IWAHORI – HECKE ALGEBRAS + MOD p LOCAL LANGLANDS PROGRAM

I

GOAL: EXPLAIN HOW HECKE ALGS ARISE IN STUDY OF LLC, THEIR STRUCTURE + PROPERTIES

TODAY: MOTIVATION + BACKGROUND

FIX A PRIME p, LET $\mathbb{Q}_p$ DENOTE THE FIELD OF p-ADIC #S.

"CLASSICAL" LOCAL LANGLANDS CORR GIVES A BIJECTION

ALG COND'N

$$\left\{ \begin{array}{l} \text{SMOOTH, IRRED REPS} \\ \quad V \quad \text{OF } GL_n(\mathbb{Q}_p) \\ \text{ON } \mathbb{C}\text{-V.S.} \end{array} \right\} \Big/ \cong \quad \xleftrightarrow{1:1} \quad \left\{ \begin{array}{l} \text{FROB SEMISIMPLE} \\ \text{WEIL-DELIGNE REPS} \\ \text{OF DIM } n \ /\mathbb{C} \end{array} \right\} \Big/ \cong$$

WHERE:

↳ "SMOOTH" = $\forall v \in V$, $\mathrm{STAB}_{GL_n(\mathbb{Q}_p)}(v)$ IS OPEN

↳ WEIL-DELIGNE REP $\overset{\text{CTS}}{\longleftrightarrow}$ ℓ-ADIC REP OF $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$

PRECISELY:

"WEIL GP"

$$\begin{array}{ccc} W_{\mathbb{Q}_p} & \xrightarrow{\text{VAL}} & \langle \text{FROB} \rangle \cong \mathbb{Z} \\ \downarrow & \square & \downarrow \\ \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) & \longrightarrow & \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \end{array}$$

WD REP: $r : W_{\mathbb{Q}_p} \longrightarrow GL_n(\mathbb{C})$ SMOOTH

$+ \ N : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ NILPOTENT S.T.

$$r(\gamma) N r(\gamma)^{-1} = p^{-\mathrm{val}(\gamma)} N \qquad \forall \gamma \in W_{\mathbb{Q}_p}$$

↳ FOR $n=1$, THIS IS LOCAL CLASS FIELD THEORY

↳ UNIQUELY DET'D BY MATCHING L-FNS & ε-FACTORS

↳ PROVED BY HARRIS-TAYLOR, HENNIART, SCHOLZE

↳ ALSO WORKS FOR EXTENSIONS OF $\mathbb{Q}_p$

↳ LLC SATISFIES INTRICATE "FINE" PROPERTIES $\Longleftrightarrow$ HIGHER RAM'N IN LCFT

GENERALIZE:

FOR SIMPLICITY

LET G = SPLIT p-ADIC REDUCTIVE GP

= GRP OF $\mathbb{Q}_p$-POINTS OF A SPLIT REDUCTIVE ALG GP $\mathbb{G}/\mathbb{Q}_p$

E.G. $G = GL_n(\mathbb{Q}_p)$, $Sp_{2n}(\mathbb{Q}_p)$, $SO_n(\mathbb{Q}_p)$, $E_8(\mathbb{Q}_p)$, ...

WANT A MAP

IMPRECISE! NEED TO ADD CONDITIONS (SHOULD BE OK W/ "TEMPERED")

$$(*)\quad \left\{\begin{array}{c}\text{SMOOTH IRREPS}\\ \text{OF } G \text{ ON}\\ \mathbb{C}\text{-V.S.}\end{array}\right\}\Big/\cong \longrightarrow \left\{\begin{array}{c}\text{ADMISSIBLE WD REPS}\\ W_F \to \widehat{G}(\mathbb{C})\end{array}\right\}\Big/\cong$$

"LANGLANDS DUAL GP"

W/ FINITE FIBERS, SATISFYING "NATURAL" COMPATIBILITIES (DESCRIPTION OF FIBERS, ETC)

↳ PRE-2021:

(G)Sp_4 : GAN-TAKEDA

CLASSICAL GPS* : ARTHUR

UNITARY GPS : MOK, KALETHA-MINGUEZ-SHIN-WHITE

↳ 2021 : FARGUES-SCHOLZE : CONSTRUCTION OF MAP FOR ALL RED'VE GPS G

CAN BE HARD TO QUANTIFY WHAT "NATURAL" MEANS (SEE HARRIS' SURVEY)

EXISTENCE OF MAP OFTEN DOESN'T GIVE FINER INFORMATION

// HOW TO OBTAIN PRECISE INFORMATION ABOUT CORRESPONDENCE ($*$)?

"STRATIFY" LHS

↳ ONE APPROACH : CONSTRUCT ($*$) FIRST FOR SUPERCUSPIDALS, THEN TRY TO EXTEND VIA PARABOLIC INDUCTION : DEBACKER-REEDER, KALETHA

↳ LOOK AT ACTION OF SMALLER & SMALLER COMPACT OPEN SUBGPS IE, PROVE ($*$) FOR A CLASS OF REPS $\forall G$, NOT FOR A FIXED G

"LEVEL 0" LET $K = G(\mathbb{Z}_p)$ $= \mathbb{Z}_p$-PTS OF G

FOR SPLIT GPS, A $\mathbb{Z}_p$-MODEL OF G ALWAYS EXISTS

EG : $K = GL_n(\mathbb{Z}_p), Sp_{2n}(\mathbb{Z}_p), E_8(\mathbb{Z}_p)\ldots$

↳ K IS A MAXIMAL COMPACT OPEN SUBGP OF G

CONSIDER THE FOLLOWING SUBSET OF LHS OF (*)

$$\left\{ \begin{matrix} \text{SMOOTH IRREPS } V \\ \text{OF } G \text{ S.T.} \\ V^K \neq 0 \end{matrix} \right\} \Big/ \cong$$

$$\left(V^K = \{ v \in V : k \cdot v = v \ \forall k \in K \} \quad \text{“}K\text{-FIXED VECTORS”} \right)$$

EX THIS SET CONTAINS

- TRIVIAL REP:

$$V = \mathbb{C} \quad w/ \quad g \cdot v = v \quad \forall g \in G, \ v \in V$$

- UNRAMIFIED PRINCIPAL SERIES:

$$(G = GL_2(\mathbb{Q}_p)) \qquad V = \mathcal{C}^{\text{LOC CST}}(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{C}) \quad w/$$

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f \right)([X:Y]) = f([dX - bY : -cX + aY])$$

$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q}_p), \quad f \in V$$

FACT 1 V^K HAS A “RESIDUAL ACTION” OF

$$\mathcal{H}(G,K) = \left\{ f : G \to \mathbb{C} : \begin{array}{l} \cdot \ f(kgk') = f(g) \ \forall g \in G, \ k,k' \in K \\ \cdot \ f \text{ HAS CPCT SUPP} \end{array} \right\}$$

“SPHERICAL HECKE ALGEBRA”

+ CONVOLUTION PRODUCT (WILL DESCRIBE THIS EXPLICITLY LATER)

FACT 2 CAN DESCRIBE ALGEBRA STRUCTURE OF $\mathcal{H}(G,K)$ EXPLICITLY: SATAKE ISOMORPHISM GIVES

$$\mathcal{H}(G,K) \xrightarrow{\sim} \mathcal{H}(T, T\cap K)^W \cong \mathbb{C}[T/T\cap K]^W \cong \mathbb{C}[X_*(T)]^W$$

$\mathbb{C}$-ALG HOM

W: WEYL GP OF T

T: MAX'L TORUS OF G

E.G.

$$T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

FOR $GL_n, Sp_{2n}, \ldots$

$X_*(T)$: COCHARACTER GP OF T $= \mathrm{Hom}(\mathbb{G}_m, T)$

IN PART'R, $\mathcal{H}(G,K)$ IS COMM.

FACT 3 IF V IRRED + $V^K \neq 0$, THEN $\dim(V^K) = 1$

SO

IRRED V w/ $V^K \neq 0$ $\longleftrightarrow$ V^K w/ $\mathcal{H}(G,K)$-ACTION

$\longleftrightarrow$ $\mathbb{C}$-ALG HOM $\mathcal{H}(G,K) \to \mathbb{C}$

$\longleftrightarrow$ $\mathbb{C}$-ALG HOM $\mathbb{C}[X_*(T)]^W \to \mathbb{C}$

$\longleftrightarrow$ ELT t OF $\hat{T}(\mathbb{C})/W$

$\hat{T}$: DUAL TORUS $\subset \hat{G}$

$$\begin{aligned}
\hat{T}(\mathbb{C}) &= X_*(\hat{T}) \otimes_{\mathbb{Z}} \mathbb{C}^\times \\
\lambda(\alpha) &\longleftarrow\!\shortmid \lambda \otimes \alpha \\
&= X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times \\
&= \mathrm{Hom}_{\mathbb{Z}}(X_*(T), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^\times \\
&= \mathrm{Hom}_{\mathbb{Z}}(X_*(T), \mathbb{C}^\times) \\
&= \mathrm{Hom}_{\mathbb{C}\text{-ALG}}(\mathbb{C}[X_*(T)], \mathbb{C})
\end{aligned}$$

$\longleftrightarrow$ $r : W_p \twoheadrightarrow \langle \text{FROB} \rangle \longrightarrow \hat{G}(\mathbb{C})$

$\text{FROB} \longmapsto t$

W-D UP TO $\hat{G}$-CONJ

THIS GIVES AN INSTANCE OF THE <u>UNRAMIFIED</u> LLC

"LEVEL 1"

NOW LET I = IWAHORI SUBGP OF G

= PREIMAGE UNDER $G(\mathbb{Z}_p) \twoheadrightarrow G(\mathbb{F}_p)$ OF BOREL SUBGP

E.G. IF $G = GL_n$, THEN

$$I = \begin{pmatrix} \mathbb{Z}_p^\times & & \mathbb{Z}_p \\ & \ddots & \\ p\mathbb{Z}_p & & \mathbb{Z}_p^\times \end{pmatrix} \equiv \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \bmod p$$

(SUBGP OF K)

NOW CONSIDER

$$\left\{ \begin{array}{c} \text{SMOOTH IRREPS } V \\ \text{OF } G \text{ S.T.} \\ V^I \neq 0 \end{array} \right\} \Big/ \cong$$

SOMETIMES CALLED "IWAHORI - SPHERICAL"

CONTAINS PREVIOUS REPS + MORE:

$$\mathcal{C}^{\text{LOC CST}}(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{C}) \,/\, \{\text{CST FNS}\} \circlearrowleft GL_2(\mathbb{Q}_p)$$

IS CONTAINED ABOVE, BUT NOT IN K - SPHERICAL REPS

<u>FACT</u> AS BEFORE, V^I OBTAINS A RESIDUAL ACTION OF

$$\mathcal{H}(G, I) = \left\{ f : G \to \mathbb{C} : \begin{array}{l} \cdot\ f(igi') = f(g) \\ \quad \forall g \in G,\ i, i' \in I \\ \cdot\ f \text{ COMPACT SUPP} \end{array} \right\}$$

+ CONVOLUTION PRODUCT (MORE LATER)

"IWAHORI – HECKE ALGEBRA"

N.B. $\mathcal{H}(G, I)$ IS NO LONGER COMMUTATIVE, BUT WE CAN STILL EXPLICITLY DESCRIBE ITS STRUCTURE (NEXT TIME)

MOREOVER, IF V IRRED AND $V^I \neq 0$, THEN

V^I IS A SIMPLE $\mathcal{H}(G, I)$-MOD

USING IDEAS FROM GEOMETRIC REP THEORY, (K-HOMOLOGY OF STEINBERG VARIETY OF TRIPLES) KAZHDAN-LUSZTIG CLASSIFIED SIMPLE $\mathcal{H}(G, I)$-MODS AND SHOWED THEY ARE NATURALLY IN BIJECTION w/ CERTAIN WEIL-DELIGNE REPS (ONLY FOR GPS w/ CONNECTED CENTER; W-D REPS WHICH ARE UNRAMIFIED ON W_p, BUT N CAN BE ANYTHING COMPATIBLE w/ $r(\text{FROB})$)

SO, IN BOTH CASES (K AND I), PASSING FROM REPS TO MODULES OVER $\mathcal{H}(G, K)$ OR $\mathcal{H}(G, I)$ ALLOWS US TO EXPLICITLY CONSTRUCT INSTANCES OF LLC (FOR "ALL" G AT ONCE)

EVEN BETTER: FOR THE IWAHORI, THE BIJECTION B/W IRREPS V w/ $V^I \neq 0$ AND SIMPLE $\mathcal{H}(G,I)$-MODS IS INDUCED FROM AN EQUIVALENCE OF CATS:

THM (BOREL, BERNSTEIN)

EQUIV OF CATS

$$\begin{aligned} \mathrm{Rep}_{\mathbb{C}}^{I}(G) := \{ V \in \mathrm{Rep}_{\mathbb{C}}(G) : \langle G \cdot V^I \rangle = V \} &\xrightarrow{\sim} \mathcal{H}(G,I)\text{-Mod} \\ V &\longmapsto V^I \end{aligned}$$

EQUIV MATCHES SIMPLE OBJECTS ON BOTH SIDES

- GIVES INFO ABOUT THE <u>CATEGORY</u> $\mathrm{Rep}_{\mathbb{C}}(G)$
- <u>NOT</u> TRUE FOR K

"<u>HIGHER LEVEL</u>": LEADS TO THEORY OF TYPES OF BUSHNELL-KUTZKO

II LAST TIME: MOTIVATION

TODAY: STRUCTURE OF $\mathcal{H}(G, I)$

NOT ESSENTIAL, ONLY FOR SIMPLICITY OF EXPOSITION

<u>RECALL</u> $G = \mathbb{G}(\mathbb{Q}_p)$ FOR $\mathbb{G}$ A SPLIT CONN. SEMISIMPLE SIMPLY CONNECTED GP $/\mathbb{Q}_q$ (E.G. SL_n)

T = $\mathbb{Q}_p$-SPLIT MAX'L TORUS

$q := p$

$B = TU$ = BOREL SUBGP

I = IWAHORI SUBGP

= PREIMAGE OF $\mathbb{B}(\mathbb{F}_p)$ UNDER

$\mathbb{G}(\mathbb{Z}_p) \twoheadrightarrow \mathbb{G}(\mathbb{F}_p)$

(THINK $I = \begin{pmatrix} \mathbb{Z}_q^\times & \mathbb{Z}_q \\ p\mathbb{Z}_q & \mathbb{Z}_q^\times \end{pmatrix}$)

$N = N_G(T)$ = NORMALIZER OF T

<u>GROUP-THEORETIC FACTS</u> (BRUHAT-TITS, IWAHORI-MATSUMOTO)

THE PAIR (I, N) IS A BN PAIR. MORE PRECISELY:

- $\widetilde{W} := N/N\cap I$ IS A COXETER GROUP, WITH GENS $\widetilde{S}$ + LENGTH FN ℓ "AFFINE WEYL GROUP"
- THE S.E.S.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T/T\cap I & \longrightarrow & N/N\cap I & \longrightarrow & N/T & \longrightarrow & 1 \\ & & \cong & & \| & & \| & & \\ & & X_*(T) & & \widetilde{W} & & W & & \end{array}$$

SPLITS, GIVING

$$\widetilde{W} \cong W \ltimes X_*(T)$$

- (BRUHAT DECOMPOSITION)
$$G = \bigsqcup_{w \in \widetilde{W}} I \dot{w} I$$
- IF $w, w' \in \widetilde{W}$ SATISFY $\ell(ww') = \ell(w) + \ell(w')$, THEN
$$I \dot{w} I \dot{w}' I = I \dot{w} \dot{w}' I$$
- IF $s \in \widetilde{S}$, THEN
$$I \dot{s} I \dot{s} I = I \dot{s} I \sqcup I$$
- $[I \dot{w} I : I] := |I w I / I| = q^{\ell(w)}$

EX FOR $G = SL_n(\mathbb{Q}_p)$,

$W \cong S_n$ = SYMMETRIC GP ON n LETTERS

W/ GENS

$$s_1 = \begin{pmatrix} \boxed{\begin{matrix} & 1 \\ -1 & \end{matrix}} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & & & & \\ & \boxed{\begin{matrix} & 1 \\ -1 & \end{matrix}} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \dots, s_{n-1} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \boxed{\begin{matrix} & 1 \\ -1 & \end{matrix}} \end{pmatrix}$$

$$\widetilde{W} = W \ltimes X_*(T)$$
$$\cong W \ltimes (T/T(\mathcal{O}))$$

NATURAL ACTION

$$= \text{“AFFINE WEYL GP OF TYPE } \widetilde{A}_{n-1}\text{”}$$

W/ GENS $\underbrace{s_1, \dots, s_{n-1}, s_0}_{\tilde{S}} = \begin{pmatrix} & & & -p^{-1} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ p & & & \end{pmatrix}$

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RECALL

$$\mathcal{H} := \mathcal{H}(G, I) = \left\{ f : G \to \mathbb{C} : \begin{array}{l} \bullet\ f \text{ HAS CPCT SUPP} \\ \bullet\ f(igi') = f(g) \\ \quad \forall i, i' \in I,\ g \in G \end{array} \right\}$$

$\Rightarrow$ BY BRUHAT DECOMP, $\mathcal{H}$ HAS A BASIS

$$\{ T_w := \mathbb{1}_{IwI} \}_{w \in \tilde{W}}$$

THE PRODUCT IS GIVEN BY CONVOLUTION: FOR $f, f' \in \mathcal{H}$,

$$(f * f')(g) = \int_G f(h) f'(h^{-1}g)\, d\mu(h)$$

$\mu(I) = 1$

$$= \sum_{h \in G/I} f(h) f'(h^{-1}g)$$

BY I-BI-INVARIANCE

FINITE SUM BY CPCT SUPP

N.B. THE PRODUCT MAKES SENSE FOR <u>ANY</u> COEFF RING R
(w/ FNS $f : G \longrightarrow R$)

WHAT IS THE PRODUCT ON THE BASIS T_w ?

LEMMA (IWAHORI–MATSUMOTO)

a) IF $w, w' \in \widetilde{W}$ SATISFY $\ell(ww') = \ell(w) + \ell(w')$, THEN

$$T_w T_{w'} = T_{ww'}$$ "BRAID RELN"

b) IF $s \in \widetilde{S}$, THEN

$$T_s^2 = (q-1) T_s + q T_1$$ "QUADRATIC RELN"

T_1: IDENTITY ELT OF $\mathcal{H}$

SKETCH

$$(T_w T_{w'})(g) = \sum_{h \in G/I} T_w(h)\, T_{w'}(h^{-1} g)$$

LHS $\neq 0$ $\Rightarrow$ SOME SUMMAND $\neq 0$

$\Rightarrow$ $h \in IwI$, $h^{-1}g \in Iw'I$

$\Rightarrow$ $g \in hIw'I \subset IwIw'I$

- IF $\ell(ww') = \ell(w) + \ell(w')$, $g \in Iww'I$

• IF $s \in \tilde{S}$, $g \in IsI \sqcup I$

SINCE $T_w T_{w'}$ IS I-BI-INV'T, SUFFICES TO TAKE $g' = ww'$, s, 1

E.G.

FOR $G = SL_2(\mathbb{Q}_p)$, $s = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, $I = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$,

WE HAVE

$$IsI/I = \bigsqcup_{x \in \mathbb{Z}_p/p\mathbb{Z}_p} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} s$$

SIZE $q = p$

So

$$(T_s^2)(1) = \sum_{h \in G/I} T_s(h)\, T_s(h^{-1})$$

$$= \sum_{h \in IsI/I} T_s(h^{-1})$$

$$= q$$

$$(T_s^2)(s) = \sum_{h \in IsI/I} T_s(h^{-1} s)$$

$$= \sum_{x \in \mathbb{Z}/p} T_s\left(s^{-1} \begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix} s\right)$$

$$= \cancel{T_s(1)} + \sum_{x \in (\mathbb{Z}/p)^\times} T_s\left(\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}\right)$$

$$= \sum_{x \neq 0} T_s\left(\underbrace{\begin{pmatrix} 1 & x^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} -x^{-1} & \\ & -x \end{pmatrix}}_{\in I} s \underbrace{\begin{pmatrix} 1 & x^{-1} \\ & 1 \end{pmatrix}}_{\in I}\right)$$

$$= \sum_{x \neq 0} T_s(s)$$

$$= q - 1$$

$$\Longrightarrow \quad T_s^2 = (q-1)T_s + q T_1$$

QUAD'C RELS FOR GENERAL SPLIT GP FOLLOWS FROM USING PRINCIPAL SL_2's

CAN PROVE BRAID RELS SIMILARLY $\square$

SUPPOSE NOW THAT V IS A SMOOTH G-REP

THEN $\mathcal{H}$ ACTS ON V^I: IF $v \in V^I$ AND $T_w \in \mathcal{H}$, THEN

$$v \cdot T_w = \sum_{h \in I / I \cap \dot{w}^{-1} I \dot{w}} h \dot{w}^{-1} \cdot v \quad \in V^I$$

$\underline{\text{EX}}$ LET $G = SL_2(\mathbb{Q}_p)$, $V = \text{IND}_B^G(1_B) = \{ f : G \to \mathbb{C} :$ TRIVIAL

$\cdot f$ LOC CST

$\cdot f(bg) = f(g)$ $\forall b \in B, g \in G \}$

THEN $V^I = \text{SPAN}\{ f_{BI}, f_{BsI} \}$ $\quad (\text{SUPP}(f_{BwI}) = BwI)$

AND USING ABOVE FORMULA, CAN CALCULATE

$$f_{BI} \cdot T_{s_1} = f_{BsI} \qquad\qquad f_{BsI} \cdot T_{s_1} = (q-1) f_{BsI} + q f_{BI}$$

$$f_{BI} \cdot T_{s_0} = (q-1) f_{BI} + q f_{BsI} \qquad\qquad f_{BsI} \cdot T_{s_0} = f_{BI}$$

IN PARTICULAR, WE HAVE A S.E.S.

$$0 \longrightarrow 1_G \longrightarrow \mathrm{Ind}_B^G(1_B) \longrightarrow ST \longrightarrow 0$$

WHICH GIVES

$$0 \longrightarrow 1_G^I \longrightarrow \mathrm{Ind}_B^G(1_B)^I \longrightarrow ST^I \longrightarrow 0$$

1_G^I IS 1-DIM'L, SPANNED BY $f_{BI} + f_{BsI}$, W/ SCALAR ACTION OF $\mathcal{H}$

$$T_{s_1} \longmapsto q \quad , \quad T_{s_0} \longmapsto q$$

ST^I IS 1-DIM'L, SPANNED BY $\overline{f_{BI}}$, W/ SCALAR ACTION OF $\mathcal{H}$

$$T_{s_1} \longmapsto -1 \quad , \quad T_{s_0} \longmapsto -1$$

<u>RMK</u> WHERE DOES PRODUCT / ACTION FORMULA COME FROM?

$$\begin{tikzcd}
V^I \times \mathcal{H} \arrow[rr] & & V^I \arrow[dd, "\wr"] \\
\mathrm{Hom}_I(1_I, V) \times \mathrm{Hom}_I(1_I, \underset{\substack{\parallel \\ X}}{\text{c-}\mathrm{Ind}_I^G(1_I)}) \arrow[u, phantom, "\wr"] \arrow[r, phantom, "\circlearrowright"] & & \\
\mathrm{Hom}_G(X, V) \times \mathrm{Hom}_G(X, X) \arrow[u, phantom, "\wr"] \arrow[rr, "\text{comp'n}"] & & \mathrm{Hom}_G(X, V)
\end{tikzcd}$$

FROBENIUS RECIPROCITY

+ SIMILAR ISOMORPHISM-CHASING FOR PRODUCT

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ALL THE CONSTRUCTIONS WE'VE DISCUSSED (DESCRIPTION OF $\mathcal{H}$, BASIS, ACTION ON V^I, ...) WORK OVER $\mathbb{Z}$ (REPLACING $\mathbb{C}$) IN PARTICULAR, WE CAN BASE CHANGE AND CONSIDER EVERYTHING w/ $\overline{\mathbb{F}}_p$-COEFFS

COEFFS = $\overline{\mathbb{F}}_p$ FROM NOW ON

IN FACT, IT IS BETTER TO WORK w/ A SLIGHTLY SMALLER COMPACT OPEN SUBGP

$$I \rightsquigarrow I_1 = \text{PRO-}p \text{ SYLOW OF } I$$

"PRO-p-IWAHORI SUBGP"

EX $G = SL_n(\mathbb{Q}_p)$

$$I = \begin{pmatrix} \mathbb{Z}_p^\times & & \mathbb{Z}_p \\ & \ddots & \\ p\mathbb{Z}_p & & \mathbb{Z}_p^\times \end{pmatrix} \rhd I_1 = \begin{pmatrix} 1+p\mathbb{Z}_p & & \mathbb{Z}_p \\ & \ddots & \\ p\mathbb{Z}_p & & 1+p\mathbb{Z}_p \end{pmatrix}$$

//

WE LET $\mathcal{H} = \mathcal{H}(G, I_1)_{/\mathbb{F}_p}$

$=$ "PRO-p-IWAHORI-HECKE ALG"

HAS SIMILAR STRUCTURE TO $\mathcal{H}(G, I)_{/\mathbb{C}}$ FROM BEFORE

EG • BASIS T_w INDEXED BY $w \in \widetilde{W}_1 := N/N \cap I_1$

$\widetilde{W}_1$ IS AN EXT'N OF $\widetilde{W}$ BY $N \cap I / N \cap I_1$

$\cong T \cap I / T \cap I_1$

$\cong \mathbb{T}(\mathbb{F}_p) \leftarrow$ FINITE TORUS

SO HAVE LOTS OF CONTROL OVER STRUCTURE

• THE T_w SATISFY BRAID REL'S AND QUADRATIC RELS

↳ FOR $G = SL_2(\mathbb{Q}_p)$

$$T_s^2 = \underbrace{\boxed{\left(\sum_{x \in \mathbb{F}_p^\times} T_{\begin{pmatrix} x & \\ & x^{-1}\end{pmatrix}}\right)}}_{q-1 \text{ TERMS}} T_s + q \boxed{T_{\begin{pmatrix} -1 & \\ & -1\end{pmatrix}}}$$

SPECIALIZE TO T_1 IN $\mathcal{H}(G, I)$

$$= \left(\sum_{x \in \mathbb{F}_p^\times} T_{\begin{pmatrix} x & \\ & x^{-1}\end{pmatrix}}\right) T_s$$

• WE UNDERSTAND THE CENTER OF $\mathcal{H}$, BERNSTEIN PRESENTATION, AND <u>ALL</u> SIMPLE MODULES

(VIGNÉRAS, OLLIVIER, ABE)

WHY IS $\mathcal{H} = \mathcal{H}(G, I_1)_{\overline{\mathbb{F}}_p}$ THE CORRECT OBJECT TO CONSIDER IN CHAR p?

LEMMA SUPPOSE V IS A NONZERO SMOOTH REP OF G (OVER $\overline{\mathbb{F}}_p$). THEN $V^{I_1} \neq 0$

RELIES ON THE FACT THAT I_1 IS PRO-p AND COEFFS $= \overline{\mathbb{F}}_p$

NOT TRUE FOR $\mathbb{C}$-COEFFS! SO EVEN mod p SUPERCUSPIDAL REPS SATISFY $V^{I_1} \neq 0$

UPSHOT V^{I_1} CARRIES MORE INFO IN CHAR p

MOREOVER, V^{I_1} HAS THE STRUCTURE OF AN $\mathcal{H}$-MODULE

AND WE HAVE mod p ANALOG OF BOREL-BERNSTEIN THM

THM (OLLIVIER, K.)

SUPPOSE $G = GL_2(\mathbb{Q}_p)$ OR $SL_2(\mathbb{Q}_p)$, AND LET $\mathcal{H} = \mathcal{H}(G, I_1)_{\overline{\mathbb{F}}_p}$.

THEN

$$\{V \in \mathrm{Rep}_{\overline{\mathbb{F}}_p}(G) : \langle G \cdot V^{I_1} \rangle = V\} = \mathrm{REP}^{I_1}_{\overline{\mathbb{F}}_p}(G) \xrightarrow{\sim} \mathrm{MOD}\text{-}\mathcal{H}$$

$$V \longmapsto V^{I_1}$$

IS AN EQUIV OF CATS

IN PARTICULAR, SINCE EVERY IRREP IS CONTAINED IN $\mathrm{Rep}^{I_1}_{\overline{\mathbb{F}}_p}(G)$,

THM GIVES A BIJECTION

$$\left\{ \begin{array}{c} \underline{\text{ALL}} \\ \text{IRREPS } V \\ \text{OF } GL_2(\mathbb{Q}_p) \\ (\text{OR } SL_2(\mathbb{Q}_p)) \end{array} \right\} \Big/ \cong \quad \xleftrightarrow{1:1} \quad \left\{ \begin{array}{c} \text{SIMPLE } \mathcal{H} \\ \text{MODULES} \end{array} \right\} \Big/ \cong$$

AGAIN, NOT TRUE FOR $\mathbb{C}$-COEFFS

<u>BUT</u>

<u>THM</u> (OLLIVIER) ABOVE EQUIV <u>FAILS</u> FOR $G = GL_2(\mathbb{Q}_{p^f})$ $(f \geq 2)$

DIFFERENT FROM CASE OF $\mathbb{C}$-COEFFS.

III LAST TIME: STRUCTURE OF $\mathcal{H}(G,I)_{\mathbb{C}}$ AND $\mathcal{H}(G,I_1)_{\overline{\mathbb{F}}_p}$

TODAY: DERIVED STRUCTURE IN CHAR p

NOTATION: $G = \mathbb{G}(F)$, $\mathbb{G}$ (SPLIT) CONNECTED REDUCTIVE GP/ $F/\mathbb{Q}_p$ FINITE

$I_1 \leq G$ PRO-p-IWAHORI SUBGP

(THINK $I_1 = \begin{pmatrix} 1+p\mathbb{Z}_p & & \mathbb{Z}_p \\ & \ddots & \\ p\mathbb{Z}_p & & 1+p\mathbb{Z}_p \end{pmatrix}$)

$\mathcal{H} = \mathcal{H}(G, I_1)_{\overline{\mathbb{F}}_p}$ = PRO-p-IWAHORI HECKE ALG

$= \left\{ f : G \to \overline{\mathbb{F}}_p : \begin{array}{l} \cdot\ f \text{ CPCT SUPP} \\ \cdot\ f(igi') = f(g) \\ \quad \forall i,i' \in I_1,\ g \in G \end{array} \right\}$

w/ CONVOLUTION PRODUCT

LAST TIME: THE FUNCTOR

$$\{ V \in \mathrm{Rep}_{\overline{\mathbb{F}}_p}(G) : \langle G \cdot V^{I_1} \rangle = V \} = \mathrm{Rep}^{I_1}_{\overline{\mathbb{F}}_p}(G) \longrightarrow \text{MOD-}\mathcal{H}$$

$$V \longmapsto V^{I_1}$$

GIVES AN EQUIV OF AB CATS FOR $G = GL_2(\mathbb{Q}_p)$ OR $G = SL_2(\mathbb{Q}_p)$, BUT FAILS IN GENERAL

REASON: • CERTAIN CHARACTER SUMS IN CALC'N ARE NONZERO $\Leftrightarrow F = \mathbb{Q}_p$ (BOTH OLLIVIER & BREUIL) w/ WITT VECTORS

IN CALC'N OF $(M \otimes_{\mathcal{H}} X)^{I_1}$

• HAVE CLASS'N RESULT FOR MODULES OVER $\overline{\mathbb{F}}_p[\![\mathcal{O}_F]\!] \Leftrightarrow F = \mathbb{Q}_p$ (PASKUNAS)

PROBLEM: SINCE I_1 IS PRO-p AND $\mathrm{char}(\overline{\mathbb{F}}_p) = p$, THE FUNCTOR

$V \longmapsto V^{I_1}$ IS NOT EXACT

SO IT MAKES SENSE TO DERIVE THIS FUNCTOR

SETUP: SET $X := \text{c-ind}_{I_1}^{G}(1_{I_1})$

$$= \left\{ f : G \longrightarrow \overline{\mathbb{F}}_p : \begin{array}{l} \cdot\ f \text{ cpct supp} \\ \cdot\ f(ig) = f(g) \\ \quad \forall i \in I_1,\ g \in G \end{array} \right\}$$

$(g \cdot f)(g') = f(g'g)$

THEN $\mathcal{H} = \text{c-ind}_{I_1}^{G}(1_{I_1})^{I_1}$

$= \mathrm{Hom}_{I_1}(1_{I_1}, X)$

FROB RECIP.

$\cong \mathrm{Hom}_G(X, X)$

WANT TO USE DERIVED CATEGORIES

NO I_1

① ON REP THEORY SIDE: $\mathrm{Rep}_{\overline{\mathbb{F}}_p}(G)$ IS A GROTHENDIECK ABELIAN CATEGORY ($\Rightarrow \exists$ ENOUGH INJECTIVES + ENOUGH K-INJECTIVES), SO $D(G) := D(\mathrm{Rep}_{\overline{\mathbb{F}}_p}(G))$

UNBOUNDED

ADMITS ALL RIGHT DERIVED FUNCTORS

($X^\bullet$ IS K-INJ $\Longleftrightarrow$ $\forall$ ACYCLIC $Y^\bullet$, $\mathrm{Hom}^\bullet(Y^\bullet, X^\bullet)$ IS ACYCLIC)

② ON HECKE ALG SIDE: NEED TO ENHANCE MOD-$\mathcal{H}$

CHOOSE AN INJECTIVE RES'N

$$0 \to X \to I^0 \to I^1 \to \cdots$$

AND DEFINE

$$\mathcal{H}^\bullet = \mathrm{Hom}_G^\bullet(I^\bullet, I^\bullet) \cong R\mathrm{Hom}_G(X,X) \text{ in } D(\mathbb{F}_p)$$

" HECKE DIFFERENTIAL GRADED ALGEBRA (DGA)" INVESTIGATED BY SCHNEIDER

- GRADED: $\mathcal{H}^i = \mathrm{Hom}_G^i(I^\bullet, I^\bullet)$
$$:= \prod_{j \in \mathbb{Z}} \mathrm{Hom}_G(I^j, I^{j+i})$$

- DIFFERENTIAL: $d_{\mathcal{H}}^i : \mathcal{H}^i \longrightarrow \mathcal{H}^{i+1}$
$$d_{\mathcal{H}}^i\big((f^j)_j\big) = \left(d_I^{j+i} \circ f^j - (-1)^i f^{j+1} \circ d_I^j \right)_j$$
THEN $d_{\mathcal{H}}^{i+1} \circ d_{\mathcal{H}}^i = 0$

- ALGEBRA: FOR $f = (f^j)_j \in \mathcal{H}^i$, $g = (g^l)_l \in \mathcal{H}^k$,

DEFINE
$$f \cdot g = (f^{m+k} \circ g^m)_m \in \mathcal{H}^{i+k}$$
THEN
$$d_{\mathcal{H}}^{i+k}(f \cdot g) = d_{\mathcal{H}}^i(f) \cdot g + (-1)^i f \cdot d_{\mathcal{H}}^j(g)$$

CAN CONSIDER MODULES OVER $\mathcal{H}^\bullet$ WHICH RESPECT THE DG STRUCTURE :

A (RIGHT) <u>DIFFERENTIAL GRADED MODULE</u> $M^\bullet$ OVER $\mathcal{H}^\bullet$ CONSISTS OF

- A COMPLEX $(M^\bullet, d_M^\bullet)$

$$\cdots \xrightarrow{d_M^{i-1}} M^i \xrightarrow{d_M^i} M^{i+1} \xrightarrow{d_M^{i+1}} \cdots$$

- A GRADED ACTION OF $\mathcal{H}^\bullet$ ON $M^\bullet$ WHICH SATISFIES THE LEIBNIZ RULE: IF $a \in \mathcal{H}^i$, $m \in M^j$, THEN

$$d_M^{i+j}(m \cdot a) = d_M^j(m) \cdot a + (-1)^j \, m \cdot d_{\mathcal{H}}^i(a)$$

SINCE $M^\bullet$ IS ALREADY A COMPLEX, CONCEPTS LIKE HOMOTOPY EQUIV AND QIS FOR DG MODULES MAKE SENSE. IN PART., WE CAN DEFINE THE CATEGORY

$$D(\text{DGMod-}\mathcal{H}^\bullet)$$

<u>N.B.</u> $\mathcal{H}^\bullet$ DEPENDS ON CHOICE OF $I^\bullet$, BUT IT IS WELL-DEFINED UP TO QUASI-ISOM, AND $D(\text{DGMod-}\mathcal{H}^\bullet)$ IS WELL-DEFINED UP TO NAT. EQUIV.

<u>THM</u> (SCHNEIDER) SUPPOSE I_1 IS TORSION-FREE. THEN

$$\begin{array}{ccc}
D(G) & \xrightarrow{\sim} & D(\text{DGMod-}\mathcal{H}^\bullet) \\
V^\bullet & \longmapsto & \mathrm{Hom}_G^\bullet(I^\bullet, i(V^\bullet)) \\
 & & \cong \mathrm{RHom}_G(X, V^\bullet) \\
p(M^\bullet) \otimes_{\mathcal{H}^\bullet} I^\bullet & \longleftarrow\!\!\shortmid & M^\bullet \\
\cong M^\bullet \otimes^L_{\mathcal{H}^\bullet} X & &
\end{array}$$

K-INJ RES OF $V^\bullet$

K-PROJ RES OF $M^\bullet$

IS AN EQUIV OF TRIANGULATED CATS

SKETCH

STEP 1 : PROVE THAT $V^\bullet \longmapsto R\Gamma(I_1, V^\bullet)$ REFLECTS ACYCLICITY

IE, $V^\bullet = 0$ IN $D(I_1)$ $\iff$ $R\Gamma(I_1, V^\bullet) = 0$ IN $D(\mathbb{F}_p)$

(USES NOETHERIANITY OF $\mathbb{F}_p[\![I_1]\!] = \varprojlim_{\substack{K \triangleleft I_1 \\ \text{OPEN}}} \mathbb{F}_p[I_1/K]$

\+ TORSION-FREENESS OF I_1 (GIVES FINITENESS OF COH DIM OF I_1))

STEP 2 : PROVE THAT X IS A COMPACT OBJECT IN $D(G)$ (USES $\mathrm{HOM}_{D(G)}(X, V^\bullet) \cong \mathbb{H}^\bullet(I_1, V^\bullet)$)

STEP 3 : PROVE THAT X IS A GENERATOR OF $D(G)$

IE, X AND ALL ITS SHIFTS $X[j]$, $j \in \mathbb{Z}$

(USES $\mathrm{HOM}_{D(G)}(X[j], V^\bullet) \cong \mathbb{H}^{-j}(I_1, V^\bullet) = 0$

$\iff$ $V^\bullet \cong 0$)

STEP 1

(ANALOGOUSLY, $\mathcal{H}^\bullet$ IS A COMPACT GENERATOR OF $D(\text{DG-MOD-}\mathcal{H}^\bullet)$)

STEP 4 : APPLY THM OF KELLER ON DG MORITA EQUIVALENCE

(IDEA : $\mathcal{H}^\bullet = R\mathrm{HOM}_G(X, X)$, SO $R\mathrm{HOM}_G(X, V^\bullet)$ GETS A RIGHT $\mathcal{H}^\bullet$-ACTION)

$\square$

COMPARE: UNDERIVED CASE:

$$\mathcal{H} \cong \mathrm{Hom}_G(X,X).$$

AND

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{C}}^{I_1}(G) & \xrightarrow{\sim} & \mathrm{Mod}\text{-}\mathcal{H} \\ V & \longmapsto & V^{I_1} \end{array}$$

IS AN INSTANCE OF MORITA EQUIVALENCE //

NOTE: IT IS HARD (IMPOSSIBLE?) TO WRITE DOWN EXPLICIT INJ RES $X \longrightarrow \mathcal{I}^\bullet \implies$ HARD TO DESCRIBE STRUCTURE OF $\mathcal{H}^\bullet$

INSTEAD: LOOK AT COHOMOLOGY OF $\mathcal{H}^\bullet$

$$h^i(\mathcal{H}^\bullet) = h^i(R\mathrm{Hom}_G(X,X)) \cong \mathrm{Ext}^i_G(X,X)$$

DEF THE PRO-p-IWAHORI-EXT ALGEBRA IS

$$E := \bigoplus_{i\in\mathbb{Z}} E^i := \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^i_G(X,X)$$

RMKS

- E IS A GRADED ALGEBRA W/ YONEDA PRODUCT
- $E^0 = \mathcal{H}$, AND EACH E^i IS AN $\mathcal{H}$-$\mathcal{H}$ BIMODULE

• WE HAVE

FROB RECIP

MACKEY

SHAPIRO

$$
\begin{aligned}
E^i &= \mathrm{EXT}^i_G(X, X) \\
&\cong \mathrm{EXT}^i_{I_1}(1_{I_1}, X|_{I_1}) \\
&\cong H^i(I_1, X) \\
&\cong \bigoplus_{w \in \widetilde{W}_1} H^i(I_1, \mathrm{IND}_{I_w}^{I_1}(1)) \\
&\cong \bigoplus_{w \in \widetilde{W}_1} H^i(I_w, \mathbb{F}_p)
\end{aligned}
$$

$I_w = I_1 \cap w^{-1} I_1 w$

$$
\begin{aligned}
&\text{c-IND}_{I_1}^{G}(1)|_{I_1} \\
&\cong \bigoplus_{w \in \widetilde{W}_1} \text{c-IND}_{I_1}^{I_1 w I_1}(1) \\
&\cong \bigoplus_{w \in \widetilde{W}_1} \mathrm{IND}_{I_1 \cap w^{-1} I_1 w}^{I_1}(1)
\end{aligned}
$$

SUPP $\subset I_1 w I_1$

• IF I_1 IS TORSION-FREE, THEN IT IS A POINCARÉ GP OF DIM $d := \dim_{\mathbb{Q}_p}(G)$.

$$\Longrightarrow \quad H^i(I_1, X) = 0 \quad \text{IF} \quad i > d$$

$$\Longrightarrow \quad E^i \text{ CONCENTRATED IN DEGS } 0, 1, \dots, d.$$

// PRODUCT STRUCTURE ON E CAN BE DESCRIBED EXPLICITLY, FOR EX:

PROP* (OLLIVIER-SCHNEIDER) SUPPOSE $v, w \in \widetilde{W}_1$ SATISFY

$$\ell(vw) = \ell(v) + \ell(w)$$

LET $\alpha \in H^i(I_1, \mathrm{ind}_{I_v}^{I_1}(1))$, $\beta \in H^j(I_1, \mathrm{ind}_{I_w}^{I_1}(1))$. THEN

$$\alpha * \beta = \underbrace{\alpha \cdot T_w}_{\in H^i(I_1, \mathrm{ind}_{I_{vw}}^{I_1}(1))} \smile \underbrace{T_v \cdot \beta}_{\in H^j(I_1, \mathrm{ind}_{I_{vw}}^{I_1}(1))}$$

PRODUCT IN E

$$\left(\in H^{i+j}(I_1, \mathrm{ind}_{I_{vw}}^{I_1}(1)) \right)$$

* CAVEAT LECTOR: [OS] USE $\mathcal{H}^{op}$ INSTEAD OF $\mathcal{H}$, SO THERE MAY BE SIGN DISCREPANCIES IN OUR FORMULA

FOLLOWS FROM EXPLICIT TECHNICAL CALCULATION w/ INJ. RES'S + GROUP COHOMOLOGY

O-S ALSO DESCRIBE THE PRECISE STRUCTURE OF THE TOP COHOMOLOGY E^d, AND A CERTAIN DUALITY OPERATION RELATING E^i TO E^{d-i}

IV

LAST TIME: DERIVED STRUCTURE OF $\mathcal{H}^\bullet$-MODS, SCHNEIDER'S EQUIVALENCE

TODAY: STRUCTURE OF DERIVED INVTS, (LOTS OF) OPEN QUESTIONS

NOTATION: $G = \mathbb{G}(F)$. $\mathbb{G}$ (SPLIT) CONNECTED REDUCTIVE GP / $\underbrace{F/\mathbb{Q}_p}_{\text{FINITE}}$

$I_1 \leq G$ PRO-p-IWAHORI SUBGP

$$\left(\text{THINK } I_1 = \begin{pmatrix} 1+p\mathbb{Z}_p & & \mathbb{Z}_p \\ & \ddots & \\ p\mathbb{Z}_p & & 1+p\mathbb{Z}_p \end{pmatrix} \right)$$

ASSUME TODAY THAT I_1 TORS-FREE

$$X = \text{c-ind}_{I_1}^G(1_{I_1})$$

$$\begin{aligned} \mathcal{H}^\bullet &= R\mathrm{Hom}_G(X,X) \\ &= \mathrm{Hom}^\bullet_G(\mathcal{I}^\bullet, \mathcal{I}^\bullet) \quad \leftarrow \text{INJ RES'N OF } X \\ &= \text{“HECKE DGA”} \end{aligned}$$

LAST TIME: ASSUME I_1 IS TORSION-FREE. THEN WE HAVE AN EQUIV OF Δ-ED CATS

$$\begin{aligned} D(\mathrm{Mod}_{\mathbb{F}_p}(G)) &\xrightarrow{\sim} D(\mathrm{DGMod}\text{-}\mathcal{H}^\bullet) \\ V^\bullet &\longmapsto R\mathrm{Hom}_G(X, V^\bullet) = \mathrm{Hom}^\bullet_G(\mathcal{I}^\bullet, \overbrace{i(V^\bullet)}^{\text{K-INJ RES'N}}) \\ \overbrace{p(M^\bullet)}^{\text{K-PROJ RESN}} \otimes_{\mathcal{H}^\bullet} \mathcal{I}^\bullet = M^\bullet \otimes^L_{\mathcal{H}^\bullet} X &\longleftarrow\!\shortmid\; M^\bullet \end{aligned}$$

RMK Let $G = SL_2(F)$, $F/\mathbb{Q}_p$ finite. How do we detect whether $V \longmapsto V^{I_1}$ is an underived equiv using Schneider's derived equiv? Not any direct way yet...

O-S define a certain torsion pair on the category Mod-$\mathcal{H}$, using the $\mathcal{H}$-modules

$$Z_m = \mathrm{Ext}^1_{\mathcal{H}}\left(H^1(I_1, X^{K_m}), \mathcal{H}\right)$$

$$\left(\{Z_m\}_m \rightsquigarrow \begin{array}{l} \mathcal{F} = \{ M : \mathrm{Hom}_{\mathcal{H}}(Z_m, M) = 0 \quad \forall m \} \\ \mathcal{T} = \{ N : \mathrm{Hom}_{\mathcal{H}}(N, M) = 0 \quad \forall M \in \mathcal{F} \} \end{array} \right)$$

Then $(M \otimes_{\mathcal{H}} X)^{I_1} \cong M \quad \forall M \in \mathcal{F}$ ⟵ full subcat ass'd to Z_m's

O-S prove

$$F = \mathbb{Q}_p \iff \mathcal{F} = \text{Mod} - \mathcal{H}$$

and this is enough to recover underived equiv for $SL_2(\mathbb{Q}_p)$

This gives a t-structure on $D^b(\text{Mod}-\mathcal{H})$

$$D^{\leq 0} = \{ X^\bullet : h^i(X^\bullet) = 0 \quad \forall i > 0, \ h^0(X^\bullet) \in \mathcal{T} \}$$
$$D^{\geq 0} = \{ X^\bullet : h^i(X^\bullet) = 0 \quad \forall i < -1, \ h^{-1}(X^\bullet) \in \mathcal{F} \}$$

Q HOW DO WE UNDERSTAND SCHNEIDER'S EQUIVALENCE?

CAN PASS TO COHOMOLOGY:

$$\begin{array}{ccccc} \mathrm{RHom}_G(X, V^\bullet) & \overset{\oplus h^i(-)}{\rightsquigarrow} & \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^i(X, V^\bullet) & = & \bigoplus_{i\in\mathbb{Z}} \mathbb{H}^i(I_1, V^\bullet) \\ \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\ \mathcal{H}^\bullet & & \bigoplus_{i\in\mathbb{Z}} h^i(\mathcal{H}^\bullet) & & \bigoplus_{i\in\mathbb{Z}} E^i =: E \end{array}$$

IE, WE CAN CONSIDER THE GRADED ACTION OF E ON THE HYPERCOHOMOLOGY $\bigoplus_{i\in\mathbb{Z}} \mathbb{H}^i(I_1, V^\bullet)$

<u>SPECIAL CASE</u> TAKE $V^\bullet = V[0]$ FOR A SMOOTH REP V

THEN THE GRP COH

$$\bigoplus_{i=0}^{d} H^i(I_1, V)$$

d ← $\dim_{\mathbb{Q}_p}(G)$

FINITE COH DIM

IS A GRADED RIGHT E-MODULE, AND EACH SPACE

$$H^i(I_1, V)$$

IS A RIGHT MODULE OVER $E^0 = \mathcal{H}$

<u>PROP</u> (K.) LET $V = \bar{\mathbb{F}}_p$ = TRIVIAL G-REP

a) $H^0(I_1, \bar{\mathbb{F}}_p) \cong \chi_{\mathrm{triv}} :=$ 1-DIM'L $\mathcal{H}$ MOD W/ T_s ACTING BY 0 $\forall$ SIMPLE AFFINE REFL'NS $s \in \tilde{S}$

THIS IS A STRAIGHTFORWARD EXERCISE, NOT DUE TO ME

b) (FOR IRRED ROOT SYSTEMS) $\bar{\alpha}(t) = \alpha(t)|\alpha(t)|_p \bmod p$, α = ! POSITIVE ROOT

$$H^1(I_1, \bar{\mathbb{F}}_p) \cong \chi_{\mathrm{TRIV}}^{\oplus m} \oplus \begin{cases} \mathrm{IND}_B^G(\bar{\alpha})^{I_1} & F = \mathbb{Q}_p \text{ AND ROOT SYSTEM } = A_1 \\ \text{SUPERSINGULAR} & \text{o/w} \end{cases}$$

AS $\mathcal{H}$ - MODS

$\mathcal{H}$-MOD ANALOGUE OF SUPERCUSPIDAL G - REPS

+ EXPLICITLY DESCRIBABLE

c)

$$H^i(I_1, \bar{\mathbb{F}}_p) \cong H^{d-i}(I_1, \bar{\mathbb{F}}_p)^\vee$$

DUAL $\mathcal{H}$-MOD

AS RIGHT $\mathcal{H}$ - MODS

(P, a) + b) + c) GIVE STRUCTURE OF $H^d(I_1, \bar{\mathbb{F}}_p)$ AND $H^{d-1}(I_1, \bar{\mathbb{F}}_p)$

EX FOR $SL_2(\mathbb{Q}_p)$, WE HAVE $d = \dim_{\mathbb{Q}_p}(G) = 3$, SO

$$\begin{aligned} H^0(I_1, \bar{\mathbb{F}}_p) &\cong \chi_{\mathrm{TRIV}} \\ H^1(I_1, \bar{\mathbb{F}}_p) &\cong \mathrm{IND}_B^G(\bar{\alpha})^{I_1} \\ H^2(I_1, \bar{\mathbb{F}}_p) &\cong (H^1)^\vee \\ &\cong \mathrm{IND}_B^G(\bar{\alpha})^{I_1} \\ H^3(I_1, \bar{\mathbb{F}}_p) &\cong (H^0)^\vee \\ &\cong \chi_{\mathrm{TRIV}} \end{aligned}$$

$\left(\bar{\alpha}\begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} = x^2 |x^2|_p \bmod p \quad \forall x \in \mathbb{Q}_p^\times \right)$

ALL $\cong$ AS $E^0 = \mathcal{H}$ - MODS

ONGOING : COMBINE THIS w/ DESCRIPTION OF $E = \bigoplus_i E^i$ FOR SL_2 (OLLIVIER - SCHNEIDER) TO CALCULATE GRADED ACTION OF E ON $\bigoplus_{i=0}^{3} H^i(I_1, \bar{\mathbb{F}}_p)$

EX FOR $G = SL_2(F)$, $F \neq \mathbb{Q}_p$, THE $\mathcal{H}$-MOD

$$H^1(I_1, \overline{\mathbb{F}}_p)$$

IS SUPERSINGULAR. SUGGESTS THAT SUPERSINGULARITY IN HIGHER COH SPACES MAY BE RELATED TO UNDERIVED EQUIV (OR LACK THEREOF)

CONJ / OPEN PROBLEM: LET G BE F-PTS OF A SPLIT RED GP $/F$.

THEN

$V \longmapsto V^{I_1}$ INDUCES AN UNDERIVED EQUIV $\Longleftrightarrow$ $H^i(I_1, V')$ IS A NONSUPERSINGULAR $\mathcal{H}$-MOD $\forall\, i \geq 0$ AND EVERY NON-SUPERCUSPIDAL IRREP V' OF G.

OPEN PROBLEM CALCULATE $H^2(I_1, \overline{\mathbb{F}}_p)$ AS AN $\mathcal{H}$-MODULE FOR GENERAL G

TRICKY TO DO THIS W/ GENS + RELS APPROACH

CAN ALSO SEE SUPERSINGULARITY IN $H^i(I_1, V)$ FOR OTHER V

PROP (K.) SUPPOSE $V = \mathrm{IND}_B^G(\chi)$ = PRINCIPAL SERIES REP

a) $$H^1(I_1, \mathrm{IND}_B^G(\chi))^{ss} \cong \left(\mathrm{IND}_B^G(\chi)^{I_1}\right)^{\oplus m} \oplus \begin{cases} \mathrm{IND}_B^G(\chi^{-1}\alpha) & \text{IF } F = \mathbb{Q}_p \text{ AND ROOT SYSTEM} = A_1 \\ \text{SUPERSINGULARS} \oplus \text{OTHER LARGER PARABOLIC INDUCTIONS} & \text{O/W} \end{cases}$$

AS $\mathcal{H}$-MODS

EXPLICITLY DESCRIBABLE

b) $$H^i(I_1, \mathrm{IND}_B^G(\chi)) \cong H^{b-i}(I_1, \mathrm{IND}_B^G(\chi^{-1}\bar{\delta}))^\vee$$

AS RIGHT $\mathcal{H}$-MODS

$b = \mathrm{DIM}_{\mathbb{Q}_p}(B)$

IF G IS $\mathbb{Q}_p$-SPLIT: $\delta = \sum_{\text{POS ROOTS}} \alpha$, $\bar{\delta}(t) = \delta(t)|\delta(t)|_p \bmod p$

OPEN PROBLEM CALCULATE HIGHER COHOMOLOGY SPACES $H^i(I_1, V)$ WHERE V IS A (GENERALIZED) STEINBERG REP OF G

// FOR $GL_2(\mathbb{Q}_p)$ OR $SL_2(\mathbb{Q}_p)$, HAVE COMPLETE INFO ABOUT $H^i(I_1, V)$

PROP (K.) IF V IS AN IRREP OF $G = GL_2(\mathbb{Q}_p)$ OR $SL_2(\mathbb{Q}_p)$, THEN WE CAN CALCULATE $H^i(I_1, V)$ AS AN $\mathcal{H}$-MODULE FOR ALL i.

COMES FROM THE FACT THAT $d = \dim_{\mathbb{Q}_p}(G)$ IS SMALL AND EXPLICIT CALCULATION, NOT FROM UNDERIVED EQUIV (SO NO CIRCULAR LOGIC)

// APPLICATIONS

LET $G = GL_2(\mathbb{Q}_p)$ OR $SL_2(\mathbb{Q}_p)$ (SO WE HAVE UNDERIVED EQUIV). SUPPOSE V, V' ARE TWO REPS OF G, AND ASSUME $\langle G \cdot V^{I_1} \rangle = V$. THEN CAN CONSTRUCT A SPECTRAL SEQ OF $\mathcal{H}$-MODS

$$(\ast) \qquad \mathrm{Ext}^i_{\mathcal{H}}(V^{I_1}, H^j(I_1, V')) \Rightarrow \mathrm{Ext}^{i+j}_G(V, V')$$

ALL $\mathcal{H}$-MODS

ESSENTIALLY A NON-NORMAL HOCHSCHILD-SERRE SS

THM WHEN $G = GL_2(\mathbb{Q}_p)$ OR $SL_2(\mathbb{Q}_p)$, WE CAN USE ⊛ AND THE STRUCTURE OF $H^i(I_1, V')$ TO CALCULATE $\mathrm{EXT}^i_G(V, V')$ FOR ALL IRRED V, V' AND ALL $i \geq 0$.

FACT THE UNDERIVED EQUIV AND THE SPECTRAL SEQ ⊛ ALSO HOLD FOR GROUPS THAT ARE ANISOTROPIC MOD CENTER

DIFFERENT d FROM BEFORE

UNDERGRADS!

THM (KEISLING-PENTLAND) SUPPOSE D/F IS A DIVISION ALGEBRA OF DIM d^2 OVER F, AND SUPPOSE $p > de(F/\mathbb{Q}_p) + 1$ ($\Rightarrow 1 + \mathfrak{m}_D$ TORS FREE). THEN WE CAN COMPUTE

$$\mathrm{EXT}^1_{D^\times}(V, V')$$

FOR ALL IRREPS V, V' OF $D^\times$.

FOR HIGHER $\mathrm{EXT}^i_{D^\times}$, NEED STRUCTURE OF HIGHER H^i

OPEN PROBLEM CALCULATE $H^i(1 + \mathfrak{m}_D, \overline{\mathbb{F}}_p)$ AS AN $\mathcal{H} = \overline{\mathbb{F}}_p[D^\times/1+\mathfrak{m}_D]$-MOD

EVEN $i = 2$ WOULD BE VERY INTERESTING (RELATED TO MORAVA STAB GPS IN K-THEORY)