{P}_2(T(n,k)) = \frac{1}{k!}\mathbb{E}[L_n(L_n-1)\dots(L_n-k+1)\widehat{L}_n(\widehat{L}_n-1)\dots(\widehat{L}_n-k+1)]$$

Since S is oscillating, both L_n and $\widehat{L}_n$ tend in probability to ∞ as $n \to \infty$, and therefore

$$\mathbb{P}_2(T(n,k)) \sim \frac{1}{k!}\mathbb{E}[(L_n\widehat{L}_n)^k]$$

On the other hand, if $(P, f\sigma)$ is a triple of the first kind. one can decompose P into $k+1$ parts: the part R_0 between time a_0 and time a_1, the part R_1, between time a_1 and time a_2 and so on. Remark that conditionally on their lengths, $R_0, \ldots, R_k$ are independent.

Morever $(P, f, \sigma) \in \mathcal{S}(n,k)$ if and only if for each $i \geq 1$, the path R_i does not reach its maximum at the starting point. In other words, if R_i has length $c_i = a_i - a_{i-1}$, we want that for every $i \geq 1$, there exist a ladder time for R_i before time c_i. Hence

$$\mathbb{P}_1(S(n,k)) = \sum_{c_0+c_1+\ldots+c_k=n} \prod_{i=1}^{k} \mathbb{P}(T_1 \leq c_i)$$

If the random walk is oscillating, there exists an M such that for $\mathbb{P}(T_1 \leq M) > 1-\epsilon$. Moreover, if n is large enough, there are more than $(n+1)^k(1-\epsilon)$ $(k+1)$-tuples $(c_0, c_1, \ldots, c_k)$ such that $c_0 + c_1 + \ldots + c_k = n$ and $c_i \geq M$ for every i. Hence as for n large enough,

$$\mathbb{P}_1(S(n,k)) \geq (n+1)^k(1-\epsilon)^{k+1}$$

Since this is true for every ϵ and since $\mathbb{P}_1(S(n,k)) \leq (n+1)^k$, we deduce that

$$\frac{\mathbb{P}_1(S(n,k))}{(n+1)^k} \to 1$$

as $n \to \infty$. This concludes the proof of (i) of Theorem 1. The proof of (ii) follows by standard arguments.

2.4 Proof of Theorem 2

Let X be a Lévy process as in Theorem 1 and let $\mathbf{e}$ be an independent, exponential random variable with parameter, say $c > 0$: $\mathbb{P}(\mathbf{e} > t) = e^{-ct}$. Let $Y, \widehat{Y}$ be the subordinators defined by (1). Then their Laplace exponents, defined by (2) and (3), satisfy

$$\phi(\lambda)\widehat{\phi}(\lambda) = \lambda$$

Moreover, the Wiener-Hopf formula (see for instance [2], chap. VI.2) entails that if we stop X at $\mathbf{e}$, then $L_{\mathbf{e}}$ and $\widehat{L}_{\mathbf{e}}$ are independent, and $L_{\mathbf{e}}$ is an exponential random variable with parameter $\phi(c)$. Put $a = \phi(c)$, $b = \widehat{\phi}(c)$. Then

$$\begin{aligned}
\mathbb{P}(L_{\mathbf{e}}\widehat{L}_{\mathbf{e}} \le x) &= \int_0^\infty \mathbb{P}(L_{\mathbf{e}} \in dy) \int_0^{x/y} \mathbb{P}(\widehat{L}_{\mathbf{e}} \in dz) \\
&= \int_0^\infty dy a e^{ay} \int_0^{x/y} dz b e^{-bz} \\
&= \int_0^\infty dy a e^{ay} - a e^{-ay - bx/y}
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\mathbb{P}(L_{\mathbf{e}}\widehat{L}_{\mathbf{e}} \in dx)}{dx} &= ab \int_0^\infty \frac{dy}{y} e^{-ay-bx/y} \\
&= \int_0^\infty \frac{dt}{t} c e^{-ct} e^{-x/t}
\end{aligned}$$

by the change of variable $t = y/b$, using the fact that $ab = c$. This latter qunatity can expressed as

$$\int_0^\infty dt c e^{-ct} f(x,t)$$

with

$$f(x,t) = \frac{1}{t} e^{-x/t}$$

Therefore we have, for every $c > 0$ and every $x > 0$:

$$\int_0^\infty dt c e^{-ct} \mathbb{P}(Z_t \in dx) = \int_0^\infty dt c e^{-ct} f(x,t) dx \tag{11}$$

This entails that for all $t, x > 0$, Z_t has a density at x and that this density is $f(x,t)$, which proves Theorem 2.

2.5 Proofs of the corollaries and of Proposition 1

Let us first examine Corollary 1. Let X be a stable process such that $Y, \widehat{Y}$, as defined in (1), are stable subordinators of index α, $1-\alpha$ respectively. It is clear that one can find such a stable process X, see [2], chap. VIII.1. Let G be the time of the maximum between 0 and 1,

$$G = \inf\{t \le 1, X_t \ge X_s,\ \forall s \le 1\}$$

Then G is arcsine distributed as in (5), see [2], chap. VI.3. Moreover, conditionally on G, using the self-similarity properties of stable processes, one has that the random variables $X = L_1/G^\alpha$ and $X' = \widehat{L}_1/(1-G)^{1-\alpha}$ are independent conditionally on G and are distributed according to (6), (7) respectively. According to Theorem 2, $L_1\widehat{L}_1 = XG^\alpha X'(1-G)^{1-\alpha}$ is exponentially distributed, which establishes the corollary.

The density of X can be expressed by

$$\mathbb{P}(X \in dt) = \frac{\Gamma(1+\alpha)}{\pi\alpha} \sum_{k\geq 1} \frac{(-1)^{k+1}}{k!} \Gamma(1+\alpha k) t^k \sin(\pi\alpha k) dt$$

and a similar expression holds for X', see [6]. Thus the corollary can also be proved by just checking the moments. Indeed,

$$\mathbb{E}(G^{n\alpha}(1-G)^{n(1-\alpha)}) = \frac{\Gamma((n+1)\alpha)\Gamma((n+1)(1-\alpha))}{\Gamma(\alpha)\Gamma(1-\alpha)n!}$$

$$\mathbb{E}(X^n) = \frac{(n+1)!\Gamma(1+\alpha)}{\Gamma((n+1)\alpha+1)}$$

$$\mathbb{E}(X'^n) = \frac{(n+1)!\Gamma(2-\alpha)}{\Gamma((n+1)(1-\alpha)+1)}$$

Remark that the product $X'G^\alpha(1-G)^{1-\alpha}$ has a simple expression:

$$\mathbb{P}(X'G^\alpha(1-G)^{1-\alpha} \in dt) = \frac{\exp(-t^{1/\alpha})}{\Gamma(1+\alpha)} dt$$

Here again, this follows by considering the moments.

Corollary 2 can be proved by iterating the factorization of Corollary 1 for $\alpha = 1/2$, using the equality above. Alternatively, it can also be proved by checking the moments. Indeed,

$$\mathbb{E}(R^p) = \Gamma(1+p/2)$$

and

$$\mathbb{E}(X_n^p) = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{p}{2^{n+2}} + \frac{1}{2}\right)$$

Using the Legendre–Gauss formula:

$$\Gamma((x+1)/2)\Gamma(x/2+1) = \frac{\Gamma(x+1)}{2^{x+1/2}}$$

one checks that $\mathbb{E}(R^p) = \prod \mathbb{E}(X_n^p)$, which yields (i). Taking the squares, we obtain (ii).

Proposition 1 is an easy application of the reflection principle [1]. Indeed, let $a \in \mathbb{R}$ and $b > a$. Then we have

$$\begin{aligned}
\mathbb{P}(S_1(S_1 - B_1) = b(b-a) | B_1 = a) &= \mathbb{P}(S_1 = b | B_1 = a) \\
&= \frac{\mathbb{P}(S_1 = b, B_1 = a)}{\mathbb{P}(B_1 = a)} \\
&= \frac{\mathbb{P}(B_1 = 2b - a)}{\mathbb{P}(B_1 = a)} \\
&= \exp(-(2b-a)^2/2 + a^2/2) \\
&= \exp(-2b(b-a))
\end{aligned}$$

where the third equality follows by reflecting B at the hitting time of b.

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