Nested regenerative sets and their associated fragmentation process

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ABSTRACT: *We give a fractal construction of nested, stable regenerative sets and study the associated inhomogeneous fragmentation process.*

1 Introduction

The purpose of this paper is to introduce an explicit construction of nested, stable regenerative sets and to study the related fragmentation process. The complete construction of regenerative sets on the whole half-line is explained in the next section. However, the intersections of these sets with the interval $[0, 1)$ admit the following description as random fractal sets. Let us introduce the

SPLITTING PROCEDURE. Given $\alpha \in [0, 1]$, this procedure takes an interval $[a, b)$, cuts it into two intervals $[a, U), [U, b)$, where U is uniformly distributed on $[a, b)$, and returns $[a, U), [U, b)$ with probability α and $[a, U)$ with probability $1-\alpha$, independently of U.

Fix $\alpha \in [0, 1]$ and use the splitting procedure to construct a

RANDOM FRACTAL SET. We construct recursively a sequence $(S_n(\alpha), n \geq 0)$ where for each n, $S_n(\alpha)$ is a collection of intervals of $[0, 1)$. Put $S_0(\alpha) = \{[0, 1)\}$. Then to obtain $S_{n+1}(\alpha)$ from $S_n(\alpha)$, apply to each interval of $S_n(\alpha)$ the splitting procedure with parameter α, independently.

Theorem 1.1 *Set*

$$S_\infty(\alpha) = \overline{\cap_{n\geq 0} S_n(\alpha)}$$

Let T be a stable subordinator with index α and let $R = \{T_t, t \geq 0\}$ be its range. Then

$$R \cap [0, 1) \stackrel{d}{=} S_\infty(\alpha)$$

A corollary is the following representation of the generalized arcsine law:

Corollary 1.2 *Let $I_0 = [0, 1]$. Define by induction a sequence of intervals $I_n = [a_n, b_n]$ as follows.*
(i) Let V_n be a uniform random variables on I_n, independent of the past.
(ii) Set $I_{n+1} = [a_n, V_n]$ (resp.$I_{n+1} = [V_n, b_n]$) with probability $1 - \alpha$ (resp. α), independently of the past.

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Then the point $I_\infty = \cap I_n$ is distributed according to the generalized arcsine law with parameter α, that is, the law on $[0,1]$ with density

$$\frac{s^{\alpha-1}(1-s)^{-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)} \tag{1}$$

Our main interest is to investigate the following

COUPLED CONSTRUCTION. We can perform the splitting procedure with parameter $\alpha \in [0,1]$ by taking an independent, uniform random variable V on $[0,1]$ and decide to return one or two sets according as $V > \alpha$ or $V \leq \alpha$. Proceeding this way, we see that we obtain a coupled construction of $(S_n(\alpha), n \geq 0)$, and therefore of $S_\infty(\alpha)$, simultaneously for all $\alpha \in [0,1]$, by starting at $\alpha = 1$.

It is clear that $(S_\infty(\alpha), \alpha \in [0,1])$ is a nested family of regenerative sets, i.e. if $\alpha < \alpha'$, $S_\infty(\alpha) \subset S_\infty(\alpha')$. It can be shown [7] that this family has the subordination property, that is, if $\alpha < \alpha'$, $S_\infty(\alpha)$ can be obtained from $S_\infty(\alpha')$ by subordination. See [1] for details on subordination.

Moreover, for each α, the regenerative set $S_\infty(\alpha)$ cuts the interval $[0,1]$ into disjoint intervals. Consider the coupled construction with α going from 1 to 0. We obtain this way a process where the intervals delimited by $S_\infty(\alpha)$ merge together. The subordination property entails [4] that this process is a Markov process known as the Bolthausen-Sznitman coalescent. This process was introduced in [5] for the study of spin glasses. See also [3, 6].

We shall focus here on the time reversal, that is, we let α go from 0 to 1, in which case we obtain a process where the intervals are broken into smaller ones. It turns out that this process has the law of an inhomogeneous fragmentation and our construction enables us to study its dislocation measure.

The whole construction of regenerative sets is explained in the next section. We give some results on random partitions in Section 3 and study more specifically the fragmentation process in Section 4.

2 The construction of stable regenerative sets

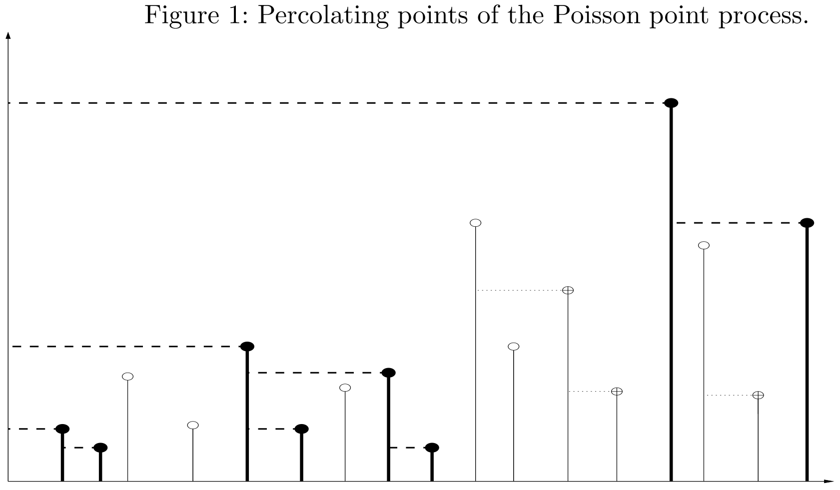
We give here the construction of stable, regenerative sets on the positive half-line. It is easy to check that if we consider the intersection of these sets with the interval $[0,1)$, we recover the fractal construction described in the introduction. In particular, Theorem 1.1 is a direct corollary of Theorem 2.1.

2.1 Regenerative sets

For a detailed account on regenerative sets, see for instance [1]. A random subset $\mathcal{R}$ of $\mathbb{R}_+$ is regenerative if and only if $0 \in \mathcal{R}$ and for every $t > 0$, conditionally on $t \in \mathcal{R}$, the subset $\mathcal{R} \cap [t, \infty)$ has the same law as $\mathcal{R} + t$ and is independent of $\mathcal{R} \cap [0,t]$. A classical result tells us that every regenerative set can be viewed as the range of a subordinator.

Moreover, if $\mathcal{R}$ is self-similar, i.e. if $\mathcal{R}$ has the same law as $c\mathcal{R}$ for every $c > 0$, then $\mathcal{R}$ is the range of a stable subordinator. In that case, to identify the index of the subordinator, one can use for instance the generalized arcsine law. For every time t set $g_t = \sup(\mathcal{R} \cap [0,t])$. Then g_1 (and more generally, by self-similarity, g_t/t)

Figure 1: Percolating points of the Poisson point process.



is distributed according to the generalized arcsine law with parameter α, i.e. the law whose density is given by (1).

2.2 The construction

Let $\mathcal{N}$ be a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $d\mu = dt \otimes x^{-2}dx$. Call C the (random) support of the measure $\mathcal{N}$. To each point $M \in C$ associate an independent, uniform random variable $U(M)$ on $[0,1]$. Note that almost surely, the coordinates of the points of C are all different, there are infinitely many points in each strip $[s,t]\times\mathbb{R}_+$ $(s<t)$ and only finitely many points in each strip $[s,t]\times[y,\infty)$ $(y>0)$. We shall implicitely assume all these properties in the sequel (that is, we shall often omit to write "almost surely").

For $s < t \in \mathbb{R}_+$ let $M(s,t)$ be the highest point of C between times s and t. Formally, $M(s,t)$ is the point of coordinates (u,y) such that $M(s,t) \in C$, $s \leq u \leq t$ and for each point $M' = (u', y') \in C$ with $s \leq u' \leq t$, one has $y' \leq y$.

One can then define $M_1(s,t) := (s_1, y_1) = M(s,t)$ and by induction, $M_{n+1}(s,t) := (s_{n+1}, y_{n+1}) = M(s_n, t)$. The sequence is finite if and only if there is a point of C with x-coordinate t. We say that $t \in \mathbb{R}_+$ percolates at parameter $\alpha \in [0,1]$ if for each n, $U(M_n(0,t)) \leq \alpha$. By convention, we say that 0 percolates.

See the figure. The black points are those which percolate, the white ones are those for which $U > \alpha$ and which, consequently, do not percolate, and the crossed ones are those for which $U \leq \alpha$ but which do not percolate. The set of percolating points is the closure of the set of x-coordinates of the black points.

Loosely speaking, if $U(M) > \alpha$, M does not percolate. In turn, when $U(M) \leq \alpha$, M "looks to the left". If the first vertical line seen by M is either the y-axis or a vertical line of a point which percolates, then M percolates. Otherwise, if this first vertical line seen by M is the line of a point which does not percolate, then M does not percolate. The procedure to determine whether M percolates is well-defined since there is only a finite number of points at a higher altitude than M and on the left of M.

2.3 Regenerativity and stability

Theorem 2.1 *The set $\mathcal{R}_\alpha$ of points percolating at parameter α has the law of the range of a stable subordinator with index α.*

Proof

(i) First we prove that the set of points percolating at parameter α is regenerative. Indeed, assume that s percolates at parameter α. Remark that for every t, the x-coordinates of the points $M_n(t)$ converge to t almost surely as n increases. Hence for every $t > s$, there exists a minimal $k = k(s,t)$ such that for $n \geq k$, $M_n(0,t) > s$. Then for $n < k$, $M_n(0,t) = M_n(0,s)$ and for $n \geq k$, $M_n(0,t) = M_{n-k}(s,t)$.

Since s percolates, one has $U(M_n(t)) \leq \alpha$ for every $n < k$. Hence conditionally on the event that s percolates, t percolates if and only if for each $n \geq k$, $U(M_{n-k}(s,t)) \leq \alpha$. In other words, the set of times $t > s$ percolating at parameter α is the set of t such that $U(M_n(s,t)) \leq \alpha$ for every n. This is independent of $\mathcal{R}_\alpha \cap [0,s]$ because of the independence properties of the Poisson point process $\mathcal{N}$ and has the same law as $\mathcal{R}_\alpha + s$ because of the translation-invariance of $\mathcal{N}$.

(ii) The self similarity is quite obvious. Indeed, write $\mathcal{N} = \sum_k \delta_{t_k,x_k}$ and set $\widehat{\mathcal{N}} = \sum_k \delta_{ct_k,cx_k}$ for some $c > 0$. Since $\mu(B) = \mu(cB)$ for every Borel set $B \in \mathbb{R}_+ \times \mathbb{R}_+$, $\widehat{\mathcal{N}}$ has the same law as $\mathcal{N}$. Therefore $\mathcal{R}_\alpha$ has the same law as $c\mathcal{R}_\alpha$ for every $c > 0$ and consequently, $\mathcal{R}_\alpha$ is the range of a stable subordinator.

(iii) It remains to prove that the parameter of the subordinator is α. Let $g_1 = \sup\{t \leq 1,\ t \in R_\alpha\}$. Define by induction the sequence $A_n := (u_n, z_n)$ by $A_0 = M_n(0,1)$ and $A_{n+1} = M(0,u_n)$. Then by construction, $g_1 \in [u_{K+1}, u_K]$ if K is defined as the smallest integer such that $U(A_{K+1}) \leq \alpha$. Remark that u_n is the product of n independent, uniform random variables on $[0,1]$ and therefore has density $(-\log t)^{n-1}/(n-1)!$. Hence the distribution of u_K is given by

$$\begin{aligned}
\mathbb{P}(u_K \in dt) &= \sum_{n\geq 1} \mathbb{P}(K = n, u_n \in dt)\\
&= \alpha(1-\alpha)\sum_{n\geq 1}(1-\alpha)^{n-1}\frac{(-\log t)^{n-1}}{(n-1)!}dt\\
&= \alpha(1-\alpha)\exp[(\alpha-1)\log t]dt = \alpha(1-\alpha)t^{\alpha-1}dt
\end{aligned}$$

and consequently $P(u_K \leq \varepsilon) \sim c\varepsilon^\alpha$ as ε goes to 0. Similar calculations show that $P(u_{K+1} \leq \varepsilon) \sim c'\varepsilon^\alpha$ as ε goes to 0. Hence $P(g_1 \leq \varepsilon) \asymp \varepsilon^\alpha$. Comparing this with the generalized arcsine law, we conclude that the index of the subordinator is α. □

3 Partitions

3.1 Partitions of integers

For more details on random partitions, as well as numerous references, we refer to Pitman's *Cours de St-Flour* [9]. We consider exchangeable random partitions of the set $\mathbb{N}$, i.e. random partitions whose law is invariant under any permutation of a finite number of integers.

An interesting construction of random partitions is the so-called "Chinese restaurant" process [9]. The model is parametrized by two reals α, θ and an initial condition $(a_1, \ldots, a_{k_0})$ where k_0 and $a_1, \ldots, a_{k_0}$ are positive integers. We must have $0 \leq \alpha \leq 1$ and $\theta \geq -k_0\alpha$.

Imagine a restaurant with infinitely many tables, labeled by the integers. There are initially a_1 customers at the first table, a_2 customers at the second table and so on. Then new customers arrive and sit at some table according to the following rule.

Suppose that at a given moment, n customers are seated and occupy k tables, the number of customers at each table being $n_1 \ldots n_k$ respectively with $n_1 + \ldots + n_k = n$. Then the $(n+1)$-th customer sits at table number i, $1 \leq i \leq k$, with probability $(n_i - \alpha)/(n+\theta)$, and at table number $k+1$ with probability $(k\alpha + \theta)/(n+\theta)$.

Associate with this process a partition of $\mathbb{N}$ by saying that i and j are in the same block of the partition if and only if the i-th and j-th customers sit at the same table. Then one can check that this random partition is exchangeable, provided that the first $a_1 + \ldots + a_{k_0}$ customers sit in an exchangeable manner. We shall call an (α, θ)-partition a partition derived from a Chinese restaurant with parameters (α, θ) and initial condition (1), that is, one has initially one customer at the first table and no other customers.

3.2 The partition associated wih the construction

We want to show that the Chinese restaurant can be embedded in the construction of Section 2. From now on we restrict ourselves to the the points of C lying in the strip $[0,1] \times \mathbb{R}_+$. Reorder these points by decreasing ordinate, denoting

$$P_1 = (t_1, y_1), \ldots P_n = (t_n, y_n) \ldots$$

with $y_1 > y_2 > \ldots$. Add by convention $P_0 = (1, \infty)$.

Say that P_m is on the left of P_n if $t_m < t_n$.

SEATING RULE. We interpret P_n as the $(n+1)$-th customer.

- The first customer sits at the first table.
- The $(n+1)$-th customer P_n sits at a new table if t_n percolates.
- Otherwise, P_n sits at the same table as the left most customer on the right of P_n.

This seating rule induces a partition $\mathcal{P}_\alpha$ and we have

Theorem 3.1 *The seating rule described above yields the Chinese restaurant with parameters $(\alpha, 0)$ and initial condition $k_0 = 1$, $a_1 = 1$.*

3.3 Proof of Theorem 3.1

For every $n \geq 2$, denote by σ_n the permutation of $\{1, 2 \ldots n\}$ such that

$$t_{\sigma_n(1)} < t_{\sigma_n(2)} \ldots < t_{\sigma_n(n)}$$

Set $s_i^n = t_{\sigma_n(i)}$ for $1 \leq i \leq n$. First we have:

Lemma 3.2 *For every n and every permutation σ of $\{1,\dots n\}$, conditionally on $\sigma=\sigma_n$, the family of reals $(s_1^n,\dots s_n^n)$ has the law $\mathcal{L}$ of the increasing reordering of n independent, uniform random variables on $[0,1]$.*

Proof By elementary results on Poisson point processes, the t_i's are independent, uniform random variables on $[0,1]$. Hence $(s_1^n,\dots s_n^n)$ has the desired law $\mathcal{L}$.

Let σ be a permutation of $\{1,\dots n\}$ and $\mathcal{L}_\sigma$ be the law of the increasing reordering conditionally on $\sigma_n=\sigma$. Remark that for every permutation τ of $\{1,\dots n\}$, $(t_1,\dots t_n)$ has the same law as $(t_{\tau(1)},\dots t_{\tau(n)})$. This entails that $\mathcal{L}_\sigma=\mathcal{L}_{\sigma\circ\tau}$ for every τ. Thus $\mathcal{L}_\sigma$ does not depend on σ and equals $\mathcal{L}$. $\square$

Let us introduce some more notation. The restaurant at parameter α induced by our construction is formally a map $f^\alpha:\mathbb{N}\to\mathbb{N}$ depending on the Poisson point process $\mathcal{N}$ and on the variables $(U(P_n),n\geq 1)$, where $f^\alpha(n)=k$ means that the n-th customer sits at the k-th table. Denote by f_n^α the restriction of f^α to $\{1,\dots n+1\}$. Set $U=(U(P_n),n\geq 1)$ and $U_n=(U(P_m),1\leq m\leq n)$.

Remark that for every n, whether or not P_n percolates is determined by (σ_n,U_n), since it only depends on the points of C on the left of P_n and at a higher altitude than P_n. Hence for every n, f_n^α is determined by (σ_n,U_n). We shall write $f_n^\alpha=H_n(\sigma_n,U_n)$. As U_n and σ_n are independent, the lemma entails that conditionally on f_n^α, the variables s_m^n, $1\leq m\leq n-1$, are the increasing reordering of $n-1$ independent, uniform random variables on $[0,1]$. In particular, conditionally on f_n^α,

$$\mathbb{P}(t_n\in(s_i^n,s_{i+1}^n))=1/n \tag{2}$$

for every $i\leq n-1$.

Observe that according to the seating rule, the set of customers sitting at a given table at a given moment is a set of consecutive points if we reorder the customers by their x-coordinates. Moreover, it is easily seen by induction that if $s_m^n,s_{m+1}^n,\dots s_{m'}^n$ is the set of x-coordinates of the customers sitting at the i-th table at time n, then $s_{m'}^n$ percolates, s_i^n does not percolate for $m\leq i<m'$ and s_{m-1}^n percolates.

As a consequence, the event that the $(n+1)$-th customer sits at the i-th table is the union of these two events:

- $\{s_m^n\leq t_n\leq s_{m'}^n\}$
- $\{s_{m-1}^n\leq t_n\leq s_m^n,\ U(P_n)>\alpha\}$

According to (2), the probability of the first event is $(m'-m)/n$ and the probability of the second event is $(1-\alpha)/n$. Hence conditionally on f_n^α, the probability that the $(n+1)$-th customer sits at the i-th table is

$$\frac{m'-m+1-\alpha}{n}=\frac{n_i-\alpha}{n}$$

where $n_i=m'-m+1$ is the number of customers sitting at table i at time n. This is the probability defining the Chinese restaurant. This being true for every table, we have proved Theorem 3.1. $\square$

4 Fragmentation

In this section, we shall have to consider cadlag functions and we denote as usually

$$f(\alpha-) = \lim_{\beta\to\alpha,\beta<\alpha} f(\beta)$$

4.1 Inhomogeneous fragmentations

We refer to [2] for more details on fragmentation. Let $S^{\downarrow}$ be the set of decreasing sequences reals $x_1 \geq x_2 \geq \ldots$, $\sum x_i \leq 1$. We call a partition of mass a random element of $S^{\downarrow}$. If a partition of integers is exchangeable, it is a classical consequence of De Finetti's theorem that each block B of the partition almost surely has an asymptotic frequency, which is the limit of $|B \cap \{1, 2, \ldots n\}|/n$. This way one can derive a partition of mass by reordering the asymptotic frequencies of the blocks in decreasing order. In particular, given a Chinese restaurant with parameters (α, θ) and initial condition $(a_1, \ldots, a_{k_0})$, we denote the law of the associated partition of mass by

$$M(\alpha, \theta; a_1, \ldots, a_{k_0}).$$

Remark that translating Theorem 3 into a result for partitions of mass (rather that partitions of the integers), we recover the Poisson-Dirichlet representation of a partition derived from a stable subordinator, as established in [8].

A general fragmentation process $(X_t, 0 \leq t \leq 1)$ is a cadlag Markov process such that:

(a) For every t, X_t is an exchangeable partition of integers. For an integer i, we denote by $B_t(i)$ the block containing i in the partition X_t.

(b) If $s < t$, X_t is a finer partition than X_s.

(c) If $B_t(i) \neq B_t(j)$, then conditionally on $(B_t(i), B_t(j))$, $(B_s(i), s \geq t)$ and $(B_s(j), s \geq t)$ are independent.

The set $Dis(i)$ of dislocation times for i is the set of times t such that $B_t(i) \neq B_{t-}(i)$. If $t \in Dis(i)$, consider the set of all blocks of X_t contained in $B_{t-}(i)$ and let $a_1(t, i) > a_2(t, i) \ldots$ be the sequence of their asymptotic frequencies. Let $b_t(i)$ be the asymptotic frequency of $B_{t-}(i)$, set $b_n(t, i) = a_n(t, i)/b(t)$ and

$$b(t, i) = (b_1(t, i), b_2(t, i), \ldots)$$

We say that a fragmentation process $(X_t, 0 \leq t \leq 1)$ is an inhomogeneous fragmentation with dislocation measure $(\mu_t, 0 \leq t \leq 1)$ if for each t, μ_t is a sigma-finite measure on $S^{\downarrow}$ and if for each integer i

$$((t, b(t, i)),\ t \in Dis(i))$$

ia a Poisson point process on $[0, 1] \times S^{\downarrow}$ with intensity measure ν given by $\nu(dtds) = dt\mu_t(ds)$. We can now state:

Theorem 4.1 *The process $(\mathcal{P}_\alpha, 0 \leq 1 \leq \alpha)$, is an inhomogeneous fragmentation. Its dislocation measure is given by*

$$\mu_\alpha = \frac{1}{1-\alpha} \sum_{n=1}^{\infty} c_n M(\alpha, -\alpha; n, 1)$$

where c_n is given by

$$\sum_{n=1}^{\infty} c_n t^n = \frac{t}{1 + \log(1-t)}$$

Remark As shown in Section 3, the partition obtained at time α is an $(\alpha, 0)$-partition. Moreover it is known [9] that if $\alpha' > \alpha$, one can obtain an $(\alpha', 0)$-partition from an $(\alpha, 0)$-partition by splitting each block of the $(\alpha, 0)$-partition according an independent $(\alpha', -\alpha)$ partition. This suggests that, loosely speaking, one could obtain $\mathcal{P}_{\alpha+}$ from $\mathcal{P}_{\alpha-}$ by an $(\alpha, -\alpha)$-partition. The expression of the dislocation measure in Thereom 4.1 provides a rigorous version of this idea, whereas an $(\alpha, -\alpha)$-partition does not make sense strictly speaking.

4.2 Proof of Theorem 4.1

It is easy to verify that (a), (b) and (c) hold for the process $(\mathcal{P}_\alpha, 0 \le 1 \le \alpha)$, using the independence properties of the construction of Section 2. We want to show the Markov property and the Poisson point process representation for the dislocations of a given block. We shall study the evolution of the block containing 1 in the partition $\mathcal{P}_\alpha$, denoted by $B(\alpha)$

Recall that the integers in $B(\alpha)$ correspond to points of the Poisson point process $\mathcal{N}$ and to the additional point $P_0 = (1, \infty)$ (see Section 2.2). We shall identify $B(\alpha)$ with these points. Rank these points by decreasing y-coordinate, denoting them by

$$Q_0(\alpha) = (x_0(\alpha), z_0(\alpha)),\ Q_1(\alpha) = (x_1(\alpha), z_1(\alpha)), \ldots$$

with $\infty = z_0(\alpha) > z_1(\alpha) > \ldots$. Set $H_1(\alpha) = 1$ and by induction $H_{k+1}(\alpha) = \min\{n, x_n(\alpha) < x_{H_k(\alpha)}(\alpha)\}$.

Also, there is natural order on $B(\alpha-)$ by ordering the points in decreasing x-coordinate. Let $\prec_\alpha$ be this order.

By construction, for every k, $U(Q_{H_k(\alpha)}) > \alpha$, and it is easily seen that $U(Q_{H_k(\alpha)})$ is uniformly distributed on $(\alpha, 1)$, independently of the fragmentation up to time α. In turn, for all $n \notin \{H_k(\alpha), k \ge 1\}$, $U(Q_n(\alpha))$ is uniformly distributed on $(0, 1)$, independently of the fragmentation up to time α.

Moreover, it is easily seen that for every n, the restriction of $\prec_\alpha$ to $Q_1(\alpha)$, $\ldots Q_n(\alpha)$ is a random total order, uniformly distributed over all possible total orders and independent of the fragmentation up to time α. We shall see that the fragmentation process is determined by $\prec_\alpha$ and the variables $U(Q_n(\alpha))$, and since they are independent of the fragmentation up to time α, this entails the Markov property.

So we have, for every k, $U(Q_{H_k(\alpha)}) > \alpha$, and $U(Q_{H_k(\alpha-)}) \ge \alpha$. There is a dislocation at time α if for some $k \ge 1$, $U(Q_{H_k(\alpha-)}) = \alpha$. In that case, we say that we have a dislocation of index $I(\alpha) = k$ and almost surely, for every $k' \ne k$, $U(Q_{H_{k'}(\alpha-)}) > \alpha$. It is easy to check the following:

Proposition 4.2 *For every $k \ge 1$, let T_k be the set of dislocation times of index k. Then the T_k's are iid Poisson point processes on $[0, 1]$ with intensity measure $dx/(1-x)$.*

Proof This is just due to the fact that, conditionally on $U(P) > \alpha$, $U(P)$ is uniformly distributed on $(\alpha, 1)$. $\square$

If α is a dislocation time, let $\mathcal{P}'(\alpha)$ be the partition of $B(\alpha-)$ into blocks of the partition $\mathcal{P}(\alpha)$.

Proposition 4.3 *Assume that there is a dislocation at time α. Then conditionally on $I(\alpha) = k$ and $H_k(\alpha-) = k'$, the partition $\mathcal{P}'(\alpha)$ has the same law as the partition derived from a Chinese restaurant with parameters $(\alpha, -\alpha)$ and initial condition $k_0 = 2$, $a_1 = k'$, $a_2 = 1$.*

Proof The proof uses the same arguments as Theorem 3.1 Suppose that $I(\alpha) = k$ and $H_k(\alpha-) = k'$. Let $\overline{P} = (x, z)$ be the left-most point percolating at α and such that $z > z_{H_k(\alpha-)}$. If there is no such point, put by convention $\overline{P} = (0, \infty)$.

We view $B(\alpha-)$ as the set of customers of the Chinese restaurant. There are initially two tables and $k' + 1$ customers. At the first table, P_0 and $Q_1(\alpha-)$,... $Q_{k'-1}(\alpha-)$ are seated. At the second table, $Q_{k'}(\alpha-)$ is seated.

We also have $k + 2$ reals, namely $x, x_1, \dots x_{k'}, 1$, yielding $k' + 1$ intervals, which we denote by $J_1(\alpha), \dots J_{k'+1}(\alpha)$ from left to right.

Let n be the minimal integer such that $P_n = (t_n, y_n)$ satisfies $t_n > x$ and $y_n < z_{H_k(\alpha-)}$. It is important to notice that we are considering P_n, as defined in Section 3.2, and not Q_n. In particular P_n may not belong to $B(\alpha-)$. Set $Q = P_n$.

By the same arguments as in the proof of Theorem 3, Q lies in one of the $k'+1$ intervals $J_1(\alpha), \dots J_{k'+1}(\alpha)$ with equal probability. Then one has to distinguish between 5 possibilities.

- $x \in J_1(\alpha)$, and $U(Q) < \alpha$. In that case, $Q \notin B(\alpha-)$.
- $x \in J_1(\alpha)$ and $U(Q) > \alpha$. In that case, Q sits at the second table.
- $x \in J_2(\alpha)$ and $U(Q) < \alpha$. In that case, Q sits at a new table.
- $x \in J_2(\alpha)$, and $U(Q) > \alpha$. Then Q sits at the first table.
- $x \notin (J_1(\alpha) \cup J_2(\alpha))$. Then Q sits at the first table.

Therefore the probabilities to sit at the first table, at the second table or at a new table are proportional to $k' - \alpha$, $1 - \alpha$, α respectively. The same scheme apply for the next customers, which proves the proposition. $\square$

Proposition 4.4 *Assume that there is a dislocation at time α. Then*

$$\sum_{k'} \mathbb{P}(H_k(\alpha-) = k' | I(\alpha) = k) t^{k'} = t(-\log(1-t))^{k-1}$$

Proof If $\prec$ is a total order on $\{1, 2, \dots n\}$, set $m_1 = 1$ and by induction, $m_{i+1} = \min\{j > m_i, j \prec m_i\}$. Now let $D(k, k')$ be set of total orders $\prec$ on $\{1, 2, \dots k'\}$ such that $m_k = k'$. As an elementary application of the theory of combinatorial species,

$$\sum_{k' \geq 1} |D(k, k')| \frac{t^{k'}}{k'!} = t(-\log(1-t))^{k-1}$$

Let us apply this to the order $\prec_\alpha$. The event that $I(\alpha)=k$, $H_k(\alpha-)=k'$ is exactly the event that

(A) $U(Q_{H_k(\alpha-)}(\alpha-))=\alpha$ and for all $i\neq k$, $U(Q_{H_i(\alpha-)}(\alpha-))>\alpha$.

(B) The order $\prec_\alpha$, restricted to $Q_1(\alpha-),\dots Q_{k'}(\alpha-)$, lies in $\in D(k,k')$.

On the other hand, the event that α is a dislocation time and that $I(\alpha)=k$ is exactly the event that (A) holds. Since the variables $U(P_n)$ are independent of the Poisson point process $\mathcal{N}$, they are independent of $\prec_\alpha$ and therefore (A) and (B) occur independently. As a consequence, the probability that (A) and (B) hold given that (A) holds is the probability of (B). Finally, remark that the restriction of $\prec_\alpha$ to $Q_1(\alpha-),\dots Q_{k'}(\alpha-)$ is uniformly distributed over the $k'!$ possible total orders. This proves the proposition. □

Propositions 4.2, 4.3 and 4.4, together with obvious independence properties of $\mathcal{N}$, entail that the process of dislocations of the block containing 1 follow the law induced by the dislocation measure ν.

Moreover, it is easily seen that the law of dislocations for the block containing i is the same for all integers i. This concludes the proof of Theorem 4.1.

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