APPENDIX B TO: K_1-INVARIANTS IN THE MOD p COHOMOLOGY OF $U(3)$ ARITHMETIC MANIFOLDS

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Appendix B. Ideal computations

B.1.1. *Ideal intersections in the special fiber of* $S^{(j)}/I^{(j)}_{T,\nabla_{\mathrm{alg}}}$.

Proof of [LLHM]*, Lemma 3.22.* We first observe that there exists $\tau \in T$ and $(\omega', a') \in r(\Sigma_0)$ such that both $\mathfrak{P}^{(j)}_{(\omega,a)}S + \sum_{j'\in\mathcal{J}\setminus\{j\}} \mathfrak{P}^{(j)}_{(\omega',a')}S$ and $\mathfrak{P}^{(j)}_{(0,0)}S + \sum_{j'\in\mathcal{J}\setminus\{j\}} \mathfrak{P}^{(j)}_{(\omega',a')}S$ are the pullback, via [LLHM, (3.8)], of a minimal prime ideal of $S/I_{\tau,\nabla_\infty}$. In particular, by the explicit description of $S/I_{\tau,\nabla_\infty}$ appearing in [LLHM, Tables 3,4], the ring $S^{(j)}/I^{b_j}_j$ is equidimensional of dimension six, and has 2 minimal primes.

We prove [LLHM, item (1)]. From [LLHM, Table 8] one immediately checks that

$$(c_{33}, c_{32}, c_{31}, c_{23}, c_{22}, c_{21}, c_{13}d_{32}-c_{12}d^*_{33}, c_{13}d_{31}-c_{11}d^*_{33}, c_{12}d_{31}-c_{11}d_{32}, (b-c)c_{12}d_{21}-(a-c)c_{11}d^*_{22})) \subseteq I^{b_j}_j \tag{B.1}$$

In particular, we obtain a surjection

$$S^{(j)}/I'^{,b_j}_j \twoheadrightarrow S^{(j)}/I^{b_j}_j \tag{B.2}$$

where we have indicated by $I'^{,b_j}_j$ the left hand side of (B.1). Moreover

$$S^{(j)}/I'^{,b_j}_j \cong \frac{\mathbb{F}[\![c_{13}, d_{21}, d_{31}, d_{32}, x^*_{11}, x^*_{22}, x^*_{33}]\!]}{c_{13}((a-c)d_{31}d^*_{22} - (b-c)d_{32}d_{21})}$$

which is evidently reduced, equidimensional of dimension six, and has two minimal prime ideals. We conclude by [LLHLM20, Lemma 3.6.11] that the surjection (B.2) is an isomorphism, hence that the inclusion (B.1) is an equality.

The proofs of [LLHM, items (2)–(5)] are analogous. □

Proof of [LLHM]*, Lemma 3.36.* The proof is analogous to that of [LLHM, Lemma 3.22]. From [LLHM, Table 9] we have an evident inclusion of ideals of $S^{(j)}$:

$$(c_{22}, c_{33}, c_{32}, e_{33}, e_{23}, d_{31}, (a-b)c_{12}c_{23} - (a-c)e_{13}d^*_{22}, d_{21}d_{32}, c_{23}d_{32}, d_{21}c_{12}) \subseteq I^{(j)}_\Lambda \tag{B.3}$$

hence a surjection

$$S^{(j)}/I'^{(j)}_\Lambda \twoheadrightarrow S^{(j)}/I^{(j)}_\Lambda \tag{B.4}$$

(where we have indicated by $I'^{(j)}_\Lambda$ the left hand side of (B.3)). An direct computation shows that

$$S^{(j)}/I'^{(j)}_\Lambda \cong \frac{\mathbb{F}[\![c_{12}, d_{21}, d_{32}, c_{13}, c_{23}, x^*_{11}, x^*_{22}, x^*_{33}]\!]}{(d_{21}d_{32}, c_{23}d_{32}, d_{21}c_{12})}$$

and the latter ring is evidently reduced, equidimensional of dimension six and has three minimal prime ideals. □

Proof of [LLHM], *Proposition 3.11.* In the following computations, we work in $\widetilde{S}^{(j)}/(\widetilde{I}^{(j)}_{\tau}+\widetilde{I}^{(j)}_{\tau'})$.

Case $\widetilde{w}^*(\overline{\rho},\tau)_j=\alpha\beta\alpha t_{\underline{1}}$ *and* $\widetilde{w}^*(\overline{\rho},\tau')_j=t_{\underline{1}}$. Using the relations $c_{22}\equiv 0$, $c_{33}\equiv -pd^*_{33}$ coming from $\widetilde{I}^{(j)}_{\alpha\beta\alpha t_{\underline{1}}}$, the last listed equation in $\widetilde{I}^{(j)}_{t_{\underline{1}}}$ becomes:

$$-pc_{12}d_{21}d^*_{33}+pc_{11}d^*_{22}d^*_{33}-p(c_{11}d^*_{22}d^*_{33}-pd^*_{11}d^*_{22}d^*_{33}) \tag{B.5}$$

On the other hand the relations $c_{21}\equiv 0$ and $c_{21}\equiv -pd_{21}$ coming from $\widetilde{I}^{(j)}_{\alpha\beta\alpha t_{\underline{1}}}$ and $\widetilde{I}^{(j)}_{t_{\underline{1}}}$ respectively give $-pc_{12}d_{21}d^*_{33}\equiv 0$, hence (B.5) becomes $pc_{11}d^*_{22}d^*_{33}-p(c_{11}d^*_{22}d^*_{33}-pd^*_{11}d^*_{22}d^*_{33})$ yelding $p^2d^*_{11}d^*_{22}d^*_{33}\equiv 0$.

Case $\widetilde{w}^*(\overline{\rho},\tau)_j=\beta\alpha t_{\underline{1}}$ *and* $\widetilde{w}^*(\overline{\rho},\tau')_j=t_{\underline{1}}$. Using the relations $c_{22}\equiv 0$ coming from $\widetilde{I}^{(j)}_{\beta\alpha t_{\underline{1}}}$, the last listed equation in $\widetilde{I}^{(j)}_{t_{\underline{1}}}$ becomes:

$$c_{12}d_{21}c_{33}-c_{11}d^*_{22}c_{33}-p(c_{11}d^*_{22}d^*_{33}+d^*_{11}d^*_{22}c_{33}) \tag{B.6}$$

and, using $d_{21}c_{33}\equiv c_{23}d_{31}$, $c_{11}c_{33}\equiv -pc_{13}d_{31}$ coming from $\widetilde{I}^{(j)}_{t_{\underline{1}}}$, equation (B.6) becomes

$$c_{12}c_{23}d_{31}+pd^*_{22}c_{13}d_{31}-p(c_{11}d^*_{22}d^*_{33}+c^*_{11}d^*_{22}c_{33})$$

which, using $c_{23}\equiv 0$, $c_{11}d^*_{33}-c_{13}d_{31}$ and $c_{33}=-pd^*_{33}$ coming from $\widetilde{I}^{(j)}_{\beta\alpha t_{\underline{1}}}$, yields $p^2d^*_{11}d^*_{22}d^*_{33}\equiv 0$.

Case $\widetilde{w}^*(\overline{\rho},\tau)_j=\beta t_{\underline{1}}$ *and* $\widetilde{w}^*(\overline{\rho},\tau')_j=\alpha t_{\underline{1}}$. Using the relation $c_{11}d^*_{33}\equiv c_{13}d_{31}$ coming from $\widetilde{I}^{(j)}_{\beta t_{\underline{1}}}$, the last listed equation in $\widetilde{I}^{(j)}_{\alpha t_{\underline{1}}}$ becomes:

$$c_{13}d_{21}d_{32}-c_{12}d_{21}d^*_{33}-pd^*_{11}d^*_{22}d^*_{33}. \tag{B.7}$$

Multiplying (B.7) by pd^*_{33}, and using $pd_{21}d^*_{33}\equiv -c_{23}d_{31}$, $pc_{12}d^*_{33}\equiv -c_{13}c_{32}$ coming from $\widetilde{I}^{(j)}_{\beta t_{\underline{1}}}$, equation (B.6) becomes

$$-c_{23}d_{31}c_{13}d_{32}+c_{13}c_{32}d_{21}d^*_{33}-p^2d^*_{11}d^*_{22}(d^*_{33})^2.$$

which, using $c_{32}\equiv -pd_{32}$ coming from $\widetilde{I}^{(j)}_{\alpha t_{\underline{1}}}$, yields

$$-c_{23}d_{31}c_{13}d_{32}-pd_{32}c_{13}d_{21}d^*_{33}-p^2d^*_{11}d^*_{22}(d^*_{33})^2$$

hence $-p^2d^*_{11}d^*_{22}(d^*_{33})^2$ noting again that $pd_{21}d^*_{33}\equiv -c_{23}d_{31}$.

Case $\widetilde{w}(\overline{\rho},\tau)_j=\beta t_{w_0(\eta)}$ *and* $\widetilde{w}(\overline{\rho},\tau')_j=w_0t_{w_0(\eta)}$. Multiplying by $-p$ the relation $-pd^*_{22}d^*_{33}\equiv c_{23}d_{32}$ coming from $\widetilde{I}^{(j)}_{\beta t_{w_0(\eta)}}$, and using $-pc_{32}\equiv e_{23}$ coming from $\widetilde{I}^{(j)}_{w_0t_{w_0(\eta)}}$ we obtain $p^2d^*_{22}d^*_{33}\equiv e_{23}d_{32}$, and the latter expression is zero in the quotient ring since $e_{23}\in\widetilde{I}^{(j)}_{\beta t_{w_0(\eta)}}$.

Case $\widetilde{w}(\overline{\rho},\tau)_j=\alpha t_{w_0(\eta)}$ *and* $\widetilde{w}(\overline{\rho},\tau')_j=w_0t_{w_0(\eta)}$. Noting that $c_{33},e_{33}\equiv 0$ (relation coming from $\widetilde{I}^{(j)}_{\alpha t_{w_0(\eta)}}$) and $-e_{33}-pc_{33}\equiv p^2d^*_{33}$ (relation coming from $\widetilde{I}^{(j)}_{w_0t_{w_0(\eta)}}$) we obtain $p^2d^*_{33}\equiv 0$.

Case $\widetilde{w}(\overline{\rho},\tau)_j=t_{w_0(\eta)}$ *and* $\widetilde{w}(\overline{\rho},\tau')_j=w_0t_{w_0(\eta)}$. It is exactly as above noticing that $c_{33},e_{33}\equiv 0$ modulo $\widetilde{I}^{(j)}_{t_{w_0(\eta)}}$).

The cases where both $\widetilde{w}^*(\overline{\rho},\tau)_j$, $\widetilde{w}^*(\overline{\rho},\tau')_j$ have length at least 2 are much easier, and give the stronger result $p\in\widetilde{I}^{(j)}_{\tau}+\widetilde{I}^{(j)}_{\tau'}$. (For instance, if $\widetilde{w}(\overline{\rho},\tau)_j=t_{w_0(\eta)}$ and $\widetilde{w}(\overline{\rho},\tau')_j=\alpha t_{w_0(\eta)}$ then $c_{22}\equiv 0$ and $c_{22}\equiv -pd^*_{22}$, relations coming from $\widetilde{I}^{(j)}_{t_{w_0(\eta)}}$ and $\widetilde{I}^{(j)}_{\alpha t_{w_0(\eta)}}$ respectively.) □

B.1.2. *Ideal intersections for $S^{(j)}/I^{(j)}_{\tau,\nabla_{\mathrm{alg}}}$, $\widetilde{w}(\overline{\rho},\tau)_j = t_{\underline{1}}$.* In this section we work in the ring $S^{(j)}/I^{(j)}_{\tau,\nabla_{\mathrm{alg}}}$. By abuse of notation we will consider $S^{(j)}$ to be the ring $\mathbb{F}[\![c_{11}, x^*_{11}, c_{12}, c_{13}, d_{21}, c_{22}, x^*_{22}, c_{23}, d_{31}, d_{32}, c_{33}, x^*_{33}]\!]$, and $I^{(j)}_{\tau,\nabla_{\mathrm{alg}}}$, $\mathfrak{P}_{(\omega,a)}$ (for $\omega \in \{0, \varepsilon_1, \varepsilon_2\}$, $a \in \{0,1\}$) ideals of $\mathbb{F}[\![c_{11}, x^*_{11}, c_{12}, c_{13}, d_{21}, c_{22}, x^*_{22}, c_{23}, d_{31}, d_{32}, c_{33}, x^*_{33}]\!]$. (In other words, we abuse notation and "neglect the variables c_{21}, c_{31}, c_{32}").

We now remark that the assignement $c_{i,j} \mapsto c_{(132)(i),(132)(j)}$, $a \mapsto c+1$, $b \mapsto a$, $c \mapsto b$ induces an automorphism of $\mathbb{F}$-algebras on $S^{(j)}/I^{(j)}_{\tau,\nabla_{\mathrm{alg}}}$, which moreover sends $\mathfrak{P}_{(0,0)}$ to $\mathfrak{P}_{(\varepsilon_1,0)}$ (resp. $\mathfrak{P}_{(0,1)}$ to $\mathfrak{P}_{(\varepsilon_2,1)}$), $\mathfrak{P}_{(\varepsilon_1,0)}$ to $\mathfrak{P}_{(\varepsilon_2,0)}$ (resp. $\mathfrak{P}_{(\varepsilon_2,1)}$ to $\mathfrak{P}_{(\varepsilon_1,1)}$) and $\mathfrak{P}_{(\varepsilon_2,0)}$ to $\mathfrak{P}_{(0,0)}$ (resp. $\mathfrak{P}_{(\varepsilon_1,1)}$ to $\mathfrak{P}_{(0,1)}$).

Proof of [LLHM]*, Lemma 3.26.* By the remark at the beginning of §B.1.2 it is enough to prove the statements for the ideals I_γ with $\ell(\gamma) = 2$, $\gamma_1 = (\varepsilon_2, 1)$, and for I_β, $\ell(\beta) = 3$, $\beta_3 = (0,1)$.

The ideal I_β, $\ell(\beta) = 3$, $\beta_3 = (0,1)$. A direct inspection of [LLHM, Table 8] gives an inclusion

$$c_{11} \in \mathfrak{P}^{(j)}_{(0,0)} \cap \mathfrak{P}^{(j)}_{(\varepsilon_2,0)} \cap \mathfrak{P}^{(j)}_{(\varepsilon_2,1)} \cap \mathfrak{P}^{(j)}_{(\varepsilon_1,0)} \cap \mathfrak{P}^{(j)}_{(\varepsilon_1,1)}.$$

Thus, we have a surjection

$$S^{(j)}/(c_{11}, I^{(j)}_{\tau,\nabla_{\mathrm{alg}}}) \twoheadrightarrow S^{(j)}/(\mathfrak{P}^{(j)}_{(0,0)} \cap \mathfrak{P}^{(j)}_{(\varepsilon_2,0)} \cap \mathfrak{P}^{(j)}_{(\varepsilon_2,1)} \cap \mathfrak{P}^{(j)}_{(\varepsilon_1,0)} \cap \mathfrak{P}^{(j)}_{(\varepsilon_1,1)}) \tag{B.8}$$

and a direct computation using [LLHM, Table 4] gives

$$S^{(j)}/(c_{11}, I^{(j)}_{\tau,\nabla_{\mathrm{alg}}}) \cong \frac{\mathbb{F}[\![c_{12}, c_{13}, c_{23}, d_{21}, d_{31}, d_{32}, x^*_{11}, x^*_{22}, x^*_{33}]\!]}{J}$$

where J is the ideal generated by

$$\begin{gathered} c_{12}d_{31},\ c_{12}(c_{23}d^*_{11} - d_{21}c_{13}),\ d_{31}(c_{23}d^*_{11} - (a-b)d_{21}c_{13}), \\ d_{32}(c_{23}d^*_{22} - d_{21}c_{13}) - (a-b-1)c_{13}d_{31}, \\ (a-c-1)c_{23}d_{32} - (a-b)(a-c-1)c_{13}d_{31} - (a-b-1)c_{12}d_{21} \end{gathered}$$

The latter ring is reduced (since the initial ideal of J is generated by squarefree monomial, as it can be checked by considering a suitable Groebner basis) and has 5 minimal primes each of dimension 6. Thus, by [LLHLM20, Lemma 3.6.11], the surjection (**??**) is an isomorphism.

Case $\gamma = ((\varepsilon_2, 1), (\varepsilon_1, 0))$. A direct inspection of [LLHM, Table 8] gives an inclusion

$$\underbrace{(c_{22}, c_{11}, (a-b)c_{13}d_{21} + (b-c-1)c_{23}d^*_{11})}_{\overset{\mathrm{def}}{=} J} \subseteq \mathfrak{P}^{(j)}_{(0,0)} \cap \mathfrak{P}^{(j)}_{(\varepsilon_2,0)} \cap \mathfrak{P}^{(j)}_{(\varepsilon_2,1)}. \tag{B.9}$$

Thus, we have a surjection

$$S^{(j)}/(J + I^{(j)}_{\tau,\nabla_{\mathrm{alg}}}) \twoheadrightarrow S^{(j)}/(\mathfrak{P}^{(j)}_{(0,0)} \cap \mathfrak{P}^{(j)}_{(\varepsilon_2,0)} \cap \mathfrak{P}^{(j)}_{(\varepsilon_2,1)}) \tag{B.10}$$

and a direct computation gives

$$S^{(j)}/(J + I^{(j)}_{\tau,\nabla_{\mathrm{alg}}}) \cong \frac{\mathbb{F}[\![c_{12}, c_{13}, d_{21}, d_{31}, d_{32}, x^*_{11}, x^*_{22}, x^*_{33}]\!]}{c_{13}(d_{31}d^*_{22} - d_{32}d_{21}), c_{12}d_{21}, c_{12}d_{31}}.$$

The latter ring is reduced, equidimensional of dimension 6 and has 3 minimal primes, and hence, by [LLHLM20, Lemma 3.6.11], the surjection (B.10) is an isomorphism. We conclude that (B.9) is an equality.

We now prove the assertion on the minimal number of generators. First of all we note that $c_{11} \in (c_{22}, (b-c-1)c_{23}d^*_{11} + (a+b)(c_{13}d_{21}))$. Indeed, using the three equations in row Mon_τ, and the equation in line 4 of row $I^{(j)}_{\tau,\nabla_{\rm alg}}$ (the "determinant" equation) in [LLHM, Table 4], we have

$$\begin{aligned} f \stackrel{\rm def}{=} &(a-b)(b-c)(a-c-1)c_{13}d_{21}d_{32} - (a-c)(a-b)(a-c-1)c_{11}d^*_{22}d^*_{33} + \\ &+ (b-c)(a-c)(b-c-1)c_{33}d^*_{22}d^*_{11} + (a-b-1)(a-b)(b-c)c_{22}d^*_{11}d^*_{33} \in I^{(j)}_{\tau,\nabla_{\rm alg}} \end{aligned}$$

and, on the other hand,

$$xc_{11}d^*_{22}d^*_{33} + yc_{22}d^*_{11}d^*_{33} + zd_{32}\left(c_{23}d^*_{11} + \frac{a+b}{b-c-1}c_{13}d_{21}\right) + \kappa d^*_{11}\mathrm{Mon}_{\tau,1} = f$$

where $z \stackrel{\rm def}{=} (b-c-1)(b-c)(a-c-1)$, $\kappa \stackrel{\rm def}{=} -(b-c-1)(b-c)$, $y = \kappa(a-b-1)$ and $x = -(a-c)(a-b)(a-c-1)$.

Hence $I_\gamma = (c_{22}, (b-c-1)c_{23}d^*_{11} + (a+b)(c_{13}d_{21}))$ and we now prove that $\overline{I}_\gamma$ has dimension 2. We first note that $\overline{c}_{22} \neq 0$, as $c_{22} \notin \mathfrak{m}_{S^{(j)}} \cdot (J + I^{(j)}_{\tau,\nabla_{\rm alg}})$ (alternatively, one can check on the explicit equations of [LLHM, Table 4] that $c_{22} \neq 0$ in $S^{(j)}/(\mathfrak{m}^2_{S^{(j)}} + I^{(j)}_{\tau,\nabla_{\rm alg}})$). Now, if we have a relation of the form $\overline{c}_2 + \kappa\overline{(a-b)c_{13}d_{21} + (b-c-1)c_{23}d^*_{11})}$ in $\overline{I}_\gamma$, for some $\kappa \in \mathbb{F}^\times$, this would imply that the natural inclusion $(c_{22}) \subseteq I_\gamma$ induces an isomorphism of 1-dimensional $\mathbb{F}$-vector spaces, and hence that $(c_{22}) \subseteq I_\gamma$ is in fact an equality. This is impossible since $(c_{22}) = I_\beta$, $I_\beta \neq I_\gamma$ (e.g. by looking at the number of minimal primes of $S^{(j)}/I^{(j)}_{\tau,\nabla_{\rm alg}}$ above them which has been computed along the proof). □

Lemma B.1. *We have the following equalities in $S^{(j)}/I^{(j)}_{\tau,\nabla_{\rm alg}}$*

(1) $-\left(\frac{a-b}{b-c}\right)c_{11}d^*_{22}d^*_{33} - \left(\frac{1-a+b}{1-a+c}\right)d^*_{11}c_{22}d^*_{33} - c_{13}(d^*_{22}d_{31} - d_{21}d_{32}) \equiv 0$;
(2) $(b-c)d^*_{11}c_{22} + (a-c)c_{11}d^*_{22} - (b-c)c_{12}d_{21} \equiv 0$;
(3) $-\left(\frac{a-c}{b-c}\right)c_{11}d^*_{22}d^*_{33} + \left(\frac{a-b-1}{a-b}\right)d^*_{11}c_{22}d^*_{33} + \left(\frac{b-c-1}{a-b}\right)d^*_{11}c_{23}d_{32} + d_{21}d_{32}c_{13} \equiv 0$;
(4) $-\left(\frac{c+1-a}{a-b}\right)c_{33}d^*_{11}d^*_{22} - \left(\frac{-c+a}{-c+b}\right)d^*_{33}c_{11}d^*_{22} - d_{32}(d^*_{11}c_{23} - c_{13}d_{21}) \equiv 0$;
(5) $(a-b)d^*_{33}c_{11} + (c+1-b)c_{33}d^*_{11} - (a-b)d_{31}c_{13} \equiv 0$;
(6) $-\left(\frac{c+1-b}{a-b}\right)c_{33}d^*_{11}d^*_{22} + \left(\frac{c-a}{c+1-a}\right)d^*_{33}c_{11}d^*_{22} + \left(\frac{a-b-1}{c+1-a}\right)d^*_{22}c_{12}d_{21} + d_{13}d_{21}c_{32} \equiv 0$.

Proof. By the remark at the beginning of §B.1.2 it is enough to prove the statements for items (1), (2) and (3).

Let $\overline{\mathrm{Mon}}_{\tau,1}$, $\overline{\mathrm{Mon}}_{\tau,2}$, $\overline{\mathrm{Mon}}_{\tau,3}$ denote the mod p-reduction of the first, second and third equations in row Mon_τ of [LLHM, Table 4]. In particular, item (2) is $\overline{\mathrm{Mon}}_{\tau,3}$.

Item (1) is deduced from $\frac{d^*_{11}}{a-c-1}\overline{\mathrm{Mon}}_{\tau,1} \equiv 0$, using the relations

$$\begin{aligned} d^*_{11}(c_{23}d_{32} - c_{33}d^*_{22}) &\equiv c_{13}d_{21}d_{32} - c_{12}d_{21}d^*_{33} - c_{13}d_{31}d^*_{22} + c_{11}d^*_{22}d^*_{33} + d^*_{11}c_{22}d^*_{33} \\ c_{12}d_{21} &\equiv d^*_{11}c_{22} + \frac{(a-b)}{(b-c)}c_{11}d^*_{22} \end{aligned}$$

(the first relation comes from the sixth equation in row $I^{(j)}_\tau$ in [LLHM, Table 4], and the second relation from $\overline{\mathrm{Mon}}_{\tau,3}$).

Item (3) is deduced from $\frac{d^*_{22}}{a-b}\overline{\mathrm{Mon}}_{\tau,2}\equiv 0$, using the relations

$$d^*_{22}(c_{23}d_{32}-c_{22}d^*_{33})\equiv c_{13}d_{21}d_{32}-c_{12}d_{21}d^*_{33}-c_{23}d_{32}d^*_{11}+c_{33}d^*_{22}d^*_{11}+d^*_{11}c_{22}d^*_{33}$$
$$c_{33}d^*_{22}\equiv c_{23}d_{32}-\frac{(a-b-1)}{(a-c-1)}c_{22}d^*_{33}$$
$$c_{12}d_{21}\equiv d^*_{11}c_{22}+\frac{(a-b)}{(b-c)}c_{11}d^*_{22}$$

(the first relation comes from the sixth equation in row $I^{(j)}_{\tau}$ in [LLHM, Table 4], and the second and third relation from $\overline{\mathrm{Mon}}_{\tau,2}$ and $\overline{\mathrm{Mon}}_{\tau,3}$). □

B.1.3. *Justification for* [LLHM, Table 6]. The justification is a direct computation, performed by exhibiting elements in $\widetilde{I}^{(j)}_{\{w_0,\alpha\beta\},\nabla_{\mathrm{alg}}}\overset{\mathrm{def}}{=}\widetilde{I}^{(j)}_{\tau_{w_0},\nabla_{\mathrm{alg}}}\cap\widetilde{I}^{(j)}_{\tau_{\alpha\beta},\nabla_{\mathrm{alg}}}$ (resp. in $\widetilde{I}^{(j)}_{\{w_0,\beta\alpha\},\nabla_{\mathrm{alg}}}=\widetilde{I}^{(j)}_{\tau_{w_0},\nabla_{\mathrm{alg}}}\cap\widetilde{I}^{(j)}_{\tau_{\beta\alpha},\nabla_{\mathrm{alg}}}$), and taking their mod-$\varpi$ reduction.

We mention that these computation can ultimately be checked by exhibiting a Groebner basis for the ideals $\widetilde{I}^{(j)}_{\tau_{w_0},\nabla_{\mathrm{alg}}}$, $\widetilde{I}^{(j)}_{\tau_{\alpha\beta},\nabla_{\mathrm{alg}}}$ and $\widetilde{I}^{(j)}_{\tau_{\beta\alpha},\nabla_{\mathrm{alg}}}$ (for the monomial ordering on $\widetilde{S}^{(j)}$ given by $c_{11}>c_{12}>c_{13}>d_{21}>c_{21}>c_{22}>c_{23}>c_{31}>d_{31}>d_{32}>c_{32}>c_{33}$), and give full detail for the most complicated equations (namely, those involving the structure constants from the monodromy).

Study of $\widetilde{I}^{(j)}_{\{w_0,\alpha\beta\},\nabla_{\mathrm{alg}}}\overset{def}{=}\widetilde{I}^{(j)}_{\tau_{w_0},\nabla_{\mathrm{alg}}}\cap\widetilde{I}^{(j)}_{\tau_{\alpha\beta},\nabla_{\mathrm{alg}}}$. We claim that the element $f\overset{\mathrm{def}}{=}(b_{\tau_{\alpha\beta},2}-b_{\tau_{\alpha\beta},3})d_{21}c_{22}d^*_{33}-(b_{\tau_{\alpha\beta},2}-b_{\tau_{\alpha\beta},3}-1)c_{23}d_{31}d^*_{22}+(b_{\tau_{\alpha\beta},,2}-b_{\tau_{\alpha\beta},1}-1)c_{21}d^*_{33}d^*_{22}$ (whose mod ϖ-reduction gives the element in the third line of row $\alpha\beta\alpha t_{\underline{1}},\alpha\beta t_{\underline{1}}$ in [LLHM, Table 6]) is in $\widetilde{I}^{(j)}_{\{w_0,\alpha\beta\},\nabla_{\mathrm{alg}}}$. Indeed, on the one hand c_{21},c_{22},c_{23} all belong to $\widetilde{I}^{(j)}_{\tau_{w_0},\nabla_{\mathrm{alg}}}$, and on the other hand

$$f=(c_{22}+pd^*_{22})(b_{\tau_{\alpha\beta},2}-b_{\tau_{\alpha\beta},3})d^*_{33}d_{21}-d^*_{22}(\mathrm{Mon}_{\tau_{\alpha\beta},2})$$

where $\mathrm{Mon}_{\tau_{\alpha\beta},2}$ denotes the second equation in row $(\alpha\beta t_{\underline{1}},\widetilde{I}^{(j)}_{\tau,\nabla_\infty},\mathrm{Mon}_\tau)$ in [LLHM, Table 3].

In a similar fashion, we have the equality

$$\begin{aligned}&((b_{\tau_{w_0},2}-b_{\tau_{w_0},3})d_{21}d_{32}+d_{31})c_{22}+((b_{\tau_{w_0},3}-b_{\tau_{w_0},1}-1+p)d_{32}+c_{32})c_{21}+\mathrm{Mon}_{\tau_{w_0},2}=\\&\quad=((b_{\tau_{w_0},2}-b_{\tau_{w_0},3})d_{21}d_{32}+d_{31})(c_{22}+pd^*_{22})+(b_{\tau_{w_0},3}-b_{\tau_{w_0},1}-1)(c_{21}d_{32}-c_{31}d^*_{22})+c_{21}(c_{32}+pd_{32})\end{aligned}$$

hence obtaining an element in $\widetilde{I}^{(j)}_{\{w_0,\alpha\beta\},\nabla_{\mathrm{alg}}}$ which reduces modulo p to the last equation in row $(\alpha\beta\alpha t_{\underline{1}},\alpha\beta t_{\underline{1}})$ of [LLHM, Table 6].

Finally, we check that

$$c_{22}(xc_{11}+yd^*_{11})+z\mathrm{Mon}_{\tau_{w_0},1}=(c_{22}+pd^*_{22})(xc_{11}+yd^*_{11})+\mathrm{Mon}_{\tau_{\alpha\beta},1}\tag{B.11}$$

where

$$\begin{cases}z\overset{\mathrm{def}}{=}\frac{(b_{\tau_{\alpha\beta},2}-b_{\tau_{\alpha\beta},3})}{(b_{\tau_{w_0},2}-b_{\tau_{w_0},3})},\\ y\overset{\mathrm{def}}{=}1-(b_{\tau_{\alpha\beta},2}-b_{\tau_{\alpha\beta},3})-\frac{(b_{\tau_{\alpha\beta},2}-b_{\tau_{\alpha\beta},3})}{(b_{\tau_{w_0},2}-b_{\tau_{w_0},3})},\\ x\overset{\mathrm{def}}{=}\frac{1}{p}\frac{(b_{\tau_{\alpha\beta},2}-b_{\tau_{\alpha\beta},3})(b_{\tau_{w_0},3}-b_{\tau_{w_0},1})-(b_{\tau_{\alpha\beta},3}-b_{\tau_{\alpha\beta},1})(b_{\tau_{w_0},2}-b_{\tau_{w_0},3})}{(b_{\tau_{w_0},2}-b_{\tau_{w_0},3})}\end{cases}$$

(note that $x,y,z\in\mathbb{Z}_p$ by the genericity assumption on $\mu_j+\eta_j$ and fact that $b_{\tau_{\alpha\beta},i}\equiv b_{\tau_{w_0},i}$ for $i=1,2,3$) and where $\mathrm{Mon}_{\tau_{w_0},1}$ (resp. $\mathrm{Mon}_{\tau_{\alpha\beta},1}$) denotes the first equation in row $(\alpha\beta\alpha t_{\underline{1}},\widetilde{I}^{(j)}_{\tau,\nabla_\infty},\mathrm{Mon}_\tau)$ (resp. $(\alpha\beta t_{\underline{1}},\widetilde{I}^{(j)}_{\tau,\nabla_\infty},\mathrm{Mon}_\tau)$) in [LLHM, Table 3]. Observing that $z\equiv 1$ and $y\equiv -(b-c)$ modulo p, equation (B.11) justifies the fourth line in row $(\alpha\beta\alpha t_{\underline{1}},\alpha\beta t_{\underline{1}})$ of [LLHM, Table 6].

The computations for the elements in the first two lines in row $\alpha\beta\alpha t_{\underline{1}}, \alpha\beta t_{\underline{1}}$ are easier (only involving finite height equations). For instance the element $c_{23}d_{32}-c_{22}d_{33}^*$ is evidently in $I^{(j)}_{\{w_0,\alpha\beta\},\nabla_{\mathrm{alg}}}$ since on the one hand $c_{22}, c_{23} \in \widetilde{I}^{(j)}_{\tau_{w_0},\nabla_{\mathrm{alg}}}$ and on the other hand $c_{22}+pd_{22}^*, c_{23}d_{32}+pd_{33}^*d_{22}^* \in \widetilde{I}^{(j)}_{\tau_{w_0},\nabla_{\mathrm{alg}}}$. □

Study of $\widetilde{I}^{(j)}_{\tau_{w_0},\nabla_{\mathrm{alg}}} \cap \widetilde{I}^{(j)}_{\tau_{\beta\alpha},\nabla_{\mathrm{alg}}}$. We explicitly construct elements in $\widetilde{I}^{(j)}_{\tau_{w_0},\nabla_{\mathrm{alg}}} \cap \widetilde{I}^{(j)}_{\tau_{\beta\alpha},\nabla_{\mathrm{alg}}}$ and compute their mod p-reductions.

The elements

$$(b_{\tau_{\beta\alpha},2} - b_{\tau_{\beta\alpha},3} + 1)(c_{32+pd_{32}}) + (b_{\tau_{\beta\alpha},1} - b_{\tau_{\beta\alpha},3})d_{31}(c_{13}d_{32} - c_{12}d_{33}^*)+ \\ + (b_{\tau_{\beta\alpha},3} - b_{\tau_{\beta\alpha},2})d_{32}(c_{13}d_{31} - c_{11}d_{33}^* + pd_{11}d_{33}^*) \tag{B.12}$$

and

$$(b_{\tau_{\beta\alpha},1} - b_{\tau_{\beta\alpha},2})(c_{13}d_{31} - c_{11}d_{33}^*) - \mathrm{Mon}_{\tau_{\beta\alpha}}. \tag{B.13}$$

are equal, and, by direct inspection of the finite height equations in [LLHM, Table 3], equation (B.12) defines an element in $\widetilde{I}^{(j)}_{\tau_{w_0},\nabla_{\mathrm{alg}}}$ and equation (B.13) defines an element in $\widetilde{I}^{(j)}_{\tau_{\beta\alpha},\nabla_{\mathrm{alg}}}$. This equation reduces mod p to the fourth line of row $\alpha\beta\alpha t_{\underline{1}}, \beta\alpha t_{\underline{1}}$ of [LLHM, Table 6].

Let $x' \stackrel{\mathrm{def}}{=} \frac{b_{\tau_{w_0},1}-b_{\tau_{w_0},2}+p(b_{\tau_{w_0},1}-b_{\tau_{w_0},3})}{1+p}$. A direct computation shows that the expressions

(B.14)

$$d_{11}^*d_{33}^*(\mathrm{Mon}_{\alpha\beta\alpha,2}) + x'c_{11}d_{21}d_{33}^*(c_{32} + pd_{32}) + (b_{\tau_{w_0},2} - b_{\tau_{w_0},3})d_{21}d_{31}(c_{13}d_{32} - c_{12}d_{33}^*)+ \\ + \Big(x'd_{31}d_{22}^* + (b_{\tau_{w_0},1} - b_{\tau_{w_0},3})c_{31}d_{22}^* - (b_{\tau_{w_0},2} - b_{\tau_{w_0},3})d_{21}d_{32}\Big)\Big(c_{13}d_{31} - c_{11}d_{33}^* + pd_{11}^*d_{33}^*\Big)+ \\ - (b_{\tau_{w_0},1} - b_{\tau_{w_0},3})d_{31}d_{22}^*(c_{13}c_{31} + pc_{11}d_{33})$$

and

(B.15)

$$\Big((b_{\tau_{w_0},1} - b_{\tau_{w_0},3})(p - c_{11}) + (b_{\tau_{w_0},1} - b_{\tau_{w_0},3} + 1)\Big)(c_{31} + pd_{31})d_{11}^*d_{22}^*d_{33}^*+ \\ + \Big(x'(d_{31}d_{22}^* - d_{21}c_{32}) - (b_{\tau_{w_0},2} - b_{\tau_{w_0},3} + px')d_{21}d_{32}\Big)(c_{13}c_{31} - c_{11}d_{33}^*)+ \\ + x'd_{21}d_{31}(c_{13}c_{32} + pc_{12}d_{33}^*) + (b_{\tau_{w_0},2} - b_{\tau_{w_0},3} + px')d_{31}(c_{13}d_{21}d_{32} - c_{12}d_{21}d_{33}^* - pd_{11}^*d_{22}^*d_{33}^*)$$

are equal, and, as above, equation (B.14) defines an element in $\widetilde{I}^{(j)}_{\tau_{w_0},\nabla_{\mathrm{alg}}}$ and (B.15) defines an element in $\widetilde{I}^{(j)}_{\tau_{\beta\alpha},\nabla_{\mathrm{alg}}}$. The mod p-reduction of such element justifies the equation in the fifth line of row $\alpha\beta\alpha t_{\underline{1}}, \beta\alpha t_{\underline{1}}$ of [LLHM, Table 6].

Finally, let

$$z' \stackrel{\mathrm{def}}{=} \frac{(b_{\tau_{w_0},2} - b_{\tau_{w_0},3}) - (b_{\tau_{\beta\alpha},2} - b_{\tau_{\beta\alpha},3})}{p} + (b_{\tau_{w_0},2} - b_{\tau_{w_0},3})$$
$$z'' \stackrel{\mathrm{def}}{=} \frac{(b_{\tau_{w_0},3} - b_{\tau_{w_0},1}) - (b_{\tau_{\beta\alpha},3} - b_{\tau_{\beta\alpha},1})}{p} + (b_{\tau_{w_0},3} - b_{\tau_{w_0},1}).$$

Again, a direct computation shows that

$$(z'c_{12}d_{21} + z''c_{11}d_{22}^* - pd_{11}d_{22}^*)(c_{13}d_{31} - c_{11}d_{33}^* + pd_{11}^*d_{33}^*) - (p+1)d_{11}^*d_{33}^*(\mathrm{Mon}_{\tau_{w_0},1}) = \\ = (z'c_{12}d_{21} + z''c_{11}d_{22}^* - pd_{11}d_{22}^*)(c_{13}d_{31} - c_{11}d_{33}^*) - d_{11}^*d_{33}^*(\mathrm{Mon}_{\tau_{\beta\alpha},1})$$

which, similarly as in the previous cases, defines an element in $\widetilde{I}^{(j)}_{\tau_{w_0},\nabla_{\mathrm{alg}}} \cap \widetilde{I}^{(j)}_{\tau_{\beta\alpha},\nabla_{\mathrm{alg}}}$ whose mod p-reduction justifies the last equation in row $\alpha\beta\alpha t_{\underline{1}}, \beta\alpha t_{\underline{1}}$ of [LLHM, Table 6]. □

B.1.4. *Justification for* [LLHM, Table 7]. As for [LLHM, Table 6], the justification is a direct computation (cf. §B.1.3). For $i \in \{1,2,3\}$ we set $b_{\mathrm{id},i} \overset{\mathrm{def}}{=} b_{\tau_{t_{w_0(\eta)}},i}$, $b_{\alpha,i} \overset{\mathrm{def}}{=} b_{\tau_{t_{w_0(\eta)}\alpha},i}$ and $b_{\beta,i} \overset{\mathrm{def}}{=} b_{\tau_{t_{w_0(\eta)}\beta},i}$ for readability in what follows.

Study of $\widetilde{I}^{(j)}_{\tau_{t_{w_0(\eta)}},\nabla_{\mathrm{alg}}} \cap \widetilde{I}^{(j)}_{\tau_{t_{w_0(\eta)}\alpha},\nabla_{\mathrm{alg}}}$. Define $z \overset{\mathrm{def}}{=} \frac{b_{\alpha,1}-b_{\alpha,2}}{b_{\mathrm{id},1}-b_{\mathrm{id},2}}$, $y \overset{\mathrm{def}}{=} b_{\alpha,2} - b_{\alpha,1} - 1 + z$ and

$$x = \frac{1}{p}\big(b_{\alpha,1} - b_{\alpha,3} - z(b_{\mathrm{id},1} - b_{\mathrm{id},3})\big)$$

(note that $z \in \mathbb{Z}_p$ as $b_{\mathrm{id},1} - b_{\mathrm{id},2} \not\equiv 0 \bmod p$ and that $x \in \mathbb{Z}_p$ as $b_{\alpha,1} - b_{\alpha,3} - z(b_{\mathrm{id},1} - b_{\mathrm{id},3}) \equiv 0 \bmod p$). A direct computation shows that the expressions

$$d_{11}^*(xc_{12}e_{23} + yc_{13}c_{22} + z\mathrm{Mon}_{t_{w_0(\eta)}})$$

and

$$xe_{13}(c_{12}d_{21} + pd_{11}^*d_{22}^*) - xc_{12}(e_{13}d_{21} - e_{23}d_{11}^*) + yd_{11}^*c_{13}(c_{22} + pd_{22}) - d_{11}^*\mathrm{Mon}_{t_{w_0(\eta)}\alpha}$$

are equal (where we denoted by $\mathrm{Mon}_{t_{w_0(\eta)}}$ and $\mathrm{Mon}_{t_{w_0(\eta)}\alpha}$ the last equation in row $t_{w_0(\eta)}$ and $t_{w_0(\eta)}\alpha$ respectively). These expressions define an element in the intersection $\widetilde{I}^{(j)}_{\tau_{t_{w_0(\eta)}},\nabla_{\mathrm{alg}}} \cap \widetilde{I}^{(j)}_{\tau_{t_{w_0(\eta)}\alpha},\nabla_{\mathrm{alg}}}$, whose mod p reduction explains the second line in row $t_{w_0(\eta)}, t_{w_0(\eta)}\alpha$ of [LLHM, Table 7]. □

Study of $\widetilde{I}^{(j)}_{\tau_{t_{w_0(\eta)}},\nabla_{\mathrm{alg}}} \cap \widetilde{I}^{(j)}_{\tau_{t_{w_0(\eta)}\beta},\nabla_{\mathrm{alg}}}$. Define

$$z' = \frac{1}{p}\big(b_{\beta,1} - b_{\beta,2} - (p+1)(b_{\mathrm{id},1} - b_{\mathrm{id},2})\big)$$
$$z'' = \frac{1}{p}\big((p+1)(b_{\mathrm{id},1} - b_{\mathrm{id},3}) - (b_{\beta,1} - b_{\beta,3})\big)$$

(note that z' and z'' are elements of $\mathbb{Z}_p$ as $b_{\beta,1} - b_{\beta,j} - (p+1)(b_{\mathrm{id},1} - b_{\mathrm{id},j}) \equiv 0 \bmod p$ for $j \in \{2,3\}$). Again a direct computation shows that the expressions

$$(z''e_{13}d_{22}^* - pc_{13}d_{22}^* + z'c_{12}c_{23})c_{33} + (p+1)d_{33}^*\mathrm{Mon}_{t_{w_0(\eta)}}$$

and

$$(z''e_{13}d_{22}^* - pc_{13}d_{22}^* + z'c_{12}c_{23})(c_{33} + p) - d_{33}^*\mathrm{Mon}_{t_{w_0(\eta)}\beta}$$

are equal (where again we denoted by $\mathrm{Mon}_{t_{w_0(\eta)}}$ and $\mathrm{Mon}_{t_{w_0(\eta)}\beta}$ the last equation in row $t_{w_0(\eta)}$ and $t_{w_0(\eta)}\beta$ respectively). These expressions define an element in the intersection $\widetilde{I}^{(j)}_{\tau_{t_{w_0(\eta)}},\nabla_{\mathrm{alg}}} \cap \widetilde{I}^{(j)}_{\tau_{t_{w_0(\eta)}\beta},\nabla_{\mathrm{alg}}}$, whose mod p reduction explains the second line in row $t_{w_0(\eta)}, t_{w_0(\eta)}\beta$ of [LLHM, Table 7]. □

B.1.5. *Computations on* $\mathrm{Tor}_1^{S^{(j)}}(\mathbb{F},(\widetilde{S}^{(j)}/\widetilde{I}^{(j)}_{T,\nabla_{\mathrm{alg}}})\otimes\mathbb{F})$. We provide details for the computations of the maps between various $\mathrm{Tor}_1^{S^{(j)}}(\mathbb{F},(\widetilde{S}^{(j)}/\widetilde{I}^{(j)}_{T,\nabla_{\mathrm{alg}}})\otimes\mathbb{F})$ appearing in the proofs of [LLHM, Lemmas 3.30, 3.33, 3.35, 3.37]. In the following computation, given an ideal $I\subseteq S^{(j)}$ we write elements of $\mathrm{Tor}_1^{S^{(j)}}(\mathbb{F},S^{(j)}/I)$ in terms of generators of I, by virtue of the canonical isomorphism $\mathrm{Tor}_1^{S^{(j)}}(\mathbb{F},S^{(j)}/I)\cong I/(\mathfrak{m}_{S^{(j)}}\cdot I)$.

Complements in the proof of [LLHM]*, Lemma 3.30.* We need to prove that the union of the images of the canonical maps

$$\mathrm{Tor}_1(\mathbb{F},(\widetilde{S}^{(j)}/(\widetilde{I}^{(j)}_{\tau_{\alpha\beta},\nabla_{\mathrm{alg}}}\cap\widetilde{I}^{(j)}_{\tau_{w_0},\nabla_{\mathrm{alg}}}))\otimes\mathbb{F})\to\mathrm{Tor}_1(\mathbb{F},(\widetilde{S}/\widetilde{I}^{(j)}_{\tau_{w_0},\nabla_\infty})\otimes\mathbb{F})\tag{B.16}$$

$$\mathrm{Tor}_1(\mathbb{F},(\widetilde{S}^{(j)}/(\widetilde{I}^{(j)}_{\tau_{\beta\alpha},\nabla_{\mathrm{alg}}}\cap\widetilde{I}^{(j)}_{\tau_{w_0},\nabla_{\mathrm{alg}}}))\otimes\mathbb{F})\to\mathrm{Tor}_1(\mathbb{F},(\widetilde{S}/\widetilde{I}^{(j)}_{\tau_{w_0},\nabla_\infty})\otimes\mathbb{F})\tag{B.17}$$

generates a spanning set for $\mathrm{Tor}_1(\mathbb{F},(S^{(j)}/I^{(j)}_{\tau_{w_0},\nabla_\infty})\otimes\mathbb{F})$, e.g. using [LLHM, Table 3], the set given by the images of the elements

$$\begin{aligned}&c_{21},\ c_{22},\ c_{23},\ c_{32},\ c_{33}\\&c_{13}d_{32}-c_{12}d^*_{33},\ c_{13}d_{31}-c_{11}d^*_{33},\ c_{13}c_{31}\\&(b-c)c_{21}d_{12}+(c-a)c_{11}d^*_{22},\ c_{31}\end{aligned}$$

(where $(a,b,c)\overset{\mathrm{def}}{=}s_j^{-1}(\mu_j+\eta_j)-(1,1,1)\equiv b_{\tau_{w_0}}$ modulo ϖ). We immediately see from row $\alpha\beta\alpha t_{\underline{1}},\alpha\beta t_{\underline{1}}$ in [LLHM, Table 6] that the elements $c_{32},c_{33},c_{13}c_{31},c_{13}d_{32}-c_{12}d^*_{33}$ are in the image of (B.16). Similarly the elements c_{21},c_{22},c_{23} are in the image of (B.17).

Writing

$$c_{13}d_{31}-c_{11}d^*_{33}=\underbrace{c_{13}d_{21}d_{32}-c_{12}d_{21}d^*_{33}-c_{13}d_{31}d^*_{22}+c_{11}d^*_{22}d^*_{33}}_{\in\text{ image of (B.17)}}-d_{21}\underbrace{(c_{13}d_{32}-c_{12}d^*_{33})}_{\in\text{ image of (B.16)}}$$

we conclude that $c_{13}d_{31}-c_{11}d^*_{33}$ is in the $\mathbb{F}$-span of the union of the images of (B.16),(B.17).

Similarly,

$$\begin{aligned}(b-c)c_{21}d_{12}+(c-a)c_{11}d^*_{22}=&\underbrace{(b-c)c_{12}d_{21}+\overline{x}c_{11}c_{22}-(a-c)c_{11}d^*_{22}-(b-c)c_{22}d^*_{11}}_{\in\text{ image of (B.16)}}+\\&-(\overline{x}c_{11}-(b-c)d^*_{11})\underbrace{c_{22}}_{\in\text{ image of (B.17)}}\end{aligned}$$

so that $(b-c)c_{21}d_{12}+(c-a)c_{11}d^*_{22}$ is in the $\mathbb{F}$-span of the union of the images of (B.16),(B.17).

Finally, note that $c_{22}(d_{21}d_{32}),c_{22}(d_{21}c_{32}),c_{22}(d_{31}d^*_{22}),c_{21}(d_{32}d^*_{22})\in I^{(j)}_{\tau_{w_0},\nabla_{\mathrm{alg}}}\cdot\mathfrak{m}_{\widetilde{S}^{(j)}}$ so that the last equation in row $\alpha\beta\alpha t_{\underline{1}},\alpha\beta t_{\underline{1}}$ in [LLHM, Table 6] is sent by the map (B.16) to $(a-c+1)c_{31}(d^*_{22})^2$ and in particular c_{31} is in the image of the map (B.16). ☐

Complements in the proof of [LLHM]*, Lemma 3.33.* The argumen is similar to that for [LLHM, Lemma 3.30]. Consider the natural maps

$$\mathrm{Tor}_1^S(\mathbb{F},(\widetilde{S}/(\widetilde{I}_{\tau_{\alpha t_{w_0(\eta)}},\nabla_\infty}\cap\widetilde{I}_{\tau_{t_{w_0(\eta)}},\nabla_\infty}))\otimes\mathbb{F})\to\mathrm{Tor}_1^S(\mathbb{F},(\widetilde{S}/\widetilde{I}_{\tau_{t_{w_0(\eta)}},\nabla_\infty})\otimes\mathbb{F})\tag{B.18}$$

$$\mathrm{Tor}_1^S(\mathbb{F},(\widetilde{S}/(\widetilde{I}_{\tau_{t_{w_0(\eta)}},\nabla_\infty}\cap\widetilde{I}_{\tau_{\beta t_{w_0(\eta)}},\nabla_\infty}))\otimes\mathbb{F})\to\mathrm{Tor}_1^S(\mathbb{F},(\widetilde{S}/\widetilde{I}_{\tau_{t_{w_0(\eta)}},\nabla_\infty})\otimes\mathbb{F}).\tag{B.19}$$

A spanning set for $\mathrm{Tor}_1^S(\mathbb{F},(\widetilde{S}/\widetilde{I}_{\tau_{t_{w_0(\eta)}},\nabla_\infty})\otimes\mathbb{F})$ (using [LLHM, Table 7]) is given by the images of the elements

$$d_{21},\ d_{32},\ c_{32},\ e_{33},\ c_{33},\ d_{32},\ e_{23},\ c_{22}$$
$$(a-c)e_{23}d_{22}^*-(a-b)c_{12}c_{23}.$$

We immediately see from row $t_{w_0(\eta)},t_{w_0(\eta)}\alpha$ in [LLHM, Table 7] that the elements d_{32}, c_{32}, e_{33}, c_{33}, d_{32}, are in the image of (B.18), and, from row $t_{w_0(\eta)},t_{w_0(\eta)}\beta$, that the element d_{21} is in the image of (B.19).

Moreover, noting that $c_{12}d_{21},e_{13}d_{21},c_{13}c_{22},c_{12}e_{23}\in\mathfrak{m}_{S^{(j)}}\cdot I^{(j)}_{\tau_{t_{w_0(\eta)}},\nabla_{\mathrm{alg}}}$ we conclude that (B.18) maps the elements $c_{12}d_{21}-c_{22}d_{11}^*$, $e_{13}d_{21}-e_{23}d_{11}^*$ and $(a-b)(c_{13}c_{22}-c_{12}c_{23})-\overline{x}c_{12}e_{23}+(a-c)e_{23}d_{22}^*$ to $c_{22}d_{11}^*$, $e_{23}d_{11}^*$ and $(a-c)e_{23}d_{22}^*-(a-b)c_{12}c_{23}$ respectively. □

Complements in the proof of [LLHM]*, Lemma 3.35.* We check that the union of the images of the canonical maps

$$\mathrm{Tor}_1^S\left(\mathbb{F},S/\big(\widetilde{I}^{(j)}_{\tau_{\alpha\beta},\nabla_{\mathrm{alg}}}\cap\widetilde{I}^{(j)}_{\tau_{w_0},\nabla_\infty},p\big)\cap\big(\widetilde{I}^{(j)}_{\tau_{w_0},\nabla_{\mathrm{alg}}}\cap\widetilde{I}^{(j)}_{\tau_{\beta\alpha},\nabla_\infty},p\big)\right)\to\mathrm{Tor}_1^S(\mathbb{F},S/I^{(j)}_\Lambda)\tag{B.20}$$

$$\mathrm{Tor}_1^S(\mathbb{F},S/\big(\widetilde{I}^{(j)}_{\tau_{\mathrm{id}},\nabla_{\mathrm{alg}}},p)\big)\to\mathrm{Tor}_1^S(\mathbb{F},S/I^{(j)}_\Lambda)\tag{B.21}$$

generates a spanning set for the target, i.e. by [LLHM, Lemma 3.34], the set given by the image of the elements $c_{33},d_{32}c_{23}-c_{22}d_{33}^*,c_{22},c_{11}d_{33}^*-c_{13}d_{31}$ of $I^{(j)}_\Lambda$. From the last row of [LLHM, Table 6] we immediately see that the elements $c_{33},d_{32}c_{23}-c_{22}d_{33}^*$ are in the image of the map (B.20). Moreover, by [LLHM, Table 4], the image of the map (B.21) contains the elements

$$(a-c-1)(c_{23}d_{32}-c_{33}d_{22}^*)-(a-b-1)c_{22}d_{33}^*$$
$$(a-b)(c_{13}d_{31}-c_{11}d_{33}^*)-(b-c-1)c_{33}d_{11}^*.$$

In particular, as $a-b\neq 0\neq b-c$, the union of the images of (B.20),(B.21) contains the elements $c_{13}d_{31}-c_{11}d_{33}^*$ and c_{22}. □

Complements in the proof of [LLHM]*, Lemma 3.37.* We check that the union of the images of the canonical maps

(B.22)
$$\mathrm{Tor}_1^{S^{(j)}}(\mathbb{F},S^{(j)}/\big(\widetilde{I}^{(j)}_{\tau_{t_{w_0(\eta)}\alpha},\nabla_\infty}\cap\widetilde{I}^{(j)}_{\tau_{t_{w_0(\eta)}},\nabla_\infty},p\big)\cap\big(\widetilde{I}_{\tau_{t_{w_0(\eta)}},\nabla_\infty}\cap\widetilde{I}_{\tau_{t_{w_0(\eta)}\beta},\nabla_\infty},p\big)\to\mathrm{Tor}_1^{S^{(j)}}(\mathbb{F},S^{(j)}/I^{(j)}_\Lambda)$$

$$\mathrm{Tor}_1^{S^{(j)}}(\mathbb{F},(\widetilde{S}/\widetilde{I}_{\tau_{t_{w_0(\eta)}w_0},\nabla_\infty})\otimes\mathbb{F})\to\mathrm{Tor}_1^{S^{(j)}}(\mathbb{F},S^{(j)}/I^{(j)}_\Lambda)\tag{B.23}$$

is a spanning set for the target. By [LLHM, Lemma 3.34] a spanning set for the target is given by

$$c_{32},e_{33},d_{31},d_{21}d_{32},\tag{B.24}$$
$$e_{23},(a-b)c_{12}c_{23}-(a-c)e_{13}d_{22}^*,c_{33}\tag{B.25}$$
$$c_{23}d_{32},c_{22},c_{12}d_{21}.\tag{B.26}$$

By the last row in [LLHM, Table 7] the elements in (B.25) are immediately checked to be in the image of (B.22). By row $t_{w_0(\eta)}w_0$ in [LLHM, Table 5], and noting further that $c_{13}c_{22},c_{13}d_{31}\in\mathfrak{m}_{S^{(j)}}I^{(j)}_\Lambda$ we immediately see that the elements in (B.25) are in the image of (B.23).

As $c_{23}d_{32}-c_{33}d_{22}^*$ is in the image of (B.22) by the last row of [LLHM, Table 7], we conclude from the above that $c_{23}d_{32}$ is in the linear span of the union of the images of (B.22) and (B.23). Moreover, as $(c-a-1)(c_{23}d_{32}-c_{33}d_{22}^*)+(a-b)c_{22}d_{33}^*$ is in the image of (B.22) by row $t_{w_0(\eta)}w_0$

in [LLHM, Table 5], we conclude by the above c_{22} is also in the linear span of the union of the images of (B.22) and (B.23). Finally, as $c_{12}d_{21} - c_{22}d_1^*$ is in the image of (B.22) by the last row of [LLHM, Table 7], we conclude from the above that $c_{12}d_{21}$ is also in the linear span of the union of the images of (B.22) and (B.23). □

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