

LOCAL MODEL THEORY FOR NON-GENERIC TAME POTENTIALLY BARSOTTI–TATE DEFORMATION RINGS

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ABSTRACT. We develop a local model theory for moduli stacks of 2-dimensional non-scalar tame potentially Barsotti–Tate Galois representations of the Galois group of an unramified extension of \mathbb{Q}_p . We derive from this explicit presentations of potentially Barsotti–Tate deformation rings, allowing us to prove structural results about them, and prove various conjectures formulated by Caruso–David–Mézard.

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1. INTRODUCTION

1.1. **Main results.** Let p be a prime number, K a p -adic field. We work with coefficient ring $\mathcal{O} = W(\mathbb{F})$ where \mathbb{F}/\mathbb{F}_p is a sufficiently large finite extension. Let $\bar{\rho} : G_K \rightarrow \mathrm{GL}_2(\mathbb{F})$ be a continuous Galois representation and $\tau = \chi \oplus \chi'$ be an inertial type where χ, χ' are *tame* characters of I_K . This gives rise to the universal framed deformation ring $R_{\bar{\rho}}^{\eta, \tau}$ classifying lifts of $\bar{\rho}$ which are potentially Barsotti–Tate (i.e. potentially crystalline with Hodge–Tate weights $0, 1$) and of type τ . Despite their prominent role in modularity questions via the Taylor–Wiles method (e.g. [Kis09]) in the last decades, their internal structure is still poorly understood. The basic reason for this, as suggested by works of Caruso–David–Mézard [CDMb],[CDMc],[CDM23], is that even when K is unramified, $R_{\bar{\rho}}^{\eta, \tau}$ exhibits a wide range of complicated behavior (in particular, it can be highly singular), especially as the inertial weights of χ/χ' become more degenerate (that is, when τ becomes more *non-generic*).

More recently, Caraiani–Emerton–Gee–Savitt [CEGSb] constructed a p -adic formal algebraic stack \mathcal{Z}^τ which interpolates the deformation rings $R_{\bar{\rho}}^{\eta, \tau}$ as $\bar{\rho}$ varies, in the sense that the latter recovers versal rings to finite type points of the former. The stacks \mathcal{Z}^τ (and its analogues for other p -adic Hodge theory conditions) are expected to be key geometric objects in the categorical p -adic Langlands conjectures formulated by Emerton–Gee–Hellmann [EGH], similar to the role played by various moduli spaces of local systems in the geometric Langlands program. Thus it is of interest to understand their geometry.

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From now on, we assume $K = \mathbb{Q}_{p^f}$ is absolutely unramified and τ is non-scalar (the scalar case $\tau = \chi \oplus \chi$ being easily handled by Fontaine–Laffaille theory). When τ is sufficiently generic, the structure of \mathcal{Z}^τ is well-understood (see e.g. [Kis09, LLHLM23]), since it can be modeled using Iwahori level local models of Shimura varieties for GL_2 , in particular its singularities are products of the singularity $XY = p$. In this paper, we introduce a method to probe the structure of \mathcal{Z}^τ which is powerful enough to handle non-generic τ . One concrete consequence of our study is the following general control on singularities:

Theorem 1.1.1 (Theorem 4.6.6). *Assume either $p \geq 7$ or $K = \mathbb{Q}_5$. Then the normalization of \mathcal{Z}^τ has rational singularities and is Gorenstein.*

Furthermore, it turns out that \mathcal{Z}^τ is almost always normal:

Theorem 1.1.2 (Theorem 4.6.10). *Assume either $p \geq 7$ or $K = \mathbb{Q}_5$. Then \mathcal{Z}^τ is normal, unless after twisting, $\tau = \chi \oplus \chi'$ is the sum of restrictions to I_K of characters of G_K , and the inertial weights of τ belong to $\{0, 1\}$. When τ is of this form, the non-normal locus consists of exactly the $\bar{\rho}$ which are Fontaine–Laffaille with specific (irregular) inertial weights determined τ .*

Remark 1.1.3. (1) Explicitly, for $K = \mathbb{Q}_p$ Theorem 1.1.2 shows that \mathcal{Z}^τ is normal unless, up to twists, $\bar{\tau} = 1 \oplus \bar{\varepsilon}$ (where $\bar{\tau}$ and $\bar{\varepsilon}$ denote the mod p -reduction of τ and the cyclotomic character respectively), in which case the non-normal locus consists of $\bar{\rho}$ is a twist of an unramified representation by $\bar{\varepsilon}$.

(2) It is proven in [CEGSa] that the special fiber of $R_{\bar{\rho}}^{\eta, \tau}$ is generically reduced. Together with Theorem 4.6.6, this implies that $R_{\bar{\rho}}^{\eta, \tau}$ is Cohen–Macaulay if and only if it is normal, in which case it is furthermore Gorenstein. Thus Theorem 1.1.2 completely classifies when $R_{\bar{\rho}}^{\eta, \tau}$ is Cohen–Macaulay. We invite the reader to compare this to the result of Hu–Paškūnas [HP19] about Cohen–Macaulayness of crystabelline deformation rings: whereas [HP19] covers the situation where $K = \mathbb{Q}_p$, τ is restricted to (possibly wildly ramified) principal series types (i.e. χ, χ' can be extended to characters of $G_{\mathbb{Q}_p}$) but allows arbitrary Hodge–Tate weights, our result allows general unramified K but restricts to tame (possibly non-principal series) types τ and Hodge–Tate weights 0, 1. We also point out that the method of [HP19] is unlikely to establish neither the Gorenstein nor the rational singularity property.

Finally, as is well-known (cf. [HP19]), Cohen–Macaulayness for (normalizations) of deformation rings allows one to upgrade $R[\frac{1}{p}] = T[\frac{1}{p}]$ theorems to integral $R = T$ theorems, hence our results give new instances of such.

(3) The fact \mathcal{Z}^τ has rather mild singularities is expected to be useful for the categorical p -adic Langlands program for $\mathrm{GL}_2(K)$, namely it suggests the conjectural functor \mathfrak{A} of [EGH] to have simple effect on a certain generating set of smooth representations of $\mathrm{GL}_2(K)$, thus giving hope that one can construct \mathfrak{A} by “generators and relations”. We point out that any generating set must necessarily involve representations of $\mathrm{GL}_2(K)$ with non-generic parameters, thus it is essential that we allow arbitrarily non-generic τ for this purpose.

Our method to probe \mathcal{Z}^τ is to construct group-theoretic local models for it. The main local model theorem has the following form:

Theorem 1.1.4 (Proposition 3.2.4, Theorem 3.3.8). *Let $\tau = \chi \oplus \chi'$ be a tame inertial type with $\chi \neq \chi'$. There exists a p -adic formal scheme $\tilde{\mathcal{Z}}^{\mathrm{mod}, \tau}$ such that*

- *If either $p \geq 7$ or $K = \mathbb{Q}_5$, \mathcal{Z}^τ/p is smooth locally isomorphic to $\tilde{\mathcal{Z}}^{\mathrm{mod}, \tau}/p$.*
- *If either $p > 16f + 7$ or $p > 7$ and $K = \mathbb{Q}_p$, then \mathcal{Z}^τ is smooth locally isomorphic to $\tilde{\mathcal{Z}}^{\mathrm{mod}, \tau}$.*

Remark 1.1.5. (Features of $\widetilde{\mathcal{Z}}^{\text{mod},\tau}$) We defer the somewhat involved definition of $\widetilde{\mathcal{Z}}^{\text{mod},\tau}$ to section 1.2 below, and instead note for now that:

- (1) Its construction involves the geometry of (mixed characteristic) loop groups.
- (2) Its geometric structure is independent of p , in the sense that it essentially arises as the p -adic completion of a natural \mathbb{Z} -scheme. In particular, this exhibits a kind of “independence-of- p ” property of tame potentially Barsotti–Tate deformation rings, as suggested in [CDM23] and [CDMa].
- (3) It admits an explicit affine cover where each affine open can be presented as (the p -saturation of) explicit equations constructed using the inertial weights of τ (see Table 5 for a sense of the presentations that show up).

Remark 1.1.6. (Bounds on p) Theorem 1.1.4 is obtained via deformation theory: we construct $\widetilde{\mathcal{Z}}^{\text{mod},\tau}$ which captures the structure of \mathcal{Z}^τ modulo some power of p , and then show this property persists when deforming to mixed characteristics. The requirement that p needs to be at least some linear bound on f arises for two related reasons:

- There are non-isomorphic charts of \mathcal{Z}^τ that are indistinguishable modulo linear powers of p . For instance, there are charts equisingular to $XY = p^k$ for any $k \leq f$. This requires us to start with a model that approximates \mathcal{Z}^τ modulo at least $p^{O(f)}$ to distinguish these charts.
- To show the approximation deforms, we need to overcome obstruction groups for certain lifting problems, whose p -torsion can have exponents as large as linear in f .

The explicit nature of our models yields, for the first time, an efficient algorithm to compute *any* given potentially Barsotti–Tate deformation ring with tame inertial type. The basic form of the algorithm is as follows (the details occur in section 4.3). To the pair $(\bar{\rho}, \tau)$ we assign

- an f -tuple $\widetilde{w}(\bar{\rho}, \tau)$ of elements in the extended affine Weyl group \widetilde{W} of GL_2 , which measures the relative position of the ι -inertial weights of $\bar{\rho}$ and τ for each $\iota : K \hookrightarrow \overline{\mathbb{Q}}_p$; and
- an f -tuple of “degeneracy types” for each $\iota : K \hookrightarrow \overline{\mathbb{Q}}_p$ which roughly measures how degenerate the ι -th inertial weight of χ/χ' is.

These datas give rise, for each ι , to a basic ring R_ι , as well as a collection of structure matrices with entries in R_ι recorded in Tables 3, 4. Both these datas are independent of f . Then $R_{\bar{\rho}}^{\eta,\tau}$ is given as a suitable completion of (the p -saturation of) the quotient of $\bigotimes_\iota R_\iota$ by the relations that certain products of the structure matrices are zero. We stress that the equations we impose will generally involve mutual interactions between R_ι ’s for arbitrary large sets of different ι . For this reason, outside of the generic case, $R_{\bar{\rho}}^{\eta,\tau}$ does not generally admit an obvious tensor product decomposition along embeddings $\iota : K \hookrightarrow \overline{\mathbb{Q}}_p$.

In the series of work [CDMb, CDMc, CDM23], Caruso–David–Mézard also investigated the problem of algorithmically computing $R_{\bar{\rho}}^{\eta,\tau}$, in the special case where $\tau = \chi \oplus \chi'$ is a principal series type (so χ, χ' extend to characters of G_K) and $\bar{\rho}$ is irreducible. In a few cases when $f \leq 3$ ([CDMb, Théorème 4.3.1] and [CDMc, 5.3.3]), they managed to determine $R_{\bar{\rho}}^{\eta,\tau}$ based on the fact that one can guess “a priori” what it is, cf. [CDMb, Remark 3.2.10]. However, their investigations also suggested that the answer becomes intrinsically complicated for large f , which made their strategy hopeless in general. We demonstrate the power of our algorithm by confirming various of their conjectural examples of [CDMc, §5.3.2, 5.3.3], as well as computing all examples for $K = \mathbb{Q}_p$, some of which are new as alluded to in [EGH, §7.5.13]. For instance, for $K = \mathbb{Q}_p$, $p \geq 7$, $R_{\bar{\rho}}^{\eta,\tau}$ is a power series ring over either \mathcal{O} , $\mathcal{O}[[X, Y]]/(XY - p)$ or $\mathcal{O}[[X, Y]]/(XY - p^2)$, except when up to

twist, $\bar{\tau} = \omega_2 \oplus \omega_2^p$ and $\bar{\rho} \otimes \bar{\varepsilon}^{-1}$ is unramified and has scalar semisimplification, or $\bar{\tau} = 1 \oplus \omega_1$ and $\bar{\rho} \otimes \bar{\varepsilon}^{-1}$ is unramified (here ω_n is Serre’s niveau n character). In these exceptional cases, assuming $p > 7$, we also give the presentation of $R_{\bar{\rho}}^{\eta, \tau}$ which turns out somewhat complicated, cf. section 5.5.1.

Besides algorithmic aspects, our theory also unifies and conceptualizes Caruso–David–Mézard’s work. More specifically, they introduced the notion of *gene* $\mathbb{X}(\tau, \bar{\rho})$ associated to $\bar{\rho}$ and τ , which is a purely combinatorial gadget, closely related to the combinatorial data inputted in our algorithm, that keeps track of the difference between the inertial weights of $\bar{\rho}$ and τ . While the motivation for $\mathbb{X}(\tau, \bar{\rho})$ was to encode geometric features of a resolution of $R_{\bar{\rho}}^{\eta, \tau}$ arising from integral p -adic Hodge theory, Caruso–David–Mézard conjectured that, surprisingly, $\mathbb{X}(\tau, \bar{\rho})$ is in fact a complete invariant, cf. [CDMc, Conjecture 5.1.6] and [CDMa, Conjecture 2]. Our model gives a geometric interpretation of $\mathbb{X}(\tau, \bar{\rho})$, allowing us to confirm this:

Theorem 1.1.7 (Theorem 5.4.16). *Assume $p > 16f + 7$. Let $\bar{\rho}$ be irreducible and τ be a non-scalar principal series tame inertial type. The deformation ring $R_{\bar{\rho}}^{\eta, \tau}$ depends only on $\mathbb{X}(\tau, \bar{\rho})$, in an explicit way, and furthermore, is an integral domain.*

In fact, our algorithm can be interpreted as giving the right generalization of Caruso–David–Mézard’s conjecture for general (i.e. not necessarily irreducible) $\bar{\rho}$ and general (i.e. not necessarily principal series) τ . It should be noted that in this more general setting, $R_{\bar{\rho}}^{\eta, \tau}$ is no longer always a domain, and one can read off when this is so from our tables.

Finally, we expect the explicit computations of the deformation ring to be useful for global applications, particularly for mod p multiplicity one questions and Breuil’s lattice conjecture in non-generic cases (see [EGS15, Theorem 10.1.1 and Theorem 8.2.1]).

1.2. Methods. When τ is sufficiently generic, a local model for \mathcal{Z}^τ can be extracted from [CEGSb] (see also [LLHLM23] for a perspective closer to the present work), owing to the fact that in the generic cases \mathcal{Z}^τ agrees with the moduli stack $Y^{\eta, \tau}$ of Breuil–Kisin modules of type τ and there are standard local models for the latter. In the non-generic cases, the essential difficulty is that \mathcal{Z}^τ is only a scheme theoretic image of a map $Y^{\eta, \tau} \rightarrow \Phi\text{-Mod}_K^{\text{ét}, 2}$ to the stack of rank 2 étale φ -modules, so that \mathcal{Z}^τ is obtained from $Y^{\eta, \tau}$ by contracting the fibers of this map (the *Kisin varieties*). This is the source of all complexities in the non-generic situation, and the main innovation of this work is to find a good group theoretic model for this contraction procedure.

The main idea in Theorem 1.1.4 is to use deformation theory to find good approximations of the map $Y^{\eta, \tau} \rightarrow \Phi\text{-Mod}_K^{\text{ét}, 2}$. Let LG (resp. L^+G) be the loop group (resp. positive loop group) for GL_2 with respect to $v(v+p)$, that is the functor $R \mapsto LG(R) \stackrel{\text{def}}{=} GL_2\left(R[v]^{\wedge(v(v+p))}\left[\frac{1}{v(v+p)}\right]\right)$ (resp. $L^+G(R) \stackrel{\text{def}}{=} GL_2\left(R[v]^{\wedge(v(v+p))}\right)$) where R is an \mathcal{O} -algebra. We also have the “Iwahori” L^+G and the first principal congruence subgroup L_1^+G which are the inverse image in L^+G of the upper triangular Borel, resp. the trivial subgroup under the mod v reduction. We set $\text{Gr}_1 \stackrel{\text{def}}{=} [L_1^+G \backslash LG]$, which is a GL_2 -torsor over a (mixed characteristic, Beilinson–Drinfeld) affine Grassmannian.

Up to a p -adic completion which we suppress for the remainder of the introduction, we have $Y^{\eta, \tau} = [LG^\tau / \varphi(L^+G)^f]$ as a quotient by a (shifted) φ -conjugation action (here φ sends v to v^p) and LG^τ is a particular closed subset of LG^f encoding certain elementary divisor bounds and the combinatorial data of τ (cf. section 3.1 for the precise definitions). The basic idea, as in [LLHLM23], is to “straighten” the φ -action in the above as much as possible: due to the contraction effect of φ , the φ -conjugation action is equivalent to the left translation action, provided one works modulo fixed powers of p and on a small enough subgroup. Unlike the generic case in *loc. cit.*, it is not

always possible to do this in characteristic p at the $L^+\mathcal{G}$ level, but if p is large enough, it is possible to do so at the $L_1^+\mathbf{G}$ -level (Lemma 3.3.7). This shows that $Y^{\eta,\tau}$ is congruent to $[\mathrm{Gr}_1^\tau/B^f\text{-sh.cnj}]$ modulo p^{cp} for some absolute constant c (here the B^f -action is via shifted conjugation, and Gr_1^τ is $[(L_1^+\mathbf{G})^f \backslash LG^\tau]$).

On the other hand, $\Phi\text{-Mod}_K^{\acute{e}t,2} = [LG^f/\varphi LG^f]$, and it is never possible to straighten the action of the larger group LG^f . However, since we are only interested in the scheme theoretic image of $Y^{\eta,\tau}$ in $\Phi\text{-Mod}_K^{\acute{e}t,2}$, it suffices to work instead with $[LG^{\mathrm{bd}}/\varphi(L^+\mathbf{G})^f]$ where LG^{bd} is a suitable bounded region in LG^f containing the orbit of LG^τ (denoted by $LG^{\mathrm{bd},(v+p)v^\mu}$ in the main text), and once again the φ -action can be straightened on $(L_1^+\mathbf{G})^f$. We remark that LG^{bd} descends to Gr_1^f , inducing a subvariety $\mathrm{Gr}_1^{\mathrm{bd}}$.

The upshot so far is that $Y^{\eta,\tau} \rightarrow \Phi\text{-Mod}_K^{\acute{e}t,2}$ is well-approximated by the natural map

$$[\mathrm{Gr}_1^\tau/B^f\text{-sh.cnj}] \rightarrow [\mathrm{Gr}_1^{\mathrm{bd}}/\mathrm{GL}_2^f\text{-sh.cnj}]$$

modulo p^{cp} . It further turns out that the scheme theoretic image $\mathcal{Z}^{\mathrm{mod},\tau}$ of this is congruent to \mathcal{Z}^τ modulo p^{cp-1} , and the model $\tilde{\mathcal{Z}}^{\mathrm{mod},\tau}$ in Theorem 1.1.4 is the pullback of $\mathcal{Z}^{\mathrm{mod},\tau}$ to the natural $(\mathrm{GL}_2)^f$ -torsor of the target. Finally, to prove Theorem 1.1.4, we need to show that the above congruences can be lifted to characteristic 0, at least locally. This is achieved by a detailed study the geometry of $\tilde{\mathcal{Z}}^{\mathrm{mod},\tau}$ to bound the p^∞ -torsion of the obstruction groups for such lifting problems (whose exponent can be as large as linear in f , as alluded to in Remark 1.1.6), and imposing bounds on p required to overcome the obstructions. A by-product of this geometric study is a control on the singularities of $\tilde{\mathcal{Z}}^{\mathrm{mod},\tau}$, which is robust enough that it can be transferred to \mathcal{Z}^τ through a mod p congruence, thus yielding Theorems 1.1.1, 1.1.2.

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1.4. Notation. We fix once and for all a separable closure \overline{K} and let $G_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}/K)$. If K is a nonarchimedean local field, we let $I_K \subset G_K$ denote the inertial subgroup. We fix a prime p . Let E be a finite extension \mathbb{Q}_p with ring of integers \mathcal{O} , uniformizer $\varpi \in \mathcal{O}$ and residue field \mathbb{F} (which we assume is large enough).

We consider the group $G \stackrel{\mathrm{def}}{=} \mathrm{GL}_2$ (defined over \mathbb{Z}). We write B for the subgroup of upper triangular matrices, $T \subset B$ for the split torus of diagonal matrices and $Z \subset T$ for the center of G . Let $X^*(T)$ be the group of characters of T which we identify with \mathbb{Z}^2 in the standard way and $\eta \in X^*(T)$ the element corresponding to $(1, 0) \in \mathbb{Z}^2$.

We write W (resp. \widetilde{W}) for the Weyl group (resp. the extended affine Weyl group) of G , which act naturally on $X^*(T)$. Thus $W = \{1, w_0\}$ is the set of permutation on 2 elements and $\widetilde{W} = X^*(T) \rtimes W$. We use the notation $t_\nu \in \widetilde{W}$ to denote the image of $\nu \in X^*(T)$ in \widetilde{W} .

Let α denote the positive root of G and $\langle \cdot, \cdot \rangle$ the duality pairing on $X^*(T) \times X_*(T)$, so a weight $\lambda \in X^*(T)$ is *dominant* if $0 \leq \langle \lambda, \alpha^\vee \rangle$. We set $X^0(T)$ to be the subgroup consisting of characters $\lambda \in X^*(T)$ such that $\langle \lambda, \alpha^\vee \rangle = 0$.

Let now K be a finite unramified extension of \mathbb{Q}_p of degree f , with ring of integers \mathcal{O}_K and residue field k . Thus $\mathcal{O}_K = W(k)$, and denote by φ the arithmetic Frobenius acting on $W(k)$ (i.e. acting by rising to the p -power on the residue field). Let $G_0 \stackrel{\text{def}}{=} \text{Res}_{\mathcal{O}_K/\mathbb{Z}_p} G/\mathcal{O}_K$, $T_0 \stackrel{\text{def}}{=} \text{Res}_{\mathcal{O}_K/\mathbb{Z}_p} T/\mathcal{O}_K$, and $Z_0 \stackrel{\text{def}}{=} \text{Res}_{\mathcal{O}_K/\mathbb{Z}_p} Z/\mathcal{O}_K$. We assume that \mathcal{O} contains the image of every ring homomorphism $\mathcal{O}_K \rightarrow \overline{\mathbb{Z}}_p$ and write $\mathcal{J} \stackrel{\text{def}}{=} \text{Hom}_{\mathbb{Z}_p}(\mathcal{O}_K, \mathcal{O})$. We define $\underline{G} \stackrel{\text{def}}{=} (G_0)_{/\mathcal{O}}$ and fix an identification of \underline{G} with the split reductive group $G_{/\mathcal{O}}^{\mathcal{J}}$. We similarly define and identify \underline{T} , and \underline{Z} . The notations \underline{W} , \widetilde{W} are clear as should be the natural isomorphisms $X^*(\underline{T}) = X^*(T)^{\mathcal{J}}$. Given an element $j \in \mathcal{J}$, we use a subscript notation to denote j -components obtained from the isomorphism $\underline{G} \cong G_{/\mathcal{O}}^{\mathcal{J}}$ (so that, for instance, given an element $\tilde{w} \in \widetilde{W}$ we write \tilde{w}_j to denote its j -th component via the induced identification $\widetilde{W} \cong \widetilde{W}^{\mathcal{J}}$). For sake of readability, we abuse notation and still write w_0 to denote the longest element in \underline{W} , and $\eta \in X^*(\underline{T})$ for the element corresponding to $(1, 0) \in \mathbb{Z}^2$ in all embeddings.

The Frobenius automorphism φ of \mathcal{O}_K induces an automorphism π on $X^*(\underline{T}) \cong X_*(\underline{T}^\vee)$ by the formula $\pi(\lambda)_\sigma = \lambda_{\sigma \circ \varphi^{-1}}$ for all $\lambda \in X^*(\underline{T})$ and $\sigma : \mathcal{O}_K \rightarrow \mathcal{O}$. We similarly define an automorphism π of \underline{W} and \widetilde{W} .

Recall that we fixed a separable closure \overline{K} of K . We choose $\pi \in \overline{K}$ such that $\pi^{p^f-1} = -p$ and let $\omega_K : G_K \rightarrow \mathcal{O}_K^\times$ be the character defined by $g(\pi) = \omega_K(g)\pi$, which is independent of the choice of π . We fix an embedding $\sigma_0 : K \hookrightarrow E$ and define $\sigma_j \stackrel{\text{def}}{=} \sigma_0 \circ \varphi^{-j}$, which identifies $\mathcal{J} = \text{Hom}(k, \mathbb{F}) = \text{Hom}_{\mathbb{Q}_p}(K, E)$ with $\mathbb{Z}/f\mathbb{Z}$. In particular, the automorphism π on $X^*(\underline{T})$ satisfies $(\pi(\lambda))_j = \lambda_{j+1}$. We write $\omega_f : G_K \rightarrow \mathcal{O}^\times$ for the character $\sigma_0 \circ \omega_K$.

Let ε denote the p -adic cyclotomic character. We fix normalization so that the p -adic cyclotomic character ε has Hodge–Tate weight $\{1\}$ for every $\kappa : K \hookrightarrow E$.

2. TAME INERTIAL TYPES AND BREUIL–KISIN MODULES

2.1. Tame inertial types and Galois representations.

2.1.1. *Tame inertial types.* An *inertial type* for K over \mathcal{O} (resp. over \mathbb{F}) is a homomorphism $\tau : I_K \rightarrow \text{GL}_2(\mathcal{O})$ (resp. $\tau : I_K \rightarrow \text{GL}_2(\mathbb{F})$) with open kernel and which extends to the Weil group of G_K . An inertial type is *tame* if it factors through the tame quotient of I_K . Given $s \in \underline{W}$ and $\mu \in X^*(\underline{T})$, we have a tame inertial type $\tau(s, \mu) : I_K \rightarrow \text{GL}_2(\mathcal{O})$ defined as follows: let r be the order of $s_0 s_1 \dots s_{f-1} \in W$, and define $\alpha_{k'} \stackrel{\text{def}}{=} (\prod_{m'=0}^{k'-1} s_{f-1-m'}^{-1})(\mu_{f-k'})$. Then

$$\tau(s, \mu) \stackrel{\text{def}}{=} \left(\sum_{i'=0}^{r-1} \alpha_{i'} p^{i'} \right) (\omega_{fr}).$$

In particular if $\nu = (\nu_j)_{j \in \mathcal{J}} \in X^*(\underline{Z}) \cong \mathbb{Z}^{\mathcal{J}}$ then

$$(2.1.1) \quad \tau(s, \mu + \nu) \cong \tau(s, \mu) \otimes_{\mathcal{O}} \omega_f^{\sum_{j \in \mathcal{J}} \nu_j p^j}.$$

Any tame inertial type is isomorphic to some $\tau(s, \mu)$.

More explicitly, if $s = (s_j)_{\mathcal{J}} \in \underline{W}$ is such that $\prod_{j=0}^{f-1} s_j = \text{id}$ then $\tau(s, \mu) \cong \omega_f^\gamma \oplus \omega_f^{\gamma'}$ where

$$(2.1.2) \quad \begin{aligned} \gamma &\stackrel{\text{def}}{=} \sum_{j=0}^{f-1} p^j \left(\mu_{f-j, (\prod_{i=0}^{f-1-j} s_i)(1)} \right) \\ \gamma' &\stackrel{\text{def}}{=} \sum_{j=0}^{f-1} p^j \left(\mu_{f-j, (\prod_{i=0}^{f-1-j} s_i)(2)} \right), \end{aligned}$$

noting that $\prod_{i=0}^{f-1-j} s_i = \prod_{i=0}^{j-1} s_{f-1-i}^{-1}$.

Similarly, if $s = (s_j)_{\mathcal{J}} \in \underline{W}$ is such that $\prod_{j=0}^{f-1} s_j = (12)$ then $\tau(s, \mu) \cong \omega_{2f}^h \oplus \omega_{2f}^{p^f h}$ where

$$(2.1.3) \quad h \stackrel{\text{def}}{=} \sum_{j=0}^{f-1} p^j \left(\mu_{f-j, (\prod_{i=0}^{f-1-j} s_i^{-1})(2)} \right) + p^f \left(\sum_{j=0}^{f-1} p^j \mu_{f-j, (\prod_{i=0}^{f-1-j} s_i^{-1})(1)} \right).$$

Remark 2.1.4. Let $\lambda, \lambda' \in X^*(\underline{T}) \xrightarrow{\sim} (\mathbb{Z}^2)^f$. Then $\lambda \equiv \lambda' \pmod{(p - \pi^{-1})X^0(\underline{T})}$ if and only if

$$\sum_{j=0}^{f-1} p^j \lambda_{j,1} + p^f \left(\sum_{j=0}^{f-1} p^j \lambda_{j,2} \right) \equiv \sum_{j=0}^{f-1} p^j \lambda'_{j,1} + p^f \left(\sum_{j=0}^{f-1} p^j \lambda'_{j,2} \right) \pmod{p^{2f} - 1}.$$

We say that (s, μ) is a *presentation* for the tame inertial type $\tau(s, \mu)$. Note that any tame inertial type will have infinitely many presentations since

$$(2.1.5) \quad \tau(s, \mu) \cong \tau(\sigma s \pi(\sigma)^{-1}, \sigma(\mu) + p\nu - \sigma s \pi(\sigma)^{-1} \pi(\nu))$$

for any $(\sigma, \nu) \in \underline{W} \times X^*(\underline{T})$. We will also record a presentation (s, μ) by the element $\tilde{w}^*(\tau) = s^{-1} t_\mu \in \widehat{\underline{W}}$.

Lemma 2.1.6. *Let $\tau : I_K \rightarrow \text{GL}_2(\mathcal{O})$ be a tame inertial type.*

Then, there exists $n \in \mathbb{Z}$, $(k_j)_{j \in \mathcal{J}} \in \{0, \dots, \frac{p+1}{2}\}^{\mathcal{J}}$ and $s \in W^{\mathcal{J}}$ such that

$$\tau \cong \tau(s, ((k_j, 0))_{j \in \mathcal{J}}) \otimes_{\mathcal{O}} \omega_f^n$$

and moreover $s_j = \text{id}$ if $k_j = 0$.

Proof. In this proof, given two tame inertial types τ and τ' we write $\tau \sim \tau'$ if $\tau \cong \tau' \otimes_{\mathcal{O}} \omega_f^n$ for some $n \in \mathbb{Z}$. Let (s', μ') be a presentation of τ . Using (2.1.1) and applying repeatedly (2.1.5) with $\sigma = \text{id}$, we see that $\tau(s', \mu') \sim \tau(s', \mu'')$ where $\mu'' \in X^*(\underline{T})$ is such that $\langle \mu''_j, \alpha^\vee \rangle \in [-\frac{p+1}{2}, \dots, \frac{p+1}{2}]$ for all $j \in \mathcal{J}$ (see also [LLHLM23, Lemma 2.3.3]). Hence, using again (2.1.1), we have $\tau(s', \mu'') \sim \tau(s', (k'_j, 0)_{j \in \mathcal{J}})$ where $|k'_j| \leq \frac{p+1}{2}$. Finally, using (2.1.1) and applying repeatedly (2.1.5) with $\nu = 0$, we obtain $\tau(s', (k'_j, 0)_{j \in \mathcal{J}}) \sim \tau(s, (k_j, 0)_{j \in \mathcal{J}})$, where at each step σ can be chosen so that $\sigma_j s_j \sigma_{j+1}^{-1} = \text{id}$ when $k'_j = 0$. \square

Definition 2.1.7. A presentation (s, μ) of a tame inertial type τ is *small* if $0 \leq \langle \mu_j, \alpha^\vee \rangle \leq \frac{p+1}{2}$ for all $j \in \mathcal{J}$ and moreover $s_j = \text{id}$ whenever $\langle \mu_j, \alpha^\vee \rangle = 0$.

If τ is a tame inertial type over \mathcal{O} we let $\bar{\tau} \stackrel{\text{def}}{=} \tau \otimes_{\mathcal{O}} \mathbb{F}$. This construction gives a bijection between isomorphism classes of tame inertial types over \mathcal{O} and tame inertial types over \mathbb{F} so that the whole discussion above holds for the latter.

Lemma 2.1.8. *Let $\tau = \tau(s, \mu)$ be a tame inertial type with small presentation (s, μ) . For $j' \in \mathcal{J}'$ define $\mathbf{a}^{(j')} \stackrel{\text{def}}{=} \sum_{i'=0}^{rf-1} \alpha_{-j'+i'} p^{i'}$. There exists a unique element $(s'_{\text{or},j'})_{j' \in \mathcal{J}'} \in W^{\mathcal{J}'}$ such that $(s'_{\text{or},j'})^{-1}(\mathbf{a}^{(j')})$ is strictly dominant for all $j' \in \mathcal{J}'$. Moreover the embedding $\sigma_0 : k \hookrightarrow \mathbb{F}$ and s can be chosen so that $(s'_{\text{or},rf-1}) = \text{id}$.*

Proof. We can assume without loss of generality that $\langle \mu_0, \alpha^\vee \rangle > 0$. As $\alpha_0 = \mu_0$ we thus have $\mathbf{a}^{(0)} = p^{rf-1} \mu_0 + \sum_{i'=0}^{rf-2} \alpha_{-j'+i'} p^{i'}$ which is dominant since $\langle \mu_0, \alpha^\vee \rangle > 0$ and $p-1 > \langle \alpha_{-j'+i'}, \alpha^\vee \rangle$ for $i' = 0, \dots, rf-2$.

We can now conclude by decreasing induction: using the relation

$$\mathbf{a}^{(rf-j')} = \frac{\mathbf{a}^{(rf-j'+1)} - \alpha_{-(rf-j'+1)}}{p} + p^{rf-1} \alpha_{j'-1}$$

we see that either $\alpha_{j'-1} \notin X^0(T)$ and hence $\mathbf{a}^{(rf-j')}$ is strictly dominant if and only if $\alpha_{j'-1}$ is strictly dominant (in which case $s'_{\text{or},rf-j'}$ is uniquely determined), or $\alpha_{j'} \in X^0(T)$ and hence $\mathbf{a}^{(rf-j')}$ is strictly dominant if and only if $\mathbf{a}^{(rf-j'+1)}$ is strictly dominant (in which case $s'_{\text{or},rf-j'} = s'_{\text{or},rf-j'+1}$ and $s'_{\text{or},rf-j'+1}$ is uniquely determined the inductive hypothesis). \square

2.1.2. Galois deformation rings. We let $\bar{\rho} : G_K \rightarrow \text{GL}_2(\mathbb{F})$ be a continuous Galois representation. Let $\mathcal{C}_{\mathcal{O}}$ be the category of Noetherian complete local \mathcal{O} -algebras with residue field \mathbb{F} and local \mathcal{O} -algebra homomorphisms. The functor that assigns to $(A, \mathfrak{m}_A) \in \mathcal{C}_{\mathcal{O}}$ the set of lifts $\rho_A : G_K \rightarrow \text{GL}_2(A)$ of $\bar{\rho}$ is representable by $R_{\bar{\rho}}^{\square}$, the (unrestricted) lifting ring of $\bar{\rho}$.

Given a tame inertial type τ over \mathcal{O} we let $R_{\bar{\rho}}^{\eta, \tau}$ be the reduced \mathcal{O} -flat quotient of $R_{\bar{\rho}}^{\square}$ such that the $\overline{\mathbb{Q}}_p$ -points of $\text{Spec } R_{\bar{\rho}}^{\eta, \tau}[1/p]$ correspond to the subset of $\rho : G_K \rightarrow \text{GL}_2(\overline{\mathbb{Q}}_p)$ (inside $\text{Spec } (R_{\bar{\rho}}[1/p])$) which are potentially Barsotti–Tate and such that the covariant Weil–Deligne inertial type is isomorphic to $\tau \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p$. The rings $R_{\bar{\rho}}^{\eta, \tau}$ are known in “generic” cases (cf. [EGS15, LLHLM23]):

Theorem 2.1.9. *Assume that τ has a presentation (s, μ) where $2 \leq \langle \mu_j, \alpha^\vee \rangle \leq p-2$. Then either $R_{\bar{\rho}}^{\eta, \tau} = 0$ or*

$$R_{\bar{\rho}}^{\eta, \tau} \cong \mathcal{O}[[Z_1, \dots, Z_{f+4-n}, X_1, Y_1, \dots, X_n, Y_n]] / (X_i Y_i - p, i = 1, \dots, n)$$

for some $n \in \{0, \dots, f\}$.

One of the main goals of this paper, accomplished in §4.6.1, is to provide the analog of Theorem 2.1.9 in highly non-generic situations.

2.2. Breuil–Kisin modules and Emerton–Gee stack.

2.2.1. We introduce the necessary background on Breuil–Kisin modules with tame descent data.

Let $\tau = \tau(s, \mu)$ be a tame inertial type with presentation (s, μ) which we fix throughout this section. Recall that $r \in \{1, 2\}$ is the order of $s_0 s_1 s_2 \cdots s_{f-1} \in W$. Let K'/K be the unramified extension of degree r contained in \overline{K} , set $f' \stackrel{\text{def}}{=} fr$, $e' \stackrel{\text{def}}{=} p^{f'} - 1$, and identify $\text{Hom}_{\mathbb{Q}_p}(K', E)$ with $\mathbb{Z}/f'\mathbb{Z}$ via $\sigma_{j'} \stackrel{\text{def}}{=} \sigma'_0 \circ \varphi^{-j'} \mapsto j'$ where $\sigma'_0 : K' \hookrightarrow E$ is a fixed choice of an embedding extending $\sigma_0 : K \hookrightarrow E$. (In particular, restriction of embeddings corresponds to reduction modulo f in the above identifications.)

We let $\pi' \in \overline{K}$ be an e' -th root of $-p$, $L' \stackrel{\text{def}}{=} K'(\pi')$ and $\Delta' \stackrel{\text{def}}{=} \text{Gal}(L'/K') \subset \Delta \stackrel{\text{def}}{=} \text{Gal}(L'/K)$. We have the character $\omega_{K'}(g) \stackrel{\text{def}}{=} \frac{g(\pi')}{\pi'}$ for $g \in \Delta'$ (which does not depend on the choice of π') and

given an \mathcal{O} -algebra R , we set $\mathfrak{S}_{L',R} \stackrel{\text{def}}{=} (W(k') \otimes_{\mathbb{Z}_p} R)[[u']]$. The latter ring is endowed with the endomorphism $\varphi : \mathfrak{S}_{L',R} \rightarrow \mathfrak{S}_{L',R}$ acting as the Frobenius on $W(k')$ and sending u' to $(u')^p$, and is endowed moreover with an action of Δ by $g'(u') = \frac{g'(\pi')}{\pi'} u' = \omega_{K'}(g')u'$ if $g' \in \Delta'$ and, letting $\sigma^f \in \Delta$ be the lift of the p^f -Frobenius on $W(k')$ which fixes π' , then σ^f acts in natural way on $W(k')$ and trivially on u' (all the endomorphism above act trivially on R by default). Finally, $v \stackrel{\text{def}}{=} (u')^{e'}$,

$$\mathfrak{S}_R \stackrel{\text{def}}{=} (\mathfrak{S}_{L',R})^{\Delta=1} = (W(k) \otimes_{\mathbb{Z}_p} R)[[v]]$$

and $E(v) \stackrel{\text{def}}{=} v + p = (u')^{e'} + p$.

Definition 2.2.1. A Breuil–Kisin module \mathfrak{M} of rank 2 over $\mathfrak{S}_{L',R}$ with descent data of type τ and height ≤ 1 is the datum of:

- (1) a rank 2 projective $\mathfrak{S}_{L',R}$ -module \mathfrak{M} ;
- (2) an injective $\mathfrak{S}_{L',R}$ -linear map $\phi_{\mathfrak{M}} : \varphi^*(\mathfrak{M}) \rightarrow \mathfrak{M}$ whose cokernel is annihilated by $E(v)$; and
- (3) a semilinear action of Δ on \mathfrak{M} which commutes with $\phi_{\mathfrak{M}}$, and such that, for each $j' \in \text{Hom}_{\mathbb{Q}_p}(K', E)$,

$$(\mathfrak{M} \otimes_{W(k') \otimes_{\mathbb{Z}_p} R, \sigma_{j'}} R) \pmod{u'} \cong \tau^\vee \otimes_{\mathcal{O}} R$$

as Δ' -representations.

We write $Y^{[0,1],\tau}$ to be the groupoid of such objects (cf. [LLHLM23, Definition 5.1.3]). We also define $Y^{\eta,\tau}(R) \subset Y^{[0,1],\tau}(R)$ to be the subgroupoid satisfying the additional *determinant condition*

$$\det(\phi_{\mathfrak{M}}) \in (R[[v]])^\times (v + p).$$

We consider $\mathfrak{M}^{(j')} \stackrel{\text{def}}{=} \mathfrak{M} \otimes_{W(k') \otimes_{\mathbb{Z}_p} R, \sigma_{j'}} R$ as a $R[[u']]$ -submodule of \mathfrak{M} in the standard way, so it is endowed with a semilinear action of Δ' . The Frobenius $\phi_{\mathfrak{M}}$ induces Δ' -equivariant morphisms $\phi_{\mathfrak{M}}^{(j')} : \varphi^*(\mathfrak{M}^{(j'-1)}) = (\varphi^*(\mathfrak{M}))^{(j')} \rightarrow \mathfrak{M}^{(j')}$ (here the pull back on the first object is with respect to the R -algebra map $\varphi : R[[u']] \rightarrow R[[u']]$ such that $u' \mapsto u'^p$). We remark that, by letting τ' denote the tame inertial type for K' obtained from τ via the identification $I_{K'} = I_K$ induced by the inclusion $K' \subseteq \overline{K}$, the semilinear action of Δ induces an isomorphism $\iota_{\mathfrak{M}} : (\sigma^f)^*(\mathfrak{M}) \cong \mathfrak{M}$ (see [LLHLM18, §6.1]) as elements of $Y^{[0,1],\tau'}(R)$.

Let $\mathfrak{M} \in Y^{[0,1],\tau}(R)$. Recall that an *eigenbasis* of \mathfrak{M} is a collection of bases $\beta^{(j')} = (f_1^{(j')}, f_2^{(j')})$ for each $\mathfrak{M}^{(j')}$ such that Δ' acts on $f_i^{(j')}$ via the character $\omega_{f_i}^{-\mathbf{a}'_i(0)}$ and such that $\iota_{\mathfrak{M}}((\sigma^f)^*(\beta^{(j')})) = \beta^{(j'+f)}$ for all $j' \in \text{Hom}_{\mathbb{Q}_p}(K', E)$. Given an eigenbasis β for \mathfrak{M} , we let $C_{\mathfrak{M},\beta}^{(j')}$ be the matrix of $\phi_{\mathfrak{M}}^{(j')} : \varphi^*(\mathfrak{M}^{(j'-1)}) \rightarrow \mathfrak{M}^{(j')}$ with respect to the bases $\varphi^*(\beta^{(j'-1)})$ and $\beta^{(j')}$ and set

$$A_{\mathfrak{M},\beta}^{(j')} \stackrel{\text{def}}{=} \text{Ad} \left((s'_{\text{or},j'})^{-1} (u')^{-\mathbf{a}'^{(j')}} \right) (C_{\mathfrak{M},\beta}^{(j')})$$

for $j' \in \text{Hom}_{\mathbb{Q}_p}(K', E)$.

Lemma 2.2.2. *Let $\mathfrak{M} \in Y^{[0,1],\tau}(R)$ with eigenbasis β . The element $A_{\mathfrak{M},\beta}^{(j')}$ has coefficients in $R[[v]]$ and is upper triangular modulo v . Finally, it only depends on the restriction of j' to K .*

Proof. By the definition of eigenbases and of the action of Δ' on $\mathfrak{S}_R \otimes_{W(k'), \sigma_{j'}} R$, we see that $(C_{\mathfrak{M},\beta}^{(j')})_\alpha \in ((u')^{\langle \mathbf{a}'^{(j')}, \alpha^\vee \rangle} R[[v]]) \cap R[[u']]$ for $\alpha \in \Phi$. Explicitly, letting $\delta_{j'} \in \{0, 1\}$ be such that $\delta_{j'} = 0$ if and only if $\mathbf{a}'^{(j')}$ is dominant, we have for $\alpha \in \Phi^+$ that $(C_{\mathfrak{M},\beta}^{(j')})_\alpha \in (u')^{e'\delta_{j'} + \langle \mathbf{a}'^{(j')}, \alpha^\vee \rangle} R[[v]]$ and

$(C_{\mathfrak{M},\beta}^{(j')})_{-\alpha} \in (u')^{e'(1-\delta_{j'})+\langle \mathbf{a}'^{(j')}, -\alpha^\vee \rangle} R[[v]]$. Thus for $\alpha \in \Phi^+$, the α -entry of $\text{Ad} \left((u')^{-\mathbf{a}'^{(j')}} \right) (C_{\mathfrak{M},\beta}^{(j')})$ is in $v^{\delta_{j'}} R[[v]]$ and the $-\alpha$ -entry is in $v^{1-\delta_{j'}} R[[v]]$. By the definition of $\delta_{j'}$ and the fact that $s'_{\text{or},j'} = \text{id}$ if and only if $\mathbf{a}'^{(j')}$ is dominant, we conclude that $A_{\mathfrak{M},\beta}^{(j')}$ is upper triangular modulo v . The fact that $A_{\mathfrak{M},\beta}^{(j')}$ depends only on j' modulo f follows from [LLHLM18, Lemma 6.2, Proposition 6.9]. \square

2.2.2. Étale Φ -modules. We recall the notion of étale Φ -modules.

Recall that $\mathcal{O}_{\mathcal{E},K}$ denotes the p -adic completion of $(W(k)[[v]])[1/v]$. It is endowed with a continuous Frobenius morphism φ extending the Frobenius on $W(k)$ and such that $\varphi(v) = v^p$. Given a p -adically complete Noetherian \mathcal{O} -algebra R we let $\Phi\text{-Mod}_K^{\text{ét},2}(R)$ be the groupoid consisting of projective modules \mathcal{M} of rank 2 over $\mathcal{O}_{\mathcal{E},K} \widehat{\otimes}_{\mathbb{Z}_p} R$ endowed with a Frobenius semilinear endomorphism $\phi_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M}$ inducing an isomorphism on the pull-back: $\text{id} \otimes_{\varphi} \phi_{\mathcal{M}} : \varphi^*(\mathcal{M}) \xrightarrow{\sim} \mathcal{M}$.

We similarly define the ring $\mathcal{O}_{\mathcal{E},L'}$, with Frobenius φ , and the groupoid $\Phi\text{-Mod}_{dd,L'}^{\text{ét},2}(R)$ of rank 2 étale $(\varphi, \mathcal{O}_{\mathcal{E},L'} \widehat{\otimes}_{\mathbb{Z}_p} R)$ -modules with descent data from L' to K .

The groupoids $\Phi\text{-Mod}_K^{\text{ét},2}$, $\Phi\text{-Mod}_{dd,L'}^{\text{ét},2}$ form fppf stacks over $\text{Spf } \mathcal{O}$ (see [CEGSb, §3.1]).

Given $\mathfrak{M} \in Y^{\eta,\tau}(R)$, $\mathfrak{M} \otimes_{\mathfrak{S}_{L',R}} (\mathcal{O}_{\mathcal{E},L'} \widehat{\otimes}_{\mathbb{Z}_p} R)$ is an object $\Phi\text{-Mod}_{dd,L'}^{\text{ét},2}(R)$, which we can descend to an étale φ -module $\mathcal{M} \in \Phi\text{-Mod}_K^{\text{ét},2}(R)$ by

$$\mathcal{M} \stackrel{\text{def}}{=} (\mathfrak{M} \otimes_{\mathfrak{S}_{L',R}} (\mathcal{O}_{\mathcal{E},L'} \widehat{\otimes}_{\mathbb{Z}_p} R))^{\Delta=1}.$$

This defines a morphism of stacks $\varepsilon_{\tau} : Y^{\eta,\tau} \rightarrow \Phi\text{-Mod}_K^{\text{ét},2}$ which is representable by algebraic spaces, proper, and of finite presentation by [CEGSb, Corollary 3.1.8(3), Proposition 3.3.5] (and the fact that taking Δ -invariants is an isomorphism of groupoids). Moreover, ε_{τ} is independent of any \widetilde{W} -presentation of τ .

Given $(\mathcal{M}, \phi_{\mathcal{M}}) \in \Phi\text{-Mod}_K^{\text{ét},2}(R)$, we decompose $\mathcal{M} = \bigoplus_{j \in \mathcal{J}} \mathcal{M}^{(j)}$ over the embeddings $\sigma_j : W(k) \rightarrow \mathcal{O}$, with induced maps $\phi_{\mathcal{M}}^{(j)} : \mathcal{M}^{(j-1)} \rightarrow \mathcal{M}^{(j)}$. The following proposition is a direct computation on the definition of the $A_{\mathfrak{M},\beta}^{(j)}$:

Proposition 2.2.3. ([LLHLM23, Proposition 5.4.2]) *Let $\mathfrak{M} \in Y^{[0,1],\tau}(R)$ and β an eigenbasis of \mathfrak{M} . Let (s, μ) be a small presentation of τ .*

Then there exists a basis \mathfrak{f} for $\varepsilon_{\tau}(\mathfrak{M})$ such that the matrix of $\phi_{\varepsilon_{\tau}(\mathfrak{M})}^{(j)}$ with respect to \mathfrak{f} is given by

$$A_{\mathfrak{M},\beta}^{(j)} s_j^{-1} v^{\mu_j} = A_{\mathfrak{M},\beta}^{(j)} \widetilde{w}^*(\tau)_j.$$

Finally, when R is a complete local Noetherian \mathcal{O} -algebra with finite residue field we have an exact functor

$$\mathbb{V}_K^* : \Phi\text{-Mod}_K^{\text{ét},2}(R) \rightarrow \text{Rep}_R^2(G_{K_{\infty}})$$

establishing an anti-equivalence of categories (by the theory of fields of norms, cf. [LLHLM18, §2.3 and §6.1]) and therefore a functor $T_{dd}^* : Y^{\eta,\tau}(R) \rightarrow \text{Rep}_R^2(G_{K_{\infty}})$ defined as the composite of ε_{τ} followed by \mathbb{V}_K^* . (Note that the formula of *loc. cit.* is inaccurate and should be modified as follows: $\mathbb{V}_K^*(\mathcal{M}) = \text{Hom}_R((\mathcal{M} \otimes_{\mathcal{O}_{\mathcal{E}} \widehat{\otimes}_{\mathbb{Z}_p} R} \mathcal{O}_{\mathcal{E}^{\text{un}}} \widehat{\otimes}_{\mathbb{Z}_p} R)^{\varphi=1}, R)$.) Finally, we recall that given $(\mathcal{M}, \phi_{\mathcal{M}}) \in \Phi\text{-Mod}_K^{\text{ét}}(R)$, we can define an étale $(\varphi^f, \mathcal{O}_{\mathcal{E},K} \widehat{\otimes}_{W(k),\sigma_0} R)$ -module obtained as the f -fold composite of the partial Frobenii acting on $\mathcal{M}^{(0)}$ (see for instance [LLHLM18, §2.3]).

Let \mathcal{Z}^{τ} be the scheme theoretic image of ε_{τ} . This is the moduli stack of tame potentially Barsotti–Tate representations of type τ constructed and studied in [CEGSb, §5.1].

A point $\bar{\rho} \in \mathcal{Z}^\tau(\mathbb{F})$ gives rise to a mod p -representation of G_K . Then $R_{\bar{\rho}}^{\eta, \tau}$, the tame potentially Barsotti–Tate deformation ring of type τ , is a versal ring to \mathcal{Z}^τ at $\bar{\rho}$ ([CEGSb, Corollary 5.2.19]).

3. MODELS

Recall from § 2.2.2 the proper, birational morphism $Y^{\eta, \tau} \rightarrow \mathcal{Z}^\tau$ which is an isomorphism on generic fibers.

3.1. Loop groups and open charts. Given a Noetherian \mathcal{O} -algebra R let $R[v]^{\wedge(v(v+p))}$ denote the $(v(v+p))$ -adic completion of $R[v]$. We denote by $R[v] \left[\frac{1}{v(v+p)} \right]_{\leq 0} \subset R[v]^{\wedge(v(v+p))} \left[\frac{1}{v(v+p)} \right]$ the subring of elements the form $\frac{P}{(v(v+p))^m}$ with $P \in R[v]$ such that $\deg P \leq 2m$.

We define:

$$\begin{aligned} LG(R) &\stackrel{\text{def}}{=} \text{GL}_2 \left(R[v]^{\wedge(v(v+p))} \left[\frac{1}{v(v+p)} \right] \right); \\ L^+G(R) &\stackrel{\text{def}}{=} \text{GL}_2 (R[v]^{\wedge(v(v+p))}); \\ L^-G(R) &\stackrel{\text{def}}{=} \left\{ A \in LG(R) \text{ and } A \text{ has coefficients in } R[v] \left[\frac{1}{v(v+p)} \right]_{\leq 0} \right\} \end{aligned}$$

Now we have surjections

$$\begin{aligned} \text{ev}^+ : L^+G &\twoheadrightarrow \text{GL}_2 \\ \text{ev}^- : L^-G &\twoheadrightarrow \text{GL}_2 \end{aligned}$$

obtained by evaluation modulo v and $1/v$ respectively.

We define

$$\begin{aligned} L_1^+G(R) &\stackrel{\text{def}}{=} \ker \text{ev}^+ \\ L_1^-G(R) &\stackrel{\text{def}}{=} \ker \text{ev}^- \\ L^+\mathcal{G}(R) &\stackrel{\text{def}}{=} (\text{ev}^+)^{-1}(B) \\ L^{--}\mathcal{G}(R) &\stackrel{\text{def}}{=} (\text{ev}^+)^{-1}(\bar{N}) \end{aligned}$$

Note that the functors $L^+\mathcal{G}$ and $L^{--}\mathcal{G}$ have a slightly different meaning than the corresponding functors in [LLHLM23], however they coincide after p -adic completion. We define a closed subfunctor $\mathcal{A}(\eta) \subset LG$ whose R -valued points consist of $A \in LG(R)$ satisfying

- (1) $\det A \in (R[v]^{\wedge(v(v+p))})^\times (v+p)$;
- (2) $A \in \text{Mat}_2(R[v]^{\wedge(v(v+p))})$ and upper triangular mod v ;
- (3) $(v+p)A^{-1}$ is upper triangular mod v .

Note that $\mathcal{A}(\eta)(\bar{\mathbb{F}})$ identifies with

$$\bigcup_{\tilde{z}_j \in \text{Adm}(\eta)} L^+\mathcal{G}(\bar{\mathbb{F}}) \tilde{z}_j L^+\mathcal{G}(\bar{\mathbb{F}})$$

where $\text{Adm}(\eta) = \left\{ \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ v & 0 \end{pmatrix} \right\}$ is the η -admissible set.

Given a type τ with small presentation $(s, \mu) \in \underline{W} \times X^*(\underline{T})$ we define

$$(3.1.1) \quad LG^\tau(R) \stackrel{\text{def}}{=} \prod_{j \in \mathcal{J}} \mathcal{A}(\eta)(R) s_j v^{\mu_j} \subset LG^{\mathcal{J}}(R)$$

We also define

$$LG^{\text{bd},(v+p)v^\mu}(R) \stackrel{\text{def}}{=} \left\{ (A_j) \in (LG(R))^{\mathcal{J}}, \text{ such that : } \begin{array}{l} \det A_j \in (R[v]^{\wedge(v(v+p))})^\times (v+p)v^{\langle \mu_j, \alpha^\vee \rangle} \\ A_j \in \text{Mat}_2(R[v]^{\wedge(v(v+p))}) \end{array} \right\}.$$

Lemma 3.1.2. (1) L_1^+G is a normal subgroup of both of L^+G and $L^+\mathcal{G}$, and

$$L^+G = L_1^+G \times \text{GL}_2$$

$$L^+\mathcal{G} = L_1^+G \times B$$

(2) The multiplication maps $L_1^+G \times L^-G \rightarrow LG$, $L^+\mathcal{G} \times L^{--}\mathcal{G} \rightarrow LG$ are formally étale monomorphisms after p -adic completion.

(3) $\mathcal{A}(\eta)^{\wedge p}$ has an affine open cover $\mathcal{A}(\tilde{z})^{\wedge p}$ where \tilde{z} runs over $\text{Adm}(\eta)$ (cf. Table 1).

Proof. Item (1) is clear.

Item (2) for the map $L^+\mathcal{G} \times L^{--}\mathcal{G} \rightarrow LG$ follows from [LLHLM23, Lemmas 3.2.2, 3.2.6] (using that $R[v]^{v(v+p)} = R[[v]]$ on rings where p is nilpotent), which implies the statement for $L_1^+G \times L^-G \rightarrow LG$.

We prove item (3). By the previous item, $\mathcal{A}^{\wedge p}(\tilde{z}) \stackrel{\text{def}}{=} L^+\mathcal{G} \cdot L\mathcal{G}^{--}\tilde{z} \cap \mathcal{A}(\eta)^{\wedge p} = L^+\mathcal{G} \cdot (L\mathcal{G}^{--}\tilde{z} \cap \mathcal{A}(\eta)^{\wedge p})$ is an open subfunctor (since $L^+\mathcal{G} \backslash \mathcal{A}(\eta)$ is finite type, cf. the proof of [LLHLM23, Corollary 3.2.10]). Consideration on $\overline{\mathbb{F}}$ -points shows that $\mathcal{A}^{\wedge p}(\tilde{z})$ for $\tilde{z} \in \text{Adm}(\eta)$ form an open cover.

Finally, if R is an \mathcal{O} -algebra where p is nilpotent, $L^{--}\mathcal{G}(R)\tilde{z}$ consists of matrices A of the form

$$A = \begin{pmatrix} 1 + \frac{1}{v}a & \frac{1}{v}b \\ c & 1 + \frac{1}{v}d \end{pmatrix} \tilde{z}$$

for $a, b, c, d, \in R[\frac{1}{v}]$, and imposing the conditions (1), (2) on it gives the explicit description in table 1. \square

TABLE 1. The affine cover of $L^+\mathcal{G} \backslash \mathcal{A}(\eta)^{\wedge p}$.

\tilde{z}	t_η	$w_0 t_\eta$	$t_{w_0 \eta}$
$\mathcal{A}^{\wedge p}(\tilde{z})$	$\begin{pmatrix} (v+p) & 0 \\ vx & 1 \end{pmatrix}$	$\begin{pmatrix} X & 1 \\ v & Y \end{pmatrix}$ $XY + p = 0$	$\begin{pmatrix} 1 & y \\ 0 & v+p \end{pmatrix}$

3.2. Loop groups and moduli of Breuil–Kisin modules. $Y^{\eta, \tau}$ has the following description as a quotient stack (cf. [LLHLM23, 5.2.1]):

Lemma 3.2.1. *Let τ be a tame inertial type with small presentation $(s, \mu) \in \underline{W} \times X^*(\underline{T})$. Then any $\mathfrak{M} \in Y^{\eta, \tau}$ has an eigenbasis Zariski locally on $Y^{\eta, \tau}$ and the assignment $\mathfrak{M} \mapsto (A_{\mathfrak{M}, \beta}^{(j)})_{j \in \mathcal{J}} \tilde{w}^*(\tau)$ defines an isomorphism of p -adic formal algebraic stacks*

$$(3.2.2) \quad Y^{\eta, \tau} \xrightarrow{\sim} \left[LG^\tau / \varphi \prod_{j \in \mathcal{J}} L^+\mathcal{G} \right]^{\wedge p}$$

and hence a morphism

$$(3.2.3) \quad Y^{\eta, \tau} \rightarrow \left[LG^{\text{bd}, (v+p)v^\mu} / \prod_{\varphi} \prod_{j \in \mathcal{J}} L^+G \right]^{\wedge_p}$$

We have a morphism

$$\iota : \left[LG^{\text{bd}, (v+p)v^\mu} / \prod_{\varphi} \prod_{j \in \mathcal{J}} L^+G \right]^{\wedge_p} \rightarrow \Phi\text{-Mod}_K^{\text{ét}, 2}$$

sending the class of $A \stackrel{\text{def}}{=} (A^{(j)})_{j \in \mathcal{J}}$ to the étale φ -module $\iota(A)$ which is free of rank 2 and such that $\phi_{\iota(A)}^{(j)} : \iota(A)^{(j-1)} \rightarrow \iota(A)^{(j)}$ has matrix $A^{(j)}$ in the standard basis.

Proposition 3.2.4. *Assume $p - 2 > \max_j \langle \mu_j, \alpha^\vee \rangle$. Let τ be a tame inertial type with a small presentation $(s, \mu) \in \underline{W} \times X^*(\underline{T})$. We have a commutative diagram of p -adic formal algebraic stacks over $\text{Spf}\mathcal{O}$:*

$$(3.2.5) \quad \begin{array}{ccc} Y^{\eta, \tau} & \xrightarrow{3.2.3} & \left[LG^{\text{bd}, (v+p)v^\mu} / \prod_{\varphi} \prod_{j \in \mathcal{J}} L^+G \right]^{\wedge_p} \\ & \searrow \varepsilon_\tau & \downarrow \exists \\ & & \mathcal{Z}^\tau \\ & \searrow \varepsilon_\tau & \downarrow \iota \\ & & \Phi\text{-Mod}_K^{\text{ét}, 2} \end{array}$$

where the hooked diagonal arrow is a closed immersion. In particular, the dotted arrow exists and makes the diagram commute.

Proof. Denote by $Y^{\text{bd}, \tau}$ the groupoid in the upper right vertex of the diagram (3.2.5).

The external triangle is commutative by Proposition 2.2.3 and Lemma 3.2.1. The factorization of ε_τ is by definition of \mathcal{Z}^τ .

Hence, the existence of the dotted arrow will follow once we prove that the diagonal hooked arrow is a closed immersion. Since ι is proper (as LG/L^+G is ind-proper and $Y^{\text{bd}, \tau}$ is a finite type p -adic formal algebraic stack), it suffices to show that it is a monomorphism.

We prove that for any Noetherian \mathcal{O}/ϖ^a -algebra R , and any pair $A_1, A_2 \in Y^{\text{bd}, \tau}(R)$, the morphism ι induces a bijection

$$\text{Hom}_{Y^{\text{bd}, \tau}}(A_1, A_2) \xrightarrow{\sim} \text{Hom}_{\Phi\text{-Mod}}(\iota(A_1), \iota(A_2)).$$

The induced map is clearly injective, and we thus prove its surjectivity. Assume that there exists $X = (X_j)_{j \in \mathcal{J}} \in LG(R)^{\mathcal{J}}$ such that

$$(3.2.6) \quad A_1^{(j)} = X_j A_2^{(j)} \left(\varphi(X_{j-1}) \right)^{-1}$$

or, equivalently,

$$(3.2.7) \quad \left(\varphi(X_{j-1}) \right) = (A_1^{(j)})^{-1} X_j A_2^{(j)}$$

for all $j \in \mathcal{J}$.

We show by induction on a that (3.2.7) forces $X_j \in L^+G(R)$ for all $j \in \mathcal{J}$.

If $a = 1$ then $p = 0$ in R and, noting that $\det((A_2^{(j)})^{-1}) \det(A_1^{(j)}) \in R^\times$ by assumption, we deduce from (3.2.7) that $\varphi^f(\det(X_j)) = u_j \det(X_j)$ for a unit $u_j \in R^\times$. This shows that $\det(X_j) \in R^\times$.

We now show that $X_j \in \text{Mat}_2(R[[v]])$. Let $\kappa_j \in \mathbb{Z}$ be the pole order of X_j at v , i.e. $\kappa_j \in \mathbb{Z}$ is minimal such that $v^{\kappa_j} X_j \in \text{Mat}_2(R[[v]])$. As $\det(A_1^{(j)}) \in R^\times v^{1+\langle \mu_j, \alpha^\vee \rangle}$ we deduce from (3.2.7) that

$$1 + \kappa_j + \langle \mu_j, \alpha^\vee \rangle \geq p\kappa_{j-1}$$

which forces $\kappa_j \leq \max_i \{ \frac{\langle \mu_i, \alpha \rangle + 1}{p-1} \} \leq \frac{p+3}{2(p-1)} < 1$. We conclude that $X_j \in L^+G(R)$ for all $j \in \mathcal{J}$ when $a = 1$.

Assume the assertion up to $a - 1$. We can thus write $X_j = \tilde{X}_j(1 + \varepsilon_j)$ where $\tilde{X}_j \in L^+G(R)$, $\varepsilon_j \in \text{Lie } LG((\varpi^{a-1}))$, for all $j \in \mathcal{J}$. Replacing $A_1^{(j)}$ by $(\tilde{X}_j)^{-1} A_1^{(j)} \left(\varphi(\tilde{X}_{j-1}) \right)$, we can assume that (3.2.6) is true with $X_j = 1 + \varepsilon_j$. In particular $A_2^{(j)} = (1 + \delta_j) A_1^{(j)}$ for some $\delta_j \in \text{Lie } LG((\omega^{a-1}))$. We now prove that actually $\varepsilon_j \in \text{Lie } L^+G((\omega^{a-1}))$. Then (3.2.6) becomes $1 = (1 + \varepsilon_j)(1 + \delta_j) \text{Ad}(A_1^{(j)})(1 - \varphi(\varepsilon_{j-1}))$ so that

$$(3.2.8) \quad (A_1^{(j)})^{-1} \varepsilon_j A_1^{(j)} + (A_1^{(j)})^{-1} (A_2^{(j)} - A_1^{(j)}) = \varphi(\varepsilon_{j-1}).$$

Let κ_j be the pole order of ε_j . Observe that

- $v^{\kappa_j + \langle \mu_j, \alpha^\vee \rangle} (v + p) \left((A_1^{(j)})^{-1} \varepsilon_j A_1^{(j)} \right) \in \text{Mat}_2(R[v]^{\wedge_{v(v+p)}})$
- $v^{\langle \mu_j, \alpha^\vee \rangle} (v + p) \left((A_1^{(j)})^{-1} (A_2^{(j)} - A_1^{(j)}) \right) \in \text{Mat}_2(R[v]^{\wedge_{v(v+p)}})$.

Since $(v + p)\varepsilon_j = v\varepsilon_j$ we deduce that

$$p\kappa_{j-1} \leq \max\{\kappa_j + \langle \mu_j, \alpha^\vee \rangle + 1, \langle \mu_j, \alpha^\vee \rangle + 1\}.$$

This forces $\kappa_j \leq \max_i \{ \frac{\langle \mu_i, \alpha^\vee \rangle + 1}{p-1} \} \leq \frac{p+3}{2(p-1)} < 1$. So that indeed $\varepsilon_j \in \text{Lie } L^+G((\omega^{a-1}))$. \square

By taking fiber product, we thus obtain a commutative diagram:

$$(3.2.9) \quad \begin{array}{ccccc} \tilde{Y}^{\eta, \tau} & \xrightarrow{\pi} & \tilde{Z}^\tau & \hookrightarrow & \left[LG^{\text{bd}, (v+p)v^\mu} / \varphi \prod_{j \in \mathcal{J}} L_1^+G \right]^{\wedge_p} \\ \downarrow \Pi_{\mathcal{J}} \text{GL}_2 & \square & \downarrow \Pi_{\mathcal{J}} \text{GL}_2 & \square & \downarrow \Pi_{\mathcal{J}} \text{GL}_2 \\ Y^{\eta, \tau} & \longrightarrow & Z^\tau & \hookrightarrow & \left[LG^{\text{bd}, (v+p)v^\mu} / \varphi \prod_{j \in \mathcal{J}} L^+G \right]^{\wedge_p} \\ & \searrow \varepsilon_\tau & \downarrow & \swarrow & \\ & & \Phi\text{-Mod}_K^{\text{ét}, 2} & & \end{array}$$

where the hooked arrows are closed immersions, the arrows decorated with $\prod_{\mathcal{J}} \text{GL}_2$ are $\prod_{\mathcal{J}} \text{GL}_2$ -torsors, and the central squares are cartesian, which defines the stacks $\tilde{Y}^{\eta, \tau}$, \tilde{Z}^τ .

3.3. Models for moduli of Breuil–Kisin modules. We now define $\text{Gr}_1^\tau \hookrightarrow \text{Gr}_1^{\text{bd}, (v+p)v^\mu}$ as the fpqc quotients $\prod_{j \in \mathcal{J}} L_1^+G \backslash LG^\tau \hookrightarrow \prod_{j \in \mathcal{J}} L_1^+G \backslash LG^{\text{bd}, (v+p)v^\mu}$. We define $Y^{\text{mod}, \eta, \tau}$ as the quotient

$$\left[\text{Gr}_1^\tau / \prod_{\mathcal{J}} B\text{-sh.cnj} \right]^{\wedge_p}$$

by the shifted conjugation action $(b_j) \cdot (g_j) = (b_j g_j b_{j-1}^{-1})$. Define also $\tilde{Y}^{\text{mod}, \eta, \tau}$ as the fiber product

$$\begin{array}{ccc} \tilde{Y}^{\text{mod}, \eta, \tau} & \longrightarrow & \left(\text{Gr}_1^{\text{bd}, (v+p)v^\mu} \right)^{\wedge_p} \\ \downarrow & \square & \downarrow \text{GL}_2^{\mathcal{J}} \\ Y^{\text{mod}, \eta, \tau} & \longrightarrow & \left[\text{Gr}_1^{\text{bd}, (v+p)v^\mu} / \prod_{\mathcal{J}} \text{GL}_2\text{-sh.cnj} \right]^{\wedge_p} \end{array}$$

Proposition 3.3.1. *The p -adic formal scheme $\tilde{Y}^{\text{mod}, \eta, \tau}$ identifies with the (closed formal) subscheme of $\left(\prod_{\mathcal{J}} (\text{Gr}_1 \times (B \setminus \text{GL}_2, \mathcal{O})) \right)^{\wedge_p}$ consisting of pairs (X, g) such that $(g_j X_j g_{j-1}^{-1})_{j \in \mathcal{J}} \in \text{Gr}_1^{\tau}$. In particular, $(X_j)_{j \in \mathcal{J}} \in \text{Gr}_1^{\text{bd}, (v+p)v^\mu}$.*

Proof. It follows from the definition that $\tilde{Y}^{\text{mod}, \eta, \tau}$ is the quotient of the space of triples $((X_j), (Y_j), (g_j)) \in \text{Gr}_1^{\text{bd}, (v+p)v^\mu} \times \text{Gr}_1^{\tau} \times \text{GL}_2^{\mathcal{J}}$ satisfying $Y_j = g_j X_j (g_{j-1})^{-1}$ by the action of $B^{\mathcal{J}}$ given by

$$(b_j) \cdot ((X_j), (Y_j), (g_j)) = ((X_j), (b_j Y_j b_{j-1}^{-1}), (b_j g_j)).$$

This finishes the proof because the (class of the) triple $((X_j), (Y_j), (g_j))$ is uniquely determined by (the class of) $((X_j), (g_j))$. \square

We define $\tilde{\mathcal{Z}}^{\text{mod}, \tau}$ as the scheme theoretic image of projection map $\text{pr} : \tilde{Y}^{\text{mod}, \eta, \tau} \rightarrow \text{Gr}_1^{\text{bd}, (v+p)v^\mu}$ sending (X, g) to X . This gives a factorization

$$(3.3.2) \quad \begin{array}{ccc} \tilde{Y}^{\text{mod}, \eta, \tau} & \xrightarrow{\pi^{\text{mod}}} & \tilde{\mathcal{Z}}^{\text{mod}, \tau} \\ & \searrow \text{pr} & \downarrow \iota \\ & & \text{Gr}_1^{\text{bd}, (v+p)v^\mu} \end{array}$$

For any $\tilde{z} \in \widetilde{W}^\vee$ by Lemma 3.1.2(2) we have a formally étale monomorphism

$$(3.3.3) \quad \text{Gr}_1^{\text{bd}, (v+p)v^\mu}(\tilde{z}) \stackrel{\text{def}}{=} [(L^- G)^{\mathcal{J}} \tilde{z}] \cap \text{Gr}_1^{\text{bd}, (v+p)v^\mu} \hookrightarrow \text{Gr}_1^{\text{bd}, (v+p)v^\mu}$$

after p -adic completion, which is an open immersion because the target is of finite type. Define $\tilde{U}(\tilde{z})$ to be the p -adic completion of the LHS of (3.3.3). Note that $\tilde{U}(\tilde{z}) = \prod_j \tilde{U}(\tilde{z}_j)$ has an obvious product structure.

Lemma 3.3.4. $\{\tilde{U}(\tilde{z})\}_{\tilde{z} \in \widetilde{W}^\vee}$ is a Zariski open covering for the p -adic completion of $\text{Gr}_1^{\text{bd}, (v+p)v^\mu}$.

Proof. Since $\text{Gr}_1^{\text{bd}, (v+p)v^\mu}$ is finite type the $\tilde{U}(\tilde{z})$ are actually Zariski open formal subschemes of its p -adic completion. Looking at $\overline{\mathbb{F}}$ -points shows that they cover. \square

In particular, by diagram (3.3.2), $\tilde{U}(\tilde{z})$ induce open substacks $\tilde{Y}^{\text{mod}, \eta, \tau}(\tilde{z}) \subseteq \tilde{Y}^{\text{mod}, \eta, \tau}$, $\tilde{\mathcal{Z}}^{\text{mod}, \tau}(\tilde{z}) \subseteq \tilde{\mathcal{Z}}^{\text{mod}, \tau}$.

Lemma 3.3.5. *The $\tilde{Y}^{\text{mod}, \eta, \tau}(\tilde{z})$ ($\tilde{\mathcal{Z}}^{\text{mod}, \tau}(\tilde{z})$) for $\tilde{z} \in \text{Adm}(\eta)^{\mathcal{J}} s^{-1} v^\mu$ form a Zariski open cover of $\tilde{Y}^{\text{mod}, \eta, \tau}$ (resp. $\tilde{\mathcal{Z}}^{\text{mod}, \eta, \tau}$).*

Proof. This follows from Lemma 3.1.2(3). \square

Lemma 3.3.6. $\tilde{Y}^{\text{mod},\eta,\tau}$ is an \mathcal{O} -flat local complete intersection of dimension $5f$ over \mathcal{O} .

Proof. It suffices to show that $Y^{\text{mod},\eta,\tau}$ is a \mathcal{O} -flat local complete intersection of dimension f over \mathcal{O} . Since Gr_1^τ is a $B^\mathcal{J}$ -torsor over $Y^{\text{mod},\eta,\tau}$, the result follows from Table 1. \square

Lemma 3.3.7. Assume that $p - 2 - \langle \mu_j, \alpha^\vee \rangle \geq N$ for all $j \in \mathcal{J}$. We have an isomorphism

$$\begin{array}{ccc} \tilde{Y}^{\eta,\tau} \otimes_{\mathcal{O}} \mathcal{O}/p^N & \longrightarrow & \left[LG^{\text{bd},(v+p)v^\mu} /_{\varphi} \prod_{j \in \mathcal{J}} L_1^+ \mathbf{G} \right] \otimes_{\mathcal{O}} \mathcal{O}/p^N \\ \downarrow \cong & & \downarrow \cong \\ \tilde{Y}^{\text{mod},\eta,\tau} \otimes_{\mathcal{O}} \mathcal{O}/p^N & \longrightarrow & \text{Gr}_1^{\text{bd},(v+p)v^\mu} \otimes_{\mathcal{O}} \mathcal{O}/p^N \end{array}$$

Proof. The fact that the left vertical arrow is an isomorphism follows from the fact that the right vertical arrow is an isomorphism. The latter fact can be proven similar to the proof of [LLHLM23, Lemma 5.2.2]: the result would follow from the fact that for any \mathcal{O}/p^N -algebra R and $A \in LG^{\text{bd},(v+p)v^\mu}(R)$ the map

$$(X_j) \mapsto (X_j \text{Ad}(A_j)\varphi(X_{j-1}))$$

is an automorphism of $(L_1^+ \mathbf{G}(R))^\mathcal{J}$. In turn this follows from the fact that

$$v^p A_j^{-1} \in \frac{v^p}{(v+p)v^{\langle \mu_j, \alpha \rangle}} \text{Mat}_2(R[[v]]) \subset v^{p-N-\langle \mu_j, \alpha \rangle} \text{Mat}_2(R[[v]]) \subset v^2 \text{Mat}_2(R[[v]]),$$

where we use that $\frac{1}{(v+p)} R[[v]] \subset \frac{1}{v^N} R[[v]]$. \square

As a consequence of Lemma 3.3.7 and diagram 3.2.9, $\tilde{U}(\tilde{z})$ also induces open substacks $\tilde{Y}^{\eta,\tau}(\tilde{z})$, $\tilde{Z}^\tau(\tilde{z})$. The following Theorem is the main result of the paper.

Theorem 3.3.8. Fix a small presentation (s, μ) of τ . Assume either $p > 8f + 3 + \max_j \langle \mu_j, \alpha^\vee \rangle$ or $p > 7$ and $K = \mathbb{Q}_p$. Then we have an isomorphism

$$\tilde{Z}^{\text{mod},\tau}(\tilde{z}) \cong \tilde{Z}^\tau(\tilde{z}).$$

Remark 3.3.9. (1) Since (s, μ) is small, $\max_j \langle \mu_j, \alpha^\vee \rangle \leq \frac{p+1}{2}$. In particular the hypothesis on p is satisfied for all τ when $p > 16f + 7$.

(2) Theorem 3.3.8 is proven under the first hypothesis in section 4.5. The proof under the improved bound $p > 7$ for \mathbb{Q}_p is completed in section 5.5.1, cf. Remark 5.5.1 for the source of the improvements.

The proof of Theorem 3.3.8 will be performed in two steps:

- (1) We first show there is a isomorphism $\tilde{Z}^\tau(\tilde{z}) \otimes_{\mathcal{O}} \mathcal{O}/p^N \xrightarrow{\sim} \tilde{Z}^{\text{mod},\tau}(\tilde{z}) \otimes_{\mathcal{O}} \mathcal{O}/p^N$ for sufficiently large N .
- (2) We bound the power of p that belongs to the ideal of singularity of $\tilde{Z}^{\text{mod},\eta,\tau}$ over \mathcal{O} . This allows us to lift the isomorphism mod p^N in the above step to an isomorphism over \mathcal{O} .

In the remainder of this section we will carry out the first step of the above strategy modulo some geometric facts about $\tilde{Y}^{\text{mod},\eta,\tau} \rightarrow \tilde{Z}^{\text{mod},\tau}$ which will be established in the later sections. The second step of the strategy is carried out in Section 4.5

3.3.1. *Modeling mod p^N .* Set $N \stackrel{\text{def}}{=} p-2-\max_j \langle \mu_j, \alpha^\vee \rangle$. Define $\tilde{\mathcal{Z}}_N^{\text{apx}}$ as the scheme theoretic image of the map

$$\tilde{Y}^{\eta, \tau} \otimes_{\mathcal{O}} \mathcal{O}/p^N \rightarrow \left[LG^{\text{bd}, (v+p)v^\mu} / \varphi \prod_{j \in \mathcal{J}} L_1^+ G \right] \otimes_{\mathcal{O}} \mathcal{O}/p^N = \text{Gr}_1^{\text{bd}, (v+p)v^\mu} \otimes_{\mathcal{O}} \mathcal{O}/p^N.$$

We have the commutative diagram

$$(3.3.10) \quad \begin{array}{ccccc} \tilde{Y}^{\eta, \tau} \otimes_{\mathcal{O}} \mathcal{O}/p^N & \xrightarrow{\sim \text{Lemma 3.3.7}} & \tilde{Y}^{\text{mod}, \eta, \tau} \otimes_{\mathcal{O}} \mathcal{O}/p^N & & \\ \downarrow \pi & \searrow & \downarrow \pi^{\text{mod}} & & \\ \tilde{\mathcal{Z}}^\tau \otimes_{\mathcal{O}} \mathcal{O}/p^N & \xleftarrow{\heartsuit} & \tilde{\mathcal{Z}}_N^{\text{apx}} & \xrightarrow{\heartsuit\heartsuit} & \tilde{\mathcal{Z}}^{\text{mod}, \tau} \otimes_{\mathcal{O}} \mathcal{O}/p^N \\ & \searrow (3.2.9) & \downarrow & & \downarrow \iota \\ & & [LG^{\text{bd}, (v+p)v^\mu} / \varphi \prod_{\mathcal{J}} L_1^+ G] \otimes_{\mathcal{O}} \mathcal{O}/p^N & \xrightarrow{\sim} & \text{Gr}_1^{\text{bd}, (v+p)v^\mu} \otimes_{\mathcal{O}} \mathcal{O}/p^N \end{array}$$

Proposition 3.3.11. *The morphisms \heartsuit , $\heartsuit\heartsuit$ have a factorization*

$$\begin{array}{ccccc} \tilde{\mathcal{Z}}^\tau \otimes_{\mathcal{O}} \mathcal{O}/p^N & \xleftarrow{\heartsuit} & \tilde{\mathcal{Z}}_N^{\text{apx}} & \xrightarrow{\heartsuit\heartsuit} & \tilde{\mathcal{Z}}^{\text{mod}, \tau} \otimes_{\mathcal{O}} \mathcal{O}/p^N \\ \uparrow & & \uparrow & & \uparrow \\ \tilde{\mathcal{Z}}^\tau \otimes_{\mathcal{O}} \mathcal{O}/p^{N-1} & \xleftarrow{\sim} & \tilde{\mathcal{Z}}_N^{\text{apx}} \otimes_{\mathcal{O}} \mathcal{O}/p^{N-1} & \xleftarrow{\sim} & \tilde{\mathcal{Z}}^{\text{mod}, \tau} \otimes_{\mathcal{O}} \mathcal{O}/p^{N-1} \end{array}$$

In particular, the natural morphisms \heartsuit , $\heartsuit\heartsuit$ induce an isomorphism

$$\tilde{\mathcal{Z}}^\tau \otimes_{\mathcal{O}} \mathcal{O}/p^{N-1} \xrightarrow{\sim} \tilde{\mathcal{Z}}^{\text{mod}, \tau} \otimes_{\mathcal{O}} \mathcal{O}/p^{N-1}.$$

The proof of Proposition 3.3.11 crucially relies on the following result, whose proof will be postponed until section 4.5.

Proposition 3.3.12. *We have:*

$$(3.3.13) \quad p \text{ coker} \left(\mathcal{O}_{\tilde{\mathcal{Z}}^{\text{mod}, \tau}} \rightarrow \pi_*^{\text{mod}} \left(\mathcal{O}_{\tilde{Y}^{\text{mod}, \eta, \tau}} \right) \right) = 0.$$

Proposition 3.3.14. *For $\ell > 0$*

$$R^\ell \pi_*^{\text{mod}} \mathcal{O}_{\tilde{Y}^{\text{mod}, \eta, \tau}} = 0.$$

Proof of Proposition 3.3.11. The statement is local on $\text{Gr}_1^{\text{bd}, (v+p)v^\mu} \otimes_{\mathcal{O}} \mathcal{O}/p^N$, so it suffices to prove it after intersecting everything with $\tilde{U}(\tilde{z})$. Write $\tilde{\mathcal{Z}}^\tau(\tilde{z}) = \text{Spf } R$, $\tilde{\mathcal{Z}}^{\text{mod}, \tau}(\tilde{z}) = \text{Spf } R^{\text{mod}}$, and $\tilde{\mathcal{Z}}_N^{\text{mod}}(\tilde{z}) = \text{Spf } R_N^{\text{apx}}$. Also let $S = \pi_* \left(\mathcal{O}_{\tilde{Y}^{\text{mod}, \eta, \tau}(\tilde{z})} \right)$ and $S^{\text{mod}} = \pi_*^{\text{mod}} \left(\mathcal{O}_{\tilde{Y}^{\text{mod}, \eta, \tau}(\tilde{z})} \right)$ and $S_N = \pi_*^{\text{mod}} \left(\mathcal{O}_{\tilde{Y}^{\text{mod}, \eta, \tau}(\tilde{z})}/p^N \right)$. Note that S and S^{mod} are p -torsion free and, $S/p^N \hookrightarrow S_N = S^{\text{mod}}/p^N$ by Proposition 3.3.14.

Since $\tilde{\mathcal{Z}}^{\text{mod}, \tau}(\tilde{z})$ is the scheme theoretic image of $\tilde{Y}^{\text{mod}, \eta, \tau} \rightarrow \tilde{U}(\tilde{z})$ we have $R^{\text{mod}} \subset S^{\text{mod}}$. Similarly $R \subset S$. Finally equation (3.3.13) shows that $S^{\text{mod}}/R^{\text{mod}}$ is p -torsion. We thus have the following commutative diagram

$$(3.3.15) \quad \begin{array}{ccccccc} S & \twoheadrightarrow & S/p^N & \hookrightarrow & S_N & = & S^{\text{mod}}/p^N & \hookleftarrow & S^{\text{mod}} \\ \uparrow \pi & & \uparrow & & \uparrow & & \uparrow & & \uparrow \pi^{\text{mod}} \\ R & \twoheadrightarrow & R/p^N & \xrightarrow{\heartsuit\heartsuit} & R_N^{\text{apx}} & \xleftarrow{\heartsuit} & R^{\text{mod}}/p^N & \hookleftarrow & R^{\text{mod}} \end{array}$$

where the hooked arrow are injective. Let C , C^{mod} denote the cokernel of the maps π , π^{mod} respectively. As S , S^{mod} are both p -flat, we deduce from (3.3.15) the following commutative diagram with exact rows (this defines C_N):

$$(3.3.16) \quad \begin{array}{ccccccc} & & \ker(\heartsuit\heartsuit) & & & & \\ & & \downarrow & & & & \\ & & \swarrow & & & & \\ 0 & \rightarrow & C^{\text{mod}}[p^N] & \rightarrow & R^{\text{mod}}/p^N & \xrightarrow{\pi^{\text{mod}}} & S^{\text{mod}}/p^N \rightarrow C^{\text{mod}}/p^N \rightarrow 0 \\ & & \downarrow & & \downarrow \heartsuit & & \parallel & & \parallel \\ 0 & \rightarrow & 0 & \rightarrow & R_N^{\text{apx}} & \hookrightarrow & S_N & \rightarrow & C_N \rightarrow 0 \\ & & \uparrow & & \uparrow \heartsuit & & \uparrow & & \uparrow \\ 0 & \rightarrow & C[p^N] & \rightarrow & R/p^N & \xrightarrow{\pi} & S/p^N & \rightarrow & C/p^N \rightarrow 0 \\ & & \swarrow & & \uparrow & & & & \\ & & \ker(\heartsuit) & & & & & & \end{array}$$

Proposition 3.3.12 imply that $pC^{\text{mod}} = 0$ and hence $C^{\text{mod}}[p^N] = C^{\text{mod}}[p]$. Thus $\ker(\heartsuit\heartsuit)$ is annihilated by p . We conclude that $p\ker(\heartsuit\heartsuit) \subseteq p^N R^{\text{mod}}$ and, as R^{mod} is p -flat, that $\ker(\heartsuit\heartsuit) \subseteq p^{N-1} R^{\text{mod}}$. This implies the factorization

$$\begin{array}{ccc} R^{\text{mod}}/p^N & \xrightarrow{\heartsuit\heartsuit} & R_N^{\text{apx}} \\ \downarrow & \searrow \exists & \\ R^{\text{mod}}/p^{N-1} & & \end{array}$$

and hence an isomorphism $R^{\text{mod}}/p^{N-1} \cong R_N^{\text{apx}}/p^{N-1}$.

Now $pC_N = 0$ so $p(C/p^N) = 0$. Hence $pC \subset p^N C \subset p^2 C$ and as C is p -adically separated we learn that $pC = 0$. We repeat the argument in the previous paragraph to obtain the factorization

$$\begin{array}{ccc} R/p^N & \xrightarrow{\heartsuit} & R_N^{\text{apx}} \\ \downarrow & \searrow \exists & \\ R/p^{N-1} & & \end{array}$$

and hence an isomorphism $R/p^{N-1} \cong R_N^{\text{apx}}/p^{N-1}$. \square

4. GEOMETRY OF LOCAL MODELS

4.1. Equations for $\tilde{Y}^{\text{mod},\eta,\tau}$. Let τ be a tame inertial type with small presentation (s, μ) . Everything we do depends on this choice but we usually suppress this dependence from the notation. Let $\tilde{w} \in \text{Adm}^\vee(\eta)^\mathcal{J}$ and set $\tilde{z} = \tilde{w}s^{-1}v^\mu$. We consider the morphism $\text{GL}_{2,\mathcal{O}} \twoheadrightarrow B \backslash \text{GL}_{2,\mathcal{O}} \xrightarrow{\sim} \mathbb{P}_{\mathcal{O}}^1$ sending $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ to $[\gamma : \delta] \in \mathbb{P}_{\mathcal{O}}^1$. Given a Noetherian \mathcal{O} -algebra R where p is nilpotent, by Proposition 3.3.1 and the definition of $\mathcal{A}(\eta)$, we see that $\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})(R)$ the groupoid of tuples $(l_j, \kappa_j, X_j)_{j \in \mathcal{J}} \in (\mathbb{P}^1 \times \text{GL}_2 \times L_1^- \text{G})^\mathcal{J}$ subject to the following conditions that for some (equivalently, any) lift $\tilde{l}_j \in \text{GL}_2(R)$ of l_j :

- (1) $X_j \tilde{w}_j \text{Ad}(s_j^{-1}v^{\mu_j})(\tilde{l}_{j-1}^{-1}) \in \text{Mat}_2(R[[v]])$

$$(2) \det(X_j \tilde{w}_j) \in R^\times(v+p)$$

$$(3) \text{ we have } l_j \kappa_j \cdot (X_j \tilde{w}_j \text{Ad}(s_j^{-1} v^{\mu_j})(\tilde{l}_{j-1}^{-1}))|_{v=0} = [0 : 1].$$

Note that $\text{Ad}(s_j^{-1} v^{\mu_j})(B(R)) \in B(R[v])$ since μ is dominant, which justifies the independence of the choice of the lift $\tilde{l}_{j-1} \in \text{GL}_2(R)$ in items (1),(2),(3) above.

In item (3) we have used the following

Convention 4.1.1. Given $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{Mat}_2(R)$ and $x, y, z, t \in R$, the equality

$$[x : y] \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = [z : t]$$

is interpreted as $t(\alpha x + \gamma y) - z(\beta x + \delta y) = 0$.

Equivalently, this can be written in matrix form

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t \\ -z \end{pmatrix} = 0.$$

We will always adopt the above convention in the tables below, note that this allows us to interpret equations as $[x : y] = [z : t]$ even if neither side are actual elements of \mathbb{P}^1 . We will also consider the following auxiliary space $\widetilde{Ba}(\tilde{z})$ of tuples $(l_j, \kappa_j, X_j, r_j)_{j \in \mathcal{J}} \in (\mathbb{P}^1 \times \text{GL}_2 \times L_1^- \text{G} \times \mathbb{P}^1)^{\mathcal{J}}$ satisfying the following variant of the conditions above:

$$(1) X_j \tilde{w}_j \text{Ad}(s_j^{-1} v^{\mu_j})(\tilde{r}_j^{-1}) \in \text{Mat}_2(R[v])$$

$$(2) \det(X_j \tilde{w}_j) \in R^\times(v+p)$$

$$(3) \text{ we have } l_j \kappa_j \cdot (X_j \tilde{w}_j \text{Ad}(s_j^{-1} v^{\mu_j})(\tilde{r}_j^{-1}))|_{v=0} = [0 : 1].$$

Clearly, we have a decomposition $\widetilde{Ba}(\tilde{z}) = \prod_{j \in \mathcal{J}} \widetilde{Ba}_j(\tilde{z}_j)$ where the j -th factor classify the quadruples $(l_j, \kappa_j, X_j, r_j)$. Furthermore $\tilde{Y}^{\text{mod}, \eta, \tau}$ is exactly the subspace of $\widetilde{Ba}(\tilde{z})$ where we impose that $r_j = l_{j-1}$ for all j .

Lemma 4.1.2. The f -tuples $(X_j, r_j)_{j \in \mathcal{J}} \in (L_1^- \text{G} \times \mathbb{P}^1)^{\mathcal{J}}(R)$ satisfying condition (1) are precisely those given in Table 2, according to $\tilde{w}_j \in \text{Adm}^\vee(\eta_j)$, $s_j \in W$ and $k_j \stackrel{\text{def}}{=} \langle \mu_j, \alpha^\vee \rangle$.

TABLE 2. Description of $X_j \in L_1^- \text{G}$

(\tilde{w}_j, s_j)	$(t_\eta, (12)), (w_0 t_\eta, (12)), (t_{w_0(\eta)}, \text{id})$	$(t_\eta, \text{id}), (w_0 t_\eta, \text{id}), (t_{w_0(\eta)}, (12))$
	$s_j w_j^{-1} X_j w_j s_j^{-1} = \begin{pmatrix} 1 + \frac{A}{v^{k_j}} & \frac{B}{v} \\ \frac{C}{v^{k_j}} + \frac{C'}{v^{k_j-1}} & 1 + \frac{D}{v} \end{pmatrix}$ $A, B, C, C', D \in R,$	$s_j w_j^{-1} X_j w_j s_j^{-1} = \begin{pmatrix} 1 + \frac{A}{v} & 0 \\ \frac{C}{v} + \frac{C'}{v^{k_j+1}} & 1 \end{pmatrix}$ $A, C, C' \in R$
$k_j = 0$	$A = C = C' = 0$	$C' = 0$
$k_j = 1$	$[A : B] = [C : D] = r_j, C' = 0$	$[C' : 1] = r_j$
$k_j > 1$	$[A : B] = [C : D] = [C' : 1] = r_j$	$[C' : 1] = r_j$

Proof. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(R)$ so that $r_j = [c : d]$. Also write $s_j w_j^{-1} X_j w_j s_j^{-1} = \begin{pmatrix} 1 + \alpha & \beta \\ \gamma & 1 + \delta \end{pmatrix}$ so that $\alpha, \beta, \gamma, \delta \in \frac{1}{v} R[\frac{1}{v}]$. We abbreviate $k = k_j$.

Letting $w_j t_{\nu_j} \stackrel{\text{def}}{=} \tilde{w}_j$ condition (1) is equivalent to

$$(4.1.3) \quad \begin{pmatrix} (1+\alpha)v^{k+\varepsilon} & \beta v^{1-\varepsilon_j} \\ \gamma v^{k+\varepsilon} & (1+\delta)v^{1-\varepsilon} \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \begin{pmatrix} v^k R[[v]] & R[[v]] \\ v^k R[[v]] & R[[v]] \end{pmatrix}$$

where $\varepsilon = 0$ (resp. $\varepsilon = 1$) if (\tilde{w}_j, s_j) is as in the first (resp. second) column of Table 2. From this we learn that $v^{1-\varepsilon}\beta$, $v^{1-\varepsilon}(1+\delta)$, $(1+\alpha)v^{k+\varepsilon}$, $\gamma v^{k+\varepsilon} \in R[[v]]$. The first two conditions show that the second column of $\begin{pmatrix} 1+\alpha & \beta \\ \gamma & 1+\delta \end{pmatrix}$ has the form specified in the table.

We now show that α, γ also have the form specified in the table.

Case $\varepsilon = 0$. Looking at the second column of (4.1.3) we get

$$(4.1.4) \quad v^k(1+\alpha)d - v\beta c \in v^k R[[v]]$$

$$(4.1.5) \quad v^k \gamma d - v(1+\delta)c \in v^k R[[v]]$$

Note that these two equations imply that αd and γd have the form specified in the table.

Since we already know $(1+\alpha)v^k$, $\gamma v^k \in R[[v]]$ we only have to consider the case $k > 1$. But then equation (4.1.5) shows that $c \in R d$ hence $d \in R^\times$ because $(c, d) = R$. But this implies α and γ are of the desired form.

Case $\varepsilon = 1$. Looking at the second column of (4.1.3) we get (using that $\beta = \delta = 0$)

$$(4.1.6) \quad v^{k+1}(1+\alpha)d \in v^k R[[v]]$$

$$(4.1.7) \quad v^{k+1}\gamma d - c \in v^k R[[v]].$$

Note that these two equations imply that αd and γd have the form specified in the table.

Since we already know $(1+\alpha)v^{k+1}$, $\gamma v^{k+1} \in R[[v]]$ we only have to consider the case $k > 0$. But then equation (4.1.7) shows that $c = 0$ hence $d \in R^\times$. Again this implies α and γ are of the desired form.

We have now shown that $\alpha, \beta, \gamma, \delta$ have the desired form. Plugging this information into equations (4.1.4), (4.1.5), (4.1.6), (4.1.7) immediately yields the remaining equations in the table. \square

Lemma 4.1.8. *For each j the spaces $\widetilde{Ba}_j(\tilde{z}_j)$ have explicit presentations given by Table 3, in terms of the corresponding spaces in Table 2.*

TABLE 3. Presentations of $\widetilde{Ba}_j(\widetilde{z}_j)$.

$\langle \mu_j, \alpha^V \rangle$	\widetilde{w}_j	s_j	t_η	$w_0 t_\eta$	$t_{w_0(\eta)}$
> 1	(12)		Variables: A, B, C, C', D $A = BC', C = pC', D = p$ $r_j = [C' : 1]$ $l_j \kappa_j = [B : -p]$	Variables: A, B, C, C', D $A = BC', C = pC', D = p$ $r_j = [C' : 1]$ $l_j \kappa_j = [-p : B]$	Variables: A, C, C' $A = p$ $r_j = [C' : 1]$ $l_j \kappa_j = [0 : 1]$
		id	Variables: A, C, C' $A = p$ $r_j = [C' : 1]$ $l_j \kappa_j = [C : -p]$	Variables: A, C, C' $A = p$ $r_j = [C' : 1]$ $l_j \kappa_j = [-p : C]$	Variables: A, B, C, C', D $A = BC', C = pC', D = p$ $r_j = [C' : 1]$ $l_j \kappa_j = [0 : 1]$
$= 1$	(12)		Variables: A, B, C, C', D $A = p - D$ $r_j = [C : D] = [p - D : B]$ $l_j \kappa_j = [D - p : C] = [-B : D]$	Variables: A, B, C, C', D $A = p - D$ $r_j = [C : D] = [p - D : B]$ $l_j \kappa_j = [C : D - p] = [-D : B]$	Variables: A, C, C' $A = p$ $r_j = [C' : 1]$ $l_j \kappa_j = [0 : 1]$
		id	Variables: A, C, C' $A = p$ $r_j = [C' : 1]$ $l_j \kappa_j = [C : -p]$	Variables: A, C, C' $A = p$ $r_j = [C' : 1]$ $l_j \kappa_j = [-p : C]$	Variables: A, B, C, C', D $A = p - D$ $r_j = [C : D] = [p - D : B]$ $l_j \kappa_j = r_j$
$= 0$	id	Variables: A, C, C' $A = p, C' = 0$ $l_j \kappa_j = r_j \begin{pmatrix} 1 & 0 \\ -C & p \end{pmatrix}$	Variables: A, C, C' $A = p, C' = 0$ $l_j \kappa_j = r_j \begin{pmatrix} 0 & -1 \\ -p & C \end{pmatrix}$	Variables: A, B, C, C', D $A = 0, C = 0, C' = 0, D = p$ $l_j \kappa_j = r_j \begin{pmatrix} p & -B \\ 0 & 1 \end{pmatrix}$	

The meaning of the variables A, B, C, C', D is in terms of the $X_j \in L_1^- G$ extracted from the corresponding entries in Table 2.

Proof. The equations involving A, B, C, C', D are exactly obtained from Table 2 by imposing the condition $\det(X_j) \in R^\times \frac{(v+p)}{v}$. The formula for r_j also follows from Table 2.

Thus the only thing we need to verify is the equation involving l_j . This is immediate in all cases except when $(\widetilde{w}_j, \langle \mu_j, \alpha^V \rangle, s_j) = (t_{w_0(\eta)}, 1, \text{id})$. In this case let $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \text{GL}_2(R)$ (resp. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(R)$) be a lift of $l_j \kappa_j$ (resp. r_j). Thus condition (3) (together with condition (1)) is the condition that the (2, 1)-entry of $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} d & aB - Ab \\ -c & aD - Cb \end{pmatrix}$ is zero, which exactly means that $l_j \kappa_j = r_j$. \square

Remark 4.1.9. So far we only considered $\widetilde{Ba}_j(\widetilde{z}_j)$, $\widetilde{U}(\widetilde{z}_j)$ as p -adic formal schemes. However Table 3 gives obvious de completions and we will work with the scheme version of these spaces in what follows.

4.2. Cohomological properties of $\widetilde{Y}^{\text{mod},\eta,\tau} \rightarrow \text{Gr}_1^{\text{bd},(v+p)v^\mu}$. Recall that diagram (3.3.2) gives a factorization

$$(4.2.1) \quad \widetilde{Y}^{\text{mod},\eta,\tau}(\widetilde{z}) \xrightarrow{\pi^{\text{mod}}} \widetilde{Z}^{\text{mod},\tau}(\widetilde{z}) \xrightarrow{\iota} \widetilde{U}(\widetilde{z})$$

In this section we will use the explicit presentations from the previous section to study $R\pi_*^{\text{mod}}\mathcal{O}_{\widetilde{Y}^{\text{mod},\eta,\tau}}$ and hence prove Propositions 3.3.12, 3.3.14.

Since ι is a closed immersion, we have $\iota_* \circ R\pi_*^{\text{mod}}\mathcal{O}_{\widetilde{Y}^{\text{mod},\eta,\tau}} = R\text{pr}_*\mathcal{O}_{\widetilde{Y}^{\text{mod},\eta,\tau}}$. In particular Proposition 3.3.12 is equivalent to

$$p \text{ coker} \left(\mathcal{O} \left(\text{Gr}_1^{\text{bd},(v+p)v^\mu} \right) \rightarrow (\text{pr})_* \left(\mathcal{O} \left(\widetilde{Y}^{\text{mod},\eta,\tau} \right) \right) \right) = 0$$

and Proposition 3.3.14 is equivalent to $R\text{pr}_*\mathcal{O}_{\widetilde{Y}^{\text{mod},\eta,\tau}}$ concentrating in degree 0. In other words we can replace π^{mod} by pr in the statements of interest. Furthermore, these statements are local on the target so it suffices to analyze the situation after intersecting with $\widetilde{U}(\widetilde{z})$ for $\widetilde{z} \in \text{Adm}^\vee(\eta)s^{-1}v^\mu$.

To analyze pr we factorize

$$\widetilde{Y}^{\text{mod},\eta,\tau}(\widetilde{z}) \xrightarrow{\quad} \widetilde{Ba}(\widetilde{z}) = \prod_j \widetilde{Ba}_j(\widetilde{z}_j) \xrightarrow{\prod \text{pr}_j} \widetilde{U}(\widetilde{z}) = \prod_j \widetilde{U}(\widetilde{z}_j)$$

where $\text{pr}_j : \widetilde{Ba}_j(\widetilde{z}_j) \rightarrow \widetilde{U}(\widetilde{z}_j)$ is the map

$$(l_j, \kappa_j, X_j, r_j) \mapsto \kappa_j X_j.$$

Observe that we have an isomorphism $\widetilde{Ba}_j(\widetilde{z}_j) \cong \text{GL}_2 \times Ba_j(\widetilde{z}_j)$ where $Ba_j(\widetilde{z}_j) = \widetilde{Ba}_j(\widetilde{z}_j) \times_{\text{GL}_2} \{1\}$, given by $(l_j, \kappa_j, X_j, r_j) \mapsto (\kappa_j, (l_j \kappa_j, X_j, r_j))$. Let p_j (resp. q_j) be the obvious projections from $Ba_j(\widetilde{z}_j) \subset \mathbb{P}^1 \times \widetilde{U}(\widetilde{z}_j) \times \mathbb{P}^1$ to the left (resp. right) \mathbb{P}^1 factor. We continue to denote pr_j the projection from $Ba_j(\widetilde{z}_j)$ to the middle factor (this is compatible with the projection from $\widetilde{Ba}_j(\widetilde{z}_j)$).

The following is immediate from Table 3

Lemma 4.2.2. *Up to isomorphism, there are the following possibilities for $Ba_j(\widetilde{z}_j)$*

- (1) $Ba_j(\widetilde{z}_j) = \mathbb{A}^2$ with p_j, q_j are constant maps from \mathbb{A}^2 to \mathbb{P}^1 and pr_j is the identity. This covers the cases $(\widetilde{w}_j, s_j, k_j) = (t_{w_0(\eta)}, \text{id}, > 1), (t_{w_0(\eta)}, (12), \geq 1)$.
- (2) $Ba_j(\widetilde{z}_j) = \text{Bl}_{(p,0)}\mathbb{A}^1 \times \mathbb{A}^1$ where $\text{Bl}_{(p,0)}\mathbb{A}^1 = \{([x : y], C) \mid Cx = py\} \subset \mathbb{P}^1 \times \mathbb{A}^1$ is the blowup of $\mathbb{A}_{\mathcal{O}}^1$ at the origin in its special fiber. Then p_j is the natural projection map from $\text{Bl}_{(p,0)}\mathbb{A}^1 \rightarrow \mathbb{P}^1$ and q_j is the natural inclusion $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ given by $C' \mapsto [C' : 1]$, and pr_j is the natural map to \mathbb{A}^2 extracting C, C' . This covers $(\widetilde{w}_j, s_j, k_j) = (t_\eta, (12), > 1), (t_\eta, \text{id}, \geq 1), (w_0 t_\eta, (12), > 1), (w_0 t_\eta, \text{id}, \geq 1)$.
- (3) $Ba_j(\widetilde{z}_j) \subset \mathbb{P}^1 \times \mathbb{A}^1 \times \mathbb{P}^1$ consists of $([x : y], C, [x' : y'])$ such that $pxy' - yx' = 0$. Then p_j, q_j are the projections to the left, resp. right \mathbb{P}^1 factor, and pr_j is the natural map to \mathbb{A}^1 extracting C . This covers the cases when $\langle \mu_j, \alpha^\vee \rangle = 0$.
- (4) $Ba_j(\widetilde{z}_j) \subset \mathbb{P}^1 \times M \times \mathbb{P}^1$ consists of $([x : y], \begin{pmatrix} A & B \\ C & D \end{pmatrix}, [x' : y'])$ such that
 - $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ has determinant 0 and trace p ;

- $(x' \ y') \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} = 0$
- $(x \ y) \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} = 0$

Then p_j, q_j are the projections to the left, resp. right \mathbb{P}^1 factor, and pr_j is the map extracting A, B, C, D . This covers $(\tilde{w}_j, s_j, k_j) = (t_\eta, (12), 1), (w_{0t_\eta}, (12), 1)$.

- (5) It is the same as the previous case, except that instead of the third item we impose $[x : y] = [x' : y']$. This correspond to the case $(\tilde{w}_j, s_j, k_j) = (t_{w_0(\eta)}, \text{id}, 1)$.

Lemma 4.2.3. (1) pr_j is proper and when $\langle \mu_j, \alpha^\vee \rangle > 0$, pr_j becomes a closed immersion after inverting p ;

- (2) $Ba_j(\tilde{z}_j)$ is a \mathcal{O} -flat local complete intersection of relative dimension 2 over \mathcal{O} . The relative dualizing sheaf of $Ba_j(\tilde{z}_j)/\mathcal{O}$ is $q_j^* \mathcal{O}_{\mathbb{P}^1}(-1) \otimes p_j^* \mathcal{O}_{\mathbb{P}^1}(-1)$.

- (3) $\mathcal{O}_{\tilde{U}(\tilde{z}_j)} \rightarrow \text{pr}_{j*} \mathcal{O}_{Ba_j(\tilde{z}_j)}$ is surjective.

The same assertions hold for $\widetilde{Ba}_j(\tilde{z}_j)$ and $\widetilde{Ba}(\tilde{z})$, but with relative dimensions 6 and 6f.

Proof. The first assertion is clear from Table 3 once we observe that (l_j, r_j) can be uniquely solved for when p is invertible and $k_j > 0$.

For the second assertion, we inspect the five cases in Lemma 4.2.2. The result is obvious for case (1) and follows from Lemma 4.2.4 below for case (2).

For case (3), the result follows from the fact that $Ba_j(\tilde{z}_j) \hookrightarrow \mathbb{P}^1 \times \mathbb{A}^1 \times \mathbb{P}^1$ is a regular immersion cut out by an equation of bidegree $(1, 1)$, which has normal bundle $q_j^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes p_j^* \mathcal{O}_{\mathbb{P}^1}(1)$, and thus relative dualizing sheaf $q_j^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes p_j^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes q_j^* \mathcal{O}_{\mathbb{P}^1}(-2) \otimes p_j^* \mathcal{O}_{\mathbb{P}^1}(-2) = q_j^* \mathcal{O}_{\mathbb{P}^1}(-1) \otimes p_j^* \mathcal{O}_{\mathbb{P}^1}(-1)$.

We turn to case (4). Set $t = x/y, s = x'/y'$. The $Ba_j(\tilde{z}_j)$ has an open cover given by

- $\text{Spec } \mathcal{O}[A, B, C, D, s, t]/(A - stD, B - tD, C - sD, p - D - stD)$.
- $\text{Spec } \mathcal{O}[A, B, C, D, s^{-1}, t]/(A - tC, B - ts^{-1}C, D - s^{-1}C, p - s^{-1}C - tC)$.
- $\text{Spec } \mathcal{O}[AB, C, D, s, t^{-1}]/(A - sB, C - st^{-1}B, D - tB, p - t^{-1}B - sB)$.
- $\text{Spec } \mathcal{O}[A, B, C, D, s, t^{-1}]/(B - s^{-1}A, C - t^{-1}A, D - s^{-1}t^{-1}A, p - A - s^{-1}t^{-1}A)$.

Thus we obtain locally a regular immersion from $Ba_j(\tilde{z}_j)$ to \mathbb{A}^6 . A simple computation shows that the determinant of the normal bundle of the regular immersion $Ba_j(\tilde{z}_j) \hookrightarrow \mathbb{P}^1 \times \mathbb{A}^4 \times \mathbb{P}^1$ is $q_j^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes p_j^* \mathcal{O}_{\mathbb{P}^1}(1)$, hence again the relative dualizing sheaf is $q_j^* \mathcal{O}_{\mathbb{P}^1}(-1) \otimes p_j^* \mathcal{O}_{\mathbb{P}^1}(-1)$.

We finally deal with case (5). Then $Ba_j(\tilde{z}_j) \hookrightarrow \mathbb{P}^1 \times \mathbb{A}^4$ is a regular immersion cut out by the equations

$$A + D = p, AD - BC = 0, (x \ y) \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} = 0$$

A similar computation as in the previous case yields the desired result.

The third assertion follows from the fact that pr_j factors through the subscheme $Z \subset \tilde{U}(\tilde{z}_j)$ such that

- Z is normal;
- $\text{pr}_{j*} \mathcal{O}_{Ba_j(\tilde{z}_j)} = \mathcal{O}_Z$ after inverting p .

The existence of such a subscheme Z follows immediately from inspecting the cases in Lemma 4.2.2: in fact $Z = \mathbb{A}^2$ with coordinate C, C' or $Z = \text{Spec } \mathcal{O}[B, C, D]/((p - D)D - BC)$ (if $\langle \mu_j, \alpha^\vee \rangle > 0$) or $Z = \mathbb{A}^1$ with coordinate C if $\langle \mu_j, \alpha^\vee \rangle = 0$. \square

We will make use of the following elementary computation.

Lemma 4.2.4. *Let $Bl_{(0,p)}\mathbb{A}^1 = \{([x : y], t) \mid xp = ty\} \subset \mathbb{P}^1 \times \mathbb{A}^1$ be the blowup of $\mathbb{A}^1_{/\mathcal{O}}$ at the ideal $(0, p)$. Let $\mathcal{O}(-k)$ be the pull back of $\mathcal{O}_{\mathbb{P}^1}(-k)$ by the projection to \mathbb{P}^1 . Then*

(1) *For $k \geq 0$*

$$H^n(Bl_{(0,p)}\mathbb{A}^1, \mathcal{O}(-k)) = \begin{cases} p^k \mathcal{O}[t] & \text{if } n = 0; \\ \text{annihilated by } p^{k-1} & \text{if } n = 1; \\ 0 & \text{if } n > 1. \end{cases}$$

(2) *The relative dualizing sheaf of $Bl_{(0,p)}\mathbb{A}^1/\mathcal{O}$ is $\mathcal{O}(-1)$.*

Proof. (1) $R\Gamma(Bl_{(0,p)}\mathbb{A}^1, \mathcal{O}(-k))$ is computed by the Čech complex:

$$\mathcal{O}\left[t, \frac{t}{p}\right] \oplus \left(\frac{p}{t}\right)^k \mathcal{O}\left[t, \frac{p}{t}\right] \rightarrow \mathcal{O}\left[t, \left(\frac{p}{t}\right)^{\pm 1}\right]$$

where all the terms are viewed as $\mathcal{O}[t]$ -submodules of $E[t^{\pm 1}]$ and the differential is given by $(f, g) \mapsto f - g$. The result now follows from an explicit computation.

(2) Since $Bl_{(0,p)}\mathbb{A}^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{A}^1$ is a regular immersion cut out by a degree 1 equation in the \mathbb{P}^1 coordinates, the normal bundle is $\mathcal{O}(1)$, hence the dualizing complex is $\mathcal{O}(-2) \otimes \mathcal{O}(1) = \mathcal{O}(-1)$. □

The following Lemma will be the key to our analysis:

Lemma 4.2.5. *Let $j \in \mathcal{J}$ and let $\varepsilon_j, \delta_j \in \{0, 1\}$.*

(1) *$\mathrm{pr}_{j*} \left(q_j^*(\mathcal{O}_{\mathbf{P}^1}(-1))^{\varepsilon_j} \otimes_{\mathcal{O}_{Ba(\tilde{z}_j)}} p_j^*(\mathcal{O}_{\mathbf{P}^1}(-1))^{\delta_j} \right)$ is p -torsion free.*

(2) *The complex $R\mathrm{pr}_{j*} \left(q_j^*(\mathcal{O}_{\mathbf{P}^1}(-1))^{\varepsilon_j} \otimes_{\mathcal{O}_{Ba(\tilde{z}_j)}} p_j^*(\mathcal{O}_{\mathbf{P}^1}(-1))^{\delta_j} \right)$ is concentrated in degree 0 if $(\varepsilon_j, \delta_j) \neq (1, 1)$, and is concentrated in degrees 0 and 1 if $(\varepsilon_j, \delta_j) = (1, 1)$.*

(3) *If $(\varepsilon_j, \delta_j) = (1, 1)$ then $R^1\mathrm{pr}_{j*} \left(q_j^*(\mathcal{O}_{\mathbf{P}^1}(-1))^{\varepsilon_j} \otimes_{\mathcal{O}_{Ba(\tilde{z}_j)}} p_j^*(\mathcal{O}_{\mathbf{P}^1}(-1))^{\delta_j} \right)$ is*

$$\begin{cases} \mathcal{O}\text{-torsion free} & \text{if } \langle \mu_j, \alpha^\vee \rangle = 0; \\ \text{isomorphic to } \mathbb{F} & \text{if } \langle \mu_j, \alpha^\vee \rangle = 1 \text{ and } (s_j, \tilde{w}_j) = (\mathrm{id}, t_{w_0(\eta)}); \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The first item is obvious because $Ba_j(\tilde{z}_j)$ is flat over \mathcal{O} . We now explain the cohomological computations.

For the remainder of the proof we set $\varepsilon \stackrel{\mathrm{def}}{=} \varepsilon_j$, $\delta \stackrel{\mathrm{def}}{=} \delta_j$ and $\mathcal{F} \stackrel{\mathrm{def}}{=} q_j^*(\mathcal{O}_{\mathbf{P}^1}(-1))^\varepsilon \otimes_{\mathcal{O}_{Ba(\tilde{z}_j)}} p_j^*(\mathcal{O}_{\mathbf{P}^1}(-1))^\delta$. We work with the five cases in Lemma 4.2.2.

The computation is trivial for case (1), and follows from Lemma 4.2.4 for case (2).

We now turn to case (4). We have an open cover $U_1 = \{y \neq 0\}$, $U_2 = \{x \neq 0\}$. Then setting $t = x/y$ we see that U_1 is the space of $(t, C, [x' : y']) \subset \mathbb{A}^2 \times \mathbb{P}^1$ such that $x'(p - Ct) - Cy' = 0$. Hence U_1 is isomorphic to $Bl_{(p,0)}\mathbb{A}^1 \times \mathbb{A}^1$ and $\mathcal{F}|_{U_1}$ is the pull-back of $\mathcal{O}(-1)^\varepsilon$ from the obvious map to \mathbb{P}^1 . Thus

- $R\Gamma(U_1, \mathcal{F}) = \mathcal{O}[C, t]p^\varepsilon$;
- Similarly $R\Gamma(U_2, \mathcal{F}) = \mathcal{O}[B, t^{-1}]p^\varepsilon$
- $R\Gamma(U_1 \cap U_2, \mathcal{F}) = \mathcal{O}[C, t, t^{-1}]p^\varepsilon$

are all concentrated in degree zero. Thus the Čech cohomology spectral sequence computing $R\Gamma(Ba_j(\tilde{z}_j), \mathcal{F})$ degenerates at E_1 and $R\Gamma(Ba_j(\tilde{z}_j), \mathcal{F})$ is computed by

$$(4.2.6) \quad \mathcal{O}[t, C] \oplus t^{-\delta} \mathcal{O}[t^{-1}, B] \xrightarrow{d} \mathcal{O}[t^{\pm 1}, C]$$

where the differential are the obvious inclusion induced by the relation $B = t(p - tC)$. In turn, this complex is quasi isomorphic to the complex

$$(4.2.7) \quad t^{-\delta} \mathcal{O}[t^{-1}, B] \xrightarrow{d} \mathcal{O}[t^{\pm 1}, C] / \mathcal{O}[t, C] = \bigoplus_{k \geq 1} t^{-k} \mathcal{O}[C].$$

Since $\delta \in \{0, 1\}$, $t^{-k} \in \text{Im}(d)$ for $k \geq 1$. Now for $\ell > 0$ and $k \geq 0$

$$(4.2.8) \quad t^{-2k-\ell} (t(p-tC))^k = (-1)^k \frac{C^k}{t^\ell} + \dots \in \text{Im}(d)$$

where \dots is a \mathcal{O} -linear combination of $\frac{C^n}{t^m}$ where $0 \leq n < k$. Hence $\frac{C^k}{t^\ell} \in \text{Im}(d)$ by induction on k .

We now deal with case (5). Similar to the previous case, we have an open cover $U_2 = \{y \neq 0\}$, $U_1 = \{x \neq 0\}$ and set $t = y/x$.

We have, after choosing a trivialization of \mathcal{F} on U_1

$$\begin{aligned} \mathcal{F}(U_1) &= \frac{\mathcal{O}[t, B, C, D]}{(D - tC, p - D - Bt)} \cong \mathcal{O}[t, C] \\ \mathcal{F}(U_2) &= t^{-(\varepsilon+\delta)} \frac{\mathcal{O}[t^{-1}, B, C, D]}{((p-D) - Bt^{-1}, C - Dt^{-1})} \cong t^{-(\varepsilon+\delta)} \mathcal{O}[t^{-1}, B] \end{aligned}$$

Thus $R\Gamma(Ba_j(\tilde{z}_j), \mathcal{F})$ is computed by the Čech complex

$$(4.2.9) \quad \mathcal{O}[t, C] \oplus t^{-\varepsilon-\delta} \mathcal{O}[t^{-1}, B] \xrightarrow{d} \mathcal{O}[t^{\pm 1}, C]$$

where the differential are the obvious inclusion induced by the relation $B = t(p - tC)$. As before this complex is quasi isomorphic to

$$t^{-\varepsilon-\delta} \mathcal{O}[t^{-1}, B] \xrightarrow{d} \mathcal{O}[t^{\pm 1}, C] / \mathcal{O}[t, C] = \bigoplus_{k \geq 1} t^{-k} \mathcal{O}[C].$$

Compared to case (4) the only new computation we have to make is when $(\varepsilon, \delta) = (1, 1)$. In this case, we have $t^{-k} \in \text{Im}(d)$ for $k \geq 2$ but only $pt^{-1} \in \text{Im}(d)$ (and $t^{-1} \notin \text{Im}(d)$). Equation (4.2.8) for $k, \ell \geq 1$ then shows that $\frac{C^k}{t^\ell} \in \text{Im}(d)$ for all $k, \ell \geq 1$. This means that the H^1 of the Čech complex is isomorphic to \mathbb{F} .

Finally we deal with case (3). Consider the open cover $U_1 = \{x \neq 0\}$, $U_2 = \{y \neq 0\}$. Write $t = y/x$, $s = x'/y'$. Then $U_2 \cong \mathbb{A}^2 = \text{Spec}(\mathcal{O}[s, C, t^{-1}]/(s - pt^{-1}))$, while $U_1 \cong \text{Bl}_{(0,p)} \mathbb{A}^1 \times \mathbb{A}^1$, where the coordinate on the blowup \mathbb{A}^1 is t . Now $\mathcal{F}|_{U_1} \cong \mathcal{O}(-1)^\varepsilon$ so $R\Gamma(U_1, \mathcal{F}) = p^\varepsilon \mathcal{O}[C, t]$ is concentrated in degree 0. On the other hand, U_2 is affine so $R\Gamma(U_2, \mathcal{F}) = s^\varepsilon t^{-\delta} \mathcal{O}[s, C, t^{-1}]/(s - pt^{-1})$. Moreover $R\Gamma(U_1 \cap U_2, \mathcal{F}) = p^\varepsilon \mathcal{O}[s, C, t^{\pm 1}]/(s - pt^{-1})$. Thus the Čech complex computing $R\Gamma(Ba_j(\tilde{z}_j), \mathcal{F})$ is given by

$$p^\varepsilon \mathcal{O}[C, t] \oplus s^\varepsilon t^{-\delta} \mathcal{O}[s, C, t^{-1}]/(s - pt^{-1}) \rightarrow p^\varepsilon \mathcal{O}[s, C, t^{\pm 1}]/(s - pt^{-1}).$$

This is quasi-isomorphic to the complex

$$t^{-\varepsilon-\delta} \mathcal{O}[C, t^{-1}] \rightarrow \mathcal{O}[C, t^{\pm 1}] / \mathcal{O}[C, t]$$

with differential induced by the natural inclusion. This has no H^1 if $(\varepsilon, \delta) \neq (1, 1)$, and has $H^1 \cong \mathcal{O}[C]$ if $(\varepsilon, \delta) = (1, 1)$. We are thus done with the cohomological computations. This finishes the proof. \square

Corollary 4.2.10. *Consider the commutative diagram:*

$$\begin{array}{ccc} \tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z}) & \xrightarrow{\Delta} & \widetilde{Ba}(\tilde{z}) \\ & \searrow \text{pr} & \downarrow \text{pr}_{\tilde{B}} \\ & & \tilde{U}(\tilde{z}) \end{array}$$

Then $R^i \text{pr}_{\tilde{B}*} \mathcal{O}_{\widetilde{Ba}(\tilde{z})} = 0$ if $i > 0$ and $\mathcal{O}_{\tilde{U}(\tilde{z})} \rightarrow \text{pr}_{\tilde{B}*} \mathcal{O}_{\widetilde{Ba}(\tilde{z})}$.

In particular, by letting $\mathcal{I}(\tilde{z})$ be the ideal sheaf defining the closed immersion Δ

- $\text{coker} \left(\mathcal{O}_{\tilde{U}(\tilde{z})} \rightarrow \text{pr}_{\tilde{B}*} \mathcal{O}_{\tilde{Y}^{\text{mod},(\eta,\tau)}(\tilde{z})} \right) = R^1 \text{pr}_{\tilde{B}*} \mathcal{I}(\tilde{z})$
- $R^i \text{pr}_{\tilde{B}*} \mathcal{O}_{\tilde{Y}^{\text{mod},(\eta,\tau)}(\tilde{z})} = R^{i+1} \text{pr}_{\tilde{B}*} \mathcal{I}(\tilde{z})$ for $i > 1$.

Proof. The first part follows from the fact that $R \text{pr}_{\tilde{B}*} \mathcal{O}_{\widetilde{Ba}(\tilde{z})} = \boxtimes_j R \text{pr}_{j*} \widetilde{Ba}_j(\tilde{z}_j)$ and hence is concentrated in degree 0 by Lemma 4.2.5 (with $\varepsilon_j = \delta_j = 0$).

For the second part, we have the exact triangle

$$R \text{pr}_{\tilde{B}*} \mathcal{I}(\tilde{z}) \rightarrow R \text{pr}_{\tilde{B}*} \mathcal{O}_{\widetilde{Ba}(\tilde{z})} \rightarrow R \text{pr}_{\tilde{B}*} \mathcal{O}_{\tilde{Y}^{\text{mod},(\eta,\tau)}(\tilde{z})} \rightarrow$$

which immediately gives the second item. For the first item, observe that the exact triangle implies $\text{coker} \left(\text{pr}_{\tilde{B}*} \mathcal{O}_{\widetilde{Ba}(\tilde{z})} \rightarrow \text{pr}_{\tilde{B}*} \mathcal{O}_{\tilde{Y}^{\text{mod},(\eta,\tau)}(\tilde{z})} \right) = R^1 \text{pr}_{\tilde{B}*} \mathcal{I}(\tilde{z})$ but also

$$\text{coker} \left(\text{pr}_{\tilde{B}*} \mathcal{O}_{\widetilde{Ba}(\tilde{z})} \rightarrow \text{pr}_{\tilde{B}*} \mathcal{O}_{\tilde{Y}^{\text{mod},(\eta,\tau)}(\tilde{z})} \right) = \text{coker} \left(\mathcal{O}_{\tilde{U}(\tilde{z})} \rightarrow \text{pr}_{\tilde{B}*} \mathcal{O}_{\tilde{Y}^{\text{mod},(\eta,\tau)}(\tilde{z})} \right)$$

because $\mathcal{O}_{\tilde{U}(\tilde{z})} \rightarrow \text{pr}_{\tilde{B}*} \mathcal{O}_{\widetilde{Ba}(\tilde{z})}$. □

Lemma 4.2.11. *Under the composite*

$$\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z}) \xrightarrow{\Delta} \widetilde{Ba}(\tilde{z}) = \prod_j \widetilde{Ba}_j(\tilde{z}_j) = \prod_j \text{GL}_2 \times Ba_j(\tilde{z}_j)$$

$\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})$ is a complete intersection defined by the zero locus of maps $\mathfrak{s}_j : \mathcal{L}_j \stackrel{\text{def}}{=} q_j^*(\mathcal{O}_{\mathbb{P}^1}(-1)) \otimes p_{j-1}^*(\mathcal{O}_{\mathbb{P}^1}(-1)) \rightarrow \mathcal{O}_{\prod_j \text{GL}_2 \times Ba_j(\tilde{z}_j)}$. In particular, the ideal sheaf $\mathcal{I}(\tilde{z})$ defining Δ identifies with

$$\tau_{<0} \left(\text{Kos}_{\bullet} \left(\bigoplus_{j \in \mathcal{J}} \mathcal{L}_j, (\mathfrak{s}_j) \right) \right)$$

Proof. It follows from the definitions that we have a diagram

$$\begin{array}{ccccc} \tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z}) & \xrightarrow{\Delta} & \prod_j \widetilde{Ba}_j(\tilde{z}_j) & \longrightarrow & \prod_j \text{GL}_2 \times Ba_j(\tilde{z}_j) \\ \downarrow & & \downarrow & & \downarrow \text{id} \times p_j \times q_j \\ \prod_{\mathcal{J}} \text{GL}_2 \times \mathbb{P}^1 & \xrightarrow{\Delta} & \prod_{\mathcal{J}} \text{GL}_2 \times \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\cong} & \prod_{\mathcal{J}} \text{GL}_2 \times \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

where the bottom left map is $(\kappa_j, l_j)_j \mapsto (\kappa_j, l_j, l_{j-1})_j$ and the bottom right map is $(\kappa_j, l_j, r_j) \mapsto (\kappa_j, l_j \kappa_j, r_j)$.

The bottom right isomorphism only commutes with projection to the right \mathbb{P}^1 factor, but not to the left \mathbb{P}^1 factor. Nevertheless, the GL_2 -equivariance of $\mathcal{O}_{\mathbb{P}^1}(-1)$ shows that its pullback via projection to the j -th left (respectively, j -th right) \mathbb{P}^1 are compatible with the bottom isomorphism.

It follows that $\tilde{Y}^{\mathrm{mod},\eta,\tau}(\tilde{z})$ is the zero common locus of f maps $\mathfrak{s}_j : \mathcal{L}_j \rightarrow \mathcal{O}_{\widetilde{Ba}(\tilde{z})}$. By Lemma 3.3.6 and Lemma 4.2.3, $\tilde{Y}^{\mathrm{mod},\eta,\tau}(\tilde{z}) \hookrightarrow \widetilde{Ba}(\tilde{z})$ has codimension f and $\widetilde{Ba}(\tilde{z})$ is local complete intersection. This implies that $\tilde{Y}^{\mathrm{mod},\eta,\tau}(\tilde{z})$ is the global complete intersection in $\widetilde{Ba}(\tilde{z})$ cut out by the \mathfrak{s}_j . \square

We compute $R\Gamma(\mathcal{I}(\tilde{z}))$ using the resolution from Lemma 4.2.11 and the Künneth formula. Let $\ell \in \{1, \dots, f\}$ and consider f -tuples $\underline{\varepsilon} = (\varepsilon_j)_{j \in \mathcal{J}}, \underline{\delta} = (\delta_j)_{j \in \mathcal{J}} \in \{0, 1\}^{\mathcal{J}}$ satisfying $\varepsilon_j = \delta_{j+1}$ and $\#\{j \in \mathcal{J}, \varepsilon_j = 1\} = \ell$. Then by Lemma 4.2.11 we have

$$\bigwedge^{\ell} \left(\bigoplus_{j \in \mathcal{J}} \mathcal{L}_j \right) = \bigoplus_{\underline{\varepsilon}, \underline{\delta}} \bigotimes_{j \in \mathcal{J}, \mathcal{O}_{\widetilde{Ba}}} \left(q_j^*(\mathcal{O}_{\mathbf{P}^1}(-1))^{\varepsilon_j} \otimes p_j^*(\mathcal{O}_{\mathbf{P}^1}(-1))^{\delta_j} \right)$$

where the direct sum runs over f -tuples $\underline{\varepsilon}, \underline{\delta}$ as above. As the $\widetilde{Ba}(\tilde{z}_j)$ are Tor-independent over \mathcal{O} , we deduce from the Künneth formula [Sta22, Tag 0FLQ] that

$$\begin{aligned} & R\Gamma \left(\bigotimes_{j \in \mathcal{J}, \mathcal{O}_{\widetilde{Ba}}} \left(q_j^*(\mathcal{O}_{\mathbf{P}^1}(-1))^{\varepsilon_j} \otimes p_j^*(\mathcal{O}_{\mathbf{P}^1}(-1))^{\delta_j} \right) \right) \\ & \cong \bigotimes_{j \in \mathcal{J}, \mathcal{O}}^{\mathbb{L}} R\Gamma \left(q_j^*(\mathcal{O}_{\mathbf{P}^1}(-1))^{\varepsilon_j} \otimes_{\mathcal{O}_{\widetilde{Ba}(\tilde{z}_j)}} p_j^*(\mathcal{O}_{\mathbf{P}^1}(-1))^{\delta_j} \right) \end{aligned}$$

From Lemma 4.2.5 we obtain the following corollaries:

Corollary 4.2.12. \bullet For $0 < \ell < f$, $R^\ell \mathrm{pr}_* \left(\bigwedge^{\ell} \left(\bigoplus_{j \in \mathcal{J}} \mathcal{L}_j \right) \right) = 0$.

- \bullet $pR^f \mathrm{pr}_* \left(\bigotimes_{j \in \mathcal{J}} \mathcal{L}_j \right) = 0$; and
- \bullet $R^f \mathrm{pr}_* \left(\bigotimes_{j \in \mathcal{J}} \mathcal{L}_j \right) \neq 0$ if and only if for each $j \in \mathcal{J}$, either $\langle \mu_j, \alpha^\vee \rangle = 0$ or $\langle \mu_j, \alpha^\vee \rangle = 1$ and $(s_j, \tilde{w}_j) = (\mathrm{id}, t_{w_0(\eta)})$.

Proof. Since the image of pr is affine, it suffices to check the statements after taking global sections, so we can replace all occurrences of $R^j \mathrm{pr}_*$ with $R^j \Gamma$.

Given our above discussion, it suffices to analyze

$$\mathrm{H}^\ell \left(\bigotimes_{j \in \mathcal{J}, \mathcal{O}}^{\mathbb{L}} R\Gamma \left(q_j^*(\mathcal{O}_{\mathbf{P}^1}(-1))^{\varepsilon_j} \otimes_{\mathcal{O}_{\widetilde{Ba}(\tilde{z}_j)}} p_j^*(\mathcal{O}_{\mathbf{P}^1}(-1))^{\delta_j} \right) \right)$$

for f -tuples $\underline{\varepsilon} = (\varepsilon_j)_{j \in \mathcal{J}}, \underline{\delta} = (\delta_j)_{j \in \mathcal{J}} \in \{0, 1\}^{\mathcal{J}}$ satisfying $\varepsilon_j = \delta_{j+1}$ and $\#\{j \in \mathcal{J}, \varepsilon_j = 1\} = \ell$. Note that these conditions imply that there are at most ℓ indices j such that $(\varepsilon_j, \delta_j) = (1, 1)$, with strict inequality if $\ell < f$. Thus the amplitude bound of Lemma 4.2.5 shows the above cohomology

group vanishes for $\ell < f$, while

$$\begin{aligned} \mathrm{H}^f \left(\bigotimes_{j \in \mathcal{J}, \mathcal{O}}^{\mathbb{L}} R\Gamma \left(q_j^*(\mathcal{O}_{\mathbf{P}^1}(-1))^{\varepsilon_j} \otimes_{\mathcal{O}_{\widetilde{Ba}(\tilde{z}_j)}} p_j^*(\mathcal{O}_{\mathbf{P}^1}(-1))^{\delta_j} \right) \right) &\cong \\ &\cong \bigotimes_{j \in \mathcal{J}, \mathcal{O}} R^1\Gamma \left(q_j^*(\mathcal{O}_{\mathbf{P}^1}(-1)) \otimes_{\mathcal{O}_{\widetilde{Ba}(\tilde{z}_j)}} p_j^*(\mathcal{O}_{\mathbf{P}^1}(-1)) \right) \end{aligned}$$

The result now follows from Lemma 4.2.5 □

We note that the proof of the above corollary also shows

Corollary 4.2.13. *For any $\ell \in \{0, \dots, f\}$ and $k > \ell$*

$$R^k \mathrm{pr}_* \left(\bigwedge^{\ell} \mathrm{pr}_{\widetilde{Ba}}^* \left(\bigoplus_{j \in \mathcal{J}} \mathcal{L}_j \right) \right) = 0.$$

Proof of Propositions 3.3.12, 3.3.14. By Lemma 4.2.11 $R\mathrm{pr}_*\mathcal{I}(\tilde{z})$ is filtered (in the derived sense) by $R\mathrm{pr}_* \left(\bigwedge^{\ell} \mathrm{pr}_{\widetilde{Ba}}^* \left(\bigoplus_{j \in \mathcal{J}} \mathcal{L}_j \right) \right) [\ell - 1]$ for $1 \leq \ell \leq f$. Then by Corollary 4.2.10, Proposition 3.3.12 follows from Corollary 4.2.12 and Proposition 3.3.14 from Corollary 4.2.13. □

We also record the following, which will be used in subsection 4.6.2

Proposition 4.2.14. *The relative dualizing sheaf of $\widetilde{Y}^{\mathrm{mod}, \eta, \tau}(\tilde{z})/\mathcal{O}$ is trivial.*

Proof. By Lemma 4.2.11, $\widetilde{Y}^{\mathrm{mod}, \eta, \tau}(\tilde{z}) \hookrightarrow \widetilde{Ba}(\tilde{z})$ is a regular immersion with normal bundle

$$\bigoplus q_j^* \mathcal{O}_{\mathbf{P}^1}(1) \otimes p_{j-1}^* \mathcal{O}_{\mathbf{P}^1}(1)$$

which thus has determinant $\bigotimes_{\mathcal{J}} q_j^* \mathcal{O}_{\mathbf{P}^1}(1) \otimes p_{j-1}^* \mathcal{O}_{\mathbf{P}^1}(1)$. The result now follows from Lemma 4.2.3 (2) □

4.3. The naive models. Recall that $\widetilde{\mathcal{Z}}^{\mathrm{mod}, \tau}(\tilde{z})$ is the scheme theoretic image of $\widetilde{Y}^{\mathrm{mod}, \eta, \tau}(\tilde{z})$ under the map π^{mod} which forgets the elements $r_j = l_{j-1} \in \mathbb{P}^1$. In other words the equations for $\widetilde{\mathcal{Z}}^{\mathrm{mod}, \tau}(\tilde{z})$ are obtained by eliminating the $r_j = l_{j-1}$ from the defining equations of $\widetilde{Y}^{\mathrm{mod}, \eta, \tau}(\tilde{z})$ which are extracted from Table 3. The goal of this section is to construct a slight enlargement $\widetilde{\mathcal{Z}}^{\mathrm{nv}, \tau}(\tilde{z})$ of $\widetilde{\mathcal{Z}}^{\mathrm{mod}, \tau}(\tilde{z})$, which has the advantage of being given by explicit equations.

We first introduce some auxilliary notation. We view $\mathcal{J} = \mathbb{Z}/f\mathbb{Z}$ as an oriented graph with edges going from j to $j - 1$.

Definition 4.3.1. Given the data (s, μ, \tilde{w}) , and $\tilde{z} \stackrel{\mathrm{def}}{=} \tilde{w}s^{-1}v^{\mu}$

- (1) Define $M_j(\tilde{z}_j)$ to be the scheme theoretic image of $Ba_j(\tilde{z}_j) \rightarrow \widetilde{U}(\tilde{z}_j)$.
- (2) Let $j \in \mathcal{J}$. We say
 - j is of type *II* if either $k_j > 1$ or $(\tilde{w}_j, s_j, k_j) = (t_{\eta}, \mathrm{id}, 1), (w_0 t_{\eta}, \mathrm{id}, 1), (t_{w_0(\eta)}, (12), 1)$;
 - j is of type *I* if $(\tilde{w}_j, s_j, k_j) = (t_{\eta}, (12), 1), (w_0 t_{\eta}, (12), 1), (t_{w_0(\eta)}, \mathrm{id}, 1)$;
 - j is of type *0* if $k_j = 0$.
- (3) A *fragmentation* of \mathcal{J} is the decomposition $\mathcal{J} = \bigcup \mathcal{J}_k$ into subsets \mathcal{J}_k such that:
 - \mathcal{J}_k is an oriented path in \mathcal{J} , i.e an ordered subset of the form $[j, j + \ell] = (j, j - 1, \dots, j - \ell)$ for some $\ell \leq f$.
 - The endpoints of the path \mathcal{J}_k are not of type 0.

- The interior points of the the path \mathcal{J}_k are of type 0.
- \mathcal{J}_k is not a singleton unless $f = 1$.

Each \mathcal{J}_k is called a fragment of \mathcal{J} .

- Remark 4.3.2.* (1) The scheme theoretic image of $\widetilde{Ba}_j(\tilde{z}_j) \rightarrow \widetilde{U}(\tilde{z}_j)$ is $\mathrm{GL}_2 \times M_j(\tilde{z}_j)$.
- (2) Under our running assumption that τ is regular, \mathcal{J} must have a vertex not of type 0. This implies \mathcal{J} has a unique fragmentation, obtained by the minimal paths joining the vertices not of type 0. We also note that each fragment \mathcal{J}_k has a well-defined starting point and ending point (which may coincide, in which case the fragment is all of \mathcal{J}).
- (3) It follows from Table 3 that
- If j is type II: $M_j(\tilde{z}_j) \cong \mathbb{A}^2$.
 - If j is type I: $M_j(\tilde{z}_j) \cong M$, the space of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ which has determinant 0 and trace p .
 - If j is type 0: $M_j(\tilde{z}_j) \cong \mathbb{A}^1$.

We now define a subspace of $\prod_{\mathcal{J}} \mathrm{GL}_2 \times M_j(\tilde{z}_j)$ using the fragmentation $\mathcal{J} = \bigcup \mathcal{J}_k$.

Definition 4.3.3. Let $\mathcal{J} = \bigcup \mathcal{J}_k$ be the fragmentation of \mathcal{J} . We define $\widetilde{\mathcal{Z}}^{\mathrm{nv},\tau}(\tilde{z})$ to be the closed subscheme of $\prod_{\mathcal{J}} \mathrm{GL}_2 \times M_j(\tilde{z}_j) \hookrightarrow \mathrm{GL}_2^{\mathcal{J}} \times \widetilde{U}(\tilde{z})$ cut out by the matrix equations

$$M_{\mathrm{out},o} \left(\prod_{\ell=o+1}^{i-1} T_{\ell} \right) M_{\mathrm{in},i} = 0$$

for each fragment $\mathcal{J}_k = \{i, \dots, o\}$, where

- $M_{\mathrm{in},i}$ is the initial matrix for $j = i$ in Table 4
- $M_{\mathrm{out},o}$ is the final matrix for $j = o$ in Table 4
- T_{ℓ} is the transition matrix for $j = \ell$ (which are of type 0) in Table 4.

We also define $\mathcal{Z}^{\mathrm{nv},\tau}(\tilde{z})$ to be the fiber of $\widetilde{\mathcal{Z}}^{\mathrm{nv},\tau}(\tilde{z})$ above $1 \in \mathrm{GL}_2^{\mathcal{J}}$.

Proposition 4.3.4. (1) The inclusion $\widetilde{\mathcal{Z}}^{\mathrm{mod},\tau}(\tilde{z}) \hookrightarrow \widetilde{U}(\tilde{z})$ factors through $\widetilde{\mathcal{Z}}^{\mathrm{nv},\tau}(\tilde{z})$.

(2) We have $\widetilde{\mathcal{Z}}^{\mathrm{mod},\tau}(\tilde{z})[\frac{1}{p}] = \widetilde{\mathcal{Z}}^{\mathrm{nv},\tau}(\tilde{z})[\frac{1}{p}] = \widetilde{Y}^{\mathrm{mod},\eta,\tau}(\tilde{z})[\frac{1}{p}]$.

In particular, $\widetilde{\mathcal{Z}}^{\mathrm{mod},\tau}(\tilde{z})$ is the p -saturation (synonymously, the \mathcal{O} -flat part) of $\widetilde{\mathcal{Z}}^{\mathrm{nv},\tau}(\tilde{z})$.

Proof. The first assertion follows from the fact that $\widetilde{\mathcal{Z}}^{\mathrm{mod},\tau}(\tilde{z})$ obey the defining equations of $\widetilde{\mathcal{Z}}^{\mathrm{nv},\tau}(\tilde{z})$, which is a consequence of the relations in Table 3 and the relation $r_j = l_{j-1}$ in $\widetilde{Y}^{\mathrm{mod},\eta,\tau}$. Indeed, these defining equations were obtained by repeatedly substituting the relations between l_j and r_j when j is type 0 and $r_j = l_{j-1}$ until it becomes a relation between r_a and l_b where a, b are not type 0, in which case one substitutes for r_a, l_b an expression in the variables on $\mathrm{GL}_2 \times M_a(\tilde{z}_a), \mathrm{GL}_2 \times M_b(\tilde{z}_b)$.

We now establish the second assertion. First, we show that the map $\widetilde{Y}^{\mathrm{mod},\eta,\tau}(\tilde{z}) \rightarrow \widetilde{U}(\tilde{z})$ is a closed immersion after inverting p , i.e. we need to show r_j, l_j are determined by the remaining variables. For each j not of type 0, we can solve for r_j, l_j when p is invertible. Using the relation $r_j = l_{j-1}$ and the relations in Table 3, we can solve for the $r_{j'}, l_{j'}$ where j' is of type 0. Thus $\widetilde{\mathcal{Z}}^{\mathrm{mod},\tau}(\tilde{z})[\frac{1}{p}] = \widetilde{Y}^{\mathrm{mod},\eta,\tau}(\tilde{z})[\frac{1}{p}]$.

To finish the proof, we need to show that $\widetilde{Y}^{\mathrm{mod},\eta,\tau}(\tilde{z})[\frac{1}{p}]$ surjects onto $\widetilde{\mathcal{Z}}^{\mathrm{nv},\tau}(\tilde{z})[\frac{1}{p}]$, i.e. we need to produce r_j, l_j satisfying all requisite relations. We use the same procedure to define r_j, l_j as in the previous paragraph. The only potential issue is that for a fragment $\mathcal{J}_k = \{i, \dots, o\}$, the

TABLE 4.

$\langle \mu_j, \alpha^\vee \rangle$	\tilde{w}_j	t_η	$w_0 t_\eta$	$t_{w_0(\eta)}$
	s_j			
> 1	(12)	Type II Initial Matrix: $\begin{pmatrix} 1 \\ -C' \end{pmatrix}$ Final Matrix: $(B \ -p) \kappa_j^{-1}$	Type II Initial Matrix: $\begin{pmatrix} 1 \\ -C' \end{pmatrix}$ Final Matrix: $(-p \ B) \kappa_j^{-1}$	Type II Initial Matrix: $\begin{pmatrix} 1 \\ -C' \end{pmatrix}$ Final Matrix: $(0 \ 1) \kappa_j^{-1}$
	id	Type II Initial Matrix: $\begin{pmatrix} 1 \\ -C' \end{pmatrix}$ Final Matrix: $(C \ -p) \kappa_j^{-1}$	Type II Initial Matrix: $\begin{pmatrix} 1 \\ -C' \end{pmatrix}$ Final Matrix: $(-p \ C) \kappa_j^{-1}$	Type II Initial Matrix: $\begin{pmatrix} 1 \\ -C' \end{pmatrix}$ Final Matrix: $(0 \ 1) \kappa_j^{-1}$
$= 1$	(12)	Type I Initial Matrix: $\begin{pmatrix} D & -B \\ -C & p-D \end{pmatrix}$ Final Matrix: $\begin{pmatrix} p-D & -C \\ -B & D \end{pmatrix} \kappa_j^{-1}$	Type I Initial Matrix: $\begin{pmatrix} D & -B \\ -C & p-D \end{pmatrix}$ Final Matrix: $\begin{pmatrix} D & -B \\ -C & p-D \end{pmatrix} \kappa_j^{-1}$	Type II Initial Matrix: $\begin{pmatrix} 1 \\ -C' \end{pmatrix}$ Final Matrix: $(0 \ 1) \kappa_j^{-1}$
	id	Type II Initial Matrix: $\begin{pmatrix} 1 \\ -C' \end{pmatrix}$ Final Matrix: $(C \ -p) \kappa_j^{-1}$	Type II Initial Matrix: $\begin{pmatrix} 1 \\ -C' \end{pmatrix}$ Final Matrix: $(-p \ C) \kappa_j^{-1}$	Type I Initial Matrix: $\begin{pmatrix} D & -B \\ -C & p-D \end{pmatrix}$ Final Matrix: $\begin{pmatrix} p-D & B \\ C & D \end{pmatrix} \kappa_j^{-1}$
$= 0$	id	Type 0 Transition Matrix: $\begin{pmatrix} 1 & 0 \\ -C & p \end{pmatrix} \kappa_j^{-1}$	Type 0 Transition Matrix: $\begin{pmatrix} 0 & -1 \\ -p & C \end{pmatrix} \kappa_j^{-1}$	Type 0 Transition Matrix: $\begin{pmatrix} p & -B \\ 0 & 1 \end{pmatrix} \kappa_j^{-1}$

The meaning of the variables in this table are the same as that of the corresponding entry in Table 3.

procedure gives two definitions of r_i : one by recursion in terms of l_o (and some variables at $j \in \{i-1, \dots, o+1\}$), the other by directly solving in terms of the variables at i . However the defining equations of $\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})$ exactly guarantee that these two definitions coincide. \square

4.4. Obstruction bounds for naive models. We wish to establish a quantitative bound for the singular ideal of $\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})/\mathcal{O}$. This will be used in the next subsection to finish the proof of Theorem 3.3.8.

To save notation, in this section we abbreviate $\mathcal{Z} = \tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})$, $M_j = M_j(\tilde{z}_j)$ and $\mathcal{M} = \prod_{\mathcal{J}} \text{GL}_2 \times M_j$. We also define affine spaces \mathcal{A}_j with closed immersions $M_j \hookrightarrow \mathcal{A}_j$ as follows:

- $\mathcal{A}_j = M_j$ if j is not type I .
- If j is type I , then M_j is the space M of matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with determinant 0 and trace p , and we define $\mathcal{A}_j = \mathbb{A}^3$ and $M_j \hookrightarrow \mathcal{A}_j$ to be the map

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto (B, C, D)$$

so that M_j identifies with the hypersurface $D(p - D) = BC$ in \mathcal{A}_j .

We thus have an inclusion $\iota : \mathcal{Z} \hookrightarrow \mathcal{A} \stackrel{\text{def}}{=} \prod_{\mathcal{J}} \text{GL}_2 \times \mathcal{A}_j$. Let \mathcal{I} be the ideal of \mathcal{A} defining \mathcal{Z} .

We first analyze a generating set of \mathcal{I} . To do this, recall the fragmentation $\mathcal{J} = \bigcup_{k \in \mathcal{K}} \mathcal{J}_k$. We also denote by $\mathcal{J}_I \subset \mathcal{J}$ the subset of j that are type I . For each j of type I , we also let A_j, B_j, C_j, D_j denote the natural coordinates on M_j , so that $A_j + D_j = p$ and $A_j D_j = B_j C_j$.

It follows from Definition 4.3.3 that we have a decomposition

$$(4.4.1) \quad \mathcal{I} = \sum_k \mathcal{I}_{\mathcal{J}_k} + \sum_{j \in \mathcal{J}_I} (D_j(p - D_j) - B_j C_j)$$

where $\mathcal{I}_{\mathcal{J}_k}$ is the ideal generated by the entries of the matrix equation associated to \mathcal{J}_k in Definition 4.3.3. This gives a presentation of \mathcal{Z} in terms of \mathcal{A} . Note that $\mathcal{Z}[\frac{1}{p}]$ has codimension $c \stackrel{\text{def}}{=} |\mathcal{K}| + |\mathcal{J}_I|$ in $\mathcal{A}[\frac{1}{p}]$.

Proposition 4.4.2. *Let J_c denote the ideal generated by the $c \times c$ minors of the Jacobian matrix of the presentation $\mathcal{O}(\mathcal{A})/\mathcal{I}$. Then $p^{2|\mathcal{J}_I|+f+|\mathcal{K}|} \in J_c$. In particular $p^{4f} \in J_c$.*

Remark 4.4.3. Proposition 4.4.2 implies $\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})[\frac{1}{p}]$ is smooth over E . Of course this can also be seen directly from the explicit description of $\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})[\frac{1}{p}]$.

We observe the following structural properties of $\mathcal{I}_{\mathcal{J}_k}$:

Lemma 4.4.4. *Suppose the fragment $\mathcal{J}_k = \{i, \dots, o\}$. Let $\kappa_o = \begin{pmatrix} a_o & b_o \\ c_o & d_o \end{pmatrix}$ record the o -th GL_2 factor of \mathcal{A} .*

There exists $\alpha, \beta, \gamma, \delta$ only involving coordinates on the $(i-1)$ -th to $(o+1)$ -th factor of \mathcal{A} with

$$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \pm p^{|\mathcal{J}_k \setminus \{i,o\}|}$$

such that:

(1) *If i, o are type II then $\mathcal{I}_{\mathcal{J}_k}$ is principal, generated by an element of the form*

$$(Y_o \quad -p) \kappa_o \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 \\ -X_i \end{pmatrix}$$

where Y_o, X_i are coordinates of $\mathcal{A}_o, \mathcal{A}_i$.

(2) *If i, o are type I then $\mathcal{I}_{\mathcal{J}_k}$ is generated by the entries of a matrix either of the form*

$$\begin{pmatrix} D_o & -B_o \\ -C_o & A_o \end{pmatrix} \kappa_o \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} D_i & -B_i \\ -C_i & A_i \end{pmatrix};$$

or

$$\begin{pmatrix} D_o & C_o \\ B_o & A_o \end{pmatrix} \kappa_o \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} D_i & -B_i \\ -C_i & A_i \end{pmatrix};$$

(3) *If i is type II and o is type I then $\mathcal{I}_{\mathcal{J}_k}$ is generated by the entries of a matrix of the form*

$$(Y_o \quad -p) \kappa_o \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} D_i & -B_i \\ -C_i & A_i \end{pmatrix}$$

where Y_o is a coordinate in \mathcal{A}_o .

(4) *If i is type I and o is type II then $\mathcal{I}_{\mathcal{J}_k}$ is generated by the entries of a matrix either of the form*

•

$$\begin{pmatrix} D_o & -B_o \\ -C_o & A_o \end{pmatrix} \kappa_o \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 \\ -X_i \end{pmatrix};$$

or

•

$$\begin{pmatrix} D_o & C_o \\ B_o & A_o \end{pmatrix} \kappa_o \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 \\ -X_i \end{pmatrix};$$

where X_i is a coordinate on \mathcal{A}_i .

Proof. The form of $\mathcal{I}_{\mathcal{J}_k}$ follows from Table 4, after possibly rearranging the entries of the matrix equations. □

Proof of Proposition 4.4.2. We will show that for any choices $G_j, H_j \in \{A_j, D_j\}$ with $j \in \mathcal{J}_I$, the element $p^{2|\mathcal{J}_I| + \sum_k |\mathcal{J}_k \setminus \{i, o\}|} \prod_{j \in \mathcal{J}_I} G_j H_j$ belongs to \mathcal{J}_c . This finishes the proof, since this implies \mathcal{J}_c contains

$$p^{2|\mathcal{J}_I| + \sum_k |\mathcal{J}_k \setminus \{i, o\}|} \prod_{j \in \mathcal{J}_I} (A_j, D_j) \ni p^{2|\mathcal{J}_I| + \sum_k |\mathcal{J}_k \setminus \{i, o\}| + 2}$$

Since the role of A_j, D_j is essentially symmetric in our argument, we will deal with the case $G_j = H_j = D_j$ for all $j \in \mathcal{J}$.

Write \mathbf{J} for the Jacobian matrix of our chosen presentation of \mathcal{Z} . Our convention is that the columns are named by the variables and rows are labeled by (the chosen) generators of \mathcal{I} . Then clearly \mathcal{J}_c contains any $c \times c$ minor of any matrix of the form $\mathbf{J}\mathbf{U}$. In other words, it suffices to find, after modifying \mathbf{J} by column operations, a $c \times c$ minor equal to $p^{2|\mathcal{J}_I| + \sum_k |\mathcal{J}_k \setminus \{i, o\}|} \prod_{j \in \mathcal{J}_I} G_j H_j$.

For $k \in \mathcal{K}$, we let g_k denote the generator of $\mathcal{I}_{\mathcal{J}_k}$ that correspond to the $(1, 1)$ -th entry of the matrix equation described in Lemma 4.4.4 (this choice correspond to our choice of $G_j = H_j = D_j$). We will choose our $c \times c$ minor to have rows corresponding to the g_k with $k \in \mathcal{K}$ and the generators $(p - D_j)D_j - B_j C_j$ for $j \in \mathcal{J}_I$.

We now explain the column operations we will perform on \mathbf{J} .

First, consider $k \in \mathcal{K}$, giving the fragment $\mathcal{J}_k = \{i, \dots, o\}$. Write the matrix generating $\mathcal{I}_{\mathcal{J}_k}$ in factorized form

$$X \begin{pmatrix} a_o & b_o \\ c_o & d_o \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} Y$$

as prescribed by Lemma 4.4.4. Then the entries of \mathcal{J}_k corresponding to row g_k and column a_o, b_o, c_o, d_o are exactly the $(1, 1)$ -th entry of the matrices

$$X \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} Y, \quad X \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} Y, \quad X \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} Y, \quad X \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} Y$$

respectively. Furthermore, no other row of \mathbf{J} has a non-zero entry at the columns a_o, b_o, c_o, d_o . Hence, given $x, y, z, t \in \mathcal{O}(\mathcal{A})$, by modifying the columns \mathbf{J} to $\mathbf{J}\mathbf{U}$ where we take linear combinations of columns a_o, b_o, c_o, d_o but do not modify the remaining columns, we can guarantee that the resulting matrix has a column which has vanishing entry for any row other than g_k , and at row g_k the entry is the $(1, 1)$ -th entry of

$$X \begin{pmatrix} x & y \\ z & t \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} Y$$

Now $\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \pm p^{|\mathcal{J}_k \setminus \{i, o\}|}$, so choosing x, y, z, t appropriately, we can make $\begin{pmatrix} x & y \\ z & t \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ become any matrix among

$$p^{|\mathcal{J}_k \setminus \{i, o\}|} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Using the explicit form of X, Y given in Lemma 4.4.4, we thus learn that after taking a linear combination of columns a_o, b_o, c_o, d_o , we can make the entry in row g_k become $p^{|\mathcal{J}_k \setminus \{i, o\}|} p, p^{|\mathcal{J}_k \setminus \{i, o\}|} D_o D_i, p^{|\mathcal{J}_k \setminus \{i, o\}|} p D_i$ or $p^{|\mathcal{J}_k \setminus \{i, o\}|} D_o$ (corresponding to the four cases in that Lemma, respectively).

Next, we consider $j \in \mathcal{J}_I$, giving a generator $(p - D_j)D_j - B_j C_j$ of \mathcal{I} . The part of J corresponding to row $(p - D_j)D_j - B_j C_j$ and column B_j, C_j, D_j is

$$\begin{pmatrix} -C_j & -B_j & p - 2D_j \end{pmatrix}$$

Since

$$\begin{pmatrix} -C_j & -B_j & p - 2D_j \end{pmatrix} \begin{pmatrix} -4C_j \\ 0 \\ p - 2D_j \end{pmatrix} = (p - 2D_j)^2 + 4B_j C_j = p^2$$

we see that taking a linear combination of column B_j, C_j, D_j produces the entry p^2 in row $(p - D_j)D_j - B_j C_j$. Moreover that the entries of this linear combination at rows $(p - D_{j'})D_{j'} - B_{j'} C_{j'}$ are 0, where $j' \in \mathcal{J}_I$ but $j' \neq j$ (note however that we have no control on the entries on the remaining rows).

To summarize, by taking appropriate linear combination among columns a_o, b_o, c_o, d_o (for each $\mathcal{J}_k = \{i, \dots, o\}$) and among columns B_j, C_j, D_j (for each $j \in \mathcal{J}_I$), and look at the rows $g_k, (p - D_j)D_j - B_j C_j$, we can find a $c \times c$ submatrix with rows $g_k, (p - D_j)D_j - B_j C_j$ of block triangular form

$$\begin{pmatrix} P & * \\ 0 & Q \end{pmatrix}$$

where

- P is diagonal of size $|\mathcal{K}| \times |\mathcal{K}|$ whose entries belong to

$$\{p^{|\mathcal{J}_k \setminus \{i, o\}|} p, p^{|\mathcal{J}_k \setminus \{i, o\}|} D_o D_i, p^{|\mathcal{J}_k \setminus \{i, o\}|} p D_i, p^{|\mathcal{J}_k \setminus \{i, o\}|} D_o\};$$

- Q is p^2 the identity matrix of size $|\mathcal{J}_I|$.

It follows that this $c \times c$ minor divides

$$p^{2|\mathcal{J}_I| + \sum_k |\mathcal{J}_k \setminus \{i, o\}|} \prod_{j \in \mathcal{J}_I} D_j^2$$

□

4.5. Proof of Theorem 3.3.8. Set $N = p - 2 - \max_j \langle \mu_j, \alpha^\vee \rangle \geq \frac{p-7}{2}$. Recall from Proposition 3.3.11 that we have a diagram

$$\begin{array}{ccc} \tilde{\mathcal{Z}}^\tau \otimes_{\mathcal{O}} \mathcal{O}/p^{N-1} & \xrightarrow{\sim} & \tilde{\mathcal{Z}}^{\text{mod}, \tau} \otimes_{\mathcal{O}} \mathcal{O}/p^{N-1} \\ \downarrow & & \downarrow \\ \left[LG^{\text{bd}, (v+p)v^\mu} / \varphi \prod_{\mathcal{J}} L_1^+ \mathbf{G} \right] \otimes_{\mathcal{O}} \mathcal{O}/p^{N-1} & \xrightarrow{\sim} & \text{Gr}_1^{\text{bd}, (v+p)v^\mu} \otimes_{\mathcal{O}} \mathcal{O}/p^{N-1} \end{array}$$

In particular, from Proposition 4.3.4 we get a closed immersion

$$\iota : \tilde{\mathcal{Z}}^\tau(\tilde{z}) \otimes_{\mathcal{O}} \mathcal{O}/p^{N-1} \xrightarrow{\sim} \tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z}) \otimes_{\mathcal{O}} \mathcal{O}/p^{N-1} \hookrightarrow \tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z}) \otimes_{\mathcal{O}} \mathcal{O}/p^{N-1}$$

We now invoke [Elk73, Lemme 1] as in the proof of [LLHLM23, Proposition 3.3.9]: By Proposition 4.4.2, the integer h in *loc.cit.* can be taken to be $4f$ while the integer k is 0 since $\tilde{\mathcal{Z}}^\tau(\tilde{z})$ is p -torsion free. It follows that if $N-1 > 8f$, we can produce a map $\tilde{\iota} : \tilde{\mathcal{Z}}^\tau(\tilde{z}) \rightarrow \tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})^{\wedge p}$ which agrees with ι modulo p^{N-1-4f} . In particular, this implies $\tilde{\iota}$ is also a closed immersion. Since $\tilde{\mathcal{Z}}^\tau(\tilde{z})$ is \mathcal{O} -flat and the \mathcal{O} -flat part of $\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})^{\wedge p}$ is exactly $\tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z})$, we can factorize $\tilde{\iota}$

$$\tilde{\mathcal{Z}}^\tau(\tilde{z}) \hookrightarrow \tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z}) \hookrightarrow \tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})^{\wedge p}$$

such that the first map is an isomorphism modulo p^{N-1-4f} . The following Lemma then implies that the inclusion $\tilde{\mathcal{Z}}^\tau(\tilde{z}) \hookrightarrow \tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z})$ is in fact an isomorphism, thus finishing the proof of Theorem 3.3.8.

Lemma 4.5.1. *Suppose we are given a surjection $\pi : R \rightarrow S$ of Noetherian p -adically complete \mathcal{O} -algebras. Assume that π induces an isomorphism $\pi : R/\varpi \cong S/\varpi$. Then π is an isomorphism.*

Proof. Let $I = \ker \pi$, then since S is \mathcal{O} -flat we get a short exact sequence

$$0 \longrightarrow I/\varpi I \longrightarrow R/\varpi \longrightarrow S/\varpi \longrightarrow 0$$

Since π induces an isomorphism modulo ϖ , we learn that $I/\varpi I = 0$. But I is p -adically separated, so $I = 0$. \square

The following is immediate from our discussion

Corollary 4.5.2. *Assume that either $p > 8f + 3 + \max_j \langle \mu_j, \alpha^\vee \rangle$ or $p > 7$ and $K = \mathbb{Q}_p$. Then $\tilde{\mathcal{Z}}^\tau(\tilde{z})$ is isomorphic to the p -adic completion of the p -saturation of $\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})$.*

In particular, this realizes $\tilde{\mathcal{Z}}^\tau(\tilde{z})$ as the p -saturation of an explicitly presented affine p -adic formal scheme.

4.6. Applications.

4.6.1. *Galois deformation rings.* Recall that we have a shifted conjugation action map

$$\text{GL}_2^{\mathcal{J}} \times \text{Gr}_1^{\mathcal{J}} \rightarrow \text{Gr}_1^{\mathcal{J}}$$

given by the formula

$$(g_j, A_j) \mapsto g_j A_j g_{j-1}^{-1}$$

This action of $\text{GL}_2^{\mathcal{J}}$ clearly factors through the quotient $\text{GL}_2^{\mathcal{J}}/\Delta Z$ by the diagonally embedded copy of the center Z of GL_2 . Furthermore since

$$\prod \det g_j A_j g_{j-1}^{-1} = \prod \det A_j$$

we see that the quantity $\prod \det A_j$ modulo $\det L_1^+ G$ is invariant along orbits.

Lemma 4.6.1. *Let $\tilde{z} = (\tilde{z}_j) = (z_j v^{\nu_j}) \in \widetilde{W}^\vee$ such that $\langle \nu_j, \alpha^\vee \rangle \neq 0$ for some j .*

(1) Suppose $\prod z_j = (12)$. Let $\bar{x} = ((1, \dots, 1), (\tilde{z}_j)_j) \in \mathrm{GL}_2^{\mathcal{J}}(\mathbb{F}) \times \prod_{\mathcal{J}} L_1^- G(\mathbb{F}) \tilde{z}_j$ and \bar{y} its image under the shifted conjugation action. The shifted conjugation action map induces an isomorphism on completions at \bar{x} :

$$\left(\mathrm{GL}_2^{\mathcal{J}} / \Delta Z \times \prod_{\mathcal{J}} L_1^- G \tilde{z}_j \right)_{\bar{x}}^{\wedge} \cong \left(\mathrm{GL}_2^{\mathcal{J}, \det=1} \times \prod_{\mathcal{J}} L_1^- G \tilde{z}_j \right)_{\bar{y}}^{\wedge}$$

(Here $\mathrm{GL}_2^{\mathcal{J}, \det=1}$ denotes the kernel of the product of determinant map $\mathrm{GL}_2^{\mathcal{J}} \rightarrow \mathbb{G}_m$.)

(2) Suppose $\prod z_j = 1$. Choose a transversal slice V to T^{\vee} near 1 in GL_2 such that the tangent space $T_1 V = \mathfrak{n} \oplus \bar{\mathfrak{n}} = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \right\} \subset \mathfrak{gl}_2$. Let $\bar{x} \in \mathrm{GL}_2^{\mathcal{J}}(\mathbb{F}) \times T^{\vee}(\mathbb{F}) \tilde{z}_0 \times \prod_{j \neq 0} L_1^- G(\mathbb{F}) \tilde{z}_j$ be the tuple $((1, 1, \dots, 1), (\bar{t} \tilde{z}_0, (\tilde{z}_j)_{j \neq 0}))$, and \bar{y} its image under the shifted conjugation action. Then the shifted conjugation action map induces an isomorphism on completion at \bar{x}

$$\left(V \times \mathrm{GL}_2^{\mathcal{J} \setminus \{0\}} \times (T^{\vee} \times L_1^- G \tilde{z}_0) \times \prod_{j \neq 0} L_1^- G \tilde{z}_j \right)_{\bar{x}}^{\wedge} \cong \left(\mathrm{GL}_2^{\mathcal{J}} \times \prod_{\mathcal{J}} L_1^- G \tilde{z}_j \right)_{\bar{y}}^{\wedge}$$

Remark 4.6.2. The case $\langle \nu_j, \alpha^{\vee} \rangle = 0$ for all j happens exactly when $\tilde{z}_j \in W^{\vee} Z(\mathbb{F}(\!(v)\!))$ for all j . In this case, the same method of proof shows that for $\bar{x} = ((1, 1, \dots, 1), (\bar{\kappa} \tilde{z}_0, (\tilde{z}_j)_{j \neq 0}))$ with image \bar{y} under the shifted conjugation action

$$\left(\mathrm{GL}_2^{\mathcal{J} \setminus \{0\}} \times (\mathrm{GL}_2 \times L_1^- G \tilde{z}_0) \times \prod_{j \neq 0} L_1^- G \tilde{z}_j \right)_{\bar{x}}^{\wedge} \cong \left(\mathrm{GL}_2^{\mathcal{J}} \times \prod_{\mathcal{J}} L_1^- G \tilde{z}_j \right)_{\bar{y}}^{\wedge}$$

Proof. (1) On a tangent vector $(1 + \varepsilon K_j, (1 + \varepsilon L_j) \tilde{z}_j)$ at \bar{x} , the formula for the action is

$$(1 + \varepsilon K_j)(1 + \varepsilon L_j) \tilde{z}_j (1 - \varepsilon K_{j-1}) = (1 + \varepsilon K_j - \varepsilon \mathrm{Ad}(\tilde{z}_j)(K_{j-1}) + \varepsilon L_j) \tilde{z}_j$$

where $K_j \in \mathrm{LieGL}_2 = \mathfrak{gl}_2$, $L_j \in \mathrm{Lie}L_1^- G = \frac{1}{v} \mathfrak{gl}_2[\frac{1}{v}]$.

Hence we need to show the map

$$S : (\mathfrak{gl}_2 \times \frac{1}{v} \mathfrak{gl}_2[\frac{1}{v}])^{\mathcal{J}} \rightarrow (v \mathfrak{gl}_2[[v]] \backslash \mathfrak{gl}_2(\!(v)\!)) \cong (\mathfrak{gl}_2 \oplus \frac{1}{v} \mathfrak{gl}_2[\frac{1}{v}])^{\mathcal{J}}$$

given by

$$(K_j, L_j) \mapsto K_j - \mathrm{Ad}(\tilde{z}_j)(K_{j-1}) + L_j$$

is an isomorphism onto the subspace of the target consisting of tuples (Y_j) whose projection to $\mathfrak{gl}_2^{\mathcal{J}}$ factors through the subspace $\mathfrak{gl}_2^{\mathcal{J}, \mathrm{tr}=0}$ with sum of tuples whose sum of traces is 0.

Define $\Pi_j : \mathfrak{gl}_2 \rightarrow \mathfrak{gl}_2$ to be the linear endomorphism given by sending X to the projection of $\mathrm{Ad}(\tilde{z}_j)X \in \mathfrak{gl}_2(\!(v)\!)$ onto the \mathfrak{gl}_2 summand, i.e. extracting the v -degree 0 part of $\mathrm{Ad}(\tilde{z}_j)X$. Since $\tilde{z}_j = z_j v^{\nu_j}$ we see that

- $\Pi_j \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \mathrm{Ad}(z_j) \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ if $\langle \nu_j, \alpha^{\vee} \rangle = 0$; and
- $\Pi_j \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \mathrm{Ad}(z_j) \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ if $\langle \nu_j, \alpha^{\vee} \rangle \neq 0$

We first show that S is injective. Let $((K_j)_j, (L_j)_j) \in \ker S$ then

$$K_0 = \Pi_0 \Pi_{-1} \cdots \Pi_{-f+1}(K_0)$$

Set $K_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then our hypothesis on \tilde{z}_j shows that the composite $\Pi_0 \Pi_{-1} \cdots \Pi_{-f+1}$ sends K_0 to $\begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}$. Thus we get $K_0 \in \text{Lie}(Z)$, and hence $K_j = K_0 \in \text{Lie}(Z)$ for all j , and then $L_j = 0$. This shows the injectivity of S .

For surjectivity, it suffices to prove the surjectivity after projecting to the $\mathfrak{gl}_2^{\mathcal{J}}$ summand. Let $(Y_j) \in \mathfrak{gl}_2^{\mathcal{J}, \text{tr}=0}$. By repeated substitution in the system

$$K_j - \Pi_j(K_{j-1}) = Y_j$$

we see that the system has a solution if and only if

$$K_0 - \Pi_0 \Pi_{-1} \cdots \Pi_{-f+1}(K_0) = \Pi_0(Y_{-1}) + \Pi_0 \Pi_{-1}(Y_{-2}) + \cdots + \Pi_0 \cdots \Pi_{-f+1}(Y_{-f})$$

has a solution $K_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Note that the right-hand side has trace 0. But

$$K_0 - \Pi_0 \Pi_{-1} \cdots \Pi_{-f+1}(K_0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a-d & b \\ c & d-a \end{pmatrix}$$

hence $K_0 - \Pi_0 \Pi_{-1} \cdots \Pi_{-f+1}(K_0)$ can become any trace 0 matrix. This gives surjectivity of S .

- (2) The argument is similar to the previous case. The only difference is that at $j = 0, 1$, the tangent vector equations (on the v -degree 0 part) we get are modified to

$$V_0 + T_0 - \Pi_0(K_{-1}) = Y_0$$

$$K_1 - \Pi_1(V_0) = Y_1$$

where $V_0 \in T_1 V = \mathfrak{n} \oplus \bar{\mathfrak{n}}$ and $T_0 \in \mathfrak{t}^\vee = \text{Lie} T^\vee$.

The existence and uniqueness of solutions to the system now boils down to solvability (in terms of V_0, T_0) of

$$V_0 + T_0 - \Pi_0 \Pi_{-1} \cdots \Pi_{-f+1}(V_0) = \Pi_0(Y_{-1}) + \Pi_0 \Pi_{-1}(Y_{-2}) + \cdots + \Pi_0 \cdots \Pi_{-f+1}(Y_{-f}),$$

which now always has a unique solution since the left-hand side reduces to $V_0 + T_0$. \square

Theorem 4.6.3. *Let τ be a regular tame type with small presentation (s, μ) and assume that either $p > 8f + 3 + \max_j \langle \mu_j, \alpha^\vee \rangle$ or $p \geq 11$ and $K = \mathbb{Q}_p$. Suppose $\tilde{z}_j = \tilde{w}_j s_j^{-1} v^{\mu_j}$ with $\tilde{w} = (\tilde{w}_j) \in \text{Adm}^\vee(\eta)$. Assume that for at least one j , $\tilde{z}_j \notin W^\vee Z(\mathbb{F}((v)))$. Let $\bar{t} \in T^\vee(\mathbb{F})$ and \bar{p} be the unique semisimple Galois representation such that the matrix of the associated φ^f -module is $\bar{t} \tilde{z}_0 \varphi(\tilde{z}_{f-1}) \cdots \varphi^{f-1}(\tilde{z}_1)$ (see §2.2.2).*

- (1) *Suppose \bar{p} is absolutely irreducible, and let $R_{\bar{p}}^{\eta, \tau}$ the potentially Barsotti-Tate deformation ring of type τ . Then, up to enlarging \mathbb{F} , $R_{\bar{p}}^{\eta, \tau}$ is isomorphic to the completion of the p -saturation of $\mathcal{Z}^{\text{uv}, \tau}(\tilde{z})$ at \tilde{z} (see Definition 4.3.3).*
- (2) *Suppose \bar{p} is reducible, and let $R_{\bar{p}}^{\eta, \tau}$ be the framed potentially Barsotti-Tate deformation ring of type τ . Then up to adding formal variables, $R_{\bar{p}}^{\eta, \tau}$ is isomorphic to the completion of the p -saturation of $(\tilde{\mathcal{Z}}^{\text{mod}, \tau}(\tilde{z}) \times_{\text{GL}_2^{\mathcal{J}}} (T \times \{1\}^{\mathcal{J} \setminus \{0\}}))$ at $(\bar{t} \tilde{z}_0, \tilde{z}_1, \dots, \tilde{z}_{f-1})$.*

Proof. By Corollary 4.5.2, Lemma 4.6.1 and the fact that $\tilde{\mathcal{Z}}^{\text{mod}, \tau}$ is invariant under the shifted conjugation action, $R_{\bar{p}}^{\eta, \tau}$ has the desired description up to adding formal power series variables. This immediately gives the second item, and the first item up to adding power series variables. To

remove the potential extra power series variables, we use the following fact [Ham75, Theorem 5]: if R, S are complete local rings such that $R[[X]] \cong S[[X]]$ then $R \cong S$. \square

Remark 4.6.4. It follows from the definitions that up to isomorphism, $\mathcal{Z}^{\text{nv},\tau}(\tilde{z})$ is described as follows:

Given $(\tilde{w}_j)_{j \in \mathcal{J}} \in \text{Adm}^\vee(\eta)$, the fragmentation $\mathcal{J} = \bigcup_{k \in \mathcal{K}} \mathcal{J}_k$, and the type of the endpoints of each \mathcal{J}_k , $\mathcal{Z}^{\text{nv},\tau}(\tilde{z})$ is the spectrum of $\otimes_{j \in \mathcal{J}} R_j / \sum_k \mathcal{I}_{\mathcal{J}_k}$ where for each fragment $\mathcal{J}_k = (i, i-1, \dots, o+1, o)$, $\mathcal{I}_{\mathcal{J}_k}$ is the ideal of $\otimes_{j \in \mathcal{J}_k} R_j$ generated by the equations

$$(4.6.5) \quad M_{\text{out},o} \cdot \left(\prod_{\ell=o+1}^{i-1} T_\ell \right) \cdot M_{\text{in},i} = 0,$$

and $M_{\text{out},o}$ (resp. $M_{\text{in},i}$, resp. T_ℓ) is the final matrix (resp. initial matrix, resp. transition matrix) appearing in the corresponding entry of Table 5 according to the type of i (resp. o).

TABLE 5. Equations for $\mathcal{Z}^{\text{nv},\tau}(\tilde{z})$

type \ \tilde{w}_j	t_η	$w_0 t_\eta$	$t_{w_0(\eta)}$
II	$R_j = \mathcal{O}[X_j, Y_j]$ Initial matrix: $\begin{pmatrix} 1 \\ -X_j \end{pmatrix}$ Final matrix: $\begin{pmatrix} Y_j & -p \end{pmatrix}$	$R_j = \mathcal{O}[X_j, Y_j]$ Initial matrix: $\begin{pmatrix} 1 \\ -X_j \end{pmatrix}$ Final matrix: $\begin{pmatrix} -p & Y_j \end{pmatrix}$	$R_j = \mathcal{O}[X_j, Y_j]$ Initial matrix: $\begin{pmatrix} 1 \\ -X_j \end{pmatrix}$ Final matrix: $\begin{pmatrix} 0 & 1 \end{pmatrix}$
I	$R_j = \frac{\mathcal{O}[X_j, Y_j, Z_j]}{(p-Y_j)Y_j - X_j Z_j}$ Initial matrix: $\begin{pmatrix} Y_j & -X_j \\ -Z_j & p-Y_j \end{pmatrix}$ Final matrix: $\begin{pmatrix} p-Y_j & -Z_j \\ -X_j & Y_j \end{pmatrix}$	$R_j = \frac{\mathcal{O}[X_j, Y_j, Z_j]}{(p-Y_j)Y_j - X_j Z_j}$ Initial matrix: $\begin{pmatrix} Y_j & -X_j \\ -Z_j & p-Y_j \end{pmatrix}$ Final matrix: $\begin{pmatrix} Y_j & -X_j \\ -Z_j & p-Y_j \end{pmatrix}$	$R_j = \frac{\mathcal{O}[X_j, Y_j, Z_j]}{(p-Y_j)Y_j - X_j Z_j}$ Initial matrix: $\begin{pmatrix} Y_j & -X_j \\ -Z_j & p-Y_j \end{pmatrix}$ Final matrix: $\begin{pmatrix} p-Y_j & X_j \\ Z_j & Y_j \end{pmatrix}$
0	$R_j = \mathcal{O}[X_j]$ Transition Matrix: $\begin{pmatrix} 1 & 0 \\ -X_j & p \end{pmatrix}$	$R_j = \mathcal{O}[X_j]$ Transition Matrix: $\begin{pmatrix} 0 & -1 \\ -p & X_j \end{pmatrix}$	$R_j = \mathcal{O}[X_j]$ Transition Matrix: $\begin{pmatrix} p & -X_j \\ 0 & 1 \end{pmatrix}$

4.6.2. Rational smoothness.

Theorem 4.6.6. (1) Let $\tilde{\mathcal{Z}}^{\text{mod},\text{nm}}$ be the normalization of $\tilde{\mathcal{Z}}^{\text{mod},\tau}$. Then $\tilde{\mathcal{Z}}^{\text{mod},\text{nm}}$ is resolution-rational (cf. [Kov, Definition 9.1]) and $\tilde{\mathcal{Z}}^{\text{mod},\text{nm}}$ is Gorenstein.

(2) Assume $p > 2 + \max_j \langle \mu_j, \alpha^\vee \rangle$. Then the same statements hold for the normalization $\tilde{\mathcal{Z}}^{\text{nm}}$ of $\tilde{\mathcal{Z}}^\tau$.

Proof. (1) The statement is local so it suffices to check it for $\tilde{\mathcal{Z}}^{\text{mod},\text{nm}}(\tilde{z})$.

Since

- $\tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z})$ is the scheme theoretic image of the proper map $\pi^{\text{mod}} : \tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z}) \rightarrow \tilde{U}(\tilde{z})$,
- π^{mod} is a closed immersion after inverting p (cf. Proposition 4.3.4),

- $\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})$ is normal,

it follows that $\pi_*^{\text{mod}} \mathcal{O}_{\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})} = \mathcal{O}_{\tilde{\mathcal{Z}}^{\text{mod},\text{nm}}(\tilde{z})}$. But Proposition 3.3.14 shows that in fact

$$R\pi_*^{\text{mod}} \mathcal{O}_{\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})} = \mathcal{O}_{\tilde{\mathcal{Z}}^{\text{mod},\text{nm}}(\tilde{z})}$$

By Grothendieck duality and the properness of π^{mod} we have

$$(4.6.7) \quad \omega_{\tilde{\mathcal{Z}}^{\text{mod},\text{nm}}(\tilde{z})/\mathcal{O}} = R\pi_*^{\text{mod}} \omega_{\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})/\mathcal{O}}$$

and by Proposition 4.2.14 and Proposition 3.3.14, the RHS of (4.6.7) is

$$R\pi_*^{\text{mod}} \mathcal{O}_{\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})/\mathcal{O}} = \mathcal{O}_{\tilde{\mathcal{Z}}^{\text{mod},\text{nm}}(\tilde{z})}.$$

Thus we learn that $\tilde{\mathcal{Z}}^{\text{mod},\text{nm}}(\tilde{z})$ is Gorenstein. Since $\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})$ is easily seen to be resolution-rational (it is locally isomorphic to a product of $\text{Spec } \mathcal{O}[X, Y]/(XY-p)$), $\tilde{\mathcal{Z}}^{\text{mod},\text{nm}}(\tilde{z})$ is also resolution-rational.

- (2) Let $\tilde{\mathcal{Z}}^{\text{nm}}(\tilde{z})$ be the normalization of $\tilde{\mathcal{Z}}^\tau$. The same argument as above shows that

$$\pi_* \mathcal{O}_{\tilde{Y}^{\eta,\tau}(\tilde{z})} = \mathcal{O}_{\tilde{\mathcal{Z}}^{\text{nm}}(\tilde{z})}$$

By Lemma 3.3.7

$$R\pi_* \mathcal{O}_{\tilde{Y}^{\eta,\tau}(\tilde{z})} \otimes_{\mathcal{O}}^{\mathbb{L}} \mathbb{F} = R\pi_* (\mathcal{O}_{\tilde{Y}^{\eta,\tau}(\tilde{z})}/\varpi) = R\pi_*^{\text{mod}} (\mathcal{O}_{\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})}/\varpi) = R\pi_*^{\text{mod}} \mathcal{O}_{\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})} \otimes_{\mathcal{O}}^{\mathbb{L}} \mathbb{F}$$

which concentrates in degree 0 by Proposition 3.3.14. Since π is proper, it follows that

$$(4.6.8) \quad R\pi_* \mathcal{O}_{\tilde{Y}^{\eta,\tau}(\tilde{z})} = \mathcal{O}_{\tilde{\mathcal{Z}}^{\text{nm}}(\tilde{z})}$$

$$(4.6.9) \quad R\pi_* (\mathcal{O}_{\tilde{Y}^{\eta,\tau}(\tilde{z})}/\varpi) = \mathcal{O}_{\tilde{\mathcal{Z}}^{\text{nm}}(\tilde{z})}/\varpi = \mathcal{O}_{\tilde{\mathcal{Z}}^{\text{mod},\text{nm}}(\tilde{z})}/\varpi$$

Applying Grothendieck duality to (4.6.9) as in the previous part, we conclude that $\omega_{(\tilde{\mathcal{Z}}^{\text{nm}}(\tilde{z})/\varpi)/\mathbb{F}} = \mathcal{O}_{\tilde{\mathcal{Z}}^{\text{nm}}(\tilde{z})}/\varpi$ is trivial. Since $\tilde{\mathcal{Z}}^{\text{nm}}(\tilde{z})$ is an affine p -adic formal scheme, we also get $\omega_{\tilde{\mathcal{Z}}^{\text{nm}}(\tilde{z})/\mathcal{O}} = \mathcal{O}_{\tilde{\mathcal{Z}}^{\text{nm}}(\tilde{z})}$ so $\tilde{\mathcal{Z}}^{\text{nm}}(\tilde{z})$ is Gorenstein. This fact, (4.6.8) and the fact that $\tilde{Y}^{\eta,\tau}$ is resolution-rational now implies $\tilde{\mathcal{Z}}^{\text{nm}}(\tilde{z})$ is resolution rational. \square

We can also completely classify the non-normal locus of $\tilde{\mathcal{Z}}^\tau$:

Theorem 4.6.10. *Suppose $p > 2 + \max_j \langle \mu_j, \alpha^\vee \rangle$. Let τ have small presentation (s, μ) . A Galois representation $\bar{\rho}$ gives a non-normal point of $\tilde{\mathcal{Z}}^\tau$ exactly when both of the following holds:*

- (1) For each j , $s_j = \text{id}$ and $\langle \mu_j, \alpha^\vee \rangle \in \{0, 1\}$.
- (2) $\bar{\rho}$ is Fontaine-Laffaille with inertial Hodge-Tate weights at embedding $j \in \mathcal{J}$ given by
 - $(1, 0)$ if $\langle \mu_j, \alpha^\vee \rangle = 0$.
 - $(1, 1)$ if $\langle \mu_j, \alpha^\vee \rangle = 1$.

Proof. It follows from the proof of Theorem 4.6.6 that the non-normal locus is exactly the support of

$$\text{coker}(\mathcal{O}_{\tilde{\mathcal{Z}}^\tau} \rightarrow \pi_* \mathcal{O}_{\tilde{Y}^{\eta,\tau}})$$

and when intersecting with $\tilde{U}(\tilde{z})$, it equals the support of

$$\text{coker}(\mathcal{O}_{\tilde{U}(\tilde{z})}/\varpi \rightarrow \pi_* \mathcal{O}_{\tilde{Y}^{\eta,\tau}}/\varpi) = \text{coker}(\mathcal{O}_{\tilde{U}(\tilde{z})}/\varpi \rightarrow \pi_*^{\text{mod}} \mathcal{O}_{\tilde{Y}^{\text{mod},\eta,\tau}}/\varpi)$$

Thus we reduce to investigating the non-normal locus of $\tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z})$. It follows from the proof of Proposition 3.3.12 and Theorem 4.6.6 that the non-normal locus is exactly the support of $R^1 \text{pr}_{\tilde{B}^*} \mathcal{I}(\tilde{z})$ in Corollary 4.2.10. But Corollary 4.2.12 shows this support is non-empty exactly

when for each j , either $\langle \mu_j, \alpha^\vee \rangle = 0$ or $(\langle \mu_j, \alpha^\vee \rangle, s_j, \tilde{w}_j) = (1, \text{id}, t_{w_0(\eta)})$. This shows that (s, μ) has the desired form.

We now suppose (s, μ) has the requisite form. Decompose $\mathcal{J} = \mathcal{J}_0 \amalg \mathcal{J}_1$ where $\mathcal{J}_0 = \{j | \langle \mu_j, \alpha^\vee \rangle = 0\}$, $\mathcal{J}_1 = \{j | \langle \mu_j, \alpha^\vee \rangle = 1\}$. Furthermore, if (s, μ) is of the right form, the proof of Lemma 4.2.5 furthermore identifies the support of $R^1 \text{pr}_{\tilde{B}^*} \mathcal{I}(\tilde{z})$ as the locus of tuples $(A_j) \in \tilde{U}(\tilde{z})(\mathbb{F})$ such that

- $A_j = g_j \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}$ if $j \in \mathcal{J}_1$.
- $A_j \in \{g_j \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}, g_j \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}\}$ if $j \in \mathcal{J}_0$;

for some $g_j \in \text{GL}_2(\mathbb{F})$ (and $B, C \in \mathbb{F}$). After modifying (A_j) by the shifted conjugation action of $\text{GL}_2^{\mathcal{J}}$, we can arrange so that $A_j \in \text{GL}_2(\mathbb{F}) \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}$ for $j \in \mathcal{J}_0$.

Now set $\lambda \in X_*(T)^\vee$ be such that $\lambda_j = (1, 0)$ for $j \in \mathcal{J}_0$ and $\lambda_j = (1, 1)$ for $j \in \mathcal{J}_1$. Analogous to [LLHM⁺, Proposition 2.2.6], the moduli space FL_λ of mod p Fontaine-Laffaille modules of weight λ has the following description: Let $H \cong \text{GL}_2^{|\mathcal{J}_1|} \times B^{|\mathcal{J}_0|} \subset \text{GL}_2^{\mathcal{J}}$ be the subgroup of tuples (X_j) such that $X_{j-1} \in \text{GL}_2$ if $j \in \mathcal{J}_1$ and $X_{j-1} \in B$ if $j \in \mathcal{J}_0$. Then FL_λ is the quotient $[\text{GL}_2^{\mathcal{J}}/H]$ given by the action

$$(X_j) \cdot (g_j) \mapsto (X_j g_j \overline{X_{j-1}}^{-1})$$

where the overline denotes the projection $B \rightarrow T$ when $j \in \mathcal{J}_0$, and is the identity map when $j \in \mathcal{J}_1$.

But then, analogous to [LLHM⁺, Proposition 8.2.4], the natural embedding $\text{FL}_\lambda \hookrightarrow \Phi\text{-Mod}_K^{\text{ét}, 2}$ identifies with

$$(g_j) \mapsto \left((g_j \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix})_{j \in \mathcal{J}_1}, (g_j \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix})_{j \in \mathcal{J}_0} \right).$$

□

Remark 4.6.11. By the definition of small presentation, Theorems 4.6.6, 4.6.10 hold whenever $p > 5$. However when $K = \mathbb{Q}_p$, the only non-trivial case of these theorems are when $\langle \mu, \alpha^\vee \rangle \leq 1$, and thus and we only need to impose $p > 3$ for those cases.

5. EQUATIONS OF THE DEFORMATION RING

In this section we apply the main results of Sections 3 and 4 (namely Theorems 3.3.8 and 4.6.3) to prove the conjectures of the series of papers [CDMb, CDMc, CDM23], in particular that tamely potentially crystalline Barsotti-Tate deformation rings R_p^τ only depend on the combinatorial gene $\mathbb{X}(\tau, \bar{\rho}|_{I_K})(\text{Theorem 5.4.16})$. Throughout this section, except for §5.5, we assume that $K \neq \mathbb{Q}_p$.

5.1. Genetics. We recall and formalize into an abstract setup the notion of genes as introduced in [CDMc, CDM23], and recall the main conjectures of *loc. cit.*

5.1.1. Combinatorial genes. Inspired by the terminology of [CDMc, CDM23] we now define the notion of *combinatorial gene* associated to a pair $(\gamma, h) \in \mathbb{Z}/(p^f - 1) \times \mathbb{Z}/(p^{2f} - 1)$.

Let $\gamma \in \mathbb{Z}/(p^f - 1)$ and $h \in \mathbb{Z}/(p^{2f} - 1)$ such that $h \not\equiv 0$ modulo $q+1$ and $h - 2\gamma - (\sum_{j=0}^{f-1} p^j) \not\equiv 0$ modulo $p^f - 1$. Consider the p -expansions

$$(5.1.1) \quad h - (p^f + 1) \left(h - \gamma - \sum_{j=0}^{f-1} p^j \right) \equiv p^{2f-1} v_0 + p^{2f-2} v_1 + \cdots + p v_{2f-2} + v_{2f-1} \pmod{p^{2f} - 1}$$

with $v_{j'} \in \{0, \dots, p-1\}$ for all $j' \in \mathcal{J}'$. A *combinatorial gene associated to (γ, h)* is a \mathcal{J}' -tuple $\mathbb{X} = \mathbb{X}(\gamma, h) \in \{\mathbf{A}, \mathbf{B}, \mathbf{AB}, \mathbf{0}\}^{\mathcal{J}'}$ which satisfies the following properties (see [CDM23, Lemma B.1.3]):

- (1) if $v_{j'} = 0$ and $\mathbb{X}_{j'+1} = \mathbf{0}$, then $\mathbb{X}_{j'} = \mathbf{AB}$;
- (2) if $v_{j'} = 0$ and $\mathbb{X}_{j'+1} \neq \mathbf{0}$, then $\mathbb{X}_{j'} = \mathbf{A}$;
- (3) if $v_{j'} = 1$ and $\mathbb{X}_{j'+1} = \mathbf{0}$, then $\mathbb{X}_{j'} = \mathbf{0}$;
- (4) if $v_{j'} = 1$ and $\mathbb{X}_{j'+1} \neq \mathbf{0}$, then $\mathbb{X}_{j'} = \mathbf{B}$;
- (5) if $v_{j'} \geq 2$, then $\mathbb{X}_{j'} = \mathbf{0}$.

By [CDM23, Lemma 1.3.3, Lemma B.1.7] a combinatorial gene \mathbb{X} associated to (γ, h) is well defined, and is unique by the proof of [CDM23, Proposition 1.4.4]. Moreover, by [CDM23, Proposition 1.3.2, Corollary 1.3.4], the \mathcal{J}' -tuple $\mathbb{X} = \mathbb{X}(\gamma, h)$ satisfies the following conditions

- ♣1 if $\mathbb{X}_{j'+1} = \mathbf{0}$, then $\mathbb{X}_{j'} \in \{\mathbf{AB}, \mathbf{0}\}$;
- ♣2 if $\mathbb{X}_{j'+1} \neq \mathbf{0}$, then $\mathbb{X}_{j'} \in \{\mathbf{A}, \mathbf{B}, \mathbf{0}\}$,
- ♣3 there exists an integer $j' \in \mathcal{J}'$ such that $\mathbb{X}_{j'} = \mathbf{0}$ or $\mathbb{X}_{j'} \neq \mathbb{X}_{j'+1}$.

The discussion after [CDMc, Lemme 2.1.7] shows that:

Lemma 5.1.2. *Assume that $\gamma + \gamma' + (\sum_{j=0}^{f-1} p^j) \equiv h$ modulo $p^f - 1$, $h \not\equiv 0 \pmod{p^f + 1}$. Then:*

$$\mathbb{X}(\gamma', h)_{j'} = \begin{cases} \mathbf{A} & \text{if } \mathbb{X}(\gamma, h)_{j'} = \mathbf{B}, \\ \mathbf{B} & \text{if } \mathbb{X}(\gamma, h)_{j'} = \mathbf{A}, \\ \mathbb{X}(\gamma, h)_{j'} & \text{otherwise,} \end{cases}$$

and

$$\mathbb{X}(\gamma, p^f h)_{j'} = \mathbb{X}(\gamma, h)_{j'+f}.$$

Following the conditions (♣1)–(♣3) and Lemma 5.1.2 we define an *abstract combinatorial gene* as follows:

Definition 5.1.3. An *abstract combinatorial gene* is an equivalence class of a \mathcal{J}' -tuple $\mathbb{X} \in \{\mathbf{A}, \mathbf{B}, \mathbf{AB}, \mathbf{0}\}^{\mathcal{J}'}$ satisfying conditions (♣1), (♣2), (♣3), by the equivalence relation generated by

$$(5.1.4) \quad \mathbb{X}' \sim \mathbb{X} \quad \text{if} \quad \begin{cases} \mathbb{X}'_{j'} = \{\mathbf{A}, \mathbf{B}\} \setminus \{\mathbb{X}_{j'}\} & \text{when } \mathbb{X}_{j'} \in \{\mathbf{A}, \mathbf{B}\} \\ \mathbb{X}'_{j'} = \mathbb{X}_{j'} & \text{otherwise.} \end{cases}$$

$$(5.1.5) \quad \mathbb{X}' \sim \mathbb{X} \quad \text{if} \quad \begin{cases} \mathbb{X}'_{j'} & = \mathbb{X}_{j'+1} \\ \mathbb{X}'_{j'+f} & = \mathbb{X}_{j'+f+1} \end{cases}$$

Note that the relation (5.1.5) implies that $(\mathbb{X}_{j'})_{j' \in \mathcal{J}'} \sim (\mathbb{X}_{j'+f})_{j' \in \mathcal{J}'}$, and that the notion of abstract combinatorial gene is independent of p .

Let $(\tau, \bar{\tau}')$ be a pair of tame inertial types satisfying the following determinant condition:

$$(\det) \quad \det(\tau) \otimes_{\mathcal{O}} \mathbb{F} \equiv \det(\bar{\tau}') \otimes_{\mathbb{F}} \omega.$$

If $\tau = \tau(s, \mu) = \omega_f^\gamma \oplus \omega_f^{\gamma'}$ and $\bar{\tau}' = \bar{\tau}(\sigma, \nu) = \omega_{2f}^h \oplus \omega_{2f}^{p^f h}$ are a tame inertial type of niveau f and a tame inertial \mathbb{F} -type of level $2f$ as in §2.1.1, then condition (det) translates into the condition of Lemma 5.1.2. We thus define the gene $\mathbb{X}(\tau, \bar{\tau}')$ of the pair $(\tau, \bar{\tau}')$ as the abstract combinatorial gene associated to the \mathcal{J}' -tuple $\mathbb{X}(\gamma, h)$. This gives the motivation behind Definition 5.1.3: the relation (5.1.4) is imposed by the isomorphism $\omega_f^\gamma \oplus \omega_f^{\gamma'} \cong \tau \cong \omega_f^{\gamma'} \oplus \omega_f^\gamma$ and the relation (5.1.5) by the fact that the isomorphism class of $\tau, \bar{\tau}'$ does not depend on the choice of the embedding σ_0 . In

particular, $\mathbb{X}(\tau, \bar{\tau}')$ only depends on the isomorphism class of τ and $\bar{\tau}'$, and is insensitive to twist by characters $\chi : I_K \rightarrow \mathcal{O}^\times$.

5.1.2. *Genetic conjectures.* The computations of [CDMc, CDM23] showed that the combinatorial genes contains non-trivial information on the generic and special fiber of Galois deformation rings with p -adic Hodge theory conditions.

Let $\bar{\rho} : G_K \rightarrow \mathrm{GL}_2(\mathbb{F})$ be irreducible and τ a tame inertial type of niveau f . Then $\bar{\rho}|_{I_K}$ defines a tame inertial \mathbb{F} -type, so that the combinatorial gene $\mathbb{X}(\tau, \bar{\rho}|_{I_K})$ is defined. The authors of [CDMc, CDM23] propose the following conjecture (which is an integral version of [CDMc, Conjecture 5.1.5])

Conjecture 5.1.6 (Conjecture 2 in [CDM23]). *The deformation ring $R_{\bar{\rho}}^{\eta, \tau}$ is determined by $\mathbb{X}(\tau, \bar{\rho}|_{I_K})$.*

They furthermore refine Conjecture 5.1.6 into the following

Conjecture 5.1.7 (Conjecture 5.2.7 [CDMc], Conjecture 3.1.2 [CDM23]). *There exists a decomposition $\mathbb{X}(\tau, \bar{\rho}|_{I_K}) = \cup_{i=0}^r (\mathbb{X}_{j_i}, \mathbb{X}_{j_i+f})_{j_i \leq j \leq j_{i+1}}$ such that*

$$R_{\bar{\rho}}^{\eta, \tau} \cong \widehat{\otimes}_{i=0}^r R_i$$

where R_i is a complete local Noetherian \mathcal{O} algebra depending only on $(\mathbb{X}_{j_i}, \mathbb{X}_{j_i+f})_{j_i \leq j \leq j_{i+1}}$.

Moreover, even if not stated as a conjecture, they suggest that

Conjecture 5.1.8 ([CDM23, CDMa]). *The deformation ring $R_{\bar{\rho}}^{\eta, \tau}$ is “independent of p ”.*

Conjectures 5.1.6, 5.1.7 and 5.1.8 are proven in Theorem 5.4.16 for $p > 8f + 3 + \max_j \langle \mu_j, \alpha^\vee \rangle$. In particular throughout sections 5.2, 5.3 and 5.4 we assume that $p > 8f + 3 + \max_j \langle \mu_j, \alpha^\vee \rangle$.

Prior to this work, Conjecture 5.1.6 was known when either τ has a presentation (s, μ) with $2 < \langle \mu_j, \alpha^\vee \rangle < p - 2$ for all $j \in \mathcal{J}$ or if $(\mathbb{X}(\tau, \bar{\rho}|_{I_K})_{j+f}, \mathbb{X}(\tau, \bar{\rho}|_{I_K})_j) = (0, 0)$ for some $j \in \mathcal{J}$. In the first case, we have $0 \in \{\mathbb{X}_j, \mathbb{X}_{j+f}\}$ for all $j \in \mathcal{J}$ which in turn implies that the Kisin variety of type (η, τ) attached to $\bar{\rho}$ is either empty or a single point by [CDMc, Théorème 2.2.1(A)]. Conjecture 5.1.6 is true by Theorem 2.1.9 in this case follow from Theorem 2.1.9 and [CDM23, Theorem 3]. In the second case, the Kisin variety is empty and the deformation ring is zero.

We now elaborate on Conjecture 5.1.8. Given an abstract combinatorial gene \mathbb{X} we prove the conjecture by constructing rings $R_{\mathbb{X}}$ which are (possibly zero) quotients of polynomial rings over $\mathbb{Z}[t]$ modulo an ideal $I_{\mathbb{X}} \subset R_{\mathbb{X}}$. These rings are independent of p (since abstract combinatorial genes are) and the Conjecture 5.1.8 would be proven by explicitly showing that $R_{\bar{\rho}}^{\eta, \tau}$ is isomorphic to the completion of $R_{\mathbb{X}}/(t - p) \otimes \mathcal{O}$ at the ideal generated by t and the variables of $R_{\mathbb{X}}$, where $\mathbb{X} = \mathbb{X}(\tau, \bar{\rho}|_{I_K})$. Again, by Theorem 2.1.9, the conjecture is known to be true when τ has a 2-generic lowest alcove presentation.

We conclude with the following observation. By [CDMc, Proposition 4.1.3], we have $R_{\bar{\rho}}^{\eta, \tau} = 0$ as soon as $(\mathbb{X}(\tau, \bar{\rho}|_{I_K})_{j+f}, \mathbb{X}(\tau, \bar{\rho}|_{I_K})_j) = (0, 0)$ for some $j \in \mathcal{J}$. Hence, in what follows, we will be interested in combinatorial genes \mathbb{X} satisfying the further condition

$$\clubsuit 4 \quad (\mathbb{X}_{j+f}, \mathbb{X}_j) \neq (0, 0) \text{ for all } j \in \mathcal{J}.$$

5.2. **Genetic translation.** We fix our setup as in Theorem 4.6.3. Hence, let $\tau : I_K \rightarrow \mathrm{GL}_2(\mathcal{O})$ be a regular tame inertial type of niveau f with a small presentation (s, μ) and let $\tilde{w} = (w_j t_{\nu_j})_{j \in \mathcal{J}} \in \mathrm{Adm}^\vee(\eta)$. Up to twist, we can furthermore assume that $\mu_{j,2} = 0$ for all $j \in \mathcal{J}$. We abbreviate $k_j \stackrel{\mathrm{def}}{=} \langle \mu_j, \alpha^\vee \rangle$ and set $\tilde{z} \stackrel{\mathrm{def}}{=} \tilde{w} s^{-1} v^\mu$ in what follows.

In order to analyze genetic data associated to τ and \tilde{w} define $\lambda \in X^*(\underline{T})$ by the condition

$$(5.2.1) \quad (z_0 z_{f-1} \cdots z_1) v^{(\sum_{j \in \mathcal{J}} p^j \lambda_j)} = \tilde{z}_0 \varphi(\tilde{z}_{f-1}) \cdots \varphi^{f-1}(\tilde{z}_1).$$

and the $2f$ -tuple $(v'_{j'})_{j' \in \mathcal{J}'}$ by

$$\begin{pmatrix} v'_{2f-1-j} \\ v'_{f-1-j} \end{pmatrix} \stackrel{\text{def}}{=} \lambda_j - \begin{pmatrix} k_{f-j} \\ k_{f-j} \end{pmatrix} \delta_{s_{\text{or}, f-1-j} \neq \text{id}}$$

where $j \in \{0, \dots, f-1\}$ and $s_{\text{or}, f-1-j} = \prod_{i=0}^{f-1-j} s_i$ (Lemma 2.1.8). By smallness of (s, μ) , we have $v'_j, v'_{f+j} \in \{-\frac{p+1}{2}, \dots, \frac{p+3}{2}\}$ for all $j \in \{0, \dots, f-1\}$. Note that the $2f$ -tuple $(v'_{j'})_{j' \in \mathcal{J}'}$ depends on the triple (\tilde{w}, s, μ) , but we omit this dependence for sake of readability.

We assume from now on that $\prod_{j \in \mathcal{J}} z_j = w_0$, $\sum_{j \in \mathcal{J}} p^j (\mu_{j,1} + \mu_{j,2}) \equiv \sum_{j \in \mathcal{J}} p^j (\lambda_{j,1} + 1 + (\lambda_{j,2} + 1))$ modulo $p^f - 1$ and $\sum_{j \in \mathcal{J}} p^j (\lambda_{j,1} + p^f \lambda_{j,2}) \not\equiv 0$ modulo $p^f + 1$. By construction, the $2f$ -tuple $(v'_{j'})_{j' \in \mathcal{J}'}$ extracts precisely the LHS of equation (5.1.1) (using equations (2.1.2) and (2.1.3)), and hence produces a gene $\mathbb{X}(v')$, satisfying items $(\clubsuit 1)$, $(\clubsuit 2)$, $(\clubsuit 3)$ thanks to the assumptions in the previous sentence.

Tables 6 and 7 give respectively the explicit description of \tilde{z}_i and $\begin{pmatrix} v'_{f+j} \\ v'_j \end{pmatrix}$ according to (s, μ) and \tilde{w} and are directly obtained from the definitions.

TABLE 6. **Genetic Translation-I**

$s_i \backslash \tilde{w}_i$	t_η	$w_0 t_\eta$	$t_{w_0(\eta)}$
id	$\begin{pmatrix} k_i + 1 \\ 0 \end{pmatrix}$	$(12) \begin{pmatrix} k_i + 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} k_i \\ 1 \end{pmatrix}$
(12)	$(12) \begin{pmatrix} k_i \\ 1 \end{pmatrix}$	$\begin{pmatrix} k_i \\ 1 \end{pmatrix}$	$(12) \begin{pmatrix} k_i + 1 \\ 0 \end{pmatrix}$

This table records $\tilde{z}_i \stackrel{\text{def}}{=} \tilde{w}_i s_i^{-1} t_{\mu_i}$.

5.3. Fibers of the map $(\tilde{w}, s, \mu) \mapsto \mathbb{X}(v')$. Given a triple (\tilde{w}, s, μ) we have produced a tuple $((s_{j+1}, s_{\text{or}, j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type at } j), (v'_{f+j}, v'_j))_{j \in \mathcal{J}}$ and a gene $\mathbb{X}(v')$ attached to $(v'_{f+j}, v'_j)_{j \in \mathcal{J}}$, hence a map

$$(5.3.1) \quad (\tilde{w}, s, \mu) \mapsto ((s_{j+1}, s_{\text{or}, j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type at } j), (v'_{f+j}, v'_j))_{j \in \mathcal{J}} \mapsto (v'_{f+j}, v'_j)_{j \in \mathcal{J}} \mapsto \mathbb{X}(v').$$

In this and the following section we analyze the shapes and types appearing in the fiber of the map (5.3.1). *From now onwards, we furthermore assume that $\mathbb{X}(v')$ satisfies condition $(\clubsuit 4)$.*

As a preliminary step, we record in Table 8 the fiber of the map

$$(5.3.2) \quad ((s_{j+1}, s_{\text{or}, j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type at } j), (v'_{f+j}, v'_j))_{j \in \mathcal{J}} \mapsto (v'_{f+j}, v'_j)_{j \in \mathcal{J}}$$

Table 8 is obtained directly from Table 7, and the only relevant property to obtain the Table 8 is whether $v'_j \geq 2$, $v'_j = 1$, $v'_j = 0$ or $v'_j < 0$.

Recall that we identify \mathcal{J}' with $\mathbb{Z}/2f$, and in what follows objects such as $\Sigma_j, \mathbb{X}_j, v'_j$ are indexed by \mathcal{J}' . For objects that are previously indexed by \mathcal{J} we extend by f periodicity, with the exception of Σ_j where the extension is given by $\Sigma_{j+f} = w_0 \Sigma_j$.

TABLE 7. Genetic Translation-II

\tilde{w}_{j+1}	s_{j+1} $s_{\text{or},j}$	id	(12)
$t_\eta, w_0 t_\eta$	id	$\Sigma_j \begin{pmatrix} k_{j+1} + 1 \\ 0 \end{pmatrix}$	$\Sigma_j \begin{pmatrix} 1 \\ -k_{j+1} \end{pmatrix}$
	(12)	$\Sigma_j \begin{pmatrix} k_{j+1} \\ 1 \end{pmatrix}$	$\Sigma_j \begin{pmatrix} 0 \\ 1 - k_{j+1} \end{pmatrix}$
$t_{w_0 \eta}$	id	$\Sigma_j \begin{pmatrix} k_{j+1} \\ 1 \end{pmatrix}$	$\Sigma_j \begin{pmatrix} 0 \\ 1 - k_{j+1} \end{pmatrix}$
	(12)	$\Sigma_j \begin{pmatrix} k_{j+1} + 1 \\ 0 \end{pmatrix}$	$\Sigma_j \begin{pmatrix} 1 \\ -k_{j+1} \end{pmatrix}$

The entries of the table record $\begin{pmatrix} v'_{j+f} \\ v'_j \end{pmatrix}$ for $j \in \{0, \dots, f-1\}$, where

$\Sigma_j \stackrel{\text{def}}{=} \prod_{i=1}^j z_i^{-1} = (\prod_{i=1}^j s_i)(w_0)^{N_j}$ and $N_j \stackrel{\text{def}}{=} \#\{i \in \{1, \dots, j\}, \tilde{w}_i = w_0 t_\eta\}$. Note that $\Sigma_0 = \text{id}$ by definition.

5.3.1. *Step one: fiber of the map $(v'_{f+j}, v'_j)_{j \in \mathcal{J}} \mapsto \mathbb{X}(v')$.* Let $\mathbb{X} \in \{\mathbf{A}, \mathbf{B}, \mathbf{AB}, \mathbf{0}\}^{\mathcal{J}'}$ satisfy conditions $(\clubsuit 1), (\clubsuit 2), (\clubsuit 3)$. In this subsection we determine the fiber above \mathbb{X} of the map $(v'_{f+j}, v'_j)_{j \in \mathcal{J}} \mapsto \mathbb{X}(v')$.

Lemma 5.3.3. *Assume that $\mathbb{X} = \mathbb{X}(v')$ for some $(v'_{f+j}, v'_j)_{j \in \mathcal{J}}$ associated to a triple (\tilde{w}, s, μ) as above (in particular \mathbb{X} satisfies conditions $(\clubsuit 1), (\clubsuit 2), (\clubsuit 3)$).*

Then, for each $j' \in \mathcal{J}'$, the values of $v'_{j'}$ are constraint by the third row of Table 9 according to the pair $\mathbb{X}_{f+j'}, \mathbb{X}_{f+j'+1}$.

Proof. Let $(v_{j'})_{j' \in \mathcal{J}'} \in \{0, \dots, p-1\}^{\mathcal{J}'}$ be the tuple defined by

$$\sum_{j'=0}^{2f-1} v_{j'} p^{2f-1-j'} \equiv \sum_{j'=0}^{2f-1} v'_{j'} p^{2f-1-j'} \pmod{p^{2f} - 1}.$$

We will show that:

- if $\mathbb{X}_i \in \{\mathbf{A}, \mathbf{B}\}$ then $v_i = v'_i$;
- if $\mathbb{X}_i = \mathbf{AB}$ then $v_i \in \{v'_i, v'_i - 1\}$;
- if $\mathbb{X}_i = \mathbf{0}$ and $\mathbb{X}_{i+1} \neq \mathbf{0}$ then $v_i \in \{v'_i, v'_i + p\}$; and
- if $\mathbb{X}_i = \mathbf{0}$ and $\mathbb{X}_{i+1} = \mathbf{0}$ then $v_i \in \{v'_i, v'_i - 1, v'_i + p, v'_i + p - 1\}$.

By definition of combinatorial gene (equations (1)-(5)) we deduce that:

- if $\mathbb{X}_i \in \{\mathbf{AB}, \mathbf{A}\}$ then $v_i = 0$;
- if $\mathbb{X}_i = \mathbf{B}$ then $v_i = 1$;

TABLE 8. **Fibre of the map**

$$(s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type}_j) \mapsto \left\{ \begin{pmatrix} 0 \\ \geq 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ < 0 \end{pmatrix} \right\}$$

s_{j+1}	$s_{\text{or},j}$	Σ_j	$\begin{pmatrix} v'_{j+f} \\ v'_j \end{pmatrix} = \begin{pmatrix} 0 \\ \geq 2 \end{pmatrix}$	$\begin{pmatrix} v'_{j+f} \\ v'_j \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} v'_{j+f} \\ v'_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} v'_{j+f} \\ v'_j \end{pmatrix} = \begin{pmatrix} 0 \\ < 0 \end{pmatrix}$
id	id	id		$(\text{id}, t_{w_0(\eta)}, 0)$		
id	id	w_0	(id, t_η, II) $(w_0, w_0 t_\eta, II)$	$(\text{id}, t_\eta, 0)$ $(w_0, w_0 t_\eta, 0)$		
id	w_0	id		$(\text{id}, t_{w_0(\eta)}, 0)$	$(\text{id}, t_{w_0(\eta)}, I)$	$(\text{id}, t_{w_0(\eta)}, II)$
id	w_0	w_0		$(\text{id}, t_\eta, 0)$ $(w_0, w_0 t_\eta, 0)$	$(\text{id}, t_{w_0(\eta)}, I)$	
w_0	id	id				
w_0	id	w_0	$(w_0, t_{w_0(\eta)}, II)$			
w_0	w_0	id			$(\text{id}, w_0 t_\eta, I)$ (w_0, t_η, I)	$(\text{id}, w_0 t_\eta, II)$ (w_0, t_η, II)
w_0	w_0	w_0			$(\text{id}, w_0 t_\eta, I)$ (w_0, t_η, I)	

Given (v'_{f+j}, v'_j) and $(s_{j+1}, s_{\text{or},j}, \Sigma_j)$, the table entries record the triples $(z_{j+1}, \tilde{w}_j, \text{type at } j)$ such that $((s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type at } j), (v'_{f+j}, v'_j))_{j \in \mathcal{J}}$ is a 7-tuple associated to some (\tilde{w}, s, μ) . (Here $j \in \mathbb{Z}/f\mathbb{Z}$.) The table records the cases where $v'_{f+j} = 0$, for $j \in \mathbb{Z}/f\mathbb{Z}$, but we note that we can replace (v'_{f+j}, v'_j) with (v'_j, v'_{f+j}) at the cost of replacing Σ_j with $w_0 \Sigma_j$, and, similarly, we can replace (v'_{j+f}, v'_j) with $(1 - v'_{j+f}, 1 - v'_j)$ at the cost of replacing Σ_j and $s_{\text{or},j}$ with $w_0 \Sigma_j$ and $w_0 s_{\text{or},j}$. The blank boxes in the table correspond to configurations of $(s_{j+1}, s_{\text{or},j}, \Sigma_j, (v'_{j+f}, v'_j))$ which can not arise.

- if $\mathbb{X}_i = \mathbb{X}_{i+1} = \mathbf{0}$ then $v_i \in [1, p-1]$; and
- if $\mathbb{X}_i = \mathbf{0}$ and $\mathbb{X}_{i+1} \neq \mathbf{0}$ then $v_i \in [2, p-1]$.

By $\clubsuit 3$) there exists $\ell \in \{0, \dots, 2f-1\}$ such that $v_\ell \neq 0$. Let $\varepsilon \in \mathbb{Z}$ be such that

$$\sum_{i=0}^{2f-1} v_i p^{2f-1-i} = \sum_{i=0}^{2f-1} v'_i p^{2f-1-i} + \varepsilon(p^{2f} - 1)$$

As $v'_i \in \{-(p+1)/2, \dots, (p+3)/2\}$ we have

$$-p^{2f} + 1 < \sum_{i=0}^{2f-1} v'_i p^{2f-1-i} < p^{2f} - 1$$

TABLE 9. Genetic Translation-II

$\mathbb{X}_{f+j'}$	A	B	AB	0	0
$\mathbb{X}_{f+j'+1}$	$\neq 0$	$\neq 0$	0	$\neq 0$	0
$v'_{j'}$	0	1	$\{1, 0\}$	$\{\geq 2, < 0\}$	Any

hence $\varepsilon \in \{0, 1\}$.

Let $i_0 \stackrel{\text{def}}{=} \min\{i \geq 0, v'_i \neq 0\}$ (this is well defined by $(\clubsuit 3)$). As

$$-p^{2f-i_0-1} < \sum_{j=i_0+1}^{2f-1} v'_j p^{2f-1-j} < p^{2f-i_0-1}$$

we have

$$(5.3.4) \quad \varepsilon = \begin{cases} 0 & \text{if } v'_{i_0} > 0, \\ 1 & \text{if } v'_{i_0} < 0. \end{cases}$$

If $\varepsilon = 1$ then $v_i = p - 1$ for $0 \leq i \leq i_0 - 1$ and moreover $v_{i_0} = p + v'_{i_0} \geq 2$ if $i_0 < 2f - 1$ and $v_{i_0} = p - 1 + v'_{i_0} \geq 2$ if $i_0 = 2f - 1$. We conclude that $\mathbb{X}_i = 0$ for all $0 \leq i \leq i_0$ if $\varepsilon = 1$.

We have the following relation between v_{2f-1} and v'_{2f-1} , and we define $\varepsilon_{2f-1} \in \{0, 1\}$ as follows:

- if $v'_{2f-1} < 0$ and $\varepsilon = 0$ then $v_{2f-1} = p + v'_{2f-1} \geq 2$, $\mathbb{X}_{2f-1} = 0$ and we define $\varepsilon_{2f-1} \stackrel{\text{def}}{=} 1$,
- if $v'_{2f-1} \geq 0$ and $\varepsilon = 0$ then $v_{2f-1} = v'_{2f-1}$, and we define $\varepsilon_{2f-1} \stackrel{\text{def}}{=} 0$,
- if $v'_{2f-1} < 1$ and $\varepsilon = 1$ then $v_{2f-1} = p + v'_{2f-1} - 1 \geq 1$, $\mathbb{X}_{2f-1} = 0$ and we define $\varepsilon_{2f-1} = 1$,
- if $v'_{2f-1} \geq 1$ and $\varepsilon = 1$ then $v_{2f-1} = v'_{2f-1} - 1$ and we define $\varepsilon_{2f-1} \stackrel{\text{def}}{=} 0$.

By decreasing induction, we deduce for $i \in \{2f - 2, \dots, 0\}$ the following relations between v_i and v'_i , and define $\varepsilon_i \in \{0, 1\}$ as follows:

- if $v'_i < 0$ and $\varepsilon_{i+1} = 0$ then $v_i = p + v'_i \geq 2$, $\mathbb{X}_i = 0$ and we define $\varepsilon_i \stackrel{\text{def}}{=} 1$,
- if $v'_i \geq 0$ and $\varepsilon_{i+1} = 0$ then $v_i = v'_i$ and we define $\varepsilon_i \stackrel{\text{def}}{=} 0$,
- if $v'_i < 1$ and $\varepsilon_{i+1} = 1$ then $v_i = p + v'_i - 1 \geq 1$, $\mathbb{X}_i = 0$ and we define $\varepsilon_i \stackrel{\text{def}}{=} 1$;
- if $v'_i \geq 1$ and $\varepsilon_{i+1} = 1$ then $v_i = v'_i - 1$ and we define $\varepsilon_i \stackrel{\text{def}}{=} 0$.

By (5.3.4) we have $\varepsilon = \varepsilon_0$ and hence the relations (a)–(d) between v_i and v'_i hold for any $i \in \mathcal{J}'$, and define an element $(\varepsilon_i)_{\mathcal{J}'} \in \{0, 1\}^{\mathcal{J}'}$.

For any $i \in \{0, \dots, 2f - 1\}$ such that $\varepsilon_{i+1} = 1$, we have $\mathbb{X}_{i+1} = 0$. Hence $v_i = v'_i - 1$ only occurs for $\mathbb{X}_i \in \{0, \text{AB}\}$ by $(\clubsuit 1)$. The conclusion follows now from a direct application of conditions (1)–(5) (for instance if $\mathbb{X}_i = \text{AB}$ then $v_i = 0$ by 1, and from (b), (d) above we conclude that $v'_i \in \{0, 1\}$). \square

The following lemma improves Lemma 5.3.3.

Lemma 5.3.5. *Keep the assumptions of Lemma 5.3.3 let $i \in \{0, \dots, 2f - 1\}$.*

- (a) If $\mathbb{X}_i = \mathbf{0}$ and $v'_i = 0$, then there exists $j \geq 0$ such that $v'_{j+i} < 0$ and $v'_{i+\ell} = 0$ for all $0 \leq \ell < j$.
- (b) If $\mathbb{X}_i = \mathbf{0}$ and $v'_i = 1$, then there exists $j \geq 0$ such that $v'_{j+i} \geq 2$ and $v'_{i+\ell} = 1$ for all $0 \leq \ell < j$.
- (c) If $\mathbb{X}_i = \mathbf{AB}$ and $v'_i = 1$, then there exists $j \geq 0$ such that $v'_{j+i} < 0$ and $v'_{i+\ell} = 0$ for all $0 \leq \ell < j$.
- (d) If $\mathbb{X}_i = \mathbf{AB}$ and $v'_i = 0$, then $v'_{i+1} \geq 1$.

Proof. In the notation of the proof of Lemma 5.3.3 we have:

- (a) If $\mathbb{X}_i = \mathbf{0}$ and $v'_i = 0$, then $\varepsilon_{i+1} = 1$ and $\mathbb{X}_{i+1} = \mathbf{0}$ and $v'_{i+1} < 1$. The claimed result follows now by induction.
- (b) This is similar to (a).
- (c) If $\mathbb{X}_i = \mathbf{AB}$ and $v'_i = 1$ then $v_i = v'_i - 1$, $\varepsilon_{i+1} = 1$ and $\mathbb{X}_{i+1} = \mathbf{0}$. Thus $v'_{i+1} \leq 0$. We conclude by (a).
- (d) If $\mathbb{X}_i = \mathbf{AB}$ and $v'_i = 0$, then $v_i = v'_i$, $\varepsilon_{i+1} = 0$ and $\mathbb{X}_{i+1} = \mathbf{0}$. Thus $v'_{i+1} \geq 1$.

□

5.3.2. *Step 2: Types and shapes in the fibers.* In this section we conclude our analysis on the fibers of (5.3.2). The main result is Proposition 5.3.6.

In the following we assume that $\mathbb{X} \in \{\mathbf{A}, \mathbf{B}, \mathbf{AB}, \mathbf{0}\}^{\mathcal{J}'}$ satisfies conditions $(\clubsuit 1)$ – $(\clubsuit 4)$. The following proposition analyzes the fiber of the composite map

$$((s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type at } j), (v'_{f+j}, v'_j))_{j \in \mathcal{J}} \mapsto (v'_{f+j}, v'_j)_{j \in \mathcal{J}} \mapsto \mathbb{X}(v')$$

where the domain is the set of $-$ -tuples attached to triples (\tilde{w}, s, μ) as in the end of Section 5.2.

Proposition 5.3.6. *Let $\mathbb{X} \in \{\mathbf{A}, \mathbf{B}, \mathbf{AB}, \mathbf{0}\}^{\mathcal{J}'}$ satisfy conditions $(\clubsuit 1)$ – $(\clubsuit 4)$. Assume that $\mathbb{X} = \mathbb{X}(v')$ for some $(s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type of } j, (v'_{j+f}, v'_j))$ associated to a triple (\tilde{w}, s, μ) .*

- (1) If $\binom{\mathbb{X}_{j+f}}{\mathbb{X}_j} = \binom{\mathbf{A}}{\mathbf{0}}$ then $(s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type of } j, (v'_{j+f}, v'_j))$ belongs to the set

$$\left\{ \begin{array}{l} (-, w_0, \text{id}, -, -, II, (0, < 0)), (\text{id}, w_0, \text{id}, \text{id}, t_{w_0(\eta)}, I, (0, 0)), \\ (\text{id}, \text{id}, w_0, \text{id}, t_\eta, 0, (0, 1)), (-, \text{id}, w_0, -, -, II, (0, \geq 2)) \end{array} \right\}.$$

- If $\binom{\mathbb{X}_{j+f}}{\mathbb{X}_j} = \binom{\mathbf{B}}{\mathbf{0}}$ then $(s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type of } j, (v'_{j+f}, v'_j))$ belongs to the set

$$\left\{ \begin{array}{l} (-, w_0, \text{id}, -, -, II, (1, < 0)), (\text{id}, w_0, \text{id}, \text{id}, t_\eta, 0, (1, 0)), \\ (\text{id}, \text{id}, w_0, \text{id}, t_{w_0(\eta)}, I, (1, 1)), (-, \text{id}, w_0, -, -, II, (1, \geq 2)) \end{array} \right\}.$$

- (2) If $\binom{\mathbb{X}_{j+f}}{\mathbb{X}_j} = \binom{\mathbf{0}}{\mathbf{AB}}$ then $(s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type of } j, (v'_{j+f}, v'_j))$ belongs to the set

$$\left\{ \begin{array}{l} (\text{id}, \text{id}, \text{id}, w_0, w_0 t_\eta, II, (\geq 2, 0)), (w_0, \text{id}, \text{id}, \text{id}, w_0 t_\eta, II, (\geq 2, 1)), \\ (w_0, w_0, w_0, \text{id}, w_0 t_\eta, II, (< 0, 0)), (\text{id}, w_0, w_0, w_0, w_0 t_\eta, II, (< 0, 1)) \end{array} \right\}.$$

- (3) If $\binom{\mathbb{X}_{j+f}}{\mathbb{X}_j} = \binom{\mathbf{A}}{\mathbf{AB}}$ then $(s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type of } j, (v'_{j+f}, v'_j))$ belongs to the set

$$\left\{ \begin{array}{l} (w_0, w_0, \text{id}, w_0, t_\eta, I, (0, 0)), (\text{id}, w_0, \text{id}, \text{id}, t_{w_0(\eta)}, 0, (0, 1)), \\ (w_0, w_0, w_0, \text{id}, w_0 t_\eta, I, (0, 0)), (\text{id}, w_0, w_0, w_0, w_0 t_\eta, 0, (0, 1)) \end{array} \right\}.$$

- If $\binom{\mathbb{X}_{j+f}}{\mathbb{X}_j} = \binom{\mathbf{B}}{\mathbf{AB}}$, then $(s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type of } j, (v'_{j+f}, v'_j))$ belongs to the set

$$\left\{ \begin{array}{l} (w_0, \text{id}, w_0, w_0, t_\eta, I, (1, 1)), (\text{id}, \text{id}, w_0, \text{id}, t_{w_0(\eta)}, 0, (1, 0)), \\ (w_0, \text{id}, \text{id}, \text{id}, w_0 t_\eta, I, (1, 1)), (\text{id}, \text{id}, \text{id}, w_0, w_0 t_\eta, 0, (1, 0)) \end{array} \right\}.$$

(4) If $\begin{pmatrix} \mathbb{X}_{j+f} \\ \mathbb{X}_j \end{pmatrix} \in \left\{ \begin{pmatrix} \mathbf{A} \\ \mathbf{A} \end{pmatrix}, \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}, \begin{pmatrix} \mathbf{B} \\ \mathbf{A} \end{pmatrix}, \begin{pmatrix} \mathbf{B} \\ \mathbf{B} \end{pmatrix} \right\}$ then $s_{j+1}, s_{\text{or},j}, (v'_{j+f}, v'_j)$ and type of j are determined by \mathbb{X} .
 Moreover if $\begin{pmatrix} \mathbb{X}_{j+f} \\ \mathbb{X}_j \end{pmatrix} \in \left\{ \begin{pmatrix} \mathbf{A} \\ \mathbf{A} \end{pmatrix}, \begin{pmatrix} \mathbf{B} \\ \mathbf{B} \end{pmatrix} \right\}$, then \mathbb{X} determines if either $\tilde{w}_j = t_{w_0(\eta)}$ or $\tilde{w}_j \in \{w_0 t_\eta, t_\eta\}$.

The other cases are deduced by the transformation $\begin{pmatrix} \mathbb{X}_{j+f} \\ \mathbb{X}_j \end{pmatrix} \mapsto \begin{pmatrix} \mathbb{X}_j \\ \mathbb{X}_{j+f} \end{pmatrix}$ (see the caption of Table 8).

Proof. Proof of (1). Assume $\mathbb{X}_j = \mathbf{0}$ and $v'_j = 1$. Let $j_1 = \max\{j' \geq j \text{ such that } v'_\ell = 1, j_0 \leq \ell \leq j'\}$. Then $v'_{j_1+1} \geq 2$ and for all $j \leq j' \leq j_1$ we have $\begin{pmatrix} \mathbb{X}_{j+f} \\ \mathbb{X}_{j'} \end{pmatrix} \in \left\{ \begin{pmatrix} \mathbf{A} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{B} \\ \mathbf{0} \end{pmatrix} \right\}$, $(s_{j'+1}, s_{\text{or},j'}, \Sigma_{j'}, z_{j'+1}) = (\text{id}, \text{id}, w_0, \text{id})$ and

$$(\tilde{w}_{j'}, \text{type of } j', (v'_{f+j'}, v'_{j'})) = \begin{cases} (t_\eta, \mathbf{0}, (\mathbf{0}, \mathbf{1})) & \text{if } \mathbb{X}_{f+j'} = \mathbf{A}, \\ (t_{w_0(\eta)}, I, (\mathbf{1}, \mathbf{1})) & \text{if } \mathbb{X}_{f+j'} = \mathbf{B}. \end{cases}$$

Indeed by $\clubsuit 4$ and Lemma 5.3.5, j_1 is well defined, $\mathbb{X}_{j'} = \mathbf{0}$, for all $j \leq j' \leq j_1 + 1$ and $v'_{j_1+1} \geq 2$. Hence $\begin{pmatrix} v'_{j_1+1+f} \\ v'_{j_1+1} \end{pmatrix} \in \left\{ \begin{pmatrix} \mathbf{0} \\ \geq 2 \end{pmatrix}, \begin{pmatrix} \mathbf{1} \\ \geq 2 \end{pmatrix} \right\}$.

By Table 8 and symmetry, the index $j_1 + 1$ is of type II and $s_{\text{or},j_1+1} = \text{id}$, $\Sigma_{j_1+1} = w_0$. By $\clubsuit 1$ and $\clubsuit 4$, for all $j \leq j' \leq j_1$ we have $\mathbb{X}_{j'+f} \in \{\mathbf{A}, \mathbf{B}\}$, hence $v'_{f+j'} \in \{0, 1\}$ is determined by $\mathbb{X}_{f+j'}$. That is, for $j \leq j' \leq j_1$, $\begin{pmatrix} v'_{j'+f} \\ v'_{j'} \end{pmatrix} \in \left\{ \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix}, \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix} \right\}$.

By decreasing induction and Table 8, since $s_{\text{or},j_1+1} = \text{id}$ and $\Sigma_{j_1+1} = w_0$, we have for $j \leq j' \leq j_1$

- if $\begin{pmatrix} v'_{j'+f} \\ v'_{j'} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix}$, $(s_{j'+1}, s_{\text{or},j'}, \Sigma_{j'}, z_{j'+1}, \tilde{w}_{j'}, \text{type of } j') = (\text{id}, \text{id}, w_0, \text{id}, t_\eta, \mathbf{0})$,
- if $\begin{pmatrix} v'_{j'+f} \\ v'_{j'} \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}$, $(s_{j'+1}, s_{\text{or},j'}, \Sigma_{j'}, z_{j'+1}, \tilde{w}_{j'}, \text{type of } j') = (\text{id}, \text{id}, w_0, \text{id}, t_{w_0(\eta)}, I)$.

The proof for $v'_j = 0$ can be deduced directly by symmetry. The cases $v'_j < 0$ and $v'_j \geq 2$ follow from Table 8.

Proof of (2). Assume $\begin{pmatrix} \mathbb{X}_{j+f} \\ \mathbb{X}_j \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{AB} \end{pmatrix}$. By $\clubsuit 1$ and item $\clubsuit 4$ of this Proposition we have $\mathbb{X}_{j+1+f} \neq \mathbf{0}$. Hence $v'_{f+j} \geq 2$ or < 0 . By Table 8 and (1) we have $(s_{\text{or},j+1}, \Sigma_{j+1}) \in \{(w_0, \text{id}), (\text{id}, w_0)\}$.

Then

$$(s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}) \in \{(w_0, \text{id}, \text{id}, \text{id}), (\text{id}, \text{id}, \text{id}, w_0), (\text{id}, w_0, w_0, w_0), (w_0, w_0, w_0, \text{id})\}$$

and $\tilde{w}_j = w_0 t_\eta$ and the type of j is II .

Proof of (3). If $\begin{pmatrix} \mathbb{X}_{j+f} \\ \mathbb{X}_j \end{pmatrix} \in \left\{ \begin{pmatrix} \mathbf{A} \\ \mathbf{AB} \end{pmatrix}, \begin{pmatrix} \mathbf{B} \\ \mathbf{AB} \end{pmatrix} \right\}$, then $\begin{pmatrix} v'_{j+f} \\ v'_j \end{pmatrix} = \left\{ \begin{pmatrix} \mathbf{0} \\ \mathbf{0,1} \end{pmatrix}, \begin{pmatrix} \mathbf{1} \\ \mathbf{0,1} \end{pmatrix} \right\}$, $\mathbb{X}_{j+1} = \mathbf{0}$ and

$$v'_{j+1} = \begin{cases} \geq 1 & \text{if } v'_j = 0, & \text{hence } (s_{\text{or},j+1}, \Sigma_{j+1}) = (\text{id}, w_0), \\ \leq 0 & \text{if } v'_j = 1, & \text{hence } (s_{\text{or},j+1}, \Sigma_{j+1}) = (w_0, \text{id}). \end{cases}$$

We conclude from Table 8.

Proof of (4). Assume $\mathbb{X}_j, \mathbb{X}_{j+f} \in \{\mathbf{A}, \mathbf{B}\}$.

- Assume that there exists j_1 such that

$$\begin{cases} \mathbb{X}_{j'}, \mathbb{X}_{j'+f} \in \{\mathbf{A}, \mathbf{B}\}, j \leq j' \leq j_1 - 1, \\ \mathbf{AB} \in \{\mathbb{X}_{j_1}, \mathbb{X}_{j_1+f}\}. \end{cases}$$

By (3) we see that s_{or,j_1} is determined by $\begin{pmatrix} \mathbb{X}_{j_1+f} \\ \mathbb{X}_{j_1} \end{pmatrix}$. Since $\mathbb{X}_{j'}, \mathbb{X}_{j'+f} \in \{\mathbf{A}, \mathbf{B}\}$, for $j \leq j' \leq j_1 - 1$ we deduce by symmetry and decreasing induction from Table 8 that $s_{j'+1}, s_{\text{or},j'}$ and the type at j' are determined by $\begin{pmatrix} \mathbb{X}_{j'+f} \\ \mathbb{X}_{j'} \end{pmatrix}$ for $j \leq j' \leq j_1 - 1$.

- Assume that such j_1 does not exist, i.e. that $\mathbb{X}_{j'} \in \{\mathbf{A}, \mathbf{B}\}$ for all $j' \in \mathcal{J}'$. Then for all $j' \in \mathcal{J}'$ either $s_{j'+1}$ or $s_{\text{or},j'}$ is determined by $\begin{pmatrix} \mathbb{X}_{j'+f} \\ \mathbb{X}_{j'} \end{pmatrix}$ (cf. Table 8). If there exists $i \in \mathcal{J}$ such that $\begin{pmatrix} \mathbb{X}_{i+f} \\ \mathbb{X}_i \end{pmatrix} \in \left\{ \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}, \begin{pmatrix} \mathbf{B} \\ \mathbf{A} \end{pmatrix} \right\}$, there is an unique choice of $s_{\text{or},i}$ such that $s_{\text{or},f-1} = \text{id}$. If $\begin{pmatrix} \mathbb{X}_{i+f} \\ \mathbb{X}_i \end{pmatrix} \in \left\{ \begin{pmatrix} \mathbf{A} \\ \mathbf{A} \end{pmatrix}, \begin{pmatrix} \mathbf{B} \\ \mathbf{B} \end{pmatrix} \right\}$ for all $i \in \mathbb{Z}/f\mathbb{Z}$ then the \mathcal{J} -tuple $(s_{\text{or},i})$ is determined by \mathbb{X} . By $(\clubsuit 3)$ there exists $j' \in \mathcal{J}'$ with $\mathbb{X}_{j'} \neq \mathbb{X}_{j'+1}$. By Tables 8 and symmetry we obtain $s_{j'+1} = w_0$. By induction we conclude that $(s_{j+1}, s_{\text{or},j})_{j \in \mathcal{J}}$ and the type of j are determined by \mathbb{X} .

Moreover, for $\begin{pmatrix} \mathbb{X}_{j+f} \\ \mathbb{X}_j \end{pmatrix} \in \left\{ \begin{pmatrix} \mathbf{A} \\ \mathbf{A} \end{pmatrix}, \begin{pmatrix} \mathbf{B} \\ \mathbf{B} \end{pmatrix} \right\}$, we deduce from Table 8 (and symmetry) that the data of $(s_{j+1}, s_{\text{or},j})$ determines whether $\tilde{w}_j = t_{w_0(\eta)}$ or $\tilde{w}_j \in \{w_0 t_\eta, t_\eta\}$ \square

5.4. Naive equations associated to a gene. Let $\mathbb{X} \in \{\mathbf{A}, \mathbf{B}, \mathbf{AB}, \mathbf{0}\}^{\mathcal{J}'}$ be a gene satisfying conditions $(\clubsuit 1)$ – $(\clubsuit 4)$. Assume that $\mathbb{X} = \mathbb{X}(v')$ for some $(v'_{f+j}, v'_j)_{j \in \mathcal{J}}$ associated to a triple (\tilde{w}, s, μ) .

Definition 5.4.1. Let $\mathbb{X} = (\mathbb{X}_j)_{j \in \mathcal{J}'}$ be a gene satisfying $(\clubsuit 1)$ – $(\clubsuit 4)$. A *cluster* for \mathbb{X} is a sequence $(\mathbb{X}_{j+f}, \mathbb{X}_j)_{j_0 \leq j \leq j_1}$ such that there exists $\ell \in \{j_0, \dots, j_1\}$ satisfying

- $\mathbb{X}_j = \mathbf{0}$ for all $j \in \{\ell, \dots, j_1\}$, $\mathbb{X}_{j_1+1} \neq \mathbf{0}$ and $\mathbb{X}_{\ell-1} = \mathbf{AB}$,
- $\mathbf{0} \notin \{\mathbb{X}_j, \mathbb{X}_{j+f}\}$, $j_0 + 1 \leq j \leq \ell - 1$,
- $\mathbf{0} \in \{\mathbb{X}_{j_0}, \mathbb{X}_{j_0+f}\}$.

We remark that $\mathbb{X} = (\mathbb{X}_j)_{j \in \mathcal{J}'}$ has either a *unique* decomposition $\mathbb{X} = \cup_{i=0}^r (\mathbb{X}_{j_i}, \mathbb{X}_{j_i+f})_{j_i \leq j \leq j_{i+1}}$ into clusters (with the convention $j_{r+1} = j_0$), or does not have *any* cluster (in which case $\mathbb{X}_j \in \{\mathbf{A}, \mathbf{B}\}$ for all $j \in \mathcal{J}'$). Note finally that j_i are of type *II* for all $i = 0, \dots, r$ by Table 9.

Let $\cup_{k \in \mathcal{K}} \mathcal{J}_k$ be the fragmentation associated to the triple (\tilde{w}, s, μ) (Definition 4.3.12). Assume \mathbb{X} admits a decomposition into clusters $\mathbb{X} = \cup_{i=0}^r (\mathbb{X}_{j_i}, \mathbb{X}_{j_i+f})_{j_i \leq j \leq j_{i+1}}$. Note that given a cluster $(\mathbb{X}_{j+f}, \mathbb{X}_j)_{j_i \leq j \leq j_{i+1}}$ the sequence (j_i, \dots, j_{i+1}) is a union of fragments of \mathcal{K} , as both j_i and j_{i+1} are of type *II*. For $i \in \{0, \dots, r\}$, we denote

$$R_{[j_i, j_{i+1}]} \stackrel{\text{def}}{=} \mathcal{O}[Y_{j_i}] \otimes (\otimes_{j_i+1 \leq j \leq j_{i+1}-1} R_j) \otimes \mathcal{O}[X_{j_{i+1}}] \text{ and } \mathcal{I}_{[j_i, j_{i+1}]} \stackrel{\text{def}}{=} \sum_{k \in \mathcal{K} \cap [j_i, j_{i+1}]} \mathcal{I}_k$$

where for all $k \in \mathcal{K} \cap [j_i, j_{i+1}]$, the ideal $\mathcal{I}_k \subset R_{[j_i, j_{i+1}]}$ is generated by the entries of the matrix equation (4.6.5) associated to \mathcal{J}_k .

Proposition 5.4.2. *Let $\mathbb{X} = (\mathbb{X}_j, \mathbb{X}_{j+f})_{j \in \mathcal{J}}$ be a gene satisfying $(\clubsuit 1)$ – $(\clubsuit 3)$.*

- (1) *Assume that \mathbb{X} has a decomposition into clusters $\mathbb{X} = \cup_{i=0}^r (\mathbb{X}_{j_i}, \mathbb{X}_{j_i+f})_{j_i \leq j \leq j_{i+1}}$. Let (\tilde{w}, s, μ) be a triple in the fiber above \mathbb{X} of the map (5.3.1). Then*

$$\mathcal{Z}^{\text{nv}, \tau}(\tilde{z}) = \text{Spec} \left(\otimes_{i=0}^r R_{[j_i, j_{i+1}]} / \mathcal{I}_{[j_i, j_{i+1}]} \right).$$

Moreover, the p -saturation of $\mathcal{Z}^{\text{nv}, \tau}(\tilde{z})$ depends only on \mathbb{X} .

- (2) *Assume \mathbb{X} does not admit a decomposition into fragment (i.e. $\mathbb{X} \in \{\mathbf{A}, \mathbf{B}\}^{\mathcal{J}'}$). Then $\mathcal{Z}^{\text{nv}, \tau}(\tilde{z})$ depends only on \mathbb{X} . In particular, the p -saturation of $\mathcal{Z}^{\text{nv}, \tau}(\tilde{z})$ depends only on \mathbb{X} .*

The proof of Proposition 5.4.2 relies on the analysis of each of the ideals $\mathcal{I}_{[j_i, j_{i+1}]}$. This analysis is performed, for a fixed cluster, in Lemmas 5.4.3, 5.4.9 and 5.4.13 below, which deal with the $\mathbf{0}$, the \mathbf{AB} , and the (\mathbf{A}, \mathbf{B}) -part of the cluster.

Thus, we fix once and for all a cluster $(\mathbb{X}_j, \mathbb{X}_{j+f})_{j_0 \leq j \leq j_1}$ of \mathbb{X} . We let $\{k_0, \dots, k_s\} \stackrel{\text{def}}{=} \mathcal{K} \cap [j_0, j_1]$ so that

- for all $j = 0, \dots, s-1$, $\mathcal{J}_{k_j} = (i_{k_j}, i_{k_j}-1, \dots, o_{k_j}+1, o_{k_j}) \subset (j_1, j_1-1, \dots, j_0)$ and $o_{k_j} = i_{k_{j+1}}$
- $i_{k_0} = j_1$, $o_{k_s} = j_0$.

Let $s' \in [0, s]$ such that $\ell - 1 \in (o_{k_{s'}}, \dots, i_{k_{s'}} - 1)$.

Given $j \in [0, s]$, we define $\bar{\mathcal{I}}_{k_j}$ as the image of \mathcal{I}_{k_j} in $R_{[j_0, j_1]} / \sum_{j'=0}^{j-1} \mathcal{I}_{k_{j'}}$. We still denote by X_i, Y_i, Z_i etc. the variables of $R_{[j_0, j_1]}$ in the quotient $R_{[j_0, j_1]} / \sum_{j'=0}^{j-1} \mathcal{I}_{k_{j'}}$.

Lemma 5.4.3 (the 0-part of a cluster). *For any $0 \leq i \leq s' - 1$, the ideal $\bar{\mathcal{I}}_{k_i}$ satisfies the following property: there exists $r_{k_i} \in \mathbb{N}$ such that*

- (1) $\bar{\mathcal{I}}_{k_i} = (Y_{o_{k_i}})$ if the type of both o_{k_i} and i_{k_i} is *II*,
- (2) $\bar{\mathcal{I}}_{k_i} = (f_{k_i}, g_{k_i})$ if the type of o_{k_i} is *I* and the type of i_{k_i} is *II*,
- (3) $p^{r_{k_i}} \bar{\mathcal{I}}_{k_i} = p^{r_{k_i}} (f_{k_i}, g_{k_i})(p, X_{i_{k_i}})$ if the type of both o_{k_i} and i_{k_i} is *I*,
- (4) $p^{r_{k_i}} \bar{\mathcal{I}}_{k_i} = p^{r_{k_i}} (Y_{o_{k_i}})(p, X_{i_{k_i}})$ if the type of o_{k_i} is *II* and the type of i_{k_i} is *I*,

where

$$\begin{cases} f_{k_i} \stackrel{\text{def}}{=} p - Y_{o_{k_i}} - X_{o_{k_i}+1} X_{o_{k_i}}, \\ g_{k_i} \stackrel{\text{def}}{=} Z_{o_{k_i}} - X_{o_{k_i}+1} (p - X_{o_{k_i}+1} X_{o_{k_i}}). \end{cases}$$

Proof. If $s' = 0$, there is nothing to prove. We assume now $s' \geq 1$.

By definition of ℓ , $\mathbb{X}_j = \mathbf{0}$ for all $j \in \{\ell, \dots, j_1\}$ and $\mathbb{X}_{j+f} \in \{\mathbf{A}, \mathbf{B}\}$ for all $j \in \{\ell, \dots, j_1 - 1\}$. Thus by Proposition 5.3.6(1) and an induction (using Table 8 to deal when j is of type *II*) we have for $j \in \{\ell, \dots, j_1 - 1\}$:

$$(5.4.4) \quad (\tilde{w}_j, \text{type of } j) \in \{(t_\eta, 0), (t_{w_0(\eta)}, I), (t_\eta, II), (t_{w_0(\eta)}, II)\}.$$

Recall that for any $k \in \mathcal{K}$ and fragment $(i_k, i_k + 1, \dots, o_k - 1, o_k)$, we know that i_k and o_k are of type either *I* or *II*, and $i_k - 1, \dots, o_k + 1$ are of type 0. Thus for any $i \in \{0, \dots, s' - 1\}$ we get from (5.4.4) and Table 5:

$$\prod_{l=o_{k_i}+1}^{i_{k_i}-1} T_l = \prod_{l=o_{k_i}+1}^{i_{k_i}-1} \begin{pmatrix} 1 & 0 \\ -X_l & p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \sum_{l=0}^{i_{k_i}-o_{k_i}-2} -p^l X_{o_{k_i}+1+l} & p^{i_{k_i}-o_{k_i}-1} \end{pmatrix}.$$

To prove the lemma we proceed by induction on $i \in \{0, \dots, s' - 1\}$ analyzing the matrices $M_{\text{out}, o_{k_i}}$ and $M_{\text{in}, i_{k_i}}$ on the fragments \mathcal{J}_{k_i} . We abbreviate $k \stackrel{\text{def}}{=} k_i$ in what follows. For $i = 0$, only the items (1) and (2) can happen (as i_{k_0} is of type *II*).

Proof of item (1). Assume o_k and i_k are of type *II*. Then by (5.4.4) and Table 5:

$$M_{\text{out}, o_k} = \begin{cases} (Y_{o_k}, -p) & \text{if } \tilde{w}_{o_k} = t_\eta, \\ (0, 1) & \text{if } \tilde{w}_{o_k} = t_{w_0(\eta)}, \end{cases} \quad \text{and } M_{\text{in}, i_k} = \begin{pmatrix} 1 \\ -X_{i_k} \end{pmatrix}.$$

After the change of variable $X_{o_k+1} \mapsto \sum_{l=0}^{i_k-o_k-1} p^l X_{o_k+1+l}$ in $R_{[j_0, j_1]}$ we have

$$(5.4.5) \quad \left(\prod_{l=o_k+1}^{i_k-1} T_l \right) M_{\text{in}, i_k} = \begin{pmatrix} 1 \\ -X_{o_k+1} \end{pmatrix}.$$

(In particular, note that the ‘‘inner’’ variables of $R_{[j_0, j_1]}$ in the outcome of the Lemma are not the same as those used in Table 5. This change of variable can be checked not to be

relevant in the gluing of Proposition 5.4.2, which uses equations from Table 5 involving only the “outer” variables of $R_{[j_0, j_1]}$ for each cluster.) Then

$$\mathcal{I}_k = \begin{cases} (Y_{o_k} - pX_{o_k+1}) & \text{if } \tilde{w}_{o_k} = t_\eta, \\ (X_{o_k+1}) & \text{if } \tilde{w}_{o_k} = t_{w_0(\eta)}. \end{cases}$$

Thus, up to the change of variable $Y_{o_k} \mapsto Y_{o_k} - pX_{o_k+1}$ if $\tilde{w}_{o_k} = t_\eta$ or $Y_{o_k} \mapsto X_{o_k+1}$, $X_{o_k+1} \mapsto Y_{o_k}$ if $\tilde{w}_{o_k} = t_{w_0(\eta)}$, we have $\mathcal{I}_k = (Y_{o_k})$ and item (1) holds.

Proof of item (2). Assume o_k is of type *I* and i_k is of type *II*. As before we have (5.4.5) after replacing X_{o_k+1} by $\sum_{l=0}^{i_k - o_k - 1} p^l X_{o_k+1+l}$ and hence, using again (5.4.4) and Table 5

$$(5.4.6) \quad M_{\text{out}, o_k} \left(\prod_{l=o_k+1}^{i_k-1} T_l \right) M_{\text{in}, i_k} = \begin{pmatrix} p - Y_{o_k} & X_{o_k} \\ Z_{o_k} & Y_{o_k} \end{pmatrix} \begin{pmatrix} 1 \\ -X_{o_k+1} \end{pmatrix} = \begin{pmatrix} p - Y_{o_k} - X_{o_k+1}X_{o_k} \\ Z_{o_k} - Y_{o_k}X_{o_k+1} \end{pmatrix}$$

Thus $\mathcal{I}_k = (p - Y_{o_k} - X_{o_k+1}X_{o_k}, Z_{o_k} - X_{o_k+1}(p - X_{o_k+1}X_{o_k}))$ and item (2) holds.

Proof of item (3). Assume o_k and i_k are both of type *I*. Since j_1 is of type *II* and $i > 0$, there exist $i' \in [0, i-1]$ such that $o_{k_l} = i_{k_{l+1}}$ is of type *I* for $i' \leq l \leq i-1$ and $i_{k_{i'}}$ is of type *II*. By item (2) applied to $k_{i'}$ and noting that $o_{k_{i'}} = i_{k_{i'+1}}$ we see that $M_{\text{in}, i_{k_{i'+1}}}$ equals:

$$(5.4.7) \quad \begin{pmatrix} Y_{i_{k_{i'+1}}} & -X_{i_{k_{i'+1}}} \\ -Z_{i_{k_{i'+1}}} & p - Y_{i_{k_{i'+1}}} \end{pmatrix} \equiv \begin{pmatrix} 1 \\ -X_{i_{k_{i'+1}+1}} \end{pmatrix} (p - X_{i_{k_{i'+1}+1}}X_{i_{k_{i'+1}}}, -X_{i_{k_{i'+1}}}) \quad \text{modulo } \mathcal{I}_{k_{i'}}.$$

We conclude by (5.4.4) and Table 5) that the image $\bar{\mathcal{I}}_{k_{i'+1}}$ of $\mathcal{I}_{k_{i'+1}}$ in $R_{[j_0, j_1]} / (\sum_{l=0}^{i'} \mathcal{I}_{k_l})$ is generated by the equations

$$(5.4.8) \quad \begin{pmatrix} p - Y_{o_{k_{i'+1}}} & X_{o_{k_{i'+1}}} \\ Z_{o_{k_{i'+1}}} & Y_{o_{k_{i'+1}}} \end{pmatrix} \begin{pmatrix} 1 \\ -(\sum_{l=0}^{i_{k_{i'+1}} - o_{k_{i'+1}} - 2} p^l X_{o_{k_{i'+1}+1+l}} - p^{o_{k_{i'+1}} - o_{k_{i'+1}} - 1} X_{i_{k_{i'+1}}}) \end{pmatrix} (p - X_{i_{k_{i'+1}+1}}X_{i_{k_{i'+1}}}, -X_{i_{k_{i'+1}}}) = 0.$$

Hence, after the change of variable $X_{o_{k_{i'+1}+1}} \mapsto \sum_{l=0}^{i_{k_{i'+1}} - o_{k_{i'+1}} - 2} p^l X_{o_{k_{i'+1}+1+l}} - p^{o_{k_{i'+1}} - o_{k_{i'+1}} - 1} X_{i_{k_{i'+1}}}$, the image $\bar{\mathcal{I}}_{k_{i'+1}}$ of $\mathcal{I}_{k_{i'+1}}$ in $R_{[j_0, j_1]} / (\sum_{i=0}^{i'} \mathcal{I}_{k_i})$ satisfies $\bar{\mathcal{I}}_{k_{i'+1}} = (f_{k_{i'+1}}, g_{k_{i'+1}})(p, X_{i_{k_{i'+1}}})$ where $f_{k_{i'+1}} = p - Y_{o_{k_{i'+1}}} - X_{o_{k_{i'+1}}}X_{o_{k_{i'+1}+1}}$ and $g_{k_{i'+1}} = Z_{o_{k_{i'+1}}} - X_{o_{k_{i'+1}+1}}(p - X_{o_{k_{i'+1}}}X_{o_{k_{i'+1}+1}})$. We now induct on $l \in \{0, \dots, i - i' - 1\}$ the case $l = 0$ being covered above. Indeed, assuming by induction that $p^{l-1} \bar{\mathcal{I}}_{k_{i'+l-1}} = p^{l-1} (f_{k_{i'+l-1}}, g_{k_{i'+l-1}})(p, X_{i_{k_{i'+l-1}}})$ (note that the term $(p, X_{i_{k_{i'+l-1}}})$ only appears for $l > 1$) we can repeat the same computations above (multiplying bot side of equations (5.4.7), (5.4.8) by p^{l-1}) to show that the image $\bar{\mathcal{I}}_{k_{i'+l}}$ of $\mathcal{I}_{k_{i'+l}}$ in $R_{[j_0, j_1]} / \sum_{l'=0}^{l-1} \mathcal{I}_{k_{i'+l'}}$ satisfies $p^l \bar{\mathcal{I}}_{k_{i'+l}} = p^l (f_{k_{i'+l}}, g_{k_{i'+l}})(p, X_{i_{k_{i'+l}}})$, where $f_{k_{i'+l}} = p - Y_{o_{k_{i'+l}}} - X_{o_{k_{i'+l}}}X_{o_{k_{i'+l}+1}}$ and $g_{k_{i'+l}} = Z_{o_{k_{i'+l}}} - X_{o_{k_{i'+l}+1}}(p - X_{o_{k_{i'+l}}}X_{o_{k_{i'+l}+1}})$.

Proof of item (4). Assume i_k is of type *I* and o_k is of type *II*. Then k_{i-1} satisfies the hypotheses of (2) or (3) Similar computations as in (3) show now that the image of $p^{r_k} \mathcal{I}_k$ in $R_{[j_0, j_1]} / (\sum_{l=0}^{i-1} \mathcal{I}_{k_l})$ is generated by the equations

$$p^{r_k} M_{\text{out}, o_k} \begin{pmatrix} 1 \\ -(\sum_{l=0}^{i_k - o_k - 2} p^l X_{o_k+1+l} - p^{o_{k_{i-1}} - o_k - 1} X_{i_{k_{i-1}}}) \end{pmatrix} (p - X_{i_{k+1}}X_{i_k}, -X_{i_k}) = 0$$

where

$$M_{\text{out}, o_k} = \begin{cases} (Y_{o_k}, -p) & \text{if } \tilde{w}_{o_k} = t_\eta, \\ (0, 1) & \text{if } \tilde{w}_{o_k} = t_{w_0(\eta)}, \end{cases}$$

Then (after exchanging and/or replacing the variables in the same way as we did for (1)), we have $r_k \in \mathbb{N}$ such that $p^{r_k} \bar{\mathcal{I}}_k = p^{r_k} (Y_{o_k})(p, X_{i_k})$. □

Lemma 5.4.9 (the AB-part of a cluster). *Let $k \stackrel{\text{def}}{=} k_{s'}$.*

(1) *If the type of $\ell - 1$ is I then the image $\bar{\mathcal{I}}_k$ of \mathcal{I}_k in $R_{[j_0, j_1]} / \sum_{i=0}^{s'-1} \mathcal{I}_{k_i}$ satisfies the following property: there exists $r_k \in \mathbb{N}$ such that*

$$\begin{cases} \bar{\mathcal{I}}_k = (f_k, g_k) & \text{if } i_k \text{ is of type II,} \\ p^{r_k} \bar{\mathcal{I}}_k = p^{r_k} (f_k, g_k)(p, X_{i_k}) & \text{if } i_k \text{ is of type I,} \end{cases}$$

with $f_k \stackrel{\text{def}}{=} p - Y_{o_k} - X_{o_k+1} X_{o_k}$, $g_k \stackrel{\text{def}}{=} Z_{o_k} - X_{o_k+1}(p - X_{o_k+1} X_{o_k})$. Moreover there exists $r_{k_{s'+1}} \in \mathbb{N}$ such that $p^{r_{k_{s'+1}}} \bar{\mathcal{I}}_{k_{s'+1}}$ is generated by the equations

$$p^{r_{k_{s'+1}}} M_{\text{out}, o_k} \left(\prod_{l=o_k+1}^{\ell-2} T_l \right) M_{\ell-1} = 0$$

where,

$$M_{\ell-1} = \begin{cases} \Sigma_{\ell-1} \begin{pmatrix} -X_{\ell-1} \\ p - X_{\ell-1} X_{\ell-1} \end{pmatrix} & \text{if } \mathbb{X}_{\ell-1+f} = \mathbf{A}, \\ w_0 \Sigma_{\ell-1} \begin{pmatrix} -X_{\ell-1} \\ p - X_{\ell-1} X_{\ell-1} \end{pmatrix} & \text{if } \mathbb{X}_{\ell-1+f} = \mathbf{B}. \end{cases}$$

(2) *If the type $\ell - 1$ is 0, then the image $\bar{\mathcal{I}}_k$ of \mathcal{I}_k in $R_{[j_0, j_1]} / \sum_{i=0}^{s'-1} \mathcal{I}_{k_i}$ satisfies the following property: there exists $r_k \in \mathbb{N}$ such that $p^{r_k} \bar{\mathcal{I}}_k$ is generated by the equations*

$$p^{r_k} M_{\text{out}, o_k} \left(\prod_{l=o_k+1}^{\ell-2} T_l \right) M_{\ell-1} = 0$$

where

$$M_{\ell-1} \stackrel{\text{def}}{=} \begin{cases} (w_0)^{\delta_{\mathbb{X}_{f+\ell-1}=\mathbf{B}}} \Sigma_{\ell-1} \begin{pmatrix} -X_{\ell-1} \\ p - X_{\ell-1} X_{\ell-1} \end{pmatrix} & \text{if } i_k \text{ is of type II,} \\ (w_0)^{\delta_{\mathbb{X}_{f+\ell-1}=\mathbf{B}}} \Sigma_{\ell-1} \begin{pmatrix} -X_{\ell-1} \\ p - X_{\ell-1} X_{\ell-1} \end{pmatrix} (X_{\ell+1}, p) & \text{if } i_k \text{ is of type I,} \end{cases}$$

(3) *If the type of $\ell - 1$ is II, then the image $\bar{\mathcal{I}}_k$ of \mathcal{I}_k in $R_{[j_0, j_1]} / \sum_{i=0}^{s'-1} \mathcal{I}_{k_i}$ satisfies the following property: there exists $r_k \in \mathbb{N}$ such that*

$$\begin{cases} \bar{\mathcal{I}}_k = (h_k) & \text{if } i_k \text{ is of type II,} \\ p^{r_k} \bar{\mathcal{I}}_k = p^{r_k} (h_k)(p, X_{i_k}) & \text{if } i_k \text{ is of type I,} \end{cases}$$

where $h_k = (p + X_{o_k+1} Y_{o_k})$.

Proof. If the type of $\ell - 1$ is I or 0, then $\ell - 1 > j_0$ (as j_0 is of type II) and $(\mathbb{X}_{\ell-1}, \mathbb{X}_{\ell-1+f}) \in \{(\mathbf{AB}, \mathbf{A}), (\mathbf{AB}, \mathbf{B})\}$. By Proposition 5.3.63,

$$(\tilde{w}_{\ell-1}, \text{type of } \ell - 1) \in \{(t_\eta, I), (w_0 t_\eta, I), (w_0 t_\eta, 0), (t_{w_0(\eta)}, 0)\}.$$

Proof of Item (1). If $\ell - 1$ is of type I , then $o_k = \ell - 1$. Using 5.4.3 (2)-(3) to describe $R_{[j_0, j_1]} / \sum_{l=0}^{s'-1} \mathcal{I}_{k_l}$ and performing the reasoning of the proof of *loc. cit.* (replacing the final matrix at o_k in equations (5.4.6), (5.4.8) with the final matrix associated to either (t_η, I) or $(w_0 t_\eta, I)$ according to \tilde{w}_{o_k}) there exists $r_k \in \mathbb{N}$ such that the image $\bar{\mathcal{I}}_k$ of \mathcal{I}_k in $R_{[j_0, j_1]} / \sum_{l=0}^{s'-1} \mathcal{I}_{k_l}$ satisfies

$$\begin{cases} \bar{\mathcal{I}}_k = (f_k, g_k) & \text{if } i_k \text{ is of type } II, \\ p^{r_k} \bar{\mathcal{I}}_k = p^{r_k} (f_k, g_k)(p, X_{i_k}) & \text{if } i_k \text{ is of type } I, \end{cases}$$

where

$$(5.4.10) \quad \begin{cases} f_k \stackrel{\text{def}}{=} p - Y_{o_k} - X_{o_k} X_{o_k+1}, \\ g_k \stackrel{\text{def}}{=} Z_{o_k} - X_{o_k+1} (p - X_{o_k} X_{o_k+1}) \end{cases}$$

Again, note that the above equations are obtained by replacing X_{o_k+1} by $-X_{o_k+1}$ and, more importantly in what follows, exchanging X_{o_k} and Z_{o_k} (resp. replacing Y_{o_k} by $p - Y_{o_k}$) when $\tilde{w}_{\ell-1} = t_\eta$ (resp. $\tilde{w}_{\ell-1} = w_0 t_\eta$).

Letting $r_{k_{s'+1}}$ be $r_{k_{s'}} + 1$ if i_k is of type I and be 0 if i_k is of type II , we furthermore claim that $p^{r_{k_{s'+1}}} \bar{\mathcal{I}}_{k_{s'+1}}$ is generated by the equations

$$p^{r_{k_{s'+1}}} M_{\text{out}, o_{k_{s'+1}}} \left(\prod_{l=o_{k_{s'+1}}+1}^{\ell-2} T_l \right) M_{\ell-1} = 0$$

where,

$$M_{\ell-1} \stackrel{\text{def}}{=} (w_0)^{\delta_{(X_{\ell-1} = B)}} \Sigma_{\ell-1} \begin{pmatrix} -X_{\ell-1} \\ p - X_{\ell-1} X_{\ell-1} \end{pmatrix}.$$

Indeed, keeping in mind the change of variables $X_{o_k} \leftrightarrow Z_{o_k}$ (resp. $Y_{o_k} \leftrightarrow p - Y_{o_k}$) when $\tilde{w}_{\ell-1} = t_\eta$ (resp. $\tilde{w}_{\ell-1} = w_0 t_\eta$) we obtain from row I of Table 5 (recall $i_{k_{s'+1}} = o_{k_{s'}} = \ell - 1$ is of type I) and (5.4.10)

$$\begin{aligned} p^{r_{k_{s'+1}}} M_{\text{in}, i_{k_{s'+1}}} &\equiv p^{r_{k_{s'}}+1} w_{\ell-1} \begin{pmatrix} p - X_{\ell-1} X_{\ell} & -X_{\ell} (p - X_{\ell-1} X_{\ell}) \\ -X_{\ell-1} & X_{\ell-1} X_{\ell} \end{pmatrix} w_{\ell-1}^{-1} \\ &= p^{r_{k_{s'+1}}} \begin{pmatrix} p - X_{\ell-1} X_{\ell} \\ -X_{\ell-1} \end{pmatrix} (1, -X_{\ell}) w_{\ell-1}^{-1} \end{aligned}$$

in $R_{[j_0, j_1]} / \sum_{l=0}^{s'} \mathcal{I}_{k_l}$, where $w_{\ell-1}$ is the permutation part of $\tilde{w}_{\ell-1}$. As $i_{k_{s'+1}}$ is of type I we deduce from Proposition 5.3.6(3) that $w_{\ell-1} = (w_0)^{\delta_{(X_{\ell-1} = B)}} \Sigma_{\ell-1}$ and the claim follows noting that the term $(1, -X_{\ell}) w_{\ell-1}^{-1}$ can be ignored when imposing condition (4.6.5).

Proof of item (2). A similar reasoning as in the proof of Lemma 5.4.3 shows that there exists $r_k \in \mathbb{N}$ and a change of variable for X_{ℓ} such that we have the following equality in $R_{[j_0, j_1]} / \sum_{i=0}^{s'-1} \mathcal{I}_{k_i}$:

$$(5.4.11) \quad p^{r_k} \left(\prod_{l=\ell}^{i_k-1} T_l \right)_{\text{in}, i_k} = \begin{cases} p^{r_k} \begin{pmatrix} 1 \\ -X_{\ell} \end{pmatrix} & \text{if } i_k \text{ is of type } II \\ p^{r_k} \begin{pmatrix} 1 \\ X_{\ell} \end{pmatrix} (p - X_{i_k+1} X_{i_k}, -X_{i_k}) & \text{if } i_k \text{ is of type } I \end{cases}$$

Moreover if $\ell - 1$ of type 0, then $(\tilde{w}_{\ell-1}, \text{ type of } \ell - 1) \in \{(w_0 t_\eta, 0), (t_{w_0(\eta)}, 0)\}$ and by Table 5 we have

$$(5.4.12) \quad T_{\ell-1} = w_{\ell-1} \begin{pmatrix} p & -X_{\ell-1} \\ 0 & 1 \end{pmatrix}$$

where $w_{\ell-1}$ is the permutation part of $\tilde{w}_{\ell-1}$.

As in the previous item, we now claim that there exists $r_k \in \mathbb{N}$ such that $p^{r_k} \bar{\mathcal{I}}_k$ is generated by the equations

$$p^{r_k} M_{\text{out}, o_k} \left(\prod_{l=o_k+1}^{\ell-2} T_l \right) M_{\ell-1} = 0$$

where

$$M_{\ell-1} = \begin{cases} (w_0)^{\delta_{\mathbb{X}^{f+\ell-1}=\mathbb{B}} \Sigma_{\ell-1}} \begin{pmatrix} p-X_{\ell-1} X_\ell \\ -X_{\ell-1} \end{pmatrix} & \text{if } i_k \text{ is of type II,} \\ (w_0)^{\delta_{\mathbb{X}^{f+\ell-1}=\mathbb{B}} \Sigma_{\ell-1}} \begin{pmatrix} p-X_{\ell-1} X_\ell \\ -X_{\ell-1} \end{pmatrix} (p, X_{\ell+1}) & \text{if } i_k \text{ is of type I.} \end{cases}$$

But this is clear from (5.4.11), (5.4.12) noting that from Proposition 5.3.6(3) we have $w_{\ell-1} = (w_0)^{\delta_{\mathbb{X}^{f+\ell-1}=\mathbb{B}} \Sigma_{\ell-1}}$ and that, when i_k is of type I, we can replace $(p - X_{i_k+1} X_{i_k}, -X_{i_k})$ by $(p, -X_{i_k})$ to obtain a system of equations equivalent to (4.6.5).

Proof of item (3). If $\ell - 1$ is of type II then $\ell - 1 = j_0 = o_k$ and $\begin{pmatrix} \mathbb{X}_{o_k+f} \\ \mathbb{X}_{o_k} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbb{A}\mathbb{B} \end{pmatrix}$. The argument in the proof of Lemma 5.4.3(1) and (4) shows that there exists $r_k \in \mathbb{N}$ such that $p^{r_k} \bar{\mathcal{I}}_k$ is generated by the equations (after an adequate change of variables)

$$\begin{cases} p^{r_k} M_{\text{out}, o_k} \begin{pmatrix} 1 \\ -X_{o_k+1} \end{pmatrix} = 0, r_k = 0 & \text{if } i_k \text{ is of type II,} \\ p^{r_k} M_{\text{out}, o_k} \begin{pmatrix} 1 \\ -X_{o_k+1} \end{pmatrix} (p - X_{i_k+1} X_{i_k}, -X_{i_k}) = 0 & \text{if } i_k \text{ is of type I} \end{cases}$$

where $M_{\text{out}, o_k} = (-p, Y_{o_k})$ (since $\tilde{w}_{o_k} = w_0 t_\eta$ by Proposition 5.3.6(2)). The result follows. \square

Lemma 5.4.13 (the A, B-part of a cluster). *Assume $j_0 \neq \ell - 1$.*

(1) *There exists $r_{k_{s'}} \in \mathbb{N}$ such that*

$$p^{r_{k_{s'}}} \bar{\mathcal{I}}_{k_{s'}} = \begin{cases} p^{r_{k_{s'}}} \bar{\mathcal{I}}'_{k_{s'}} & \text{if type of } i_{k_{s'}} \text{ is II,} \\ p^{r_{k_{s'}}} \bar{\mathcal{I}}'_{k_{s'}} (p, X_{\ell+1}) & \text{if type of } i_{k_{s'}} \text{ is I,} \end{cases}$$

where $\bar{\mathcal{I}}'_{k_{s'}}$ is an ideal of $R_{[j_0, j_1]} / \sum_{i=0}^{s'-1} \mathcal{I}_{k_i}$ generated by elements which do not depend (up to an automorphism induced by an explicit change of variables at $o_{k_{s'}}$) on the choice of (\tilde{w}, s, μ) in the fiber above \mathbb{X} of the map (5.3.1).

(2) *For all $s'' \in \{s' + 1, \dots, s\}$, $\bar{\mathcal{I}}_{k_{s''}}$ is an ideal of $R_{[j_0, j_1]} / \sum_{l=0}^{s''-1} \mathcal{I}_{k_l}$ generated by elements which do not depend on the choice of (\tilde{w}, s, μ) in the fiber above \mathbb{X} of the map (5.3.1).*

Proof. Since $j_0 \neq \ell - 1$, $\mathbb{X}_{\ell-1} = \mathbb{A}\mathbb{B}$, $\mathbb{X}_{\ell-1+f} \in \{\mathbb{A}, \mathbb{B}\}$ and $\mathbb{X}_j, \mathbb{X}_{j+f} \in \{\mathbb{A}, \mathbb{B}\}$ for all $j \in \{j_0 + 1, \dots, \ell - 2\}$. By Proposition 5.3.6 the triple $(s_{j+1}, s_{\text{or}, j}, \text{type of } j)$ are determined by \mathbb{X} , for all $j \in \{j_0 + 1, \dots, \ell - 2\}$.

Proof of item (1). The case $o_{k_{s'}} = \ell - 1$ follows from Lemma 5.4.9 (1).

Assume $o_{k_{s'}} < \ell - 1$. By Lemma 5.4.9 (2), there exists $r_{k_{s'}} \in \mathbb{N}$ such that $p^{r_{k_{s'}}} \bar{\mathcal{I}}_{k_{s'}}$ is generated by the equations

$$(5.4.14) \quad p^{r_{k_{s'}}} M_{\text{out}, o_{k_{s'}}} \left(\prod_{l=o_k+1}^{\ell-2} T_l \right) M_{\ell-1} = 0$$

where

$$M_{\ell-1} \stackrel{\text{def}}{=} \begin{cases} (w_0)^{\delta_{\mathbb{X}f+\ell-1}=\mathbb{B}} \Sigma_{\ell-1} \binom{-X_{\ell-1}}{p-X_{\ell-1}X_{\ell}} & \text{if } i_{k_{s'}} \text{ is of type } II, \\ (w_0)^{\delta_{\mathbb{X}f+\ell-1}=\mathbb{B}} \Sigma_{\ell-1} \binom{-X_{\ell-1}}{p-X_{\ell-1}X_{\ell}} (X_{\ell+1}, p) & \text{if } i_{k_{s'}} \text{ is of type } I. \end{cases}$$

We first prove, by decreasing induction on $j \in \{o_{k_{s'}} + 1, \dots, \ell - 1\}$, that in (5.4.14) we can replace $\left(\prod_{l=j}^{\ell-2} T_l\right) M_{\ell-1}$ by

$$\begin{cases} \Sigma_j M'_j(p, X_{\ell+1}) & \text{if } i_{k_{s'}} \text{ is of type } I, \\ \Sigma_j M'_j & \text{if } i_{k_{s'}} \text{ is of type } II, \end{cases}$$

where M'_j depends only on \mathbb{X} .

The result is true for $\ell - 1$. For the inductive step, using the relation $M'_j = \Sigma_j^{-1} T_j \Sigma_{j+1} M'_{j+1}$, it is enough to prove that up to sign the matrix $\Sigma_j^{-1} T_j \Sigma_{j+1} = \Sigma_j^{-1} T_j \Sigma_j z_{j+1}$ only depends on \mathbb{X} for $j \in (o_{k_{s'}} + 1, \dots, \ell - 2)$. This is a casewise check using Table 8 and Table 5. Indeed, as j is of type 0 for all $j \in (o_{k_{s'}} + 1, \dots, \ell - 2)$ we have from Table 8 that

$$((w_0)^{\delta_{\mathbb{X}f+j}=\mathbb{B}} \Sigma_j, \tilde{w}_j, z_{j+1}) \in \{(\text{id}, t_{w_0(\eta)}, \text{id}), (w_0, t_\eta, w_0), (w_0, t_\eta, \text{id})\}$$

(The factor $(w_0)^{\delta_{\mathbb{X}f+j}=\mathbb{B}}$ is justified by Table 9) and an elementary computation from Table 5 shows that

$$\Sigma_j^{-1} T_j \Sigma_j z_{j+1} = (\pm) (w_0)^{\delta_{\mathbb{X}f+j}=\mathbb{B}} \begin{pmatrix} p & -X_j \\ 0 & 1 \end{pmatrix} (w_0)^{\delta_{\mathbb{X}f+j}=\mathbb{B}}.$$

We conclude that $p^{r_{k_{s'}}} \bar{\mathcal{L}}_{k_{s'}}$ is generated by the equations

$$(5.4.15) \quad p^{r_{k_{s'}}} M_{\text{out}, o_{k_{s'}}} M_{o_{k_{s'}}} = 0$$

where

$$M_{o_{k_{s'}}} = \begin{cases} \Sigma_{o_{k_{s'}}+1} M'_{o_{k_{s'}}+1}(p, X_{\ell+1}) & \text{if } i_{k_{s'}} \text{ is of type } I, \\ \Sigma_{o_{k_{s'}}+1} M'_{o_{k_{s'}}+1} & \text{if } i_{k_{s'}} \text{ is of type } II, \end{cases}$$

and $M'_{o_{k_{s'}}+1}$ depends only on \mathbb{X} .

We now perform a casewise analysis according to the type of $j \stackrel{\text{def}}{=} o_{k_{s'}}$.

- ♠1 Assume j is of type I . Then by Tables 9 and 8 we have $\binom{\mathbb{X}_{j+f}}{\mathbb{X}_j} \in \left\{ \binom{\mathbb{A}}{\mathbb{A}}, \binom{\mathbb{B}}{\mathbb{B}} \right\}$. The system of equations (5.4.15) is equivalent to

$$p^{r_{k_{s'}}} \Sigma_j M_{\text{out}, j} \Sigma_{j+1} M'_{j+1}(p, X_{\ell+1})^{\delta_{i_{k_{s'}}=I}} = 0.$$

A direct check on Table 8 and Table 5 shows that $\Sigma_j M_{\text{out}, j} \Sigma_j z_{j+1}$ only depends on whether $\tilde{w}_j = t_{w_0\eta}$ or $\tilde{w}_j \in \{t_\eta, w_0 t_\eta\}$ up to the change of variables $X_j \longleftrightarrow Z_j, Y_j \longleftrightarrow p - Y_j$ (change of variables which happens exactly when $\Sigma_j = w_0$). By Proposition 5.3.6(4) we conclude that the system of equation (5.4.15) only depends on \mathbb{X} up to the change of variables $X_j \longleftrightarrow Z_j, Y_j \longleftrightarrow p - Y_j$.

- ♠2 Assume $j = o_{k_{s'}}$ is of type II . By definition of cluster and the fact that $o_{k_{s'}} \leq \ell - 1$, we conclude that j is of type II . A direct check on Table 8 (which provides the possible choices for $(\tilde{w}_j, \Sigma_j, z_j)$) and Table 5 shows that $M_{\text{out}, j} \Sigma_j z_{j+1}$ only depends on whether $\tilde{w}_j = t_{w_0\eta}$ or $\tilde{w}_j \in \{t_\eta, w_0 t_\eta\}$. We conclude from Proposition 5.3.6(4) that the system of equation (5.4.15) only depends on \mathbb{X} .

Proof of item (2). In this case we necessarily have $s' < s$. Starting from case (\spadesuit 1) above, we inductively analyze, for $s \geq s'' > s'$, the system of equations

$$M_{\text{out}, o_{k_{s''}}} \left(\prod_{l=o_{k_{s''}+1}}^{i_{k_{s''}}-1} T_l \right) \Sigma_{o_{k_{s''}-1}} M_{\text{in}, o_{k_{s''}-1}} \Sigma_{o_{k_{s''}-1}} = 0$$

where the presence of the $\Sigma_{o_{k_{s''}-1}}$ -conjugation on the matrix $M_{\text{in}, o_{k_{s''}-1}}$ (defined in Table 5) is explained by the change of variables $X_{o_{k_{s''}-1}} \longleftrightarrow Z_{o_{k_{s''}-1}}$, $Y_{o_{k_{s''}-1}} \longleftrightarrow p - Y_{o_{k_{s''}-1}}$ when $\Sigma_{o_{k_{s''}-1}} = w_0$. Since the expression $M_{\text{in}, o_{k_{s''}-1}}$ is independent of \mathbb{X} , we can now perform the same argument appearing in the proof of item (1) (where we replace $M'_{\ell-1}$ by $M_{\text{in}, o_{k_{s''}-1}}$ in the initial inductive argument there). \square

Proof of Proposition 5.4.2. . Let $(\mathbb{X}_j, \mathbb{X}_{j+f})_{j \in \mathcal{J}}$ be a gene satisfying (\clubsuit 1)–(\clubsuit 3). If there exists $i \in \mathcal{J}$ such that $(\mathbb{X}_i, \mathbb{X}_{i+f}) = (0, 0)$ then the deformation ring is zero and the result is obvious. In what follows we assume that $(\mathbb{X}_j, \mathbb{X}_{j+f})_{j \in \mathcal{J}}$ satisfies (\clubsuit 1)–(\clubsuit 4).

Assume that there exists $i \in \mathcal{J}'$ such that $\mathbb{X}_i = 0$ and let $\mathbb{X} = \cup_{i=0}^r (\mathbb{X}_{j_i}, \mathbb{X}_{j_i+f})_{j_i \leq j \leq j_{i+1}}$ be the decomposition of \mathbb{X} into clusters.

Proof if item (1). By Lemmas 5.4.3–5.4.13 applied on each fragments of \mathbb{X} we have

$$\mathcal{Z}^{\text{nv}, \tau}(\tilde{z}) = \text{Spec} \left(\otimes_{i=0}^r R_{[j_i, j_{i+1}]} / \mathcal{I}_{[j_i, j_{i+1}]} \right).$$

and each $\mathcal{I}_{[j_i, j_{i+1}]}$ is a sum of the ideals described in

- Lemma 5.4.3, so that after p -saturation we can solve the variables Y_i, Z_i , while the variable X_i is free,
- Lemma 5.4.9 and Lemma 5.4.13 (1) so that after p -saturation these ideals produce equations which do not depend on the choice of (\tilde{w}, s, μ) in the fiber of the map (5.3.1) at \mathbb{X} .
- Lemma 5.4.13 (2), and these ideals admits generators which do not depend on the choice of (\tilde{w}, s, μ) in the fiber of the map (5.3.1) at \mathbb{X} .

We conclude that the p -saturation of $\mathcal{Z}^{\text{nv}, \tau}(\tilde{z})$ depends only on \mathbb{X} .

Proof of item (2). The proof is similar to the proof of Proposition 5.3.6(4) If $\mathbb{X}_{j'} \in \{\mathbb{A}, \mathbb{B}\}$ for all $j' \in \mathcal{J}'$ then either s_{j+1} or $s_{\text{or}, i}$ is determined by $\mathbb{X}_{j'}$, for all $j' \in \mathcal{J}$. If there exists $j_0 \in \mathcal{J}$ such that $\begin{pmatrix} \mathbb{X}_{j_0} \\ \mathbb{X}_{j_0+f} \end{pmatrix} \in \left\{ \begin{pmatrix} \mathbb{A} \\ \mathbb{B} \end{pmatrix}, \begin{pmatrix} \mathbb{B} \\ \mathbb{A} \end{pmatrix} \right\}$ then s_{j_0+1} is uniquely determined (cf. Table 8). As $s_{\text{or}, f-1} = \text{id}$ by Lemma 2.1.8 we conclude that $(s_{j+1}, s_{\text{or}, j})_{j \in \mathcal{J}}$ is uniquely determined by $(\mathbb{X}_j, \mathbb{X}_{j+f})_{j \in \mathcal{J}}$.

If $\begin{pmatrix} \mathbb{X}_j \\ \mathbb{X}_{j+f} \end{pmatrix} \in \left\{ \begin{pmatrix} \mathbb{A} \\ \mathbb{A} \end{pmatrix}, \begin{pmatrix} \mathbb{B} \\ \mathbb{B} \end{pmatrix} \right\}$ for all $j \in \mathcal{J}$ then $(s_{\text{or}, j})_{j \in \mathcal{J}}$ is determined by \mathbb{X} . By (\clubsuit 3) there exists $j' \in \mathcal{J}'$ with $\mathbb{X}_{j'} \neq \mathbb{X}_{j'+1}$. By Table 8, we obtain $s_{j'+1} = (12)$ and by induction $(s_{j+1}, s_{\text{or}, j})_{j \in \mathcal{J}}$ are determined.

As $(s_{j+1}, s_{\text{or}, j}, \text{type of } j)$ are determined for all $j \in \mathcal{J}$, $\mathcal{Z}^{\text{nv}, \tau}(\tilde{z})$ is determined by \mathbb{X} . \square

Theorem 5.4.16. *Assume that $p > 8f + 3 + \max_j \langle \mu_j, \alpha^\vee \rangle$. Let τ be a regular tame inertial type of niveau f and $\bar{\rho} : G_K \rightarrow \text{GL}_2(\mathbb{F})$ be absolutely irreducible and such that $\det(\bar{\rho}) \otimes_{\mathbb{F}} \omega = \det(\tau) \otimes_{\mathcal{O}} \mathbb{F}$. Then $R_{\bar{\rho}}^{\eta, \tau}$ depends only on $\mathbb{X}(\tau, \bar{\rho}|_{I_K})$. Moreover, there exists an integer $r \geq 0$ and a decomposition $\mathbb{X}(\tau, \bar{\rho}|_{I_K}) = \cup_{i=0}^r (\mathbb{X}_{j_i}, \mathbb{X}_{j_i+f})_{j_i \leq j \leq j_{i+1}}$ such that*

$$R_{\bar{\rho}}^{\eta, \tau} \cong \widehat{\otimes}_{i=0}^r R_i$$

where R_i is a complete local Noetherian \mathcal{O} algebra depending only on $(\mathbb{X}_{j_i}, \mathbb{X}_{j_i+f})_{j_i \leq j \leq j_{i+1}}$.

Proof. This follows from Theorem 4.6.3 and Proposition 5.4.2. \square

5.5. Examples. We collect some examples computing potentially Barsotti–Tate deformation rings using the techniques of this article.

In what follows, given $\bar{x} \in \mathbb{F}^\times$ we denote by $\text{un}_{\bar{x}}$ the unramified character of G_K sending p to \bar{x} .

5.5.1. Examples when $f=1$. By Theorem 4.6.3, we see immediately that if we are in a Type II situation of Table 4 then $R_{\bar{\rho}}^{\eta, \tau}$ is formally smooth over \mathcal{O} or $\mathcal{O}[[X, Y]]/(XY - p)$. We are thus left with the following cases

- (1) Case 1: $\tau = \tau((12), (1, 0))$ (thus $\tau = \omega_2 \oplus \omega_2^p$) and $\tilde{w} \in \{t_\eta, w_0 t_\eta\}$, with $\tilde{z} \in \{w_0 t_{(1,1)}, t_{(1,1)}\}$.
- (2) Case 2: $\tau = \tau(\text{id}, (1, 0))$ (thus $\tau = 1 \oplus \omega$) and $\tilde{w} = t_{w_0(\eta)}$, $\tilde{z} = t_{(1,1)}$.

We wish to compute the p -saturation of $\tilde{\mathcal{Z}}^{\text{nv}, \tau}(\tilde{z})$ in those cases.

We start with case 1. Since the two subcases give isomorphic spaces, we will work with $\tilde{w} = w_0 t_\eta$, so that $\tilde{z} = t_{(1,1)}$. From Table 4, $\tilde{\mathcal{Z}}^{\text{nv}, \tau}(\tilde{z})$ is presented as the quotient of $\mathcal{O}[B, C, D, \alpha, \beta, \gamma, \delta]$ subject to $D(p - D) = BC$, $\alpha\delta - \beta\gamma$ invertible, and the relation

$$\begin{pmatrix} D & -B \\ -C & (p - D) \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} D & -B \\ -C & (p - D) \end{pmatrix} = 0$$

Using $D(p - D) = BC$, a simple manipulation shows that the above matrix equation is equivalent to $DF = BF = CF = (p - D)F = 0$ where $F = \gamma B + \beta C - \alpha D - \delta(p - D)$. It follows that the p -saturation $\tilde{\mathcal{Z}}^{\text{mod}, \tau}(\tilde{z})$ is given by the equations

$$D(p - D) = BC, \gamma B + \beta C = \alpha D + \delta(p - D)$$

We check on Macaulay 2 that the ideal of 2×2 minors of the Jacobian matrix of the above relations together with the relations themselves contains p^2 , and has radical (p, B, C, D) . Thus the non-smooth locus of $\tilde{\mathcal{Z}}^{\text{mod}, \tau}(\tilde{z})/\mathcal{O}$ is $p = B = C = D = 0$, which correspond to $\bar{\rho} \otimes \bar{\varepsilon}^{-1}$ unramified. After twisting $\bar{\rho}$, we see that $R_{\bar{\rho}}^{\eta, \tau}$ is isomorphic to the completion of

$$\mathcal{O}[B, C, D, \alpha, \beta, \gamma, \delta]/(D(p - D) - BC, \gamma B + \beta C - \alpha D - \delta(p - D))$$

at either $(p, B, C, D, \alpha - 1, \delta - 1, \beta, \gamma)$, $(p, B, C, D, \alpha - s, \delta - t, \beta, \gamma)$ (with $s \neq t \in \mathbb{F}^\times$) or $(p, B, C, D, \alpha - 1, \delta - 1, \beta - 1, \gamma)$.

We note that the second ideal correspond to the point $\bar{\rho} = \text{un}_s \bar{\varepsilon} \oplus \text{un}_t \bar{\varepsilon}$ with $s \neq t$, and after eliminating D using the fact that $\alpha - \delta$ is unit, and making a change of variable on B, C , we see that the completion of $\tilde{\mathcal{Z}}^{\text{mod}, \tau}(\tilde{z})$ at $\bar{\rho}$ is a power series ring over $\mathcal{O}[[X, Y]]/(XY - p^2)$.

We now move to case 2. Reading off from Table 4 like in case 1, $\tilde{\mathcal{Z}}^{\text{nv}, \tau}(\tilde{z})$ is presented as the quotient of $\mathcal{O}[B, C, D, \alpha, \beta, \gamma, \delta]$ subject to $D(p - D) = BC$, $\alpha\delta - \beta\gamma$ invertible, and the relations

$$BC\alpha + DC\beta + DB\gamma - BC\delta - pC\beta = 0,$$

$$DC\alpha - C^2\beta - BC\gamma - DC\delta + pD\gamma = 0$$

$$DB\alpha - BC\beta - B^2\gamma - DB\delta - pB\alpha - pD\beta + pB\delta + p^2\beta = 0.$$

One can check that $\tilde{\mathcal{Z}}^{\text{nv}, \tau}(\tilde{z})$ is already p -saturated, hence agrees with $\tilde{\mathcal{Z}}^{\text{mod}, \tau}(\tilde{z})$. We check on Macaulay 2 that the ideal of 2×2 minors of the Jacobian matrix of the above relations together with the relations themselves contains p^3 , and has radical (p, B, C, D) . Thus the non-smooth locus of $\tilde{\mathcal{Z}}^{\text{mod}, \tau}(\tilde{z})/\mathcal{O}$ is $p = B = C = D = 0$, which correspond to $\bar{\rho} \otimes \bar{\varepsilon}^{-1}$ unramified. We again conclude that after twisting $\bar{\rho}$, $R_{\bar{\rho}}^{\eta, \tau}$ is isomorphic to the completion of

$$\mathcal{O}[B, C, D, \alpha, \beta, \gamma, \delta] \left/ \left(\begin{array}{l} D(p - D) - BC, BC\alpha + DC\beta + DB\gamma - BC\delta - pC\beta, \\ DC\alpha - C^2\beta - BC\gamma - DC\delta + pD\gamma, \\ DB\alpha - BC\beta - B^2\gamma - DB\delta - pB\alpha - pD\beta + pB\delta + p^2\beta \end{array} \right) \right.$$

at either $(p, B, C, D, \alpha-1, \delta-1, \beta, \gamma)$, $(p, B, C, D, \alpha-s, \delta-t, \beta, \gamma)$ (with $s \neq t \in \mathbb{F}^\times$) or $(p, B, C, D, \alpha-1, \delta-1, \beta-1, \gamma)$.

Remark 5.5.1. (Bounds on p for $f = 1$) Our explicit computations in this section allows us to slightly relax the requirement on p in Theorem 3.3.8. Specifically, the improvement in the proof of Theorem 3.3.8 comes from two sources

- We can get a map out of $\mathcal{O}[[X, Y]]/(XY - p^k)$ as soon as we have a map modulo p^{k+1} , instead of using the general Elkik bound which would have required a map modulo p^{2k+1} . This shows that our above models are valid for $p \geq 7$, unless (up to twists) $\bar{\rho} \otimes \bar{\varepsilon}^{-1}$ is unramified and has scalar semisimplification and $\tau = \omega_2 \oplus \omega_2^p$, or $\bar{\rho} \otimes \bar{\varepsilon}^{-1}$ is unramified and $\tau = 1 \oplus \omega$. We remark that these computations justify the claims made in [EGH, §7.5.13].
- In the remaining cases, we get a small saving in the bound required to apply Elkik’s approximation theorem compared to the general bound of Proposition 4.4.2. This in particular shows that our models for these cases are valid for $p > 7$.

5.5.2. *Examples with $f = 3$.* Tables 11, 10 record several examples of deformation rings corresponding to the examples of the left column (resp. right column) of [CDMc, Table 4] (by completion at the ideal generated by the variables $X_0, X_1, X_2, Y_0 \dots$). In particular, contrary to the expectations of [CDMc, §5.3.2], the deformation rings extracted from the first three rows of Table 11 are *not* all of the form “ $XY + p^2$ ”, despite the fact that these examples share the same stratified Kisin variety and the type τ is non degenerate in the sense of [CDMc]. This gives a counterexample to [CDMc, Conjecture 5.1.5] when the coefficient ring \mathcal{O}_E of *loc. cit.* is absolutely unramified (but not after enlarging it).

TABLE 10. **Examples from [CDMc, §5.3] for $f = 3$**

	$(II, w_0 t_\eta), (0, t_{w_0(\eta)}), (0, t_{w_0(\eta)})$	$R = \mathcal{O}[X_0, Y_0, X_1, X_2]$ $I^{\text{nv}} = I^{\text{p-sat}} = (X_0 Y_0 + p^3)$
	$(II, w_0 t_\eta), (I, w_0 t_\eta), (0, w_0 t_\eta)$	$R = \mathcal{O}[X_0, Y_0, X_1, Y_1, Z_1, X_2]$ $I^{\text{nv}} = \left(\begin{array}{ccc} Y_1(p - Y_1) - X_1 Z_1, & p Y_1 + Y_0 Z_1, & p X_1 + Y_0(p - Y_1), \\ Y_1 X_0 + X_1(p + X_0 X_2), & Z_1 X_0 + (p - Y_1)(p + X_0 X_2) \end{array} \right)$ $I^{\text{p-sat}} = \left(\begin{array}{ccc} Y_0 Z_1 + p Y_1, & Y_1^2 + X_1 Z_1 - p Y_1, & Y_0 Y_1 - p Y_0 - p X_1, \\ X_0 X_1 X_2 + X_0 Y_1 + p X_1, & X_0 Y_1 X_2 - X_0 Z_1 - p X_0 X_2 + p Y_1 - p^2 \end{array} \right)$
	$(I, w_0 t_\eta), (0, t_{w_0(\eta)}), (I, t_\eta)$	$R = \mathcal{O}[X_0, Y_0, Z_0, X_1, X_2, Y_2, Z_2]$ $I^{\text{nv}} = I^{\text{p-sat}} = \left(\begin{array}{ccc} (p - Y_2) Y_0 + Z_2 Z_0, & (p - Y_2) X_0 + Z_2(p - Y_0) \\ X_2 Y_0 + Z_0 Y_2, & X_2 X_0 + Y_2(p - Y_0), \\ Y_0(p Y_2 + X_1 Z_2) + X_0 Z_2, & Z_0(p Y_2 + X_1 Z_2) + Z_2(p - Y_0), \\ Y_0(p X_2 + X_1(p - Y_2)) + X_0(p - Y_2), & Z_0(p X_2 + X_1(p - Y_2)) + (p - Y_0)(p - Y_2), \\ Y_2(p - Y_2) - X_2 Z_2, & Y_0(p - Y_0) - X_0 Z_0 \end{array} \right)$

TABLE 11. Examples from [CDMc, §5.3] for $f = 3$

Gene	(type of j, \tilde{w}_j) $_{j=0,1,2}$	Equations for $\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})$ and $(\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z}))^{p\text{-sat}}$
$\begin{array}{ c c c } \hline \text{O} & \text{B} & \text{A} \\ \hline \text{A} & \text{AB} & \text{O} \\ \hline \end{array}$	$(II, t_\eta), (0, w_0 t_\eta), (II, t_\eta)$	$R = \mathcal{O}[X_0, X_1, X_2, Y_0]$ $I^{\text{nv}} = I^{p\text{-sat}} = (X_2 Y_0 + p^2)$
$\begin{array}{ c c c } \hline \text{O} & \text{B} & \text{B} \\ \hline \text{A} & \text{AB} & \text{O} \\ \hline \end{array}$	$(II, t_\eta), (0, w_0 t_\eta), (II, t_{w_0 \eta})$	$R = \mathcal{O}[X_0, X_1, X_2, Y_0]$ $I^{\text{nv}} = I^{p\text{-sat}} = (X_2 Y_0 + p^2)$
$\begin{array}{ c c c } \hline \text{O} & \text{B} & \text{B} \\ \hline \text{B} & \text{A} & \text{AB} \\ \hline \end{array}$	$(II, t_\eta), (0, t_\eta), (0, w_0 t_\eta)$	$R = \mathcal{O}[X_0, X_1, X_2, Y_0]$ $I^{\text{nv}} = I^{p\text{-sat}} = (X_0 Y_0 + p^3)$
$\begin{array}{ c c c } \hline \text{A} & \text{B} & \text{B} \\ \hline \text{A} & \text{A} & \text{A} \\ \hline \end{array}$	$(I, w_0 t_\eta), (0, t_\eta), (0, w_0 t_\eta)$	$R = \mathcal{O}[X_0, X_1, X_2, Y_0, Z_0]$ $I^{\text{nv}} = \left(\begin{array}{ccc} (Y_0(Z_0 + p^2 Y_0 + (p - Y_0)X_1), & Z_0(Z_0 + p^2 Y_0 + (p - Y_0)X_1), & X_0(Z_0 - p^2 X_0 + (p - Y_0)X_1), \\ (p - Y_0)(Z_0 - p^2 X_0 + (p - Y_0)X_1), & (p - Y_0)Y_0 - X_0 Z_0 & \end{array} \right)$ $I^{p\text{-sat}} = \left(\begin{array}{c} Y_0^2 + X_0 Z_0 - p Y_0, \quad p X_0 Y_0 - X_1 Y_0 - p X_0 Z_0 - p^2 X_0 + p X_1 + p^2 Y_0 + Z_0 \\ p X_1 Y_0 + X_0 Z_0 + Y_0 Z_0 + p^3 X_0 - p^2 X_1 - p Z_0, \\ X_0^2 Y_0 + X_0 Z_0^2 + X_1 Y_0 + 2p X_0 Z_0 + p^2 X_0 - p X_1 - p^2 Y_0 - Z_0 \\ p X_0^3 - X_0^2 X_1 + p Y_0 Z_0^2 + p^2 X_0^2 - p X_0 X_1 + (p^4 - 2p^2)X_0 Z_0 + (p^4 + 1)Y_0 Z_0 + p^3 Y_0 \end{array} \right)$
$\begin{array}{ c c c } \hline \text{A} & \text{B} & \text{A} \\ \hline \text{A} & \text{A} & \text{B} \\ \hline \end{array}$	$(II, t_{w_0 \eta}), (0, t_{w_0 \eta}), (0, w_0 t_\eta)$	$R = \mathcal{O}[X_0, X_1, X_2, Y_0, Z_0]$ $I^{\text{nv}} = \left(\begin{array}{ccc} p Z_0(p - Y_0) + (X_0 - X_1(p - Y_0))(-p Y_0 - Z_0 X_2), & -p(p - Y_0)^2 + (X_0 - X_1(p - Y_0))(p X_0 + X_2(p - Y_0)), \\ p Z_0^2 + (-Z_0 X_1 + Y_0)(-p Y_0 - Z_0 X_2), & -p Z_0(p - Y_0) + (-Z_0 X_1 + Y_0)(p X_0 + X_2(p - Y_0)), \\ & (p - Y_0)Y_0 - X_0 Z_0 \end{array} \right)$ $I^{p\text{-sat}} = \left(\begin{array}{c} Y_0^2 + X_0 Z_0 - p Y_0, \quad X_1 X_2 Z_0^2 + p X_1 Y_0 Z_0 - X_2 Y_0 Z_0 + p X_0 Z_0 + p Z_0^2 - p^2 Y_0, \\ X_1 X_2 Y_0 Z_0 - p X_0 X_1 Z_0 + X_0 X_2 Z_0 - p X_1 X_2 Z_0 + p X_0 Y_0 + p Y_0 Z_0 - p^2 Z_0, \\ X_0 X_1 X_2 Z_0 + p X_0 X_1 Y_0 - X_0 X_2 Y_0 + p X_1 X_2 Y_0 + p X_0^2 - p^2 X_0 X_1 + p X_0 X_2 - p^2 X_1 X_2 + p X_0 Z_0 + p^2 Y_0 - p^3 \end{array} \right)$
$\begin{array}{ c c c } \hline \text{O} & \text{B} & \text{B} \\ \hline \text{A} & \text{B} & \text{AB} \\ \hline \end{array}$	$(II, t_\eta), (I, t_{w_0 \eta}), (0, w_0 t_\eta)$	$R = \mathcal{O}[X_0, Y_0, X_1, Y_1, Z_1, X_2]$ $I^{\text{nv}} = \left(\begin{array}{ccc} Y_0 Y_1 + p Z_1, & Y_0 X_1 + p(p - Y_1), & X_1(p + X_0 X_2) - X_0(p - Y_1), \\ Y_1(p + X_0 X_2) - X_0 Z_1, & (p - Y_1)Y_1 - X_1 Z_1 & \end{array} \right)$ $I^{p\text{-sat}} = \left(\begin{array}{ccc} Y_1^2 + X_1 Z_1 - p Y_1, & Y_0 Y_1 + p Z_1, & Y_0 X_1 - p Y_1 + p^2 \\ X_0 Y_0 + p X_0 X_2 + p^2, & X_0 Y_1 X_2 - X_0 Z_1 + p Y_1, & X_0 X_1 X_2 + X_0 Y_1 - p X_0 + p X_1 \end{array} \right)$
$\begin{array}{ c c c } \hline \text{O} & \text{A} & \text{B} \\ \hline \text{AB} & \text{O} & \text{AB} \\ \hline \end{array}$	$(II, w_0 t_\eta), (II, t_\eta), (0, t_{w_0 \eta})$	$R = \mathcal{O}[X_0, Y_0, X_1, Y_1, X_2]$ $I^{\text{nv}} = I^{p\text{-sat}} = (p + Y_0 X_1, X_0 X_2 Y_1 + p(X_0 + Y_1))$
$\begin{array}{ c c c } \hline \text{O} & \text{A} & \text{B} \\ \hline \text{A} & \text{B} & \text{AB} \\ \hline \end{array}$	$(II, t_\eta), (0, t_{w_0 \eta}), (0, w_0 t_\eta)$	$R = \mathcal{O}[X_0, Y_0, X_1, X_2]$ $I^{\text{nv}} = I^{p\text{-sat}} = (p X_0 Y_0 + (p + X_1 Y_0)(p + X_0 X_2))$

In the third column we express $\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})$ as $\text{Spec}(R/I^{\text{nv}})$, where $R = \bigotimes_{j=0}^2 R_j$ and $I^{\text{nv}}, (R_j)_{j=0,1,2}$ are extracted from the data of the second column and Table 5

REFERENCES

- [CDMa] Xavier Caruso, Agnès David, and Ariane Mézard, *Can we dream of a 1-adic langlands correspondance?*, Lecture Notes in Mathematics Festschrift in honour of Catriona Byrne, to appear.
- [CDMb] ———, *Un calcul d’anneaux de déformations potentiellement Barsotti-Tate*, Trans. AMS, 370 (9), 2018, 6041-6096.
- [CDMc] ———, *Variétés de Kisin stratifiées et déformations potentiellement Barsotti-Tate*, J. Inst. Math. Jussieu, 2018.
- [CDM23] Xavier Caruso, Agnès David, and Ariane Mézard, *Combinatorics of Serre weights in the potentially Barsotti-Tate setting*, Mosc. J. Comb. Number Theory **12** (2023), no. 1, 1–56 (English).
- [CEGSa] Ana Caraiani, Matthew Emerton, Toby Gee, and David Savitt, *The geometric Breuil-Mézard conjecture for two-dimensional potentially Barsotti-Tate Galois representations*, <https://arxiv.org/abs/2207.05235>, preprint (2022).
- [CEGSb] ———, *Local geometry of moduli stacks of two-dimensional Galois representations*, Proceedings of the International Colloquium on ‘Arithmetic Geometry’, TIFR Mumbai, Jan. 6–10, 2020, (to appear).
- [EGH] Matthew Emerton, Toby Gee, and Eugen Hellmann, *An introduction to the categorical p -adic Langlands program*, <https://arxiv.org/pdf/2210.01404.pdf>, preprint (2022).
- [EGS15] Matthew Emerton, Toby Gee, and David Savitt, *Lattices in the cohomology of Shimura curves*, Invent. Math. **200** (2015), no. 1, 1–96. MR 3323575
- [Elk73] Renée Elkik, *Solutions d’équations à coefficients dans un anneau hensélien*, Ann. Sci. École Norm. Sup. (4) **6** (1973), 553–603 (1974). MR 345966
- [Ham75] Eloise Hamann, *On power-invariance*, Pac. J. Math. **61** (1975), 153–159 (English).
- [HP19] Yongquan Hu and Vytautas Paškūnas, *On crystabelline deformation rings of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ (with an appendix by Jack Shotton)*, Math. Ann. **373** (2019), no. 1-2, 421–487 (English).
- [Kis09] Mark Kisin, *Moduli of finite flat group schemes, and modularity*, Ann. of Math. (2) **170** (2009), no. 3, 1085–1180. MR 2600871 (2011g:11107)
- [Kov] Sandor Kovacs, *Rational singularities*, <https://arxiv.org/pdf/1703.02269.pdf>, preprint (2017).
- [LLHLM18] Daniel Le, Bao Viet Le Hung, Brandon Levin, and Stefano Morra, *Potentially crystalline deformation rings and Serre weight conjectures: shapes and shadows*, Invent. Math. **212** (2018), no. 1, 1–107. MR 3773788
- [LLHLM23] Daniel Le, Bao V. Le Hung, Brandon Levin, and Stefano Morra, *Local models for Galois deformation rings and applications*, Invent. Math. **231** (2023), no. 3, 1277–1488 (English).
- [LLHM⁺] Daniel Le, Bao Le Hung, Stefano Morra, Chol Park, and Zicheng Qian, *Moduli of fontaine-laffaille representations and a mod- p local-global compatibility result*, Memoirs of the American Mathematical Society (to appear).
- [Sta22] The Stacks project authors, *The stacks project*, <https://stacks.math.columbia.edu>, 2022.

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