K_1-INVARIANTS IN THE MOD p COHOMOLOGY OF $U(3)$ ARITHMETIC MANIFOLDS

DANIEL LE, BAO V. LE HUNG, AND STEFANO MORRA

ABSTRACT. Let F/F^+ be a CM extension and $H_{/F^+}$ a definite unitary group in three variables that splits over F. We describe Hecke isotypic components of mod p algebraic modular forms on H at first principal congruence level at p and "minimal" level away from p in terms of the restrictions of the associated Galois representation to decomposition groups at p when these restrictions are tame and sufficiently generic. This confirms an expectation of local-global compatibility in the mod p Langlands program. To prove our result, we develop a local model theory for multitype deformation rings and new methods to work with patched modules that are not free over their scheme-theoretic support.

CONTENTS

1. INTRODUCTION

1.1. **The main result.** In this paper, we describe some Hecke isotypic components of spaces of algebraic modular forms at first principal congruence level for definite unitary groups in three variables. We begin by motivating this problem. Let p be a prime and $\mathbb{F}/\mathbb{F}_p$ be a (sufficiently large) finite extension. Let F/F^+ be a CM extension for which p is inert in F^+ and splits in F. Let n be a positive integer and $H_{/F^+}$ be an outer form of GL_n which splits over F and is definite at infinity i.e. $H(F^+\otimes_{\mathbb{Q}}\mathbb{R})$ is compact. Let $U^p\subset H(\mathbb{A}^{\infty p}_{F^+})$ be a compact open subgroup. We define a space of mod p modular forms

$$S(U^p,\mathbb{F})\stackrel{\mathrm{def}}{=}\{f: H(F^+)\backslash H(\mathbb{A}^{\infty}_{F^+})/U^p\to\mathbb{F}\text{ locally constant}\}$$

at infinite level at p. This has a faithful action of a Hecke algebra $\mathbb{T}=\mathbb{F}[T^{(j)}_{\widetilde{v}}]_{1\le j\le n, v\in\mathcal{P}}$ over $\mathbb{F}$ at "good places" $v\in\mathcal{P}$ where $\mathcal{P}$ is a cofinite subset of the places of F^+ that split in F (the indexing of the Hecke operators depends on a choice of a place $\widetilde{v}$ of F lying over v). The ring $\mathbb{T}$ is semilocal, and for a homomorphism $\alpha:\mathbb{T}\to\mathbb{F}$ with kernel $\mathfrak{m}$ there is a continuous semisimple representation

$\overline{r} : G_F \to \mathrm{GL}_n(\mathbb{F})$ whose conjugacy class is characterized by the equations

$$\det(xI_n - \overline{r}(\mathrm{Frob}_{\widetilde{v}})) = \sum_{i=0}^{n}(-1)^i(\mathbf{N}\widetilde{v})^{\binom{i}{2}}\alpha(T_{\widetilde{v}}^{(j)})x^{n-i}$$

for each $v \in \mathcal{P}$ (see [CHT08, Proposition 3.4.2]). Suppose from now on that $\overline{r}$ is irreducible. Motivated by the (classical) local Langlands correspondence and local-global compatibility, there is the following central conjecture in the mod p Langlands program.

Conjecture 1.1. *For a finite extension $K/\mathbb{Q}_p$, there is an injection*

$$\mathrm{LLC} : \{G_K \to \mathrm{GL}_n(\mathbb{F})\}_{/\cong} \hookrightarrow \{\textit{finite length smooth admissible } \mathrm{GL}_n(K)\textit{-representations}/\mathbb{F}\}_{/\cong}$$

such that for a place $w|p$ of F, $S(U^p, \mathbb{F})[\mathfrak{m}] \cong \mathrm{LLC}(\overline{r}|_{G_{F_w}})^{\oplus d}$ as $H(F_p^+) \cong \mathrm{GL}_n(F_w)$-representations for some positive integer $d = d(U^p)$.

This injection should be the atom with which one builds a deformation theoretic p-adic local Langlands correspondence that mediates Langlands reciprocity in p-adic families. Such an injection has been constructed for $n = 2$ and $K = \mathbb{Q}_p$, but the situation appears to be substantially more complicated if $n > 2$ or $K \neq \mathbb{Q}_p$. One reason is that there does not seem to be a simple classification of irreducible smooth admissible $\mathrm{GL}_n(K)$-representations, not to mention a (conjectural) local characterization of LLC or a simple (conjectural) description of the image of LLC. Given these difficulties, it is natural to study the space $S(U^p, \mathbb{F})[\mathfrak{m}]$ directly.

At present, the strongest evidence towards Conjecture 1.1 is a description of the constituents of the $\mathrm{GL}_n(\mathcal{O}_{F_w})$-socle of $S(U^p, \mathbb{F})[\mathfrak{m}]$ when $\overline{r}$ is tamely ramified and sufficiently generic at p [LLHLM23]. When $n = 2$, there is stronger evidence—the invariants of $\pi(\overline{r}) \overset{\mathrm{def}}{=} S(U^p, \mathbb{F})[\mathfrak{m}]$ under the pro-p Iwahori and first principal congruence subgroups can be described in terms of $\overline{\rho}|_{G_{F_w}}$ [EGS15, HW18, LMS22, Le] when $\overline{\rho}|_{G_{F_w}}$ is sufficiently generic, $\overline{r}$ satisfies a Taylor–Wiles hypothesis, and U^p is "minimal" (in particular, this implies $d = 1$). There are a number of reasons to single out these two compact open subgroups, even for general n. First, they are pro-p subgroups so that their invariants are necessarily nonzero. Second, their normalizers jointly generate the group $\mathrm{GL}_n(F_w)$ which is a central feature in the theory of coefficient systems on buildings [SS97, Pas04, BP13]. Finally, it is expected that the invariants under the first principal subgroup $U(p) \overset{\mathrm{def}}{=} \ker(\mathrm{GL}_n(\mathcal{O}_{F_w}) \to \mathrm{GL}_n(k_w))$ generate $\pi(\overline{r})$ (this is known for $n = 2$ and $\overline{\rho}$ sufficiently generic by [HW22, Theorem 1.6] and [BHH$^+$21, Theorem 1.3.8]). Far less is known when $n > 2$, and current evidence suggests that the situation is very complicated. When $n = 3$ and the level is "minimal", the authors and Levin described the invariants of $\pi(\overline{r})$ under the pro-p Iwahori subgroup in terms of $\overline{\rho}|_{G_{F_w}}$ when this representation is sufficiently generic [LLHLM20, LLHLMb]. In this paper, we build on these results to describe the invariants of $\pi(\overline{r})$ under the first principal congruence subgroup $U(p)$ when $n = 3$ and $\overline{r}$ is sufficiently generic and tamely ramified at p.

Theorem 1.1 (Theorem 6.14)**.** *Suppose that $n = 3$. Moreover, suppose that*

- *$\overline{r}$ satisfies a Taylor–Wiles hypothesis;*
- *$\overline{r}$ is tamely ramified and sufficiently generic at p; and*
- *U^p is "minimal".*

Then the $\mathrm{GL}_3(k_w)$-representation $\pi(\overline{r})^{U(p)}$ is uniquely determined by $\overline{r}|_{G_{F_w}}$.

Remark 1.2. (1) In fact, we show that $\pi(\overline{r})^{U(p)}$ is uniquely determined by the restriction $\overline{r}|_{I_{F_w}}$ to the inertial subgroup. We also explicitly determine the semisimplification of $\pi(\overline{r})^{U(p)}$ (see Theorem 6.13).

(2) The assumption that p is inert in F^+ can be replaced by an assumption that p is unramified in F^+.
(3) It seems likely that one could remove the hypothesis that $\overline{r}$ is tamely ramified at p by breaking the problem into cases based on the set of extremal weights [LLHLMb, LLHLMa].
(4) In stark contrast to $n = 2$, many constituents of $\operatorname{soc}\pi(\overline{r})^{U(p)}$ reappear beyond the socle. Theorem 6.14 thus has the flavor of characterizing a specific point in a (continuous) moduli of representations with the same discrete invariants. This does not seem to have been previously anticipated in the literature, even though the appearance of constituents of $\operatorname{soc}\pi(\overline{r})^{U(p)}$ beyond the socle was expected by F. Herzig (private communication).

[BHH+21] gives a number of refinements of Conjecture 1.1 and proves some of these when $n = 2$ in part by using results on invariants of $\pi(\overline{r})$. We expect that Theorem 1.1 will have similar applications, and we hope to return to this in future work.

1.2. **Methods.** The proof of Theorem 1.1 can be broken into three steps. Let $n = 3$ and U^p and $\overline{r}$ be as in Theorem 1.1. Let $\overline{\rho} \overset{\text{def}}{=} \overline{r}|_{G_{F_w}}$. It was shown in [LLHLM20] that the $\mathrm{GL}_3(\mathcal{O}_{F_w})$-socle of $\pi(\overline{r})$ is $\oplus_{\sigma\in W^?(\overline{\rho})}\sigma$ where $W^?(\overline{\rho})$ is an explicit set of irreducible $\mathrm{GL}_3(\mathcal{O}_{F_w})$-representations over $\mathbb{F}$ (equivalently irreducible $\mathrm{GL}_3(k_w)$-representations) predicted in [Her09, GHS18]. First one calculates the dimension of $\mathrm{Hom}_{\mathrm{GL}_3(k_w)}(P_\sigma, \pi(\overline{r})^{U(p)})$ for each $\sigma \in W^?(\overline{\rho})$ where P_σ denotes a $\mathrm{GL}_3(k_w)$-projective cover of σ. This gives the multiplicity of each $\sigma \in W^?(\overline{\rho})$ as a Jordan–Hölder factor of $\pi(\overline{r})^{U(p)}$. Next, we construct a certain $\mathrm{GL}_3(k_w)$-representation D_{m} depending on $\overline{r}|_{I_{F_w}}$ such that $\mathrm{JH}(D_{\mathrm{m}}) = W^?(\overline{\rho})$, D_{m} injects into $\pi(\overline{r})^{U(p)}$, and both $\pi(\overline{r})^{U(p)}$ and D_{m} contain each $\sigma \in W^?(\overline{\rho})$ as a Jordan–Hölder factor with the same multiplicity. Finally, a simple argument shows that $\pi(\overline{r})^{U(p)}$ is the unique maximal representation containing D_{m} such that they have the same $\mathrm{GL}_3(k_w)$-socle and contain each $\sigma \in W^?(\overline{\rho})$ as a Jordan–Hölder factor with the same multiplicity.

The most difficult step is calculating the dimensions of the multiplicity spaces $\mathrm{Hom}_{\mathrm{GL}_3(k_w)}(P_\sigma, \pi(\overline{r})^{U(p)})$ for $\sigma \in W^?(\overline{\rho})$. The basic idea, going back to Taylor and Wiles, is to replace the dual of $\pi(\overline{r})^{U(p)}$ with a space M_∞ (we caution that this is slightly different than the meaning of M_∞ later in the text) obtained by taking truncated Witt vectors as coefficients, adding auxiliary level (at Taylor–Wiles places), and then taking a noncanonical limit. This object, and more generally its multiplicity spaces (the "patched modules"), are maximal Cohen–Macaulay modules over local Galois deformation spaces with p-adic Hodge theory conditions. While these objects are very non-canonical, taking fiber at the closed point recovers their original versions. Thus, dimensions of mod p multiplicity spaces can be computed as minimal number of generators of patched modules. Diamond and Fujiwara observed that when a patched module is supported over a local deformation space that happens to be regular, then it is free by theorems of Serre and Auslander–Buchsbaum. This shows a mod p multiplicity agrees with a characteristic 0 multiplicity, which are often known by automorphic methods. In [EGS15], Emerton, Gee and Savitt pioneered methods for studying mod p multiplicity questions when the relevant deformation spaces are not regular by taking advantage of the fact that M_∞ is a projective $W(\mathbb{F})[\mathrm{GL}_3(k_w)]$-module i.e. $M_\infty(-) \overset{\text{def}}{=} \mathrm{Hom}_{\mathrm{GL}_3(k_w)}(M_\infty, (-)^\vee)^\vee$ defines an exact functor where $(-)^\vee$ denotes Pontrjagin duality. This implies:

(1) (Nakayama gluing) If $V \subset W$ such that $M_\infty(V) \subset \mathfrak{m}M_\infty(W)$, then $M_\infty(W)$ and $M_\infty(W/V)$ have the same minimal number of generators.
(2) (Fiber product gluing) A quasi-isomorphism $V \to P^\bullet$ of complexes of $W(\mathbb{F})[\mathrm{GL}_3(k_w)]$-modules gives a quasi-isomorphism $M_\infty(V) \to M_\infty(P^\bullet)$.

By combining these two properties, one has a potential strategy to compute the minimal number of generators of complicated patched modules from simpler ones. We will first illustrate the strategy for $n = 2$ and then describe what changes for $n = 3$.

Suppose that $n = 2$, that $\overline{\rho}$ is tame and sufficiently generic, and that $\sigma \in W^?(\overline{\rho})$. Let $\overline{P}_\sigma \stackrel{\text{def}}{=} P_\sigma/\mathrm{rad}^2(P_\sigma)$ where $\mathrm{rad}^\bullet$ denotes the radical filtration. Then $\overline{P}_\sigma$ sits in a short exact sequence

$$0 \to \overline{P}_\sigma \to \bigoplus_i Q_i \to (\bigoplus_i \sigma)/\Delta\sigma \to 0,$$

where Q_i denote the different length two quotients of $\overline{P}_\sigma$ and $\Delta\sigma \subset \bigoplus_i \sigma$ is the diagonally embedded copy. Using the Diamond–Fujiwara trick, one shows $M_\infty(Q_i)$ is cyclic. This allows us to compute $M_\infty(Q_i)$ and check that $M_\infty(\overline{P}_\sigma)$ is also cyclic via fiber product gluing. Finally, one shows that $M_\infty(P_\sigma/\mathrm{rad}^m P_\sigma)$ is cyclic for all m inductively using Nakayama gluing, the crucial ingredient being that $M_\infty(\mathrm{rad}^{m-1} P_\sigma/\mathrm{rad}^{m+1} P_\sigma)$ and $M_\infty(\mathrm{rad}^{m-1} P_\sigma/\mathrm{rad}^m P_\sigma)$ have the same minimal number of generators. This last fact follows from the cyclicity of $M_\infty(\overline{P}_\kappa)$ for *all* $\kappa \in W^?(\overline{\rho}) \cap \mathrm{JH}(\mathrm{rad}^{m-1} P_\sigma/\mathrm{rad}^m P_\sigma)$, and the following "covering" property of $\mathrm{rad}^{m-1} P_\sigma/\mathrm{rad}^m P_\sigma$ with respect to $W^?(\overline{\rho})$:

Property 1.3. The cokernel of the evaluation map

$$\bigoplus_{\kappa \in W^?(\overline{\rho}) \cap \mathrm{JH}(\mathrm{rad}^{m-1} P_\sigma/\mathrm{rad}^m P_\sigma)} P_\kappa \otimes \mathrm{Hom}_{\mathrm{GL}_3(k_w)}(P_\kappa, \mathrm{rad}^{m-1} P_\sigma/\mathrm{rad}^{m+1} P_\sigma) \to \mathrm{rad}^{m-1} P_\sigma/\mathrm{rad}^{m+1} P_\sigma$$

has no Jordan-Holder factors in $W^?(\overline{\rho})$.

There are two major obstacles to carrying out an analogous strategy when $n > 2$. First, one would like to show an analogue of Property 1.3. Unfortunately, the structure of projective $\mathbb{F}[\mathrm{GL}_n(k_w)]$-modules is poorly understood. While it is known that generic projective indecomposable $\mathbb{F}[\mathrm{GL}_n(k_w)]$-modules P_σ are the restrictions of certain tilting modules for (a product of) $\mathrm{GL}_{n/\mathbb{F}}$, describing the submodule structure of such tilting modules appears to be hopeless in general. Fortunately, when $n = 3$, tilting modules for $\mathrm{GL}_{3/\mathbb{F}}$ are fairly well-understood (in part because Ext^1 groups for simple modules have dimension at most one) [BDM15], from which we are able to deduce Property 1.3 (Proposition 4.11). Much of the first part of §4 is devoted to these considerations.

The second obstacle is that $M_\infty(P_\sigma/\mathrm{rad}^2(P_\sigma))$ is often not cyclic, which prevents us from Nakayama gluing as when $n = 2$. The remedy is to instead consider larger quotients of P_σ. We first elaborate when $F^+ = \mathbb{Q}$ as the general case involves substantial complication in a different direction. Here, the highest weight of a Serre weight is in one of two p-restricted alcoves, an upper alcove and a lower alcove. Moreover, the radical layers of a projective indecomposable $\mathbb{F}[\mathrm{GL}_3(\mathbb{F}_p)]$-module contain Serre weights in a single p-alcove of alternating type. One can compute $M_\infty(P_\sigma/\mathrm{rad}^2(P_\sigma))$ for $\sigma \in W^?(\overline{\rho})$ using fiber product gluing and see that it is not cyclic exactly when σ is in the lower alcove, which precludes the verbatim inductive argument in the $n = 2$ case. Fortunately, it turns out that $M_\infty(P_\sigma/\mathrm{rad}^3(P_\sigma))$ is cyclic when σ is in the upper alcove. Then a modified inductive argument using Property 1.3 and Nakayama gluing along upper alcove layers shows that $M_\infty(P_\sigma)$ and $M_\infty(P_\sigma/\mathrm{rad}^2(P_\sigma))$ are minimally generated by the same number of elements.

To show that $M_\infty(P_\sigma/\mathrm{rad}^3(P_\sigma))$ is cyclic, we decompose it as an iterated fiber product. Since some Jordan–Hölder factors appear with multiplicity greater than one, $P_\sigma/\mathrm{rad}^3(P_\sigma)$ cannot be presented in terms of the mod p reductions of a single irreducible $\mathrm{GL}_3(\mathbb{F}_p)$-representation. Instead, we use fiber products of lattices in several different Deligne–Lusztig representations. In parallel, $M_\infty(P_\sigma/\mathrm{rad}^3(P_\sigma))$ is not supported on a single tame type Galois deformation space. This forces us to work in some highly singular "multitype" deformation rings, which first appeared in print in [Le19] and were later used in [BHH+23], but were first developed in the course of this project. We develop a local model theory for such multitype deformation rings building on [LLHLM23, LLHLMb], cf. §3.1. With this in hand, we deduce the desired cyclicity from the transversality of certain scheme theoretic intersections in these models.

The $F^+ \neq \mathbb{Q}$ case brings many further complications. Now, our arguments require fiber product gluing situations involving noncyclic patched modules, which makes it hard to control the end result. However, we give new criteria for the minimal number of generators of a fiber product to be well-behaved in terms of the transversality of (only) infinitesimal properties of its factors. These criteria are essential because while the local models for single tame type deformation rings have tensor product structures over embeddings $F_w \hookrightarrow W(\mathbb{F})[1/p]$, the local models for multitype stacks do not seem to have such product structures. A key observation here is that the product structure persists infinitesimally, yielding the sought-after transversality statements.

Finally, we come to the construction of D_{m}, which is the minimal subrepresentation of $\pi(\overline{r})^{U(p)}$ containing all Jordan–Hölder factors in $W^?(\overline{\rho})$. When $n = 2$, cyclicity of certain patched modules implies that $D_{\mathrm{m}} \cong \oplus_{\sigma \in W^?(\overline{\rho})}\sigma$—in particular it is determined uniquely from multiplicity information. When $n = 3$, there is a moduli of representations with the same $\mathbb{F}[\mathrm{GL}_3(k_w)]$-socle and multiset of Jordan–Hölder factors as D_{m}, but only D_{m} injects into $\pi(\overline{r})^{U(p)}$. To single out this point, we give "coordinates" on this moduli space using categories of lattices in various Deligne–Lusztig representations. Then we compute $M_\infty(D)$ for $\mathbb{F}[\mathrm{GL}_3(k_w)]$-modules D in (a part of) this moduli space and characterize $M_\infty(D_{\mathrm{m}})$ among them.

Remark 1.4. Unlike when $n = 2$, our approach falls short of giving a purely local characterization of $M_\infty(P_\sigma)$ in general, since there is a non-trivial moduli of non-cyclic Cohen–Macaulay modules with prescribed support. In principle, one could attempt such a characterization by presenting indecomposable projective $W(\mathbb{F})[\mathrm{GL}_3(k_w)]$-modules $\widetilde{P}_\sigma$ in terms of sums of lattices in Deligne–Lusztig representations and use the fact that patched modules for such lattices were computed in [LLHLM20]. However, it seems challenging to find explicit presentations and worse, the noncyclicity of some patched lattices makes it unclear how to compute the effect of patching on the *maps* that arise in such a presentation.

1.3. **Outline.** §2 is a brief summary of background on tame inertial types, Deligne–Lusztig representations, the inertial local Langlands, and Serre weights largely following [LLHLM20, LLHLMb]. §3 and §4 collect a number of technical results needed for the proof of the main result which spans §5 and §6. We defer several tedious ideal intersection computations to [LLHM], which can be ignored by a reader willing to take these computations on faith. We urge the reader skip §3 and §4 on a first reading, only referring back as needed. §3.1 develops a local model theory for some multitype deformation rings building on [LLHLM23, LLHLMb]. §3.2 collects a number of commutative algebra lemmas. §3.4 contains technical computations with deformation rings used in §5 and 6. §4 establishes results in modular and integral representation theory building on [BDM15, LLHLM20] needed to prove the main result. §5 computes multiplicities of Serre weights in an axiomatic context by combining results in §3 and 4. §6 proves Theorem 1.1 first in an axiomatic context and then in a global context.

Acknowledgements. This work has been developed over several years starting in the fall of 2016. We apologize for the long delay in writing up this. The authors thank the various institutions where they have been hosted while working on this project (University of Toronto, Institute for Advanced Studies, Université de Montpellier, Università degli Studi di Padova, the Hausdorff Institute of Mathematics) and thank Florian Herzig for his comments on an earlier draft of this paper. B.LH.

acknowledges support from the National Science Foundation under grant Nos. DMS-1128155, DMS-1802037, DMS-2302619 and the Alfred P. Sloan Foundation. D.L. was supported by the National Science Foundation under agreements Nos. DMS-1128155, DMS-1703182, and DMS-2302623, an AMS-Simons travel grant, and a start-up grant from Purdue University. S.M. is member of the Institut Universitaire de France and acknowledges its support.

1.4. **Notation.** For any given field K we fix once and for all a separable closure $\overline{K}$ and define $G_K \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{K}/K)$. If K is a nonarchimedean local field, we let $I_K \subset G_K$ denote the inertial subgroup. We fix a prime p. Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Let $\overline{\mathbb{F}}_p$ denote its residue field. Let $\mathbb{F} \subset \overline{\mathbb{F}}_p$ be a finite subfield which we will assume is sufficiently large for our purposes. Let $\mathcal{O}$ be $W(\mathbb{F})$ and E be the fraction field of $\mathcal{O}$. Then we have a natural embedding $E \subset \overline{\mathbb{Q}}_p$.

Let $G \stackrel{\text{def}}{=} \mathrm{GL}_{3/\mathbb{Z}}$, $B \subset G$ the subgroup of upper triangular matrices, $T \subset B$ the split torus of diagonal matrices and $Z \subset T$ the center of G. Let $\Phi^+ \subset \Phi$ denote the subset of positive roots in the set of roots for (G, B, T). Let $X^*(T)$ be the group of characters of T which we identify with $\mathbb{Z}^3$ in the standard way.

We write W (resp. W_a, resp. $\widetilde{W}$) for the Weyl group (resp. the affine Weyl group, resp. the extended affine Weyl group) of G. If $\Lambda_R \subset X^*(T)$ denotes the root lattice for G we then have

$$W_a = \Lambda_R \rtimes W, \quad \widetilde{W} = X^*(T) \rtimes W$$

and use the notation $t_\nu \in \widetilde{W}$ to denote the image of $\nu \in X^*(T)$. Let $\eta = (2,1,0) \in X^*(T)$. We define the *p-dot action* by $t_\lambda w \cdot \mu = p\lambda + w(\mu + \eta) - \eta$. Let w_0 denote the longest element in W and define $\widetilde{w}_h \stackrel{\text{def}}{=} w_0 t_{-\eta}$.

Let $\langle\ ,\ \rangle$ denote the duality pairing on $X^*(T) \times X_*(T)$. A weight $\lambda \in X^*(T)$ is *dominant* if $0 \le \langle \lambda, \alpha^\vee \rangle$ for all simple root $\alpha \in \Phi$. Set $X^0(T)$ to be the subgroup consisting $\lambda \in X^*(T)$ such that $\langle \lambda, \alpha^\vee \rangle = 0$ for all $\alpha \in \Phi$, and $X_1(T)$ to be the set of $\lambda \in X^*(T)$ such that $0 \le \langle \lambda, \alpha^\vee \rangle < p$ for all simple root $\alpha \in \Phi$.

A p-alcove is a connected component of

$$X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \ \setminus \ \Big(\bigcup_{(\alpha,n)} \{\lambda \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \ : \ \langle \lambda + \eta, \alpha^\vee \rangle = np\} \Big)$$

where (α, n) runs over $\Phi^+ \times \mathbb{Z}$. A p-alcove C is p-restricted (resp. dominant) if $0 < \langle \lambda + \eta, \alpha^\vee \rangle < p$ (resp. $0 < \langle \lambda + \eta, \alpha^\vee \rangle$) for all simple roots $\alpha \in \Phi$ and $\lambda \in C$. The group W_a acts simply transitively on the set of alcoves and throughout this paper the dot action of $\widetilde{W}$ on the alcoves will always be the p-dot action. We let $C_0 \subset X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ denote the dominant base alcove (i.e. the alcove defined by $0 < \langle \lambda + \eta, \alpha^\vee \rangle < p$ for all $\alpha \in \Phi^+$), and set

$$\widetilde{W}^+ \stackrel{\text{def}}{=} \{\widetilde{w} \in \widetilde{W} : \widetilde{w} \cdot C_0 \text{ is dominant}\}, \quad \widetilde{W}_1^+ \stackrel{\text{def}}{=} \{\widetilde{w} \in \widetilde{W}^+ : \widetilde{w} \cdot C_0 \text{ is } p\text{-restricted}\}.$$

We sometimes refer to C_0 as the *lower alcove* and $\widetilde{w}_h \cdot C_0$ as the *upper alcove*. Given $N \in \mathbb{N}$ we say that $\lambda \in X^*(T)$ is *N-deep in alcove* C_0 if $N < \langle \lambda + \eta, \alpha^\vee \rangle < p - N$ for all $\alpha \in \Phi^+$.

Let $\mathcal{O}_p$ be a finite étale $\mathbb{Z}_p$-algebra and fix an isomorphism $\mathcal{O}_p \cong \prod_{v \in S_p} \mathcal{O}_v$ where S_p is a finite set and $\mathcal{O}_v$ is the ring of integers of a finite unramified extension F_v^+ of $\mathbb{Q}_p$. We define $G_0 \stackrel{\text{def}}{=} \mathrm{Res}_{\mathcal{O}_p/\mathbb{Z}_p} G_{/\mathcal{O}_p}$, and similarly $T_0 \subset B_0 \subset G_0$. We assume that $\mathcal{O}$ contains the image of any ring homomorphism $\mathcal{O}_p \to \overline{\mathbb{Z}}_p$ and write $\mathcal{J} \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_p, \mathcal{O})$. We define $\underline{G} \stackrel{\text{def}}{=} (G_0)_{/\mathcal{O}}$, and similarly $\underline{T} \subset \underline{B} \subset \underline{G}$. We use underlined notations for the objects introduced above for G and now relative to $\underline{G}$ (hence $\underline{\Phi}^+ \subset \underline{\Phi}$, $\underline{W}$, $\underline{W}_a$, $\underline{\widetilde{W}}$, $\underline{\widetilde{W}}^+$, $\underline{\widetilde{W}}_1^+$, $\underline{C}_0$ and the like) and we have a notion of N-deepness in alcove $\underline{C}_0$ for elements of $X^*(\underline{T})$. The natural isomorphism $\underline{G} \cong G^{\mathcal{J}}_{/\mathcal{O}}$ induces compatible isomorphisms $X^*(\underline{T}) = X^*(T)^{\mathcal{J}}$ and the like. Given an element $j \in \mathcal{J}$, we use a subscript notation to denote j-components obtained from the isomorphism $\underline{G} \cong G^{\mathcal{J}}_{/\mathcal{O}}$ (e.g. given $\widetilde{w} \in \underline{\widetilde{W}}$ we write $\widetilde{w}_j$ to denote its j-th component via the induced isomorphism $\underline{\widetilde{W}} \cong \widetilde{W}^{\mathcal{J}}$).

For sake of readability, we abuse notation and still write w_0 to denote the longest element in $\underline{W}$, and $\eta \in X^*(\underline{T})$ for the element $\sum_{j\in\mathcal{J}}(2,1,0)_j$. The meaning of w_0, η and $\widetilde{w}_h \stackrel{\text{def}}{=} w_0 t_{-\eta}$ should be clear from the context.

The absolute Frobenius automorphism on $\mathcal{O}_p/p$ lifts canonically to an automorphism φ of $\mathcal{O}_p$. We define an automorphism π of $X^*(\underline{T})$ by $\pi(\lambda)_\sigma = \lambda_{\sigma\circ\varphi^{-1}}$ for all $\lambda \in X^*(\underline{T})$ and $\sigma : \mathcal{O}_p \to \mathcal{O}$. We similarly define an automorphism π of $\underline{W}$ and $\underline{\widetilde{W}}$.

When $S_p = \{v\}$ is a singleton we simplify notation and let $K \stackrel{\text{def}}{=} F^+_p$, $f \stackrel{\text{def}}{=} [K : \mathbb{Q}_p]$, ring of integers $\mathcal{O}_K$, residue field k. Let $W(k)$ be ring of Witt vectors of k, which is also the ring of integers of K. We denote the arithmetic Frobenius automorphism on $W(k)$ by φ (it acts as raising to p-th power on the residue field).

Recall that we fixed a separable closure $\overline{K}$ of K. We choose $\pi \in \overline{K}$ such that $\pi^{p^f-1} = -p$ and let $\omega_K : G_K \to \mathcal{O}_K^\times$ be the character defined by $g(\pi) = \omega_K(g)\pi$, which is independent of the choice of π. We fix an embedding $\sigma_0 : K \hookrightarrow E$ and define $\sigma_j = \sigma_0 \circ \varphi^{-j}$, which identifies $\mathcal{J} = \mathrm{Hom}(k, \mathbb{F}) = \mathrm{Hom}_{\mathbb{Q}_p}(K, E)$ with $\mathbb{Z}/f\mathbb{Z}$. We write $\omega_f : G_K \to \mathcal{O}^\times$ for the character $\sigma_0 \circ \omega_K$.

Let ε denote the p-adic cyclotomic character. We normalize the definitions of labelled Hodge–Tate weights so that ε has Hodge–Tate weight $\{1\}$ for every embedding $K \hookrightarrow E$ (this convention is opposite of that of [EG23, CEG+16] and agrees with that of [GHS18]), and the functor from potentially semistable representation to Weil–Deligne representations is *covariant*.

A potentially semistable representation $\rho : G_K \to \mathrm{GL}_3(E)$ has type (μ, τ) if ρ has labeled Hodge–Tate weights $\mu \in X^*(\underline{T})$ (note that this differs from the conventions of [GHS18] via a shift by η) and the restriction to I_K of the Weil-Deligne representation attached to ρ is isomorphic to τ.

Let Γ be a group. If V is a finite length Γ-representation, we let $\mathrm{JH}(V)$ be the (finite) set of Jordan–Hölder factors of V. If V° is a finite $\mathcal{O}$-module with a Γ-action, we write $\overline{V}^\circ$ for the Γ-representation $V^\circ \otimes_{\mathcal{O}} \mathbb{F}$ over $\mathbb{F}$.

2. Preliminaries

2.1. **Tame inertial types, inertial local Langlands, Serre weights.** Unless otherwise stated, we assume throughout this section that $S_p = \{v\}$. We write $\mathcal{O}_p = \mathcal{O}_K$ (the ring of integers of a finite unramified extension K of $\mathbb{Q}_p$ of degree f) and let $G_0 = \mathrm{Res}_{\mathcal{O}_K/\mathbb{Z}_p} G_{/\mathcal{O}_K}$. We drop subscripts v from notation and we identify $\mathcal{J} = \mathrm{Hom}_{\mathbb{Q}_p}(K, E)$ with $\mathbb{Z}/f\mathbb{Z}$ via $\sigma_j \stackrel{\text{def}}{=} \sigma_0 \circ \varphi^{-j} \mapsto j$.

2.1.1. *Tame inertial types, Deligne–Lusztig representations and inertial local Langlands.* An inertial type (for K, over E) is the $\mathrm{GL}_3(E)$-conjugacy class of an homomorphism $\tau : I_K \to \mathrm{GL}_3(E)$ with open kernel and which extends to an homomorphism $W_K \to \mathrm{GL}_3(E)$. We will identify a tame inertial type with a fixed choice of a representative in its $\mathrm{GL}_3(E)$-conjugacy class, and say that the type is *tame* if the homomorphism τ factors through the tame quotient of I_K (this notion is independent of the choice of the representative in the conjugacy class). Given $s \in \underline{W}$, $\mu \in X^*(\underline{T}) \cap \underline{C}_0$ and $j' \in \{0, \dots, 6f-1\}$ we set $\boldsymbol{\alpha}'_{j'} \stackrel{\text{def}}{=} s^{-1}_{6f-1} s^{-1}_{6f-2} \cdots s^{-1}_{6f-j'}(\mu_{6f-j'} + \eta_{6f-j'}) \in X^*(T)$, where the subscripts are taken modulo f, and let $\mathbf{a}'^{(j')}$ be $\sum_{i=0}^{6f-1} \boldsymbol{\alpha}'_{-j+i} p^i \in X^*(T)$. We thus define the tame inertial type $\tau(s, \mu+\eta)$ to be $(\omega_{6f})^{\mathbf{a}'^{(0)}}$, and say that (s, μ) is a *lowest alcove presentation* for $\tau(s,\mu+\eta)$. Given $N\in\mathbb{N}$ we say that a tame inertial type τ is N-generic if there exists a pair (s,μ) as above such that $\tau\cong\tau(s,\mu+\eta)$ and μ is N-deep in alcove $\underline{C}_0$. In this case, we say that τ has an N-generic lowest alcove presentation (s,μ) with associated element $\widetilde{w}(\tau)\overset{\text{def}}{=}t_{\mu+\eta}s\in\underline{\widetilde{W}}$.

Replacing E with $\mathbb{F}$ in the preceding paragraph we have the notion of inertial $\mathbb{F}$-type, tame inertial $\mathbb{F}$-type (typically denoted with the overlined notation $\overline{\tau}$), lowest alcove presentation and associated element $\widetilde{w}(\overline{\tau})$ for a tame inertial $\mathbb{F}$-type $\overline{\tau}$, and N-genericity for $N\in\mathbb{N}$.

Given a pair $(s,\mu-\eta)\in\underline{W}\times X^*(\underline{T})\cap\underline{C}_0$ we can attach to it a virtual $G_0(\mathbb{F}_p)$-representation $R_s(\mu)$ over E by [GHS18, Definition 9.2.2] and the paragraph above *loc. cit.*, where in *loc. cit.* the representation $R_s(\mu)$ is denoted by $R(s,\mu)$. If $\mu-\eta$ is moreover 1-deep in $\underline{C}_0$ then $R_s(\mu)$ is an irreducible representation of $G_0(\mathbb{F}_p)$ and the pair (s,μ) is said to be a lowest alcove presentation of $R_s(\mu)$. Finally, given $N\in\mathbb{N}$ we say that a Deligne–Lusztig representation R is N-generic if there exist $(s,\mu-\eta)\in\underline{W}\times X^*(\underline{T})\cap\underline{C}_0$ such that $\mu-\eta$ is N-deep in $\underline{C}_0$ and $R\cong R_s(\mu)$, in which case $(s,\mu-\eta)$ is a lowest alcove presentation for R with corresponding element $\widetilde{w}(R)\overset{\text{def}}{=}t_\mu s\in\underline{\widetilde{W}}$.

By [CEG+16, Theorem 3.7] we can attach to a tame inertial type $\tau: I_K\to\mathrm{GL}_3(E)$ an irreducible $G_0(\mathbb{Z}_p)$-representation $\sigma(\tau)$ over E, satisfying results towards the inertial local Langlands correspondence. In this paper we take $\sigma(\tau)$ to be the inflation of $R_s(\mu+\eta)$ ([LHL19, Corollary 2.3.5]).

If $\mu\in\underline{C}_0$ is 1-deep and $\tau\cong\tau(s,\mu+\eta)$ then let $\sigma(\tau)$ be the inflation of $R_s(\mu+\eta)$ to $G_0(\mathbb{Z}_p)$. This representation $\sigma(\tau)$ satisfies the properties described in [CEG+16, Theorem 3.7], see [LHL19, Corollary 2.3.5] (this is a form of inertial local Langlands).

2.1.2. *Serre weights.* A *Serre weight* for $G_0(\mathbb{F}_p)$ is (an isomorphism class of) an absolutely irreducible $G_0(\mathbb{F}_p)$-representation over $\mathbb{F}$. Any Serre weight is the restriction to $G_0(\mathbb{F}_p)$ of an irreducible algebraic representation of G_0 of highest weight in $X_1(\underline{T})$, and given $\lambda\in X_1(\underline{T})$ we write $F(\lambda)$ for the Serre weight associated to the irreducible algebraic representation $L(\lambda)$ of G_0 over $\mathbb{F}_p$. The assignment $\lambda\mapsto F(\lambda)$ gives a bijection between $X_1(\underline{T})/(p-\pi)X^0(\underline{T})$ and the set of Serre weight for $G_0(\mathbb{F}_p)$.

Let $\mathcal{A}$ denote the set of p-restricted alcoves in $X^*(\underline{T})\otimes_{\mathbb{Z}}\mathbb{R}$. As in [LLHLMb, LLHLM20] it will be more convenient to parametrize (regular) Serre weights by a subset of $\underline{\Lambda}_W\times\mathcal{A}$. Specifically let $\lambda\in X^*(\underline{T})\cap\underline{C}_0+\eta$ and define $\underline{\Lambda}_W^\lambda$ to be the subset of $X^*(\underline{T})/X^0(\underline{T})$ whose elements $\overline{\omega}$ satisfy $\omega+\lambda-\eta\in\underline{C}_0$ for some (equivalently, for all) lift $\omega\in X^*(\underline{T})$ of $\overline{\omega}$. Then *loc. cit.* produces an injection $\mathfrak{Tr}_\lambda:\underline{\Lambda}_W^\lambda\times\mathcal{A}\hookrightarrow X_1(\underline{T})/(p-\pi)X^0(\underline{T})$ whose image consists of regular Serre weights with central character $(\lambda-\eta)|_{\underline{Z}}$ modulo $(p-\pi)X^*(\underline{Z})$ (see *loc. cit.* Proposition 2.1.3 and 2.1.4.). The reason this parametrization is preferred is that it decouples contributions of different elements of $\mathcal{J}$ in the representation theory of G_0.

2.1.3. *Combinatorics of types and weights.* We identify $\mathcal{A}$ with $\{0,1\}^{\mathcal{J}}$ where 0 and 1 stand for the lower and upper alcove, respectively. The standard action of $\underline{\widetilde{W}}$ on $X^*(\underline{T})$ descends to an action on $X^*(\underline{T})/X^0(\underline{T})$ hence on $X^*(\underline{T})/X^0(\underline{T})\times\mathcal{A}$ by letting $\underline{\widetilde{W}}$ act trivially on $\mathcal{A}$. Let ε_1 (resp. ε_2) denote the image of $(1,0,0)\in X^*(T)$ (resp. $(0,0,-1)\in X^*(T)$) in $X^*(T)/X^0(T)$ and set

$$\Sigma_0\overset{\text{def}}{=}\left\{\begin{matrix}(\varepsilon_1+\varepsilon_2,0),(\varepsilon_1-\varepsilon_2,0),(\varepsilon_2-\varepsilon_1,0)\\ (0,1),(\varepsilon_1,1),(\varepsilon_2,1)\\ (0,0),(\varepsilon_1,0),(\varepsilon_2,0)\end{matrix}\right\}$$

and $\Sigma\overset{\text{def}}{=}\Sigma_0^{\mathcal{J}}\subset X^*(\underline{T})/X^0(\underline{T})\times\mathcal{A}$. We define $(\omega,a),(\nu,b)\in\Sigma_0$ to be *adjacent* if $\omega-\nu\in\{0,\pm\varepsilon_1,\pm\varepsilon_2,\pm\varepsilon_1-\varepsilon_2\}$ and $a\neq b$. With this notion of adjacency Σ_0 is a connected graph with a distance function d.

Assume that $\lambda-\eta$ is 0-deep in $\underline{C}_0$ and let $(s,\mu-\eta)\in \underline{W}\times X^*(\underline{T})\cap\underline{C}_0$ be a pair such that $\mu-\eta$ is 2-deep and $\mu+\eta-\lambda\in\underline{\Lambda}_R$. Then the set set $\mathrm{JH}(R_s(\mu))$ is given by $F(\mathfrak{Tr}_\lambda(t_{\mu-\eta}s(\Sigma))$, see [LLHLMb, Proposition 2.1.1].

Moreover, we *define* $W^?(\overline{\tau}(s,\mu+\eta))$ to be the set of Serre weights $F\Big(\mathfrak{Tr}_\lambda\Big(t_{\mu+\eta-\lambda}s\big(r(\Sigma)\big)\Big)\Big)$, where $r(\Sigma)$ is defined by swapping the digits of $a\in\{0,1\}^{\mathcal{J}}$ in the elements $(\varepsilon,a)\in\Sigma$.

2.1.4. *L-groups and L-parameters.* In this subsection we let S_p have arbitrary finite cardinality.

Let F_p^+ be $\mathcal{O}_p[1/p]$ so that $F_p^+\cong\prod_{v\in S_p}F_v^+$ where $F_v^+\stackrel{\text{def}}{=}\mathcal{O}_v[1/p]$ for each $v\in S_p$. Let

$$\underline{G}^\vee_{/\mathbb{Z}}\stackrel{\text{def}}{=}\prod_{F_p^+\to E}G^\vee_{/\mathbb{Z}}$$

be the dual group of $\underline{G}$ so that the Langlands dual group of G_0 is ${}^L\underline{G}_{/\mathbb{Z}}\stackrel{\text{def}}{=}\underline{G}^\vee\rtimes\mathrm{Gal}(E/\mathbb{Q}_p)$ where $\mathrm{Gal}(E/\mathbb{Q}_p)$ acts on the set of homomorphisms $F_p^+\to E$ by post-composition.

An *L-homomorphism* (over E) is a continuous homomorphism $\rho: G_{\mathbb{Q}_p}\to {}^L\underline{G}(E)$ which is compatible with the projection to $\mathrm{Gal}(E/\mathbb{Q}_p)$. The $\underline{G}^\vee(E)$-conjugacy class of an L-homomorphism is called L-parameter. An inertial L-parameter is a $\underline{G}^\vee(E)$-conjugacy class of a homomorphism $\tau: I_{\mathbb{Q}_p}\to\underline{G}^\vee(E)$ with open kernel, and which admits an extension to an L-homomorphism. An (inertial) L-parameter is *tame* if some (equivalently, any) representative in its equivalence class factors through the tame quotient of $I_{\mathbb{Q}_p}$. Fixing isomorphisms $\overline{F_v^+}\stackrel{\sim}{\to}\overline{\mathbb{Q}}_p$ for all $v\in S_p$, we have a bijection between L-parameters (resp. tame inertial L-parameters) and collections $(\rho_v)_{v\in S_p}$ (resp. $(\tau_v)_{v\in S_p}$) where $\rho_v: G_{F_v^+}\to\mathrm{GL}_3(E)$ is a continuous Galois representation (resp. $\tau_v: I_{F_v^+}\to\mathrm{GL}_3(E)$ is a tame inertial type for F_v^+) for all $v\in S_p$.

We have similar notions when E is replaced by $\mathbb{F}$. Abusing terminology, we dentify (tame inertial) L-parameters with a fixed choice of a representative in its class. Nothing in what follows will depend on this choice.

The definitions and results of §2.1.1–2.1.3 generalize for tame inertial L-parameters and L-homomorphism. In particular, given a tame inertial L-parameter τ corresponding to the collection $(\tau_v)_{v\in S_p}$, we let $\sigma(\tau)$ be $\otimes_{v\in S_p}\sigma(\tau_v)$ (an irreducible smooth $G_0(\mathbb{Z}_p)$-representation over E).

3. Geometry of multi-type deformation rings

3.1. **Multi-type deformation rings.** The goal of this section is to introduce –and give the first tools to analyze– deformation rings for $\overline{\rho}: G_K\to\mathrm{GL}_3(\mathbb{F})$, with p-adic Hodge theory conditions defined by a finite set of tame inertial types.

Let $\overline{\rho}: G_K\to\mathrm{GL}_3(\mathbb{F})$ be a continuous Galois representation, and let $R_{\overline{\rho}}^\square$ denote the universal lifting ring for $\overline{\rho}$.

If τ is an inertial type for K, let $R_{\overline{\rho}}^{\leq\eta,\tau}$ (resp. $R_{\overline{\rho}}^{\eta,\tau}$) be the reduced quotient of $R_{\overline{\rho}}^\square$ parametrizing potentially crystalline representations of type τ and Hodge–Tate weights $\leq(2,1,0)$ (resp. equal to $(2,1,0)$).

Definition 3.1. Let $T\stackrel{\text{def}}{=}\{\tau: I_K\to\mathrm{GL}_3(E)\}$ be a finite set of inertial types for K. We define $R_{\overline{\rho}}^{\leq\eta,T}$ to be the image of $R_{\overline{\rho}}^\square\to\prod_T R_{\overline{\rho}}^{\leq\eta,\tau}$. We say that $R_{\overline{\rho}}^{\leq\eta,T}$ is the $(\leq\eta,T)$ *multi-type deformation ring.* We similarly define $R_{\overline{\rho}}^{\eta,T}$ by replacing "$\leq\eta$" with η everywhere.

From now on we assume $\overline{\rho}: G_K\to\mathrm{GL}_3(\mathbb{F})$ is tame, and that $\overline{\rho}|_{I_K}=\overline{\tau}(s,\mu)$ where μ is 6-deep in alcove $\underline{C}_0$. Set $\widetilde{w}(\overline{\rho})\stackrel{\text{def}}{=}t_{\mu+\eta}s$.

3.1.1. *Presentations of single type deformation rings.* Suppose $T = \{\tau\}$ consists of a single tame inertial type. In this case, the local model theory of [LLHLMb], [LLHLM23] produces an explicit presentation of $R^{\leq \eta,\tau}$ which we now recall. In order for $R^{\leq\eta,\tau}$ to be non-zero, τ must admit a lowest alcove presentation such that

$$\widetilde{w}^*(\overline{\rho},\tau) \stackrel{\text{def}}{=} (\widetilde{w}(\tau)^{-1}\widetilde{w}(\overline{\rho}))^*$$

belongs to $\mathrm{Adm}^\vee(\eta)$ by [LLHLM23, Corollary 5.5.8] (in particular, this gives τ a 4-generic lowest alcove presentation). We abbreviate $\widetilde{z} = (z_j t_{\nu_j})_{j\in\mathcal{J}} \stackrel{\text{def}}{=} \widetilde{w}^*(\overline{\rho},\tau)$ for the remainder of this subsection.

We recall the following objects from [LLHLMb],[LLHLM23]:

- The Emerton–Gee stack $\mathcal{X}^{\leq\eta,\tau}$ parametrizing potentially crystalline representations with Hodge–Tate weights $\leq \eta$ and inertial type τ (see [LLHLMb, §3.2]).
- The moduli stack $Y^{\leq\eta,\tau}$ of Breuil-Kisin modules for $K_\infty \stackrel{\text{def}}{=} \bigcup_{n\in\mathbb{N}} K(\sqrt[p^n]{-p})$ with tame descent data of type τ and elementary divisors $\leq \eta$. It admits an open substack $Y^{\leq\eta,\tau}(\widetilde{z})$ (see [LLHLMb, Proposition 3.1.1]).
- We also have the stack $\Phi\text{-}\mathrm{Mod}_K^{\text{ét},3}$ parametrizing rank 3 étale φ modules over K_∞ (see [LLHLM23, §5.4.1]).
- (cf. [LLHLMb, §3.1]) $M(\widetilde{z}) \stackrel{\text{def}}{=} \prod_{j\in\mathcal{J}} M_j(\widetilde{z}_j)$ the affine scheme over $\mathcal{O}$ parametrizing $\mathcal{J}$-tuples of matrices 3×3 matrices $A \stackrel{\text{def}}{=} (A^{(j)})_{j\in\mathcal{J}}$ whose entries are polynomials in v with the following constraints for all $j \in \mathcal{J}$:
  - $A^{(j)}_{ik}$ is divisible by v for $i > k$.
  - $\deg A^{(j)}_{ik} \leq \nu_{j,k} - \delta_{i<z_j(k)}$, with equality when $i = z_j(k)$.
  - The leading coefficient of $A^{(j)}_{z_j(k)k}$ is a unit.
  - $\det A^{(j)}$, which is cubic with unit top coefficient, is a unit times $(v+p)^3$.

  We furthermore have the closed subscheme $\widetilde{U}(\leq\eta,\widetilde{z}) \subset M(\widetilde{z})$ obtained as the $\mathcal{O}$-flat closure of the locus where for all $j\in\mathcal{J}$ the 2×2 minors of $A^{(j)}$ are divisible by $(v+p)$ (i.e. they vanish when setting $v = -p$).

The relationship between these objects is summarized in the following diagram with cartesian squares [LLHLMb, Theorem 3.2.2]:

$$\begin{array}{ccccccc}
\widetilde{U}(\widetilde{z},\leq\eta)^{\wedge_p} & \longrightarrow & Y^{\leq\eta,\tau}(\widetilde{z}) & \hookrightarrow & Y^{\leq\eta,\tau} & \hookrightarrow & \Phi\text{-}\mathrm{Mod}_K^{\text{ét},3} \\
\uparrow & \square & \uparrow & \square & \uparrow & \nearrow & \\
\widetilde{\mathcal{X}}^{\leq\eta,\tau}(\widetilde{z}) & \longrightarrow & \mathcal{X}^{\leq\eta,\tau}(\widetilde{z}) & \hookrightarrow & \mathcal{X}^{\leq\eta,\tau} & &
\end{array} \tag{3.1}$$

whose relevant features for us are:

- The top left horizontal arrow identifies
$$Y^{\leq\eta,\tau}(\widetilde{z}) = [\widetilde{U}(\widetilde{z},\leq\eta)^{\wedge_p}/_\tau T^{\vee,\mathcal{J}}]$$
where the action of $(t_j) \in T^{\vee,\mathcal{J}}$ is the τ-twisted shifted conjugation action defined by:
$$A^{(j)} \mapsto t_j A^{(j)} \sigma_j^{-1}(t_{j-1})$$
where (σ,κ) is the lowest alcove presentation of τ determined at the beginning of §3.1.1.
- $\widetilde{\mathcal{X}}^{\leq\eta,\tau}(\widetilde{z})$ identifies with an explicit closed, $T^{\vee,\mathcal{J}}$-stable formal subscheme $\widetilde{U}(\widetilde{z},\leq\eta,\nabla_{\tau,\infty})$ which is obtained as the p-saturation of a natural "monodromy condition" on the Breuil–Kisin module living over $\widetilde{U}(\widetilde{z},\leq\eta)^{\wedge_p}$ cf. [LLHLM23, Definition 7.1.2 & Corollary 7.1.5]. We also recall that there is a closed immersion $\widetilde{U}_{\mathrm{reg}}(\widetilde{z},\leq\eta,\nabla_{\tau,\infty}) \hookrightarrow \widetilde{U}(\widetilde{z},\leq\eta,\nabla_{\tau,\infty})$ [LLHLM23,

Theorem 7.3.2] which correspond to the part $\mathcal{X}^{\eta,\tau}(\widetilde{z}) \hookrightarrow \mathcal{X}^{\leq\eta,\tau}(\widetilde{z})$ with Hodge-Tate weights exactly η.
- The edges of the triangle are closed immersions.
- The composite of the top horizontal arrow assigns to $A = (A^{(j)})$ the free étale φ-module with matrix of Frobenius given by $A\widetilde{w}^*(\tau)$.

Our chosen tame $\overline{\rho}$ gives a point in $\mathcal{X}^{\leq\eta,\tau}(\widetilde{z})(\mathbb{F})$ and a point $\overline{\mathfrak{M}}_\tau \in Y^{\leq\eta,\tau}(\mathbb{F})$, whose lifts to $\widetilde{U}(\widetilde{z}, \leq \eta)(\mathbb{F})$ has the form $\overline{A} = \overline{D}_\tau\widetilde{z}$ for $\overline{D}_\tau \in T^{\vee,\mathcal{J}}(\mathbb{F})$ well-defined up to τ-shifted conjugation. Then $R^{\leq\eta,\tau}_{\overline{\rho}}$ is a versal ring to $\mathcal{X}^{\leq\eta,\tau}$ at $\overline{\rho}$, thus diagram (3.1) identifies $R^{\leq\eta,\tau}_{\overline{\rho}}$, up to formal variables, with the completion of the explicit (formal) scheme $\widetilde{U}(\widetilde{z}, \leq\eta, \nabla_{\tau,\infty})$ at $\overline{A}$.

3.1.2. *Presentations of multi-type deformation rings.* We now let T be a finite set of tame inertial types endowed with lowest alcove presentations such that $R^{\leq\eta,\tau}_{\overline{\rho}} \neq 0$ for all $\tau \in T$, thus we get $\widetilde{w}^*(\overline{\rho},\tau) \in \mathrm{Adm}^\vee(\eta)$ for each $\tau \in T$. For each $j \in \mathcal{J}$ let $T^{(j)} \overset{\text{def}}{=} \{\widetilde{w}^*(\overline{\rho},\tau)_j, \tau \in T\}$. Throughout this paper, we restrict ourselves to T satisfying the following:

Hypothesis 3.2. For each $j \in \mathcal{J}$, either:
(I) $T^{(j)} \subseteq \{t_{\underline{1}}, \alpha\beta t_{\underline{1}}, \beta\alpha t_{\underline{1}}, \alpha\beta\alpha t_{\underline{1}}, \alpha\beta\alpha\gamma t_{\underline{1}}\}$; or
(II) $T^{(j)} \subseteq \{t_{w_0(\eta)}, t_{w_0(\eta)}\alpha, t_{w_0(\eta)}\beta, t_{w_0(\eta)}w_0\}$; or
(III) $T^{(j)}$ is a singleton which does not fall in either of the cases above, $\widetilde{w}^*(\overline{\rho},\tau)_j \in \mathrm{Adm}^\vee(\eta_j)$ and $\ell(\widetilde{w}(\overline{\rho},\tau)_j) \geq 2$.

The basic idea to probe $R^{\leq\eta,T}_{\overline{\rho}}$ is to glue diagrams (3.1) along $\Phi\text{-}\mathrm{Mod}^{\text{ét},3}_K$ as τ varies over T. To this end, we consider the following:

Definition 3.3. (1) Let $M^T \overset{\text{def}}{=} \prod_{j\in\mathcal{J}} M^{T,(j)}$ be the affine scheme such that
- For $T^{(j)}$ as in (I), $M^{T,(j)}$ consists of

$$\begin{pmatrix} d^{*(j)}_{11}(v+p) + c^{(j)}_{11} + \frac{e^{(j)}_{11}}{v} & c^{(j)}_{12} & c^{(j)}_{13} \\ d^{(j)}_{21}(v+p) + c^{(j)}_{21} & d^{*(j)}_{22}(v+p) + c^{(j)}_{22} & c^{(j)}_{23} \\ d^{(j)}_{31}(v+p) + c^{(j)}_{31} & d^{(j)}_{32}(v+p) + c^{(j)}_{32} & d^{*(j)}_{33}(v+p) + c^{(j)}_{33} \end{pmatrix}$$

with determinant $(v+p)^3 d^{*(j)}_{11} d^{*(j)}_{22} d^{*(j)}_{33}$ and $d^{*(j)}_{ii}$ invertible.
- For $T^{(j)}$ as in (II), $M^{T,(j)}$ consists of

$$\begin{pmatrix} d^{*(j)}_{11} & c^{(j)}_{12} & c^{(j)}_{13}(v+p) + e^{(j)}_{13} \\ d^{(j)}_{21} & d^{*(j)}_{22}(v+p) + c^{(j)}_{22} & c^{(j)}_{23}(v+p) + e^{(j)}_{23} \\ d^{(j)}_{31} & d^{(j)}_{32}(v+p) + c^{(j)}_{32} & d^{*(j)}_{33}(v+p)^2 + c^{(j)}_{33}(v+p) + e^{(j)}_{33} \end{pmatrix}$$

with determinant $(v+p)^3 d^{*(j)}_{11} d^{*(j)}_{22} d^{*(j)}_{33}$ and $d^{*(j)}_{ii}$ invertible.
- For $T^{(j)}$ as in (III), $M^{T,(j)} = M_j(\widetilde{w}^*(\overline{\rho},\tau)_j)$ as in §3.1.1.

(2) Define the element $\widetilde{w}^{*,T}(\overline{\rho}) \in \widetilde{W}^{\mathcal{J}}$ as

$$\widetilde{w}^{*,T}(\overline{\rho})_j = \begin{cases} t_{-\underline{1}}\widetilde{w}^*(\overline{\rho})_j & \text{if } T^{(j)} \text{ is as in item (I)} \\ t_{-w_0(\eta)}\widetilde{w}^*(\overline{\rho})_j & \text{if } T^{(j)} \text{ is as in item (II)} \\ \widetilde{w}^*(\tau)_j & \text{else.} \end{cases}$$

The significance of the above definition is explained by:

Proposition 3.4. *For each $\tau \in T$, we have the diagram*

$$\begin{array}{ccccc} \widetilde{U}(\widetilde{w}^*(\tau,\overline{\rho}), \leq \eta)^{\wedge p} & \xhookrightarrow{r_{\widetilde{w}^*(\tau)}} & (M^T)^{\wedge p}\widetilde{w}^{*,T}(\overline{\rho}) & \longrightarrow & \Phi\text{-}\mathrm{Mod}_K^{\text{ét},3} \\ \big\downarrow f.s. & & \big\downarrow f.s. & \nearrow & \\ [\widetilde{U}(\widetilde{w}^*(\tau,\overline{\rho}), \leq \eta)^{\wedge p}/_{\tau}T^{\vee,\mathcal{J}}] & \xhookrightarrow{r_{\widetilde{w}^*(\tau)}} & \left[(M^T)^{\wedge p}\widetilde{w}^{*,T}(\overline{\rho})/T^{\vee,\mathcal{J}}\text{-sh.cnj}\right] & & \end{array} \tag{3.2}$$

where

- *The vertical arrows are quotient maps by $T^{\vee,\mathcal{J}}$.*
- *The top right arrow assigns to the tuple $(A^{(j)}\widetilde{w}^{*,T}(\overline{\rho})_j)_{j\in\mathcal{J}}$ the corresponding free étale φ-module with corresponding matrix of Frobenius.*
- *The hooked horizontal arrows are induced by right multiplication by $\widetilde{w}^*(\tau)$ on the family of matrices. They are closed immersions, equivariant for the τ-shifted conjugation action on the source and shifted conjugation action on the target.*
- *The diagonal arrow is a monomorphism.*

Remark 3.5. By [EG21, Theorem 5.4.20], $\Phi\text{-}\mathrm{Mod}_K^{\text{ét},3}$ can be written as $\mathrm{colim}_h \Phi\text{-}\mathrm{Mod}_K^{\leq h,3}$ of Noetherian p-adic formal substacks, where $\Phi\text{-}\mathrm{Mod}_K^{\leq h,3}$ is the moduli stack of étale φ-modules with height $\leq h$ with respect to the choice of polynomial $F(v) = v(v+p)$. Then in (3.1), we may replace $\Phi\text{-}\mathrm{Mod}_K^{\text{ét},3}$ by $\Phi\text{-}\mathrm{Mod}_K^{\leq h,3}$ so that all the objects involved are now p-adic formal algebraic stacks, and notions such as scheme theoretic images behave as expected.

Proof. The last item follows from the proof of [LLHLM23, Proposition 5.4.4], while all the other items follow from the definitions. □

Recall from §3.1.1 the closed subscheme $\widetilde{U}(\widetilde{w}^*(\tau,\overline{\rho}), \leq \eta, \nabla_{\tau,\infty}) \subset \widetilde{U}(\widetilde{w}^*(\tau,\overline{\rho}), \leq \eta)^{\wedge p}$ characterized by $\mathcal{X}^{\leq \eta,\tau} = [\widetilde{U}(\widetilde{w}^*(\tau,\overline{\rho}), \leq \eta, \nabla_{\tau,\infty})/_{\tau}T^{\vee,\mathcal{J}}]$.

Definition 3.6. Let $M^{T,\nabla_\infty} \subset (M^T)^{\wedge p}$ so that $M^{T,\nabla_\infty}\widetilde{w}^{*,T}(\overline{\rho})$ is the scheme theoretic image of

$$\coprod_{\tau\in T} r_{\widetilde{w}^*(\tau)} : \coprod_{\tau\in T} \widetilde{U}(\widetilde{w}^*(\tau,\overline{\rho}), \leq \eta, \nabla_{\tau,\infty}) \to M^T\widetilde{w}^{*,T}(\overline{\rho}).$$

We now turn to formal completions. Our given $\overline{\rho}$ gives rise to a point $\mathcal{X}^{\leq\eta,\tau}(\mathbb{F})$ for each $\tau \in T$, whose image $\overline{\rho}_\infty \overset{\text{def}}{=} \overline{\rho}|_{G_{K_\infty}}$ in $\mathrm{Mod}_K^{\text{ét},3}(\mathbb{F})$ is independent of τ. By Proposition 3.4, $\overline{\rho}_\infty$ arises from a point $\overline{A} \in M^T(\mathbb{F})$ which is unique up to the $T^{\vee,\mathcal{J}}(\mathbb{F})$ action. We fix the choice of such a point $\overline{x}_{\overline{\rho}}$. We note that the semisimplicity of $\overline{\rho}$ is equivalent to the fact that $\overline{A}\widetilde{w}^{*,T}(\overline{\rho}) = \overline{D}\widetilde{w}^*(\overline{\rho})$ for $\overline{D} \in T^{\vee,\mathcal{J}}(\mathbb{F})$, so that its pre-image under $r_{\widetilde{w}^*(\tau)}$ is $\overline{D}\widetilde{w}^*(\overline{\rho},\tau)$. In particular the entries of $\overline{D}$ correspond to the numbers $\overline{d}_{ii}^{*(j)} \in \mathbb{F}$.

In the following Proposition, we adopt the convention that if X is a stack and $x \in X(\mathbb{F})$ then X_x stands for the formal completion of X at x, i.e. the restriction of X to Artinian test rings:

Proposition 3.7. *Let $\mathcal{X}^{\leq\eta,T} \overset{def}{=} \bigcup_{\tau\in T}\mathcal{X}^{\leq\eta,\tau}$ be the scheme theoretic union of the $\mathcal{X}^{\leq\eta,\tau}$ for $\tau \in T$ inside $\Phi\text{-}\mathrm{Mod}_K^{\text{ét},3}$. Then*

- *$R_{\overline{\rho}}^{\leq\eta,T}$ is a versal ring of $\mathcal{X}^{\leq\eta,T}$ at $\overline{\rho}_\infty$;*
- *$\mathcal{X}_{\overline{\rho}_\infty}^{\leq\eta,T} = [M_{\overline{x}_{\overline{\rho}}}^{T,\nabla_\infty}\widetilde{w}^{*,T}(\overline{\rho})/T_1^{\vee,\mathcal{J}}\text{-sh.cnj}]$ as subfunctors of $\Phi\text{-}\mathrm{Mod}_{K,\overline{\rho}_\infty}^{\text{ét},3}$.*

Proof. As in [EG23, §3.6], the G_{K_∞} lifting ring $R_{\overline{\rho}_\infty}^{\square}$ is a versal ring for $\Phi\text{-}\mathrm{Mod}_K^{\text{ét},3}$ at $\overline{\rho}_\infty$. After pulling back to $\mathrm{Spf}\, R_{\overline{\rho}_\infty}^{\square}$, $\mathcal{X}^{\leq\eta,\tau}$ becomes $\mathrm{Spf}\, R_{\overline{\rho}}^{\leq\eta,\tau}$, and hence $\mathcal{X}^{\leq\eta,T}$ becomes the formal spectrum of $\mathrm{im}(R^{\square}_{\overline{\rho}_\infty} \to \prod_{\tau\in T} R^{\leq\eta,\tau}_{\overline{\rho}})$. However by [LLHLM18, Proposition 3.12], restriction to G_{K_∞} induces a surjection $R^{\square}_{\overline{\rho}_\infty} \twoheadrightarrow R^{\square}_{\overline{\rho}}$, hence this image ring coincides with $R^{\leq\eta,T}_{\overline{\rho}}$. This gives the first item.

For the second item, by Proposition 3.4, after pulling back to Spf $R^{\square}_{\overline{\rho}_\infty}$, $[M^T\widetilde{w}^{*,T}(\overline{\rho})/T^{\vee,\mathcal{J}}\text{-sh.cnj}]$, $[M^{T,\nabla_\infty}\widetilde{w}^{*,T}(\overline{\rho})/T^{\vee,\mathcal{J}}\text{-sh.cnj}]$ becomes Spf R', Spf R, where $R^{\square}_{\overline{\rho}_\infty} \twoheadrightarrow R' \twoheadrightarrow R^{\leq\eta,\tau}_{\overline{\rho}}$ for each $\tau\in T$, and by definition R is the image of the natural map $R' \to \prod_{\tau\in T} R^{\leq\eta,\tau}_{\overline{\rho}}$. But this shows $R = R^{\leq\eta,T}_{\overline{\rho}}$. □

Proposition 3.8. *Recall that T satisfies Hypothesis 3.2 and $\overline{\rho}$ is tame, 6-generic. Then there is an isomorphism*

$$R^{\eta,T}_{\overline{\rho}}[\![x_1,\cdots,x_{3f}]\!] \cong \widetilde{S}/\widetilde{I}_{T,\nabla_\infty}[\![\{y_i\}_{1\leq i\leq 9}]\!]$$

where

- $\widetilde{S} \overset{def}{=} \widehat{\otimes}_{j\in\mathcal{J}}\widetilde{S}^{(j)}$*, where $\widetilde{S}^{(j)}$ is the ring in Tables 3, 4 , 5 in cases (I)-(II), and Table* [LLHLMb, Table 1] *in case (III).*
- *For each $\tau\in T$, $\widetilde{I}_{\tau,\nabla_\infty} \overset{def}{=} \sum_{j\in\mathcal{J}}\widetilde{I}^{(j)}_{\tau,\nabla_\infty}$ where for each $j\in\mathcal{J}$ the ideal $\widetilde{I}^{(j)}_{\tau,\nabla_\infty}\subseteq\widetilde{S}$ has the form described in Tables 3, 4 and 5 if $T^{(j)}$ is as in item (I)–(II), and are described in* [LLHLMb, Table 2][1] *if $T^{(j)}$ is as in item (III).*
- $\widetilde{I}_{T,\nabla_\infty} \overset{def}{=} \bigcap_{\tau\in T}\widetilde{I}_{\tau,\nabla_\infty}$.

Warning 3.9. We emphasize that:

- the $O(p^{N-4})$-tails appearing in the $\widetilde{I}^{(j)}_{\tau,\nabla_\infty}$-rows of tables 3, 4, 5 *involve variables in all embedding*;
- the structure constants $b_{\tau,1}, b_{\tau,2}, b_{\tau,3}\in\mathbb{Z}_p$ appearing in the $\widetilde{I}^{(j)}_{\tau,\nabla_\infty}$-rows of tables 3, 4, 5 *depend on the whole f-tuple $\widetilde{w}(\overline{\rho},\tau)$.*

In particular $\widetilde{I}^{(j)}_{\tau,\nabla_\infty}$ is *not* an ideal of $\widetilde{S}^{(j)}$, and $\widetilde{I}^{(j)}_{\tau,\nabla_\infty}+O(p^{N-4})$ *does not* depend only on $\widetilde{w}(\overline{\rho},\tau)_j$ in general.

Proof. This follows from Proposition 3.7, and the following observations:

- $\widetilde{S}$ is the formal power series ring on the coefficients of the entries of the matrices that M^T parametrizes (where the variables corresponding to unit entries are characterized by $d^*_{ii} = [\overline{d}^*_{ii}](1+x^*_{ii})$), and thus $M^T_{\overline{x}_{\overline{\rho}}}$ is the locus in Spf$\widetilde{S}$ where the determinant conditions on the matrices are imposed.
- By [LLHLMb, Theorem 3.2.2] and the discussion in [LLHLM20, §3.6.1], quotiening by $\widetilde{I}_{\tau,\nabla_\infty}$ corresponds to $\widetilde{U}_{\mathrm{reg}}(\widetilde{w}^*(\overline{\rho},\tau),\leq\eta,\nabla_{\tau_\infty})$ via the closed immersion $r_{\widetilde{w}^*(\tau)}$ in diagram (3.2). Note that a priori we only know $\widetilde{I}_{\tau,\nabla_\infty}$ is the p-saturation of $\sum_{j\in\mathcal{J}}\widetilde{I}^{(j)}_{\tau,\nabla_\infty}$. However by [LLHLM20, §3.6.1], the first bullet point at page 54, the natural surjection $\widetilde{S}/\sum_{j\in\mathcal{J}}\widetilde{I}^{(j)}_{\tau,\nabla_\infty} \twoheadrightarrow \widetilde{S}/\widetilde{I}_{\nabla_{\tau,\infty}}$ is an isomorphism, and hence we don't need to p-saturate.

□

[1]and moreover adding a term $O(p^{N-1})$ to each generator of the ideals $\widetilde{I}^{(j)}_{\tau,\nabla_\infty}$ of [LLHLMb, Table 2]

Remark 3.10. (Compatibility under shrinking T) Suppose $T' \subset T$ satisfy Hypothesis 3.2. Then we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Spf}\, R_{\overline{\rho}}^{\leq\eta,T'} & \longrightarrow & \left[M^{T',\nabla_\infty} \cdot \widetilde{w}^{*,T'}(\overline{\rho})/T^{\vee,\mathcal{J}}\text{-sh.cnj}\right] \\ \big\downarrow & & \big\downarrow \\ \mathrm{Spf}\, R_{\overline{\rho}}^{\leq\eta,T} & \longrightarrow & \left[M^{T,\nabla_\infty} \cdot \widetilde{w}^{*,T}(\overline{\rho})/T^{\vee,\mathcal{J}}\text{-sh.cnj}\right] \\ \big\downarrow & & \big\downarrow \\ \mathrm{Spf}\, R_{\overline{\rho}_\infty}^{\square} & \longrightarrow & \Phi\text{-}\mathrm{Mod}_K^{\text{ét},3} \end{array}$$

with formally smooth horizontal arrows. It follows that the isomorphism in Proposition 3.8 for $T' \subset T$ can be chosen to be compatible with the inclusion of ideals $\widetilde{I}_{T,\nabla_\infty} \subset \widetilde{I}_{T',\nabla_\infty}$.

We also record the following result, which will be later used to analyze $\widetilde{I}_{T,\nabla_\infty}$:

Proposition 3.11. *Let $\widetilde{I}_\tau = \sum_{j\in\mathcal{J}} \widetilde{I}_\tau^{(j)}$ where for each $j \in \mathcal{J}$ the ideal $\widetilde{I}_\tau^{(j)} \subseteq \widetilde{S}^{(j)}$ is described in Tables 3, 4, 5. Then*

(1) $\widetilde{I}_\tau$ correspond to the closed immersion $r_{\widetilde{w}^*(\tau)} : \widetilde{U}(\widetilde{w}^*(\overline{\rho},\tau), \leq\eta) \hookrightarrow M^T \widetilde{w}^{*,T}(\overline{\rho})$.

(2) If $\widetilde{w}(\overline{\rho},\tau)_j \neq \widetilde{w}(\overline{\rho},\tau')_j$ then

$$p^2 \in \widetilde{I}_\tau^{(j)} + \widetilde{I}_{\tau'}^{(j)}.$$

Moreover if $\ell\big(\widetilde{w}(\overline{\rho},\tau)_j\big) \geq 2$, $\ell\big(\widetilde{w}(\overline{\rho},\tau')_j\big) \geq 2$ and $\{\widetilde{w}(\overline{\rho},\tau)_j, \widetilde{w}(\overline{\rho},\tau')_j\} \neq \{\alpha\beta\alpha\gamma t_{\underline{1}}, \alpha\beta\alpha t_{\underline{1}}\}$ we have

$$p \in \widetilde{I}_\tau^{(j)} + \widetilde{I}_{\tau'}^{(j)}.$$

Proof. The first item is due to the fact the top group of generators of $\widetilde{I}_\tau^{(j)}$ (appearing in the corresponding row of Table 3, 4, 5) cut out the degree bound conditions, while the bottom group cut out the elementary divisor conditions of $\widetilde{U}(\widetilde{w}^*(\overline{\rho},\tau), \leq\eta)$.

The second item follows from inspecting the tables. We give a sample computation for the case $\widetilde{w}^*(\overline{\rho},\tau)_j = \alpha\beta t_{\underline{1}}$, $\widetilde{w}^*(\overline{\rho},\tau')_j = t_{\underline{1}}$ (for other cases see [LLHM, §B.1.1]). In the ring $\widetilde{S}^{(j)}/(\widetilde{I}_\tau^{(j)} + \widetilde{I}_{\tau'}^{(j)})$ we have:

$$c_{12}d_{21}c_{33} = c_{12}c_{23}d_{31} = \frac{c_{13}d_{32}c_{23}d_{31}}{d_{33}^*} = -pc_{13}d_{31}d_{22}^* = -pd_{22}^*d_{33}^*$$

Using this, and the relations $c_{22} = -pd_{22}^*$, $c_{33} = -pd_{33}^*$, the last generator in $\widetilde{I}_{t_{\underline{1}}}^{(j)}$ becomes

$$c_{12}d_{21}c_{33} - c_{11}c_{22}d_{33}^* - c_{11}d_{22}^*c_{33} - d_{11}^*c_{22}c_{33} - p(c_{11}d_{22}^*d_{33}^* + d_{11}^*c_{22}d_{33}^* + d_{11}^*d_{22}^*c_{33}) = 2p^2d_{11}^*d_{22}^*d_{33}^*.$$

$\square$

3.2. **Commutative algebra.**

In this section, we collect various commutative algebra results that we require in later sections.

Lemma 3.12. *Let R be a Noetherian ring and M a finitely generated R-module. Suppose that we have an exact sequence*

$$0 \to L \to M \to N \to 0$$

of R-modules such that $\mathrm{Supp}(L)$ *and* $\mathrm{Ass}(N)$ *are disjoint. Then L is the kernel of the natural map*

$$\lambda : M \to \bigoplus_{\mathfrak{p}\in\mathrm{Ass}(N)} M_{\mathfrak{p}}.$$

Proof. Let L' be $\ker\lambda$. Then $L \subset L'$ since $L_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathrm{Ass}(N)$ by assumption. Now let $a \in L'$ and denote by $\overline{a}$ the image in N. For any $\mathfrak{p} \in \mathrm{Ass}(N)$, the image of a in $M_{\mathfrak{p}}$ is 0, and so the image of $\overline{a}$ in $N_{\mathfrak{p}}$ is 0. We conclude from [Sta23, Tag 0311] that $\overline{a} = 0$ or equivalently that $a \in L$. Thus, $L = L'$. □

Throughout the rest of this section $(R, \mathfrak{m})$ denotes a local ring with residue field $\mathbb{F}$. (In later applications, $\mathbb{F}$ is as in §1.4, though we do not require this here.)

Corollary 3.13. *Let R be a Noetherian local ring and M a finitely generated R-module. Suppose that we have an exact sequence*

$$0 \to L \to M \to N \to 0$$

of R-modules where N is a finitely generated maximal Cohen–Macaulay R-module and $\mathrm{Supp}(L) \cap \mathrm{Supp}(N)$ *contains no minimal primes of R. Then the image of L in M is the kernel of the natural map*

$$M \to \bigoplus_{\mathfrak{p}\in \mathrm{Supp}(N)\ \textit{minimal}} M_{\mathfrak{p}}.$$

Proof. By [Mat86, Theorem 17.3(i)], N has no embedded primes so that $\mathrm{Ass}(N)$ is precisely the set of minimal primes of R in $\mathrm{Supp}(N)$. Thus $\mathrm{Ass}(N)$ and $\mathrm{Supp}(L)$ are disjoint by assumption. The result then follows from Lemma 3.12. □

Lemma 3.14. *Let R be a local ring with residue field $\mathbb{F}$. Let $I, J \subset K \subset R$ be ideals with K finitely generated. Let M be the kernel of the map*

$$R/I \oplus R/J \to R/K$$

which is the difference of the natural surjections. Then the following are equivalent:
- *(1) M is a cyclic R-module;*
- *(2) $I + J = K$; and*
- *(3) the induced map* $\mathrm{Tor}_1^R(\mathbb{F}, R/I) \oplus \mathrm{Tor}_1^R(\mathbb{F}, R/J) \to \mathrm{Tor}_1^R(\mathbb{F}, R/K)$ *is surjective.*

If these equivalent conditions hold then:
- *(i) $M \cong R/I \cap J$; and*
- *(ii) the image of* $\mathrm{Tor}_1^R(\mathbb{F}, M) \to \mathrm{Tor}_1^R(\mathbb{F}, R/K)$ *induced by the composition $M \to R/I \to R/K$ is the intersection of the images of* $\mathrm{Tor}_1^R(\mathbb{F}, R/I) \to \mathrm{Tor}_1^R(\mathbb{F}, R/K)$ *and* $\mathrm{Tor}_1^R(\mathbb{F}, R/J) \to \mathrm{Tor}_1^R(\mathbb{F}, R/K)$.

Proof. We first show that (2) implies (1) and (i). Suppose that $I + J = K$. Then the natural map $\psi : R/(I \cap J) \to R/I \oplus R/J$ is injective with cokernel $R/(I + J)$ using that $R/I \oplus R/J = (0 \oplus R/J) + \mathrm{im}(\psi)$. This identifies M with the cyclic R-module $R/(I \cap J)$.

The inclusion $I + J \subset K$ is an equality if and only the map $I \oplus J \to K$ obtained by taking the difference is a surjection. By Nakayama's lemma, this is equivalent to the surjectivity of the induced map $I \otimes_R \mathbb{F} \oplus J \otimes_R \mathbb{F} \to K \otimes_R \mathbb{F}$. As there is a functorial identification of $\mathrm{Tor}_1^R(\mathbb{F}, R/L) \cong L \otimes_R \mathbb{F}$ for ideals $L \subset R$, we see that (2) and (3) are equivalent.

Finally we show that (1) implies (3) and (ii). If M is a cyclic R-module, then the sequence

$$0 \to M \otimes_R \mathbb{F} \to (R/I) \otimes_R \mathbb{F} \oplus (R/J) \otimes_R \mathbb{F} \to (R/K) \otimes_R \mathbb{F} \to 0$$

is exact. The Tor^R-long exact sequence gives the exact sequence

$$\mathrm{Tor}_1^R(\mathbb{F}, M) \to \mathrm{Tor}_1^R(\mathbb{F}, R/I) \oplus \mathrm{Tor}_1^R(\mathbb{F}, R/J) \to \mathrm{Tor}_1^R(\mathbb{F}, R/K) \to 0$$

where the second map is the difference of the natural maps $\mathrm{Tor}_1^R(\mathbb{F}, R/I) \to \mathrm{Tor}_1^R(\mathbb{F}, R/K)$ and $\mathrm{Tor}_1^R(\mathbb{F}, R/J) \to \mathrm{Tor}_1^R(\mathbb{F}, R/K)$. This gives (3). As the image of $\mathrm{Tor}_1^R(\mathbb{F}, M)$ in $\mathrm{Tor}_1^R(\mathbb{F}, R/I) \oplus$

$\mathrm{Tor}_1^R(\mathbb{F}, R/J)$ consists of pairs (a,b) such that the images of a and b in $\mathrm{Tor}_1^R(\mathbb{F}, R/K)$ coincide, (ii) follows. □

Lemma 3.15. *Let R be a local ring with residue field $\mathbb{F}$. Let $1 \leq n$ and $I_i \subset K$ be ideals of R for $i = 1, \dots, n$. Let M be a finitely generated R-module with a fixed surjection to R/K. Let N be the kernel of the map*

$$M \oplus \oplus_{i=1}^n R/I_i \to (R/K)^{\oplus(n+1)}/\Delta(R/K)$$

induced by the sum of the natural maps $R/I_i \twoheadrightarrow R/K$ and the fixed map $M \twoheadrightarrow R/K$, and where $\Delta(R/K)$ denotes the diagonally embedded copy of R/K.

Write V_i (resp. W) for the image of the induced map

$$\mathrm{Tor}_1^R(\mathbb{F}, R/I_i) \to \mathrm{Tor}_1^R(\mathbb{F}, R/K)$$

for $1 \leq i \leq n$ (resp. $\mathrm{Tor}_1^R(\mathbb{F}, M) \to \mathrm{Tor}_1^R(\mathbb{F}, R/K)$). Assume that for all $1 \leq j \leq n$,

$$V_j + W \cap \cap_{i \neq j} V_i = \mathrm{Tor}_1^R(\mathbb{F}, R/K).$$

Then the projection map $N \to M$ induces an isomorphism $N \otimes_R \mathbb{F} \to M \otimes_R \mathbb{F}$.

Proof. From the Tor-exact sequence and using that $\oplus_{i=1}^n R/I_i \to (R/K)^{\oplus(n+1)}/\Delta(R/K)$ is an isomorphism after applying $- \otimes_R \mathbb{F}$, it suffices to show that the map

$$\mathrm{Tor}_1^R(\mathbb{F}, M \oplus \oplus_{i=1}^n R/I_i) \to \mathrm{Tor}_1^R(\mathbb{F}, R/K)^{\oplus(n+1)}/\Delta(\mathrm{Tor}_1^R(\mathbb{F}, R/K)) \tag{3.3}$$

is surjective. Writing an element of $\mathrm{Tor}_1^R(\mathbb{F}, R/K)^{\oplus(n+1)}$ as $(x_i)_{i=0}^n$, it suffices to show that for any $0 < j \leq n$ and $a_j \in \mathrm{Tor}_1^R(\mathbb{F}, R/K)$ and setting $a_i = 0$ for $i \neq j$, $(a_i)_{i=0}^n + \Delta(\mathrm{Tor}_1^R(\mathbb{F}, R/K))$ is in the image of (3.3). By assumption, we can write $a_j = b_j + c_j$ where $b_j \in V_j$ and $c_j \in W \cap \cap_{i \neq j} V_i$. Then $(a_i - c_j)_{i=0}^n \in (a_i)_{i=0}^n + \Delta(\mathrm{Tor}_1^R(\mathbb{F}, R/K))$ and $(a_i - c_j)_{i=0}^n$ is in the image of the map

$$\mathrm{Tor}_1^R(\mathbb{F}, \oplus_{i=1}^n R/I_i \oplus M) \to \mathrm{Tor}_1^R(\mathbb{F}, (R/K)^{\oplus(n+1)}).$$

□

In a similar fashion using the Tor-exact sequence we obtain

Lemma 3.16. *Let R be a local ring with residue field $\mathbb{F}$. Let $1 \leq n$ and $I_i \subset K$ be ideals of R for $i = 1, \dots, n$. Let N be the kernel of the map*

$$\oplus_{i=1}^n R/I_i \to (R/K)^{\oplus n}/\Delta(R/K)$$

induced by the sum of the natural maps $R/I_i \twoheadrightarrow R/K$, where $\Delta(R/K)$ denotes the diagonally embedded copy of R/K.

Then

$$\dim_{\mathbb{F}}(N \otimes_R \mathbb{F}) = 1 + \dim_{\mathbb{F}}\left(\mathrm{coker}\left(\oplus_{i=1}^n \mathrm{Tor}_1^R(\mathbb{F}, R/I_i) \to \mathrm{Tor}_1^R(\mathbb{F}, (R/K)^{\oplus n}/\Delta(R/K))\right)\right).$$

Lemma 3.17 ("Distortion" Lemma)**.** *Let $(R, \mathfrak{m})$ be a local $\mathcal{O}$-algebra with residue field $\mathbb{F}$ with $\mathcal{O}$ and $\mathbb{F}$ as in §1.4. Let $k \geq 2$ and $\{I_1, \dots, I_k\}$ be p-saturated ideals of R.*

Let $f \in R$ and assume that:

(1) for each $\ell = 1, \dots, k$ there exists $\varepsilon_\ell \in \mathfrak{m}$ such that $f + p\varepsilon_\ell \in I_\ell$
(2) we have

$$p \in \sum_{\ell=1}^k \bigcap_{i \neq \ell} I_i.$$

Then there exist $a_\ell \in \bigcap_{i \neq \ell} I_i$ for all $\ell = 1, \dots, k$ such that

- $f + \sum_\ell a_\ell \varepsilon_\ell \in \bigcap_{i=1}^k I_i$
- $\sum_\ell a_\ell \varepsilon_\ell \in \mathfrak{m}\Big(\bigcap_{i=1}^k (p) + I_i\Big)$.

Proof. By (2) we can find $a_\ell \in \bigcap_{i \neq \ell} I_i$ such that $p = \sum_\ell a_\ell$. Then $p(f + \sum_\ell a_\ell \varepsilon_\ell) = \sum_\ell a_\ell (f + p\varepsilon_\ell) \in \cap_{i=1}^k I_i$ hence $f + \sum_\ell a_\ell \varepsilon_\ell \in \cap_{i=1}^k I_i$ (because $\cap_{i=1}^k I_i$ is p-saturated).

Furthermore, for each ℓ, $a_\ell \in \bigcap_{i=1}^k \big((p) + I_i\big)$, hence the second item follows. ☐

Lemma 3.18. *Let R and S be complete local Noetherian $\mathbb{F}$-algebras. Let I be a proper ideal of R and M be a finitely generated S-module. Then for any $\psi \in \mathrm{Aut}_{R\widehat{\otimes}S}((R/I)\widehat{\otimes}M)$, there exists $\widetilde{\psi} \in \mathrm{Aut}_{R\widehat{\otimes}S}(R\widehat{\otimes}M)$ such that the diagram*

$$\begin{array}{ccc} R\widehat{\otimes}M & \xrightarrow{\ \widetilde{\psi}\ } & R\widehat{\otimes}M \\ \downarrow & & \downarrow \\ (R/I)\widehat{\otimes}M & \xrightarrow{\ \psi\ } & (R/I)\widehat{\otimes}M \end{array}$$

commutes where the vertical maps are the natural projections. (All completed tensor products are taken over $\mathbb{F}$.)

Proof. Observe that $\mathrm{Hom}_{R\widehat{\otimes}S}(R\widehat{\otimes}M, R\widehat{\otimes}-) \cong R\widehat{\otimes}\mathrm{Hom}_S(M,-)$ as functors on finitely generated S-modules, since they are left exact in M and are isomorphic when $M = S$. Hence $\mathrm{End}_{R\widehat{\otimes}S}(R\widehat{\otimes}M) \cong R\widehat{\otimes}\mathrm{End}_S(M)$ surjects onto $\mathrm{End}_{(R/I)\widehat{\otimes}S}((R/I)\widehat{\otimes}M) \cong (R/I)\widehat{\otimes}\mathrm{End}_S(M)$.

Thus any element $a \in \mathrm{Aut}_{R\widehat{\otimes}S}((R/I)\widehat{\otimes}M)$ can be lifted to an element $\widetilde{a} \in \mathrm{End}_{R\widehat{\otimes}S}(R\widehat{\otimes}M)$. We claim that any such lift is in fact in $\mathrm{Aut}_{R\widehat{\otimes}S}(R\widehat{\otimes}M)$. Consider the left ideal $\mathrm{End}_{R\widehat{\otimes}S}(R\widehat{\otimes}M)\cdot\widetilde{a}$. Since this ideal surjects onto $\mathrm{End}_{R\widehat{\otimes}S}((R/I)\widehat{\otimes}M)$, we have that

$$\mathrm{End}_{R\widehat{\otimes}S}(R\widehat{\otimes}M)\cdot\widetilde{a} + I\widehat{\otimes}S\cdot\mathrm{End}_{R\widehat{\otimes}S}(R\widehat{\otimes}M) = \mathrm{End}_{R\widehat{\otimes}S}(R\widehat{\otimes}M).$$

Since every term in this equation is a finitely generated left $R\widehat{\otimes}S$-module, Nakayama's lemma implies that $\mathrm{End}_{R\widehat{\otimes}S}(R\widehat{\otimes}M)\cdot\widetilde{a} = \mathrm{End}_{R\widehat{\otimes}S}(R\widehat{\otimes}M)$ so that $\widetilde{a} \in \mathrm{Aut}_{R\widehat{\otimes}S}(R\widehat{\otimes}M)$. ☐

3.3. **Special fiber.** Recall from 3.1 that (s,μ) is a fixed lowest alcove presentation for $\overline{\rho}$ and we assume from now on that μ is $N \geq 6$-deep. We write S, $S^{(j)}$, I_{T,∇_∞} etc. for the mod p reduction of $\widetilde{S}$, $\widetilde{S}^{(j)}$, $\widetilde{I}_{T,\nabla_\infty}$ etc.

Let $\tau \in T$. Then by [LLHLMb, Theorem 3.3.2], the minimal primes of $\overline{R}^{\eta,\tau}_{\overline{\rho}}$ are in bijection with the Serre weights $\sigma \in W^?(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}(\tau))$. Since the underlying topological space of Spec $\overline{R}^{\eta,T}_{\overline{\rho}}$ and $\bigcup_{\tau\in T}$ Spec $\overline{R}^{\eta,\tau}_{\overline{\rho}}$ coincide, and Spec $(S/I_{T,\nabla_\infty})$ is a formally smooth modification of Spec $\overline{R}^{\eta,T}_{\overline{\rho}}$ by Proposition 3.8, we learn that the minimal primes of $S/I_{T,\nabla_\infty}$ are in bijection with $W^?(\overline{\rho}) \cap \bigcup_{\tau\in T}\mathrm{JH}(\overline{\sigma}(\tau))$. We denote by $\mathfrak{P}_\sigma$ the miminal prime of $S/I_{T,\nabla_\infty}$ corresponding to σ.

We now give an approximation of $\overline{R}^{\eta,T}_{\overline{\rho}}$. Let $\widetilde{I}^{(j)}_{\tau,\nabla_{\mathrm{alg}}}$ be the ideal of $\widetilde{S}^{(j)}$ generated by the elements listed in in row $\widetilde{I}^{(j)}_{\tau,\nabla_\infty}$ of Tables 3, 4, 5 *without their $O(p^{N-4})$-tails* if $T^{(j)}$ is as in item (I)–(II) (resp. the ideal appearing in [LLHLMb, Table 2] row $\widetilde{z}_j$ if $T^{(j)} = \{\widetilde{z}_j\}$ is as in item (III)), and write $\widetilde{I}_{T,\nabla_{\mathrm{alg}}}$ for the ideal $\bigg(\bigcap_{\tau\in T}\Big(\sum_{j\in\mathcal{J}}\widetilde{I}^{(j)}_{\tau,\nabla_{\mathrm{alg}}}\widetilde{S}\Big)\bigg)$ of $\widetilde{S}$. Write $I_{T,\nabla_{\mathrm{alg}}}$ for the image of $\widetilde{I}_{T,\nabla_{\mathrm{alg}}}$ in S, and similarly define $I^{(j)}_{\tau,\nabla_{\mathrm{alg}}}$ for $\tau \in T$, $j \in \mathcal{J}$.

Proposition 3.19. *Fix $j \in \mathcal{J}$ and assume T has the following form:*
(1) $\#T^{(j')} = 1$ if $j' \neq j$;
(2) either $T^{(j)} \subseteq \{\alpha\beta\alpha t_{\underline{1}}, \beta\alpha t_{\underline{1}}, \alpha\beta t_{\underline{1}}, t_{\underline{1}}\}$ or $T^{(j)} \subseteq \{t_{w_0(\eta)}, \alpha t_{w_0(\eta)}, \beta t_{w_0(\eta)}, w_0 t_{w_0(\eta)}\}$.
Assume that μ is $N > 10$ deep in $\underline{C}_0$. Then we have a surjection

$$S/I_{T,\nabla_{\mathrm{alg}}} \twoheadrightarrow S/I_{T,\nabla_\infty}. \tag{3.4}$$

For each $\sigma \in W^?(\overline{\rho}) \cap \bigcup_{\tau\in T} \mathrm{JH}(\overline{\sigma}(\tau))$, $\overline{\mathfrak{P}}_\sigma$ pulls back to the prime ideal $\sum_{j=0}^{f-1} \mathfrak{P}^{(j)}_{(\varepsilon_j,a_j)} S$ of S where:
- *$(\varepsilon, a) = ((\varepsilon_j, a_j))_{j\in\mathcal{J}} \in r(\Sigma)$ is such that $\sigma = F(\mathfrak{Tr}_{\mu+2\eta}(s(\varepsilon, a)))$;*
- *the ideal $\mathfrak{P}^{(j)}_{(\varepsilon_j,a_j)}$ is the prime ideal of $S^{(j)}$ described in Table 8 (resp. Table 9) if $T^{(j)} \subseteq \{\alpha\beta\alpha t_{\underline{1}}, \beta\alpha t_{\underline{1}}, \alpha\beta t_{\underline{1}}, t_{\underline{1}}\}$ (resp. if $T^{(j)} \subseteq \{t_{w_0(\eta)}, \alpha t_{w_0(\eta)}, \beta t_{w_0(\eta)}, w_0 t_{w_0(\eta)}\}$).*

Proof. Letting $\widetilde{I}_{\tau,\nabla_{\mathrm{alg}}} \overset{\mathrm{def}}{=} \sum_j \widetilde{I}^{(j)}_{\tau,\nabla_{\mathrm{alg}}} \widetilde{S}$, the existence of the surjection (3.4) is equivalent to the inclusion

$$\bigcap_{\tau\in T} \widetilde{I}_{\tau,\nabla_{\mathrm{alg}}} + (p) \subseteq \bigcap_{\tau\in T} \widetilde{I}_{\tau,\nabla_\infty} + (p)$$

For each $j \in \mathcal{J}$ and $\tau \in T$ let $\{g^{(j)}_{\tau,i,\mathrm{alg}}\}_i$ be the set of generators of $\widetilde{I}^{(j)}_{\tau,\nabla_{\mathrm{alg}}}$ listed in row $\widetilde{w}^*(\overline{\rho}, \tau)_j$ of Tables 3, 4, 5 without their $\mathcal{O}(p^{N-4})$-tails, so that we can write $g^{(j)}_{\widetilde{z}_j,i,\mathrm{alg}} = g^{(j)}_{\tau,i,\infty} - \mathcal{O}(p^{N-4})$, where $\mathcal{O}(p^{N-4})$ is an element of $p^{N-4}\widetilde{S}$ (depending on τ) and $\{g^{(j)}_{\tau,i,\infty}\}_i \subset \widetilde{I}_{\tau,\nabla_\infty}$.

By Proposition 3.11 and the fact that $\widetilde{I}^{(j)}_{\tau,\nabla_{\mathrm{alg}}} \subseteq \widetilde{S}^{(j)}$ for any $j \in \mathcal{J}$ we have

$$p^6\Big(\bigcap_{\tau\in T} \widetilde{I}_{\tau,\nabla_{\mathrm{alg}}}\Big) = p^6\Big(\sum_{j\in\mathcal{J}} \Big(\bigcap_{\tau\in T} \widetilde{I}^{(j)}_{\tau,\nabla_{\mathrm{alg}}}\Big)\widetilde{S}\Big) \subseteq \sum_{j\in\mathcal{J}} \Big(\prod_{\tau\in T} \widetilde{I}^{(j)}_{\tau,\nabla_{\mathrm{alg}}}\Big)\widetilde{S}. \tag{3.5}$$

Given $f \in \bigcap_{\tau\in T} \widetilde{I}_{\tau,\nabla_{\mathrm{alg}}}$ we can write $p^6 f$ as a $\widetilde{S}$-linear combination of multiples of $\prod_{\tau\in T} g^{(j)}_{\tau,i,\mathrm{alg}}$. Setting $\widetilde{f}_\infty$ to be the same linear combination as $p^6 f$ but replacing $\prod_{\tau\in T} g^{(j)}_{\tau,i,\mathrm{alg}}$ by $\prod_{\tau\in T} g^{(j)}_{\tau,i,\infty}$, we thus have

$$p^6 f + \mathcal{O}(p^{N-4}) = \widetilde{f}_\infty \in \bigcap_{\tau\in T} \widetilde{I}_{\tau,\nabla_\infty}.$$

yielding $p^6(f + \mathcal{O}(p^{N-4-6})) \in \bigcap_{\tau\in T} \widetilde{I}_{\tau,\nabla_\infty}$ by the assumption on N. As the intersection of the p-saturated ideals $\widetilde{I}_{\tau,\nabla_\infty}$ is again p-saturated we conclude that $f + \mathcal{O}(p^{N-4-6}) \in \bigcap_{\tau\in T} \widetilde{I}_{\tau,\nabla_\infty}$. The labeling and the explicit equations for the ideals $\mathfrak{P}^{(j)}_{(\varepsilon_j,a_j)}$ follow from in [LLHLMb, Theorem 3.3.2]. □

Remark 3.20. Fix $\tau \in T$. The statement of Proposition 3.19 can be improved, replacing condition (1) by

(1') if $j' \neq j$ then either $T^{(j')} \subseteq \{\alpha\beta\alpha\gamma t_{\underline{1}}, \alpha\beta\alpha t_{\underline{1}}, \beta\alpha t_{\underline{1}}, \alpha\beta t_{\underline{1}}, t_{\underline{1}}\}$ or $T^{(j')} \subseteq \{t_{w_0(\eta)}, t_{w_0(\eta)}\alpha, t_{w_0(\eta)}\beta, t_{w_0(\eta)}w_0\}$

and with the following more precise condition on N:

(3.6)

$$N-4 > \max_{j\in\mathcal{J}} \Big\{\big(\#\{\tau' \neq \tau \mid \ell\big(\widetilde{w}^*(\overline{\rho}, \tau)_j^{-1}\widetilde{w}(\overline{\rho}, \tau')_j\big) \leq 1\}\big) + 2\big(\#\{\tau' \neq \tau \mid \ell\big(\widetilde{w}^*(\overline{\rho}, \tau)_j^{-1}\widetilde{w}(\overline{\rho}, \tau')_j\big) > 1\}\big)\Big\}.$$

This is because in the proof of Proposition 3.19 the inclusion (3.5) and the reasoning following it still holds true when replacing 6 by the right hand side of (3.6).

Moreover, under the stronger assumption
(3.7)
$$N-4 > \Big(\#\{\tau'\neq\tau \mid \max_j \ell\big(\widetilde{w}^*(\overline{\rho},\tau)_j^{-1}\widetilde{w}(\overline{\rho},\tau')_j\big)\leq 1\}\Big)+2\Big(\#\{\tau'\neq\tau \mid \max_j \ell\big(\widetilde{w}^*(\overline{\rho},\tau)_j^{-1}\widetilde{w}(\overline{\rho},\tau')_j\big)> 1\}\Big).$$
the surjection (3.4) is actually an isomorphism. Indeed, again Proposition 3.11 gives the inclusion $p^{(\text{RHS of (3.7)})}\Big(\bigcap_{\tau\in T}\widetilde{I}_{\tau,\nabla_\infty}\Big)\subseteq\prod_{\tau\in T}\widetilde{I}_{\tau,\nabla_\infty}$ and the argument of Proposition 3.19 can be now performed by reversing the roles of $\widetilde{I}_{\tau,\mathrm{alg}}$ and $g^{(j)}_{\tau,i,\mathrm{alg}}$ with $\widetilde{I}_{\tau,\infty}$ and $g^{(j)}_{\tau,i,\infty}$, and again replacing 6 by the RHS of (3.7).

In particular whenever $N-4$ satisfies condition (3.6) we have a commutative diagram

(3.8)
$$\begin{array}{ccc} S/I_{T,\nabla_{\mathrm{alg}}} & \twoheadrightarrow & S/I_{T,\nabla_\infty} \\ \downarrow & & \downarrow \\ S/I_{\tau,\nabla_{\mathrm{alg}}} & \xrightarrow{\sim} & S/I_{\tau,\nabla_\infty} \end{array}$$

where horizontal arrow is an isomorphism (by applying the previous paragraph to the case $\#T=1$) and the vertical maps are the canonical surjections.

3.4. **Ideal relations in multi-type deformation rings.** Let $\overline{\rho}:G_K\to\mathrm{GL}_3(\mathbb{F})$ be a continuous semisimple Galois representation together with a lowest alcove presentation (s,μ) where μ is $N>10$-deep. In this section we record facts about $R^{\eta,T}_{\overline{\rho}}$ that we will need later.

Fix $j\in\mathcal{J}$. Recall the running assumption that T satisfies Hypothesis 3.2. We now assume additionally that

(IV) $\alpha\beta\alpha\gamma t_{\underline{1}}\notin T^{(j)}$;

(V) For $j'\neq j$, $\#T^{(j')}=1$ and $\widetilde{w}^*(\overline{\rho},\tau)_{j'}\in\{\alpha\beta\alpha t_{\underline{1}},\ \beta\gamma\beta t_{\underline{1}},\ \gamma\alpha\gamma t_{\underline{1}},\ \}$.

By (IV) we can thus replace the ring $\widetilde{S}^{(j)}$ appearing in Tables 3, 4 by $\widetilde{S}^{(j)}/(e_{11})$, and omit the variable e_{11} in the computations of these sections. Throughout this section let $(a,b,c)\in\mathbb{F}_p^3$ be the mod p reduction of $-(s_j^{-1}(\mu_j+\eta))$.

3.4.1. *Analysis of $S^{(j)}/I^{(j)}_{T,\nabla_{\mathrm{alg}}}$.* In what follows, we assume that $T^{(j)}$ is as in item (I). Given $b_j\in\{B,F_s,E_s,F_o,E_o\}$ we define $I_j^{b_j}\subset S^{(j)}$ to be the intersection $\mathfrak{P}^{(j)}_{(0,0)}\cap\mathfrak{P}^{(j)}_{(\omega,a)}$, where $(\omega,a)=(0,1)$, $(\varepsilon_1,0)$, $(\varepsilon_2,0)$, $(\varepsilon_2-\varepsilon_1,1)$, $(\varepsilon_1-\varepsilon_2,1)$ respectively if $b_j=B$, F_s, E_s, F_o, E_o.

Lemma 3.21. *Let $b_j\in\{B,E_o,F_o,E_s,F_s\}$. Then the ideal $I_j^{b_j}$ is given by:*

(1) $(c_{33},c_{32},c_{31},c_{23},c_{22},c_{21},c_{13}d_{32}-c_{12}d^*_{33},c_{13}d_{31}-c_{11}d^*_{33},c_{12}d_{31}-c_{11}d_{32},(b-c)c_{12}d_{21}-(a-c)c_{11}d^*_{22}))$
if $b_j=B$;

(2) $c_{33},c_{32},c_{31},c_{21},c_{11},c_{23}d_{31},c_{22}d_{31},c_{13}d_{31},c_{12}d_{31},c_{13}c_{22}-c_{12}c_{23},c_{13}d_{21}-c_{23}d^*_{11},c_{12}d_{21}-c_{22}d^*_{11},(a-c-1)c_{23}d_{32}-(a-b-1)c_{22}d^*_{33},(a-c-1)c_{13}d_{32}-(a-b-1)c_{12}d^*_{33}$
if $b_j=F_s$;

(3) $(c_{32},c_{31},c_{22},c_{21},c_{12},c_{11},c_{23}d_{32}-c_{33}d^*_{33},c_{23}d_{31}-d_{21}c_{33},(a-b)c_{13}d_{31}+(b-c-1)c_{33}d^*_{11},(a-b)c_{13}d_{21}+(b-c-1)c_{23}d^*_{11},c_{13}d_{21}d_{32}-c_{13}d_{31}d^*_{22})$
if $b_j=E_s$;

(4) $(c_{33},c_{32},c_{31},c_{22},c_{13},c_{12},c_{11},c_{23}d_{32},c_{21}d_{32},(b-c-1)c_{23}d_{31}+(a-b+1)c_{21}d^*_{33})$
if $b_j=F_o$;

(5) $(c_{33}, c_{31}, c_{23}, c_{22}, c_{21}, c_{13}, c_{11}, d_{21}c_{32}, c_{12}d_{21}, (a-c)c_{12}d_{31} + (-b+c-1)c_{32}d_{11}^*)$ *if* $b_j = E_o$.

Proof. See [LLHM, §B.1.1]. □

For $b_j \in \{B, F_s, E_s, E_o, F_o\}$ let M^{b_j} be the $S^{(j)}$-module $S^{(j)}/I_j^{b_j}$. Let $M^{\emptyset}$ be the $S^{(j)}$-module $S^{(j)}/\mathfrak{P}^{(j)}_{(0,0)}$. For $a_j = \{B, F_s, E_s, E_o, F_o\}$ (resp. $a_j = \{B, F_s, E_s\}$) let M^{a_j} be the kernel of the natural surjective map

$$\oplus_{b_j \in a_j} M^{b_j} \to (M^{\emptyset})^{\oplus \# a_j}/\Delta(M^{\emptyset}) \to 0$$

where $\Delta(M^{\emptyset})$ denotes the diagonally embedded copy of $M^{\emptyset}$ in $(M^{\emptyset})^{\oplus \# a_j}$.

Proposition 3.22. *For either* $a_j = \{B, F_s, E_s, E_o, F_o\}$ *or* $a_j = \{B, F_s, E_s\}$ *we have*

$$\dim_{\mathbb{F}} \left(M^{a_j} \otimes_{S^{(j)}} S^{(j)}/\mathfrak{m}_{S^{(j)}} \right) = 3.$$

Proof. By Lemma 3.16 it is enough to check that the cokernel of the natural map

$$\oplus_{b_j \in a_j} \mathrm{Tor}_1^{S^{(j)}}(\mathbb{F}, M^{b_j}) \to \mathrm{Tor}_1^{S^{(j)}}(\mathbb{F}, (M^{\emptyset})^{\oplus \# a_j}/\Delta(M^{\emptyset})) \tag{3.9}$$

is two dimensional. Note that given an ideal $I \subseteq S^{(j)}$ we have $\mathrm{Tor}_1^{S^{(j)}}(\mathbb{F}, S^{(j)}/I) \cong I/(\mathfrak{m}_{S^{(j)}} \cdot I)$, which allows us to write elements of $\mathrm{Tor}_1^{S^{(j)}}(\mathbb{F}, S^{(j)}/I)$ in terms of generators of I. In particular we see that $\mathrm{Tor}_1^{S^{(j)}}(\mathbb{F}, M^{\emptyset})$ has a basis consisting of the image of the elements c_{ik} for $1 \leq i, k \leq 3$. An immediate check on the generators of $I_j^{b_j}$ (described in Lemma 3.21) shows that the image of the natural map $\mathrm{Tor}_1^{S^{(j)}}(\mathbb{F}, M^{b_j}) \to \mathrm{Tor}_1^{S^{(j)}}(\mathbb{F}, M^{\emptyset})$ has the following description according to b_j:

(1) if $b_j \in \{B, F_s, E_s\}$ it is the subspace of $\mathrm{Tor}_1^{S^{(j)}}(\mathbb{F}, M^{\emptyset})$ generated by c_{ik}, $1 \leq i, k \leq 3$, $(ik) \neq (13)$;
(2) if $b_j = F_o$ it is the subspace of $\mathrm{Tor}_1^{S^{(j)}}(\mathbb{F}, M^{\emptyset})$ generated by c_{ik} for $1 \leq i, k \leq 3$, $(ik) \neq (23)$; and
(3) if $b_j = E_o$ it is the subspace of $\mathrm{Tor}_1^{S^{(j)}}(\mathbb{F}, M^{\emptyset})$ generated by c_{ik} for $1 \leq i, k \leq 3$, $(ik) \neq (12)$.

We deduce that the cokernel of the natural map

$$\oplus_{b_j \in a_j} \mathrm{Tor}_1^{S^{(j)}}(\mathbb{F}, M^{b_j}) \to \mathrm{Tor}_1^{S^{(j)}}(\mathbb{F}, (M^{\emptyset})^{\# a_j}) \tag{3.10}$$

has dimension $\# a_j$, with basis given by (the image of) the elements $\{c_{13}^{b_j}\}_{b_j \in \{B, F_s, E_s,\}}$, and $c_{23}^{F_o}$, $c_{12}^{E_o}$ if $\{E_o, F_o\} \subset a_j$ (the superscripts on the c_{ik} denote which copy of $\mathrm{Tor}_1^{S^{(j)}}(\mathbb{F}, M^{\emptyset}) \subseteq \mathrm{Tor}_1^{S^{(j)}}(\mathbb{F}, (M^{\emptyset})^{\oplus \# a_j})$ the element c_{ik} lives in). On the other hand the image of $\Delta(\mathrm{Tor}_1^{S^{(j)}}(\mathbb{F}, M^{\emptyset}))$ in the cokernel of (3.10) is generated by $c_{13}^B + c_{13}^{E_s} + c_{13}^{F_s}$, $c_{23}^{F_o}$, $c_{12}^{E_o}$ if $\# a_j = 5$ and by $c_{13}^B + c_{13}^{E_s} + c_{13}^{F_s}$ if $\# a_j = 3$. In both cases we conclude that the cokernel of (3.9) has dimension 2. □

3.4.2. *Surgery for* $T^{(j)} = \{t_{\underline{1}}\}$. Suppose that $T^{(j)} = \{t_{\underline{1}}\}$ for all $j \in \mathcal{J}$. We write τ for the unique element in T. We now fix $j \in \mathcal{J}$. We abbreviate $\widetilde{R} \overset{\text{def}}{=} \widetilde{S}^{(j)}/\widetilde{I}^{(j)}_{\tau, \nabla_{\text{alg}}}$, and $R \overset{\text{def}}{=} \widetilde{R}/(p)$ so that a presentation for $\widetilde{R}$ is given in row $t_{\underline{1}}$ of Table 4. The ring $\widetilde{R}$ is normal and Cohen–Macaulay by [LLHLM18, Corollary 8.9]. Let $\mathrm{j} : U \hookrightarrow \mathrm{Spec}\, \widetilde{R}$ be the complement of the vanishing locus of

$$\prod_{(\omega,a) \neq (\nu,b) \in \Sigma_0} \mathfrak{P}^{(j)}_{(\omega,a)} + \mathfrak{P}^{(j)}_{(\nu,b)}.$$

Then U is a regular scheme by the proof of [LLHLM20, Lemma 5.2.1]. For a coherent reflexive (i.e. coherent, S_2, and torsion-free) $\widetilde{R}$-module $\widetilde{M}$ and an effective divisor $D = \sum_{(\omega,a)\in\Sigma_0} n_{(\omega,a)}[\mathfrak{P}^{(j)}_{(\omega,a)}]$ of U supported in the special fiber (if $\mathfrak{P}^{(j)}_{(\omega,a)} = R$, then take $[\mathfrak{P}^{(j)}_{(\omega,a)}] = 0$), define

$$\widetilde{M}(-D) \overset{\text{def}}{=} \mathfrak{j}_*\mathfrak{j}^* \prod_{(\omega,a)\in\Sigma_0} (\mathfrak{P}^{(j)}_{(\omega,a)})^{n_{(\omega,a)}}\widetilde{M}. \tag{3.11}$$

Since $\mathfrak{j}^* \prod_{(\omega,a)\in\Sigma_0}(\mathfrak{P}^{(j)}_{(\omega,a)})^{n_{(\omega,a)}}$ is locally free on U, $\mathfrak{j}^* \prod_{(\omega,a)\in\Sigma_0}(\mathfrak{P}^{(j)}_{(\omega,a)})^{n_{(\omega,a)}}\widetilde{M}$ is coherent and reflexive so that $\widetilde{M}(-D) = \mathfrak{j}_*\mathfrak{j}^*\widetilde{M}(-D)$ is its unique (up to isomorphism) coherent and reflexive extension by [Sta23, Tag 0EBJ].

Lemma 3.23. *If $D = \sum_{(\omega,a)\in\Sigma_0} n_{(\omega,a)}[\mathfrak{P}^{(j)}_{(\omega,a)}]$ with all $0 \le n_{(\omega,a)} \le 1$, then*

$$\widetilde{R}(-D) = \cap_{\substack{(\omega,a)\in\Sigma_0\\ n_{(\omega,a)}=1}} \mathfrak{P}^{(j)}_{(\omega,a)}.$$

Proof. Since $\widetilde{R}/\widetilde{R}(-D)$ is S_1 (as $\widetilde{R}$ and $\widetilde{R}(-D)$ are both S_2) and R_0, it is reduced. Thus $\widetilde{R}(-D)$ is the intersection of prime ideals. Since $\mathfrak{j}^*\widetilde{R}(-D) = \mathfrak{j}^* \cap_{\substack{(\omega,a)\in\Sigma_0\\ n_{(\omega,a)}=1}} \mathfrak{P}^{(j)}_{(\omega,a)}$, the result follows. □

Let $\widetilde{M}^{(j)}_{(0,1)}$ be a free $\widetilde{R}$-module of rank one. Recall from §2.1.3 that Σ_0 is a connected graph endowed with a distance function which we denote d. For $(\omega,a) \in \Sigma_0$, we define

$$\widetilde{M}^{(j)}_{(\omega,a)} = \widetilde{M}^{(j)}_{(0,1)}\Big(- \sum_{(\nu,b)\in\Sigma_0} \frac{1}{2}\Big(d((\omega,a),(0,1)) + d((\omega,a),(\nu,b)) - d((0,1),(\nu,b))\Big)[\mathfrak{P}^{(j)}_{(\nu,b)}]\Big). \tag{3.12}$$

Lemma 3.24. *We have $\widetilde{M}^{(j)}_{(\varepsilon_1,1)} = (-c_{23}d_{32} + c_{22}d^*_{33} + c_{33}d^*_{22} + pd^*_{22}d^*_{33})\widetilde{M}^{(j)}_{(0,1)}$ and $\widetilde{M}^{(j)}_{(\varepsilon_2,1)} = (c_{33} + pd^*_{33})\widetilde{M}^{(j)}_{(0,1)}$, where we have omitted (j)-superscripts in the variables.*

Proof. We will show that $M_1 \overset{\text{def}}{=} \widetilde{M}^{(j)}_{(\varepsilon_2,1)}$ and $M_2 \overset{\text{def}}{=} (c_{33} + pd^*_{33})\widetilde{M}^{(j)}_{(0,1)}$ are equal as the proof of the other claim is similar. Let M be $\widetilde{M}^{(j)}_{(0,1)}$. As M_2 is free of rank one, it is coherent and reflexive. By [Sta23, Tag 0EBJ], it suffices to show that the locally free sheaves $\mathfrak{j}^*M_1$ and $\mathfrak{j}^*M_2$ of rank 1 on U are equal, or that M/M_1 and M/M_2 have the same length after localization at the minimal primes of R. First, for a minimal prime $\mathfrak{P}$ of Spec R, $M_{\mathfrak{P}}/(c_{33},p)M_{\mathfrak{P}}$ has length 0 or 1 with length equal to 1 if and only if $c_{33} \in \mathfrak{P}$ since the special fiber is reduced. We see that the length is 0 if and only $\mathfrak{P} = \mathfrak{P}^{(j)}_{(\varepsilon_2,1)}$.

Second, we claim that

$$(c_{33} + pd^*_{33})(-c_{12}d_{21} + c_{11}d^*_{22} + c_{22}d^*_{11} + pd^*_{11}d^*_{22}) = p^2 d^*_{11}d^*_{22}d^*_{33} \tag{3.13}$$

so that p annihilates $(c_{33},p)M_{\mathfrak{P}}/(c_{33} + pd^*_{33})M_{\mathfrak{P}}$. Indeed, since the determinant of $M^{T,(j)}$ equals $(v+p)^3 d^*_{11}d^*_{22}d^*_{33}$, specializing at $v = 0$ gives

$$\prod_{i=1}^{3}(c_{ii} + pd^*_{ii}) = p^3 d^*_{11}d^*_{22}d^*_{33}.$$

Moreover, the top-left 2×2 minor is divisible by $v+p$ and the $(2,1)$-entry is divisible by v so that specializing at $v = -p$ gives $c_{11}c_{22} = -pc_{12}d_{21}$. Combining these and dividing both sides by p gives the claim.

By the claim, $(c_{33}, p)M_{\mathfrak{P}}/(c_{33} + pd^*_{33})M_{\mathfrak{P}}$ has length 0 or 1 with length equal to 0 if and only if $\frac{p}{c_{33}+pd^*_{33}} \in \widetilde{R}_{\mathfrak{P}}$ or equivalently, by (3.13),

$$-c_{12}d_{21} + c_{11}d^*_{22} + c_{22}d^*_{11} \in \mathfrak{P}.$$

We see from Table 8 that the length is 1 if and only if $\mathfrak{P} = \mathfrak{P}^{(j)}_{(0,1)}$.

It is now easy to check the desired length statement from (3.12). □

We define a *path* to be a sequence of elements $\gamma = (\gamma_k)_{k\geq 1}^{\ell(\gamma)}$ in $\{(0,0), (\varepsilon_1, 0), (\varepsilon_2, 0), (0,1), (\varepsilon_1, 1), (\varepsilon_2, 1)\}$ of length $\ell(\gamma) = 2$ or 3 such that

- the γ_k are distinct;
- γ_k and γ_{k+1} are adjacent for $1 \leq k \leq \ell(\gamma) - 1$; and
- $\gamma_1 = (\varepsilon_1, 1)$ or $(\varepsilon_2, 1)$.

For paths β and γ, we write $\beta \leq \gamma$ if $\ell(\beta) \leq \ell(\gamma)$ and $\beta_k = \gamma_k$ for $1 \leq k \leq \ell(\beta)$. For a path γ, we define subsets $\Sigma_\gamma \subset \Sigma_0$ as follows:

- If $\ell(\gamma) = 2$, then $\Sigma_\gamma = \{\gamma_2, (0,1), (\varepsilon_1, 1), (\varepsilon_2, 1)\} \setminus \{\gamma_1\}$.
- If $\ell(\gamma) = 3$, then $\Sigma_\gamma = \{\gamma_3\}$.

Given a path γ, we define $M^{(j)}_\gamma$ to be $\widetilde{M}^{(j)}_{\gamma_1}(-D_\gamma)/p\widetilde{M}^{(j)}_{\gamma_1}$ where

$$D_\gamma \stackrel{\text{def}}{=} \sum_{(\varepsilon,a)\notin\Sigma_\gamma} [\mathfrak{P}^{(j)}_{(\varepsilon,a)}].$$

Then by Lemmas 3.23 and 3.24, $M^{(j)}_\gamma$ is naturally identified with $I_\gamma(\widetilde{M}^{(j)}_{\gamma_1} \otimes_{\mathcal{O}} \mathbb{F})$ where we define I_γ to be the image of $\bigcap_{(\varepsilon,a)\notin\Sigma_\gamma} \mathfrak{P}^{(j)}_{(\varepsilon,a)}$ in R. Note that $I_\gamma \supseteq I_\beta$ if $\beta \geq \gamma$. Moreover, note that different paths of length three can give rise to the same ideal I_γ (e.g. I_γ depends only on γ_3 and not on γ_2). In fact, if $\ell(\beta) = 3 = \ell(\gamma)$ and $\beta_3 = \gamma_3$, then $M^{(j)}_\beta$ and $M^{(j)}_\gamma$ are naturally identified. This follows easily from the definition (3.11) if $\beta_3 = \gamma_3 \neq (0,1)$. If $\beta_3 = \gamma_3 = (0,1)$, then $M^{(j)}_\beta$ and $M^{(j)}_\gamma$ are both identified with $p^2\widetilde{M}^{(j)}_{(0,1)}/p^2\widetilde{M}^{(j)}_{(0,1)}(-[\mathfrak{P}^{(j)}_{(0,1)}])$. Indeed, there are inclusions $p^2\widetilde{M}^{(j)}_{(0,1)} \subset \widetilde{M}^{(j)}_{(\varepsilon_i,1)}$ but $p^2\widetilde{M}^{(j)}_{(0,1)} \not\subset p\widetilde{M}^{(j)}_{(\varepsilon_i,1)}$ for $i = 1$ and 2 by (3.12). Then for $\alpha = \beta$ and γ, we claim that the natural map $p^2\widetilde{M}^{(j)}_{(0,1)} \to M^{(j)}_\alpha$ is surjective and induces an isomorphism $p^2\widetilde{M}^{(j)}_{(0,1)}/p^2\widetilde{M}^{(j)}_{(0,1)}(-[\mathfrak{P}^{(j)}_{(0,1)}]) \cong M^{(j)}_\alpha$. We explain for $\alpha_1 = (\varepsilon_1, 1)$, the other case being similar. The surjectivity follows from the fact that $p^2\widetilde{M}^{(j)}_{(0,1)} = (c_{11} + pd^*_{11})\widetilde{M}^{(j)}_{(\varepsilon_1,1)}$ and Lemma 3.25 below. The kernel of this surjection is determined by the fact that the image of the map is isomorphic to $R/\mathfrak{P}^{(j)}_{(0,1)}$.

The computations for Lemma 3.25, similar to those of [LLHLM20, §3.6.3], are recorded in [LLHM, §B.1.2].

Lemma 3.25. *Let γ be a path. Then I_γ is minimally generated by $4-\ell(\gamma)$ elements, and a minimal set of generators for I_γ is given by:*

$$\begin{array}{ll}
(c_{22}\dfrac{d_{11}^*}{d_{22}^*},\ d_{31}d_{22}^* - d_{21}d_{32}) & \text{if } \gamma = \big((\varepsilon_2,1),(0,0)\big);\\
(c_{22}\dfrac{d_{11}^*}{d_{22}^*}, c_{12}) & \text{if } \gamma = \big((\varepsilon_2,1),(\varepsilon_2,0)\big);\\
(c_{22}\dfrac{d_{11}^*}{d_{22}^*}, (a-b)c_{13}d_{21} + (b-c-1)c_{23}d_{11}^*) & \text{if } \gamma = \big((\varepsilon_2,1),(\varepsilon_1,0)\big);\\
(c_{33}\dfrac{d_{11}^*}{d_{33}^*},\ d_{31}) & \text{if } \gamma = \big((\varepsilon_1,1),(0,0)\big);\\
(c_{33}\dfrac{d_{11}^*}{d_{33}^*}, (c+1-a)c_{13}d_{32} + (a-b-1)c_{12}d_{33}^*) & \text{if } \gamma = \big((\varepsilon_1,1),(\varepsilon_2,0)\big);\\
(c_{33}\dfrac{d_{11}^*}{d_{33}^*}, c_{13}d_{21} - c_{23}d_{11}^*) & \text{if } \gamma = \big((\varepsilon_1,1),(\varepsilon_1,0)\big);\\
(c_{11}) & \text{if } \ell(\gamma)=3, \gamma_3 = (0,1);\\
(c_{22}) & \text{if } \ell(\gamma)=3, \gamma_3 = (\varepsilon_1,1);\\
(c_{33}) & \text{if } \ell(\gamma)=3, \gamma_3 = (\varepsilon_2,1).
\end{array}$$

If γ is a path of length 3, $\gamma > \beta$, and $\kappa_\gamma \in \mathbb{F}^\times$, we define the map

$$\iota^\gamma_{\kappa_\gamma} : M_\gamma^{(j)} \longrightarrow M_\beta^{(j)} \tag{3.14}$$

to be κ_γ times the natural inclusion. If one chooses a generator of $\widetilde{M}^{(j)}_{\gamma_1} \otimes_{\mathcal{O}} \mathbb{F}$, then (3.14) is identified with the map

$$\iota^\gamma_{\kappa_\gamma} : I_\gamma \longrightarrow I_\beta \tag{3.15}$$

defined in an analogous way. For $\kappa = (\kappa_\gamma)_{\gamma,\ell(\gamma)=3} \in (\mathbb{F}^\times)^{12}$, we get a collection of maps $\iota^\gamma_{\kappa_\gamma}$ which induce a map

$$\iota_\kappa : \bigoplus_{\gamma,\ell(\gamma)=3} M_\gamma^{(j)} \to \bigoplus_{\beta,\ell(\beta)=2} M_\beta^{(j)}.$$

We let $(\bigoplus_{\gamma,\ell(\gamma)=3} M_\gamma^{(j)})^0$ be the subspace cut out by the conditions

$$\sum_{\substack{\gamma,\ell(\gamma)=3,\\ \gamma_3=(\omega,1)}} a_\gamma = 0$$

for $\omega \in \{0,\varepsilon_1,\varepsilon_2\}$ (recall that the M_γ with $\ell(\gamma)=3$ and $\gamma_3=(\omega,1)$ are identified). We define ι^0_κ to be the restriction of ι_κ to $(\bigoplus_{\gamma,\ell(\gamma)=3} M_\gamma^{(j)})^0$, and let M_κ be the cokernel of ι^0_κ.

In order to to compute M_κ in certain instances, we choose generators for $\widetilde{M}^{(j)}_{(\varepsilon_i,1)}$. Fix a generator $m \in \widetilde{M}^{(j)}_{(0,1)}$ and generators $(-c_{23}d_{32}+c_{22}d_{33}^*+c_{33}d_{22}^*+pd_{22}^*d_{33}^*)m$ and $\frac{b-a}{b-c}d_{22}^*(c_{33}+pd_{33}^*)m$ of $\widetilde{M}^{(j)}_{(\varepsilon_1,1)}$ and $\widetilde{M}^{(j)}_{(\varepsilon_2,1)}$, respectively. Using these generators, we get identifications of $M_\gamma^{(j)} = I_\gamma$ such that if $\ell(\gamma') = 3 = \ell(\gamma)$ and $\gamma'_3 = \gamma_3$, then the identification of $M^{(j)}_{\gamma'}$ and $M^{(j)}_\gamma$ is compatible with the equality $I_{\gamma'} = I_\gamma$. (When $\gamma_3 = (0,1)$, then $I_\gamma = (c_{11}, p)\widetilde{R}/p\widetilde{R}$. One checks from Table 4 that

$$\frac{b-a}{b-c} c_{11} d^*_{22} = -c_{12}d_{21} + c_{11}d^*_{22} + c_{22}d^*_{11}$$

in R so that modulo $p(c_{33} + pd^*_{33})\widetilde{M}^{(j)}_{(0,1)}$, we have

$$\begin{aligned}(c_{11} + pd^*_{11})\frac{b-a}{b-c} d^*_{22}(c_{33} + pd^*_{33})m &\equiv p^2 d^*_{11}d^*_{22}d^*_{33}m \\ &= (c_{11} + pd^*_{11})(-c_{23}d_{32} + c_{22}d^*_{33} + c_{33}d^*_{22} + pd^*_{22}d^*_{33})m\end{aligned}$$

and the equality is an analogue of (3.13).) Then M_κ is isomorphic to the cokernel of the map

$$\iota^0_\kappa : \left(\bigoplus_{\gamma,\ell(\gamma)=3} I_\gamma\right)^0 \to \bigoplus_{\beta,\ell(\beta)=2} I_\beta \tag{3.16}$$

where the notation $(-)^0$ denotes the subspace cut out by the conditions

$$\sum_{\substack{\gamma,\ell(\gamma)=3,\\ \gamma_3=(\omega,1)}} a_\gamma = 0$$

for $\omega \in \{0, \varepsilon_1, \varepsilon_2\}$.

In what follows $\kappa = (\kappa_\gamma)_{\gamma,\ell(\gamma)=3} \in (\mathbb{F}^\times)^{12}$ is a tuple such that $\kappa_\gamma = 1$ if $\gamma_2 = (0,0)$ or $\gamma_3 = (0,1)$. We abbreviate $\lambda_1 \overset{\text{def}}{=} -\kappa_{((\varepsilon_2,1),(\varepsilon_2,0),(\varepsilon_1,1))}$, $\lambda_2 \overset{\text{def}}{=} -\kappa_{((\varepsilon_2,1),(\varepsilon_1,0),(\varepsilon_1,1))}$, $\lambda_3 \overset{\text{def}}{=} -\kappa_{((\varepsilon_1,1),(\varepsilon_2,0),(\varepsilon_2,1))}$, $\lambda_4 \overset{\text{def}}{=} -\kappa_{((\varepsilon_1,1),(\varepsilon_1,0),(\varepsilon_2,1))}$. (There are 12 paths of length 3 and 8 of these paths have the property that $\gamma_2 = (0,0)$ or $\gamma_3 = (0,1)$.) We now analyze the module M_κ and certain maps $I_\gamma \to M_\kappa$ (Proposition 3.26 and Corollary 3.27). If M is a finitely generated $S^{(j)}$-module and $m \in M$ we use overlined notations $\overline{M}$ and $\overline{m}$ for $M \otimes_{S^{(j)}} S^{(j)}/\mathfrak{m}_{S^{(j)}}$ and the image of m in $\overline{M}$, respectively. Similar notation apply for $S^{(j)}$-linear maps between finitely generated $S^{(j)}$-modules.

We define the following $\mathbb{F}$-basis on $\bigoplus_{\gamma,\ell(\gamma)=3} \overline{I}_\gamma$:

$$\underline{e} \overset{\text{def}}{=} \Bigg(\underbrace{\overline{c}_{11}, \quad \overline{c}_{22}}_{\substack{\bigoplus_{\gamma>\beta,\ \beta=((\varepsilon_2,1),(0,0))} \overline{I}_\gamma}}, \underbrace{\overline{c}_{11}, \quad \overline{c}_{22}}_{\substack{\bigoplus_{\gamma>\beta,\ \beta=((\varepsilon_2,1),(\varepsilon_2,0))} \overline{I}_\gamma}}, \underbrace{\overline{c}_{11}, \quad \overline{c}_{22}}_{\substack{\bigoplus_{\gamma>\beta,\ \beta=((\varepsilon_2,1),(\varepsilon_1,0))} \overline{I}_\gamma}}, \underbrace{\overline{c}_{11}, \quad \overline{c}_{33}}_{\substack{\bigoplus_{\gamma>\beta,\ \beta=((\varepsilon_1,1),(0,0))} \overline{I}_\gamma}}, \underbrace{\overline{c}_{11}, \quad \overline{c}_{33}}_{\substack{\bigoplus_{\gamma>\beta,\ \beta=((\varepsilon_1,1),(\varepsilon_2,0))} \overline{I}_\gamma}}, \underbrace{\overline{c}_{11}, \quad \overline{c}_{33}}_{\substack{\bigoplus_{\gamma>\beta,\ \beta=((\varepsilon_1,1),(\varepsilon_1,0))} \overline{I}_\gamma}} \Bigg)$$

and, on $\left(\bigoplus_{\gamma,\ell(\gamma)=3} I_\gamma\right)^0$, the basis

$$\underline{e}^0 \overset{\text{def}}{=} \underline{e} \cdot \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Finally, on $\bigoplus_{\beta,\ell(\beta)=2} \overline{I}_\beta$ we consider the basis $\underline{f}$ deduced from Lemma 3.25.

Proposition 3.26. *Using the bases $\underline{e}^0$ and $\underline{f}$ described above, the matrix A_κ associated to $\bar{\iota}^0_\kappa$ is given by:*

$$\begin{pmatrix}
1 & 1 & \tfrac{(b-c)(a-b-1)}{(a-b)(c+1-a)} & \tfrac{(b-c)(a-b-1)}{(a-b)(c+1-a)} & \tfrac{(b-c)(a-b-1)}{(a-b)(c+1-a)} & \tfrac{(b-c)(a-b-1)}{(a-b)(c+1-a)} & \tfrac{(b-c)(a-b-1)}{(a-b)(c+1-a)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\lambda_1 & 0 & \tfrac{(b-c)}{(a-c)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & \tfrac{(a-b-1)(b-c)}{(b-a)(a-c)} & & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \tfrac{(b-c-1)}{(b-a)} & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \tfrac{(b-c-1)(c-a)}{(b-a)(a-c-1)} & 0 & \lambda_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \tfrac{(a-c-1)(b-c)}{(b-a)(a-c)} & 0 & \lambda_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Proof. By Lemmas 3.25 and [LLHM, Lemma B.1], the following congruences hold in $S^{(j)}$:

$$c_{11} \equiv -\frac{(b-c)(1-a+b)}{(a-b)(1-a+c)}\frac{d^*_{11}}{d^*_{22}}c_{22} \qquad \text{modulo } \mathfrak{m}_{S^{(j)}} \cdot I_\beta \qquad \text{if } \beta = (\varepsilon_2, 1), (0, 0);$$

$$c_{11} \equiv -\frac{(b-c)}{(a-c)}\frac{d^*_{11}}{d^*_{22}}c_{22} \qquad \text{modulo } \mathfrak{m}_{S^{(j)}} \cdot I_\beta \qquad \text{if } \beta = (\varepsilon_2, 1), (\varepsilon_2, 0);$$

$$c_{11} \equiv -\frac{(b-c)(1-a+b)}{(a-c)(a-b)}\frac{d^*_{11}}{d^*_{22}}c_{22} \qquad \text{modulo } \mathfrak{m}_{S^{(j)}} \cdot I_\beta \qquad \text{if } \beta = (\varepsilon_2, 1), (\varepsilon_1, 0);$$

$$c_{11} \equiv \frac{(b-c-1)}{(a-b)}\frac{d^*_{11}}{d^*_{33}}c_{33} \qquad \text{modulo } \mathfrak{m}_{S^{(j)}} \cdot I_\beta \qquad \text{if } \beta = (\varepsilon_1, 1), (0, 0);$$

$$c_{11} \equiv \frac{(b-c-1)(c-a)}{(a-b)(a-c-1)}\frac{d^*_{11}}{d^*_{33}}c_{33} \qquad \text{modulo } \mathfrak{m}_{S^{(j)}} \cdot I_\beta \qquad \text{if } \beta = (\varepsilon_1, 1), (\varepsilon_2, 0);$$

$$c_{11} \equiv \frac{(b-c)(a-c-1)}{(a-c)(a-b)}\frac{d^*_{11}}{d^*_{33}}c_{33} \qquad \text{modulo } \mathfrak{m}_{S^{(j)}} \cdot I_\beta \qquad \text{if } \beta = (\varepsilon_1, 1), (\varepsilon_1, 0).$$

Thus, the matrix associated to $\bar{\iota}_\kappa$, in the bases $\underline{e}$, $\underline{f}$ is

$$\begin{pmatrix}
\tfrac{(b-c)(a-b-1)}{(a-b)(c+1-a)} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \tfrac{(c-b)}{(a-c)} & -\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \tfrac{(a-b-1)(b-c)}{(a-b)(a-c)} & -\lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \tfrac{(b-c-1)}{(a-b)} & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tfrac{(b-c-1)(c-a)}{(a-b)(a-c-1)} & -\lambda_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tfrac{(a-c-1)(b-c)}{(a-b)(a-c)} & -\lambda_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

and the conclusion follows by the definition of $\underline{e}^0$. □

The following result will be used in §6:

Corollary 3.27. *Assume that $\kappa = \kappa_{\min}(a, b, c) \in (\mathbb{F}^\times)^{12}$ is such that $\kappa_\gamma = 1$ if $\gamma_2 = (0, 0)$ or $\gamma_3 = (0, 1)$, and*

$$(-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4) = \\ = \left(\frac{(a-b)(a-c-1)}{(a-b-1)(a-c)}, \frac{a-c-1}{a-c}, -\frac{(a-c)}{(a-c-1)}, -\frac{(b-c)(a-c-1)}{(b-c-1)(a-c)}\right).$$

Then for any two paths $\gamma > \beta$ the composite

$$\overline{I}_\gamma \to \overline{I}_\beta \to \overline{M}_\kappa^{(j)}$$

is nonzero.

Proof. By Proposition 3.26 and Lemma 3.25 it is equivalent to show that for any $i \in \{1, 3, 5, 7, 9, 11\}$ the linear system $A_\kappa \underline{X} = \underline{e}_i$ has no solutions, where $\underline{e}_i = (\dots, 0, 1, 0 \dots)$ has a unique non-zero entry at position i. Equivalently, we show that for any $i = 1, \dots, 6$ the linear system

$$\begin{pmatrix} 1 & 1 & \frac{(b-c)(a-b-1)}{(a-b)(c+1-a)} & \frac{(b-c)(a-b-1)}{(a-b)(c+1-a)} & \frac{(b-c)(a-b-1)}{(a-b)(c+1-a)} & \frac{(b-c)(a-b-1)}{(a-b)(c+1-a)} & \frac{(b-c)(a-b-1)}{(a-b)(c+1-a)} & 0 & 0 \\ \lambda_1 & 0 & \frac{(b-c)}{(a-c)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & \frac{(a-b-1)(b-c)}{(b-a)(a-c)} & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(b-c-1)}{(b-a)} & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{(b-c-1)(c-a)}{(b-a)(a-c-1)} & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{(a-c-1)(b-c)}{(b-a)(a-c)} & 0 & \lambda_4 \end{pmatrix} \underline{X} = \underline{e}_i$$

has no solutions, where again $\underline{e}_i = (\dots, 0, 1, 0 \dots)$ has the unique nonzero entry at position i. Writing $\underline{e}_i = (e_{i,j})_{1 \leq j \leq 6}$, and letting a_{ij} denote the relevant entries of the previous matrix, the linear system above is equivalent after row reduction to

$$\begin{pmatrix} 0 & 0 & a_{13} - \lambda_1^{-1} a_{23} & a_{14} - \lambda_2^{-1} a_{34} & 0 & a_{16} + \frac{a_{15} a_{56}}{\lambda_3 a_{45}} & a_{17} + \frac{a_{15} a_{67}}{\lambda_4 a_{45}} & 0 & 0 \\ \lambda_1 & 0 & a_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & a_{34} & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{45} & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & a_{56} & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{57} & 0 & \lambda_4 \end{pmatrix} \underline{X} = \tag{3.17}$$

$$= \begin{pmatrix} e_{i,1} - \lambda_1^{-1} e_{i,2} - \lambda_2^{-1} e_{i,3} - \frac{a_{15}}{a_{45}} (e_{i,4} - \lambda_3^{-1} e_{i,5} - \lambda_4^{-1} e_{i,6}) \\ e_{i,2} \\ e_{i,3} \\ e_{i,4} \\ e_{i,5} \\ e_{i,6} \end{pmatrix}$$

By definition of $\kappa_{\min}(a, b, c)$, the first row of the matrix in the RHS of (3.17) is zero, which implies that the linear system (3.17) has no solution since $e_{i,j} \neq 0$ exactly when $j = i$, and $\lambda_i \neq 0$ for $i = 1, 2, 3, 4$. □

We conclude with a result on the support cycle of M_κ. We employ the terminology and notations of [EG14, §2.2], in particular Definition 2.2.5 from *loc. cit.* where we take $\mathcal{X}$ to be Spec (R) (so that $d = 6$ in the notation of *loc. cit.*).

Corollary 3.28. *For any $\kappa = (\kappa_\gamma)_{\gamma, \ell(\gamma)=3} \in (\mathbb{F}^\times)^{12}$ such that $\kappa_\gamma = 1$ if $\gamma_2 = (0,0)$ or $\gamma_3 = (0,1)$ we have*

$$Z_d(M_\kappa) = \sum_{\omega \in \{0, \varepsilon_1, \varepsilon_2\}} \mathfrak{P}_{(\omega,1)} + 2 \sum_{\omega \in \{0, \varepsilon_1, \varepsilon_2\}} \mathfrak{P}_{(\omega,0)}.$$

Proof. Let γ be a path. By definition of I_γ, the minimal prime ideals of R in the support of I_γ are exactly $\mathfrak{P}_{(\omega,a)}$ for $(\omega, a) \in \Sigma_\gamma$. Hence, for any path γ we have $Z_d(I_\gamma) = \sum_{(\omega,a) \in \Sigma_\gamma} \mathfrak{P}_{(\omega,a)}$.

The conclusion now follows from the additivity of cycles in short exact sequences ([EG14, Lemma 2.2.7]), the fact that M_κ is the cokernel of the injective map ι_κ^0 and noting that we have an exact sequence

$$0 \to \left(\bigoplus_{\gamma, \ell(\gamma)=3} I_\gamma \right)^0 \to \bigoplus_{\gamma, \ell(\gamma)=3} I_\gamma \to \oplus_{\gamma \in F} I_\gamma \to 0$$

with $F \stackrel{\text{def}}{=} \Big\{ ((\varepsilon_2, 1), (0,0), (\varepsilon_1, 1)), ((\varepsilon_1, 1), (0,0), (\varepsilon_2, 1)), ((\varepsilon_2, 1), (0,0), (0,1)) \Big\}$. □

3.4.3. *Analysis for* $T^{(j)} = \{w_0 t_{\underline{1}}, \alpha\beta t_{\underline{1}}, \beta\alpha t_{\underline{1}}\}$. We now have $\#T = 3$ and write $\tau_{w_0}, \tau_{\alpha\beta}, \tau_{\beta\alpha}$ for the elements of T distinguished by the conditions $\widetilde{w}^*(\overline{\rho}, \tau_{w_0})_j = \alpha\beta\alpha t_{\underline{1}}$, $\widetilde{w}^*(\overline{\rho}, \tau_{\alpha\beta})_j = \alpha\beta t_{\underline{1}}$ and $\widetilde{w}^*(\overline{\rho}, \tau_{\beta\alpha})_j = \beta\alpha t_{\underline{1}}$ respectively.

Note that the canonical surjections $\widetilde{S}/\big(\widetilde{I}_{\tau_*,\nabla_\infty} \cap \widetilde{I}_{\tau_{w_0},\nabla_\infty}\big) \twoheadrightarrow \widetilde{S}/\widetilde{I}_{\tau_{w_0},\nabla_\infty}$ induce canonical maps

$$\mathrm{Tor}_1^S(\mathbb{F}, (\widetilde{S}/\big(\widetilde{I}_{\tau_*,\nabla_\infty} \cap \widetilde{I}_{\tau_{w_0},\nabla_\infty}\big)) \otimes \mathbb{F}) \to \mathrm{Tor}_1^S(\mathbb{F}, (\widetilde{S}/\widetilde{I}_{\tau_{w_0},\nabla_\infty}) \otimes \mathbb{F}) \tag{3.18}$$

for $* \in \{\alpha\beta, \beta\alpha\}$.

Lemma 3.29. *The union of the images of* $\mathrm{Tor}_1^S(\mathbb{F}, (\widetilde{S}/\big(\widetilde{I}_{\tau_{\alpha\beta},\nabla_\infty} \cap \widetilde{I}_{\tau_{w_0},\nabla_\infty}\big)) \otimes \mathbb{F})$ *and* $\mathrm{Tor}_1^S(\mathbb{F}, (\widetilde{S}/\big(\widetilde{I}_{\tau_{w_0},\nabla_\infty} \cap \widetilde{I}_{\tau_{\beta\alpha},\nabla_\infty}\big)) \otimes \mathbb{F})$ *in* $\mathrm{Tor}_1^S(\mathbb{F}, (\widetilde{S}/\widetilde{I}_{\tau_{w_0},\nabla_\infty}) \otimes \mathbb{F})$ *is spanning.*

Proof. By the first row of diagram (3.8) it is enough to prove the statement with ∇_∞ replaced by ∇_{alg}. We have

$$\mathrm{Tor}_1^S(\mathbb{F}, (\widetilde{S}/\widetilde{I}_{\tau_{w_0},\nabla_\infty}) \otimes \mathbb{F}) \cong \mathrm{Tor}_1^S(\mathbb{F}, S/I_{\tau_{w_0},\nabla_{\mathrm{alg}}}) \cong \bigoplus_j I^{(j)}_{\tau_{w_0},\nabla_{\mathrm{alg}}}/(I^{(j)}_{\tau_{w_0},\nabla_{\mathrm{alg}}} \cdot \mathfrak{m}_{S^{(j)}}).$$

where the first equality is from the second row of diagram (3.8), and the second equality follows from the fact that $I_{\tau_{w_0},\nabla_{\mathrm{alg}}} = \sum_{j' \in \mathcal{J}} I^{(j)}_{\tau_{w_0},\nabla_{\mathrm{alg}}} S$ and $I^{(j')}_{\tau_{w_0},\nabla_{\mathrm{alg}}} \subseteq S^{(j')}$ for all $j' \in \mathcal{J}$. Let $j' \neq j$ and $* \in \{\alpha\beta, \beta\alpha\}$. We thus have $\widetilde{w}(\overline{\rho}, \tau_*)_{j'} = \widetilde{w}(\overline{\rho}, \tau_{w_0})_{j'}$ and the elements listed in column 4, row $\widetilde{w}(\overline{\rho}, \tau_{w_0})_{j'}$ of Table 3 are independent of τ modulo $p \cdot \mathfrak{m}_{S^{(j')}}$, as , $b^{(j')}_{\tau_*} \equiv b^{(j')}_{\tau_{w_0}}$ modulo p and $\overline{\rho}$ is tame (so that $\mathfrak{m}_{S^{(j')}} \supseteq (c^{(j')}_{ik}, 1 \leq i,k \leq 3, d^{(j')}_{21}, d^{(j')}_{31}, d^{(j')}_{32})$).

Hence applying repeatedly Lemma 3.17 with $k = 2$, $I_1 = \widetilde{I}_{\tau_*,\nabla_\infty}$, $I_2 = \widetilde{I}_{\tau_{w_0},\nabla_\infty}$ and f an element of $\widetilde{I}^{(j')}_{\tau_{w_0},\nabla_{\mathrm{alg}}}$ with $j' \neq j$, we conclude that the image of the map (3.18) contains $\sum_{j' \neq j} I^{(j')}_{\tau_{w_0},\nabla_{\mathrm{alg}}}/(I^{(j')}_{\tau_{w_0},\nabla_{\mathrm{alg}}} \cdot \mathfrak{m}_{S^{(j')}})$. Thus the desired statement will follow once we prove that the union of the images of

$$\mathrm{Tor}_1^S(\mathbb{F}, (\widetilde{S}^{(j)}/\big(\widetilde{I}^{(j)}_{\tau_*,\nabla_{\mathrm{alg}}} \cap \widetilde{I}^{(j)}_{\tau_{w_0},\nabla_{\mathrm{alg}}}\big)) \otimes \mathbb{F}) \to \mathrm{Tor}_1^S(\mathbb{F}, (\widetilde{S}/\widetilde{I}^{(j)}_{\tau_{w_0},\nabla_\infty}) \otimes \mathbb{F})$$

for $* \in \{\alpha\beta, \beta\alpha\}$ is spanning in $\mathrm{Tor}_1(\mathbb{F}, (\widetilde{S}^{(j)}/\widetilde{I}^{(j)}_{\tau_{w_0},\nabla_\infty}) \otimes \mathbb{F})$. This follows from an inspection of Table 6 (see [LLHM, §B.1.5] for details). □

Lemma 3.30. *We have* $p \in \widetilde{I}^{(j)}_{\tau_{\alpha\beta}} \cap \widetilde{I}^{(j)}_{\tau_{w_0}} + \widetilde{I}^{(j)}_{\tau_{\beta\alpha}} \cap \widetilde{I}^{(j)}_{\tau_{w_0}} + \widetilde{I}^{(j)}_{\tau_{\alpha\beta}} \cap \widetilde{I}^{(j)}_{\tau_{\beta\alpha}}$.

In particular, $p \in \widetilde{I}_{\tau_{\alpha\beta},\nabla_\infty} \cap \widetilde{I}_{\tau_{w_0},\nabla_\infty} + \widetilde{I}_{\tau_{\beta\alpha},\nabla_\infty} \cap \widetilde{I}_{\tau_{w_0},\nabla_\infty} + \widetilde{I}_{\tau_{\alpha\beta},\nabla_\infty} \cap \widetilde{I}_{\tau_{\beta\alpha},\nabla_\infty}$.

Proof. From Table 3 we have $c_{22} \in \widetilde{I}^{(j)}_{\tau_{\beta\alpha}} \cap \widetilde{I}^{(j)}_{\tau_{w_0}}$ and $c_{11}d^*_{33} - d_{31}c_{13} - pd^*_{11}d^*_{33} \in \widetilde{I}^{(j)}_{\tau_{\alpha\beta}} \cap \widetilde{I}^{(j)}_{\tau_{w_0}}$. Moreover, using the first and last equation in row $(\alpha\beta t_{\underline{1}}, \widetilde{I}^{(j)}_\tau)$ of Table 3 we obtain $c_{11}d^*_{22}d^*_{33} - d_{31}c_{13}d^*_{22} - c_{22}d^*_{11}d^*_{33} \in \widetilde{I}^{(j)}_{\tau_{\alpha\beta}}$ and hence $c_{11}d^*_{22}d^*_{33} - d_{31}c_{13}d^*_{22} - c_{22}d^*_{11}d^*_{33} \in \widetilde{I}^{(j)}_{\tau_{\alpha\beta}} \cap \widetilde{I}^{(j)}_{\tau_{\beta\alpha}}$ since $c_{22}, c_{11}d^*_{33} - d_{31}c_{13}$ are both in $\widetilde{I}^{(j)}_{\tau_{\beta\alpha}}$. Thus,

$$\begin{aligned} pd^*_{11}d^*_{22}d^*_{33} &= (c_{11}d^*_{22}d^*_{33} - d_{31}c_{13}d^*_{22} - c_{22}d^*_{11}d^*_{33}) + c_{22}d^*_{11}d^*_{33} - d^*_{22}(c_{11}d^*_{33} - d_{31}c_{13} - pd^*_{11}d^*_{33}) \\ &\in \widetilde{I}^{(j)}_{\tau_{\alpha\beta}} \cap \widetilde{I}^{(j)}_{\tau_{\beta\alpha}} + \widetilde{I}^{(j)}_{\tau_{\beta\alpha}} \cap \widetilde{I}^{(j)}_{\tau_{w_0}} + \widetilde{I}^{(j)}_{\tau_{\alpha\beta}} \cap \widetilde{I}^{(j)}_{\tau_{w_0}} \end{aligned}$$

which implies the statement as $d^*_{11}d^*_{22}d^*_{33}$ is a unit in $\widetilde{S}^{(j)}$. □

Remark 3.31. Even though $p \notin \widetilde{I}_{\tau_{\alpha\beta}} + \widetilde{I}_{\tau_{\beta\alpha}}$, the situation changes after imposing monodromy, i.e. we do have $p \in \widetilde{I}_{\tau_{\alpha\beta},\nabla_\infty} + \widetilde{I}_{\tau_{\beta\alpha},\nabla_\infty}$. To see this, for $\tau \in \{\tau_{\alpha\beta}, \tau_{\beta\alpha}\}$ write $\mathrm{Mon}_{\tau,1}$ for the first element in row Mon_τ of Table 3. Then $\mathrm{Mon}_{\tau_{\beta\alpha},1} - \mathrm{Mon}_{\tau_{\alpha\beta},1} \in \widetilde{I}^{(j)}_{\tau_{\alpha\beta},\nabla_\infty} + \widetilde{I}^{(j)}_{\tau_{\beta\alpha},\nabla_\infty}$ and, by inspection on the equations $\mathrm{Mon}_{\tau_{\alpha\beta},1}, \mathrm{Mon}_{\tau_{\beta\alpha},1}$ and using that $(a,b,c) \stackrel{\mathrm{def}}{=} (b_{\tau,1}, b_{\tau,2}, b_{\tau,3}) \mod p$ is independent of $\tau \in \{\tau_{\alpha\beta}, \tau_{\beta\alpha}\}$, we deduce

$$p\left((b-c)d_{11}^*d_{22}^* + pd_{11}^*d_{22}^* + xd_{21}c_{21} + yc_{11}d_{22}^* + O(p^{N-5})\right)$$

for some $x, y \in \mathbb{Z}_p$. The factor in parenthesis is a unit in $\widetilde{S}^{(j)}$ since $p, d_{21}c_{21}, c_{11} \in \mathfrak{m}_{\widetilde{S}^{(j)}}$, $N > 5$ and $b - c \not\equiv 0$ modulo p and hence $p \in \widetilde{I}^{(j)}_{\tau_{\alpha\beta},\nabla_\infty} + \widetilde{I}^{(j)}_{\tau_{\beta\alpha},\nabla_\infty}$.

3.4.4. *Analysis for $T^{(j)} = \{t_{w_0(\eta)}, t_{w_0(\eta)}\alpha, t_{w_0(\eta)}\beta\}$.* The analysis is similar to that of §3.4.3, replacing $w_0t_{\underline{1}}, \alpha\beta t_{\underline{1}}$ and $\beta\alpha t_{\underline{1}}$ by $t_{w_0(\eta)}$, $t_{w_0(\eta)}\alpha$ and $t_{w_0(\eta)}\beta$ respectively. A proof analogous to that of Lemma 3.29, using now Table 7 instead of Table 6, gives us the following result:

Lemma 3.32. *The union of the images of* $\mathrm{Tor}_1^S(\mathbb{F}, (\widetilde{S}/(\widetilde{I}_{\tau_{t_{w_0(\eta)}\alpha},\nabla_\infty} \cap \widetilde{I}_{\tau_{t_{w_0(\eta)}},\nabla_\infty})) \otimes \mathbb{F})$ *and of* $\mathrm{Tor}_1^S(\mathbb{F}, (\widetilde{S}/(\widetilde{I}_{\tau_{t_{w_0(\eta)}},\nabla_\infty} \cap \widetilde{I}_{\tau_{t_{w_0(\eta)}\beta},\nabla_\infty})) \otimes \mathbb{F})$ *in* $\mathrm{Tor}_1^S(\mathbb{F}, (\widetilde{S}/\widetilde{I}_{\tau_{t_{w_0(\eta)}},\nabla_\infty}) \otimes \mathbb{F})$ *is spanning.*

(Note that, for consistency with 3.4.3, we should replace η by $(1,0,-1)$ in the statement of Lemma 3.32; we used η instead for ease of notation.)

3.4.5. *Analysis for $T^{(j)} = \{w_0t_{\underline{1}}, \alpha\beta t_{\underline{1}}, \beta\alpha t_{\underline{1}}, t_{\underline{1}}\}$.* We now have $\#T = 4$ and write $\tau_{w_0}, \tau_{\alpha\beta}, \tau_{\beta\alpha}, \tau_{\mathrm{id}}$ for the elements of T distinguished by the conditions $\widetilde{w}^*(\overline{\rho}, \tau_{w_0})_j = w_0t_{\underline{1}}$, $\widetilde{w}^*(\overline{\rho}, \tau_{\alpha\beta})_j = \alpha\beta t_{\underline{1}}$, $\widetilde{w}^*(\overline{\rho}, \tau_{\beta\alpha})_j = \beta\alpha t_{\underline{1}}$ and $\widetilde{w}^*(\overline{\rho}, \tau_{\beta\alpha})_j = t_{\underline{1}}$ respectively.

Recall that $\widetilde{I}_{T,\nabla_\infty} = \widetilde{I}_{\tau_{\alpha\beta},\nabla_\infty} \cap \widetilde{I}_{\tau_{w_0},\nabla_\infty} \cap \widetilde{I}_{\tau_{\beta\alpha},\nabla_\infty} \cap \widetilde{I}_{\tau_{\mathrm{id}},\nabla_\infty}$. Define $I_\Lambda^{(j)}$ to be the intersection $\mathfrak{P}^{(j)}_{(0,1)} \cap \mathfrak{P}^{(j)}_{(0,0)} \cap \mathfrak{P}^{(j)}_{(\varepsilon_1,0)} \cap \mathfrak{P}^{(j)}_{(\varepsilon_1,0)}$ in $S^{(j)}/I^{(j)}_{\tau_{\mathrm{id}},\nabla_{\mathrm{alg}}}$, and let $\widetilde{I}_\Lambda$ be the pullback in $\widetilde{S}$ of the ideal $\sum_{j'\in\mathcal{J},j'\neq j} I^{(j')}_{\tau_{\mathrm{id}},\nabla_{\mathrm{alg}}}S + I_\Lambda^{(j)}S \subseteq S$ via $\widetilde{S}/\widetilde{I}_{T,\nabla_\infty} \twoheadrightarrow S/I_{\tau_{\mathrm{id}},\nabla_\infty} \cong S/I_{\tau_{\mathrm{id}},\nabla_{\mathrm{alg}}}$, where the isomorphism follows from the bottom line of (3.8). (Note that in the setting of this subsection the ideal $I^{(j')}_{\tau,\nabla_{\mathrm{alg}}}$ is independent of $\tau \in T$ when $j' \neq j$, in particular in the definition of $\widetilde{I}_\Lambda$ we can replace $\sum_{j'\in\mathcal{J},j'\neq j} I^{(j')}_{\tau_{\mathrm{id}},\nabla_{\mathrm{alg}}}S$ with $\sum_{j'\in\mathcal{J},j'\neq j} I^{(j')}_{\tau,\nabla_{\mathrm{alg}}}S$ for any choice of $\tau \in T$.) The proof of the following Lemma is analogous to that of Lemma 3.21 (see also [LLHM, §B.1.1]).

Lemma 3.33. *Under the current assumption we have*

$$I_\Lambda^{(j)} = (c_{22},\, c_{33},\, d_{32}c_{23}, c_{13}d_{31} - c_{11}d_{33}^*).$$

We have $\widetilde{I}^{(j)}_{\tau_{\beta\alpha},\nabla_\infty} \subseteq I^{(j)}_{\tau_{\beta\alpha},\nabla_{\mathrm{alg}}} \subseteq \mathfrak{P}^{(j)}_{(0,1)} \cap \mathfrak{P}^{(j)}_{(0,0)} \cap \mathfrak{P}^{(j)}_{(\varepsilon_1,0)}$ by Proposition 3.19 and [LLHLM20, Table 3], and similarly $\widetilde{I}^{(j)}_{\tau_{\alpha\beta},\nabla_\infty} \subseteq I^{(j)}_{\tau_{\alpha\beta},\nabla_{\mathrm{alg}}} \subseteq \mathfrak{P}^{(j)}_{(0,1)} \cap \mathfrak{P}^{(j)}_{(0,0)} \cap \mathfrak{P}^{(j)}_{(\varepsilon_2,0)}$. Hence we have canonical surjections $\widetilde{S}/\big(\widetilde{I}_{\tau_{\alpha\beta},\nabla_\infty} \cap \widetilde{I}_{\tau_{w_0},\nabla_\infty}, p\big) \cap (\widetilde{I}_{\tau_{w_0},\nabla_\infty} \cap \widetilde{I}_{\tau_{\beta\alpha},\nabla_\infty}, p) \twoheadrightarrow \widetilde{S}/\widetilde{I}_\Lambda$ and $\widetilde{S}/(\widetilde{I}_{\tau_{\mathrm{id}},\nabla_\infty}, p) \twoheadrightarrow \widetilde{S}/\widetilde{I}_\Lambda$ which induce canonical maps

$$\mathrm{Tor}_1^S(\mathbb{F}, \widetilde{S}/\big(\widetilde{I}_{\tau_{\alpha\beta},\nabla_\infty} \cap \widetilde{I}_{\tau_{w_0},\nabla_\infty}, p\big) \cap (\widetilde{I}_{\tau_{w_0},\nabla_\infty} \cap \widetilde{I}_{\tau_{\beta\alpha},\nabla_\infty}, p)) \to \mathrm{Tor}_1^S(\mathbb{F}, (\widetilde{S}/\widetilde{I}_\Lambda) \otimes \mathbb{F})$$

and

$$\mathrm{Tor}_1^S(\mathbb{F}, (\widetilde{S}/I_{\tau_{\mathrm{id}},\nabla_\infty}) \otimes \mathbb{F}) \to \mathrm{Tor}_1^S(\mathbb{F}, (\widetilde{S}/\widetilde{I}_\Lambda) \otimes \mathbb{F}).$$

Lemma 3.34. *The union of the images of* $\mathrm{Tor}_1^S(\mathbb{F}, (\widetilde{S}/\widetilde{I}_{\tau_{\mathrm{id}},\nabla_\infty}) \otimes \mathbb{F})$ *and* $\mathrm{Tor}_1^S(\mathbb{F}, \widetilde{S}/\big(\widetilde{I}_{\tau_{\alpha\beta},\nabla_\infty} \cap \widetilde{I}_{\tau_{w_0},\nabla_\infty}, p\big) \cap (\widetilde{I}_{\tau_{w_0},\nabla_\infty} \cap \widetilde{I}_{\tau_{\beta\alpha},\nabla_\infty}, p))$ *in* $\mathrm{Tor}_1^S(\mathbb{F}, (\widetilde{S}/\widetilde{I}_\Lambda) \otimes \mathbb{F})$ *is spanning.*

Proof. The proof is very similar to that of Lemma 3.29. As in *loc. cit.* it suffices to prove the statement with ∇_∞ replaced by $\nabla_{\rm alg}$ everywhere. We have

$$\mathrm{Tor}_1^S(\mathbb{F}, \widetilde{S}/\widetilde{I}_{\tau_{\rm id},\nabla_{\rm alg}}) \otimes \mathbb{F}) = \bigoplus_j I^{(j)}_{\tau_{\rm id},\nabla_{\rm alg}}/(I^{(j)}_{\tau_{\rm id},\nabla_{\rm alg}} \cdot \mathfrak{m}_{S^{(j)}})$$

since $I_{\tau_{\rm id},\nabla_{\rm alg}} = \sum_{j'\in\mathcal{J}} I^{(j')}_{\tau_{\rm id},\nabla_{\rm alg}} S$ with $I^{(j')}_{\tau_{\rm id},\nabla_{\rm alg}} \subseteq S^{(j')}$ for all $j' \in \mathcal{J}$, and we have a similar decompositions

$$\mathrm{Tor}_1^S(\mathbb{F}, (\widetilde{S}/\widetilde{I}_\Lambda) \otimes \mathbb{F}) = I^{(j)}_\Lambda/(I^{(j)}_{\tau_{\rm id},\nabla_{\rm alg}} \cdot \mathfrak{m}_{S^{(j)}}) \oplus \bigoplus_{j'\neq j} I^{(j')}_{\tau_{\rm id},\nabla_{\rm alg}}/(I^{(j')}_{\tau_{\rm id},\nabla_{\rm alg}} \cdot \mathfrak{m}_{S^{(j')}})$$

and for $\mathrm{Tor}_1^S(\mathbb{F}, S/\big(\widetilde{I}_{\tau_{\alpha\beta},\nabla_{\rm alg}} \cap \widetilde{I}_{\tau_{w_0},\nabla_\infty}, p\big) \cap (\widetilde{I}_{\tau_{w_0},\nabla_{\rm alg}} \cap \widetilde{I}_{\tau_{\beta\alpha},\nabla_\infty}, p))$.

Hence the desired statement will follow once we prove that union of the images of

$$\mathrm{Tor}_1^{S^{(j)}}(\mathbb{F}, S^{(j)}/\big(\widetilde{I}^{(j)}_{\tau_{\alpha\beta},\nabla_{\rm alg}} \cap \widetilde{I}^{(j)}_{\tau_{w_0},\nabla_\infty}, p\big) \cap (\widetilde{I}^{(j)}_{\tau_{w_0},\nabla_{\rm alg}} \cap \widetilde{I}^{(j)}_{\tau_{\beta\alpha},\nabla_\infty}, p)) \to \mathrm{Tor}_1^{S^{(j)}}(\mathbb{F}, S^{(j)}/I^{(j)}_\Lambda)$$

and

$$\mathrm{Tor}_1^{S^{(j)}}(\mathbb{F}, S^{(j)}/\big(\widetilde{I}^{(j)}_{\tau_{\rm id},\nabla_{\rm alg}}, p)) \to \mathrm{Tor}_1^{S^{(j)}}(\mathbb{F}, S^{(j)}/I^{(j)}_\Lambda)$$

is spanning. This follows from the last row in Table 6 (see [LLHM, §B.1.5] for details). □

3.4.6. *Analysis for* $T^{(j)} = \{t_{w_0(\eta)}, t_{w_0(\eta)}\alpha, t_{w_0(\eta)}\beta, t_{w_0(\eta)}w_0\}$. The analysis is similar to that of §3.4.5, replacing $w_0 t_{\underline{1}}$, $\alpha\beta t_{\underline{1}}$, $\beta\alpha t_{\underline{1}}$ and $t_{\underline{1}}$ by $t_{w_0(\eta)}$, $t_{w_0(\eta)}\alpha$, $t_{w_0(\eta)}\beta$ and $t_{w_0(\eta)}w_0$ respectively, and $\mathfrak{P}^{(j)}_{(0,0)} \cap \mathfrak{P}^{(j)}_{(0,1)}$ by $\mathfrak{P}^{(j)}_{(\varepsilon_1+\varepsilon_2,1)}$. The following Lemma is proved in [LLHM, §B.1.1].

Lemma 3.35. *In the current assumptions we have:*

$$I^{(j)}_\Lambda = (c_{22}, c_{33}, c_{32}, e_{33}, e_{23}, d_{31}, (a-b)c_{12}c_{23} - (a-c)e_{13}d^*_{22}, d_{21}d_{32}, c_{23}d_{32}, d_{21}c_{12}).$$

A proof analogous to that of Lemma 3.34, using now Table 7 instead of Table 6, yields the following:

Lemma 3.36. *The union of the image of* $\mathrm{Tor}_1^S(\mathbb{F}, (\widetilde{S}/\widetilde{I}_{\tau_{t_{w_0(\eta)}w_0},\nabla_\infty}) \otimes \mathbb{F})$ *and of* $\mathrm{Tor}_1^S(\mathbb{F}, \widetilde{S}/\big(\widetilde{I}_{\tau_{t_{w_0(\eta)}\alpha},\nabla_\infty} \cap \widetilde{I}_{\tau_{t_{w_0(\eta)}},\nabla_\infty}, p\big) \cap (\widetilde{I}_{\tau_{t_{w_0(\eta)}},\nabla_\infty} \cap \widetilde{I}_{\tau_{t_{w_0(\eta)}\beta},\nabla_\infty}, p))$ *in* $\mathrm{Tor}_1^S(\mathbb{F}, (\widetilde{S}/\widetilde{I}_\Lambda) \otimes \mathbb{F})$ *is spanning.*

4. Representations of GL_3

4.1. **Some tilting modules for** GL_3**.** We label some alcoves for GL_3 as in Figure 1. For a

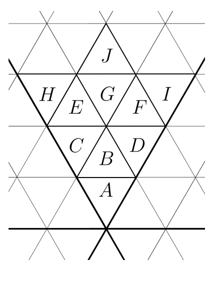


Figure 1. Labelling of alcoves for GL_3

dominant alcove X, let $\widetilde{w}_X \in W_a$ denote the corresponding element of the affine Weyl group. We write $\widetilde{w}_X = t_{\omega_X}\widetilde{w}^0_X$ for some $\widetilde{w}^0_X \in \widetilde{W}^+_1$ and dominant $\omega_X \in X^*(T)$ which are unique up to $X^0(T)$. If $\lambda \in X^*(T)$ is 0-deep in an alcove, λ_X denotes the unique weight in alcove X that is linked to λ

(cf. [Jan03, II.6.5]). Then $\lambda_X = \lambda_X^0 + p\omega_X$ with $\lambda_X^0 \in X_1(T)$ and ω_X as above. Moreover, if $\lambda \in C_0$ then $\lambda_X = \widetilde{w}_X \cdot \lambda = \widetilde{w}_X^0 \cdot \lambda + p\omega_X$ so that $\lambda_X^0 = \widetilde{w}_X^0 \cdot \lambda$.

Let $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$. For a dominant $\lambda \in X^*(T)$, recall from [Jan03, §II.2.4] the simple $\mathrm{GL}_{3/\mathbb{F}}$-representation $L(\lambda)$ with highest weight λ. In this section we write Ext^1 for $\mathrm{Ext}^1_{\mathrm{GL}_{3/\mathbb{F}}}$. From [Yeh, Theorem 4.2.3(i)] we have:

Proposition 4.1. *For any $\lambda, \mu \in X^+(T)$ we have $\dim_{\mathbb{F}} \mathrm{Ext}^1(L(\lambda), L(\mu)) \leq 1$.*

Proposition 4.1 implies that if a nonsplit extension of two simple $\mathrm{GL}_{3/\mathbb{F}}$-modules exists, then it is unique up to isomorphism. Recall from [Jan03, §II.6.17] that $\mathrm{Ext}^1(L(\lambda), L(\mu)) = 0$ unless λ and μ are linked i.e. $\mu = \widetilde{w} \cdot \lambda$ for some $\widetilde{w} \in W_a$.

For $\lambda \in X_1^*(T)$, recall from [LLHLM20, Theorem 4.2.1] the $\mathrm{GL}_{3/\mathbb{F}}$-representation $Q_1(\lambda)$ (see also [Jan03, §II.11.3 and §II.11.11]). As the following results explain, the module $Q_1(\lambda)$ acts like an injective and projective module in the full subcategory of modules whose Jordan–Hölder factors have p-bounded highest weight.

Proposition 4.2. *If $\mu \in X^+(T)$ is p-bounded, i.e. $\langle \mu, \alpha^\vee \rangle < 4p$ for all roots α, then $\mathrm{Ext}^1(L(\mu), Q_1(\lambda)) = 0$ and $\mathrm{Ext}^1(Q_1(\lambda), L(\mu)) = 0$.*

Proof. The second vanishing statement follows from the first by duality and the first vanishing statement follows from the proof of [LLHLM20, (4.8)]. □

By dévissage, Proposition 4.2 yields the following.

Corollary 4.3. *If M is a (finite length) module with only p-bounded Jordan–Hölder factors, then $\mathrm{Ext}^1(M, Q_1(\lambda)) = 0$ and $\mathrm{Ext}^1(Q_1(\lambda), M) = 0$.*

For $\lambda \in X^+(\underline{T})$, we define $V(\lambda)$ as in [Jan03, II.2.13(1)], and write $W(\lambda)$ for its dual. By [Jan03, II.2.14(1)] $V(\lambda)$ (resp. $W(\lambda)$) has ireducible cosocle (resp. socle) isomorphic to $L(\lambda)$, and we call it the Weyl module (resp. dual Weyl module) associated to λ. A Weyl (resp. dual Weyl) filtration on a $\mathrm{GL}_{3/\mathbb{F}}$-module M is an exhaustive filtration whose graded pieces are direct sum of Weyl (resp. dual Weyl) modules (cf. [Jan03, II.4.19]).

The following Proposition is a reformulation of [BDM15, Theorem B(b)].

Proposition 4.4 ([BDM15]). *(1) The module $Q_1(\lambda)$ is rigid with Loewy layers given by the rows in Figures 2, 3 (a row with alcove labels $X_1, \cdots, X_n$ denotes a direct sum with multiplicity of the simple $\mathrm{GL}_{3/\mathbb{F}}$-representations $L(\lambda_{X_1}), \cdots, L(\lambda_{X_n})$).*

(2) The module $Q_1(\lambda)$ has a Weyl (resp. dual Weyl) filtration such that $\mathrm{gr}(Q_1(\lambda))$ is a direct sum of multiplicity free Weyl modules (resp. dual Weyl modules) with Jordan–Hölder factors given by the connected components of the graphs in Figures 2 (resp. 3).

(3) The socle and cosocle filtrations on these Weyl modules coincide with the filtrations induced from the Loewy filtration on $Q_1(\lambda)$ (up to shift).

(4) Each edge in Figure 2 (resp. 3) indicates the existence of a subquotient of a Weyl module (resp. dual Weyl module) in $\mathrm{gr}(Q_1(\lambda))$, which is a nonsplit extension (unique up to isomorphism by Proposition 4.1) of the indicated simple modules.

Let $\mathrm{rad}^\bullet Q_1(\lambda)$ denote the (decreasing) radical filtration. We let $Q_1(\lambda)^{\widehat{k}}$ denote $Q_1(\lambda)/\mathrm{rad}^k Q_1(\lambda)$. Informally speaking, $Q_1(\lambda)^{\widehat{k}}$ is the maximal quotient of $Q_1(\lambda)$ obtained by removing the k-th layer.

FIGURE 2. Weyl and dual Weyl filtrations for $Q_1(\lambda)$, case $\lambda \in B$

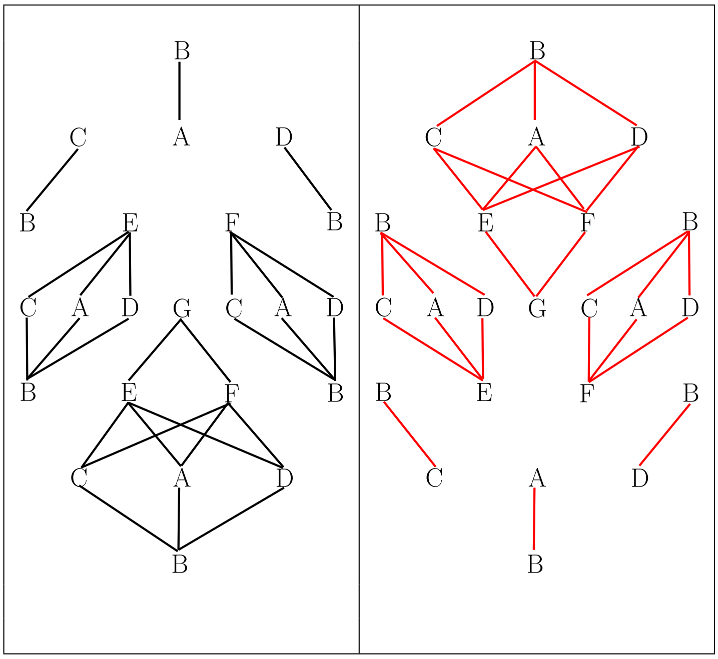
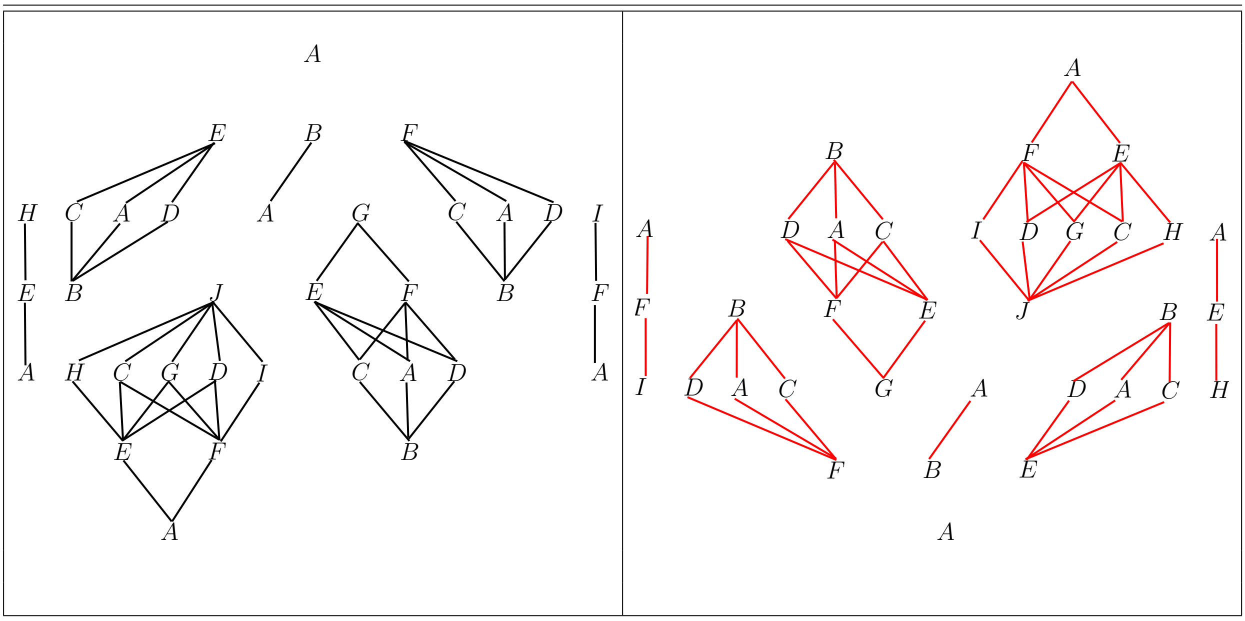


FIGURE 3. Weyl and dual Weyl filtrations for $Q_1(\lambda)$, case $\lambda \in A$



Proposition 4.5 (Translation principle)**.** *Let $\lambda \in X_1^*(T)$ be m-deep in its alcove, and $\varepsilon \in X^*(T)$ such that $\langle \varepsilon, \alpha^\vee \rangle \leq m$ for all $\alpha \in \Phi$. Then*

$$L(\varepsilon) \otimes \mathrm{rad}^k Q_1(\lambda) \cong \bigoplus_{\nu \in L(\varepsilon)} \mathrm{rad}^k Q_1(\lambda + \nu)$$

(where $\nu \in L(\varepsilon)$ means that $\nu \in X^*(T)$ is in the weight space of $L(\varepsilon)$, counted with multiplicity).

Proof. Noting that $L(\varepsilon) \otimes L(\lambda)$ is semisimple by the assumption on ε ([Hum06, Proposition 6.4]), the statement follows by dévissage from [AK11, Lemma 4.6] and Proposition 4.4. □

Restriction to rational points. Recall from §1.4 the algebraic group G_0 and the natural isomorphism $\underline{G}_{/\mathbb{F}} \stackrel{\text{def}}{=} G_0 \times_{\mathbb{Z}_p} \mathbb{F} \cong \prod_{j\in\mathcal{J}} \mathrm{GL}_{3/\mathbb{F}}$. Let G be the finite group $G_0(\mathbb{F}_p)$.

The following proposition expresses the Jordan–Hölder factors of $L(\mu)|_{\mathrm{G}}$, for suitable $\mu \in X^*(\underline{T})$, in terms of the extension graph.

Proposition 4.6. *Let m, t be nonnegative integers such that $4t \le m$ and $\lambda \in A^{\mathcal{J}}$ be m-deep. Assume that for each $j \in \mathcal{J}$, X_j is a dominant alcove such that $\max_{\mu\in X_j,\alpha\in\Phi}\langle\mu,\alpha^\vee\rangle < 4pt$. Then*

$$\otimes_{j\in\mathcal{J}} L(\widetilde{w}_{X_j}\cdot\lambda_j)|_{\mathrm{G}} \cong \bigoplus_{\omega\in L(\sum_{j\in\mathcal{J}}\omega_{X_j})} \mathfrak{Tr}_{\lambda+\eta}(\pi\omega, \pi(\widetilde{w}^0_{X_j}(A))_{j\in\mathcal{J}}).$$

Proof. By [LLHLM20, Lemma 4.2.4(1)] (see also [LLHLM20, proof of Lemma 4.2.5]) we have

$$\otimes_{j\in\mathcal{J}} L(\widetilde{w}_{X_j}\cdot\lambda_j)|_{\mathrm{G}} \cong \bigoplus_{\omega\in L(\sum_j\omega_{X_j})} F\Big(\pi\omega + \sum_{j\in\mathcal{J}} \widetilde{w}^0_{X_j}\cdot\lambda_j\Big)$$

and the conclusion follows directly from the definition of $\mathfrak{Tr}_{\lambda+\eta}$ (see [LLHLMb, equation (2.3)]). □

4.2. **G-projective covers and modular Serre weights.** For $\lambda \in X_1(\underline{T})$, let $Q_1(\lambda)$ be the $\underline{G}_{/\mathbb{F}}$-module $\otimes_{j\in\mathcal{J}} Q_1(\lambda_j)$. For $a = (a_j)_j \in \mathbb{Z}^{\mathcal{J}}_{\ge 0}$ and $\lambda \in X_1(\underline{T})$, let $\mathrm{rad}^a Q_1(\lambda)$ (resp. $Q_1(\lambda)^{\widehat{a}}$) be the tensor product $\otimes_{j\in\mathcal{J}}\mathrm{rad}^{a_j} Q_1(\lambda_j)$ (resp. $\otimes_{j\in\mathcal{J}} Q_1(\lambda_j)^{\widehat{a_j}}$). We also let $\mathrm{rad}^{>a} Q_1(\lambda)$ be $\sum_{b\ge a, b\ne a}\mathrm{rad}^b Q_1(\lambda)$, where $\ge$ denotes the product partial order on $\mathbb{Z}^{\mathcal{J}}_{\ge 0}$, and let $\mathrm{gr}^a Q_1(\lambda)$ be $\mathrm{rad}^a Q_1(\lambda)/\mathrm{rad}^{>a} Q_1(\lambda)$. Note that $\mathrm{rad}^{>a} Q_1(\lambda)$ is $\mathrm{rad}(\mathrm{rad}^a Q_1(\lambda))$. For a tuple $a = (a_j)_{j\in\mathcal{J}}$ with $0 \le a_j \le 7$, let $|a| = \sum_j a_j$. If $n \in \mathbb{N}$ we define

$$\mathrm{rad}^n Q_1(\lambda) \stackrel{\text{def}}{=} \sum_{\substack{a\in\mathbb{Z}^{\mathcal{J}}_{\ge 0}\\ |a|=n}} \mathrm{rad}^a Q_1(\lambda).$$

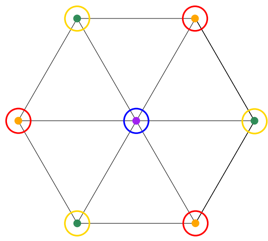
In fact, $\{\mathrm{rad}^n Q_1(\lambda)\}_{n\in\mathbb{Z}_{\ge 0}}$ coincides with the radical filtration for the $\mathbb{F}[\underline{G}_{/\mathbb{F}}]$-module $Q_1(\lambda)$, though we will not use this.

If σ is a Serre weight, then let $\widetilde{P}_\sigma$ and P_σ denote a $\mathcal{O}[\mathrm{G}]$-projective cover and $\mathbb{F}[\mathrm{G}]$-projective cover of σ, respectively. Then $P_\sigma \cong \widetilde{P}_\sigma \otimes_{\mathcal{O}} \mathbb{F}$. Recall that $\mathbb{F}[\mathrm{G}]$ is a Frobenius algebra, so that P_σ is isomorphic to the injective envelope of σ.

Proposition 4.7. *Assume that $p \ge 5$. If $\lambda \in X_1(\underline{T})$ is 1-deep, then the projective* G*-module $P_{F(\lambda)}$ is isomorphic to $Q_1(\lambda)|_{\mathrm{G}}$. In particular, if λ is 4-deep, the Jordan–Hölder factors of $P_{F(\lambda)}$ are described by Proposition 4.6 (with $X_j \in \{A,B,C,D,E,F,G\}$ if λ_j is in alcove B and $X_j \in \{A,B,C,D,E,F,G,H,I,J\}$ if λ_j is in alcove A).*

Proof. This is a particular case of [LLHLM20, Theorem 4.2.1]. As for the last statement, we note that the alcoves appearing in Figure 1 are all p-bounded. □

Let $\lambda \in X_1(\underline{T})$ be 1-deep and $\sigma = F(\lambda)$. For $a = (a_j)_j \in \mathbb{Z}^{\mathcal{J}}_{\ge 0}$ and $n \in \mathbb{Z}_{\ge 0}$, let $\mathrm{rad}^a P_\sigma$, $\mathrm{rad}^{>a} P_\sigma$, $P_\sigma^{\widehat{a}}$, $\mathrm{gr}^a P_\sigma$ and $\mathrm{rad}^n P_\sigma$ be the subquotients of P_σ corresponding to the restrictions $\mathrm{rad}^a Q_1(\lambda)|_{\mathrm{G}}$, $\mathrm{rad}^{>a} Q_1(\lambda)|_{\mathrm{G}}$, $Q_1(\lambda)^{\widehat{a}}|_{\mathrm{G}}$, $\mathrm{gr}^a Q_1(\lambda)|_{\mathrm{G}}$ and $\mathrm{rad}^n Q_1(\lambda)|_{\mathrm{G}}$ under a fixed choice of isomorphism in Proposition 4.7 (the isomorphism classes of the subquotients do not depend on this choice of the



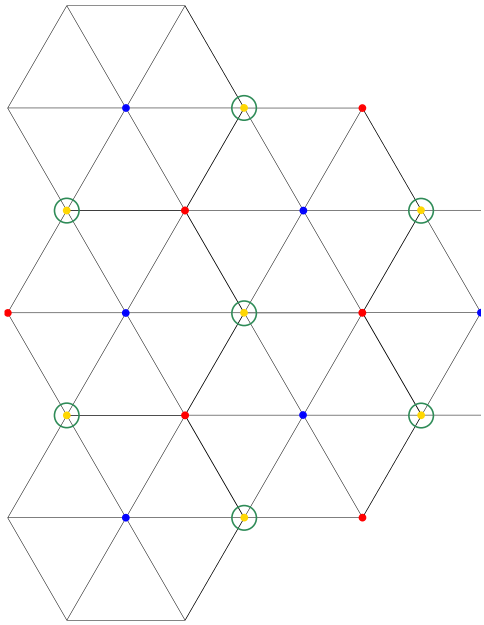


FIGURE 4. In this picture we represent the elements in the $\pi(j)$-th coordinate of $(\omega, (\widetilde{w}^0_{X_j}(A))_{j\in\mathcal{J}})$ for $\omega \in L(\sum_{j\in\mathcal{J}} \omega_{X_j})$, where $X_j \in \{A, B, C, D, E, F, G, J, I, H\}$. (Informally speaking, these elements pictures the $\pi(j)$-th component of the restriction of $L(\sum_{j\in\mathcal{J}} \omega_{X_j})$ to G.) On the left we consider the case where $X_j \in \{A, B, C, D, E, F\}$. The gold (resp. red, resp. blue) circles represent the case $X_j = E$ (resp. $X_j = F$, resp. $X_j = B$). The green (resp. orange, resp. purple) dots represent the case $X_j = C$ (resp. $X_j = D$, resp. $X_j = A$). On the right we consider the case where $X_j \in \{G, J, I, H\}$. The green circles represent the case where $X_j = J$. The gold (resp. red, resp. blue) dots represent the case $X_j = G$ (resp. $X_j = H$, resp. $X_j = I$).

isomorphism). Note that if σ is 8-deep then $\{\mathrm{rad}^n P_\sigma\}_{n\in\mathbb{N}}$ coincides with the radical filtration for the $\mathbb{F}[\mathrm{G}]$-module P_σ by [LLHLM20, Corollary 4.2.3] and the analogous discussion for $Q_1(\lambda)$ before Proposition 4.7. This description of the radical filtration of P_σ also follows from the next result.

We first introduce some notation. If $a = (a_j)_{j\in\mathcal{J}}$ and $b = (b_j)_{j\in\mathcal{J}}$ are tuples with $a_j, b_j \in \mathbb{Z}$ for all $j \in \mathcal{J}$, let $a \pm b$ be the tuple $(a_j \pm b_j)_{j\in\mathcal{J}}$. For an integer n and $i \in \mathcal{J}$, let $n_i \in \mathbb{Z}^{\mathcal{J}}$ be the tuple with $n_{i,j} = n\delta_{i=j}$.

Proposition 4.8. *Assume that σ is 8-deep. Let $S \subset \{a \in \mathbb{Z}^{\mathcal{J}}_{\geq 0} \mid |a| = n\}$ be a nonempty subset. Then*

$$\mathrm{rad} \sum_{a\in S} \mathrm{rad}^a P_\sigma = \sum_{a\in S} \mathrm{rad}^{>a} P_\sigma.$$

Proof. Since $\sum_{a\in S} \mathrm{rad}^a P_\sigma / \sum_{a\in S} \mathrm{rad}^{>a} P_\sigma \cong \oplus_{a\in S} \mathrm{gr}^a P_\sigma$, $\mathrm{rad} \sum_{a\in S} \mathrm{rad}^a P_\sigma \subset \sum_{a\in S} \mathrm{rad}^{>a} P_\sigma$.

To show the reverse inclusion, we need an intermediate result. Let $\sigma = F(\lambda)$, $b \in \mathbb{Z}^{\mathcal{J}}_{\geq 0}$, and $i \in \mathcal{J}$. Then

$$\mathrm{rad}^b P_\sigma / \mathrm{rad}^{>b+1_i} P_\sigma \tag{4.1}$$

is the restriction to G of

$$\mathrm{rad}^{b_i} Q_1(\lambda_i)/\mathrm{rad}^{b_i+2} Q_1(\lambda_i) \otimes \otimes_{j\neq i} \mathrm{gr}^{b_j} Q_1(\lambda).$$

By Proposition 4.4 and [LLHLM20, Lemma 4.2.2], the radical of (4.1) is $\mathrm{rad}^{>b} P_\sigma/\mathrm{rad}^{>b+1_i} P_\sigma$.

We now show the reverse inclusion. If $a \in S$ and $b > a$, we will show that $\mathrm{rad}^b P_\sigma \subset \mathrm{rad}\,\mathrm{rad}^a P_\sigma$ by reverse induction on $|b|$. If $|b| > 6\#\mathcal{J}$, then $\mathrm{rad}^b P_\sigma = 0$ and we are done. Suppose now that $|b| \leq 6\#\mathcal{J}$. Since $b > a$, $b - 1_i \geq a$ for some $i \in \mathcal{J}$. Then the kernel of the map $\mathrm{rad}^a P_\sigma \to \mathrm{cosoc}\,\mathrm{rad}^a P_\sigma$ contains $\mathrm{rad}^{>b} P_\sigma \subset \mathrm{rad}\,\mathrm{rad}^a P_\sigma$ by the inductive hypothesis.

Thus the kernel of the induced map $\mathrm{rad}^{b-1_i} P_\sigma \to \mathrm{cosoc}\,\mathrm{rad}^a P_\sigma$ factors through $\mathrm{rad}^{b-1_i} P_\sigma/\mathrm{rad}^{>b} P_\sigma$. By the above paragraph, $\mathrm{rad}^b P_\sigma$ is in the kernel of the map $\mathrm{rad}^{b-1_i} P_\sigma \to \mathrm{cosoc}\,\mathrm{rad}^a P_\sigma$ which gives the desired inclusion. □

Corollary 4.9. *Assume that σ is 8-deep. Let $a = (a_j)_{j\in\mathcal{J}}$ with $a_j = 2$ for all $j \in \mathcal{J}$. Suppose that Λ is a multiplicity free $\mathbb{F}[\mathrm{G}]$-module with cosocle isomorphic to σ such that $\mathrm{JH}(\Lambda) \subset \mathrm{JH}(P_\sigma^{\widehat{a}})$. Then Λ is isomorphic to a unique quotient of $P_\sigma^{\widehat{a}}$.*

Proof. Let Λ be as in the statement of the corollary. Since $\mathrm{cosoc}\,\Lambda \cong \sigma$, we can and do fix a surjection $P_\sigma \twoheadrightarrow \Lambda$. We claim that this map factors through $P_\sigma^{\widehat{a}}$. Suppose otherwise. Then the restriction $\mathrm{rad}^{2_i} P_\sigma \to \Lambda$ is nonzero for some $i \in \mathcal{J}$. By Proposition 4.8, this implies that $\mathrm{JH}(\mathrm{gr}^{2_i} P_\sigma) \cap \mathrm{JH}(\mathrm{rad}\,\Lambda) \neq \emptyset$. This contradicts the assumption that Λ is a multiplicity free $\mathbb{F}[\mathrm{G}]$-module and $\mathrm{JH}(\Lambda) \subset \mathrm{JH}(P_\sigma^{\widehat{a}})$. The uniqueness of the quotient follows from the fact that $P_\sigma^{\widehat{a}}$ is multiplicity free. □

The following proposition shows that maps between projective envelopes are compatible with the filtrations $\mathrm{rad}^a P_{F(\lambda)}$.

Proposition 4.10. *Let σ and κ be 8-deep Serre weights and $P_\kappa \to \mathrm{rad}^a P_\sigma$ be a map whose image is not contained in $\mathrm{rad}^{>a} P_\sigma$. Then the image of $\mathrm{rad}^b P_\kappa$ is contained in $\mathrm{rad}^{a+b} P_\sigma$ but not in $\mathrm{rad}^{>a+b} P_\sigma$.*

Proof. We proceed by induction with respect to the partial ordering on b. The case $b_j = 0$ for all $j \in \mathcal{J}$ is clear. Suppose that the image of $\mathrm{rad}^b P_\kappa$ is contained in $\mathrm{rad}^{a+b} P_\sigma$ but not in $\mathrm{rad}^{>a+b} P_\sigma$. Let $i \in \mathcal{J}$. Thus, the alcoves of all of the constituents of $\mathrm{gr}^b P_\kappa$ and $\mathrm{gr}^{a+b} P_\sigma$ coincide. This implies that $\mathrm{JH}(\mathrm{gr}^{b+1_i} P_\kappa)$ and $\mathrm{JH}(\mathrm{rad}^{a+b} P_\sigma/\mathrm{rad}^{a+b+1_i} P_\sigma)$ are disjoint by alcove considerations in embedding i. Hence the map $\mathrm{gr}^{b+1_i} P_\kappa \to \mathrm{rad}^{a+b} P_\sigma/(\mathrm{rad}^{a+b+1_i} P_\sigma + \mathrm{Im}(\mathrm{rad}^{>b+1_i} P_\kappa))$ is zero. Thus the image of $\mathrm{rad}^{b+1_i} P_\kappa$ and $\mathrm{rad}^{>b+1_i} P_\kappa$ in $\mathrm{rad}^{a+b} P_\sigma/\mathrm{rad}^{a+b+1_i} P_\sigma$ coincide. By Nakayama's lemma these images are zero so that the image of $\mathrm{rad}^{b+1_i} P_\kappa$ is contained in $\mathrm{rad}^{a+b+1_i} P_\sigma$.

It suffices to show that the induced map $\varphi : \mathrm{gr}^{b+1_i} P_\kappa \to \mathrm{gr}^{a+b+1_i} P_\sigma$ is nonzero. In fact, by [LLHLM20, Lemma 4.2.2] this map is the socle of the the map

$$\mathrm{rad}^b P_\kappa/(\mathrm{rad}^{>b+1_i} P_\kappa + \sum_{j\neq i} \mathrm{rad}^{b+1_j} P_\kappa) \to \mathrm{rad}^{a+b} P_\sigma/(\mathrm{rad}^{>a+b+1_i} P_\sigma + \sum_{j\neq i} \mathrm{rad}^{a+b+1_j} P_\sigma), \tag{4.2}$$

which is nonzero by the inductive hypothesis. If $\varphi = 0$, then (4.2) factors through a map $\mathrm{gr}^b P_\kappa \to \mathrm{gr}^{a+b+1_i} P_\sigma$ which must be 0 by alcove considerations at embedding i. This is a contradiction. □

4.3. **The covering property.** For $\lambda \in X_1(\underline{T})$, let $A(\lambda)$ be the set $\{j \in \mathcal{J} \mid \lambda_j \in A\}$. If $\sigma \cong F(\lambda)$, then we define $A(\sigma)$ to be $A(\lambda)$.

Fix a tame L-parameter $\overline{\rho}$ with a lowest alcove presentation (s, μ) for it. The main result of this section is the following result which plays a key role in §5.2.

Proposition 4.11. *Let $\sigma \in W^?(\overline{\rho})$ be 8-deep. Fix $a \in \mathbb{Z}_{\geq 0}^{\mathcal{J}}$ and let $N \subset \mathrm{rad}^a P_\sigma$ be a submodule such that the cokernel of the induced map $N \to \mathrm{gr}^a P_\sigma$ has no Jordan–Hölder factors in $W(\overline{\rho})$. Then no Jordan–Hölder factors of $\mathrm{rad}^a P_\sigma/N$ are in $W(\overline{\rho})$.*

The proof of Proposition 4.11 follows inductively from the following lemma.

Lemma 4.12. *Let $\sigma \in W^?(\overline{\rho})$ be 8-deep. Fix $a, b \in \mathbb{Z}_{\geq 0}^{\mathcal{J}}$ such that for some $i \in \mathcal{J}$, $b_i = a_i + 1$ and $b_j = a_j$ for all $j \neq i$. Let $N \subset \mathrm{rad}^a P_\sigma/\mathrm{rad}^{>b} P_\sigma$ be a submodule such that the cokernel of the induced map $N \to \mathrm{gr}^a P_\sigma$ has no Jordan–Hölder factors in $W(\overline{\rho})$. Then no Jordan–Hölder factors of $(\mathrm{rad}^a P_\sigma/\mathrm{rad}^{>b} P_\sigma)/N$ are in $W(\overline{\rho})$.*

Proof of Proposition 4.11. We proceed by induction. For convenience, for any submodule $M \subset \mathrm{rad}^a P_\sigma$, we write $\overline{M}$ for the image of M in $\mathrm{rad}^a P_\sigma/N$. Similarly, for submodules $M' \subset M \subset \mathrm{rad}^a P_\sigma$, we write $\overline{M/M'}$ for $\overline{M}/\overline{M'}$. Proposition 4.11 holds for a sufficiently large. Let $a \in \mathbb{Z}_{\geq 0}^{\mathcal{J}}$ and suppose that Proposition 4.11 holds for any $b \in \mathbb{Z}_{\geq 0}^{\mathcal{J}}$ as in Lemma 4.12. Suppose that N is as in Proposition 4.11. By the exact sequence

$$0 \to \sum_{\substack{b > a \\ b \text{ as in L. } 4.12}} \overline{\mathrm{rad}^b P_\sigma} \to \overline{\mathrm{rad}^a P_\sigma} \to \overline{\mathrm{gr}^a P_\sigma} \to 0,$$

it suffices to show that for any $b \in \mathbb{Z}_{\geq 0}^{\mathcal{J}}$ as in Lemma 4.12, $\overline{\mathrm{rad}^b P_\sigma}$ contains no Jordan–Hölder factors in $W^?(\overline{\rho})$.

Let $b \in \mathbb{Z}_{\geq 0}^{\mathcal{J}}$ be as in the above exact sequence, i.e., as in Lemma 4.12. By Lemma 4.12, $\overline{\mathrm{rad}^a P_\sigma/\mathrm{rad}^{>b} P_\sigma}$ contains no Jordan–Hölder factors in $W^?(\overline{\rho})$. Thus $\overline{\mathrm{gr}^b P_\sigma} \subset \overline{\mathrm{rad}^a P_\sigma/\mathrm{rad}^{>b} P_\sigma}$ contains no Jordan–Hölder factors in $W^?(\overline{\rho})$. By the inductive hypothesis, $\overline{\mathrm{rad}^b P_\sigma}$ contains no Jordan–Hölder factors in $W^?(\overline{\rho})$. ☐

The proof of Lemma 4.12 requires a series of results. Let $\lambda = (\lambda_j)_j \in X_1(\underline{T})$ be 8-deep in alcove $A^{\mathcal{J}}$. If $X = (X_j)_j$ is a dominant alcove, then let $\lambda_X \overset{\mathrm{def}}{=} \sum_j \lambda_{j,X_j}$ (using notation from §4.1). In particular we have $\lambda_{j,X_j} = \widetilde{w}_{X_j} \cdot \lambda_j = \lambda^0_{j,X_j} + p\omega_{X_j}$ where $\lambda^0_{j,X_j} \overset{\mathrm{def}}{=} \widetilde{w}^0_{X_j} \cdot \lambda_j \in X_1^*(T)$. For the remainder of this section, $M = M_i \otimes \otimes_{j \neq i} L(\lambda_{j,X_j})$ where $X_j \in \{A, B, C, D, E, F, G, H, I, J\}$ for all $j \neq i$ and M_i is a rigid module with Loewy length two. Then M is rigid of Loewy length two and we let M^1 and M^0 be the cosocle and socle, respectively. Unless otherwise stated, M_i is a nonsplit extension of $L(\lambda_{i,X_i^1})$ by $L(\lambda_{i,X_i^0})$. We will require the following lemmas.

Lemma 4.13. *Let $\sigma \in \mathrm{JH}(M^1|_{\mathrm{G}})$. Let $N \subset N' \subset M|_{\mathrm{G}}$ be submodules such that the cokernel of the natural map $N \to \mathrm{cosoc}\, N'$ does not contain σ as a Jordan–Hölder factor. Then N'/N does not contain σ as a Jordan–Hölder factor.*

Proof. If $\varphi : N \to \mathrm{cosoc}\, N'$ is the natural map, there is an exact sequence

$$0 \to N'' \to N'/N \to \mathrm{coker}\,\varphi \to 0$$

for some subquotient N'' of $M^0|_{\mathrm{G}}$. Since $\sigma \notin \mathrm{JH}(M^0|_{\mathrm{G}})$ by alcove considerations, and $\sigma \notin \mathrm{JH}(\mathrm{coker}\,\varphi)$ by assumption, $\sigma \notin \mathrm{JH}(N'/N)$. ☐

Lemma 4.14. *There exist a finite set S, 6-deep weights $\lambda^k \in X_1(\underline{T})$ in alcove $A^{\mathcal{J}}$ for each $k \in S$ with $\lambda^k_{\pi(i)} = \lambda_{\pi(i)}$, and alcoves $Y_j \in \{A, B\}$ for each $j \neq i$ such that with $M_{k,i}$ a nonsplit extension of $L(\lambda^k_{i,X_i^1})$ by $L(\lambda^k_{i,X_i^0})$ and $M_k = M_{k,i} \otimes \otimes_{j \neq i} L(\lambda^k_{j,Y_j})$ for each $k \in S$, $M|_{\mathrm{G}} \cong \oplus_{k \in S} M_k|_{\mathrm{G}}$.*

Proof. See the proof of [LLHLM20, Proposition 4.2.10]. □

Proposition 4.15. *Suppose that $X_i^1, X_i^0 \in \{A, B, C, D, E, F, G\}$ or*

$$(X_i^1, X_i^0) \in \{(C, J), (D, J), (J, C), (J, D), (E, H), (F, I), (H, E), (I, F)\}.$$

Let σ^n be a Jordan–Hölder factor in $M^n|_{\mathrm{G}}$ with multiplicity one for $n = 0$ and 1 and fix nonzero maps (unique up to scalar) $P_{\sigma^1} \to M|_{\mathrm{G}}$ and $M|_{\mathrm{G}} \to P_{\sigma^0}$. Then the composition $P_{\sigma^1} \to M|_{\mathrm{G}} \to P_{\sigma^0}$ is nonzero if and only if $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{G}]}(\sigma^1, \sigma^0) \neq 0$.

Proof. This follows from the proofs of [LLHLM20, Propositions 4.2.10 and 4.2.12]. Indeed, if $X_i^1, X_i^0 \in \{A, B, C, D, E, F, G\}$, then the result follows from [LLHLM20, Proposition 4.2.10]. We will consider the cases $(X_i^1, X_i^0) \in \{(J, C), (J, D), (H, E), (I, F)\}$. The remaining cases follow by duality. One reduces to the cases where $X_j \in \{A, B\}$ for $j \neq i$ by Lemma 4.14 (see the proof of [LLHLM20, Proposition 4.2.10]). The cases $(X_i^1, X_i^0) = (J, C)$ or (J, D) are similar to the cases (G, E) and (G, F) covered in [LLHLM20, Proposition 4.2.12]. The cases (X_i^1, X_i^0) is (H, E) or (I, F) are also similar to the cases just mentioned. Indeed, in the last step of the proof of [LLHLM20, Proposition 4.2.12] one uses instead that $L(2\omega_{X_i^1})$ is isomorphic to the submodule of $L(\omega_{X_i^1}) \otimes L(\omega_{X_i^1})$ on which the involution given by switching tensor factors acts by 1. □

Lemma 4.16. *If $\{X_i^1, X_i^0\} = \{G, J\}$, then $M|_{\mathrm{G}}$ is a direct sum of indecomposable length two modules.*

Proof. Decompose $M^1|_{\mathrm{G}} = \oplus_k \sigma_k$ (we identify σ_k with a submodule of $M^1|_{\mathrm{G}}$ even though various σ_k may be isomorphic). Let $N_k \subset M|_{\mathrm{G}}$ be the image of a map $P_{\sigma_k} \to M|_{\mathrm{G}}$ lifting a projective cover $P_{\sigma_k} \twoheadrightarrow \sigma_k \subset M^1|_{\mathrm{G}}$ of σ_k. For each k, the length of N_k is at least two since $\sigma_k \notin \mathrm{JH}(\mathrm{soc}\, M|_{\mathrm{G}})$ by [LLHLM20, Lemma 4.2.2]. Moreover, since there is a unique $\sigma'_k \in \mathrm{JH}(M^0|_{\mathrm{G}})$ such that $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{G}]}(\sigma_k, \sigma'_k)$ is nonzero and $\dim_{\mathbb{F}} \mathrm{Ext}^1_{\mathbb{F}[\mathrm{G}]}(\sigma_k, \sigma'_k) = 1$ by [LLHLM20, Lemma 4.2.6], the length of N_k is exactly two. The natural map $\oplus_k N_k \to M|_{\mathrm{G}}$ is a surjective map between objects of the same length, and is thus an isomorphism. □

We now let M^A and M^B be $L(\lambda_{i,A}) \otimes \otimes_{j\neq i} L(\lambda_{j,X_j})$ and $L(\lambda_{i,B}) \otimes \otimes_{j\neq i} L(\lambda_{j,X_j})$, respectively. Fix $\sigma \in \mathrm{JH}(M^A|_{\mathrm{G}})$.

Proposition 4.17. *Suppose $(X_i^1, X_i^0) = (G, E)$ or (G, F). If N is a submodule of $M|_{\mathrm{G}}$ with $\sigma \in \mathrm{JH}(N)$, then N contains at least two weights in $M^0|_{\mathrm{G}}$ adjacent to σ.*

Proof. Let N be as in the statement. The projection of N to some $M_k|_{\mathrm{G}}$ in Lemma 4.14 contains σ as a Jordan–Hölder factor, and so we reduce to the case where $X_j \in \{A, B\}$ for all $j \neq i$. If N contains only one weight in $M^0|_{\mathrm{G}}$ adjacent to σ, then there is a map $\varphi : P_\sigma \to N \subset M|_{\mathrm{G}}$ such that $\mathrm{Im}(\varphi) \cap \mathrm{soc}(M|_{\mathrm{G}})$ is simple. Considering the composition of φ with the injective envelope of $M|_{\mathrm{G}}$, the argument of [LLHLM20, Proposition 4.2.12] implies that there is a nonzero element of the 0-weight space of $L(\omega_{X_i^0} - w_0\omega_{X_i^0})$ whose image in $L(\omega_{X_i^0}) \otimes L(-w_0\omega_{X_i^0})$ is a pure tensor of weight eigenvectors. This contradicts Lemma 4.18. □

Lemma 4.18. *Let $\omega \in X^*(T)$ be a fundamental weight. Under the inclusion $L(\omega - w_0\omega) \subset L(\omega) \otimes L(-w_0\omega)$, any element in the 0 weight space of $L(\omega - w_0\omega)$ is not a pure tensor of weight eigenvectors in $L(\omega) \otimes L(-w_0\omega)$.*

Proof. The (2-dimensional) 0-weight space of $L(\omega - w_0\omega)$ is stable under the Weyl group symmetry. However, the Weyl group orbit of a weight 0 pure tensor of (nonzero) weight eigenvectors in $L(\omega) \otimes L(-w_0\omega)$ spans the entire (3-dimensional) 0-weight space of $L(\omega) \otimes L(-w_0\omega)$. □

Proposition 4.19. *Suppose that $(X_i^1, X_i^0) = (E, G)$ or (F, G). If N is a submodule of $M|_{\mathrm{G}}$ containing two distinct Serre weights adjacent to σ as Jordan–Hölder factors, then N contains the σ-isotypic part of $M^0|_{\mathrm{G}}$.*

Proof. Suppose that N is as above. Define N' be the kernel of natural surjection $(M|_{\mathrm{G}})^* \twoheadrightarrow N^*$, where $(-)^*$ denotes the contragredient representation. If $\sigma^* \in \mathrm{JH}(N')$, then two of the weights adjacent to σ^* are Jordan–Hölder factors of N' by Proposition 4.17, and thus not of N^* since $M^1|_{\mathrm{G}}$ is a multiplicity free representation. This contradicts the assumption that N contains two distinct Serre weights adjacent to σ as Jordan–Hölder factors. Thus $\sigma^* \notin \mathrm{JH}(N')$ and N contains the σ-isotypic part of $M^0|_{\mathrm{G}}$. □

Recall our standing assumption that λ is 8-deep in alcove $A^{\mathcal{J}}$.

Proposition 4.20. *Suppose that either*
(1) *at least one of X_i^1 or X_i^0 is in $\{A, B\}$;*
(2) *$(X_i^1, X_i^0) = (C, F)$ or (D, E); or*
(3) *$X_i^1 = E$ or F.*

Assume further that $F(\lambda)$ is linked to a modular Serre weight, or equivalently, that $F(\sum_j \lambda_{j,B}) \in W^?(\overline{\rho})$. If $X_i^1 = A$, assume further that $F(\lambda_{i,A} + \sum_{j\neq i} \lambda_{j,B}) \in W^?(\overline{\rho})$. Suppose that N is a submodule of $M|_{\mathrm{G}}$ such that the cokernel of the projection of N onto $M^1|_{\mathrm{G}}$ contains no Serre weights in $W^?(\overline{\rho})$. Then $M|_{\mathrm{G}}/N$ contains no Jordan–Hölder factors in $W^?(\overline{\rho})$.

Proof. In this proof we repeatedly use the description of $W^?(\overline{\rho})$ in terms of the extension graph given in §2.1.3. With $M|_{\mathrm{G}} \cong \oplus_{k\in S} M_k|_{\mathrm{G}}$ as in Lemma 4.14, $M|_{\mathrm{G}}/N$ is a quotient of $\oplus_{k\in S} M_k|_{\mathrm{G}}/(M_k|_{\mathrm{G}} \cap N)$. It suffices to show the following claim: for each $k \in S$, $M_k|_{\mathrm{G}}/(M_k|_{\mathrm{G}} \cap N)$ contains no Jordan–Hölder factors in $W^?(\overline{\rho})$. Note that M_k has simple cosocle as a $\underline{G}_{/\mathbb{F}}$-module, whose (6-deep) highest weight is linked to the weight $\lambda^k \in A^{\mathcal{J}}$ appearing in Lemma 4.14. Note that if $\lambda = \mathfrak{Tr}_{\mu+\eta}(\nu, \underline{0})$ then $\lambda^k = \mathfrak{Tr}_{\mu+\eta}(\nu + \omega, \underline{0})$ for some $\omega \in L(\sum_{j\neq i} \pi\omega_{X_j})$ by Proposition 4.6. Thus if $F(\lambda^k)$ is not linked to a modular weight then $(\nu_{\pi(j)} + \omega_{\pi(j)}, 1)$ is not in $s_{\pi(j)}(r(\Sigma_0))$ for some $j \neq i$. Then $\mathrm{JH}(M_k|_{\mathrm{G}}) \cap W^?(\overline{\rho})$ is empty, and the claim holds. Assume now that $F(\lambda^k)$ is linked to a modular weight. We will show that the remaining hypotheses of the proposition hold for $M_k|_{\mathrm{G}} \cap N \subset M_k|_{\mathrm{G}}$ so that we reduce to the case where $X_j \in \{A, B\}$ for $j \neq i$. Clearly, the enumerated condition that holds for M holds for any M_k. It remains to show that the cokernel of the natural map $M_k|_{\mathrm{G}} \cap N \to M_k^1|_{\mathrm{G}} \overset{\mathrm{def}}{=} \mathrm{cosoc}\, M_k|_{\mathrm{G}}$ contains no Jordan–Hölder factors in $W^?(\overline{\rho})$. Let $\sigma \in \mathrm{JH}(M_k^1|_{\mathrm{G}}) \cap W^?(\overline{\rho})$. By Lemma 4.13 taking $N' = M|_{\mathrm{G}}$, $\sigma \notin \mathrm{JH}(M|_{\mathrm{G}}/N)$. Thus, in fact, $\sigma \notin \mathrm{JH}(M_k|_{\mathrm{G}}/(M_k|_{\mathrm{G}} \cap N))$.

We now establish the proposition assuming that $X_j \in \{A, B\}$ for $j \neq i$. We assume that $F(\lambda_{i,B} + \sum_{j\neq i} \lambda_{j,X_j}) \in W^?(\overline{\rho})$ since otherwise $\mathrm{JH}(M|_{\mathrm{G}}) \cap W^?(\overline{\rho}) = \emptyset$, and the proposition follows immediately. Our analysis now breaks into the three cases for X_i^0 and X_i^1 described in the enumerated conditions of the proposition. If X_i^1 or X_i^0 is A or B, then it is easy to see that N necessarily contains $M^0|_{\mathrm{G}}$, and the proposition follows. (Note that if $X_i^1 = A$, then $F(\lambda_{i,A} + \sum_{j\neq i} \lambda_{j,X_j}) \in W^?(\overline{\rho})$.)

We next consider case where (2) holds. We assume that $(X_i^1, X_i^0) = (C, F)$ as the case (D, E) is symmetric. Suppose $\sigma^0 \subset M^0|_{\mathrm{G}}$ is simple and in $W^?(\overline{\rho})$. Then one checks that there is a unique $\sigma^1 \in \mathrm{JH}(M^1|_{\mathrm{G}}) \cap W^?(\overline{\rho})$ and moreover that σ^1 is adjacent to σ^0. Then $\sigma^0 \subset N$ by Proposition 4.15.

Finally, we consider the case where (3) holds. We assume that $X_i^1 = E$ as the case $X_i^1 = F$ is symmetric. Then X_i^0 is A, C, D, G, H, or I. The case where $X_i^0 = A$ was already considered in (1). Let $\sigma^0 \subset M^0|_{\mathrm{G}}$ be simple and in $W^?(\overline{\rho})$. Since $F(\lambda_{i,B} + \sum_{j\neq i} \lambda_{j,X_j}) \in W^?(\overline{\rho})$ by assumption, it is easy to check that there is $\sigma^1 \in \mathrm{JH}(M^1|_{\mathrm{G}}) \cap W^?(\overline{\rho})$. Suppose first that X_i^0 is D or H. Then σ^0 and σ^1 must be adjacent. (The contrapositive is easier to see: If $\sigma^0 = \mathfrak{Tr}_{\mu+\eta}(s_j(\omega^0), a^0)$ and $\sigma^1 = \mathfrak{Tr}_{\mu+\eta}(s_j(\omega^1), a^1)$ are not adjacent, then $\omega_i^0 - \omega_i^1$ is an element of Λ_R using that $(\omega_j^0, a_j^0) = (\omega_j^1, a_j^1)$ for all $j \neq i$. This implies that $\omega_E - \omega_X \in \Lambda_R$ for $X = D$ or H which is false.) The result then follows from Proposition 4.15. Suppose next that X_i^0 is C or I. If σ^0 and σ^1 are not adjacent, then the weight in $\mathrm{JH}(M^1|_{\mathrm{G}})$ linked to σ^0 is also modular and the result follows again from Proposition 4.15 (see Figure 5).

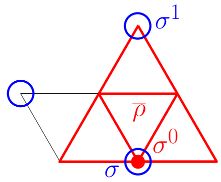


FIGURE 5. σ^0 and σ^1 are nonadjacent weights in $W^?(\overline{\rho})$. The blue circles represent $\mathrm{JH}(M^1|_{\mathrm{G}})$. The weight σ linked to σ^0 is in $\mathrm{JH}(M^1|_{\mathrm{G}}) \cap W^?(\overline{\rho})$.

Suppose finally that X_i^0 is G. If $\sigma^0 \in \mathrm{JH}(M^A|_{\mathrm{G}})$ (resp. $\mathrm{JH}(M^0|_{\mathrm{G}}) \setminus \mathrm{JH}(M^A|_{\mathrm{G}})$), then σ^0 is adjacent to exactly two modular weights (resp. one modular weight) of $\mathrm{JH}(M^1|_{\mathrm{G}})$. The result then follows from Proposition 4.19 (resp. Proposition 4.15). □

Proposition 4.21. *Let M_i be a (unique up to isomorphism) rigid nonsplit extension of $L(\lambda_{C_i}) \oplus L(\lambda_{G_i})$ by $L(\lambda_{J_i})$. Assume further that $F(\lambda)$ is linked to a modular Serre weight $F(\lambda')$ with $\lambda'_i = \lambda_i$. If N is a submodule of $M|_{\mathrm{G}}$ such that the cokernel of the projection of N onto $M^1|_{\mathrm{G}}$ contains no modular Serre weights, then $M|_{\mathrm{G}}/N$ contains no modular Serre weights (as Jordan–Hölder factors).*

Proof. Suppose that σ^0 is a weight in $W^?(\overline{\rho}) \cap (\mathrm{JH}(M^0|_{\mathrm{G}}) \setminus \mathrm{JH}(M^B|_{\mathrm{G}}))$. Then, using that $F(\lambda)$ is linked to a modular Serre weight, $L(\lambda_{C_i}) \otimes \otimes_{j\neq i} L(\lambda_{j,X_j})|_{\mathrm{G}}$ contains a modular weight adjacent to σ^0, and N contains σ^0 by Proposition 4.15 with $(X_i^1, X_i^0) = (C, J)$. If σ^0 is a weight in $W^?(\overline{\rho}) \cap \mathrm{JH}(M^B|_{\mathrm{G}})$, then the Jordan–Hölder factor of $M^A|_{\mathrm{G}}$ adjacent to σ^0 is modular (by the assumption that $\lambda'_i = \lambda_i$) and appears with multiplicity two in $M^1|_{\mathrm{G}}$. Then σ^0 appears in N with multiplicity two by Lemma 4.16. □

Proof of Lemma 4.12. Let σ, a, b, i, and N be as in Lemma 4.12. For convenience, for any submodule $M \subset P_\sigma/\mathrm{rad}^{>b}P_\sigma$, we write $\overline{M}$ for the image of M in $(P_\sigma/\mathrm{rad}^{>b}P_\sigma)/N$. Similarly, if $M' \subset M \subset P_\sigma/\mathrm{rad}^{>b}P_\sigma$ are submodules, then we write $\overline{M/M'}$ for $\overline{M}/\overline{M'}$. Let σ be $F(\mu)$ so that $\mathrm{rad}^a P_\sigma/\mathrm{rad}^{>b}P_\sigma$ is isomorphic to $\mathrm{rad}^a Q_1(\mu)/\mathrm{rad}^{>b}Q_1(\mu)|_{\mathrm{G}}$ where

$$\mathrm{rad}^a Q_1(\mu)/\mathrm{rad}^{>b}Q_1(\mu) \cong \mathrm{rad}^{a_i}Q_1(\mu_i)/\mathrm{rad}^{a_i+2}Q_1(\mu_i) \otimes \otimes_{j\neq i} \mathrm{gr}^{a_j} Q_1(\mu_j). \tag{4.3}$$

We will show that $\mathrm{JH}(\overline{\mathrm{rad}^a P_\sigma/\mathrm{rad}^{>b}P_\sigma}) \cap W^?(\overline{\rho})$ is empty. Since $\mathrm{JH}(\overline{\mathrm{gr}^a P_\sigma}) \cap W^?(\overline{\rho})$ is empty by assumption, it suffices to show that $\mathrm{JH}(\overline{\mathrm{gr}^b P_\sigma}) \cap W^?(\overline{\rho})$ is empty. Let σ^0 be in $\mathrm{JH}(\mathrm{gr}^b P_\sigma) \cap W^?(\overline{\rho})$. We claim that there is a finite, separated, and exhaustive increasing filtration $F^k \mathrm{gr}^b Q_1(\mu)$ on $\mathrm{gr}^b Q_1(\mu)$ and submodules

$$M_k \subset (\mathrm{rad}^a Q_1(\mu)/\mathrm{rad}^{>b}Q_1(\mu))/F^{k-1}\mathrm{gr}^b Q_1(\mu) \tag{4.4}$$

for each k such that

(1) $F^k \mathrm{gr}^b Q_1(\mu)/F^{k-1}\mathrm{gr}^b Q_1(\mu) \subset M_k$; and
(2) $\sigma^0 \notin \mathrm{JH}(\overline{M_k|_{\mathrm{G}}})$.

Admitting this claim, we can inductively show for all k that $\sigma^0 \notin \mathrm{JH}(\overline{F^k \operatorname{gr}^b Q_1(\mu)|_{\mathrm{G}}})$. For k sufficiently small, $F^k \operatorname{gr}^b Q_1(\mu) = 0$ and we are done. If $\sigma^0 \notin \mathrm{JH}(\overline{F^{k-1} \operatorname{gr}^b Q_1(\mu)|_{\mathrm{G}}})$ for some k, then $\sigma^0 \notin \mathrm{JH}(\overline{F^k \operatorname{gr}^b Q_1(\mu)|_{\mathrm{G}}})$ since

$$\overline{F^k \operatorname{gr}^b Q_1(\mu)/F^{k-1} \operatorname{gr}^b Q_1(\mu)|_{\mathrm{G}}} \subset \overline{M_k|_{\mathrm{G}}}$$

by (1) and $\sigma^0 \notin \mathrm{JH}(\overline{M_k|_{\mathrm{G}}})$ by (2).

It suffices to prove the claim. We set $F^0 \operatorname{gr}^b Q_1(\mu) = M_0 = 0$ and $F^1 \operatorname{gr}^b Q_1(\mu) = M_1$ to be the largest submodule $M \subset \operatorname{gr}^b Q_1(\mu)$ such $\sigma^0 \notin \mathrm{JH}(M|_{\mathrm{G}})$. Now, by Proposition 4.4, we can recursively find M_k as in (4.4) to be a subquotient of a graded piece in either the Weyl filtration or the dual Weyl filtration of the form

$$M_k = M_{i,k} \otimes \otimes_{j\neq i} L(\lambda_{j,X_j})$$

where λ is the unique element in $A^{\mathcal{J}} \subset X^*(\underline{T})$ linked to μ and

- $\operatorname{soc} M_{i,k} \cong L(\lambda_{i,X_i^0})$ and $\operatorname{cosoc} M_{k,i} \cong L(\lambda_{i,X_i^1})$ with (X_i^1, X_i^0) in the appropriate column of Table 1 or 2; or
- $a_i = 2$, $\mu_i \in A$, $\operatorname{soc} M_{i,k} \cong L(\lambda_{i,J})$, and $\operatorname{cosoc} M_{k,i} \cong L(\lambda_{i,C}) \oplus L(\lambda_{i,G})$ (as indicated by $(C \oplus G, J)$ in Table 2).

We define $F^k \operatorname{gr}^b Q_1(\mu)$ to be the preimage of $\operatorname{soc} M_k$. Then (1) holds by construction, and it suffices to check (2) for each M_k. This is clear for $k = 0$ and 1 and follows from Propositions 4.20 and 4.21 for $k > 1$. □

TABLE 1. **Subquotients of** $\operatorname{rad}^{b_i-1}Q_1(\lambda_i)/\operatorname{rad}^{b_i+1}Q_1(\mu_i)$**,** $\mu_i \in B$

a_i	0	1	2	3	4	5
Weyl		(C,B), (D,B)	(E,C), (E,A), (E,D) (F,C), (F,A), (F,D)	$(C,B)\times 2$	(E,A)	(C,B)
Dual Weyl	(B,C), (B,A), (B,D)	(D,E), (C,F)	(E,G)	(D,E), (C,F)	(B,C), (B,D)	

This table records length two subquotients of $\operatorname{rad}^{b_i-1}Q_1(\lambda_i)/\operatorname{rad}^{b_i+1}Q_1(\mu_i)$ for $\mu_i \in B$ which are subquotients of either a Weyl or dual Weyl module in the associated graded of the Weyl or dual Weyl filtration. The notation (X^1, X^0) denotes a nonsplit extension of X^1 by X^0 (unique up to isomorphism).

4.4. **Presentations of some quotients of** P_σ**.** In this section, we introduce some quotients of P_σ and give presentations which will be used in §5.2.

Let $a = (a_j)_{j\in\mathcal{J}}$ be a tuple such that a_j is a subset of $\{B, E_o, E_s, F_o, F_s\}$ for $j \in A(\sigma)$ (which we write simply as a string in any order) and $a_j = \widehat{1}$ or $\widehat{2}$ if $j \notin A(\sigma)$. Let $b = (b_j)_{j\in\mathcal{J}}$ be a tuple such that $b_j = \widehat{2}$ if $j \in A(\sigma)$ and $b_j = a_j$ otherwise. We now define a quotient P_σ^a of P_σ^b.

Write $\sigma = F(\lambda)$ for a 4-deep weight $\lambda \in X_1(\underline{T})$ so that P_σ^b is isomorphic to $Q_1(\lambda)^b|_{\mathrm{G}}$ and Proposition 4.7 applies. For each $i \in A(\sigma)$, $Q_1(\lambda)^b$ has a submodule

$$\begin{aligned}\operatorname{gr}^1 Q_1(\lambda_i) \otimes \otimes_{j\neq i} Q_1(\lambda_j)^{b_j} &\cong (L(\lambda_{i,B}) \oplus L(\lambda_{i,E}) \oplus L(\lambda_{i,F})) \otimes \otimes_{j\neq i} Q_1(\lambda_j)^{b_j} \\ &\cong (L(\lambda^0_{i,B}) \otimes L(p\omega_{i,B}) \oplus L(\lambda^0_{i,E}) \otimes L(p\omega_{i,E}) \oplus L(\lambda^0_{i,F}) \otimes L(p\omega_{i,F})) \otimes \otimes_{j\neq i} Q_1(\lambda_j)^{b_j}\end{aligned}$$

Table 2. **Subquotients of** $\mathrm{rad}^{b_i-1}Q_1(\lambda_i)/\mathrm{rad}^{b_i+1}Q_1(\mu_i)$**,** $\mu_i \in A$

a_i	0	1	2	3	4	5
Weyl		$(E,A),\ (B,A),\ (F,A)$	$(A,B)\times 2$	$(E,A)\times 2,\ (F,A)$		(E,A)
Dual Weyl	$(A,E),\ (A,F)$	$(B,D),\ (B,C),\ (F,D),\ (E,C)$ $(F,I),\ (F,G),\ (E,H)$	$(A,E)\times 2,\ (A,F)\times 2$ $(C\oplus G,J)$	$(F,I),\ (F,G),\ (E,H)$ $(B,D)\times 2,\ (B,C)\times 2$	$(A,F),\ (A,B),\ (A,E)$	
Other	(A,B)					

This table records subquotients of $\mathrm{rad}^{b_i-1}Q_1(\lambda_i)/\mathrm{rad}^{b_i+1}Q_1(\mu_i)$ for $\mu_i \in A$ as in Table 1. The nonsplit extension (A,B) in the last row is the quotient $Q_1(\lambda_i)^B$ though it is not a quotient of either the quotient Weyl or dual Weyl module. $(C\oplus G,J)$ denotes the length 3 rigid nonsplit extension (unique up to isomorphism) of the direct sum of C and G by J.

(recall from §4.1 that $\lambda_{i,X} = \lambda^0_{i,X} + p\omega_{i,X}$ with $\lambda^0_{i,X} \in X_1(T)$ for $X = B,E,F$). Since

$$\begin{aligned}&(L(\lambda^0_{i,B})\otimes L(p\omega_{i,B})\oplus L(\lambda^0_{i,E})\otimes L(p\omega_{i,E})\oplus L(\lambda^0_{i,F})\otimes L(p\omega_{i,F}))\otimes\otimes_{j\neq i}Q_1(\lambda_j)^{b_j}|_{\mathrm{G}}\\ \cong&(L(\lambda^0_{i,B})\otimes L(\pi\omega_{i,B})\oplus L(\lambda^0_{i,E})\otimes L(\pi\omega_{i,E})\oplus L(\lambda^0_{i,F})\otimes L(\pi\omega_{i,F}))\otimes\otimes_{j\neq i}Q_1(\lambda_j)^{b_j}|_{\mathrm{G}},\end{aligned}$$

for each $X\in\{B,E,F\}$ and torus weight $\varepsilon_{i,X}$ in $L(\omega_{i,X})$ the translation principle (Proposition 4.5) produces submodules

$$L(\lambda^0_{i,X})\otimes Q_1(\lambda_{\pi(i)}+\pi\varepsilon_{i,X})^{b_{\pi(i)}}\otimes\otimes_{j\neq i,\pi(i)}Q_1(\lambda_j)^{b_j}|_{\mathrm{G}} \tag{4.5}$$

of P^b_σ if $\pi(i)\neq i$ and

$$L(\lambda^0_{i,X}+\varepsilon_{i,X})\otimes\otimes_{j\neq i}Q_1(\lambda_j)^{b_j}|_{\mathrm{G}} \tag{4.6}$$

if $\pi(i)=i$. For each $j\in A(\sigma)$, let $X_j\in\{B,E,F\}$, and let

$$\lambda^0 = \sum_{j\notin A(\sigma)}\lambda_j + \sum_{j\in A(\sigma)}\lambda^0_{j,X_j}.$$

For $j\in A(\sigma)$, if $X_j = B$ (resp. E and F), let $c_j = B$ (resp. $c_j\in\{E_o,E_s\}$ and $c_j\in\{F_o,F_s\}$). Then for each $j\in A(\sigma)$, there is a unique torus weight ε_{j,c_j} in $L(\omega_{j,X_j})$ such that if $c_j = E_o$ or F_o (resp. E_s or F_s), then $F(\lambda^0+\sum_{j\in A(\sigma)}\pi\varepsilon_{j,c_j})$ is obvious (resp. shadow) in embedding $\pi(j)$.

Finally, we let P^a_σ be the quotient of P^b_σ by the sum of the submodules in (4.5) or (4.6) with $i\in A(\sigma)$ and ε_i a torus weight in $L(\omega_{i,X})$ for $X = B$, E, or F but not in $\{\varepsilon_{i,c_i}\mid c_i\in a_i\}$. Informally speaking, P^a_σ is the non-zero quotient of P^b_σ containing precisely σ and the Jordan–Hölder factors labelled by the elements in a_j for $j\in A(\sigma)$ (see Figure 6).

Lemma 4.22. *Let $\mathcal{C}$ be an abelian category. Suppose that we have an exact sequence*

$$0\to\oplus_{i\in I}N_i\to M\to N\to 0$$

in $\mathcal{C}$ where I is a finite set. Letting M_i be the cokernel of the map $\oplus_{j\in I,j\neq i}N_j\to M$ for each $i\in I$, there is an exact sequence

$$0\to M\to\oplus_{i\in I}M_i\to\oplus_{i\in I}N/\Delta(N)\to 0$$

where the second map is (the composition of the diagonal map and) the sum of the natural projections, the third map is induced by the sum of the natural quotient maps, and Δ denotes the diagonally embedded copy of N in $\oplus_{i\in I}N$.

Proof. There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \oplus_{i\in I}N_i & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \oplus_{i\in I}N_i & \longrightarrow & \oplus_{i\in I}M_i & \longrightarrow & \oplus_{i\in I}N & \longrightarrow & 0 \end{array}$$

where the first row is as in the statement of the lemma, the second row is the direct sum of the exact sequences $0 \to N_i \to M_i \to N \to 0$, the second vertical map is the identity, the third vertical map is the the composition of the diagonal map and the sum of the natural projections, and the fourth vertical map is the diagonal map. The snake lemma gives the desired exact sequence. ☐

The following results are special cases of Lemma 4.22. As usual we assume that σ is 8-deep so that [LLHLM20, Lemma 4.2.2] applies.

Proposition 4.23. *Assume that σ is 8-deep. Let $a = (a_j)_{j\in\mathcal{J}}$ be a tuple such that a_j is a subset of $\{B, E_o, E_s, F_o, F_s\}$ for $j \in A(\sigma)$ and either $a_j = \widehat{1}$ for all $j \notin A(\sigma)$ or $a_j = \widehat{2}$ for all $j \notin A(\sigma)$. Fix $i \in A(\sigma)$. For $X \subset a_i$, let $a_X = (a_{X,j})_{j\in\mathcal{J}}$ be the tuple such that $a_{X,j} = a_j$ if $j \neq i$ and $a_{X,i} = X$. Then there is an exact sequence*

$$0 \to P_\sigma^a \to \oplus_{X\in a_i} P_\sigma^{a_X} \to (\oplus_{X\in a_i} P_\sigma^{a_\emptyset})/\Delta(P_\sigma^{a_\emptyset}) \to 0$$

where $\Delta(P_\sigma^{a_\emptyset})$ denotes the diagonally embedded copy of $P_\sigma^{a_\emptyset}$ in $\oplus_{X\in a_i} P_\sigma^{a_\emptyset}$ and the second and third maps are the sums of the natural surjections.

Proof. For $X \in a_i$, let N_X be $\ker(P_\sigma^{a_X} \to P_\sigma^{a_\emptyset})$ and L be $\ker(P_\sigma^a \to P_\sigma^{a_\emptyset})$. Then there are surjections $L \to N_X$, and the natural map $L \to \oplus_{X\in a_i} N_X$ is surjective because the Jordan–Hölder factors of N_X are pairwise disjoint by extension graph considerations at embedding i. This map $L \to \oplus_{X\in a_i} N_X$ is then an isomorphism by length considerations. The result now follows from Lemma 4.22 setting $M = P_\sigma^a$ and $N = P_\sigma^{a_\emptyset}$. ☐

Let

$$\overline{P}_\sigma \stackrel{\text{def}}{=} P_\sigma / \big(\sum_a \mathrm{rad}^a P_\sigma + \sum_b \mathrm{rad}^b P_\sigma \big) \qquad (4.7)$$

where a varies over tuples with

$$a_i = \begin{cases} 2 & \text{if } i \in A(\sigma) \\ 3 & \text{if } i \notin A(\sigma) \end{cases}$$

for some $i \in \mathcal{J}$ and $a_j = 0$ for $j \neq i$ and b varies over tuples with $b_i = b_{i'} = 1$ for some $i \notin A(\sigma)$ and $i' \in \mathcal{J}$ distinct from i, and $b_j = 0$ for $j \neq i, i'$ (the set of b is empty if $\mathcal{J} = A(\sigma)$ or $\#\mathcal{J} = 1$). The following proposition gives another characterization of $\overline{P}_\sigma$.

Proposition 4.24. *Assume that σ is 8-deep. There is an exact sequence*

$$0 \to \overline{P}_\sigma \to \oplus_c P_\sigma^c \to (\oplus_c \sigma)/\Delta(\sigma) \to 0 \qquad (4.8)$$

where c runs over tuples $(c_j)_{j\in\mathcal{J}}$ with $c_i = \widehat{3}$ for some $i \notin A(\sigma)$ and $c_j = \widehat{1}$ for all $j \neq i$ and the tuple $(c_j)_{j\in\mathcal{J}}$ with $j = \widehat{2}$ for all $j \in A(\sigma)$ and $j = \widehat{1}$ for all $j \notin A(\sigma)$, $\Delta(\sigma) \subset \oplus_c \sigma$ denotes the diagonally embedded copy, and the maps are the natural projections.

Proof. For a tuple c as in (4.8), let N_c be $\ker(P^c_\sigma \twoheadrightarrow \sigma)$, and let L be $\ker(\overline{P}_\sigma \to \sigma)$. Then we have natural surjections $L \to N_c$. We claim that the natural map $L \to \oplus_c N_c$ is surjective. It suffices to show that it is surjective after taking cosocles. This surjectivity follows from the fact that $\operatorname{cosoc} L \to \operatorname{cosoc} N_c$ is surjective for each c and that the sets $\mathrm{JH}(\operatorname{cosoc} N_c)$ are pairwise disjoint by alcove considerations (as usual, we use [LLHLM20, Lemma 4.2.2]). Finally, the map $L \to \oplus_c N_c$ is an isomorphism by length considerations. The result now follows from Lemma 4.22 setting $M = \overline{P}_\sigma$ and $N = \sigma$. □

4.5. **Lattices in direct sums of Deligne–Lusztig representations.** Let σ be an 8-deep Serre weight. We now study certain quotients of P_σ arising from reductions of lattices in direct sums of Deligne–Lusztig representations. Recall that $\widetilde{P}_\sigma \to \sigma$ is a $\mathcal{O}[\mathrm{G}]$-projective cover of σ. Since the multiplicity of an irreducible $E[\mathrm{G}]$-module R in $\widetilde{P}_\sigma \otimes_{\mathcal{O}} E$ is the multiplicity of σ in $\overline{R}$, we have that

$$\widetilde{P}_\sigma \otimes_{\mathcal{O}} E \cong \oplus_{\sigma \in \mathrm{JH}(\overline{R})} R$$

where R runs over Deligne–Lusztig representations.

Let T be a set of Deligne–Lusztig representations whose reduction contains σ as a Jordan–Hölder factor. Let $\widetilde{P}^T_\sigma$ be the quotient of $\widetilde{P}_\sigma$ which is isomorphic to the image of the composition

$$\widetilde{P}_\sigma \hookrightarrow \widetilde{P}_\sigma \otimes_{\mathcal{O}} E \cong \oplus_{\sigma \in \mathrm{JH}(\overline{R})} R \twoheadrightarrow \oplus_{R \in T} R.$$

The isomorphism class of $\widetilde{P}^T_\sigma$ does not depend on the choice of isomorphism above. Let $P^T_\sigma \overset{\text{def}}{=} \widetilde{P}^T_\sigma \otimes_{\mathcal{O}} \mathbb{F}$. Then we have natural surjections

$$P_\sigma \twoheadrightarrow P^T_\sigma \twoheadrightarrow P^S_\sigma \tag{4.9}$$

for any subset $S \subset T$. If T is the set of all Deligne–Lusztig representations whose reduction contains σ as a Jordan–Hölder factor, then $\widetilde{P}^T_\sigma \cong \widetilde{P}_\sigma$.

We now study P^T_σ for specific subsets T. We begin by labelling certain Deligne–Lusztig representations. Suppose that $A(\sigma) \neq \mathcal{J}$, and fix $i \in \mathcal{J} \setminus A(\sigma)$. Identify S_3 with the i-th component of $\underline{W}$. Let (s, μ) be a lowest alcove presentation for a Deligne–Lusztig representation such that $\sigma \in \mathrm{JH}(\overline{R}_s(\mu + \eta))$ is an outer weight. For $w \in S_3 \subset \underline{W}$, let R_w be the Deligne–Lusztig representations $R_{sw}(\mu + \eta)$.

Lemma 4.25. *Fix $i' \neq i$ in $\mathcal{J}$. Define $a = (a_j)_{j \in \mathcal{J}}$ by $a_j = 0$ for all $j \neq i'$ and $a_{i'} = 1$. Then the images of the maps*

$$\operatorname{rad}^a P_\sigma \to P^{\{R_{\alpha\beta}, R_{w_0}\}}_\sigma \quad \text{and} \quad \operatorname{rad}^a P_\sigma \to P^{\{R_{w_0}, R_{\beta\alpha}\}}_\sigma \tag{4.10}$$

induced by (4.9) *do not contain σ as a Jordan–Hölder factor.*

Proof. Let S be $\{R_{\alpha\beta}, R_{w_0}\}$ or $\{R_{w_0}, R_{\beta\alpha}\}$. Suppose that $\sigma = F(\lambda)$. First suppose that $i' \in A(\sigma)$. Using Proposition 4.6 we see that for any $\nu \in X_1(\underline{T})$, P^S_σ contains no Jordan–Hölder factors of $L(\lambda_{i',X}) \otimes \otimes_{j \neq i'} L(\nu_j)|_{\mathrm{G}}$ if $X = B$ and at most one Jordan–Hölder factor if $X = E$ or F. Then using the Weyl filtration in Proposition 4.4 and Proposition 4.15 with $(X^1_{i'}, X^0_{i'}) = (B, A), (E, A), (F, A)$, and the dual Weyl filtration in Proposition 4.4 and the dual of Proposition 4.17, we see that the kernel of (4.10) contains all Jordan–Hölder factors of the form $L(\lambda_{i',A}) \otimes \otimes_{j \neq i'} L(\nu_j)|_{\mathrm{G}}$.

Next suppose that $i' \notin A(\sigma)$. Again from Proposition 4.6 we see that for any $\nu \in X_1(\underline{T})$, P^S_σ contains at most two Jordan–Hölder factors of $L(\lambda_{i',X}) \otimes \otimes_{j \neq i'} L(\nu_j)|_{\mathrm{G}}$ if $X = C$ or D. Then using the Weyl filtration in Proposition 4.4 and Proposition 4.15 with $(X^1_{i'}, X^0_{i'}) = (C, B), (D, B)$, we see that the kernel of (4.10) contains all Jordan–Hölder factors of the form $L(\lambda_{i',B}) \otimes \otimes_{j \neq i'} L(\nu_j)|_{\mathrm{G}}$. □

Lemma 4.26. *Define $a = (a_j)_{j\in\mathcal{J}}$ by $a_j = 0$ for all $j \neq i$ and $a_i = 4$. The images of the maps*

$$\mathrm{rad}^a P_\sigma \to P_\sigma^{\{R_{\alpha\beta},R_{w_0}\}} \quad \text{and} \quad \mathrm{rad}^a P_\sigma \to P_\sigma^{\{R_{w_0},R_{\beta\alpha}\}} \tag{4.11}$$

induced by (4.9) *do not contain σ as a Jordan–Hölder factor.*

Proof. The proof is similar to that of Lemma 4.25. For any $\nu \in X_1(\underline{T})$ and S equal to $\{R_{\alpha\beta}, R_{w_0}\}$ or $\{R_{w_0}, R_{\beta\alpha}\}$, P_σ^S contains at most two Jordan–Hölder factors of $L(\lambda_{i,X}) \otimes \otimes_{j\neq i} L(\nu_j)|_{\mathrm{G}}$ if $X = E$ or F. Then using the Weyl filtration in Proposition 4.4 and Proposition 4.15 with $(X_i^1, X_i^0) = (E, A), (F, A), (A, B)$, we see that the kernel of (4.11) contains all Jordan–Hölder factors of the form $L(\lambda_{i,B}) \otimes \otimes_{j\neq i} L(\nu_j)|_{\mathrm{G}}$. □

Lemma 4.27. *Define $a = (a_j)_{j\in\mathcal{J}}$ by $a_j = 0$ for all $j \neq i$ and $a_i = 2$. Let P be a quotient of P_σ and let M be the kernel of the map $\mathrm{rad}^a P_\sigma \to P$. Recall that if $\sigma = F(\lambda)$, then $\mathrm{gr}^a P_\sigma$ is isomorphic to*

$$\mathrm{gr}^a P_\sigma \cong \mathrm{gr}^2 Q_1(\lambda_i) \otimes \otimes_{j\neq i} \mathrm{gr}^0 Q_1(\lambda_j)|_{\mathrm{G}} \tag{4.12}$$

$$\cong \sigma^{\oplus 2} \oplus (L(\lambda_{i,E}) \otimes \otimes_{j\neq i} L(\lambda_j))|_{\mathrm{G}} \oplus (L(\lambda_{i,F}) \otimes \otimes_{j\neq i} L(\lambda_j))|_{\mathrm{G}} \tag{4.13}$$

so that there is a projection map

$$\mathrm{gr}^a P_\sigma \to (L(\lambda_{i,E}) \otimes \otimes_{j\neq i} L(\lambda_j))|_{\mathrm{G}} \oplus (L(\lambda_{i,F}) \otimes \otimes_{j\neq i} L(\lambda_j))|_{\mathrm{G}}. \tag{4.14}$$

If κ is a Jordan–Hölder factor of the image of M under (4.14)*, then κ is not a Jordan–Hölder factor of P.*

Proof. Suppose that κ is a Jordan–Hölder factor of the image of M under (4.14). Using Proposition 4.6, there is a $\overline{\rho}$ such that

$$\mathrm{JH}(\mathrm{gr}^a P_\sigma) \cap W^?(\overline{\rho}) = \{\kappa\}.$$

As the LHS of (4.14) is multiplicity free, we conclude that the cokernel of $M \to \mathrm{gr}^a P_\sigma$ does not contain any elements of $W^?(\overline{\rho})$. By Proposition 4.11, $\kappa \notin \mathrm{JH}(\mathrm{rad}^a P_\sigma/M)$. Since $\kappa \notin \mathrm{JH}(P_\sigma/\mathrm{rad}^a P_\sigma)$ (again using Proposition 4.6), κ is not a Jordan–Hölder factor of the image of the surjective map $P_\sigma \to P$ and thus not in $\mathrm{JH}(P)$. □

For $n \geq 0$ define

$$\mathrm{Fil}^{n_i} P_\sigma = \mathrm{rad}^{n_i} P_\sigma + \sum_{j\neq i} \mathrm{rad}^{1_j} P_\sigma.$$

Note that $\mathrm{Fil}^{n_i} P_\sigma / \mathrm{Fil}^{(n+1)_i} P_\sigma$ is $\mathrm{gr}^{n_i} P_\sigma$.

Lemma 4.28. *The kernel of the map $\mathrm{Fil}^{2_i} P_\sigma \to P_\sigma^{\{R_{\alpha\beta},R_{w_0}\}} \oplus P_\sigma^{\{R_{w_0},R_{\beta\alpha}\}} \oplus P_\sigma^{R_{\mathrm{id}}}$ induced by* (4.9) *is contained in $\mathrm{Fil}^{3_i} P_\sigma$.*

Proof. Suppose that κ is a simple submodule of the projection of the kernel of

$$\mathrm{Fil}^{2_i} P_\sigma \to P_\sigma^{\{R_{\alpha\beta},R_{w_0}\}} \oplus P_\sigma^{\{R_{w_0},R_{\beta\alpha}\}} \oplus P_\sigma^{R_{\mathrm{id}}}$$

to $\mathrm{Fil}^{2_i} P_\sigma / \mathrm{Fil}^{3_i} P_\sigma \cong \mathrm{gr}^{2_i} P_\sigma$.

Let $\sigma = F(\lambda)$. First suppose that κ is a Jordan–Hölder factor of

$$(L(\lambda_{i,E}) \otimes \otimes_{j\neq i} L(\lambda_j))|_{\mathrm{G}} \oplus (L(\lambda_{i,F}) \otimes \otimes_{j\neq i} L(\lambda_j))|_{\mathrm{G}}. \tag{4.15}$$

Then Lemma 4.27 with $P = P_\sigma^{\{R_{\alpha\beta},R_{w_0}\}}, P_\sigma^{\{R_{w_0},R_{\beta\alpha}\}}$, and $P_\sigma^{R_{\mathrm{id}}}$ implies that κ is not a Jordan–Hölder factor of $P_\sigma^{\{R_{\alpha\beta},R_{w_0}\}} \oplus P_\sigma^{\{R_{w_0},R_{\beta\alpha}\}} \oplus P_\sigma^{R_{\mathrm{id}}}$. However, every Jordan–Hölder factor of (4.15) is in

$$\mathrm{JH}(P_\sigma^{\{R_{\alpha\beta},R_{w_0}\}} \oplus P_\sigma^{\{R_{w_0},R_{\beta\alpha}\}} \oplus P_\sigma^{R_{\mathrm{id}}}) = \mathrm{JH}(\overline{R}_{\mathrm{id}} \oplus \overline{R}_{\alpha\beta} \oplus \overline{R}_{w_0} \oplus \overline{R}_{\beta\alpha}),$$

which is a contradiction.

We conclude that κ must isomorphic to σ. Let P_σ^b be the quotient of $P_\sigma/\operatorname{Fil}^{3_i} P_\sigma$ by (4.15). Then there is a natural injection $\kappa \hookrightarrow P_\sigma^b$, and we identify κ with its image. Recall from the Weyl filtration of Proposition 4.4 that P_σ^b contains the direct sum of two submodules M_C and M_D which are the restrictions to G of extensions of $L(\lambda_{i,C}) \otimes \otimes_{j\neq i} L(\lambda_j)$ and $L(\lambda_{i,D}) \otimes \otimes_{j\neq i} L(\lambda_j)$, respectively, by $L(\lambda)$. Then κ is in the direct sum $M_C \oplus M_D$. Suppose without loss of generality that the projection of κ to M_C is nonzero. One of $P_\sigma^{\{R_{\alpha\beta},R_{w_0}\}}$ and $P_\sigma^{\{R_{w_0},R_{\beta\alpha}\}}$, say P_σ^S, does not contain all of the Jordan–Hölder factors of $L(\lambda_{i,D}) \otimes \otimes_{j\neq i} L(\lambda_j)|_{\mathrm{G}}$. We conclude from Proposition 4.15 with $(X_i^1, X_i^0) = (D, B)$ that the image of the map $\operatorname{Fil}^{2_i} P_\sigma \to P_\sigma^S$ does not contain σ as a Jordan–Hölder factor. Combined with Lemmas 4.25 and 4.26, we see that the image of the surjection $P_\sigma \twoheadrightarrow P_\sigma^S$ contains σ as a Jordan–Hölder factor with multiplicity at most 1. This contradicts the fact that P_σ^S contains σ as a Jordan–Hölder factor with multiplicity 2. $\square$

Proposition 4.29. *The kernel of the map*

$$P_\sigma \to P_\sigma^{\{R_{\alpha\beta},R_{w_0}\}} \oplus P_\sigma^{\{R_{w_0},R_{\beta\alpha}\}} \oplus P_\sigma^{R_{\mathrm{id}}} \tag{4.16}$$

induced by (4.9) *is contained in* $\operatorname{Fil}^{3_i} P_\sigma$*.*

Proof. We claim that the map

$$P_\sigma/\operatorname{Fil}^{3_i} P_\sigma \to (P_\sigma^{\{R_{\alpha\beta},R_{w_0}\}} \oplus P_\sigma^{\{R_{w_0},R_{\beta\alpha}\}} \oplus P_\sigma^{R_{\mathrm{id}}})/\operatorname{im}(\operatorname{Fil}^{3_i} P_\sigma) \tag{4.17}$$

is injective, where $\operatorname{im}(\operatorname{Fil}^{3_i} P_\sigma)$ denotes the image of $\operatorname{Fil}^{3_i} P_\sigma$ under (4.16). Note that $\operatorname{Fil}^{2_i} P_\sigma/\operatorname{Fil}^{3_i} P_\sigma \subset P_\sigma/\operatorname{Fil}^{3_i} P_\sigma$ is the socle by [LLHLM20, Lemma 4.2.2]. It thus suffices to show that the restriction of (4.17) to $\operatorname{Fil}^{2_i} P_\sigma/\operatorname{Fil}^{3_i} P_\sigma$ is injective. Suppose that $a + \operatorname{Fil}^{3_i} P_\sigma$ is in the kernel of (4.17) for $a \in \operatorname{Fil}^{2_i} P_\sigma$. Then $a-b$ is in the kernel of (4.16) for some $b \in \operatorname{Fil}^{3_i} P_\sigma$. By Lemma 4.28, $a \in \operatorname{Fil}^{3_i} P_\sigma$, and the claim follows.

Suppose now that a is in the kernel of (4.16). Then $a \in \operatorname{Fil}^{3_i} P_\sigma$ by the injectivity of (4.17). $\square$

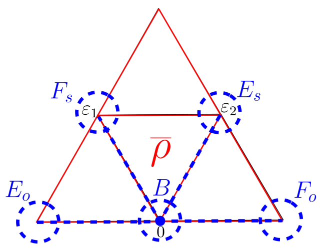


FIGURE 6. Using the extension graph with origin in $\lambda_{\underline{A}} + \eta$ (where $\lambda \in X_1(\underline{T})$ is such that $\sigma = F(\lambda)$) we describe the Jordan–Hölder factors of P_σ^a at $j \in \mathcal{J}$, for different configurations of $a_j \subset \{B, E_o, E_s, F_o, F_s\}$: given $b \in \{B, E_o, E_s, F_o, F_s\}$ the dashed circle appears if and only if $b \in a_j$.

5. Patched modules

5.1. **Patching axioms.** Recall from §1.4 that $\mathcal{O}_p = \prod_{v\in S_p} \mathcal{O}_v$ where S_p is a finite set and for each $v \in S_p$, $\mathcal{O}_v$ is the ring of integers in a finite unramified extension F_v^+ of $\mathbb{Q}_p$.

Let $\overline{\rho}$ be an $\mathbb{F}$-valued L-homomorphism and let $(\overline{\rho}_v)_{v\in S_p}$ be the collection of continuous Galois representations corresponding to it (see §2.1.4). Let $R_\infty \stackrel{\text{def}}{=} R_{\overline{\rho}}\widehat{\otimes}_{\mathcal{O}} R^p$ where

$$R_{\overline{\rho}} \stackrel{\text{def}}{=} \widehat{\bigotimes}_{v\in S_p,\mathcal{O}} R^{\square}_{\overline{\rho}_v}$$

and R^p is a (nonzero) complete local Noetherian equidimensional flat $\mathcal{O}$-algebra with residue field $\mathbb{F}$ such that each irreducible component of Spec R^p and of Spec $\overline{R}^p$ is geometrically irreducible. A finite set T of tame inertial L-parameters gives rise to a collection $(T_v)_{v\in S_p}$ of finite sets of tame inertial types and we let $R_\infty(T) \stackrel{\text{def}}{=} R_\infty \otimes_{R^{\square}_{\overline{\rho}}} R^{\eta,T}_{\overline{\rho}}$ where

$$R^{\eta,T}_{\overline{\rho}} \stackrel{\text{def}}{=} \widehat{\bigotimes}_{v\in S_p,\mathcal{O}} R^{\eta,T_v}_{\overline{\rho}_v}.$$

Write X_∞, $X_\infty(T)$, and $\overline{X}_\infty(T)$ for Spec R_∞, Spec $R_\infty(T)$, and Spec $\overline{R}_\infty(T)$ respectively. Finally, let $\mathrm{Mod}(X_\infty)$ be the category of coherent sheaves over X_∞, and $\mathrm{Rep}_{\mathcal{O}}(\mathrm{GL}_3(\mathcal{O}_p))$ the category of topological $\mathcal{O}[\mathrm{GL}_3(\mathcal{O}_p)]$-modules which are finitely generated over $\mathcal{O}$.

Definition 5.1. A *weak patching functor* for an L-homomorphism $\overline{\rho} : G_{\mathbb{Q}_p} \to {}^L\underline{G}(\mathbb{F})$ is a nonzero covariant exact functor $M_\infty : \mathrm{Rep}_{\mathcal{O}}(\mathrm{GL}_3(\mathcal{O}_p)) \to \mathrm{Mod}(X_\infty)$ satisfying the following axioms: if τ is an inertial L-parameter and $\sigma^\circ(\tau)$ is an $\mathcal{O}$-lattice in $\sigma(\tau)$ then

(1) $M_\infty(\sigma^\circ(\tau))$ is either zero or a maximal Cohen–Macaulay sheaf on $X_\infty(\tau)$; and
(2) for all $\sigma \in \mathrm{JH}(\overline{\sigma}^\circ(\tau))$, $M_\infty(\sigma)$ is a maximal Cohen–Macaulay sheaf on $\overline{X}_\infty(\tau)$ (or is 0).

(We identify the tame inertial L-parameter with the finite set $T = \{\tau\}$ in the above definition.) A weak patching functor M_∞ is *minimal* if R^p is formally smooth over $\mathcal{O}$ and whenever τ is an inertial L-parameter, $M_\infty(\sigma^\circ(\tau))[p^{-1}]$, which is locally free over the regular scheme Spec $R_\infty(\tau)[p^{-1}]$, has rank at most one on each connected component.

Remark 5.2. In §6.3 we will further assume that M_∞ has the form $\mathrm{Hom}_{\mathrm{GL}_3(\mathcal{O}_p)}(-, M_\infty^\vee)^\vee$ for some pseudocompact $\mathcal{O}[\![\mathrm{GL}_3(\mathcal{O}_p)]\!]$-module M_∞. Note that the M_∞ that arise from Taylor–Wiles patching always have this property.

Given a finite set T of tame inertial L-parameters we have a surjective homomorphism $R^{\square}_{\overline{\rho}} \twoheadrightarrow R^{\eta,T}_{\overline{\rho}}$. In particular, for any $T' \subseteq T$ we have a surjective homomorphism

$$R_\infty \twoheadrightarrow R_\infty(T) \twoheadrightarrow R_\infty(T')$$

whose kernel (in either R_∞ or $R_\infty(T)$) will be denoted as $I_\infty(T')$. (This abuse of notation will not create confusion in what follows, since it will always be clear from the context which ring the ideal $I_\infty(T')$ is in.)

5.2. **Minimal number of generators.** As in §4.2, P_σ denotes a $\mathbb{F}$[G]-projective cover of a Serre weight σ, and we have fixed $\lambda \in X_1(\underline{T})$ such that $\sigma \cong F(\lambda)$. We will use notation from §4.2 in what follows. Recall that $A(\sigma) \stackrel{\text{def}}{=} \{j \in \mathcal{J} \mid \lambda_j \in A\}$. The following theorem is the main result of the section.

Theorem 5.3. *Let $\overline{\rho}$ be a 11-generic tame L-homomorphism and let $\sigma \in W^?(\overline{\rho})$.*
Then $M_\infty(P_\sigma)$ is minimally generated by $3^{\#A(\sigma)}$ elements.

One can reduce this theorem to Lemma 5.4 below. Recall from (4.7) that

$$\overline{P}_\sigma \stackrel{\text{def}}{=} P_\sigma/\big(\sum_a \mathrm{rad}^a P_\sigma + \sum_b \mathrm{rad}^b P_\sigma\big)$$

where a varies over tuples with

$$a_i = \begin{cases} 2 & \text{if } i \in A(\sigma) \\ 3 & \text{if } i \notin A(\sigma) \end{cases}$$

for some $i \in \mathcal{J}$ and $a_j = 0$ for $j \neq i$ and b varies over tuples with $b_i = b_{i'} = 1$ for some $i \notin A(\sigma)$ and $i' \in \mathcal{J}$ distinct from i, and $b_j = 0$ for $j \neq i, i'$ (the set of b is empty if $\mathcal{J} = A(\sigma)$ or $\#\mathcal{J} = 1$).

Lemma 5.4. *Let $\overline{\rho}$ be a 11-generic tame L-homomorphism and let $\sigma \in W^?(\overline{\rho})$.*

(1) Let $a = (a_j)_{j \in \mathcal{J}}$ be a tuple such that $a_j = \widehat{2}$ if $j \in A(\sigma)$ and $a_j = \widehat{1}$ or $\widehat{2}$ if $j \notin A(\sigma)$. Then $M_\infty(P_\sigma^a)$ is minimally generated by $3^{\# A(\sigma)}$ elements.

(2) $M_\infty(\overline{P}_\sigma)$ is minimally generated by $3^{\# A(\sigma)}$ elements.

The following is a consequence of Lemma 5.4.

Corollary 5.5. *Let $\overline{\rho}$ be a 11-generic tame L-homomorphism and let $\sigma \in W^?(\overline{\rho})$.*

Let a be a tuple as in Lemma 5.4(1) and b be a tuple with $b_j > 0$ for some $j \notin A(\sigma)$. Then the following hold.

(1) The image of the natural map

$$M_\infty(\mathrm{rad}^b P_\sigma) \to M_\infty(P_\sigma^a)$$

is contained in $\mathfrak{m} M_\infty(P_\sigma^a)$.

(2) The image of the natural map

$$M_\infty(\mathrm{rad}^b P_\sigma) \to M_\infty(\overline{P}_\sigma)$$

is contained in $\mathfrak{m} M_\infty(\overline{P}_\sigma)$.

Proof. We use a similar argument as in the proof of [Le19, Lemma 4.5]. Let $a_{\min}$ be the tuple $(a_{\min,j})_{j \in \mathcal{J}}$ with $a_{\min,j} = \widehat{2}$ if $j \in A(\sigma)$ and $a_{\min,j} = \widehat{1}$ if $j \notin A(\sigma)$. The main observation is that the images of $\mathrm{rad}^b P_\sigma$ in P_σ^a and $\overline{P}_\sigma$ are contained in the kernels of the natural surjections to $P_\sigma^{a_{\min}}$, respectively. Since these surjections are isomorphisms after applying $M_\infty(-) \otimes_{R_\infty} R_\infty/\mathfrak{m}$ by dimension considerations using Lemma 5.4, the result follows. $\square$

Before proving Lemma 5.4, we first use it to prove Theorem 5.3. We first introduce some notation.

Definition 5.6. For a tuple $a = (a_j)_{j \in \mathcal{J}}$ with $0 \leq a_j \leq 7$, let

$$A(\mathrm{gr}^a P_\sigma) = \{j \in \mathcal{J} \mid (j \notin A(\sigma) \text{ and } 2 \nmid a_j) \text{ or } (j \in A(\sigma), a_j \geq 2, \text{ and } 2 \mid a_j)\}.$$

The definition of $A(\mathrm{gr}^a P_\sigma)$ means that for $j \in \mathcal{J}$, $j \in A(\mathrm{gr}^a P_\sigma)$ if and only if $a_j > 0$ and for some (equivalently for all) $\kappa \in \mathrm{JH}(\mathrm{gr}^a P_\sigma)$, $j \in A(\kappa)$.

Lemma 5.7. *In the setup of Theorem 5.3 suppose that $c = (c_j)_{j \in \mathcal{J}}$ with $0 \leq c_i \leq 7$ and*

$$c_i \geq \begin{cases} 2 & \text{if } i \in A(\sigma) \\ 1 & \text{if } i \notin A(\sigma) \end{cases} \tag{5.1}$$

for some $i \in \mathcal{J}$. Then the injection

$$M_\infty(\mathrm{gr}^c P_\sigma) \hookrightarrow M_\infty\Big(P_\sigma / \big(\mathrm{rad}^{|c|+1} P_\sigma + \sum_{\substack{|c|=|d| \\ \# A(\mathrm{gr}^d P_\sigma) > \# A(\mathrm{gr}^c P_\sigma)}} \mathrm{rad}^d P_\sigma\big)\Big). \tag{5.2}$$

factors through $\mathfrak{m} M_\infty\Big(P_\sigma / \big(\mathrm{rad}^{|c|+1} P_\sigma + \sum_{\substack{|c|=|d| \\ \# A(\mathrm{gr}^d P_\sigma) > \# A(\mathrm{gr}^c P_\sigma)}} \mathrm{rad}^d P_\sigma\big)\Big)$.

Proof. First suppose that $A(\mathrm{gr}^c P_\sigma) = \emptyset$. Then we claim that $c_\ell \geq 2$ for some $\ell \in \mathcal{J}$. Indeed, if $c_j < 2$ for all $j \in \mathcal{J}$, then (5.1) implies that for some $j \notin A(\sigma)$, $c_j = 1$. Then $j \in A(\mathrm{gr}^c P_\sigma)$ which is a contradiction.

Suppose that $c_\ell \geq 2$. Let I be the image of $M_\infty(\mathrm{rad}^{c-2_\ell} P_\sigma)$ in the codomain of (5.2). Then I contains (the image under (5.2) of) $M_\infty(\mathrm{gr}^c P_\sigma)$. We will show that $\mathfrak{m}I$ contains $M_\infty(\mathrm{gr}^c P_\sigma)$.

Let $N \subset \mathrm{rad}^{c-2_\ell} P_\sigma$ be the minimal submodule for which the cokernel of the map $N \to \mathrm{gr}^{c-2_\ell} P_\sigma$ contains no Jordan–Hölder factors in $W(\overline{\rho})$. Let M' be the image of $M_\infty(N)$ in the codomain of (5.2). Then $M' = I$ by Proposition 4.11. (Note that σ is 9-deep by [LLHLM20, Lemma 4.2.13].) We will show that $M_\infty(\mathrm{gr}^c P_\sigma) \subset \mathfrak{m}M'$.

Let $P \overset{\mathrm{def}}{=} \oplus_{\kappa \in \mathrm{JH}(\mathrm{gr}^{c-2_\ell} P_\sigma) \cap W(\overline{\rho})} P_\kappa^{\oplus [\mathrm{gr}^{c-2_\ell} P_\sigma : \kappa]}$. By projectivity, the natural map $P \to \mathrm{gr}^{c-2_\ell} P_\sigma$ lifts to a map $P \to N$ which we now fix. Since the cokernel of the composite $P \to N \hookrightarrow \mathrm{rad}^{c-2_\ell} P_\sigma \twoheadrightarrow \mathrm{gr}^{c-2_\ell} P_\sigma$ contains no Jordan–Hölder factors in $W(\overline{\rho})$, by Proposition 4.11 and minimality of N we conclude that the map $P \to N$ is surjective. Moreover, the preimage of $\mathrm{gr}^c P_\sigma$ in P under the map

$$P \to N \to P_\sigma / \Big(\mathrm{rad}^{|c|+1} P_\sigma + \sum_{\substack{|d|=|c| \\ \#A(\mathrm{gr}^d P_\sigma) > \#A(\mathrm{gr}^c P_\sigma)}} \mathrm{rad}^d P_\sigma\Big) \tag{5.3}$$

is contained in

$$\mathrm{rad}^1 P = \oplus_{\kappa \in \mathrm{JH}(\mathrm{gr}^{c-2_\ell} P_\sigma) \cap W(\overline{\rho})} \mathrm{rad}^1 P_\kappa^{\oplus [\mathrm{gr}^{c-2_\ell} P_\sigma : \kappa]}.$$

Even more, by alcove considerations, the preimage of $\mathrm{gr}^c P_\sigma$ in P is contained in

$$\oplus_{\kappa \in \mathrm{JH}(\mathrm{gr}^{c-2_\ell} P_\sigma) \cap W(\overline{\rho})} \sum_{j \in \mathcal{J}} \mathrm{rad}^{2_j} P_\kappa^{\oplus [\mathrm{gr}^{c-2_\ell} P_\sigma : \kappa]}. \tag{5.4}$$

It suffices to show that the image of $M_\infty(5.4)$ in M' is contained in $\mathfrak{m}M'$.

We claim that the map (5.3) factors through

$$\oplus_{\kappa \in \mathrm{JH}(\mathrm{gr}^{c-2_\ell} P_\sigma) \cap W(\overline{\rho})} \overline{P}_\kappa^{\oplus [\mathrm{gr}^{c-2_\ell} P_\sigma : \kappa]} \tag{5.5}$$

in the notation of (4.7). Let $\kappa \in \mathrm{JH}(\mathrm{gr}^{c-2_\ell} P_\sigma) \cap W(\overline{\rho})$. Suppose that $a = (a_j)_{j \in \mathcal{J}}$ is a tuple as in the definition of (4.7), i.e., $a = 2_j$ for $j \in A(\kappa)$ or $a = 3_j$ for $j \notin A(\kappa)$. If $a = 3_j$ for $j \notin A(\kappa)$, then the image of $\mathrm{rad}^a P_\kappa$ in P_σ is contained in $\mathrm{rad}^{c-2_\ell+3_j} P_\sigma \subset \mathrm{rad}^{|c|+1} P_\sigma$ by Proposition 4.10 (note that κ is 9-deep by [LLHLM20, Lemma 4.2.13]). If $a = 2_j$ for $j \in A(\kappa)$, then the image of $\mathrm{rad}^a P_\kappa$ in P_σ is contained in $\mathrm{rad}^{c-2_\ell+2_j} P_\sigma$. Since $|c - 2_\ell + 2_j| = |c|$ and $j \in A(\mathrm{rad}^{c-2_\ell+2_j} P_\sigma)$, we conclude that $\mathrm{rad}^a P_\kappa$ maps to 0 in the codomain of (5.3).

Now suppose that $b = (b_j)_{j \in \mathcal{J}}$ is a tuple as in the definition of (4.7). In particular, $b_i = 1$ for some $i \notin A(\kappa)$. Then $A(\mathrm{gr}^{c-2_\ell+b} P_\sigma) \neq \emptyset$ (in fact $i \in A(\mathrm{gr}^{c-2_\ell+b} P_\sigma)$) so that the image of $\mathrm{rad}^b P_\kappa$ in P_σ is contained in

$$\sum_{\substack{|d|=|c| \\ \#A(\mathrm{gr}^d P_\sigma) > \#A(\mathrm{gr}^c P_\sigma)}} \mathrm{rad}^d P_\sigma.$$

This establishes the claim.

Now for any generic Serre weight κ and $i \in A(\kappa)$, the image of $\mathrm{rad}^{2_i} P_\kappa$ in $\overline{P}_\kappa$ is 0. Thus the image of (5.4) in (5.5) is equal to the the image of (5.4) where j instead ranges over $j \notin A(\kappa)$. Thus the image of $M_\infty(5.4)$ in $M_\infty(5.5)$ is contained in $\mathfrak{m}M_\infty(5.5)$ by Corollary 5.5(2). We conclude that the image of $M_\infty(5.4)$ in M' is contained in $\mathfrak{m}M'$. This finishes the proof of the lemma when $A(\mathrm{gr}^c P_\sigma) = \emptyset$.

For $A(\mathrm{gr}^c P_\sigma) \neq \emptyset$, we employ a similar argument using Corollary 5.5(1) instead of Corollary 5.5(2). If $A(\mathrm{gr}^c P_\sigma) \neq \emptyset$, let $\ell \in A(\mathrm{gr}^c P_\sigma)$. Then we replace $c - 2_\ell$ in the above argument with $c - 1_\ell$. We otherwise define I, N, M', and $P \to N$ as before. In this case, the map (5.3) factors as

$$P \to P/\mathrm{rad}^2 P \to \mathrm{rad}^{|c|-1} P_\sigma / \big(\mathrm{rad}^{|c|+1} P_\sigma + \sum_{\substack{|d|=|c| \\ \#A(\mathrm{gr}^d P_\sigma) > \#A(\mathrm{gr}^c P_\sigma)}} \mathrm{rad}^d P_\sigma\big) \tag{5.6}$$

since the codomain of (5.6) has a length two semisimple filtration. By alcove considerations, the preimage of $\mathrm{gr}^c P_\sigma$ in $P/\mathrm{rad}^2 P$ under the map in (5.6) is contained in

$$\oplus_{\kappa \in \mathrm{JH}(\mathrm{gr}^{c-1_\ell} P_\sigma) \cap W(\overline{\rho})} \mathrm{gr}^{1_\ell} P_\kappa^{\oplus [\mathrm{gr}^{c-1_\ell} P_\sigma : \kappa]} \tag{5.7}$$

(recall that $\ell \in A(c)$ was fixed above). As $\ell \notin A(\kappa)$, $M_\infty(5.7)$ is contained in $\mathfrak{m} M_\infty(P/\mathrm{rad}^2 P)$ by Corollary 5.5(1), noting that $P_\kappa/\mathrm{rad}^2 P_\kappa$ is a quotient of P_κ^a with $a = (\widehat{2})_{j \in \mathcal{J}}$. □

Proof of Theorem 5.3. First, we claim that for a tuple $c = (c_j)_{j \in \mathcal{J}}$ with $0 \leq c_j \leq 7$ and

$$c_i \geq \begin{cases} 2 & \text{if } i \in A(\sigma) \\ 1 & \text{if } i \notin A(\sigma) \end{cases}$$

for some $i \in \mathcal{J}$,

$$M_\infty(\mathrm{rad}^c P_\sigma) \subset \mathfrak{m} M_\infty(P_\sigma).$$

Indeed, we can partially order ($\succ$) such tuples c lexicographically using $|\bullet|$ and $\#A(\mathrm{gr}^\bullet P_\sigma)$ and proceed by induction. If $c_j = 7$ for all $j \in \mathcal{J}$, then the result follows from Lemma 5.7 (as the right hand side of (5.2) in this case is simply $M_\infty(P_\sigma)$). In general,

$$M_\infty(\mathrm{rad}^c P_\sigma) \subset \mathfrak{m} M_\infty(P_\sigma) + \sum_{c' \succ c} M_\infty(\mathrm{rad}^{c'} P_\sigma) \subset \mathfrak{m} M_\infty(P_\sigma)$$

where the first inclusion follows from Lemma 5.7 and the second inclusion follows by the inductive hypothesis.

Now let $a = (a_j)_{j \in \mathcal{J}}$ be the tuple with

$$a_j = \begin{cases} \widehat{2} & \text{if } j \in A(\sigma) \\ \widehat{1} & \text{if } j \notin A(\sigma). \end{cases}$$

By the previous paragraph, the map $P_\sigma \to P_\sigma^a$ induces an isomorphism $M_\infty(P_\sigma)/\mathfrak{m} \to M_\infty(P_\sigma^a)/\mathfrak{m}$. By Lemma 5.4(1), $M_\infty(P_\sigma^a)/\mathfrak{m}$ is minimally generated by $3^{\#A(\sigma)}$ elements. □

5.3. **Proof of Lemma 5.4.** We prove Lemma 5.4 using intersection computations in multitype deformation spaces. Throughout this section we are in the setup of Lemma 5.4. Thus, $\overline{\rho}$ is a 11-generic tame L-homomorphism and $\lambda \in X_1(\underline{T})$ is such that $\sigma \overset{\mathrm{def}}{=} F(\lambda) \in W^?(\overline{\rho})$. We fix a point $\overline{x}_{\overline{\rho}} \in M^T(\mathbb{F})$ corresponding to $\overline{\rho}$ (see §3.1.2) and we fix a weak minimal patching functor M_∞ for $\overline{\rho}$.

5.3.1. *Preliminaries.* We introduce some notation that will be used in the proofs appearing in this section. Let T be a finite set of tame inertial types satyisfying Hypothesis 3.2 and recall from §3.1.2 that $\overline{\rho}$ gives rise to a point $\overline{x}_{\overline{\rho}} \in M^T(\mathbb{F})$ which we now fix. By Definition 3.6 and (3.2) we have

$$M^{T,\nabla_\infty}_{\overline{x}_{\overline{\rho}}} \cdot \widetilde{w}^{*,T}(\overline{\rho}) \hookrightarrow M^T_{\overline{x}_{\overline{\rho}}} \cdot \widetilde{w}^{*,T}(\overline{\rho}) \to \Phi\text{-}\mathrm{Mod}^{\text{ét},3}_{K,\overline{\rho}_\infty}. \tag{5.8}$$

Recall that $\widetilde{S} = \widehat{\otimes}_{j \in \mathcal{J}, \mathcal{O}} \widetilde{S}^{(j)}$ is the formal power series ring on the natural coordinates of $M^T_{\overline{x}_{\overline{\rho}}} \cdot \widetilde{w}^{*,T}(\overline{\rho})$ and hence $M^{T,\nabla_\infty}_{\overline{x}_{\overline{\rho}}} \cdot \widetilde{w}^{*,T}(\overline{\rho})$ corresponds to a quotient $\widetilde{S}^{\nabla_\infty}$ of $\widetilde{S}$. Pulling back to $\mathrm{Spf}\, R^{\square}_{\overline{\rho}_\infty}$ along

(5.8) gives a formally smooth map $R_{\overline{\rho}}^{\leq\eta,T} \to \widetilde{S}^{\nabla_\infty,\square}$, where $\widetilde{S}^{\square}$ is a suitable power series ring over $\widetilde{S}$ and $\widetilde{S}^{\square} \twoheadrightarrow \widetilde{S}^{\nabla_\infty,\square}$ is the pullback of $\widetilde{S} \twoheadrightarrow \widetilde{S}^{\nabla_\infty}$.

If V is a finite $\mathcal{O}[\![K]\!]$-module such that the $R_{\overline{\rho}}^{\square}$-action on $M_\infty(V)$ factors through $R_{\overline{\rho}}^{\leq\eta,T}$, then we define

$$M'_\infty(V) \stackrel{\text{def}}{=} M_\infty(V)\widehat{\otimes}_{R_{\overline{\rho}}^{\leq\eta,T}}\widetilde{S}^{\nabla_\infty,\square}. \tag{5.9}$$

Let $R'_\infty(T) \stackrel{\text{def}}{=} R_\infty(T)\widehat{\otimes}_{R_{\overline{\rho}}^{\leq\eta,T}}\widetilde{S}^{\nabla_\infty,\square}$. If $T \subset T'$ are as in Proposition 3.8 and the R_∞-action on $M_\infty(V)$ factors through $R_\infty(T)$ then the $\widetilde{S}^{\nabla_\infty,\square}$-actions defined with respect to T and T' on $M'_\infty(V)$ are compatible by Remark 3.10.

As $R'_\infty(T)$ is a formally smooth $\widetilde{S}^{\nabla_\infty,\square}$-algebra, we can choose an isomorphism $\overline{R}'_\infty(T) \cong S^{\nabla_\infty,\square}\widehat{\otimes}_{\mathbb{F}}\mathcal{A}$ where $S^{\nabla_\infty,\square} \stackrel{\text{def}}{=} \widetilde{S}^{\nabla_\infty,\square} \otimes_{\mathcal{O}} \mathbb{F}$ and $\mathcal{A}$ denotes a formally smooth $\mathbb{F}$-algebra. If V is moreover an $\mathbb{F}$-vector space, then $M'_\infty(V)$ is a module for the formally smooth S-algebra $S_\infty^{\square} \stackrel{\text{def}}{=} S^{\square}\widehat{\otimes}_{\mathbb{F}}\mathcal{A}$ where $S^{\square} \stackrel{\text{def}}{=} \widetilde{S}^{\square} \otimes_{\mathcal{O}} \mathbb{F}$. When $M'_\infty(V)$ is obtained from an S-module by extension of scalars along $S \to S_\infty^{\square} \twoheadrightarrow \overline{R}'_\infty(T)$, calculations in S can be substituted for those in $S_\infty^{\square}$. We caution that we have suppressed the dependence on the set T in the notation $\widetilde{S}$, $\widetilde{S}^{\square}$, $S^{\nabla_\infty,\square}$, $S_\infty^{\square}$, etc. In practice, the set T will change from from proof to proof. When this notation is used, we will indicate which T we take.

Finally, note that the genericity assumption on $\overline{\rho}$ implies that any $\tau \in T$ is 9-generic (see for instance [LLHLM23, §5.5]) and any $\kappa \in W^?(\overline{\rho})$ is 9-deep ([LLHLM20, Lemma 4.2.13]), so that in particular [LLHLMb, Proposition 5.3.2] (which improves [LLHLM20, Theorem 4.1.9]) and Proposition 4.11 apply.

Recall that we have fixed $\sigma = F(\lambda) \in W^?(\overline{\rho})$. Let τ be a tame inertial type such that $\sigma \in \mathrm{JH}(\overline{\sigma}(\tau))$ and let $\ell \notin A(\sigma)$. For $w \in S_3$, let $\tau_{w,\ell}$ be the tame inertial L-parameter such that $\widetilde{w}(\overline{\rho}, \tau_{w,\ell})_j = \widetilde{w}(\overline{\rho}, \tau)_j$ for all $j \neq \ell$ and $\widetilde{w}(\overline{\rho}, \tau_{w,\ell})_\ell = ww_0\widetilde{w}(\overline{\rho}, \tau)_\ell$. Given a subset $\Sigma \subset S_3$, let $T_{\Sigma,\ell}$ be the set $\{\tau_{w,\ell} \mid w \in \Sigma\}$ and $\sigma(T_{\Sigma,\ell})$ the set $\{\sigma(\tau_{w,\ell}) \mid w \in \Sigma\}$ of Deligne–Lusztig representations. Note that we suppressed the type τ from the notation of $T_{\Sigma,\ell}$ and $\sigma(T_{\Sigma,\ell})$.

5.3.2. *Cyclic patched modules.* In this subsection we assume that $A(\sigma) \subsetneq \mathcal{J}$ and fix $i \notin A(\sigma)$. The main result of this subsection is the following proposition.

Proposition 5.8. *Let $a = (a_j)_{j\in\mathcal{J}}$ be the tuple with $a_i = \widehat{3}$ and $a_j = \widehat{1}$ for all $j \neq i$. Then $M_\infty(P_\sigma^a)$ is a cyclic R_∞-module.*

Let $\tau_{\min}$ be the minimal tame inertial L-parameter of σ with respect to $\overline{\rho}$ in the sense of [LLHLM20, Remark 3.5.10]. Up to changing lowest alcove presentations of $\overline{\rho}$, we assume that $t_{-\underline{1}}\widetilde{w}(\overline{\rho}, \tau_{\min})_i = t_{w_0\eta}$ or w_0. In the remainder of this subsection, given a subset $\Sigma \subset S_3$ and $w \in \Sigma$, define τ_w, T_Σ and $\sigma(T_\Sigma)$ as in §5.3.1 with respect to $\tau = \tau_{\min}$ and $\ell = i$, omitting the subscript i from the notation for readability. Let W be the Weyl module over $\mathbb{F}$ with cosocle isomorphic to σ. Then there are surjections $P_\sigma^{\sigma(\tau_w)} \twoheadrightarrow W$ for all $w \in S_3$ by [LLHLMb, Proposition 5.3.2]. These surjections are unique up to scalar. This induces a unique up to scalar surjection $P_\sigma^{\sigma(T_\Sigma)} \to W$ for any nonempty $\Sigma \subset S_3$.

We will repeatedly use the following result.

Proposition 5.9. *Let $\widetilde{P}_\sigma \twoheadrightarrow P_1$ and $\widetilde{P}_\sigma \twoheadrightarrow P_2$ be nonzero surjections. Then there is an $\mathcal{O}[\mathrm{G}]$-module C and an exact sequence*

$$\widetilde{P}_\sigma \to P_1 \oplus P_2 \to C \to 0$$

where the restriction of the second map to each summand is surjective.

Moreover, C satisfies the following property. If D is an $\mathcal{O}[\mathrm{G}]$-module such that σ is a Jordan–Hölder factor of D with multiplicity one and there are surjections $P_1 \twoheadrightarrow D$ and $P_2 \twoheadrightarrow D$, then there is a surjection $C \twoheadrightarrow D$.

Proof. We let C be the cokernel of the induced map $\widetilde{P}_\sigma \to P_1 \oplus P_2$. The induced maps $P_1 \to C$ and $P_2 \to C$ are surjective since they are surjective after passing to cosocles (using that formation of cosocle is right exact).

Now suppose that we have D and surjections $P_1 \twoheadrightarrow D$ and $P_2 \twoheadrightarrow D$ as in the statement of the proposition. After rescaling the surjection $P_1 \to D$, we can assume that the two compositions $\widetilde{P}_\sigma \to D$ induce the same maps on cosocles. Thus their difference factors through $\operatorname{rad} D$. Since σ is not a Jordan–Hölder factor of $\operatorname{rad} D$, this difference is 0. In other words, taking a difference, we get a surjective map $\delta : P_1 \oplus P_2 \to D$ whose composition with the map $\widetilde{P}_\sigma \to P_1 \oplus P_2$ is 0. This gives the desired surjection $C \twoheadrightarrow D$. □

Lemma 5.10. *With $\overline{\rho}$, σ, and T_Σ as above, $M_\infty(P_\sigma^{\sigma(T_{w_0,\alpha\beta})})$ and $M_\infty(P_\sigma^{\sigma(T_{w_0,\beta\alpha})})$ are cyclic R_∞-modules.*

Proof. As the two cases are similar, we prove the proposition for $T \overset{\text{def}}{=} T_{w_0,\alpha\beta}$. We claim that $M_\infty(P_\sigma^{\sigma(T)})$ is isomorphic to $R_\infty/I_\infty(\tau_{\alpha\beta}) \cap I_\infty(\tau_{w_0})$. Proposition 5.9 gives an exact sequence

$$0 \to \widetilde{P}_\sigma^{\sigma(T)} \to \widetilde{P}_\sigma^{\sigma(\tau_{\alpha\beta})} \oplus \widetilde{P}_\sigma^{\sigma(\tau_{w_0})} \to C \to 0.$$

Indeed, since the maps $\widetilde{P}_\sigma^{\sigma(\tau_{\alpha\beta})} \to \widetilde{P}_\sigma^{\sigma(\tau_{\alpha\beta})} \otimes_{\mathcal{O}} E$ and $\widetilde{P}_\sigma^{\sigma(\tau_{w_0})} \to \widetilde{P}_\sigma^{\sigma(\tau_{w_0})} \otimes_{\mathcal{O}} E$ are injective, the image of $\widetilde{P}_\sigma$ in $\widetilde{P}_\sigma^{\sigma(\tau_{\alpha\beta})} \oplus \widetilde{P}_\sigma^{\sigma(\tau_{w_0})}$ is isomorphic to its image in $\widetilde{P}_\sigma^{\sigma(\tau_{\alpha\beta})} \otimes_{\mathcal{O}} E \oplus \widetilde{P}_\sigma^{\sigma(\tau_{w_0})} \otimes_{\mathcal{O}} E$. Applying the exact functor M_∞, we have the exact sequence

$$0 \to M_\infty(\widetilde{P}_\sigma^{\sigma(T)}) \to M_\infty(\widetilde{P}_\sigma^{\sigma(\tau_{\alpha\beta})}) \oplus M_\infty(\widetilde{P}_\sigma^{\sigma(\tau_{w_0})}) \to M_\infty(C) \to 0.$$

Moreover, the restriction of the third map to each summand is a surjection. By [LLHLMb, Theorem 5.3.1] (which improves the genericity condition in [LLHLM20, Theorem 5.1.1]), $M_\infty(\widetilde{P}_\sigma^{\sigma(\tau_w)})$ is isomorphic to $R_\infty(\tau_w)$ for all $w \in S_3$. We can choose isomorphisms to obtain an exact sequence

$$0 \to M_\infty(\widetilde{P}_\sigma^{\sigma(T)}) \to R_\infty/I_\infty(\tau_{\alpha\beta}) \oplus R_\infty/I_\infty(\tau_{w_0}) \to R_\infty/I_\infty(C) \to 0$$

for some ideal $I_\infty(C)$ with $I_\infty(\tau_{\alpha\beta}) + I_\infty(\tau_{w_0}) \subset I_\infty(C)$ where the third map is the difference of the natural surjections. By Lemma 3.14, it suffices to show that $I_\infty(C) \subset I_\infty(\tau_{\alpha\beta}) + I_\infty(\tau_{w_0})$.

Then there are surjections $P_\sigma^{\sigma(\tau_{\alpha\beta})} \twoheadrightarrow W$ and $P_\sigma^{\sigma(\tau_{w_0})} \twoheadrightarrow W$ and thus a surjection $C \twoheadrightarrow W$ by the second part of Proposition 5.9. The induced map $P_\sigma^{\sigma(\tau_{w_0})} \twoheadrightarrow W$ becomes an isomorphism after applying $M_\infty(-)$ by [LLHLMb, Proposition 5.3.2] and [LLHLM20, Corollary 2.3.11, Theorem 3.5.2]. We conclude that $I_\infty(C) \subset (p) + I_\infty(\tau_{w_0})$. By Proposition 3.11, $(p) + I_\infty(\tau_{w_0}) \subset I_\infty(\tau_{\alpha\beta}) + I_\infty(\tau_{w_0})$ so that $I_\infty(C) \subset I_\infty(\tau_{\alpha\beta}) + I_\infty(\tau_{w_0})$. □

Let $\overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha})}$ be the image of the map $P_\sigma \to P_\sigma^{\sigma(T_{w_0,\alpha\beta})} \oplus P_\sigma^{\sigma(T_{w_0,\beta\alpha})}$ induced by the natural surjections $P_\sigma \to P_\sigma^{\sigma(T_{w_0,\beta\alpha})}$ and $P_\sigma \to P_\sigma^{\sigma(T_{w_0,\alpha\beta})}$.

Lemma 5.11. *With $\overline{\rho}$, σ, $\overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha})}$ as above, $M_\infty(\overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha})})$ is a cyclic R_∞-module.*

Proof. We have an exact sequence

$$0 \to \overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha})} \to P_\sigma^{\sigma(T_{w_0,\alpha\beta})} \oplus P_\sigma^{\sigma(T_{w_0,\beta\alpha})} \to C \to 0$$

for some C as in Proposition 5.9. Since there are surjective maps $P_\sigma^{\sigma(T_{w_0,\alpha\beta})} \to W$ and $P_\sigma^{\sigma(T_{w_0,\beta\alpha})} \to W$, the second part of Proposition 5.9 gives a surjective map $C \to W$. This gives an exact sequence

$$0 \to \overline{P}'_\sigma \to P_\sigma^{\sigma(T_{w_0,\alpha\beta})} \oplus P_\sigma^{\sigma(T_{w_0,\beta\alpha})} \to W \to 0$$

where the third map is a difference of the surjections and $\overline{P}'_\sigma$ is the kernel of this map. Then we have $\overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha})} \subset \overline{P}'_\sigma$.

Applying M_∞ to the second exact sequence, we have

$$0 \to M_\infty(\overline{P}'_\sigma) \to M_\infty(P_\sigma^{\sigma(T_{w_0,\alpha\beta})}) \oplus M_\infty(P_\sigma^{\sigma(T_{w_0,\beta\alpha})}) \to M_\infty(W) \to 0.$$

Let Σ be $\{\alpha\beta, \beta\alpha, w_0, \mathrm{id}\}$ and define $M'_\infty(-)$ using $T = T_\Sigma$ (see (5.9)). By Lemma 5.10 and its proof (which shows that $M_\infty(P_\sigma^{\sigma(\tau_{w_0})}) \cong M_\infty(W)$), we can and do choose isomorphisms to get the exact sequence

$$0 \to M'_\infty(\overline{P}_\sigma^{T,\prime}) \to R'_\infty(T)/\big(\widetilde{I}_{\tau_{w_0},\nabla_\infty} \cap \widetilde{I}_{\tau_{\alpha\beta},\nabla_\infty}, p\big) \oplus R'_\infty(T)/\big(\widetilde{I}_{\tau_{w_0},\nabla_\infty} \cap \widetilde{I}_{\tau_{\beta\alpha},\nabla_\infty}, p\big) \to R'_\infty(T)/\big(\widetilde{I}_{\tau_{w_0},\nabla_\infty}, p\big) \to 0.$$

As the union of the images of $\mathrm{Tor}_1^{S_\infty^\square}\Big(\mathbb{F}, R'_\infty(T)/\big(\widetilde{I}_{\tau_{w_0},\nabla_\infty} \cap \widetilde{I}_{\tau_{\alpha\beta},\nabla_\infty}, p\big)\Big)$ and $\mathrm{Tor}_1^{S_\infty^\square}\Big(\mathbb{F}, R'_\infty(T)/\big(\widetilde{I}_{\tau_{w_0},\nabla_\infty} \cap \widetilde{I}_{\tau_{\beta\alpha},\nabla_\infty}, p\big)\Big)$ in $\mathrm{Tor}_1^{S_\infty^\square}\Big(\mathbb{F}, R'_\infty(T)/\big(\widetilde{I}_{\tau_{w_0},\nabla_\infty}, p\big)\Big)$ is spanning by Lemma 3.29, Lemma 3.14 implies that $M'_\infty(\overline{P}'_\sigma)$ is a cyclic $R'_\infty(T)$-module so that $M_\infty(\overline{P}'_\sigma)$ is a cyclic R_∞-module.

Finally, we claim that the inclusion $M_\infty(\overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha})}) \hookrightarrow M_\infty(\overline{P}'_\sigma)$ is an isomorphism. As $M_\infty(\overline{P}'_\sigma)$ is a cyclic R_∞-module, it suffices to show that the map is nonzero after applying $-\otimes_{R_\infty} \mathbb{F}$ by Nakayama's lemma. This follows from the fact that applying the functor $M_\infty(-) \otimes_{R_\infty} \mathbb{F}$ to the composition $\overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha})} \subset \overline{P}'_\sigma \twoheadrightarrow P_\sigma^{\sigma(T_{w_0,\alpha\beta})} \twoheadrightarrow \sigma$ (with all maps the natural ones) induces a surjection and that $M_\infty(\sigma) \neq 0$. □

Let $\overline{P}_\sigma^{\sigma(T_{\mathrm{id},w_0,\alpha\beta,\beta\alpha})}$ be the image of the map $P_\sigma \to P_\sigma^{\sigma(\tau_{\mathrm{id}})} \oplus \overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha})}$ induced by the natural surjections $P_\sigma \to P_\sigma^{\sigma(\tau_{\mathrm{id}})}$ and $P_\sigma \to \overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha})}$. Equivalently, $\overline{P}_\sigma^{\sigma(T_{\mathrm{id},w_0,\alpha\beta,\beta\alpha})}$ is the image of the map

$$P_\sigma \to P_\sigma^{\{R_{\alpha\beta},R_{w_0}\}} \oplus P_\sigma^{\{R_{w_0},R_{\beta\alpha}\}} \oplus P_\sigma^{R_{\mathrm{id}}}$$

appearing in (4.16).

Lemma 5.12. *With $\overline{\rho}$, σ, $\overline{P}_\sigma^{\sigma(T_{\mathrm{id},w_0,\alpha\beta,\beta\alpha})}$ as above, $M_\infty(\overline{P}_\sigma^{\sigma(T_{\mathrm{id},w_0,\alpha\beta,\beta\alpha})})$ is a cyclic R_∞-module.*

Proof. The proof is similar to that of Lemma 5.11. We have an exact sequence

$$0 \to \overline{P}_\sigma^{\sigma(T_{\mathrm{id},w_0,\alpha\beta,\beta\alpha})} \to P_\sigma^{\sigma(\tau_{\mathrm{id}})} \oplus \overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha})} \to C \to 0$$

for some C as in Proposition 5.9. Let $\lambda \in X_1(\underline{T})$ such that $\sigma = F(\lambda)$. By [LLHLMb, Proposition 5.3.2], there exists a quotient Λ of $P_\sigma^{\sigma(\tau_{\mathrm{id}})}$ whose Jordan–Hölder factors are precisely $\kappa = \mathfrak{Tr}_{\lambda_A+\eta}(\omega, a) \in P_\sigma^{\sigma(\tau_{\mathrm{id}})}$ with $(\omega_i, a_i) \in \{(0,0), (\varepsilon_1, 0), (\varepsilon_2, 0), (\varepsilon_1 - \varepsilon_2, 0), (\varepsilon_2 - \varepsilon_1, 0), (0,1)\}$. Though we will not use it, one can check that Λ is the cokernel of the composition

$$\mathrm{rad}^{2i} P_\sigma \subset P_\sigma \to P_\sigma^{\sigma(\tau_{\mathrm{id}})}.$$

We will show that the natural surjection $P_\sigma \twoheadrightarrow \Lambda$ factors through $\overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha})}$. Again by [LLHLMb, Proposition 5.3.2], for each $w \in \{w_0, \alpha\beta, \beta\alpha\}$ there is a unique quotient Q_w of $P_\sigma^{\sigma(\tau_w)}$ whose Jordan–Hölder factors are precisely $\mathrm{JH}(P_\sigma^{\sigma(\tau_w)}) \cap \mathrm{JH}(\Lambda)$. Moreover, there is a surjection $\Lambda \to Q_w$ whose kernel we denote K_w. Then the image of $N_w \overset{\mathrm{def}}{=} \ker(P_\sigma \twoheadrightarrow P_\sigma^{\sigma(\tau_w)})$ in Λ is contained in K_w. As

$\mathrm{JH}(K_{\alpha\beta}) \cap \mathrm{JH}(K_{\beta,\alpha})$ is empty, the intersection of the images of $N_{\alpha\beta}$ and $N_{\beta\alpha}$ in Λ is 0. Since each N_w contains the kernel of the map $P_\sigma \twoheadrightarrow \overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha})}$, we obtain the desired factorization. Then the second part of Proposition 5.9 gives a surjective map $C \to \Lambda$ and an exact sequence

$$0 \to \overline{P}'_\sigma \to P_\sigma^{\sigma(\tau_{\mathrm{id}})} \oplus \overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha})} \to \Lambda \to 0 \tag{5.10}$$

where the third map is a difference of the surjections and $\overline{P}'_\sigma$ is the kernel of this map (which is different from what is denoted $\overline{P}'_\sigma$ in the proof of Lemma 5.11). Then we have $\overline{P}_\sigma^{\sigma(T_{\mathrm{id},w_0,\alpha\beta,\beta\alpha})} \subset \overline{P}'_\sigma$.

Applying M_∞ to (5.10), we have

$$0 \to M_\infty(\overline{P}'_\sigma) \to M_\infty(P_\sigma^{\sigma(\tau_{\mathrm{id}})}) \oplus M_\infty(\overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha})}) \to M_\infty(\Lambda) \to 0.$$

By [LLHLMb, Theorem 5.3.1], $M_\infty(P_\sigma^{\sigma(\tau_{\mathrm{id}})})$ is a cyclic R_∞-module. This implies that $M_\infty(\Lambda)$ is a cyclic R_∞-module as well. Let T and M'_∞ as in the proof of Lemma 5.11. By the proof of [LLHLM20, Lemma 3.6.2] the annihilator of $M'_\infty(\Lambda)$ corresponds to the ideal $(\widetilde{I}_\Lambda, p) \subseteq \widetilde{S}$, with $\widetilde{I}_\Lambda$ defined as in §3.4.5. Let $I'_\infty(T_{w_0,\alpha\beta,\beta\alpha})$ be the image of $(\widetilde{I}_{\tau_{w_0},\nabla_\infty} \cap \widetilde{I}_{\tau_{\alpha\beta},\nabla_\infty}, p) \cap (\widetilde{I}_{\tau_{w_0},\nabla_\infty} \cap \widetilde{I}_{\tau_{\beta\alpha},\nabla_\infty}, p) \subseteq \widetilde{S}$ in $\widetilde{S}^{\nabla_\infty,\square}$. By Lemma 3.14 $M'_\infty(\overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha})})$ is isomorphic to $R'_\infty(T)/I'_\infty(T_{w_0,\alpha\beta,\beta\alpha})$ and we can and do choose isomorphisms so that we have the exact sequence

$$0 \to M'_\infty(\overline{P}'_\sigma) \to R'_\infty(T)/\big(\widetilde{I}_{\tau_{\mathrm{id}}\nabla_\infty}, p\big) \oplus R'_\infty(T)/I'_\infty(T_{w_0,\alpha\beta,\beta\alpha}) \to R'_\infty(T)/\big(\widetilde{I}_\Lambda, p\big) \to 0$$

where the third map is the difference of the natural surjections. Since the union of the images of $\mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, R'_\infty(T)/\big(\widetilde{I}_{\tau_{\mathrm{id}}\nabla_\infty}, p\big))$ and $\mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, R'_\infty(T)/I'_\infty(T_{w_0,\alpha\beta,\beta\alpha}))$ in $\mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, R'_\infty(T)/(\widetilde{I}_\Lambda, p))$ is spanning by Lemma 3.34, Lemma 3.14 implies that $M_\infty(\overline{P}'_\sigma)$ is a cyclic R_∞-module. Then one shows that $M_\infty(\overline{P}_\sigma^{\sigma(T_{\mathrm{id},w_0,\alpha\beta,\beta\alpha})}) = M_\infty(\overline{P}'_\sigma)$ as in the proof of Lemma 5.11. □

Proof of Proposition 5.8. By Proposition 4.29, with a as in the statement of Proposition 5.8 (note that $P_\sigma^a = P_\sigma/\mathrm{Fil}^{3_i} P_\sigma$), the surjection $P_\sigma \to P_\sigma^a$ factors through $\overline{P}_\sigma^{\sigma(T_{\mathrm{id},w_0,\alpha\beta,\beta\alpha})}$. Since $M_\infty(\overline{P}_\sigma^{\sigma(T_{\mathrm{id},w_0,\alpha\beta,\beta\alpha})})$ is a cyclic R_∞-module by Lemma 5.12, we conclude that $M_\infty(P_\sigma^a)$ is as well. □

5.3.3. *A* Tor *computation.* In this subsection, fix $i \notin A(\sigma)$ and let $a = (a_j)_{j\in\mathcal{J}}$ be the tuple with $a_i = \widehat{3}$ and $a_j = \widehat{1}$ for all $j \neq i$. In order to prove Lemma 5.4, we will need to find a lower bound for the image of the map

$$\mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, M'_\infty(P_\sigma^a)) \to \mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, M'_\infty(\sigma))$$

induced by the surjection $P_\sigma^a \twoheadrightarrow \sigma$ (see Lemma 5.12). Let $P_\sigma^{\sigma(T_{w_0,\alpha\beta}),a}$ be the cokernel of the composition

$$\mathrm{Fil}^{3_i} P_\sigma \subset P_\sigma \twoheadrightarrow P_\sigma^{\sigma(T_{w_0,\alpha\beta})}$$

where $P_\sigma^{\sigma(T_{w_0,\alpha\beta})}$ is defined as in §5.3.2. Similarly, we define $P_\sigma^{\sigma(\tau_{\mathrm{id}}),a}$, $\overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha}),a}$, $\overline{P}_\sigma^{\sigma(T_{\mathrm{id},w_0,\alpha\beta,\beta\alpha}),a}$, and Λ^a (with Λ defined as in the proof of Lemma 5.12).

Lemma 5.13. *We have an exact sequence*

$$0 \to M_\infty(\overline{P}_\sigma^{\sigma(T_{\mathrm{id},w_0,\alpha\beta,\beta\alpha}),a}) \to M_\infty(P_\sigma^{\sigma(\tau_{\mathrm{id}}),a}) \oplus M_\infty(\overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha}),a}) \to M_\infty(\Lambda^a) \to 0.$$

Proof. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Fil}^{3_i} P_\sigma & \longrightarrow & \mathrm{Fil}^{3_i} P_\sigma \oplus \mathrm{Fil}^{3_i} P_\sigma & \longrightarrow & \mathrm{Fil}^{3_i} P_\sigma & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \overline{P}_\sigma^{\sigma(T_{\mathrm{id},w_0,\alpha\beta,\beta\alpha})} & \longrightarrow & P_\sigma^{\sigma(\tau_{\mathrm{id}})} \oplus \overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha})} & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

where the rows are exact, the bottom row is as in Lemma 5.12, and the top row has nonzero maps given by diagonal and difference maps. Since $\ker(\mathrm{Fil}^{3_i} P_\sigma \oplus \mathrm{Fil}^{3_i} P_\sigma) \to \ker(\mathrm{Fil}^{3_i} P_\sigma)$ is surjective (as both $P_\sigma^{\sigma(\tau_{\mathrm{id}})} \to C$, $\overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha})} \to C$ are factorizations of $P_\sigma \twoheadrightarrow C$), the snake lemma furnishes an exact sequence

$$0 \to \overline{P}_\sigma^{\sigma(T_{\mathrm{id},w_0,\alpha\beta,\beta\alpha}),a} \to P_\sigma^{\sigma(\tau_{\mathrm{id}}),a} \oplus \overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha}),a} \to C^a \to 0$$

where C^a denotes the cokernel of the natural map $\mathrm{Fil}^{3_i} P_\sigma \to C$. The result follows by applying $M_\infty(-)$ and noting that the natural map $M_\infty(C) \to M_\infty(\Lambda)$ is an isomorphism by the proof of Lemma 5.12. □

The following is the main result of the subsection.

Lemma 5.14. *Recall from* (5.9) *the definition of $M'_\infty(-)$ with respect to $T_{\mathrm{id},w_0,\alpha\beta,\beta\alpha}$. The image of* $\mathrm{Tor}_1(\mathbb{F}, M'_\infty(\overline{P}_\sigma^{\sigma(T_{\mathrm{id},w_0,\alpha\beta,\beta\alpha}),a})) \to \mathrm{Tor}_1(\mathbb{F}, M'_\infty(\Lambda^a))$ *is the intersection of the images of the maps*

$$\mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, M'_\infty(\overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha}),a})) \to \mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, M'_\infty(\Lambda^a))$$

and

$$\mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, M'_\infty(P_\sigma^{\sigma(\tau_{\mathrm{id}}),a})) \to \mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, M'_\infty(\Lambda^a)).$$

Proof. The result follows from combining Lemmas 5.13 and 3.14 using that $M_\infty(\overline{P}_\sigma^{\sigma(T_{\mathrm{id},w_0,\alpha\beta,\beta\alpha}),a})$ is a cyclic R_∞-module by Lemma 5.12 so that $M'_\infty(\overline{P}_\sigma^{\sigma(T_{\mathrm{id},w_0,\alpha\beta,\beta\alpha}),a})$ is a cyclic $S_\infty^\square$-module. □

5.3.4. *Noncyclic patched modules.* Recall that at the beginning of §5.3 we have fixed $\lambda \in X_1(\underline{T})$ such that $\sigma = F(\lambda) \in W^?(\overline{\rho})$. Throughout this subsection we let τ_{min}^B be the minimal tame inertial L-parameter of $F(\sum_j \lambda_{j,B})$ with respect to $\overline{\rho}$ in the sense of [LLHLM20, Remark 3.5.10]. Note that, up to changing lowest alcove presentation of $\overline{\rho}$, we can and do assume that $t_{-\underline{1}}\widetilde{w}(\overline{\rho}, \tau_{\mathrm{min}}^B)_j = t_{w_0\eta}$ or $t_{-\underline{1}}\widetilde{w}(\overline{\rho}, \tau_{\mathrm{min}}^B)_j = w_0$, for all $j \in \mathcal{J}$.

In this subsection, let T be the set of tame inertial types τ with $\widetilde{w}(\overline{\rho}, \tau)_j = w_j w_0 \widetilde{w}(\overline{\rho}, \tau_{\mathrm{min}}^B)_j$ with $w_j \in \{\mathrm{id}, \alpha\beta, \beta\alpha, w_0, t_{-\underline{1}}t_{w_0\eta}\}$ if $t_{-\underline{1}}\widetilde{w}(\overline{\rho}, \tau_{\mathrm{min}}^B)_j = w_0$, and $w_j \in \{\mathrm{id}, \alpha\beta, \beta\alpha, w_0\}$ if $t_{-\underline{1}}\widetilde{w}(\overline{\rho}, \tau_{\mathrm{min}}^B)_j = t_{w_0\eta}$. With this choice of T, we define $M'_\infty(-)$ as in (5.9) (with the implicit assertion that the R_∞-action on $M_\infty(-)$ factors through $R_\infty(T)$). We also let a denote a tuple $(a_j)_{j\in\mathcal{J}}$ with $a_j \subset \{B, E_o, F_o, E_s, F_s\}$ if $j \in A(\sigma)$ and $a_j = \widehat{1}$ or $\widehat{2}$ if $j \notin A(\sigma)$. For $j \in A(\sigma)$, b_j will denote an element of $\{B, E_o, F_o, E_s, F_s\}$, $I_j^{b_j}$ is as defined in §3.4.1, and $M_j^{b_j} \overset{\mathrm{def}}{=} S^{(j)}/I_j^{b_j}$. From [LLHLMb, Lemma 5.3.3], [LLHLM20, Lemma 3.6.2] we deduce that $M'_\infty(\sigma) \cong \overline{R}'_\infty(T)/(\mathfrak{P}_\sigma)$ where $\mathfrak{P}_\sigma = \sum_j \mathfrak{P}_\sigma^{(j)} S$ for prime ideals $\mathfrak{P}_\sigma^{(j)} \subset S^{(j)}$, and $\mathfrak{P}_\sigma$ is the pullback to S of the ideal $\overline{\mathfrak{P}}_\sigma \subset S/I_{T,\nabla_\infty}$ defined in §3.3. For $j \notin A(\sigma)$, let $M_j^{\widehat{1}} = S^{(j)}/\mathfrak{P}_\sigma^{(j)}$.

Proposition 5.15. *(1) Let $b = (b_j)_{j\in\mathcal{J}}$ with $b_j \in \{B, E_o, F_o, E_s, F_s\}$ if $j \in A(\sigma)$ and $b_j = \widehat{1}$ if $j \notin A(\sigma)$. Then $M'_\infty(P_\sigma^b) \cong (\widehat{\otimes}_j M_j^{b_j})\widehat{\otimes}_{\widetilde{S}} R'_\infty(T)$ with $M_j^{b_j}$ defined as above.*

(2) For $j \notin A(\sigma)$, there are ideals $I_j^{\widehat{2}}$ such that for any $b = (b_j)_{j\in\mathcal{J}}$ with $b_j \in \{B, E_o, F_o, E_s, F_s\}$ if $j \in A(\sigma)$ and $b_j = \widehat{2}$ if $j \notin A(\sigma)$, $M'_\infty(P_\sigma^b) \cong (\widehat{\otimes}_j M_j^{b_j})\widehat{\otimes}_{\widetilde{S}} R'_\infty(T)$ where $M_j^{b_j} \overset{def}{=} S^{(j)}/I_j^{\widehat{2}}$ if $j \notin A(\sigma)$.

Proof. We prove (2) as (1) is similar but easier. There exists $\tau \in T$ with $\widetilde{w}(\overline{\rho}, \tau)_j = w_0\widetilde{w}(\overline{\rho}, \tau_{\min}^B)_j$ for $j \notin A(\sigma)$ and $\widetilde{w}(\overline{\rho}, \tau)_j \neq w_0\widetilde{w}(\overline{\rho}, \tau_{\min}^B)_j$ for $j \in A(\sigma)$ such that $\mathrm{JH}(P_\sigma^b) \cap W^?(\overline{\rho}) \subset \mathrm{JH}(\overline{\sigma}(\tau))$. By [LLHLMb, Proposition 5.3.2], there is a quotient Q of $P_\sigma^{\sigma(\tau)}$ such that $\mathrm{JH}(Q) = \mathrm{JH}(P_\sigma^b) \cap W^?(\overline{\rho})$. By Corollary 4.9, there is a surjection $P_\sigma^b \twoheadrightarrow Q$. Since $\mathrm{JH}(Q) = \mathrm{JH}(P_\sigma^b) \cap W^?(\overline{\rho})$ and P_σ^b is multiplicity free, the induced map $M_\infty(P_\sigma^b) \twoheadrightarrow M_\infty(Q)$ is an isomorphism. In particular the R_∞-action on $M_\infty(P_\sigma^b)$ factors through $R_\infty(T)$. As $M'_\infty(Q)$ is a cyclic $R'_\infty(T)$-module by [LLHLMb, Theorem 5.3.1], so is $M'_\infty(P_\sigma^b)$. The result now follows from [LLHLM20, Lemma 3.6.2]. □

If $j \in A(\sigma)$, and $\emptyset \neq a_j \subset \{B, E_o, F_o, E_s, F_s\}$, define an $S^{(j)}$-module $M_j^{a_j}$ by the exact sequence

$$0 \to M_j^{a_j} \to \oplus_{b_j\in a_j} S^{(j)}/I_j^{b_j} \to (\oplus_{b_j\in a_j} S^{(j)}/\mathfrak{P}_\sigma^{(j)})/\Delta(S^{(j)}/\mathfrak{P}_\sigma^{(j)}) \to 0, \tag{5.11}$$

where $I_j^{b_j}$ is as in §3.4.1, the third map is induced by the sum of the natural projections, and $\Delta(S^{(j)}/\mathfrak{P}_\sigma^{(j)})$ denotes the diagonally embedded submodule.

Lemma 5.16. *Let a be $(a_j)_{j\in\mathcal{J}}$ with $a_j \subset \{B, E_o, F_o, E_s, F_s\}$ if $j \in A(\sigma)$ and $a_j = \widehat{1}$ for all $j \notin A(\sigma)$ or $a_j = \widehat{2}$ for all $j \notin A(\sigma)$. Then $M'_\infty(P_\sigma^a) \cong (\widehat{\otimes}_j M_j^{a_j})\widehat{\otimes}_{\widetilde{S}} R'_\infty(T)$.*

Proof. We induct on $k \overset{\mathrm{def}}{=} \#\{j \in A(\sigma) : \#a_j > 1\}$. The case $k = 0$ follows from Proposition 5.15. Now suppose that $k > 0$ and $\#a_i > 1$ for $i \in A(\sigma)$. For $X \in a_i$, let a_X be the tuple with $a_{X,i} = X$ and $a_{X,j} = a_j$ for $j \neq i$. Then Proposition 4.23 gives an exact sequence

$$0 \to P_\sigma^a \to \oplus_{X\in a_i} P_\sigma^{a_X} \to (\oplus_{X\in a_i} P_\sigma^{a_\emptyset})/\Delta(P_\sigma^{a_\emptyset}) \to 0,$$

where the third map is induced by the sum of the natural projections. Since the R_∞-action on $M_\infty(P_\sigma^{a_X})$ factors through $R_\infty(T)$ for each $X \in a_i$ by the inductive hypothesis, the same is true for $M_\infty(P_\sigma^a)$. This induces the exact sequence

$$0 \to M'_\infty(P_\sigma^a) \to \oplus_{X\in a_i} M'_\infty(P_\sigma^{a_X}) \to (\oplus_{X\in a_i} M'_\infty(P_\sigma^{a_\emptyset}))/\Delta(M'_\infty(P_\sigma^{a_\emptyset})) \to 0.$$

By the induction hypothesis, the modules $M'_\infty(P_\sigma^{a_X})$ and $M'_\infty(P_\sigma^{a_\emptyset})$ are isomorphic to modules $(\widehat{\otimes}_j M_j^{a_{X,j}})\widehat{\otimes}_{\widetilde{S}} R'_\infty(T)$ and $(\widehat{\otimes}_j M_j^{a_{\emptyset,j}})\widehat{\otimes}_{\widetilde{S}} R'_\infty(T)$, respectively. Thus we have an exact sequence

$$0 \to M'_\infty(P_\sigma^a) \to \oplus_{X\in a_i}(\widehat{\otimes}_j M_j^{a_{X,j}})\widehat{\otimes}_{\widetilde{S}} R'_\infty(T) \to (\oplus_{X\in a_i}(\widehat{\otimes}_j M_j^{a_{\emptyset,j}})\widehat{\otimes}_{\widetilde{S}} R'_\infty(T))/\Delta \to 0 \tag{5.12}$$

where Δ is short for $\Delta((\widehat{\otimes}_j M_j^{a_{\emptyset,j}})\widehat{\otimes}_{\widetilde{S}} R'_\infty(T))$.

The third map of (5.12) is induced by a sum of surjective maps

$$(\widehat{\otimes}_j M_j^{a_{X,j}})\widehat{\otimes}_{\widetilde{S}} R'_\infty(T) \twoheadrightarrow (\widehat{\otimes}_j M_j^{a_{\emptyset,j}})\widehat{\otimes}_{\widetilde{S}} R'_\infty(T) \tag{5.13}$$

for each $X \in a_i$. By consideration of scheme-theoretic supports, (5.13) factors through the surjection

$$(\widehat{\otimes}_j M_j^{a_{X,j}})\widehat{\otimes}_{\widetilde{S}} R'_\infty(T) \twoheadrightarrow (\widehat{\otimes}_j M_j^{a_{\emptyset,j}})\widehat{\otimes}_{\widetilde{S}} R'_\infty(T)$$

induced by $M_i^{a_{X,i}} \twoheadrightarrow M_i^{a_{\emptyset,i}}$. The resulting surjective endomorphism of the Cohen–Macaulay module $(\widehat{\otimes}_j M_j^{a_{\emptyset,j}})\widehat{\otimes}_{\widetilde{S}} R'_\infty(T)$ must be an isomorphism since the kernel of the map must have support of smaller dimension by cycle considerations and the Cohen–Macaulay module $(\widehat{\otimes}_j M_j^{a_{\emptyset,j}})\widehat{\otimes}_{\widetilde{S}} R'_\infty(T)$ cannot have embedded primes by [Mat86, Theorem 17.3]. We conclude that the map (5.13) is, up to postcomposing with an automorphism of the codomain, induced by a surjection $M_i^{a_{X,i}} \twoheadrightarrow M_i^{a_{\emptyset,i}}$. By Lemma 3.18 applied to $M_i^{a_{X,i}} \widehat{\otimes} \widehat{\otimes}_{j\neq i} M_j^{a_{X,j}}$ (using that $M_i^{a_{X,i}}$ and $M_i^{a_{\emptyset,i}}$ are cyclic), up to precomposing with an automorphism of the domain, (5.13) is the map induced by a surjection $M_i^{a_{X,i}} \twoheadrightarrow M_i^{a_{\emptyset,i}}$. Fixing an isomorphism $M_i^{a_{\emptyset,i}} \cong S^{(i)}/\mathfrak{P}_\sigma^{(i)}$, we can choose isomorphisms $M_i^{a_{X,i}} \cong S^{(i)}/I_i^X$ for each $X \in a_i$ so that the surjection $S^{(i)}/I_i^X \cong M_i^{a_{X,i}} \twoheadrightarrow M_i^{a_{\emptyset,i}} \cong S^{(i)}/\mathfrak{P}_\sigma^{(i)}$ takes 1 to 1. (5.12) is then obtained from (5.11) by taking completed tensor products over $\mathbb{F}$ and then applying $-\otimes_{\widetilde{S}} R'_\infty(T)$ which is exact here. (Each module is the completion of a module over a polynomial ring over $\mathbb{F}$. These completed tensor products are obtained by usual tensor product of these decompleted modules over $\mathbb{F}$ and then completion—each step is exact. Applying $-\otimes_{\widetilde{S}} R'_\infty(T)$ simply has the effect of adding formal variables.) The result now follows. □

Proof of Lemma 5.4(1). Let $\widehat{(2)}$ be the tuple $\widehat{(2)}_{j\in\mathcal{J}}$ and b be the tuple $(b_j)_{j\in\mathcal{J}}$ with $b_j = \{B, E_o, E_s, F_o, F_s\}$ if $j \in A(\sigma)$ and $b_j = \widehat{2}$ otherwise. Then the kernel of the surjective map $P_\sigma^{\widehat{(2)}} \to P_\sigma^b$ has no modular Serre weights. This induces an isomorphism $M_\infty(P_\sigma^{\widehat{(2)}}) \to M_\infty(P_\sigma^b)$. By Lemma 5.16 and Proposition 3.22, $M'_\infty(P_\sigma^b)$ and thus $M_\infty(P_\sigma^{\widehat{(2)}})$ is minimally generated by $3^{\#A(\sigma)}$ elements.

Let $c = (c_j)_{j\in\mathcal{J}}$ with $c_j = \widehat{2}$ if $j \in A(\sigma)$ and $c_j = \widehat{1}$ otherwise. A similar argument as in the previous paragraph implies that $M_\infty(P_\sigma^c)$ is minimally generated by $3^{\#A(\sigma)}$ elements. In particular, the natural surjection $M_\infty(P_\sigma^{\widehat{(2)}})/\mathfrak{m} \to M_\infty(P_\sigma^c)/\mathfrak{m}$ is an isomorphism. For a tuple a as in Lemma 5.4(1), the surjection $P_\sigma^{\widehat{(2)}} \to P_\sigma^c$ factors through P_σ^a and Lemma 5.4(1) follows. □

We record the following result for use in §6.

Proposition 5.17. *Let $a = (a_j)_{j\in\mathcal{J}}$ be the tuple with $a_j = \{B, E_s, F_s\}$ if $j \in A(\sigma)$ and $a_j = \widehat{1}$ otherwise. Then $M_\infty(P_\sigma^a)$ is minimally generated by $3^{\#A(\sigma)}$ elements.*

Proof. This follows from Lemma 5.16 and Proposition 3.22. □

Proof of Lemma 5.4(2). If $A(\sigma) = \mathcal{J}$, then the result follows from Lemma 5.4(1). We now assume that $A(\sigma) \neq \mathcal{J}$. Recall from Proposition 4.24 the exact sequence (4.8):

$$0 \to \overline{P}_\sigma \to \oplus_c P_\sigma^c \to (\oplus_c \sigma)/\Delta(\sigma) \to 0$$

where c runs over tuples $(c_j)_{j\in\mathcal{J}}$ with $c_i = \widehat{3}$ for some $i \notin A(\sigma)$ and $c_j = \widehat{1}$ for all $j \neq i$ and the tuple $(c_j)_{j\in\mathcal{J}}$ with $j = \widehat{2}$ for all $j \in A(\sigma)$ and $j = \widehat{1}$ for all $j \notin A(\sigma)$, $\Delta(\sigma) \subset \oplus_c \sigma$ denotes the diagonally embedded copy, and the maps are the natural projections. Since the R_∞-action on each $M_\infty(P_\sigma^c)$ factors through $R_\infty(T)$, the same is true for $M_\infty(\overline{P}_\sigma)$. Using Remark 3.10 we have the exact sequence

$$0 \to M'_\infty(\overline{P}_\sigma) \to \oplus_c M'_\infty(P_\sigma^c) \to (\oplus_c M'_\infty(\sigma))/\Delta(M'_\infty(\sigma)) \to 0. \tag{5.14}$$

For the tuples c with $c_i = \widehat{3}$ for some $i \notin A(\sigma)$ and $c_j = \widehat{1}$ for all $j \neq i$, $M'_\infty(P_\sigma^c)$ is a cyclic module, i.e. isomorphic to $S_\infty^\square/I_i$ for some ideal I_i, by Proposition 5.8. Let M be $M'_\infty(P_\sigma^c)$ for the tuple $(c_j)_{j\in\mathcal{J}}$ with $j = \widehat{2}$ for all $j \in A(\sigma)$ and $j = \widehat{1}$ for all $j \notin A(\sigma)$. We can choose isomorphisms so that (5.14) becomes

$$0 \to M'_\infty(\overline{P}_\sigma) \to M \oplus \oplus_{j\notin A(\sigma)} S_\infty^\square/I_j \to (\oplus_c S_\infty^\square/(\mathfrak{P}_\sigma))/\Delta(S_\infty^\square/(\mathfrak{P}_\sigma)) \to 0.$$

We claim that the hypotheses of Lemma 3.15 hold, from which we deduce that $M_\infty(\overline{P}_\sigma)$ is minimally generated by the same number of elements as M which is $3^{\#A(\sigma)}$ by Lemma 5.4(1).

We now verify the hypotheses of Lemma 3.15. Let V_j (resp. U) be the image of the induced map

$$\mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, S_\infty^\square/I_j) \to \mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, S_\infty^\square/(\mathfrak{P}_\sigma))$$

for $j \notin A(\sigma)$ (resp. $\mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, M) \to \mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, S_\infty^\square/(\mathfrak{P}_\sigma))$). We need to show that for all $i \notin A(\sigma)$,

$$V_i + U \cap \cap_{j\neq i} V_j = \mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, S_\infty^\square/(\mathfrak{P}_\sigma)). \tag{5.15}$$

As the natural map $\oplus_{j\in\mathcal{J}}\mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, S_\infty^\square/(\mathfrak{P}_\sigma^{(j)})) \to \mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, S_\infty^\square/(\mathfrak{P}_\sigma))$ is surjective, it is enough to show for all $i \notin A(\sigma)$ and $j_0 \in \mathcal{J}$, the LHS of (5.15) contains the image of the natural map

$$\mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, S_\infty^\square/(\mathfrak{P}_\sigma^{(j_0)})) \to \mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, S_\infty^\square/(\mathfrak{P}_\sigma)). \tag{5.16}$$

In what follows, for an ideal $I \subset S$, we identify $\mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, S_\infty^\square/(I))$ with $I \otimes_S \mathbb{F}$ (recall that $\mathbb{F} = S/\mathfrak{m}_S$).

Fix $i \notin A(\sigma)$ and $j_0 \in \mathcal{J}$. We claim that U and V_j for each $j \notin A(\sigma)$ with $j \neq j_0$ contain the image of (5.16). This would imply that (5.15) contains the image of (5.16). (If $j_0 \neq i$ then take $j = i$ and V_i would contain the image of (5.16). If $j_0 = i$ then $U \cap \cap_{j\neq i} V_j$ would contain the image of (5.16).)

We first consider U. By Lemma 5.16, M is isomorphic to $M'_\infty(P_\sigma^a) \cong (\widehat{\otimes}_{\jmath\in\mathcal{J}} M_\jmath^{a_\jmath})\widehat{\otimes}_S R'_\infty(T)$ where $a_\jmath = \{B, E_o, F_o, E_s, F_s\}$ if $\jmath \in A(\sigma)$ and $a_\jmath = \widehat{1}$ for all $\jmath \notin A(\sigma)$. Since σ occurs as a Jordan–Hölder factor in P_σ^a with multiplicity one, the cycle of M is multiplicity free. Corollary 3.13 implies that the kernel of the map $M \to M'_\infty(\sigma)$ is the kernel of $M \to M_{\mathfrak{P}_\sigma}$. Thus, up to postcomposing with an automorphism, the map $M \to M'_\infty(\sigma)$ is unique and thus induced by $S^{(\jmath)}$-module surjections $M_\jmath^{a_\jmath} \twoheadrightarrow S^{(\jmath)}/\mathfrak{P}_\sigma^{(\jmath)}$. As $M_{j_0}^{a_{j_0}}$ is isomorphic to $S^{(j_0)}/\mathfrak{P}_\sigma^{(j_0)}$, the surjection $M_{j_0}^{a_{j_0}} \twoheadrightarrow S^{(j_0)}/\mathfrak{P}_\sigma^{(j_0)}$ is an isomorphism. We conclude that U contains the image of (5.16).

We next turn to V_j. Let $j \in \mathcal{J}$ with $j \notin A(\sigma)$ and $j \neq j_0$. Let $\tau_{\min}$ be the minimal type with respect to σ as in [LLHLM20, Remark 3.5.10]. For the rest of this proof, for a subset $\Sigma \subset S_3$ and an element $w \in \Sigma$ define $\tau_{w,j}^B$, $T_{\Sigma,j}^B$ (resp. $\sigma(T_{\Sigma,j})$) as in 5.3.1 with respect to $\tau_{\min}^B$ (resp. $\tau_{\min}$) and $\ell = j$.

We first suppose that $t_{-\underline{1}}\widetilde{w}(\overline{\rho}, \tau_{\min}^B)_{j_0} = t_{w_0\eta}$. For each $\tau_{w,j}^B$ with $w \in \{\mathrm{id}, \alpha\beta, \beta\alpha, w_0\}$ and each generator $c^{(j_0)}$ in the $(\varepsilon_1+\varepsilon_2, 1)$-entry of Table 9, one sees from the $t_{w_0\eta}$-entries in Table 5 that $c^{(j_0)} + p\varepsilon_{w,j,c^{(j_0)}} \in \widetilde{I}_{\tau_{w,j},\nabla_\infty}$ for some $\varepsilon_{w,j,c^{(j_0)}} \in \mathfrak{m}_{\widetilde{S}}$. By Lemmas 3.17 and 3.30, each generator $c^{(j_0)}$ is in $(\widetilde{I}_{\tau_{w_0,j},\nabla_\infty} \cap \widetilde{I}_{\tau_{\alpha\beta,j},\nabla_\infty} \cap \widetilde{I}_{\tau_{\beta\alpha,j},\nabla_\infty}, p) + \mathfrak{m}_{S_\infty^\square}\mathfrak{P}_\sigma$. Thus the surjective map $S_\infty^\square/(\widetilde{I}_{\tau_{w_0,j},\nabla_\infty} \cap \widetilde{I}_{\tau_{\alpha\beta,j},\nabla_\infty} \cap \widetilde{I}_{\tau_{\beta\alpha,j},\nabla_\infty}, p) \to S_\infty^\square/(\mathfrak{P}_\sigma)$ induces a map on $\mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, -)$ whose image contains each generator $c^{(j_0)}$. Let $a = (a_\ell)_{\ell\in\mathcal{J}}$ be the tuple with $a_j = \widehat{3}$ and $a_\ell = \widehat{1}$ for each $\ell \neq j$ as in §5.3.3. Since the action of $S_\infty^\square$ on $M'_\infty(\overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha,j}),a})$ factors through $(\widetilde{I}_{\tau_{w_0,j},\nabla_\infty} \cap \widetilde{I}_{\tau_{\alpha\beta,j},\nabla_\infty} \cap \widetilde{I}_{\tau_{\beta\alpha,j},\nabla_\infty}, p)$, we conclude by Lemma 5.14 that V_j contains each generator $c^{(j_0)}$.

Finally, we suppose that $t_{-\underline{1}}\widetilde{w}(\overline{\rho}, \tau_{\min}^B)_{j_0} = w_0$. Similar arguments as before (using the $\alpha\beta\alpha t_{\underline{1}}$-entry of Table 3) show that, for $j \neq j_0$, V_j contains the images of the generators of the ideal in Lemma 3.21(1). Looking at the $(0,1)$-entry of Table 8 if $j_0 \notin A(\sigma)$ (resp. the $(0,0)$-entry of Table 8 if $j_0 \in A(\sigma)$), it suffices to show that the image of $c^{(j_0)} \stackrel{\text{def}}{=} (b-c)d_{21}^{(j_0)}d_{32}^{(j_0)} - (a-c)d_{31}^{(j_0)}d_{22}^{*,(j_0)}$ (resp. $c_{13}^{(j_0)}$) in $\mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, S_\infty^\square/(\mathfrak{P}_\sigma))$ is in V_j if $j_0 \notin A(\sigma)$ (resp. $j_0 \in A(\sigma)$; note in this case that if V_j contains $c_{13}^{(j_0)}$ *and* the images of the generators of the ideal in Lemma 3.21(1), then V_j contains the images of the generators of the ideal in the $(0,0)$-entry of Table 8). Let $a = (a_\ell)_{\ell\in\mathcal{J}}$ be the tuple with $a_j = \widehat{3}$ and $a_\ell = \widehat{1}$ for $\ell \neq j$. The module $M'_\infty(\overline{P}_\sigma^{\sigma(\tau_{\mathrm{id},j}),a})$ is cyclic by [LLHLM20, Theorem 5.1.1] with scheme-theoretic support determined by [LLHLM20, Lemma 3.6.2], from which we see that $c^{(j_0)}$ (resp. $c_{13}^{(j_0)}$) is in the image of $\mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, M'_\infty(\overline{P}_\sigma^{\sigma(\tau_{\mathrm{id},j}),a})) \to \mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, M'_\infty(\sigma))$, and so it suffices by Lemma 5.14 to show that the image of $c^{(j_0)}$ (resp. $c_{13}^{(j_0)}$) is contained in the image of

$$\mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, M'_\infty(\overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha,j}),a})) \to \mathrm{Tor}_1^{S_\infty^\square}(\mathbb{F}, M'_\infty(\sigma)). \tag{5.17}$$

Thus, we are left to show that $c^{(j_0)}$ (resp. $c_{13}^{(j_0)}$) annihilates $M'_\infty(\overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha,j}),a})$, since this module is cyclic over $S_\infty^\square$ by Lemma 5.11.

For each $w \in \{\mathrm{id}, \alpha\beta, \beta\alpha, w_0\}$,

$$(\widetilde{z}_w^*)^{-1} c_{13}^{(j_0)}((\widetilde{b}_w - \widetilde{c}_w) d_{21}^{(j_0)} d_{32}^{(j_0)} - (\widetilde{a}_w - \widetilde{c}_w) d_{31}^{(j_0)} d_{22}^{*,(j_0)}) - p \in \widetilde{I}_{\tau_{w,j},\nabla_\infty}$$

where $\widetilde{m}_w \in \mathbb{Z}$ is a specific lift of m for $m = a, b, c$ and $\widetilde{z}_w^* \in \widetilde{S}^\times$ is a specific unit (all depending *a priori* on w) by [LLHLM18, §5.3.1]. Fixing lifts $\widetilde{m} \in \mathbb{Z}$ of m for $m = a, b, c$, and a lift $\widetilde{z}^*$ of the reductions of $\widetilde{z}_w^*$ modulo p (note that $\widetilde{z}_w^*$ modulo p is independent of w), we have for each $w \in \{\mathrm{id}, \alpha\beta, \beta\alpha, w_0\}$ that

$$(\widetilde{z}^*)^{-1} c_{13}^{(j_0)}((\widetilde{b} - \widetilde{c}) d_{21}^{(j_0)} d_{32}^{(j_0)} - (\widetilde{a} - \widetilde{c}) d_{31}^{(j_0)} d_{22}^{*,(j_0)}) - p + p\varepsilon_{w,j} \in \widetilde{I}_{\tau_{w,j},\nabla_\infty}$$

for some $\varepsilon_{w,j} \in c_{13}^{(j_0)} \mathfrak{m}_{\widetilde{S}}$. Taking $f \stackrel{\mathrm{def}}{=} (\widetilde{z}^*)^{-1} c_{13}^{(j_0)}((\widetilde{b} - \widetilde{c}) d_{21}^{(j_0)} d_{32}^{(j_0)} - (\widetilde{a} - \widetilde{c}) d_{31}^{(j_0)} d_{22}^{*,(j_0)}) - p$, we conclude from Lemmas 3.17 and 3.30 that

$$c_{13}^{(j_0)} \widetilde{c}^{(j_0)} - p \in \widetilde{I}_{\tau_{w_0,j},\nabla_\infty} \cap \widetilde{I}_{\tau_{\alpha\beta,j},\nabla_\infty} \cap \widetilde{I}_{\tau_{\beta\alpha,j},\nabla_\infty}$$

for some $\widetilde{c}^{(j_0)} \in (\widetilde{b} - \widetilde{c}) d_{21}^{(j_0)} d_{32}^{(j_0)} - (\widetilde{a} - \widetilde{c}) d_{31}^{(j_0)} d_{22}^{*,(j_0)} + \mathfrak{m}_{\widetilde{S}}$. Filtering $\overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha,j}),a}$ with irreducible subquotients κ induces a filtration on $M'_\infty(\overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha,j}),a})$ with subquotients $M'_\infty(\kappa)$. For each κ and $m \in M'_\infty(\kappa)$, the support of $\widetilde{c}^{(j_0)} m$ (resp. $c_{13}^{(j_0)} m$) has positive codimension in the scheme-theoretic support of $M'_\infty(\kappa)$ as $c_{13}^{(j_0)}$ (resp. $\widetilde{c}^{(j_0)}$) is $M'_\infty(\kappa)$-regular. Thus the same is true for $\widetilde{c}^{(j_0)} m$ (resp. $c_{13}^{(j_0)} m$) with $m \in M'_\infty(\overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha,j}),a})$. Since $M'_\infty(\overline{P}_\sigma^{\sigma(T_{w_0,\alpha\beta,\beta\alpha,j}),a})$ is maximal Cohen–Macaulay over its support, it has no embedded primes from which we conclude that it is annihilated by $\widetilde{c}^{(j_0)}$ (resp. $c_{13}^{(j_0)}$). □

6. Locality results

6.1. Subquotients of Deligne–Lusztig representations and presentations. We begin by defining some quotients of reductions of generic Deligne–Lusztig representations of G. Fix a Deligne–Lusztig representation $R_s(\mu - \underline{1})$ with $\mu - \eta \in \underline{C}_0$ being 9 deep, so that $\mathrm{JH}(\overline{R}_s(\mu - \underline{1})) = F(\mathfrak{Tr}_\mu(s(\Sigma)))$ and [LLHLMb, Proposition 5.3.2] applies to $R_s(\mu - \underline{1})$.

Given two $\mathcal{O}$-lattices $\Lambda_1, \Lambda_2 \subset R_s(\mu - \underline{1})$, there is a unique $n \in \mathbb{Z}$ so that $p^n \Lambda_1 \subset \Lambda_2$ and $p^{n-1}\Lambda_1 \not\subset \Lambda_2$. We denote by ι the composition

$$\iota : \Lambda_1 \overset{\times p^n}{\underset{\sim}{\to}} p^n \Lambda_1 \subset \Lambda_2. \tag{6.1}$$

By construction $\iota \otimes_{\mathcal{O}} \mathbb{F} : \Lambda_1 \otimes_{\mathcal{O}} \mathbb{F} \to \Lambda_2 \otimes_{\mathcal{O}} \mathbb{F}$ is nonzero.

Recall from [EGS15, Lemma 4.1.1] that, as $R_s(\mu - \underline{1})$ is residually multiplicity free, given $\sigma \in \mathrm{JH}(\overline{R}_s(\mu - \underline{1}))$, there is a unique up to scaling $\mathcal{O}$-lattice in $R_s(\mu - \underline{1})^\sigma \subset R_s(\mu - \underline{1})$ with cosocle isomorphic to σ. We will define various quotients of reductions of the lattices $R_s(\mu - \underline{1})^\sigma$ whose structure is given by [LLHLMb, Proposition 5.3.2].

Recall from §3.4.2 that we defined a *path* to be a sequence of elements

$$\gamma = (\gamma_k)_{k \geq 1}^{\ell(\gamma)} \in \{(0,0), (\varepsilon_1, 0), (\varepsilon_2, 0), (0,1), (\varepsilon_1, 1), (\varepsilon_2, 1)\}^{\ell(\gamma)}$$

satisfying certain properties. We also defined subsets $\Sigma_\gamma \subset \Sigma_0$ for each path γ and a partial ordering $\leq$ on the set of paths.

Fix a subset $J \subset \mathcal{J}$. For each $j \in \mathcal{J} \setminus J$, fix $(\omega_j, a_j) \in \Sigma_0$. For a tuple $\gamma_J = (\gamma^{(j)})_{j\in J}$ of paths, let

$$\ell(\gamma_J) = \sum_{j\in J} \ell(\gamma^{(j)})$$

and $\sigma(\gamma_J) = F(\mathfrak{Tr}_\mu(s\omega, a))$ where (ω_j, a_j) is the fixed element in Σ_0 if $j \notin J$ and is $\gamma^{(j)}_{\ell(\gamma^{(j)})}$ otherwise. By [LLHLMb, Proposition 5.3.2], there is a unique quotient Q_{γ_J} of $R_s(\mu - \underline{1})^{\sigma(\gamma_J)} \otimes_{\mathcal{O}} \mathbb{F}$ such that

$$\mathrm{JH}(Q_{\gamma_J}) = \times_{j\in J} \Sigma_{\gamma^{(j)}} \times \times_{j\notin J} \{(\omega_j, a_j)\}.$$

We define a complex using two choices. First, choose a complete ordering of J. Second, for each path γ of length 3 and $j \in J$, let $\kappa^{(j)}_\gamma$ be an element in $\mathbb{F}^\times$ with $\kappa^{(j)}_\gamma = 1$ if $\gamma_2 = (0, 0)$ or $\gamma_3 = (0, 1)$. (There are 12 paths of length 3 and 8 of these paths have the property that $\gamma_2 = (0, 0)$ or $\gamma_3 = (0, 1)$.) It is the second choice that is more consequential. Consider the complex

$$0 \to \bigoplus_{\ell(\gamma_J)=3\#J} Q_{\gamma_J} \to \bigoplus_{\ell(\gamma_J)=3\#J-1} Q_{\gamma_J} \to \cdots \to \bigoplus_{\ell(\gamma_J)=2\#J} Q_{\gamma_J}, \tag{6.2}$$

where each map is a direct sum of maps $Q_{\gamma_J} \to Q_{\beta_J}$ which is nonzero if and only if $\beta^{(j)} \leq \gamma^{(j)}$ for all $j \in J$. If $\beta^{(j)} \leq \gamma^{(j)}$ for all $j \in J$, $\ell(\beta_J) = \ell(\gamma_J) - 1$ and $i \in \mathcal{J}$ is defined by $\beta^{(i)} \neq \gamma^{(i)}$, then the map $Q_{\gamma_J} \to Q_{\beta_J}$ is induced by

$$\Big(\prod_{j\in J, \ell(\gamma^{(j)})=3} (-1)^{\delta_{j<i}} \kappa^{(j)}_{\gamma^{(j)}} \Big) \iota$$

with ι in (6.1) and $\delta_{j<i} = 1$ if $j < i$ and 0 if $j \geq i$. Though we will not use it, one can check that (6.2) is exact.

We now define a subcomplex of (6.2). For each $2\#J \leq \ell \leq 3\#J$, let

$$\Big(\bigoplus_{\ell(\gamma_J)=\ell} Q_{\gamma_J} \Big)^0 \subset \bigoplus_{\ell(\gamma_J)=\ell} Q_{\gamma_J}$$

be the submodule of elements $(a_{\gamma_J})_{\ell(\gamma_J)=\ell}$ such that for any tuple β_J with $\ell(\beta_J) = \ell$ and $i \in J$ such that $\ell(\beta^{(i)}) = 3$, we have

$$\sum_{\substack{\gamma^{(j)}=\beta^{(j)}\,\forall\, j\neq i \\ \ell(\gamma^{(i)})=3, \gamma^{(i)}_3=\beta^{(i)}_3}} a_{\gamma_J} = 0.$$

(Note that Q_{γ_J} are the same quotient for all γ_J in the sum above, and the sum takes place in this same quotient.) Then

$$0 \to \Big(\bigoplus_{\ell(\gamma_J)=3\#J} Q_{\gamma_J} \Big)^0 \to \Big(\bigoplus_{\ell(\gamma_J)=3\#J-1} Q_{\gamma_J} \Big)^0 \to \cdots \to \Big(\bigoplus_{\ell(\gamma_J)=2\#J} Q_{\gamma_J} \Big)^0 \tag{6.3}$$

is a subcomplex of (6.2). Again, though we will not use it, one can check that (6.3) is exact.

6.2. **Patching and presentations.** In this section, we fix an 11-generic L-homomorphism $\overline{\rho}$ and a patching functor M_∞ for it. Let ν be a weight in $W(\eta)$. Then $\nu_j \in S_3(\eta_j) \cup \{(1,1,1)_j\}$ for each $j \in \mathcal{J}$. Let $J = J(\nu) \stackrel{\text{def}}{=} \{j \in \mathcal{J} \mid \nu_j = (1,1,1)_j\}$. For $j \notin J$, if $\nu_j = w_j\eta_j$ for $w_j \in S_3$, then we let $(\omega(\nu)_j, a(\nu)_j) = (-\varepsilon_{w_j}, \delta_{(-1)^{w_j}=-1})$ where $\varepsilon_{w_j} = \varepsilon'_{w_j} + X^0(T) \in X^*(T)/X^0(T)$ and $w_j t_{\varepsilon'_{w_j}} \in \widetilde{W}_1^+$ and $\delta_{(-1)^{w_j}=-1} = 1$ if w_j is an odd permutation and is 0 if w_j is an even permutation.

Let $\widetilde{w} \in \mathrm{Adm}(\eta)$ be such that $\widetilde{w}_j = t_{\underline{1}}$ for all $j \in J$ and $(\omega(\nu)_j, a(\nu)_j) \in \widetilde{w}_j(\Sigma_0)$ and $\ell(\widetilde{w}_j) \geq 2$ for all $j \notin J$. Let τ be the tame inertial L-parameter such that $\widetilde{w}(\overline{\rho}, \tau) = \widetilde{w}$. Given $\sigma(\tau) = R_s(\mu - \underline{1})$, J, and $(\omega_j, a_j) \stackrel{\text{def}}{=} (\widetilde{w}_j^{-1}\omega(\nu)_j, a(\nu)_j) \in \Sigma_0$ for $j \notin J$, we have the subquotients Q_{γ_J} of $\sigma(\tau)$ for each tuple γ_J of paths as defined in §6.1. (Note that $\mu - \eta$ can be chosen to be 9-deep in alcove $\underline{C}_0$.)

We will use the notation M'_∞ as in (5.9) with $T = \{\tau\}$. If $(\omega, a) = (\omega_j, a_j)_{j\in\mathcal{J}}$ with (ω_j, a_j) as above if $j \notin J$ and $(\omega_j, a_j) \in \{(\varepsilon_1, 1), (\varepsilon_2, 1), (0, 1)\}$ if $j \in J$, and letting $\sigma \stackrel{\text{def}}{=} F(\mathfrak{Tr}_\mu(s\omega, a)) \in \mathrm{JH}(\overline{R}_s(\mu - \underline{1}))$, then [LLHLM20, Theorem 5.1.1] implies that $M'_\infty(R_s(\mu - \underline{1})^\sigma)$ is a free $R'_\infty(\tau)$-module of rank one. Let $M^{(j)}_{(\omega_j,a_j)}$ be $\widetilde{M}^{(j)}_{(\omega_j,a_j)} \otimes_{\mathcal{O}} \mathbb{F}$ where $\widetilde{M}^{(j)}_{(\omega_j,a_j)}$ is defined in §3.4.2. For a pair (ω_j, a_j) and (ω'_j, a'_j) in $\{(0,0), (\varepsilon_1, 0), (\varepsilon_2, 0), (0,1), (\varepsilon_1, 1), (\varepsilon_2, 1)\}$, there is a unique $n((\omega_j, a_j), (\omega'_j, a'_j)) \stackrel{\text{def}}{=} n \in \mathbb{Z}$ such that $p^n\widetilde{M}^{(j)}_{(\omega'_j,a'_j)} \subset \widetilde{M}^{(j)}_{(\omega_j,a_j)}$ and $p^{n-1}\widetilde{M}^{(j)}_{(\omega'_j,a'_j)} \not\subset \widetilde{M}^{(j)}_{(\omega_j,a_j)}$. By [LLHLM20, Theorem 5.2.3], the finite category consisting of maps

$$M'_\infty(\overline{R}_s(\mu - \underline{1})^{F(\mathfrak{Tr}_\mu(s\omega', a'))}) \to M'_\infty(\overline{R}_s(\mu - \underline{1})^{F(\mathfrak{Tr}_\mu(s\omega, a))})$$

induced by ι in (6.1) is identified with the maps

$$\widehat{\otimes}_{j\in\mathcal{J}} M^{(j)}_{(\omega'_j,a'_j)} \widehat{\otimes}_S \overline{R}'_\infty(\tau) \to \widehat{\otimes}_{j\in\mathcal{J}} M^{(j)}_{(\omega_j,a_j)} \widehat{\otimes}_S \overline{R}'_\infty(\tau) \tag{6.4}$$

given by maps $M^{(j)}_{(\omega'_j,a'_j)} \to M^{(j)}_{(\omega_j,a_j)}$ which are induced by multiplication by $p^{n((\omega_j,a_j),(\omega'_j,a'_j))}$. (Tensor products without a subscript are taken over $\mathbb{F}$.)

Now let γ_J be a tuple of paths and σ be $F(\mathfrak{Tr}_\mu(s\omega, a))$ with (ω_j, a_j) the fixed choice for $j \notin J$ and $(\omega_j, a_j) = \gamma_1^{(j)}$ for $j \in J$. Then, up to scaling by a unique power of p, each Q_{γ_J} is a subquotient of $R_s(\mu - \underline{1})^\sigma$. Moreover, by [LLHLM20, Lemma 3.6.2] or by Corollary 3.13, there is an ideal $I_{\gamma^{(j)}} \subset S^{(j)}$ for each $j \in J$ and path $\gamma^{(j)}$ and a prime ideal $\mathfrak{P}^{(j)}_{(\omega_j,a_j)}$ for each $j \notin J$ so that $M'_\infty(Q_{\gamma_J})$ is identified with

$$(\widehat{\bigotimes_{j\notin J}} M^{(j)}_{(\omega_j,a_j)}/\mathfrak{P}^{(j)}_{(\omega_j,a_j)} M^{(j)}_{(\omega_j,a_j)}) \widehat{\otimes} (\widehat{\bigotimes_{j\in J}} I_{\gamma^{(j)}} M^{(j)}_{\gamma_1^{(j)}}) \widehat{\otimes}_S \overline{R}'_\infty(\tau)$$

and $M'_\infty(\iota) : M'_\infty(Q_{\gamma_J}) \to M'_\infty(Q_{\beta_J})$ for $\beta \leq \gamma$ is induced by (6.4).

With the above identifications, applying M'_∞ to (6.3) yields the complex

$$(\widehat{\bigotimes_{j\notin J}} M^{(j)}_{(\omega_j,a_j)}/\mathfrak{P}^{(j)}_{(\omega_j,a_j)} M^{(j)}_{(\omega_j,a_j)}) \widehat{\otimes} (\widehat{\bigotimes_{j\in J}} C^{(j)}_{\kappa^{(j)}}) \widehat{\otimes}_S \overline{R}'_\infty(\tau) \tag{6.5}$$

where $C^{(j)}_{\kappa^{(j)}}$ is the complex (3.16) and the signs for the tensor product of the $C^{(j)}_{\kappa^{(j)}}$ are given by the complete ordering on J. Let $Q_{\nu,\kappa}$ be the cokernel of the last map in (6.3). By exactness of M'_∞, we have an identification

$$M'_\infty(Q_{\nu,\kappa}) \cong (\widehat{\otimes}_{j\in\mathcal{J}} M^{(j)}_{\kappa^{(j)}}) \widehat{\otimes}_S \overline{R}'_\infty(\tau) \tag{6.6}$$

where $M^{(j)}_{\kappa^{(j)}}$ is the cokernel of (3.16) as defined in §3.4.2. (To see that (6.5) is exact, note that each tensor factor is the completion of a complex of modules over a polynomial ring. Then (6.5) is the completion of a tensor product of these complexes. The exactness of (6.5) follows from the exactness of completion for modules over a Noetherian ring.)

Proposition 6.1. *A Serre weight $F(\mathfrak{Tr}_\mu(s\omega, a))$ is a Jordan–Hölder factor of $Q_{\nu,\kappa}$ if and only if (ω_j, a_j) is the fixed element for $j \notin J$ and $(\omega_j, a_j) \in \{(0,0), (\varepsilon_1, 0), (\varepsilon_2, 0), (0,1), (\varepsilon_1, 1), (\varepsilon_2, 1)\}$ for $j \in J$. In this case, its multiplicity is $2^{\#\{j\in J | a_j=0\}}$.*

Proof. By the exactness of $M'_\infty(-)$ and the fact that $\mathrm{JH}(Q_{\nu,\kappa}) \subseteq W^?(\overline{\rho})$ we conclude that

$$Z_d(M'_\infty(Q_{\nu,\kappa})) = \sum_{\sigma\in\mathrm{JH}(Q_{\nu,\kappa})} [Q_{\nu,\kappa} : \sigma]\mathfrak{P}_\sigma \overline{R}'_\infty(\tau).$$

The result now follows from the identification (6.6), Proposition 3.19 and Corollary 3.28. □

For each $j \in J$ let $(a_j, b_j, c_j) \in \mathbb{F}_p^3$ be the mod p reduction of $-s_j^{-1}(\mu_j)$. From now on we take κ to be $\kappa(\overline{\rho}, \nu) \overset{\text{def}}{=} \Big(\kappa_{\min}(a_j, b_j, c_j)\Big)^J$ (see §3.4.2 for the definition of the tuple $\kappa_{\min}(a_j, b_j, c_j)$).

Proposition 6.2. *If $\ell(\gamma_J) = 2\#J$, then the composition*

$$Q_{\gamma_J} \to \bigoplus_{\ell(\gamma_J)=2\#J} Q_{\gamma_J} = \Big(\bigoplus_{\ell(\gamma_J)=2\#J} Q_{\gamma_J}\Big)^0 \to Q_{\nu,\kappa} \tag{6.7}$$

is injective.

Proof. It suffices to show that for path $\beta_J \geq \gamma_J$ with $\ell(\beta_J) = 3\#J$, the composition of (6.7) and the natural inclusion $Q_{\beta_J} \hookrightarrow Q_{\gamma_J}$ is nonzero as Q_{γ_J} is multiplicity free and the map $\oplus_\beta Q_{\beta_J} \hookrightarrow Q_{\gamma_J}$, summing over all such β_J, identifies the domain with the socle of the codomain. We will show that $M'_\infty(Q_{\beta_J}) \to M'_\infty(Q_{\gamma_J}) \to M'_\infty(Q_{\nu,\kappa})$ is nonzero.

Applying M'_∞ to (6.7) and using the above identifications yields a map

$$\begin{aligned}
&(\widehat{\bigotimes_{j\notin J}} M^{(j)}_{(\omega_j,a_j)}/\mathfrak{P}^{(j)}_{(\omega_j,a_j)} M^{(j)}_{(\omega_j,a_j)})\widehat{\otimes}(\widehat{\bigotimes_{j\in J}} I_{\gamma^{(j)}} M^{(j)}_{\gamma_1^{(j)}})\widehat{\otimes}_S \overline{R}'_\infty(\tau)\\
\to&(\widehat{\bigotimes_{j\notin J}} M^{(j)}_{(\omega_j,a_j)}/\mathfrak{P}^{(j)}_{(\omega_j,a_j)} M^{(j)}_{(\omega_j,a_j)})\widehat{\otimes}(\widehat{\bigotimes_{j\in J}} M^{(j)}_{\kappa^{(j)}})\widehat{\otimes}_S \overline{R}'_\infty(\tau)
\end{aligned}$$

induced by the compositions

$$(I_{\gamma^{(j)}} M^{(j)}_{\gamma_1^{(j)}}) \to \bigoplus_{\ell(\beta^{(j)})=2} I_{\alpha^{(j)}} M^{(j)}_{\beta_1^{(j)}} \to M^{(j)}_{\kappa^{(j)}} \tag{6.8}$$

for each $j \in J$ where the first map is the natural inclusion and the second map is the natural projection. Similarly, the map $M'_\infty(Q_{\beta_J}) \hookrightarrow M'_\infty(Q_{\gamma_J})$ is given by

$$\begin{aligned}
&(\widehat{\bigotimes_{j\notin J}} M^{(j)}_{(\omega_j,a_j)}/\mathfrak{P}^{(j)}_{(\omega_j,a_j)} M^{(j)}_{(\omega_j,a_j)})\widehat{\otimes}(\widehat{\bigotimes_{j\in J}} I_{\beta^{(j)}} M^{(j)}_{\gamma_1^{(j)}})\widehat{\otimes}_S \overline{R}'_\infty(\tau)\\
\to&(\widehat{\bigotimes_{j\notin J}} M^{(j)}_{(\omega_j,a_j)}/\mathfrak{P}^{(j)}_{(\omega_j,a_j)} M^{(j)}_{(\omega_j,a_j)})\widehat{\otimes}(\widehat{\bigotimes_{j\in J}} I_{\gamma^{(j)}} M^{(j)}_{\gamma_1^{(j)}})\widehat{\otimes}_S \overline{R}'_\infty(\tau)
\end{aligned}$$

The desired nonvanishing now follows from Corollary 3.27. □

Proposition 6.3. *A Serre weight $F(\mathfrak{Tr}_\mu(s\omega, a))$ is a Jordan–Hölder factor of* $\operatorname{soc} Q_{\nu,\kappa}$ *if and only if (ω_j, a_j) is the fixed element for $j \notin J$ and $(\omega_j, a_j) \in \{(0,1), (\varepsilon_1, 1), (\varepsilon_2, 1)\}$ for $j \in J$. Morever,* $\operatorname{soc} Q_{\nu,\kappa}$ *is multiplicity free.*

Proof. If (ω_j, a_j) is the fixed element for $j \notin J$ and $(\omega_j, a_j) \in \{(0,1), (\varepsilon_1, 1), (\varepsilon_2, 1)\}$ for $j \in J$, then $\sigma \stackrel{\text{def}}{=} F(\mathfrak{Tr}_\mu(s\omega, a)) \in \mathrm{JH}(\operatorname{soc} Q_{\gamma_J})$ for some tuple of paths γ_J with $\ell(\gamma_J) = 2\#J$. By Proposition 6.2, there an inclusion $\operatorname{soc} Q_{\gamma_J} \hookrightarrow \operatorname{soc} Q_{\nu,\kappa}$ so that $\sigma \in \mathrm{JH}(\operatorname{soc} Q_{\nu,\kappa})$.

Now let $\theta \stackrel{\text{def}}{=} F(\mathfrak{Tr}_\mu(s\omega, a)) \in \mathrm{JH}(Q_{\nu,\kappa})$. Then (ω_j, a_j) is the fixed element for $j \notin J$ and $(\omega_j, a_j) \in \{(0,0), (\varepsilon_1, 0), (\varepsilon_2, 0), (0,1), (\varepsilon_1, 1), (\varepsilon_2, 1)\}$ for $j \in J$. Suppose that $a_i = 0$ for some $i \in J$ and that we have a nonzero map $P_\theta \to Q_{\nu,\kappa}$. Consider a lift

$$\begin{array}{ccc} & & \bigoplus_{\ell(\gamma_J)=2\#J} Q_{\gamma_J} \\ & \nearrow & \downarrow \\ P_\theta & \longrightarrow & Q_{\nu,\kappa} \end{array}$$

where the vertical map is the natural quotient map. The composition

$$P_\theta \to \bigoplus_{\ell(\gamma_J)=2\#J} Q_{\gamma_J} \twoheadrightarrow \bigoplus_{\substack{\ell(\gamma_J)=2\#J \\ \gamma_1^{(i)}=(\varepsilon_k,1)}} Q_{\gamma_J},$$

where the second map is the natural projection, is nonzero for $k = 1$ and 2. Moreover, this composition factors as the composition

$$P_\theta \to \bigoplus_{\substack{\ell(\gamma_J)=2\#J \\ \gamma^{(i)}=((\varepsilon_k,1),(\omega_i,0))}} Q_{\gamma_J} \subset \bigoplus_{\substack{\ell(\gamma_J)=2\#J \\ \gamma_1^{(i)}=(\varepsilon_k,1)}} Q_{\gamma_J} \tag{6.9}$$

as θ is not a Jordan–Hölder factor of the cokernel of the inclusion in (6.9). Let $\theta' = F(\mathfrak{Tr}_\mu(s\omega', a'))$ where $(\omega'_j, a'_j) = (\omega_j, a_j)$ for all $j \neq i$ and $(\omega'_i, a'_i) = (\varepsilon_{3-k}, 1)$. Then θ' is a Jordan–Hölder factor of the image of (6.9)—indeed the composition of (6.9) with the projection to some Q_{γ_J} is surjective (choosing γ_J so that $\operatorname{cosoc} Q_{\gamma_J} \cong \theta$, and using that Q_{γ_J} is multiplicity free) and $\theta' \in \mathrm{JH}(Q_{\gamma_J})$. On the other hand, we claim that the intersection of

$$\bigoplus_{\substack{\ell(\gamma_J)=2\#J \\ \gamma^{(i)}=((\varepsilon_k,1),(\omega_i,0))}} Q_{\gamma_J}$$

and the image of

$$\left(\bigoplus_{\ell(\gamma_J)=2\#J+1} Q_{\gamma_J} \right)^0 \to \bigoplus_{\ell(\gamma_J)=2\#J} Q_{\gamma_J} \twoheadrightarrow \bigoplus_{\substack{\ell(\gamma_J)=2\#J \\ \gamma_1^{(i)}=(\varepsilon_k,1)}} Q_{\gamma_J}, \tag{6.10}$$

(where the first map in (6.10) comes from (6.3)) does not contain θ' as a Jordan–Hölder factor. Indeed, the cokernel of

$$\bigoplus_{\substack{\ell(\gamma_J)=2\#J+1 \\ \gamma^{(i)}=((\varepsilon_k,1),(\omega_i,0),(\varepsilon_{3-k},1))}} Q_{\gamma_J} \subset \bigoplus_{\substack{\ell(\gamma_J)=2\#J \\ \gamma^{(i)}=((\varepsilon_k,1),(\omega_i,0))}} Q_{\gamma_J}$$

does not contain θ' as a Jordan–Hölder factor. It suffices to show that the intersection of the images of

$$\bigoplus_{\substack{\ell(\gamma_J)=2\#J+1 \\ \gamma^{(i)}=((\varepsilon_k,1),(\omega_i,0),(\varepsilon_{3-k},1))}} Q_{\gamma_J}$$

and

$$\left(\bigoplus_{\ell(\gamma_J)=2\#J+1} Q_{\gamma_J}\right)^0$$

in

$$\bigoplus_{\substack{\ell(\gamma_J)=2\#J\\ \gamma_1^{(i)}=(\varepsilon_k,1)}} Q_{\gamma_J}$$

is zero. An element

$$(a_{\gamma_J})_{\gamma_J} \in \bigoplus_{\substack{\ell(\gamma_J)=2\#J+1\\ \ell(\gamma^{(i)})=3,\,\gamma_1^{(i)}=(\varepsilon_k,1)}} Q_{\gamma_J} \subset \bigoplus_{\substack{\ell(\gamma_J)=2\#J\\ \gamma_1^{(i)}=(\varepsilon_k,1)}} Q_{\gamma_J}$$

in the intersection of these images satisfies the condition $a_{\gamma_J} = 0$ if $\gamma_2^{(i)} \neq (\omega_i, 0)$ and for each γ_J as above $\sum_{\beta_J} a_{\beta_J} = 0$ where the sum runs over β_J with $\beta^{(j)} = \gamma^{(j)}$ for all $j \neq i$ and $\beta_3^{(i)} = (\varepsilon_{3-k}, 1)$. These conditions imply that $a_{\gamma_J} = 0$ for all γ_J.

Thus θ' is a Jordan–Hölder factor of the image of the original map $P_\theta \to Q_{\nu,\kappa}$. We conclude that θ is not a Jordan–Hölder factor of $\operatorname{soc} Q_{\nu,\kappa}$.

Finally, the Jordan–Hölder factors of $\operatorname{soc} Q_{\nu,\kappa}$ appear in a composition series of $Q_{\nu,\kappa}$ with multiplicity one by Proposition 6.1. In particular, $\operatorname{soc} Q_{\nu,\kappa}$ is multiplicity free. □

Proposition 6.4. *The module $Q_{\nu,\kappa}$ is indecomposable.*

Proof. Suppose that $Q_{\nu,\kappa} = Q \oplus Q'$ where Q is nonzero. Let $F(\mathfrak{Tr}_\mu(\omega, a)) \cong \sigma \subset Q$ be a simple submodule. Let $\sigma' \subset Q_{\nu,\kappa}$ be a simple submodule. We will show that $\sigma' \subset Q$ which implies that $Q' = 0$ and thus that $Q_{\nu,\kappa}$ is indecomposable.

For a tuple of paths β_J of length $2\#J$, we identify Q_{β_J} with its image in $Q_{\nu,\kappa}$ using Proposition 6.2. We claim that if $Q_{\beta_J} \cap Q$ is nonzero, then $Q_{\beta_J} \subset Q$. Indeed, the splitting of the inclusion $Q \hookrightarrow Q_{\nu,\kappa}$ gives a splitting of the (nonzero) map $Q_{\beta_J} \cap Q \hookrightarrow Q_{\beta_J}$. On the other hand, Q_{β_J} is indecomposable since $\operatorname{cosoc} Q_{\beta_J}$ is simple. We conclude that $Q_{\beta_J} \cap Q = Q_{\beta_J}$.

Now there are tuples of paths γ_J and γ'_J of length $2\#J$ such that $\sigma \subset Q_{\gamma_J}$ and $\sigma' \subset Q_{\gamma'_J}$. Since $\sigma \subset Q$, $Q_{\gamma_J} \cap Q \neq 0$ so that $Q_{\gamma_J} \subset Q$ by the previous paragraph. On the other hand, $\mathrm{JH}(Q_{\gamma_J}) \cap \mathrm{JH}(Q_{\gamma'_J})$ contains $\sigma_0 \overset{\text{def}}{=} F(\mathfrak{Tr}_\mu(\omega, a))$ where (ω_j, a_j) is the fixed element in Σ_0 for $j \notin J$ and is $(0, 1)$ for all $j \in J$. As σ_0 is a Jordan–Hölder factor of $Q_{\nu,\kappa}$ with multiplicity one by Proposition 6.1, $Q_{\gamma_J} \cap Q_{\gamma'_J} \neq 0$. We conclude from the previous paragraph that $\sigma' \subset Q_{\gamma'_J} \subset Q$. □

6.3. **Distinguished presentations and locality.** We maintain the notation (and assumptions on) $\overline{\rho}$, M'_∞, ν, and τ from the last section. In particular, throughout this section $\overline{\rho}$ is 11-generic. Let $\kappa(\overline{\rho}, \nu) \overset{\text{def}}{=} \kappa$ be as in 6.2 so that the map $M^{(j)}_{(\omega_j,1)} \to M^{(j)}_{\gamma^{(j)}} \to M^{(j)}_{\kappa^{(j)}}$ is nonzero for $(\omega_j, 1) \neq \gamma_1^{(j)}$. Let $D_{\mathrm{m},\nu}(\overline{\rho}) \overset{\text{def}}{=} Q_{\nu,\kappa(\overline{\rho},\nu)}$ and $D_{\mathrm{m}}(\overline{\rho})$ be

$$\bigoplus_{\nu \in W(\eta)} D_{\mathrm{m},\nu}(\overline{\rho}).$$

Remark 6.5. While the definition of $D_{\mathrm{m},\nu}(\overline{\rho})$ depends on a choice of tame inertial L-parameter (see §6.2), we will see *a posteriori* that $D_{\mathrm{m},\nu}(\overline{\rho})$ does not depend on this choice (see Remark 6.12).

Proposition 6.6. *(1)* $\mathrm{JH}(D_{\mathrm{m}}(\overline{\rho})) \subset W^?(\overline{\rho})$.

(2) If $\sigma \in W^?(\overline{\rho})$, then σ is a Jordan–Hölder factor of $D_{\mathrm{m}}(\overline{\rho})$ with multiplicity $3^{\#A(\sigma)}$.

(3) The socle of $D_{\mathrm{m}}(\overline{\rho})$ is multiplicity free.

Proof. (1) follows from the fact that for each Q_{γ_J} appearing in §6.2, $\mathrm{JH}(Q_{\gamma_J}) \subset W^?(\overline{\rho})$.

For each $\sigma = F(\mathfrak{Tr}_\mu(\omega, a)) \in W^?(\overline{\rho})$, $\sigma \in \mathrm{JH}(D_{\mathrm{m},\nu}(\overline{\rho}))$ if and only if for all $j \in \mathcal{J}$, $(-\varepsilon_{w_j}, \delta_{(-1)^{w_j}=-1}) = (\widetilde{w}_j(\omega), a)$ if $\nu_j = w_j\eta_j$ and

$$(\omega_j, a_j) \in \{(0,0), (\varepsilon_1, 0), (\varepsilon_2, 0), (0,1), (\varepsilon_1, 1), (\varepsilon_2, 1)\}$$

if $\nu_j = (1,1,1)_j$. In particular, there are $2^{\#A(\sigma)}$ choices of $\nu \in W(\eta)$ so that $\sigma \in \mathrm{JH}(D_{\mathrm{m},\nu}(\overline{\rho}))$. Moreover, if σ appears as a Jordan–Hölder factor of $D_{\mathrm{m},\nu}(\overline{\rho})$, then it appears with multiplicity $2^{\#\{j \in J(\nu) | a_j = 0\}}$. The binomial expansion of $(1+2)^{\#A(\sigma)}$ yields (2).

(3) follows easily from Proposition 6.3. □

Proposition 6.7. *For each irreducible submodule $\sigma \subset D_{\mathrm{m},\nu}(\overline{\rho})$, $M_\infty(\sigma) \notin \mathfrak{m}M_\infty(D_{\mathrm{m},\nu}(\overline{\rho}))$.*

Proof. Suppose that $\sigma \overset{\mathrm{def}}{=} F(\mathfrak{Tr}_\mu(\omega, a))$ is in the image of the map $Q_{\gamma_J} \to D_{\mathrm{m},\nu}(\overline{\rho})$ for some γ_J of length $2\#J$. By §6.2, it suffices to show that the image of the composition $M^{(j)}_{(\omega_j, a_j)} \to M^{(j)}_{\gamma^{(j)}} \to M^{(j)}_{\kappa(\overline{\rho},\nu)^{(j)}}$ is not contained in $\mathfrak{m}^{(j)} M^{(j)}_{\kappa(\overline{\rho},\nu)^{(j)}}$ for all $j \in J$. This follows from Corollary 3.27. □

We now assume that M_∞ has the form $\mathrm{Hom}^{\mathrm{cont}}_{\mathrm{GL}_3(\mathcal{O}_p)}(-, M_\infty^\vee)^\vee$ for a pseudocompact $\mathcal{O}[\![\mathrm{GL}_3(\mathcal{O}_p)]\!]$-module also denoted M_∞. Let $\pi = (M_\infty/\mathfrak{m})^\vee$. Then there is a natural transformation $\mathrm{Hom}_K(-, \pi) \cong (M_\infty(-)/\mathfrak{m})^\vee$.

Proposition 6.8. *There is an injection $D_{\mathrm{m}}(\overline{\rho}) \hookrightarrow \pi^{K_1}$. Moreover, this induces an isomorphism on socles.*

Proof. First, we show that there is an injection $D_{\mathrm{m},\nu}(\overline{\rho}) \hookrightarrow \pi$ for each $\nu \in W(\eta)$. Recall that $\mathrm{soc}\, D_{\mathrm{m},\nu}(\overline{\rho})$ is multiplicity free and has length $3^{\#J}$ by Proposition 6.3. For cardinality reasons, there is an element of $(M_\infty(D_{\mathrm{m},\nu}(\overline{\rho}))/\mathfrak{m})^\vee$ which is not in the kernel of any map $(M_\infty(D_{\mathrm{m},\nu}(\overline{\rho}))/\mathfrak{m})^\vee \to M_\infty(\sigma)/\mathfrak{m}$ for any irreducible submodule $\sigma \subset D_{\mathrm{m},\nu}(\overline{\rho})$ by Proposition 6.7. This element corresponds to an injective homomorphism $D_{\mathrm{m},\nu}(\overline{\rho}) \to \pi$.

Taking a direct sum of such injections gives a map $D_{\mathrm{m}}(\overline{\rho}) \to \pi^{K_1}$. Any irreducible submodule of the kernel is a submodule of $D_{\mathrm{m},\nu}(\overline{\rho})$ for some $\nu \in W(\eta)$ since $\mathrm{soc}\, D_{\mathrm{m}}(\overline{\rho})$ is multiplicity free by Proposition 6.6(3). We conclude that the kernel is 0. To show that this injection induces an isomorphism on socles, it suffices to show that the K-socle of π is isomorphic to $\oplus_{\sigma \in W^?(\overline{\rho})}\sigma$. This follows from [LLHLMb, Lemma 5.3.3]. □

Proposition 6.9. *Let $\mathcal{W}$ be a set of Serre weights and $V \subset Q$ be an $\mathbb{F}[\mathrm{G}]$-submodule of an $\mathbb{F}[\mathrm{G}]$-module. Then there is a unique maximal submodule $U \subset Q$ such that $\mathrm{JH}(U/(V \cap U)) \cap \mathcal{W} = \emptyset$. Moreover, $V \subset U$.*

Proof. Suppose that $U, U' \subset Q$ are maximal submodules such that $\mathrm{JH}(U/(V \cap U)) \cap \mathcal{W} = \mathrm{JH}(U'/(V \cap U')) \cap \mathcal{W} = \emptyset$. Then there is a surjection $U/(V \cap U) \oplus U'/(V \cap U') \twoheadrightarrow (U + U')/(V \cap (U + U'))$ from which we conclude that $\mathrm{JH}((U + U')/(V \cap (U + U'))) \cap \mathcal{W} = \emptyset$. By maximality of U and U', we conclude that $U = U'$. Moreover, maximality implies that $V \subset U$. □

Fix injections $D_{\mathrm{m},\nu}(\overline{\rho}) \hookrightarrow \oplus_{\sigma \in \mathrm{soc}\, D_{\mathrm{m},\nu}(\overline{\rho})} P_\sigma$ for each $\nu \in W(\eta)$. Taking a direct sum gives an injection $D_{\mathrm{m}}(\overline{\rho}) \hookrightarrow \oplus_{\sigma \in W^?(\overline{\rho})} P_\sigma$. We identify the domains with their images in $\oplus_{\sigma \in W^?(\overline{\rho})} P_\sigma$. Using Proposition 6.9, we let $D_0(\overline{\rho})$ (resp. $D_{0,\nu}(\overline{\rho})$ for each $\nu \in W(\eta)$) be the maximal submodule $U \subset \oplus_{\sigma \in W^?(\overline{\rho})} P_\sigma$ such that $\mathrm{JH}(U/(D_{\mathrm{m}}(\overline{\rho}) \cap U)) \cap W^?(\overline{\rho}) = \emptyset$ (resp. $\mathrm{JH}(U/(D_{\mathrm{m},\nu}(\overline{\rho}) \cap U)) \cap W^?(\overline{\rho}) = \emptyset$).

Proposition 6.10. *We have*

$$D_0(\overline{\rho}) \cong \bigoplus_{\nu\in W(\eta)} D_{0,\nu}(\overline{\rho}).$$

Proof. Let $\nu \in W(\eta)$. Then $D_{0,\nu}(\overline{\rho})/(D_{\mathrm{m}}(\overline{\rho}) \cap D_{0,\nu}(\overline{\rho}))$ is a quotient of $D_{0,\nu}(\overline{\rho})/(D_{\mathrm{m},\nu}(\overline{\rho}) \cap D_{0,\nu}(\overline{\rho}))$ from which we conclude that $\mathrm{JH}(D_{0,\nu}(\overline{\rho})/(D_{\mathrm{m}}(\overline{\rho}) \cap D_{0,\nu}(\overline{\rho}))) \cap W^?(\overline{\rho}) = \emptyset$ and thus that $D_{0,\nu}(\overline{\rho}) \subset D_0(\overline{\rho})$. Since $\mathrm{soc}\, D_{\mathrm{m},\nu}(\overline{\rho}) \subset \mathrm{soc}\, D_{0,\nu}(\overline{\rho}) \subset \mathrm{soc}\, \oplus_{\sigma\in W^?(\overline{\rho})} P_\sigma \cong \oplus_{\sigma\in W^?(\overline{\rho})}\sigma$ and $\mathrm{JH}(\mathrm{soc}\, D_{0,\nu}(\overline{\rho})/\mathrm{soc}\, D_{\mathrm{m},\nu}(\overline{\rho}))\cap W^?(\overline{\rho}) = \emptyset$, we have that $\mathrm{soc}\, D_{\mathrm{m},\nu}(\overline{\rho}) = \mathrm{soc}\, D_{0,\nu}(\overline{\rho})$. Thus $\mathrm{JH}(\mathrm{soc}\, D_{0,\nu}(\overline{\rho}))$ are pairwise disjoint for $\nu \in W(\eta)$, and the natural map

$$\bigoplus_{\nu\in W(\eta)} D_{0,\nu}(\overline{\rho}) \to D_0(\overline{\rho})$$

is injective.

It suffices to show that for each $\nu \in W(\eta)$, the image, denoted $D_{0,\nu}$, of the projection

$$D_0(\overline{\rho}) \to \oplus_{\sigma\in \mathrm{JH}(\mathrm{soc}\, D_{0,\nu}(\overline{\rho}))} P_\sigma$$

is contained in $D_{0,\nu}(\overline{\rho})$. The image of the restriction of this projection to $D_{\mathrm{m}}(\overline{\rho})$ is $D_{\mathrm{m},\nu}(\overline{\rho})$. Thus we have submodules

$$D_{\mathrm{m},\nu}(\overline{\rho}) \subset D_{0,\nu} \subset \oplus_{\sigma\in \mathrm{JH}(\mathrm{soc}\, D_{0,\nu}(\overline{\rho}))} P_\sigma$$

with $\mathrm{JH}(D_{0,\nu}/D_{\mathrm{m},\nu}(\overline{\rho})) \cap W^?(\overline{\rho}) \subset \mathrm{JH}(D_0(\overline{\rho})/D_{\mathrm{m}}(\overline{\rho})) \cap W^?(\overline{\rho}) = \emptyset$. Maximality implies that $D_{0,\nu} \subset D_{0,\nu}(\overline{\rho})$. □

Theorem 6.11. *There is an isomorphism $\pi^{K_1} \cong D_0(\overline{\rho})$.*

Proof. The inclusion $D_{\mathrm{m}}(\overline{\rho}) \subset D_0(\overline{\rho})$ induces an isomorphism after applying M_∞. Thus we get an injection $D_0(\overline{\rho}) \hookrightarrow \pi^{K_1}$ extending the injection $D_{\mathrm{m}}(\overline{\rho}) \hookrightarrow \pi^{K_1}$. We can extend the map $D_0(\overline{\rho}) \subset \oplus_{\sigma\in W^?(\overline{\rho})} P_\sigma$ to a map

$$\pi^{K_1} \hookrightarrow \oplus_{\sigma\in W^?(\overline{\rho})} P_\sigma$$

by injectivity of P_σ and Proposition 6.8. We claim that $\mathrm{JH}(\pi^{K_1}/D_{\mathrm{m}}(\overline{\rho})) \cap W^?(\overline{\rho}) = \emptyset$. Then by maximality of $D_0(\overline{\rho})$, we conclude that $\pi^{K_1} \cong D_0(\overline{\rho})$.

It suffices to prove the claim. Suppose that $\sigma \in W^?(\overline{\rho})$. The multiplicity of σ as a Jordan–Hölder factor of π^{K_1} is $\dim_{\mathbb{F}} \mathrm{Hom}_{\mathrm{G}}(P_\sigma, \pi^{K_1}) = \dim_{\mathbb{F}} M_\infty(P_\sigma)/\mathfrak{m} = 3^{\#A(\sigma)}$ by Theorem 5.3. This is precisely the multiplicity of σ as a Jordan–Hölder factor of $D_{\mathrm{m}}(\overline{\rho})$ by Proposition 6.6(2). The claim follows. □

Remark 6.12. For $\nu \in W(\eta)$, the $\mathbb{F}[\mathrm{G}]$-modules $D_{\mathrm{m},\nu}(\overline{\rho})$ (see §6.2), and thus the modules $D_{0,\nu}(\overline{\rho})$, depend on $\overline{\rho}|_{I_{\mathbb{Q}_p}}$ (to define $W^?(\overline{\rho})$), ν, a choice of a tame inertial L-parameter τ, and a choice of lowest alcove presentation for $\overline{\rho}$ (to define κ and to parametrize the obvious weights in $W^?(\overline{\rho})$). However, by Proposition 6.4 and Theorem 6.11, $D_{0,\nu}(\overline{\rho})$ is an indecomposable summand of π^{K_1} which is determined by its socle and is independent of τ. By the Krull–Schmidt theorem, we conclude that both $D_{\mathrm{m},\nu}(\overline{\rho})$ and $D_{0,\nu}(\overline{\rho})$ depend only on $\overline{\rho}|_{I_{\mathbb{Q}_p}}$, a lowest alcove presentation of $\overline{\rho}$, and $\nu \in W(\eta)$. Moreover, since π^{K_1} is independent of a lowest alcove presentation of $\overline{\rho}$ and $\nu \in W(\eta)$, $D_0(\overline{\rho})$ depends only on $\overline{\rho}|_{I_{\mathbb{Q}_p}}$.

6.4. **Jordan–Hölder factors of $D_0(\overline{\rho})$.** We describe the Jordan–Hölder factors of $D_0(\overline{\rho})$ with multiplicity. Fix $\overline{\rho}$, $\nu \in W(\eta)$, and $\tau = \tau(s, \mu - \underline{1})$ with $\widetilde{w} = \widetilde{w}(\overline{\rho}, \tau)$ as in §6.2. We define a set Σ_{ν_j} for each $j \in \mathcal{J}$. Let

$$\Sigma_{w_j\eta_j} \stackrel{\text{def}}{=} \{(\widetilde{w}(\omega - (w_0 w_j)^{-1}(\varepsilon_1 + \varepsilon_2)), a) \mid (\omega, a) \in \Sigma_0\}$$

for $w_j \in S_3$ and

$$\Sigma_{(1,1,1)_j} \stackrel{\text{def}}{=} \Sigma_0 \cup \{(-\varepsilon_1, 0), (-\varepsilon_2, 0), (2\varepsilon_1, 0), (2\varepsilon_1 - \varepsilon_2, 0), (2\varepsilon_2, 0), (2\varepsilon_2 - \varepsilon_1, 0)\}.$$

Theorem 6.13. *Let $\nu \in W(\eta)$. Then*

$$\mathrm{JH}(D_{0,\nu}(\overline{\rho})) = F(\mathfrak{Tr}_\mu(s \prod_{j\in\mathcal{J}} \Sigma_{\nu_j})).$$

Moreover, if $\sigma = F(\mathfrak{Tr}_\mu(s\omega, a))$ and

$$n(\sigma) = \#\{j \in \mathcal{J} \mid \nu_j = (1,1,1)_j,\ a_j = 0,\ \text{and } (\omega, a) \in \Sigma_0\},$$

then σ appears in $\mathrm{JH}(D_{0,\nu}(\overline{\rho}))$ with multiplicity $2^{n(\sigma)}$.

Proof. Fix an injective envelope

$$\iota : D_{0,\nu}(\overline{\rho}) \hookrightarrow \bigoplus_{\sigma\in\mathrm{JH}(\mathrm{soc} D_{0,\nu}(\overline{\rho}))} P_\sigma.$$

Let $V_\sigma \subset P_\sigma$ be the Weyl submodule in the Weyl filtration in Proposition 4.4. If $\sigma \in \mathrm{JH}(\mathrm{soc} D_{0,\nu}(\overline{\rho}))$ and $\sigma \uparrow \kappa$ (σ and κ are linked Serre weights with κ in a higher alcove), then σ appears with multiplicity one in $\mathrm{JH}(D_{0,\nu}(\overline{\rho}))$ and $\kappa \notin \mathrm{JH}(D_{0,\nu}(\overline{\rho}))$ if $\kappa \ncong \sigma$. Using this, it is easy to show using the dual Weyl filtration in Proposition 4.4 and [LLHLM20, Lemma 4.2.2] that the $\mathbb{F}$-dual $\iota^\vee$ of ι factors through the dual Weyl quotients $V_\sigma^\vee$ of $P_{\sigma^\vee}$. For a tuple a, let $\mathrm{rad}^a V_\sigma^\vee$ be the image of $\mathrm{rad}^a P_{\sigma^\vee}$ in $V_\sigma^\vee$.

Let $J = J(\nu) = \{j \in \mathcal{J} \mid \nu_j = (1,1,1)_j\}$. If $i \in J$, we claim that $\mathrm{rad}^{2_i} V_\sigma^\vee \subset \ker \iota^\vee$. It suffices to show that the induced map

$$\mathrm{gr}^{2_i} V_\sigma^\vee \to D_{0,\nu}(\overline{\rho})^\vee / \iota^\vee(\mathrm{rad}^{>2_i} V_\sigma^\vee) \tag{6.11}$$

is 0 as $\mathrm{gr}^{2_i} V_\sigma^\vee$ is the cosocle of $\mathrm{rad}^{2_i} V_\sigma^\vee$ by [LLHLM20, Lemma 4.2.2]. We first claim that the induced map

$$\mathrm{gr}^{2_i} V_\sigma^\vee \to D_{\mathrm{m},\nu}(\overline{\rho})^\vee / \iota^\vee(\mathrm{rad}^{>2_i} V_\sigma^\vee) \tag{6.12}$$

is 0. Indeed, (6.12) factors through $\mathrm{rad} D_{\mathrm{m},\nu}(\overline{\rho})^\vee / \iota^\vee(\mathrm{rad}^{>2_i} V_\sigma^\vee)$, but $\mathrm{JH}(\mathrm{rad} D_{\mathrm{m},\nu}(\overline{\rho})^\vee) \cap \mathrm{JH}(\mathrm{gr}^{2_i} V_\sigma^\vee) = \emptyset$ by alcove considerations. Thus, if $\kappa^\vee \subset \mathrm{gr}^{2_i} V_\sigma^\vee$ with $\kappa \in W^?(\overline{\rho})$, then $\kappa^\vee$ maps to 0 under (6.11).

Now suppose that $\kappa^\vee \subset \mathrm{gr}^{2_i} V_\sigma^\vee$ is a simple G-submodule with $\kappa \notin W^?(\overline{\rho})$. (For example, if $\sigma \cong \mathfrak{Tr}_\mu(s\omega, a)$ with $(\omega_i, a_1) = (0, 1)$ and $\kappa \cong \mathfrak{Tr}_\mu(s\xi, a)$, then $(\xi_i, a_i) = (-\varepsilon, 1)$ for $\varepsilon = \varepsilon_1$ or ε_2.) To show that $\kappa^\vee$ maps to 0 under (6.11), it suffices to show that κ is not a Jordan–Hölder factor of $\iota(D_{0,\nu}(\overline{\rho}))$. Suppose otherwise. Note that κ is a Jordan–Hölder factor of P_σ for a unique $\sigma \in \mathrm{JH}(\mathrm{soc} D_{0,\nu}(\overline{\rho}))$ which we now fix. Thus, κ is a Jordan–Hölder factor of $\iota(D_{0,\nu}(\overline{\rho})) \cap V_\sigma$. Propositions 4.4 and 4.15 imply that $\iota(D_{\mathrm{m},\nu}(\overline{\rho})) \cap V_\sigma$, and in particular $D_{\mathrm{m},\nu}(\overline{\rho})$, contains an extension of σ' by σ where σ' is the weight linked to σ with alcove differing precisely at $i \in \mathcal{J}$. This contradicts the fact that $D_{\mathrm{m},\nu}(\overline{\rho})$ does not contain the extension of two simple modules in $W^?(\overline{\rho})$.

To summarize, there is a surjection

$$\iota^\vee : \bigoplus_{\sigma\in\mathrm{JH}(\mathrm{soc} D_{0,\nu}(\overline{\rho}))} (V_\sigma^\vee / \sum_{i\in J} \mathrm{rad}^{2_i} V_\sigma^\vee) \twoheadrightarrow D_{0,\nu}(\overline{\rho})^\vee. \tag{6.13}$$

Let $V^\vee$ be the domain of (6.13). Let $D_{0,\nu}^\vee$ denote the quotient of $V^\vee$ by the image of

$$\bigoplus_{\sigma\in W^?(\overline{\rho})} \ker(\mathrm{Hom}(P_\sigma, V^\vee) \to \mathrm{Hom}(P_\sigma, D_{0,\nu}(\overline{\rho})^\vee)) \otimes P_\sigma$$

under the evaluation map. Then (6.13) induces an isomorphism $D_{0,\nu}^\vee \xrightarrow{\sim} D_{0,\nu}(\overline{\rho})^\vee$ by the maximality property of $D_{0,\nu}(\overline{\rho})$. Let V and $D_{0,\nu}$ be the $\mathbb{F}$-duals of $V^\vee$ and $D_{0,\nu}^\vee$, respectively. It suffices to show that $D_{0,\nu}$ has the properties asserted in the theorem for $D_{0,\nu}(\overline{\rho})$.

First, if $\kappa \in \mathrm{JH}(V)\setminus F(\mathfrak{Tr}_\mu(s\prod_{j\in\mathcal{J}}\Sigma_{\nu_j}))$, then Propositions 4.15 and 4.19 imply that there exists $\sigma \in W^?(\overline{\rho})\setminus\mathrm{JH}(D_{\mathrm{m},\nu})$ such that the map $\mathrm{Hom}_{\mathrm{G}}(P_{\kappa^\vee}, P_{\sigma^\vee})\otimes\mathrm{Hom}_{\mathrm{G}}(P_{\sigma^\vee}, V^\vee) \to \mathrm{Hom}_{\mathrm{G}}(P_{\kappa^\vee}, V^\vee)$ induced by composition is surjective. Since the induced map $\mathrm{Hom}_{\mathrm{G}}(P_{\sigma^\vee}, V^\vee) \to \mathrm{Hom}_{\mathrm{G}}(P_{\sigma^\vee}, D_{0,\nu}^\vee) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{G}}(P_{\sigma^\vee}, D_{\mathrm{m},\nu}^\vee)$ is 0, we have that $\mathrm{Hom}_{\mathrm{G}}(P_{\kappa^\vee}, D_{0,\nu}^\vee) = 0$. We conclude that

$$\mathrm{JH}(D_{0,\nu}^\vee) \subset F(\mathfrak{Tr}_\mu(s\prod_{j\in\mathcal{J}}\Sigma_{\nu_j})).$$

Now suppose that $\kappa \in F(\mathfrak{Tr}_\mu(s\prod_{j\in\mathcal{J}}\Sigma_{\nu_j}))$. Let $N(\kappa)$ denote the set of weights in $\mathrm{JH}(D_{\mathrm{m},\nu})$ nearest to κ in the metric defined in [LLHLM20, Definition 2.1.8]. (The set $N(\kappa)$ may have more than one element.) Let d be the distance of κ to elements in $N(\kappa)$. It suffices to show that $[D_{0,\nu}^\vee : \kappa^\vee] = \sum_{\sigma\in N(\kappa)}[D_{0,\nu}^\vee : \sigma^\vee]$. For each $\sigma \in N(\kappa)$, fix a lift $P_{\kappa^\vee} \to P_{\sigma^\vee}$ of a nonzero map $P_{\kappa^\vee} \to P_{\sigma^\vee}/\mathrm{rad}^{d+1}P_{\sigma^\vee}$. Then the induced map

$$\bigoplus_{\sigma\in N(\kappa)} \mathrm{Hom}_{\mathrm{G}}(P_{\sigma^\vee}, V^\vee) \to \mathrm{Hom}_{\mathrm{G}}(P_{\kappa^\vee}, V^\vee)$$

is an isomorphism. Thus, $[D_{0,\nu}^\vee : \kappa^\vee] \leq \sum_{\sigma\in N(\kappa)}[D_{0,\nu}^\vee : \sigma^\vee]$. If $\tau \in \mathrm{JH}(D_{\mathrm{m},\nu})$ and a composition $P_{\kappa^\vee} \to P_{\tau^\vee} \to V^\vee$ is nonzero, then this composition can be written as a composition $P_{\kappa^\vee} \to P_{\sigma^\vee} \to P_{\tau^\vee} \to V^\vee$ where the first map is the one fixed above. We conclude that $[D_{0,\nu}^\vee : \kappa^\vee] = \sum_{\sigma\in N(\kappa)}[D_{0,\nu}^\vee : \sigma^\vee]$. ☐

6.5. **Global applications.** We now apply the results of §6.3 to obtain instances of local–global compatibility in the mod-p Langlands correspondence for GL_3. We follow the setup (and most of the notation) of [LLHLM20, §5.3]. In particular F/F^+ is a CM extension which is unramified at all finite places.

We fix a totally definite outer form $H_{/F^+}$ of GL_3 which splits over F, and a compact open subgroup $U^p \leq H(\mathbb{A}_{F^+}^{\infty p})$. Given a finite smooth $\mathbb{F}[U^p]$-module W we have the space of mod p algebraic automorphic forms

$$S(U^p, W) \stackrel{\mathrm{def}}{=} \left\{f : H(F^+)\backslash H(\mathbb{A}_{F^+}^\infty) \to W \mid f(gu) = u^{-1}f(g)\ \forall\ g \in G(\mathbb{A}_{F^+}^\infty), u \in U^p\right\}.$$

This space carries commuting actions of a Hecke algebra $\mathbb{T}^{\mathrm{univ}}$ and $H(F_p^+)$. We fix a maximal ideal $\mathfrak{m} \subset \mathbb{T}^{\mathrm{univ}}$ in the support of $S(U^p, W)$ giving rise to a Galois representation $\overline{r} : G_F \to \mathrm{GL}_3(\mathbb{F})$. We now make the same assumptions as in [LLHLM20, §5.3], namely:

1. p is unramified in F^+ and all places in F^+ above p split in F;
2. $H_{/F^+}$ is quasi-split at all finite places;
3. U^p is as in [LLHLM20, §5.3, (1)-(3)] (so that U^p is hyperspecial at all but one auxiliary place);
4. $W = W_{\Sigma_0^+} \otimes_{\mathcal{O}} \mathbb{F}$ where $W_{\Sigma_0^+}$ is obtained using minimally ramified types away from p;
5. $\overline{r}$ satisfies the Taylor–Wiles conditions of [LLHLM18, Definition 7.3]; and
6. if $\overline{r}$ is ramified at a finite place w of F then $w|_{F^+}$ splits in F.

We set $\pi(\overline{r}) \overset{\text{def}}{=} S(U^p, W)[\mathfrak{m}]$. Let S_p be the set of places of F^+ above p. For each $v \in S_p$ fix a place $w|v$ of F, and isomorphisms $F_v^+ \cong F_w$, $H(F_v^+) \cong \mathrm{GL}_3(F_w)$ (see [LLHLM20, §5.3]). As explained in §2.1.4 the collection $\{\overline{r}|_{G_{F_v^+}}\}_{v \in S_p}$ gives rise to an L-homomorphism $\overline{\rho} : G_{\mathbb{Q}_p} \to {}^L\underline{G}(\mathbb{F})$.

Theorem 6.14. *Let $\overline{r} : G_F \to \mathrm{GL}_3(\mathbb{F})$ a continuous Galois representation satisfying items (1)–(6), and let $\overline{\rho}$ the L-homomorphism corresponding to $\{\overline{r}|_{G_{F_v^+}}\}_{v \in S_p}$. We further assume that $\overline{\rho}$ is tame and 11-generic. Then*

$$\pi(\overline{r})^{K_1} \cong D_0(\overline{\rho})$$

where $D_0(\overline{\rho})$ is as in §6.3.

Proof. As in [HLM17, Theorem 5.2.1], we can and do choose a weak minimal patching functor M_∞ of the form $\mathrm{Hom}^{\mathrm{cont}}_{\mathrm{GL}_3(\mathcal{O}_p)}(-, M_\infty^\vee)^\vee$ for a pseudocompact $\mathcal{O}[\![\mathrm{GL}_3(\mathcal{O}_p)]\!]$-module also denoted M_∞ satisfying Definition 5.1 so that $\pi(\overline{\rho}) \cong (M_\infty/\mathfrak{m})^\vee$. The result follows now from Theorem 6.11. $\square$

Appendix A. Tables for multi-type deformation rings

Table 3. **The ring $\widetilde{S}$ and its ideals $\widetilde{I}^{(j)}_{\tau,\nabla_\infty}$.**

$A^{T,(j)}\widetilde{w}^{*,T}(\overline{\rho})_j$	$\begin{pmatrix} d^*_{11}(v+p)+c_{11}+\frac{e_{11}}{v} & c_{12} & c_{13} \\ d_{21}(v+p)+c_{21} & d^*_{22}(v+p)+c_{22} & c_{23} \\ d_{31}(v+p)+c_{31} & d_{32}(v+p)+c_{32} & d^*_{33}(v+p)+c_{33} \end{pmatrix} \cdot s_j^{-1} v^{\mu_j+(1,0,-1)}$
$\widetilde{S}^{(j)}$	$\mathcal{O}[\![c_{11}, x^*_{11}, e_{11}, c_{12}, c_{13}, d_{21}, c_{21}, c_{22}, x^*_{22}, c_{23}, d_{31}, c_{31}, d_{32}, c_{32}, c_{33}, x^*_{33}]\!]$

$\widetilde{w}^*(\overline{\rho},\tau)_j$			
$\alpha\beta\alpha\gamma t_{\underline{1}}$	$\widetilde{I}^{(j)}_{\tau,\nabla_\infty}$	$\widetilde{I}^{(j)}_{\tau}$	$c_{13},\ c_{23}$ $c_{32}+pd_{32},\ c_{33}+pd^*_{33}$
			$c_{21},\ c_{22}$ $c_{11}-pd^*_{11},\ e_{11}-p^2d^*_{11}$
		Mon_τ	$(b_{\tau,1}-b_{\tau,3}-1)d^*_{22}c_{31}+(b_{\tau,2}-b_{\tau,3}-1)d_{21}d_{32}+pd_{31}d^*_{22}+O(p^{N-4})$
$\alpha\beta\alpha t_{\underline{1}}$	$\widetilde{I}^{(j)}_{\tau,\nabla_\infty}$	$\widetilde{I}^{(j)}_{\tau}$	$e_{11},\ c_{21},\ c_{22},\ c_{23}$ $c_{32}+pd_{32},\ c_{33}+pd^*_{33}$
			$c_{13}d_{32}-c_{12}d^*_{33},$ $c_{13}d_{31}-c_{11}d^*_{33}+pd^*_{11}d^*_{33}$ $c_{13}c_{31}+pc_{11}d^*_{33}$
		Mon_τ	$(b_{\tau,2}-b_{\tau,3})d_{21}c_{12}+(b_{\tau,3}-b_{\tau,1})c_{11}d^*_{22}-pd^*_{11}d^*_{22}+O(p^{N-4})$ $p(b_{\tau,2}-b_{\tau,3})d_{21}d_{32}+(b_{\tau,1}-b_{\tau,3}+1)c_{31}d^*_{22}+pd_{31}d^*_{22}+O(p^{N-4})$
$\alpha\beta t_{\underline{1}}$	$\widetilde{I}^{(j)}_{\tau,\nabla_\infty}$	$\widetilde{I}^{(j)}_{\tau}$	$e_{11},\ c_{22}+pd^*_{22},\ c_{32}+pd_{32},\ c_{33}+pd^*_{33}$
			$c_{13}d_{32}-c_{12}d^*_{33},$ $c_{31}d^*_{22}-d_{32}c_{21}$ $c_{23}d_{32}+pd^*_{22}d^*_{33}$ $c_{13}c_{21}-c_{23}c_{11}$ $c_{11}d^*_{33}-d_{31}c_{13}-pd^*_{11}d^*_{33}$
		Mon_τ	$(b_{\tau,2}-b_{\tau,3})d_{21}c_{12}+(b_{\tau,3}-b_{\tau,1})c_{11}d^*_{22}+p(b_{\tau,2}-b_{\tau,3}-1)d^*_{11}d^*_{22}+O(p^{N-4})$ $(b_{\tau,2}-b_{\tau,3}-1)c_{23}d_{31}+(b_{\tau,1}-b_{\tau,2}+1)c_{21}d^*_{33}+p(b_{\tau,2}-b_{\tau,3})d_{21}d^*_{33}+O(p^{N-4})$

We give a presentation of $\mathcal{O}(M_{\overline{\rho}}^{\nabla_{T,\infty}})$ when $\#T=1$. Let $zt_\nu \stackrel{\text{def}}{=} \widetilde{w}^*(\overline{\rho},\tau)$ and define $b_\tau^{(j)} \in T(\mathbb{Z}_p)$ by $z_j^{-1}(b_\tau^{(j)}) = (s'_{\mathrm{or},j})^{-1}(\mathbf{a}'^{(j)})/(p^{f'}-1)$, where $\mathbf{a}'^{(j)}$ is defined with respect to $(sz, \mu - sz(\nu)) \in \underline{W} \times X^*(\underline{T})$ (see §2.1.1). For readability, we write b_τ, c_{ik}, etc. instead of $b_\tau^{(j)}, c_{ik}^{(j)}$, etc. We assume that μ is N-deep in $\underline{C}_0$ (hence $\mu - sz(\nu)$ is $(N-2)$-deep in $\underline{C}_0$). Note that $D^{\tau,\overline{\beta}}_{\mathfrak{M}_\tau} \hookrightarrow M^T_{\overline{\rho}} \cdot \widetilde{w}^{*,T}(\overline{\rho})$ (Proposition 3.4) is given by $A^{(j)}_{\mathfrak{M},\beta} \mapsto A^{(j)}_{\mathfrak{M},\beta} \cdot z_j^{-1} \cdot \widetilde{w}^{*,T}(\overline{\rho})_j$ when $T^{(j)}$ is as in (I)-(II). Finally $(a,b,c) \stackrel{\text{def}}{=} b_\tau$ modulo ϖ and note that $(a,b,c) \equiv -\big(s_j^{-1}(\mu_j+\eta_j) - z(\nu)\big)$ modulo ϖ.

TABLE 4. **Further cases of the ideals $\widetilde{I}^{(j)}_{\tau}$, $\widetilde{I}^{(j)}_{\tau,\nabla_\infty}$ and $I^{(j)}_{\tau,\nabla_{\mathrm{alg}}}$ when $\widetilde{w}^*(\overline{\rho},\tau)_j \in \{\beta\alpha t_{\underline{1}}, t_{\underline{1}}\}$.**

$\widetilde{w}^*(\overline{\rho},\tau)_j$			
$\beta\alpha t_{\underline{1}}$	$\widetilde{I}^{(j)}_{\tau,\nabla_\infty}$	$\widetilde{I}^{(j)}_{\tau}$	$e_{11},\ c_{21},\ c_{22},\ c_{23}$ $c_{31}+pd_{31},\ c_{33}+pd^*_{33}$
			$c_{11}d^*_{33}-c_{13}d_{31},$ $c_{13}c_{32}+pd^*_{33}c_{12}$ $d_{21}(c_{13}d_{32}-c_{12}d^*_{33})-pd^*_{11}d^*_{22}d^*_{33}$
		Mon_τ	$(b_{\tau,2}-b_{\tau,3})d_{21}c_{12}+(b_{\tau,3}-b_{\tau,1})c_{11}d^*_{22}-pd^*_{11}d^*_{22}+O(p^{N-4})$ $(b_{\tau,1}-b_{\tau,3})c_{12}d_{31}+(b_{\tau,3}-b_{\tau,1})c_{11}d_{32}+(b_{\tau,3}-b_{\tau,2}-1)c_{32}d^*_{11}-pd_{32}d^*_{11}+O(p^{N-4})$
$t_{\underline{1}}$	$\widetilde{I}^{(j)}_{\tau,\nabla_\infty}$	$\widetilde{I}^{(j)}_{\tau}$	$e_{11},\ c_{21}+pd_{21},\ c_{31}+pd_{31},\ c_{32}+pd_{32}$
			$c_{23}d_{31}-d_{21}c_{33},\ c_{12}d_{31}-c_{11}d_{32},$ $c_{13}c_{22}-c_{12}c_{23},\ pc_{13}d_{32}+c_{12}c_{33},$ $c_{22}d_{31}+pd_{21}d_{32},\ pc_{13}d_{31}+c_{11}c_{33},$ $pc_{13}d_{21}+c_{11}c_{23},\ pc_{12}d_{21}+c_{11}c_{22},$ $c_{13}d_{21}d_{32}-c_{12}d_{21}d^*_{33}-c_{13}d_{31}d^*_{22}-c_{23}d_{32}d^*_{11}+c_{11}d^*_{22}d^*_{33}+d^*_{11}c_{22}d^*_{33}+d^*_{11}d^*_{22}c_{33},$ $c_{12}d_{21}c_{33}-c_{11}c_{22}d^*_{33}-c_{11}d^*_{22}c_{33}-d^*_{11}c_{22}c_{33}-p(c_{11}d^*_{22}d^*_{33}+d^*_{11}c_{22}d^*_{33}+d^*_{11}d^*_{22}c_{33})$
		Mon_τ	$(b_{1,\tau}-b_{3,\tau}-1)(c_{23}d_{32}-c_{33}d^*_{22})-(b_{1,\tau}-b_{2,\tau}-1)c_{22}d^*_{33}+pd^*_{11}d^*_{22}d^*_{33}+O(p^{N-4}),$ $(b_{1,\tau}-b_{2,\tau})(c_{13}d_{31}-c_{11}d^*_{33})+(b_{2,\tau}-b_{3,\tau}-1)c_{33}d^*_{11}-pd^*_{11}d^*_{22}d^*_{33}+O(p^{N-4}),$ $(b_{2,\tau}-b_{3,\tau})(c_{12}d_{21}-c_{11}d^*_{11})-(b_{1,\tau}-b_{3,\tau})c_{11}d^*_{22}-pd^*_{11}d^*_{22}d^*_{33}+O(p^{N-4}),$

Table 5. **The ring $\widetilde{S}$ and its ideals $\widetilde{I}^{(j)}_{\tau,\nabla_\infty}$.**

$A^{T,(j)}\widetilde{w}^{*,T}(\overline{\rho})_j$	$\begin{pmatrix} d^*_{11} & c_{12} & c_{13}(v+p)+e_{13} \\ d_{21} & d^*_{22}(v+p)+c_{22} & c_{23}(v+p)+e_{23} \\ d_{31} & d_{32}(v+p)+c_{32} & d^*_{33}(v+p)^2+c_{33}(v+p)+e_{33} \end{pmatrix} \cdot s_j^{-1}v^{\mu_j}$
$\widetilde{S}^{(j)}$	$\mathcal{O}[\![x^*_{11}, c_{12}, c_{13}, e_{13}, d_{21}, c_{22}, x^*_{22}, c_{23}, e_{23}, d_{31}, d_{32}, c_{32}, c_{33}, e_{33}, x^*_{33}]\!]$

$\widetilde{w}^*(\overline{\rho},\tau)_j$			
$t_{w_0(\eta)}$	$\widetilde{I}^{(j)}_{\tau,\nabla_\infty}$	$\widetilde{I}^{(j)}_\tau$	$d_{21},\ d_{31},\ c_{32}+pd_{32}$
			$e_{33},\ c_{33},\ d_{32},\ e_{23},\ c_{22}$
		Mon_τ	$(b_{\tau,1}-b_{\tau,2})c_{12}c_{23}+pc_{13}d^*_{22}-(b_{\tau,1}-b_{\tau,3})e_{13}d^*_{22}+O(p^{N-4})$
$t_{w_0(\eta)}\alpha$	$\widetilde{I}^{(j)}_{\tau,\nabla_\infty}$	$\widetilde{I}^{(j)}_\tau$	$c_{22}+pd^*_{22},\ c_{32}+pd_{32},\ d_{31}$
			$e_{33}, c_{33}, d_{32},$ $e_{13}d_{21}-e_{23}d^*_{11}, c_{12}d_{21}+pd^*_{11}d^*_{22}$
		Mon_τ	$(b_{\tau,2}-b_{\tau,1})c_{12}c_{23}+p(b_{\tau,2}-b_{\tau,1}-1)c_{13}d^*_{22}+(b_{\tau,1}-b_{\tau,3})e_{13}d^*_{22}+O(p^{N-1})$
$t_{w_0(\eta)}\beta$	$\widetilde{I}^{(j)}_{\tau,\nabla_\infty}$	$\widetilde{I}^{(j)}_\tau$	$d_{21},\ d_{31},\ e_{33}+pc_{33}+p^2d^*_{33}$
			$c_{32},\ e_{23},\ c_{22},\ c_{33}+pd^*_{33},\ c_{23}d_{32}+pd^*_{22}d^*_{33}$
		Mon_τ	$(b_{\tau,1}-b_{\tau,2})c_{12}c_{23}+pc_{13}d^*_{22}-(b_{\tau,1}-b_{\tau,3})e_{13}d^*_{22}+O(p^{N-4})$
$t_{w_0(\eta)}w_0$	$\widetilde{I}^{(j)}_{\tau,\nabla_\infty}$	$\widetilde{I}^{(j)}_\tau$	$pd_{32}+c_{32},\ pc_{23}+e_{23},\ e_{33}+pc_{33}+p^2d^*_{33}$
			$e_{13}d_{32}-c_{12}c_{33}-pc_{12}d^*_{33},\ c_{23}d_{31}-d_{21}c_{33}-pd_{21}d^*_{33},$ $c_{12}d_{31}+pd_{32}d^*_{11},\ e_{13}d_{21}+pc_{23}d^*_{11},\ c_{12}d_{21}-c_{22}d^*_{11},$ $c_{13}d_{21}d_{32}-c_{13}d_{31}d^*_{22}-c_{23}d_{32}d^*_{11}+c_{33}d^*_{11}d^*_{22},\ e_{13}d_{31}+pc_{33}d^*_{11}+p^2d^*_{33}d^*_{11}$
		Mon_τ	$(b_{\tau,2}-b_{\tau,1})(c_{13}c_{22}-c_{12}c_{23})+pc_{13}d^*_{22}+(b_{\tau,3}-b_{\tau,1})e_{13}d^*_{22}+O(p^{N-4}),$ $(b_{\tau,3}-b_{\tau,1}-1)(c_{23}d_{32}-c_{33}d^*_{11})-(b_{\tau,2}-b_{\tau,1})c_{22}d^*_{33}-p(b_{\tau,3}-b_{\tau,1})d^*_{11}d^*_{33}+O(p^{N-4}),$ $(b_{\tau,2}-b_{\tau,1}-1)c_{13}d_{31}+(b_{\tau,3}-b_{\tau,2})c_{33}d^*_{11}+p(b_{\tau,3}-b_{\tau,1})d^*_{11}d^*_{33}+O(p^{N-4})$

Table 6. **Special fiber for the algebraic multi-type deformation ring when $T^{(j)} = \{\alpha\beta t_{\underline{1}}, \alpha\beta\alpha t_{\underline{1}}\}$ or $T^{(j)} = \{\beta\alpha t_{\underline{1}}, \alpha\beta\alpha t_{\underline{1}}\}$.**

$\overline{A}^{T,(j)}\widetilde{w}^*(\overline{\rho})$	$\begin{pmatrix} d^*_{11}v + c_{11} + \frac{e_{11}}{v} & c_{12} & c_{13} \\ d_{21}v + c_{21} & d^*_{22}v + c_{22} & c_{23} \\ d_{31}v + c_{31} & d_{32}v + c_{32} & d^*_{33}v + c_{33} \end{pmatrix} \cdot s_j^{-1} v^{\mu_j + (1,0,-1)}$
$S^{(j)}$	$\mathbb{F}[\![c_{11}, x^*_{11}, e_{11}, c_{12}, c_{13}, d_{21}, c_{21}, c_{22}, x^*_{22}, c_{23}, d_{31}, c_{31}, d_{32}, c_{32}, c_{33}, x^*_{33}]\!]$
$T^{(j)}$	some elements of $I^{(j)}_{T,\nabla_{\mathrm{alg}}}$
$\alpha\beta\alpha t_{\underline{1}}, \alpha\beta t_{\underline{1}}$	$e_{11},\ c_{33},\ c_{32},\ c_{23}d_{32} - c_{22}d^*_{33},\ c_{13}d_{32} - c_{12}d^*_{33},\ c_{23}c_{31},\ c_{13}c_{31},$ $c_{13}d_{31} - c_{11}d^*_{33},\ c_{22}c_{23},\ c_{21}c_{22},\ c_{13}c_{21} - c_{11}c_{23},\ c^2_{21}d_{32} - c_{21}c_{31}d^*_{22},$ $(b-c)d_{21}c_{22}d^*_{33} + (-b+c+1)c_{23}d_{31}d^*_{22} + (-a+b-1)c_{21}d^*_{22}d^*_{33},$ $(b-c)c_{12}d_{21} + \overline{x}c_{11}c_{22} - (a-c)c_{11}d^*_{22} - (b-c)c_{22}d^*_{11} \quad \exists\, \overline{x} \in \mathbb{F}_p,$ $(b-c)d_{21}c_{22}d_{32} + c_{22}d_{31}d^*_{22} + c_{21}c_{32}d^*_{22} + (a-c+1)d^*_{22}(c_{31}d^*_{22} - c_{21}d_{32})$
$\alpha\beta\alpha t_{\underline{1}}, \beta\alpha t_{\underline{1}}$	$e_{11},\ c_{33},\ c_{23},\ c_{22},\ c_{21},\ c_{31}c_{32},\ c_{13}c_{32},\ c_{11}c_{32},\ c_{13}c_{31},\ c_{12}c_{31},$ $c_{13}d_{21}d_{32} - c_{12}d_{21}d^*_{33} - c_{13}d_{31}d^*_{22} + c_{11}d^*_{22}d^*_{33},$ $c^2_{13}d_{31}d_{32} - c_{12}c_{13}d_{31}d^*_{33} - c_{11}c_{13}d_{32}d^*_{33} + c_{11}c_{12}(d^*_{33})^2,$ $(a-b)c_{13}d_{31}d_{32} - (a-c)c_{12}d_{31}d^*_{33} + (b-c)c_{11}d_{32}d^*_{33} + (b-c+1)c_{32}d^*_{11}d^*_{33},$ $(b-c)c_{12}d_{21}d_{31} - (a-b)c_{13}d^2_{31} - (b-c)c_{11}d_{21}d_{32} +$ $\quad + (a-b)c_{11}(d_{31}d^*_{22} - d_{21}c_{32}) - c_{31}d^*_{22}((a-c)c_{11} - (a-c+1)d^*_{11}),$ $\overline{z}'(c_{12}c_{13}d_{21}d_{31} - c_{11}c_{12}d_{21}d^*_{33}) + \overline{z}''(c^2_{11}d^*_{22}d^*_{33} - c_{11}c_{13}d_{31}d^*_{11}) +$ $\quad + (b-c)c_{12}d_{21}d^*_{11}d^*_{33} - (a-c)c_{11}d^*_{11}d^*_{22}d^*_{33} \qquad \exists\, \overline{z}', \overline{z}'' \in \mathbb{F}$
	some elements of $I^{(j)}_{\{w_0,\alpha\beta\},\nabla_{\mathrm{alg}}} \cap I^{(j)}_{\{w_0,\beta\alpha\},\nabla_{\mathrm{alg}}}$
	$e_{11},\ c_{33},\ c_{31}c_{32},\ c_{23}c_{32},\ c_{21}c_{32},\ c_{13}c_{32},\ c_{11}c_{32},\ c_{23}d_{32} - c_{22}d^*_{33},\ c_{23}c_{31},$ $c_{13}c_{31},\ c_{12}c_{31},\ c_{22}c_{23},\ c_{21}c_{22},\ c_{13}c_{22} - c_{12}c_{23},\ c_{13}c_{21} - c_{11}c_{23},\ c_{12}c_{21} - c_{11}c_{22},$ $c^2_{21}d_{32} - c_{21}c_{31}d^*_{22},\ c_{13}d_{21}d_{32} - c_{12}d_{21}d^*_{33} - c_{13}d_{31}d^*_{22} + c_{11}d^*_{22}d^*_{33},$ $c^2_{13}d_{31}d_{32} - c_{12}c_{13}d_{31}d^*_{33} - c_{11}c_{13}d_{32}d^*_{33} + c_{11}c_{12}(d^*_{33})^2,$ $(b-c)d_{21}c_{22}d^*_{33} - (b-c-1)c_{23}d_{31}d^*_{22} - (a-b+1)c_{21}d^*_{22}d^*_{33},$ $(a-b)c_{13}d_{31}d_{32} - (a-c)c_{12}d_{31}d^*_{33} + (b-c)c_{11}d_{32}d^*_{33} + (b-c+1)c_{32}d^*_{11}d^*_{33},$ $(b-c)(c_{12}d_{21}d_{31} - c_{11}d_{21}d_{32} - c_{22}d_{31}d^*_{11} - c_{21}d_{32}d^*_{11}) + (a-b)d_{31}(c_{11}d^*_{33} - c_{13}d_{31}) +$ $\quad + (-a+c-1)c_{31}d^*_{11}d^*_{33},$ $\overline{z}'c_{12}d_{21}(c_{13}d_{31} - c_{11}d^*_{33}) + \overline{z}''c_{11}d^*_{22}(c_{11}d^*_{33} - c_{13}d_{31}) +$ $\quad - (b-c)(c_{12}d_{21} - c_{22}d^*_{11})d^*_{11}d^*_{33} - \overline{x}c_{11}c_{22}d^*_{11}d^*_{33} + (a-c)c_{11}d^*_{11}d^*_{22}d^*_{33}$

In this table we give an explicit presentation of the ring $S/I_{T,\nabla_{\mathrm{alg}}}$ in some cases when $\#T^{(j)} = 2$ (see Proposition 3.19, and Lemma 3.29), and of the ideal $\overline{S^{(j)}}/I^j_{\{w_0,\alpha\beta\},\nabla_{\mathrm{alg}}} \cap I^j_{\{w_0,\beta\alpha\},\nabla_{\mathrm{alg}}}$ needed for Lemma 3.34. We have set $(a,b,c) \overset{\mathrm{def}}{=} -\big(s_j^{-1}(\mu_j + \eta_j) - (1,1,1)\big)$. It can be shown that $\overline{x} \equiv \frac{1}{p}\frac{(b_{\tau_{\alpha\beta},2} - b_{\tau_{\alpha\beta},3})(b_{\tau_{w_0},3} - b_{\tau_{w_0},1}) - (b_{\tau_{\alpha\beta},3} - b_{\tau_{\alpha\beta},1})(b_{\tau_{w_0},2} - b_{\tau_{w_0},3})}{(b_{\tau_{w_0},2} - b_{\tau_{w_0},3})}$, $\overline{z}' = \frac{(b_{\tau_{w_0},2} - b_{\tau_{w_0},3}) - (b_{\tau_{\beta\alpha},2} - b_{\tau_{\beta\alpha},3})}{p} + (b_{\tau_{w_0},2} - b_{\tau_{w_0},3})$ and $\overline{z}'' \equiv \frac{(b_{\tau_{w_0},3} - b_{\tau_{w_0},1}) - (b_{\tau_{\beta\alpha},3} - b_{\tau_{\beta\alpha},1})}{p} + (b_{\tau_{w_0},3} - b_{\tau_{w_0},1})$ modulo p, but we will not need this fact.

Table 7. **Special fiber for the algebraic multi-type deformation ring when $\#T^{(j)}=2$.**

$\overline{A}^{T,(j)}\widetilde{w}^*(\overline{\rho})$	$\begin{pmatrix} d_{11}^* & c_{12} & c_{13}v+e_{13} \\ d_{21} & d_{22}^*v+c_{22} & c_{23}v+e_{23} \\ d_{31} & d_{32}v+c_{32} & d_{33}^*v^2+c_{33}v+e_{33}\end{pmatrix}\cdot s_j^{-1}v^{\mu_j+(1,0,-1)}$
$S^{(j)}$	$\mathbb{F}[\![x_{11}^*, c_{12}, c_{13}, e_{13}, d_{21}, x_{22}^*, c_{22}, c_{23}, e_{23}, d_{31}, d_{32}, c_{32}, x_{33}^*, c_{33}, e_{33}]\!]$
$T^{(j)}$	some elements of $I^{(j)}_{T,\nabla_{\mathrm{alg}}}$
$t_{w_0(\eta)}, t_{w_0(\eta)}\alpha$	$e_{33},\ c_{33},\ c_{32},\ d_{32},\ d_{31},\ d_{21}c_{22},\ e_{13}d_{21}-e_{23}d_{11}^*,\ c_{12}d_{21}-c_{22}d_{11}^*,$ $(a-b)(c_{13}c_{22}-c_{12}c_{23})-\overline{x}c_{12}e_{23}+(a-c)e_{13}d_{22}^*,\ \exists\overline{x}\in\mathbb{F}_p$
$t_{w_0(\eta)}, t_{w_0(\eta)}\beta$	$e_{33},\ c_{32},\ d_{31},\ e_{23},\ c_{22},\ d_{21},\ d_{32}c_{33},\ c_{23}d_{32}-c_{33}d_{22}^*,$ $\overline{z}'c_{33}c_{12}c_{23}+\overline{z}''c_{33}e_{13}d_{22}^*-(a-b)c_{12}c_{23}d_{33}^*+(a-c)e_{13}d_{22}^*d_{33}^*,\ \exists\overline{z}',\overline{z}''\in\mathbb{F}_p$
	some elements of $I^{(j)}_{\{t_{w_0(\eta)},t_{w_0(\eta)}\alpha\},\nabla_{\mathrm{alg}}}\cap I^{(j)}_{\{t_{w_0(\eta)},t_{w_0(\eta)}\beta\},\nabla_{\mathrm{alg}}}$
	$e_{33},\ c_{32},\ d_{31},\ d_{32}c_{33},\ c_{23}d_{32}-c_{33}d_{22}^*,$ $d_{21}d_{32},\ d_{21}c_{22},\ e_{13}d_{21}-e_{23}d_{11}^*,\ c_{12}d_{21}-c_{22}d_{11}^*,$ $\overline{z}'c_{33}c_{12}c_{23}+\overline{z}''c_{33}e_{13}d_{22}^*+(a-b)d_{33}^*(c_{13}c_{22}-c_{12}c_{23})+$ $-\overline{x}c_{12}e_{23}d_{33}^*+(a-c)e_{13}d_{22}^*d_{33}^*$

In this table we give an explicit presentation of the ring $S/I_{T,\nabla_{\mathrm{alg}}}$ in some cases when $\#T^{(j)}=2$ (see Proposition 3.19, and Lemma 3.32) and of the ideal $I^{(j)}_{\{t_{w_0(\eta)},t_{w_0(\eta)}\alpha\}\nabla_{\mathrm{alg}}}\cap I^{(j)}_{\{t_{w_0(\eta)},t_{w_0(\eta)}\beta\}\nabla_{\mathrm{alg}}}$ needed for Lemma 3.36. We have set $(a,b,c)\overset{\mathrm{def}}{=}-\big(s_j^{-1}(\mu_j+\eta_j)-w_0(\eta_j)\big)$. It can be shown that $\overline{x}=\frac{1}{p}\frac{(b_{\alpha,1}-b_{\alpha,3})(b_{\mathrm{id},1}-b_{\mathrm{id},2})-(b_{\mathrm{id},1}-b_{\mathrm{id},3})(b_{\alpha,1}-b_{\alpha,2})}{(b_{\mathrm{id},1}-b_{\mathrm{id},2})}$, $\overline{z}'=\frac{1}{p}(b_{\beta,1}-b_{\beta,2}-(p+1)(b_{\mathrm{id},1}-b_{\mathrm{id},2}))$ and $\overline{z}''=\frac{1}{p}((p+1)(b_{\mathrm{id},1}-b_{\mathrm{id},3})-(b_{\beta,1}-b_{\beta,3}))$, where we have set $b_{\mathrm{id},i}\overset{\mathrm{def}}{=}b_{\tau_{t_{w_0(\eta)}},i}$, $b_{\alpha,i}\overset{\mathrm{def}}{=}b_{\tau_{t_{w_0(\eta)}\alpha},i}$ and $b_{\beta,i}\overset{\mathrm{def}}{=}b_{\tau_{t_{w_0(\eta)}\beta},i}$ for readability ($i\in\{1,2,3\}$).

TABLE 8. **Minimal prime ideals of $S/I_{T,\nabla_{\mathrm{alg}}}$ when $\#T \leq 4$ and $T^{(j)} \subseteq \{\alpha\beta\alpha\gamma t_{\underline{1}}, \alpha\beta\alpha t_{\underline{1}}, \beta\alpha t_{\underline{1}}, \alpha\beta t_{\underline{1}}, t_{\underline{1}}\}$**

$(\eta_j, a_j) \in r(\Sigma_0)$	$\mathfrak{P}_{(\varepsilon_j, a_i)}$
$(0,0)$	$e_{11},\ c_{33},\ c_{32},\ c_{31},\ c_{23},\ c_{22},\ c_{21},\ c_{13},\ c_{12},\ c_{11}$
$(\varepsilon_1, 0)$	$e_{11},\ c_{33},\ c_{32},\ d_{32},\ c_{31},\ d_{31},\ c_{22},\ c_{21},\ c_{12},\ c_{11}$
$(\varepsilon_2, 0)$	$e_{11},\ c_{33},\ c_{32},\ c_{31},\ d_{31},\ c_{23},\ c_{22},\ c_{21},\ d_{21},\ c_{11}$
$(0,1)$	$e_{11},\ c_{33},\ c_{32},\ c_{31},\ c_{23},\ c_{22},\ c_{21},\ c_{13}d_{32} - c_{12}d_{33}^*,\ c_{13}d_{31} - c_{11}d_{33}^*,$ $c_{12}d_{31} - c_{11}d_{32},\ (b-c)d_{21}d_{32} - (a-c)d_{31}d_{22}^*,\ (b-c)c_{12}d_{21} - (a-c)c_{11}d_{22}^*$
$(\varepsilon_1, 1)$	$e_{11},\ c_{33},\ c_{32},\ c_{31},\ d_{31},\ c_{21},\ c_{11},\ c_{13}c_{22} - c_{12}c_{23},\ c_{13}d_{21} - c_{23}d_{11}^*,\ c_{12}d_{21} - c_{22}d_{11}^*,$ $(a-c-1)c_{23}d_{32} - (a-b-1)c_{22}d_{33}^*,\ (a-c-1)c_{13}d_{32} - (a-b-1)c_{12}d_{33}^*$
$(\varepsilon_2, 1)$	$e_{11},\ c_{32},\ c_{31},\ c_{22},\ c_{21},\ c_{12},\ c_{11},\ c_{23}d_{32} - c_{33}d_{22}^*,\ d_{21}d_{32} - d_{31}d_{22}^*,\ c_{23}d_{31} - d_{21}c_{33},$ $(a-b)c_{13}d_{31} + (b-c-1)c_{33}d_{11}^*,\ (a-b)c_{13}d_{21} + (b-c-1)c_{23}d_{11}^*$
$(\varepsilon_2 - \varepsilon_1, 1)$	$e_{11},\ c_{33},\ c_{32},\ d_{32},\ c_{31},\ c_{22},\ c_{13},\ c_{12},\ c_{11},$ $(b-c-1)c_{23}d_{31} + (a-b+1)c_{21}d_{33}^*$
$(\varepsilon_1 - \varepsilon_2, 1)$	$e_{11},\ c_{33},\ c_{31},\ c_{23},\ c_{22},\ c_{21},\ d_{21},\ c_{13},\ c_{11},$ $(a-c)c_{12}d_{31} - (b-c+1)c_{32}d_{11}^*$

We give the presentation of the minimal prime ideals for $S/I_{T,\nabla_{\mathrm{alg}}}$ when $T^{(j)} = \{\alpha\beta\alpha\gamma t_{\underline{1}}, \alpha\beta\alpha t_{\underline{1}}, \beta\alpha t_{\underline{1}}, \alpha\beta t_{\underline{1}}, t_{\underline{1}}\}$, using the parametrization of Proposition 3.19. We have set $(a,b,c) \overset{\mathrm{def}}{=} -\big(s_j^{-1}(\mu_j + \eta_j) - (1,1,1)\big)$.

TABLE 9. **Minimal prime ideals of $S/I_{T,\nabla_{\mathrm{alg}}}$ when $\#T \leq 4$ and $T^{(j)} \subseteq \{t_{w_0(\eta)}, t_{w_0(\eta)}\alpha, t_{w_0(\eta)}\beta, t_{w_0(\eta)}w_0\}$**

$(\eta_j, a_j) \in r(\Sigma_0)$	$\mathfrak{P}_{(\varepsilon_j, a_i)}$
$(0,0)$	$e_{33},\ c_{33},\ c_{32},\ e_{23},\ c_{23},\ c_{22},\ e_{13},\ c_{13},\ c_{12}$
$(\varepsilon_1, 0)$	$e_{33},\ c_{32},\ d_{31},\ d_{32},\ c_{33},\ c_{22},\ c_{12},\ e_{13},\ e_{23}$
$(\varepsilon_2, 0)$	$e_{33},\ c_{33},\ c_{32},\ d_{31},\ e_{23},\ c_{23},\ c_{22},\ d_{21},\ e_{13}$
$(\varepsilon_1, 1)$	$e_{33},\ c_{32},\ e_{23},\ c_{22},\ e_{13},\ c_{12},\ d_{21}c_{33} - c_{23}d_{31},$ $d_{21}d_{32} - d_{31}d_{22}^*,\ c_{23}d_{32} - c_{33}d_{22}^*$ $(a-b+1)c_{13}d_{31} + (b-c)c_{33}d_{11}^*,\ (a-b+1)c_{13}d_{21} + (b-c)c_{23}d_{11}^*$
$(\varepsilon_2, 1)$	$e_{33},\ c_{33},\ c_{32},\ d_{31},\ e_{23},\ e_{13},\ c_{12}c_{23} - c_{13}c_{22}$ $c_{12}d_{21} - c_{22}d_{11}^*,\ c_{13}d_{21} - c_{23}d_{11}^*$ $(a-c+1)c_{23}d_{32} - (a-b)c_{22}d_{33}^*,\ (a-c+1)c_{13}d_{32} - (a-b)c_{12}d_{33}^*$
$(\varepsilon_1 + \varepsilon_2, 1)$	$e_{33},\ c_{32},\ d_{31},\ e_{23},\ d_{32},\ c_{33},\ d_{21},\ c_{22},$ $(a-b)c_{12}c_{23} - (a-c)e_{13}d_{22}^*$

We give the presentation of the minimal prime ideals for $S/I_{T,\nabla_{\mathrm{alg}}}$ when $T^{(j)} = \{t_{w_0(\eta)}, t_{w_0(\eta)}\alpha, t_{w_0(\eta)}\beta, t_{w_0(\eta)}w_0\}$, using the parametrization of Proposition 3.19. We have set $(a,b,c) \stackrel{\mathrm{def}}{=} -\big(s_j^{-1}(\mu_j + \eta_j) - w_0(\eta_j)\big)$.

References

[AK11] Henning Haahr Andersen and Masaharu Kaneda, *Rigidity of tilting modules*, Mosc. Math. J. **11** (2011), no. 1, 1–39 (English).

[BDM15] C. Bowman, S. R. Doty, and S. Martin, *Decomposition of tensor products of modular irreducible representations for* SL_3*: the* $p \geq 5$ *case*, Int. Electron. J. Algebra **17** (2015), 105–138. MR 3310689

[BHH$^+$21] Christophe Breuil, Florian Herzig, Yongquan Hu, Stefano Morra, and Benjamin Schraen, *Conjectures and results on modular representations of* $\mathrm{GL}_n(K)$ *for a* p*-adic field* K, `https://arxiv.org/pdf/2102.06188.pdf`, 2021, preprint.

[BHH$^+$23] ———, *Gelfand-Kirillov dimension and mod p cohomology for* GL_2, Invent. Math. **234** (2023), no. 1, 1–128 (English).

[BP13] C. Breuil and V. Paskunas, *Towards a modulo p langlands correspondence for* $\mathbf{GL}_2$, vol. 2016, Memoirs of Amer. Math. Soc., 2013.

[CEG+16] Ana Caraiani, Matthew Emerton, Toby Gee, David Geraghty, Vytautas Paškūnas, and Sug Woo Shin, *Patching and the p-adic local Langlands correspondence*, Camb. J. Math. **4** (2016), no. 2, 197–287. MR 3529394

[CHT08] Laurent Clozel, Michael Harris, and Richard Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations*, Publ. Math. Inst. Hautes Études Sci. (2008), no. 108, 1–181, With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras. MR 2470687 (2010j:11082)

[EG14] Matthew Emerton and Toby Gee, *A geometric perspective on the Breuil-Mézard conjecture*, J. Inst. Math. Jussieu **13** (2014), no. 1, 183–223. MR 3134019

[EG21] ———, *"Scheme-theoretic images" of morphisms of stacks*, Algebr. Geom. **8** (2021), no. 1, 1–132 (English).

[EG23] ———, *Moduli stacks of étale* (Φ,Γ)*-modules and the existence of crystalline lifts*, Ann. Math. Stud., vol. 215, Princeton, NJ: Princeton University Press, 2023 (English).

[EGS15] Matthew Emerton, Toby Gee, and David Savitt, *Lattices in the cohomology of Shimura curves*, Invent. Math. **200** (2015), no. 1, 1–96. MR 3323575

[GHS18] Toby Gee, Florian Herzig, and David Savitt, *General Serre weight conjectures*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 12, 2859–2949. MR 3871496

[Her09] Florian Herzig, *The weight in a Serre-type conjecture for tame n-dimensional Galois representations*, Duke Math. J. **149** (2009), no. 1, 37–116. MR 2541127 (2010f:11083)

[HLM17] Florian Herzig, Daniel Le, and Stefano Morra, *On* mod p *local-global compatibility for* GL_3 *in the ordinary case*, Compos. Math. **153** (2017), no. 11, 2215–2286. MR 3705291

[Hum06] James E. Humphreys, *Modular representations of finite groups of Lie type*, London Mathematical Society Lecture Note Series, vol. 326, Cambridge University Press, Cambridge, 2006. MR 2199819 (2007f:20023)

[HW18] Yongquan Hu and Haoran Wang, *Multiplicity one for the* mod$p$ *cohomology of Shimura curves: the tame case*, Math. Res. Lett. **25** (2018), no. 3, 843–873 (English).

[HW22] ———, *On the* mod$p$ *cohomology for* GL_2*: the non-semisimple case*, Camb. J. Math. **10** (2022), no. 2, 261–431. MR 4461834

[Jan03] Jens Carsten Jantzen, *Representations of algebraic groups*, second ed., Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, Providence, RI, 2003. MR 2015057 (2004h:20061)

[Le] Daniel Le, *Lattices in the cohomology of* $U(3)$ *arithmetic manifolds*, `http://arxiv.org/abs/1507.04766`, preprint (2015).

[Le19] ———, *Multiplicity one for wildly ramified representations*, Algebra Number Theory **13** (2019), no. 8, 1807–1827. MR 4017535

[LHL19] Daniel Le, Bao V. Le Hung, and Brandon Levin, *Weight elimination in Serre-type conjectures*, Duke Math. J. **168** (2019), no. 13, 2433–2506 (English).

[LLHLMa] Daniel Le, Bao Viet Le Hung, Brandon Levin, and Stefano Morra, *Extremal weights and a tameness criterion for mod p Galois representations*, `https://arxiv.org/pdf/2206.06442.pdf`, preprint (2022).

[LLHLMb] ———, *Serre weights for three-dimensional wildly ramified Galois representations*, Algebra & Number Theory, (to appear).

[LLHLM18] ———, *Potentially crystalline deformation rings and Serre weight conjectures: shapes and shadows*, Invent. Math. **212** (2018), no. 1, 1–107. MR 3773788

[LLHLM20] ———, *Serre weights and Breuil's lattice conjectures in dimension three*, Forum Math. Pi **8** (2020), e5, 135. MR 4079756

[LLHLM23] Daniel Le, Bao V. Le Hung, Brandon Levin, and Stefano Morra, *Local models for Galois deformation rings and applications*, Invent. Math. **231** (2023), no. 3, 1277–1488 (English).

[LLHM] Daniel Le, Bao Viet Le Hung, and Stefano Morra, *Appendix B to* K_1*-invariants in the mod p cohomology of* $U(3)$ *arithmetic manifolds*, `https://www.math.univ-paris13.fr/~morra/AppendixB.pdf`, preprint (2024).

[LMS22] Daniel Le, Stefano Morra, and Benjamin Schraen, *Multiplicity one at full congruence level*, J. Inst. Math. Jussieu **21** (2022), no. 2, 637–658 (English).

[Mat86] Hideyuki Matsumura, *Commutative ring theory. Transl. from the Japanese by M. Reid*, Camb. Stud. Adv. Math., vol. 8, Cambridge University Press, Cambridge, 1986 (English).

[Pas04] V. Paskunas, *Coefficient systems and supersingular representations of* $\mathrm{GL}_2(F)$, Mém. Soc. Math. Fr. (N.S.) (2004), no. 99, vi+84. MR 2128381 (2005m:22017)

[SS97] P. Schneider and U. Stuhler, *Representation theory and sheaves on the Bruhat-Tits building*, Inst. Hautes Études Sci. Publ. Math. (1997), no. 85, 97–191. MR 1471867 (98m:22023)

[Sta23] The Stacks project authors, *The stacks project*, https://stacks.math.columbia.edu, 2023.

[Yeh] Sami El Badawi Yehia, *Extensions of simple modules for the universal Chevalley groups and its parabolic subgroups*, available at https://wrap.warwick.ac.uk/106601/1/WRAP_Theses_Yehia_1982.pdf, Ph. D. thesis, University of Warwick (1982).

Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907-2067

Email address: ledt@purdue.edu

Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208, USA

Email address: lhvietbao@googlemail.com

LAGA, UMR 7539, CNRS, Université Paris 13 - Sorbonne Paris Cité, Université de Paris 8, 99 avenue Jean Baptiste Clément, 93430 Villetaneuse, France

Email address: morra@math.univ-paris13.fr