

LOCAL MODEL THEORY FOR NON-GENERIC TAME POTENTIALLY BARSOTTI–TATE DEFORMATION RINGS

BAO V. LE HUNG, ARIANE MÉZARD, AND STEFANO MORRA
with an appendix by DAT PHAM

ABSTRACT. We develop a local model theory for moduli stacks of 2-dimensional non-scalar tame potentially Barsotti–Tate Galois representations of the Galois group of an unramified extension of \mathbb{Q}_p . We derive from this explicit presentations of potentially Barsotti–Tate deformation rings, allowing us to prove structural results about them, and prove various conjectures formulated by Caruso–David–Mézard.

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1. INTRODUCTION

1.1. Main results. Let p be a prime number, K a p -adic field. Write G_K for the absolute Galois group of K and $I_K \subset G_K$ the inertia subgroup. We work with coefficient ring $\mathcal{O} = W(\mathbb{F})$ where \mathbb{F}/\mathbb{F}_p is a sufficiently large finite extension. Let $\bar{\rho} : G_K \rightarrow \mathrm{GL}_2(\mathbb{F})$ be a continuous Galois representation and $\tau = \chi \oplus \chi'$ be an inertial type where χ, χ' are *tame* characters of I_K . This gives rise to the universal framed deformation ring $R_{\bar{\rho}}^{\eta, \tau}$ classifying lifts of $\bar{\rho}$ which are potentially Barsotti–Tate (i.e. potentially crystalline with Hodge–Tate weights $0, 1$) and of type τ . Despite their prominent role in modularity questions via the Taylor–Wiles method (e.g. [Kis09]) in the last decades, their internal structure is still poorly understood. The basic reason for this, as suggested by works of Caruso–David–Mézard [CDM18a],[CDM18b],[CDM23b], is that even when K is unramified, $R_{\bar{\rho}}^{\eta, \tau}$ exhibits a wide range of complicated behavior, in particular it can be highly singular. This happens especially as the inertial weights of χ/χ' become more degenerate (that is, when τ becomes more *non-generic*).

More recently, Caraiani–Emerton–Gee–Savitt [CEGSb] constructed a p -adic formal algebraic stack \mathcal{Z}^τ which interpolates the deformation rings $R_{\bar{\rho}}^{\eta, \tau}$ as $\bar{\rho}$ varies, in the sense that the latter recovers versal rings to finite type points of the former. The stacks \mathcal{Z}^τ (and its analogues for other p -adic Hodge theory conditions) are expected to be key geometric objects in the categorical p -adic

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Langlands conjectures formulated by Emerton–Gee–Hellmann [EGH], similar to the role played by various moduli spaces of local systems in the geometric Langlands program. Thus it is of interest to understand their geometry.

From now on, we assume $K = \mathbb{Q}_{p^f}$ is absolutely unramified and τ is non-scalar (the scalar case $\tau = \chi \oplus \chi$ being easily handled by Fontaine–Laffaille theory). When τ is sufficiently generic, the structure of \mathcal{Z}^τ is well-understood (see e.g. [Kis09, LLHLM23]), since it can be modeled using Iwahori level local models of Shimura varieties for GL_2 , in particular its singularities are products of the singularity $XY = p$. In this paper, we introduce a method to probe the structure of \mathcal{Z}^τ which is powerful enough to handle non-generic τ . One concrete consequence of our study is the following general control on singularities:

Theorem 1.1.1 (Theorem 3.3.11, Corollary 3.3.15, Theorem 4.6.6 and Theorem A.9). *Assume either $p \geq 7$ or $K = \mathbb{Q}_5$. Then the normalization of \mathcal{Z}^τ has rational singularities, and is Cohen–Macaulay.*

Furthermore, it turns out that \mathcal{Z}^τ is almost always normal:

Theorem 1.1.2 (Theorem 4.6.10). *Assume either $p \geq 7$ or $K = \mathbb{Q}_5$. Then \mathcal{Z}^τ is normal, unless after twisting, $\tau = \chi \oplus 1$ is the sum of restrictions to I_K of characters of G_K , and the inertial weights of τ belong to $\{0, 1\}$. When τ is of this form, the non-normal locus consists of exactly the $\bar{\rho}$ which are Fontaine–Laffaille with specific (irregular) inertial weights determined by τ .*

Remark 1.1.3. (1) Explicitly, for $K = \mathbb{Q}_p$ Theorem 1.1.2 shows that \mathcal{Z}^τ is normal unless, up to twists, $\bar{\tau} = \bar{\varepsilon} \oplus 1$ (where $\bar{\tau}$ and $\bar{\varepsilon}$ denote the mod p -reduction of τ and the cyclotomic character respectively), in which case the non-normal locus consists of those $\bar{\rho}$ which are twists of an unramified representation by $\bar{\varepsilon}$ (see Example 5.5.1).

(2) It is proven in [CEGSa] that the special fiber of $R_{\bar{\rho}}^{\eta, \tau}$ is generically reduced. **Together with Theorem 1.1.1, this implies that $R_{\bar{\rho}}^{\eta, \tau}$ is Cohen–Macaulay if and only if it is normal. Thus Theorem 1.1.2 completely classifies when $R_{\bar{\rho}}^{\eta, \tau}$ is Cohen–Macaulay.** We invite the reader to compare this to the result of Hu–Paškūnas [HP19] about Cohen–Macaulayness of crystabelline deformation rings: whereas [HP19] covers the situation where $K = \mathbb{Q}_p$, τ is restricted to (possibly wildly ramified) principal series types (i.e. χ, χ' can be extended to characters of $G_{\mathbb{Q}_p}$) but allows arbitrary Hodge–Tate weights, our result allows general unramified K but restricts to tame (possibly non-principal series) types τ and Hodge–Tate weights 0, 1. We also point out that the method of [HP19] is unlikely to establish neither the **Cohen–Macaulayness** nor the rational singularity property.

Finally, as is well-known (cf. [HP19]), Cohen–Macaulayness for (normalizations) of deformation rings allows one to upgrade $R[\frac{1}{p}] = \mathbb{T}[\frac{1}{p}]$ theorems to integral $R = \mathbb{T}$ theorems, hence our results give new instances of such.

(3) The fact \mathcal{Z}^τ has rather mild singularities is expected to be useful for the categorical p -adic Langlands program for $\mathrm{GL}_2(K)$, namely it suggests the conjectural functor \mathfrak{A} of [EGH] to have simple effect on a certain generating set of smooth representations of $\mathrm{GL}_2(K)$, thus giving hope that one can construct \mathfrak{A} by “generators and relations”. We point out that any generating set must necessarily involve representations of $\mathrm{GL}_2(K)$ with non-generic parameters, thus it is essential that we allow arbitrarily non-generic τ for this purpose.

Our method to probe \mathcal{Z}^τ is to construct group-theoretic local models for it. The main local model theorem has the following form:

Theorem 1.1.4. *Let $\tau = \chi \oplus \chi'$ be a tame inertial type with $\chi \neq \chi'$. There exists a p -adic formal scheme $\tilde{\mathcal{Z}}^{\mathrm{mod}, \tau}$ such that*

- (1) (Corollary 3.3.15) *If either $p \geq 7$ or $K = \mathbb{Q}_5$, \mathcal{Z}^τ/p is smooth locally isomorphic to $\widetilde{\mathcal{Z}}^{\text{mod},\tau}/p$.*
- (2) (Theorem 3.3.11) *If $p > \max\{32f - 23, 13\}$, then \mathcal{Z}^τ is smooth locally isomorphic to $\widetilde{\mathcal{Z}}^{\text{mod},\tau}$.*

Remark 1.1.5. (Features of $\widetilde{\mathcal{Z}}^{\text{mod},\tau}$) We defer the somewhat involved definition of $\widetilde{\mathcal{Z}}^{\text{mod},\tau}$ to section 1.2 below, and instead note for now that:

- (1) Its construction involves the geometry of (mixed characteristic) loop groups.
- (2) Its geometric structure is independent of p , in the sense that it essentially arises as the p -adic completion of a natural \mathbb{Z} -scheme. In particular, this exhibits a kind of “independence-of- p ” property of tame potentially Barsotti–Tate deformation rings, as suggested in [CDM23b] and [CDM23a].
- (3) It admits (see Proposition 4.3.4) an explicit affine cover where each affine open can be presented as (the p -saturation of the ideal generated by) explicit equations constructed using the inertial weights of τ (see Table 5 for a sense of the presentations that show up).

Remark 1.1.6. (Bounds on p) Theorem 1.1.4 is obtained via deformation theory: we construct $\widetilde{\mathcal{Z}}^{\text{mod},\tau}$ which captures the structure of \mathcal{Z}^τ modulo some power of p , and then show this property persists when deforming to mixed characteristics. The requirement that p needs to be at least some linear bound on f arises for two related reasons:

- There are non-isomorphic charts of \mathcal{Z}^τ that are indistinguishable modulo linear powers of p . For instance, there are charts equisingular to $XY = p^k$ for any $k \leq f$. This requires us to start with a model that approximates \mathcal{Z}^τ modulo at least $p^{O(f)}$ to distinguish these charts.
- To show the approximation deforms, we need to overcome obstruction groups for certain lifting problems, whose p -torsion can have exponents as large as linear in f .

The explicit nature of our models yields, for the first time, an efficient algorithm to compute *any* given potentially Barsotti–Tate deformation ring with tame inertial type. The basic form of the algorithm is as follows (the details occur in section 4.3). To the pair $(\bar{\rho}, \tau)$ we assign

- an f -tuple $\tilde{w}(\bar{\rho}, \tau)$ of elements in the extended affine Weyl group \widetilde{W} of GL_2 , which measures the relative position of the ι -inertial weights of $\bar{\rho}$ and τ for each $\iota : K \hookrightarrow \overline{\mathbb{Q}_p}$; and
- an f -tuple of “degeneracy types” for each $\iota : K \hookrightarrow \overline{\mathbb{Q}_p}$ which roughly measures how degenerate the ι -th inertial weight of χ/χ' is.

These data give rise, for each ι , to a basic ring R_ι , as well as a collection of structure matrices with entries in R_ι recorded in Tables 3, 4. Both these datas are independent of f . Then $R_{\bar{\rho}}^{\eta,\tau}$ is given as a suitable completion of (the p -saturation of) the quotient of $\bigotimes_\iota R_\iota$ by the relations that certain products of the structure matrices are zero. We stress that the equations we impose will generally involve mutual interactions between R_ι ’s for arbitrary large sets of different ι . For this reason, outside of the generic case, $R_{\bar{\rho}}^{\eta,\tau}$ does not generally admit an obvious tensor product decomposition along embeddings $\iota : K \hookrightarrow \overline{\mathbb{Q}_p}$.

In the series of work [CDM18a, CDM18b, CDM23b], Caruso–David–Mézard also investigated the problem of algorithmically computing $R_{\bar{\rho}}^{\eta,\tau}$, in the special case where $\tau = \chi \oplus \chi'$ is a principal series type (so χ, χ' extend to characters of G_K) and $\bar{\rho}$ is irreducible. In a few cases when $f \leq 3$ ([CDM18a, Théorème 4.3.1] and [CDM18b, 5.3.3]), they managed to determine $R_{\bar{\rho}}^{\eta,\tau}$ based on the fact that one can guess “a priori” what it is, cf. [CDM18a, Remark 3.2.10]. However, their investigations also suggested that the answer becomes intrinsically complicated for large f , which made their strategy hopeless in general. We demonstrate the power of our algorithm by confirming various of their

conjectural examples of [CDM18b, §5.3.2, 5.3.3], as well as computing all examples for $K = \mathbb{Q}_p$, some of which are new as alluded to in [EGH, §7.5.13]. For instance, for $K = \mathbb{Q}_p$, $p \geq 7$, $R_{\bar{\rho}}^{\eta, \tau}$ is a power series ring over either \mathcal{O} , $\mathcal{O}[[X, Y]]/(XY - p)$ or $\mathcal{O}[[X, Y]]/(XY - p^2)$, except when up to twist, $\bar{\tau} = \omega_2 \oplus \omega_2^p$ and $\bar{\rho} \otimes \bar{\varepsilon}^{-1}$ is unramified and has scalar semisimplification, or $\bar{\tau} = 1 \oplus \omega_1$ and $\bar{\rho} \otimes \bar{\varepsilon}^{-1}$ is unramified (here ω_n is Serre’s niveau n character). In these exceptional cases, assuming $p \geq 17$, we also give the presentation of $R_{\bar{\rho}}^{\eta, \tau}$ which turns out somewhat complicated, cf. section 5.5.1.

Besides algorithmic aspects, our theory also unifies and conceptualizes Caruso–David–Mézar’s work. More specifically, they introduced the notion of *gene* $\mathbb{X}(\tau, \bar{\rho})$ associated to $\bar{\rho}$ and τ , which is a purely combinatorial gadget, closely related to the combinatorial data inputted in our algorithm, that keeps track of the difference between the inertial weights of $\bar{\rho}$ and τ . While the motivation for $\mathbb{X}(\tau, \bar{\rho})$ was to encode geometric features of a resolution of $R_{\bar{\rho}}^{\eta, \tau}$ arising from integral p -adic Hodge theory, Caruso–David–Mézar conjectured that, surprisingly, $\mathbb{X}(\tau, \bar{\rho})$ is in fact a complete invariant, cf. [CDM18b, Conjecture 5.1.6] and [CDM23a, Conjecture 2]. Our model gives a geometric interpretation of $\mathbb{X}(\tau, \bar{\rho})$, allowing us to confirm this:

Theorem 1.1.7 (Theorem 5.4.16). *Assume $p > \max\{32f - 23, 13\}$. Let $\bar{\rho}$ be irreducible and τ be a non-scalar principal series tame inertial type. The deformation ring $R_{\bar{\rho}}^{\eta, \tau}$ depends only on $\mathbb{X}(\tau, \bar{\rho})$, in an explicit way, and furthermore, is an integral domain.*

In fact, our algorithm can be interpreted as giving the right generalization of Caruso–David–Mézar’s conjecture for general (i.e. not necessarily irreducible) $\bar{\rho}$ and general (i.e. not necessarily principal series) τ . It should be noted that in this more general setting, $R_{\bar{\rho}}^{\eta, \tau}$ is no longer always a domain, and one can read off when this is so from our tables.

Finally, we expect the explicit computations of the deformation ring to be useful for global applications, particularly for mod p multiplicity one questions and Breuil’s lattice conjecture in non-generic cases (see [EGS15, Theorem 10.1.1 and Theorem 8.2.1]).

1.2. Methods. When τ is sufficiently generic, a local model for \mathcal{Z}^τ can be extracted from [CEGSb] (see also [LLHLM23] for a perspective closer to the present work), owing to the fact that in the generic cases \mathcal{Z}^τ agrees with the moduli stack $Y^{\eta, \tau}$ of Breuil–Kisin modules of type τ and there are standard local models for the latter. In the non-generic cases, the essential difficulty is that \mathcal{Z}^τ is only a scheme theoretic image of a map $Y^{\eta, \tau} \rightarrow \Phi\text{-Mod}_K^{\text{ét}, 2}$ to the stack of rank 2 étale φ -modules, so that \mathcal{Z}^τ is obtained from $Y^{\eta, \tau}$ by contracting the fibers of this map (the *Kisin varieties*). This is the source of all complexities in the non-generic situation, and the main innovation of this work is to find a good group theoretic model for this contraction procedure.

The main idea in Theorem 1.1.4 is to use deformation theory to find good approximations of the map $Y^{\eta, \tau} \rightarrow \Phi\text{-Mod}_K^{\text{ét}, 2}$. Let LG (resp. L^+G) be the loop group (resp. positive loop group) for GL_2 with respect to $v(v+p)$, that is the functor $R \mapsto LG(R) \stackrel{\text{def}}{=} GL_2\left(R[v]^{\wedge(v(v+p))} \left[\frac{1}{v(v+p)}\right]\right)$ (resp. $L^+G(R) \stackrel{\text{def}}{=} GL_2\left(R[v]^{\wedge(v(v+p))}\right)$) where R is an \mathcal{O} -algebra, see §3.1. We also have the “Iwahori” L^+G and the first principal congruence subgroup L_1^+G which are the inverse image in L^+G of the upper triangular Borel, resp. the trivial subgroup under the mod v reduction. We set $\text{Gr}_1 \stackrel{\text{def}}{=} [L_1^+G/LG]$, which is a GL_2 -torsor over a (mixed characteristic, Beilinson–Drinfeld) affine Grassmannian.

Up to a p -adic completion which we suppress for the remainder of the introduction, we have $Y^{\eta, \tau} = [LG^\tau / \varphi(L^+G)^f]$ as a quotient by a (shifted) φ -conjugation action (here φ sends v to v^p) and LG^τ is a particular closed subset of LG^f encoding certain elementary divisor bounds and the combinatorial data of τ (cf. section 3.1 for the precise definitions). The basic idea, as in [LLHLM23],

is to “straighten” the φ -action in the above as much as possible: due to the contraction effect of φ , the φ -conjugation action is equivalent to the left translation action, provided one works modulo fixed powers of p and on a small enough subgroup. Unlike the generic case in *loc. cit.*, it is not always possible to do this in characteristic p at the $L^+\mathcal{G}$ level, but if p is large enough, it is possible to do so at the $L_1^+\mathbf{G}$ -level (Lemma 3.3.9). This shows that $Y^{\eta,\tau}$ is congruent to $[\mathrm{Gr}_1^\tau/B^f\text{-sh.cnj}]$ modulo $p^{O(p)}$ (here the B^f -action is via shifted conjugation, and Gr_1^τ is $[(L_1^+\mathbf{G})^f \backslash LG^\tau]$).

On the other hand, $\Phi\text{-Mod}_K^{\text{ét},2} = [LG^f/\varphi LG^f]$, and it is never possible to straighten the action of the larger group LG^f . However, since we are only interested in the scheme theoretic image of $Y^{\eta,\tau}$ in $\Phi\text{-Mod}_K^{\text{ét},2}$, it suffices to work instead with $[LG^{\text{bd}}/\varphi(L^+\mathbf{G})^f]$ where LG^{bd} is a suitable bounded region in LG^f containing the orbit of LG^τ (denoted by $LG^{\text{bd},(v+p)v^{\hat{\mu}}}$ in the main text), and once again the φ -action can be straightened on $(L_1^+\mathbf{G})^f$. We remark that LG^{bd} descends to Gr_1^f , inducing a subvariety $\mathrm{Gr}_1^{\text{bd}}$.

The upshot so far is that $Y^{\eta,\tau} \rightarrow \Phi\text{-Mod}_K^{\text{ét},2}$ is well-approximated by the natural map

$$[\mathrm{Gr}_1^\tau/B^f\text{-sh.cnj}] \rightarrow [\mathrm{Gr}_1^{\text{bd}}/\mathrm{GL}_2^f\text{-sh.cnj}]$$

modulo $p^{O(p)}$. It further turns out that the scheme theoretic image $\mathcal{Z}^{\text{mod},\tau}$ of this is also congruent to \mathcal{Z}^τ modulo $p^{O(p)}$, and the model $\tilde{\mathcal{Z}}^{\text{mod},\tau}$ in Theorem 1.1.4 is the pullback of $\mathcal{Z}^{\text{mod},\tau}$ to the natural $(\mathrm{GL}_2)^f$ -torsor of the target. Finally, to prove Theorem 1.1.4, we need to show that the above congruences can be lifted to characteristic 0, at least locally. This is achieved by a detailed study of the geometry of $\tilde{\mathcal{Z}}^{\text{mod},\tau}$ to bound the p^∞ -torsion of the obstruction groups for such lifting problems (whose exponent can be as large as linear in f , as alluded to in Remark 1.1.6), and imposing bounds on p is required to overcome the obstructions. A by-product of this geometric study is a control on the singularities of $\tilde{\mathcal{Z}}^{\text{mod},\tau}$, which is robust enough that it can be transferred to \mathcal{Z}^τ through a mod p congruence, thus yielding Theorems 1.1.1, 1.1.2.

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1.4. Notation. Let K be a finite extension of \mathbb{Q}_p . We fix once and for all a separable closure \overline{K} , let $G_K \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{K}/K)$ and $I_K \subset G_K$ denote the inertial subgroup. Let E be a finite extension \mathbb{Q}_p with ring of integers \mathcal{O} , uniformizer $\varpi \in \mathcal{O}$ and residue field \mathbb{F} (which we assume is large enough).

We consider the group $G \stackrel{\text{def}}{=} \mathrm{GL}_2$ (defined over \mathbb{Z}). We write B for the subgroup of upper triangular matrices, $T \subset B$ for the split torus of diagonal matrices and $Z \subset T$ for the center of G . Let \overline{N} be the subgroup of unipotent lower triangular matrices. Let $X^*(T)$ be the group of characters of T which we identify with \mathbb{Z}^2 in the standard way and $\eta \in X^*(T)$ the element corresponding to $(1, 0) \in \mathbb{Z}^2$.

We write W (resp. \widetilde{W}) for the Weyl group (resp. the extended affine Weyl group) of G , which act naturally on $X^*(T)$. Thus $W = \{1, w_0\}$ is the set of permutation on 2 elements and $\widetilde{W} = X^*(T) \rtimes W$. We use the notation $t_\nu \in \widetilde{W}$ to denote the image of $\nu \in X^*(T)$ in \widetilde{W} . Given $\tilde{z} \in \widetilde{W}$ we write z to denote the projection of \tilde{z} to W .

Let α denote the positive root of G and $\langle \cdot, \cdot \rangle$ the duality pairing on $X^*(T) \times X_*(T)$, so a weight $\lambda \in X^*(T)$ is *dominant* if $0 \leq \langle \lambda, \alpha^\vee \rangle$. We set $X^0(T)$ to be the subgroup consisting of characters $\lambda \in X^*(T)$ such that $\langle \lambda, \alpha^\vee \rangle = 0$.

Let now K be a finite unramified extension of \mathbb{Q}_p of degree f , with ring of integers \mathcal{O}_K and residue field k . Thus $\mathcal{O}_K = W(k)$, and denote by φ the arithmetic Frobenius acting on $W(k)$ (i.e. acting by rising to the p -power on the residue field). Let $G_0 \stackrel{\text{def}}{=} \text{Res}_{\mathcal{O}_K/\mathbb{Z}_p} G/\mathcal{O}_K$, $T_0 \stackrel{\text{def}}{=} \text{Res}_{\mathcal{O}_K/\mathbb{Z}_p} T/\mathcal{O}_K$, and $Z_0 \stackrel{\text{def}}{=} \text{Res}_{\mathcal{O}_K/\mathbb{Z}_p} Z/\mathcal{O}_K$. We assume that \mathcal{O} contains the image of every ring homomorphism $\mathcal{O}_K \rightarrow \overline{\mathbb{Z}}_p$ and write $\mathcal{J} \stackrel{\text{def}}{=} \text{Hom}_{\mathbb{Z}_p}(\mathcal{O}_K, \mathcal{O})$. We define $\underline{G} \stackrel{\text{def}}{=} (G_0)_{/\mathcal{O}}$ and fix an identification of \underline{G} with the split reductive group $G_{/\mathcal{O}}^{\mathcal{J}}$. We similarly define and identify \underline{T} , and \underline{Z} . The notations \underline{W} , $\widetilde{\underline{W}}$ are clear as should be the natural isomorphisms $X^*(\underline{T}) = X^*(T)^{\mathcal{J}}$. Given an element $j \in \mathcal{J}$, we use a subscript notation to denote j -components obtained from the isomorphism $\underline{G} \cong G_{/\mathcal{O}}^{\mathcal{J}}$ (so that, for instance, given an element $\tilde{w} \in \widetilde{\underline{W}}$ we write \tilde{w}_j to denote its j -th component via the induced identification $\widetilde{\underline{W}} \cong \widetilde{W}^{\mathcal{J}}$). For sake of readability, we abuse notation and still write w_0 to denote the longest element in \underline{W} , and $\eta \in X^*(\underline{T})$ for the element corresponding to $(1, 0) \in \mathbb{Z}^2$ in all embeddings.

The Frobenius automorphism φ of \mathcal{O}_K induces an automorphism π on $X^*(\underline{T}) \cong X_*(\underline{T}^\vee)$ by the formula $\pi(\lambda)_\sigma = \lambda_{\sigma \circ \varphi^{-1}}$ for all $\lambda \in X^*(\underline{T})$ and $\sigma : \mathcal{O}_K \rightarrow \mathcal{O}$. We similarly define an automorphism π of \underline{W} and $\widetilde{\underline{W}}$.

Recall that we fixed a separable closure \overline{K} of K . We choose $\pi \in \overline{K}$ such that $\pi^{p^f-1} = -p$ and let $\omega_K : G_K \rightarrow \mathcal{O}_K^\times$ be the character defined by $g(\pi) = \omega_K(g)\pi$, which is independent of the choice of π . (The context will avoid any confusion with the notation π for the automorphism on $X^*(\underline{T})$, \underline{W} and $\widetilde{\underline{W}}$.) We fix an embedding $\sigma_0 : K \hookrightarrow E$ and define $\sigma_j \stackrel{\text{def}}{=} \sigma_0 \circ \varphi^{-j}$, which identifies $\mathcal{J} = \text{Hom}(k, \mathbb{F}) = \text{Hom}_{\mathbb{Q}_p}(K, E)$ with $\mathbb{Z}/f\mathbb{Z}$. In particular, the automorphism π on $X^*(\underline{T})$ satisfies $(\pi(\lambda))_j = \lambda_{j+1}$. We write $\omega_f : G_K \rightarrow \mathcal{O}^\times$ for the character $\sigma_0 \circ \omega_K$. Given an integer $r \geq 1$ we let K'/K be the unramified extension of degree r contained in \overline{K} , set $f' \stackrel{\text{def}}{=} fr$, $e' \stackrel{\text{def}}{=} p^{f'} - 1$, k' its residue field, and identify $\mathcal{J}' \stackrel{\text{def}}{=} \text{Hom}_{\mathbb{Q}_p}(K', E)$ with $\mathbb{Z}/f'\mathbb{Z}$ via $\sigma_{j'} \stackrel{\text{def}}{=} \sigma'_0 \circ \varphi^{-j'} \mapsto j'$ where $\sigma'_0 : K' \hookrightarrow E$ is a fixed choice of an embedding extending $\sigma_0 : K \hookrightarrow E$. (In particular, restriction of embeddings corresponds to reduction modulo f in the above identifications.)

Let ε denote the p -adic cyclotomic character. We fix normalization so that the p -adic cyclotomic character ε has Hodge–Tate weight $\{1\}$ for every $\kappa : K \hookrightarrow E$.

2. TAME INERTIAL TYPES AND BREUIL–KISIN MODULES

2.1. Tame inertial types and Galois representations.

2.1.1. *Tame inertial types.* An *inertial type* for K over \mathcal{O} (resp. over \mathbb{F}) is a homomorphism $\tau : I_K \rightarrow \text{GL}_2(\mathcal{O})$ (resp. $\tau : I_K \rightarrow \text{GL}_2(\mathbb{F})$) with open kernel and which extends to the Weil group of G_K . An inertial type is *tame* if it factors through the tame quotient of I_K . Given $s \in \underline{W}$ and $\mu \in X^*(\underline{T})$, we have a tame inertial type $\tau(s, \mu) : I_K \rightarrow \text{GL}_2(\mathcal{O})$ defined as follows: let r be the order of $s_0 s_1 \dots s_{f-1} \in W$, and for $0 \leq k' \leq rf - 1$ define $\alpha_{k'} \stackrel{\text{def}}{=} (\prod_{m'=0}^{k'-1} s_{f-1-m'}^{-1})(\mu_{f-k'})$ (where

the indexes of s and μ are considered modulo f). Then

$$\tau(s, \mu) \stackrel{\text{def}}{=} \left(\sum_{i'=0}^{rf-1} \alpha_{i'} p^{i'} \right) (\omega_{fr}).$$

In particular if $\nu = (\nu_j)_{j \in \mathcal{J}} \in X^*(\underline{\mathbb{Z}}) \cong \mathbb{Z}^{\mathcal{J}}$ then

$$(2.1.1) \quad \tau(s, \mu + \nu) \cong \tau(s, \mu) \otimes_{\mathcal{O}} \omega_f^{\sum_{j \in \mathcal{J}} \nu_j p^j}.$$

Any tame inertial type is isomorphic to some $\tau(s, \mu)$.

More explicitly, if $s = (s_j)_{j \in \mathcal{J}} \in \underline{W}$ is such that $\prod_{j=0}^{f-1} s_j = \text{id}$ then $\tau(s, \mu) \cong \omega_f^\gamma \oplus \omega_f^{\gamma'}$ where

$$(2.1.2) \quad \begin{aligned} \gamma &\stackrel{\text{def}}{=} \sum_{j=0}^{f-1} p^j \left(\mu_{f-j, (\prod_{i=0}^{f-1-j} s_i^{-1})(1)} \right) \\ \gamma' &\stackrel{\text{def}}{=} \sum_{j=0}^{f-1} p^j \left(\mu_{f-j, (\prod_{i=0}^{f-1-j} s_i^{-1})(2)} \right), \end{aligned}$$

noting that $\prod_{i=0}^{f-1-j} s_i = \prod_{i=0}^{j-1} s_{f-1-i}^{-1}$.

Similarly, if $s = (s_j)_{j \in \mathcal{J}} \in \underline{W}$ is such that $\prod_{j=0}^{f-1} s_j = w_0$ then $\tau(s, \mu) \cong \omega_{2f}^h \oplus \omega_{2f}^{p^f h}$ where

$$(2.1.3) \quad h \stackrel{\text{def}}{=} \sum_{j=0}^{f-1} p^j \left(\mu_{f-j, (\prod_{i=0}^{f-1-j} s_i^{-1})(2)} \right) + p^f \left(\sum_{j=0}^{f-1} p^j \mu_{f-j, (\prod_{i=0}^{f-1-j} s_i^{-1})(1)} \right).$$

We say that (s, μ) is a *presentation* for the tame inertial type $\tau(s, \mu)$. Note that any tame inertial type will have infinitely many presentations since

$$(2.1.4) \quad \tau(s, \mu) \cong \tau(\sigma s \pi(\sigma)^{-1}, \sigma(\mu) + p\nu - \sigma s \pi(\sigma)^{-1} \pi(\nu))$$

for any $(\sigma, \nu) \in \underline{W} \times X^*(\underline{\mathbb{T}})$. We will also record a presentation (s, μ) by the element $s^{-1} t_\mu \in \widetilde{W}$ (denoted as $\widetilde{w}^*(\tau)$ in [LLHLM23]).

Lemma 2.1.5. *Assume $p \geq 3$. Let $\tau : I_K \rightarrow \text{GL}_2(\mathcal{O})$ be a tame inertial type.*

Then, there exists $n \in \mathbb{Z}$, $(k_j)_{j \in \mathcal{J}} \in \{0, \dots, \frac{p+1}{2}\}^{\mathcal{J}}$ and $s \in W^{\mathcal{J}}$ such that

$$\tau \cong \tau(s, ((k_j, 0))_{j \in \mathcal{J}}) \otimes_{\mathcal{O}} \omega_f^n$$

and moreover

- $s_j = \text{id}$ if $k_j = 0$.
- If $k_j = \frac{p+1}{2}$ for some j , then there is some j' such that $s_{j'} \neq \text{id}$.

Proof. In this proof, given two tame inertial types τ and τ' we write $\tau \sim \tau'$ if $\tau \cong \tau' \otimes_{\mathcal{O}} \omega_f^n$ for some $n \in \mathbb{Z}$. Let (s', μ') be a presentation of τ . Using (2.1.1) and applying repeatedly (2.1.4) with $\sigma = \text{id}$, we see that $\tau(s', \mu') \sim \tau(s', \mu'')$ where $\mu'' \in X^*(\underline{\mathbb{T}})$ is such that $\langle \mu''_j, \alpha^\vee \rangle \in [-\frac{p+1}{2}, \frac{p+1}{2}]$ for all $j \in \mathcal{J}$ (see also [LLHLM23, Lemma 2.3.3]). Hence, using again (2.1.1), we have $\tau(s', \mu'') \sim \tau(s', (k'_j, 0)_{j \in \mathcal{J}})$ where $|k'_j| \leq \frac{p+1}{2}$. Using (2.1.1) and applying repeatedly (2.1.4) with $\nu = 0$, we obtain $\tau(s', (k'_j, 0)_{j \in \mathcal{J}}) \sim \tau(s, (k_j, 0)_{j \in \mathcal{J}})$, where at each step σ can be chosen so that $\sigma_j s_j \sigma_{j+1}^{-1} = \text{id}$ when $k'_j = 0$. This gives the desired form for τ , except that the last item may fail.

Now suppose $k_j = \frac{p+1}{2}$ for some j but $s_j = \text{id}$ for all j . Without loss of generality assume $k_0 = \frac{p+1}{2}$. If $k_j = \frac{p+1}{2}$ for all j then by (2.1.4)

$$\tau(s, ((\frac{p+1}{2}, 0))_{j \in \mathcal{J}}) \sim \tau(s, ((-\frac{p-1}{2}, 0))_{j \in \mathcal{J}})$$

and then adjusting s as above we can arrange to get the desired form for τ with $0 \leq k_j \leq \frac{p-1}{2}$. Otherwise, there is a minimal integer $f > a \geq 0$ such that $0 \leq k_a \leq \frac{p-1}{2}$. Then (2.1.4) gives

$$\tau(s, ((\frac{p+1}{2}, 0), \dots, (\frac{p+1}{2}, 0), (k_a, 0), \dots)) \sim \tau(s, ((-\frac{p+1}{2}, 0), \dots, (-\frac{p+1}{2}, 0), (k_a + 1, 0), \dots))$$

Repeating the process we again arrive at a situation where $s_j = \text{id}$ for all j and $|k_j| \leq \frac{p-1}{2}$, and adjusting s as above we can arrange to get the desired form for τ with $0 \leq k_j \leq \frac{p-1}{2}$. \square

Definition 2.1.6. A presentation (s, μ) of a tame inertial type τ is *small* if

- $0 \leq \langle \mu_j, \alpha^\vee \rangle \leq \frac{p+1}{2}$ for all $j \in \mathcal{J}$,
- $s_j = \text{id}$ whenever $\langle \mu_j, \alpha^\vee \rangle = 0$, and
- If $\langle \mu_j, \alpha^\vee \rangle = \frac{p+1}{2}$ for some $j \in \mathcal{J}$ then $s_{j'} \neq \text{id}$ for some $j' \in \mathcal{J}$.

Given a small presentation (s, μ) we define $\hat{\mu} \stackrel{\text{def}}{=} (\langle \mu_j, \alpha^\vee \rangle, 0)_{j \in \mathcal{J}} \in X^*(\underline{T})$.

If τ is a tame inertial type over \mathcal{O} we let $\bar{\tau} \stackrel{\text{def}}{=} \tau \otimes_{\mathcal{O}} \mathbb{F}$. This construction gives a bijection between isomorphism classes of tame inertial types over \mathcal{O} and tame inertial types over \mathbb{F} so that the whole discussion above holds for the latter.

Lemma 2.1.7. *Let $\tau = \tau(s, \mu)$ be a tame inertial type with small presentation (s, μ) . For $j' \in \mathcal{J}'$ define $\mathbf{a}^{(j')} \stackrel{\text{def}}{=} \sum_{i'=0}^{rf-1} \alpha_{-j'+i'} p^{i'}$. There exists a unique element $(s'_{\text{or}, j'})_{j' \in \mathcal{J}'} \in W^{\mathcal{J}'}$ such that $(s'_{\text{or}, j'})^{-1}(\mathbf{a}^{(j)})$ is strictly dominant for all $j' \in \mathcal{J}'$. Moreover the embedding $\sigma_0 : k \hookrightarrow \mathbb{F}$ and s can be chosen so that $(s'_{\text{or}, rf-1}) = \text{id}$.*

Proof. We can assume without loss of generality that $\langle \mu_0, \alpha^\vee \rangle > 0$. As $\alpha_0 = \mu_0$ we thus have $\mathbf{a}^{(rf-1)} = p^{rf-1} \mu_0 + \sum_{i'=0}^{rf-2} \alpha_{-1+i'} p^{i'}$ which is dominant since $\langle \mu_0, \alpha^\vee \rangle > 0$ and $p-1 > \langle \alpha_{-1+i'}, \alpha^\vee \rangle$ for $i' = 0, \dots, rf-2$.

We can now conclude by induction on $j' \in \{0, \dots, rf-1\}$: using the relation

$$\mathbf{a}^{(rf-j')} = \frac{\mathbf{a}^{(rf-j'+1)} - \alpha_{-(rf-j'+1)}}{p} + p^{rf-1} \alpha_{j'-1}$$

we see that either $\alpha_{j'-1} \notin X^0(T)$ and hence $\mathbf{a}^{(rf-j')}$ is strictly dominant if and only if $\alpha_{j'-1}$ is strictly dominant (in which case $s'_{\text{or}, rf-j'}$ is uniquely determined), or $\alpha_{j'} \in X^0(T)$ and hence $\mathbf{a}^{(rf-j')}$ is strictly dominant if and only if $\mathbf{a}^{(rf-j'+1)}$ is strictly dominant (in which case $s'_{\text{or}, rf-j'} = s'_{\text{or}, rf-j'+1}$ and $s'_{\text{or}, rf-j'+1}$ is uniquely determined the inductive hypothesis). \square

2.1.2. Galois deformation rings. We let $\bar{\rho} : G_K \rightarrow \text{GL}_2(\mathbb{F})$ be a continuous Galois representation. Let $\mathcal{C}_{\mathcal{O}}$ be the category of Noetherian complete local \mathcal{O} -algebras with residue field \mathbb{F} and local \mathcal{O} -algebra homomorphisms. The functor that assigns to $(A, \mathfrak{m}_A) \in \mathcal{C}_{\mathcal{O}}$ the set of lifts $\rho_A : G_K \rightarrow \text{GL}_2(A)$ of $\bar{\rho}$ is representable by $R_{\bar{\rho}}^{\square}$, the (unrestricted) lifting ring of $\bar{\rho}$.

Given a tame inertial type τ over \mathcal{O} we let $R_{\bar{\rho}}^{\eta, \tau}$ be the reduced \mathcal{O} -flat quotient of $R_{\bar{\rho}}^{\square}$ such that the $\overline{\mathbb{Q}}_p$ -points of $\text{Spec } R_{\bar{\rho}}^{\eta, \tau}[1/p]$ correspond to the subset of $\rho : G_K \rightarrow \text{GL}_2(\overline{\mathbb{Q}}_p)$ (inside

$\text{Spec}(R_{\bar{p}}[1/p])$ which are potentially Barsotti–Tate and such that the *covariant* Weil–Deligne inertial type is isomorphic to $\tau \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p$. The rings $R_{\bar{p}}^{\eta, \tau}$ are known in “generic” cases (cf. [EGS15, LLHLM23]):

Theorem 2.1.8. *Assume that τ has a presentation (s, μ) where $2 \leq \langle \mu_j, \alpha^\vee \rangle \leq p-2$. Then either $R_{\bar{p}}^{\eta, \tau} = 0$ or*

$$R_{\bar{p}}^{\eta, \tau} \cong \mathcal{O}[[Z_1, \dots, Z_{f+4-n}, X_1, Y_1, \dots, X_n, Y_n]] / (X_i Y_i - p, i = 1, \dots, n)$$

for some $n \in \{0, \dots, f\}$.

One of the main goals of this paper, Theorem 4.6.3, is to provide the analog of Theorem 2.1.8 in highly non-generic situations.

2.2. Breuil–Kisin modules and Emerton–Gee stack.

2.2.1. We introduce the necessary background on Breuil–Kisin modules with tame descent data.

Let $\tau = \tau(s, \mu)$ be a tame inertial type with presentation (s, μ) which we fix throughout this section. Letting $r \in \{1, 2\}$ be the order of $s_0 s_1 s_2 \cdots s_{f-1} \in W$ recall from §1.4 that K'/K denotes the unramified extension of degree r contained in \overline{K} , $f' \stackrel{\text{def}}{=} fr$, $e' \stackrel{\text{def}}{=} p^{f'} - 1$, k' its residue field and we have identifications $\mathcal{J}' \stackrel{\text{def}}{=} \text{Hom}_{\mathbb{Q}_p}(K', E) \cong \mathbb{Z}/f'\mathbb{Z}$ compatible with $\mathcal{J} \stackrel{\text{def}}{=} \text{Hom}_{\mathbb{Q}_p}(K, E) \cong \mathbb{Z}/f\mathbb{Z}$.

We let $\pi' \in \overline{K}$ be an e' -th root of $-p$, $L' \stackrel{\text{def}}{=} K'(\pi')$ and $\Delta' \stackrel{\text{def}}{=} \text{Gal}(L'/K') \subset \Delta \stackrel{\text{def}}{=} \text{Gal}(L/K)$. We have the character $\omega_{K'}(g) \stackrel{\text{def}}{=} \frac{g(\pi')}{\pi'}$ for $g \in \Delta'$ (which does not depend on the choice of π') and given an \mathcal{O} -algebra R , we set $\mathfrak{S}_{L', R} \stackrel{\text{def}}{=} (W(k') \otimes_{\mathbb{Z}_p} R)[[u']]$. The latter ring is endowed with the endomorphism $\varphi : \mathfrak{S}_{L', R} \rightarrow \mathfrak{S}_{L', R}$ acting as the Frobenius on $W(k')$ and sending u' to $(u')^p$, and is endowed moreover with an action of Δ by $g'(u') = \frac{g'(\pi')}{\pi'} u' = \omega_{K'}(g') u'$ if $g' \in \Delta'$ and, letting $\sigma^f \in \Delta$ be the lift of the p^f -Frobenius on $W(k')$ which fixes π' , then σ^f acts in natural way on $W(k')$ and trivially on u' (all the endomorphisms above act trivially on R by default). Finally, $v \stackrel{\text{def}}{=} (u')^{e'}$,

$$\mathfrak{S}_R \stackrel{\text{def}}{=} (\mathfrak{S}_{L', R})^{\Delta=1} = (W(k) \otimes_{\mathbb{Z}_p} R)[[v]]$$

and $E(v) \stackrel{\text{def}}{=} v + p = (u')^{e'} + p$.

Definition 2.2.1. A Breuil–Kisin module \mathfrak{M} of rank 2 over $\mathfrak{S}_{L', R}$ with descent data of type τ and height ≤ 1 is the datum of:

- (1) a rank 2 projective $\mathfrak{S}_{L', R}$ -module \mathfrak{M} ;
- (2) an injective $\mathfrak{S}_{L', R}$ -linear map $\phi_{\mathfrak{M}} : \varphi^*(\mathfrak{M}) \rightarrow \mathfrak{M}$ whose cokernel is annihilated by $E(v)$; and
- (3) a semilinear action of Δ on \mathfrak{M} which commutes with $\phi_{\mathfrak{M}}$, and such that, for each $j' \in \text{Hom}_{\mathbb{Q}_p}(K', E)$,

$$(\mathfrak{M} \otimes_{W(k') \otimes_{\mathbb{Z}_p} R, \sigma_{j'}} R) \pmod{u'} \cong \tau^\vee \otimes_{\mathcal{O}} R$$

as Δ' -representations.

We write $Y^{[0,1], \tau}$ to be the groupoid of such objects (cf. [LLHLM23, Definition 5.1.3]). We also define $Y^{\eta, \tau}(R) \subset Y^{[0,1], \tau}(R)$ to be the subgroupoid satisfying the additional *determinant condition*

$$\det(\phi_{\mathfrak{M}}) \in (R[[v]])^\times (v + p).$$

We consider $\mathfrak{M}^{(j')} \stackrel{\text{def}}{=} \mathfrak{M} \otimes_{W(k') \otimes_{\mathbb{Z}_p} R, \sigma_{j'}} R$ as a $R[[u']]$ -submodule of \mathfrak{M} in the standard way, so it is endowed with a semilinear action of Δ' . The Frobenius $\phi_{\mathfrak{M}}$ induces Δ' -equivariant morphisms $\phi_{\mathfrak{M}}^{(j')} : \varphi^*(\mathfrak{M}^{(j'-1)}) = (\varphi^*(\mathfrak{M}))^{(j')} \rightarrow \mathfrak{M}^{(j')}$ (here the pull back on the first object is with respect to

the R -algebra map $\varphi : R[[u']] \rightarrow R[[u']]$ such that $u' \mapsto u'^p$). We remark that, by letting τ' denote the tame inertial type for K' obtained from τ via the identification $I_{K'} = I_K$ induced by the inclusion $K' \subseteq \overline{K}$, the semilinear action of Δ induces an isomorphism $\iota_{\mathfrak{M}} : (\sigma^f)^*(\mathfrak{M}) \cong \mathfrak{M}$ (see [LLHLM18, §6.1]) as elements of $Y^{[0,1],\tau'}(R)$.

Let $\mathfrak{M} \in Y^{[0,1],\tau}(R)$. Recall that an *eigenbasis* of \mathfrak{M} is a collection of bases $\beta^{(j')} = (f_1^{(j')}, f_2^{(j')})$ for each $\mathfrak{M}^{(j')}$ such that Δ' acts on $f_i^{(j')}$ via the character $\omega_{f_i}^{-\mathbf{a}'_i(0)}$ and such that $\iota_{\mathfrak{M}}((\sigma^f)^*(\beta^{(j')})) = \beta^{(j'+f)}$ for all $j' \in \text{Hom}_{\mathbb{Q}_p}(K', E)$. Given an eigenbasis β for \mathfrak{M} , we let $C_{\mathfrak{M},\beta}^{(j')}$ be the matrix of $\phi_{\mathfrak{M}}^{(j')} : \varphi^*(\mathfrak{M}^{(j'-1)}) \rightarrow \mathfrak{M}^{(j')}$ with respect to the bases $\varphi^*(\beta^{(j'-1)})$ and $\beta^{(j')}$ and set

$$A_{\mathfrak{M},\beta}^{(j')} \stackrel{\text{def}}{=} \text{Ad} \left((s'_{\text{or},j'})^{-1} (u')^{-\mathbf{a}'(j')} \right) (C_{\mathfrak{M},\beta}^{(j')})$$

for $j' \in \text{Hom}_{\mathbb{Q}_p}(K', E)$.

Lemma 2.2.2. *Let $\mathfrak{M} \in Y^{[0,1],\tau}(R)$ with eigenbasis β . The element $A_{\mathfrak{M},\beta}^{(j')}$ has coefficients in $R[[v]]$ and is upper triangular modulo v . Finally, it only depends on the restriction of j' to K .*

Proof. By the definition of eigenbases and of the action of Δ' on $\mathfrak{S}_R \otimes_{W(k'),\sigma_{j'}} R$, we see that $(C_{\mathfrak{M},\beta}^{(j')})_{\alpha} \in ((u')^{\langle \mathbf{a}'(j'), \alpha^{\vee} \rangle} R[[v]]) \cap R[[u']]$ for $\alpha \in \Phi$. Explicitly, letting $\delta_{j'} \in \{0, 1\}$ be such that $\delta_{j'} = 0$ if and only if $\mathbf{a}'(j')$ is dominant, we have for $\alpha \in \Phi^+$ that $(C_{\mathfrak{M},\beta}^{(j')})_{\alpha} \in (u')^{e'\delta_{j'} + \langle \mathbf{a}'(j'), \alpha^{\vee} \rangle} R[[v]]$ and $(C_{\mathfrak{M},\beta}^{(j')})_{-\alpha} \in (u')^{e'(1-\delta_{j'}) + \langle \mathbf{a}'(j'), -\alpha^{\vee} \rangle} R[[v]]$. Thus for $\alpha \in \Phi^+$, the α -entry of $\text{Ad} \left((u')^{-\mathbf{a}'(j')} \right) (C_{\mathfrak{M},\beta}^{(j')})$ is in $v^{\delta_{j'}} R[[v]]$ and the $-\alpha$ -entry is in $v^{1-\delta_{j'}} R[[v]]$. By the definition of $\delta_{j'}$ and the fact that $s'_{\text{or},j'} = \text{id}$ if and only if $\mathbf{a}'(j')$ is dominant, we conclude that $A_{\mathfrak{M},\beta}^{(j')}$ is upper triangular modulo v . The fact that $A_{\mathfrak{M},\beta}^{(j')}$ depends only on j' modulo f follows from [LLHLM18, Lemma 6.2, Proposition 6.9]. \square

2.2.2. Étale Φ -modules. We recall the notion of étale Φ -modules.

Recall that $\mathcal{O}_{\mathcal{E},K}$ denotes the p -adic completion of $(W(k)[[v]])[1/v]$. It is endowed with a continuous Frobenius morphism φ extending the Frobenius on $W(k)$ and such that $\varphi(v) = v^p$. Given a p -adically complete Noetherian \mathcal{O} -algebra R , we let $\Phi\text{-Mod}_K^{\text{ét},2}(R)$ be the groupoid consisting of projective modules \mathcal{M} of rank 2 over $\mathcal{O}_{\mathcal{E},K} \widehat{\otimes}_{\mathbb{Z}_p} R$ endowed with a Frobenius semilinear endomorphism $\phi_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M}$ inducing an isomorphism on the pull-back: $\text{id} \otimes_{\varphi} \phi_{\mathcal{M}} : \varphi^*(\mathcal{M}) \xrightarrow{\sim} \mathcal{M}$.

We similarly define the ring $\mathcal{O}_{\mathcal{E},L'}$, with Frobenius φ , and the groupoid $\Phi\text{-Mod}_{dd,L'}^{\text{ét},2}(R)$ of rank 2 étale $(\varphi, \mathcal{O}_{\mathcal{E},L'} \widehat{\otimes}_{\mathbb{Z}_p} R)$ -modules with descent data from L' to K .

The groupoids $\Phi\text{-Mod}_K^{\text{ét},2}$, $\Phi\text{-Mod}_{dd,L'}^{\text{ét},2}$ form fppf stacks over $\text{Spf } \mathcal{O}$ (see [CEGSb, §3.1]).

Given $\mathfrak{M} \in Y^{\eta,\tau}(R)$, $\mathfrak{M} \otimes_{\mathfrak{S}_{L',R}} (\mathcal{O}_{\mathcal{E},L'} \widehat{\otimes}_{\mathbb{Z}_p} R)$ is an object $\Phi\text{-Mod}_{dd,L'}^{\text{ét},2}(R)$, which we can descend to an étale φ -module $\mathcal{M} \in \Phi\text{-Mod}_K^{\text{ét},2}(R)$ by

$$\mathcal{M} \stackrel{\text{def}}{=} (\mathfrak{M} \otimes_{\mathfrak{S}_{L',R}} (\mathcal{O}_{\mathcal{E},L'} \widehat{\otimes}_{\mathbb{Z}_p} R))^{\Delta=1}.$$

This defines a morphism of stacks $\varepsilon_{\tau} : Y^{\eta,\tau} \rightarrow \Phi\text{-Mod}_K^{\text{ét},2}$ which is representable by algebraic spaces, proper, and of finite presentation by [CEGSb, Corollary 3.1.8(3), Proposition 3.3.5] (and the fact that taking Δ -invariants is an isomorphism of groupoids). Moreover, ε_{τ} is independent of any \widetilde{W} -presentation of τ .

Given $(\mathcal{M}, \phi_{\mathcal{M}}) \in \Phi\text{-Mod}_K^{\text{ét},2}(R)$, we decompose $\mathcal{M} = \bigoplus_{j \in \mathcal{J}} \mathcal{M}^{(j)}$ over the embeddings $\sigma_j : W(k) \rightarrow \mathcal{O}$, with induced maps $\phi_{\mathcal{M}}^{(j)} : \mathcal{M}^{(j-1)} \rightarrow \mathcal{M}^{(j)}$. The following proposition is a direct computation on the definition of the $A_{\mathfrak{M},\beta}^{(j)}$:

Proposition 2.2.3. ([LLHLM23, Proposition 5.4.2]) *Let $\mathfrak{M} \in Y^{\eta,\tau}(R)$ and β an eigenbasis of \mathfrak{M} . Let (s, μ) be a small presentation of τ .*

Then there exists a basis \mathfrak{f} for $\varepsilon_{\tau}(\mathfrak{M})$ such that the matrix of $\phi_{\varepsilon_{\tau}(\mathfrak{M})}^{(j)}$ with respect to \mathfrak{f} is given by

$$A_{\mathfrak{M},\beta}^{(j)} s_j^{-1} v^{\mu_j}.$$

Finally, when R is a complete local Noetherian \mathcal{O} -algebra with finite residue field we have an exact functor

$$\mathbb{V}_K^* : \Phi\text{-Mod}_K^{\text{ét},2}(R) \rightarrow \text{Rep}_R^2(G_{K_{\infty}})$$

establishing an anti-equivalence of categories (by the theory of fields of norms, cf. [LLHLM18, §2.3 and §6.1]) and therefore a functor $T_{dd}^* : Y^{\eta,\tau}(R) \rightarrow \text{Rep}_R^2(G_{K_{\infty}})$ defined as the composite of ε_{τ} followed by \mathbb{V}_K^* . (Note that the formula of *loc. cit.* is inaccurate and should be modified as follows: $\mathbb{V}_K^*(\mathcal{M}) = \text{Hom}_R((\mathcal{M} \otimes_{\mathcal{O}_{\varepsilon} \widehat{\otimes} R} \mathcal{O}_{\mathcal{E}^{\text{un}}} \widehat{\otimes} R)^{\varphi=1}, R)$.)

Let \mathcal{Z}^{τ} be the scheme theoretic image of ε_{τ} . This is the moduli stack of tame potentially Barsotti–Tate representations of type τ constructed and studied in [CEGSb, §5.1].

A point $\bar{\rho} \in \mathcal{Z}^{\tau}(\mathbb{F})$ gives rise to a mod p -representation of G_K . Then $R_{\bar{\rho}}^{\eta,\tau}$, the tame potentially Barsotti–Tate deformation ring of type τ , is a versal ring to \mathcal{Z}^{τ} at $\bar{\rho}$ ([CEGSb, Corollary 5.2.19]).

3. MODELS

We describe moduli of Breuil–Kisin modules in terms of loop groups (§3.2), provide models for such moduli spaces (§3.3) and state the main comparison results in Theorem 3.3.11 and Corollary 3.3.15.

3.1. Loop groups and open charts. Given a Noetherian \mathcal{O} -algebra R let $R[v]^{\wedge(v(v+p))}$ denote the $(v(v+p))$ -adic completion of $R[v]$. We denote by $R[v] \left[\frac{1}{v(v+p)} \right]_{\leq 0} \subset R[v]^{\wedge(v(v+p))} \left[\frac{1}{v(v+p)} \right]$ the subring of elements the form $\frac{P}{(v(v+p))^m}$ with $P \in R[v]$ such that $\deg P \leq 2m$.

We define:

$$\begin{aligned} LG(R) &\stackrel{\text{def}}{=} \text{GL}_2 \left(R[v]^{\wedge(v(v+p))} \left[\frac{1}{v(v+p)} \right] \right); \\ L^+G(R) &\stackrel{\text{def}}{=} \text{GL}_2 \left(R[v]^{\wedge(v(v+p))} \right); \\ L^-G(R) &\stackrel{\text{def}}{=} \text{GL}_2 \left(R \left[\frac{1}{v} \right] \right) \end{aligned}$$

(note that $R \left[\frac{1}{v} \right] = R[v] \left[\frac{1}{v(v+p)} \right]_{\leq 0}$ if p is nilpotent in R). Now we have surjections

$$\begin{aligned} \text{ev}^+ : L^+G &\rightarrow \text{GL}_2 \\ \text{ev}^- : L^-G &\rightarrow \text{GL}_2 \end{aligned}$$

obtained by evaluation modulo v and $1/v$ respectively.

We define

$$\begin{aligned} L_1^+ \mathbf{G}(R) &\stackrel{\text{def}}{=} \ker \text{ev}^+; \\ L_1^- \mathbf{G}(R) &\stackrel{\text{def}}{=} \ker \text{ev}^-; \\ L^+ \mathcal{G}(R) &\stackrel{\text{def}}{=} (\text{ev}^+)^{-1}(B); \\ L^{--} \mathcal{G}(R) &\stackrel{\text{def}}{=} (\text{ev}^-)^{-1}(\overline{N}). \end{aligned}$$

where \overline{N} denotes the subgroup of unipotent lower triangular matrices. Note that the functors $L^+ \mathcal{G}$ and $L^{--} \mathcal{G}$ have a slightly different meaning than the corresponding functors in [LLHLM23], however they coincide after p -adic completion. We define a closed subfunctor $\mathcal{A}(\eta) \subset LG$ whose R -valued points consist of $A \in LG(R)$ satisfying

- (1) $\det A \in (R[v]^{\wedge(v(v+p))})^\times (v+p)$;
- (2) $A \in \text{Mat}_2(R[v]^{\wedge(v(v+p))})$ and upper triangular mod v ;
- (3) $(v+p)A^{-1}$ is upper triangular mod v .

Note that $\mathcal{A}(\eta)(\overline{\mathbb{F}})$ identifies with

$$\bigcup_{\tilde{w} \in \text{Adm}(\eta)} L^+ \mathcal{G}(\overline{\mathbb{F}}) \tilde{w} L^+ \mathcal{G}(\overline{\mathbb{F}})$$

where $\text{Adm}(\eta) \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ v & 0 \end{pmatrix} \right\}$ is referred to as the η -admissible set.

Given a type τ with small presentation $(s, \mu) \in \underline{W} \times X^*(\underline{T})$ we define

$$(3.1.1) \quad LG^\tau(R) \stackrel{\text{def}}{=} \prod_{j \in \mathcal{J}} \mathcal{A}(\eta)(R) s_j^{-1} v^{\hat{\mu}_j} \subset LG^{\mathcal{J}}(R)$$

where $\hat{\mu} = (\hat{\mu}_j)_{j \in \mathcal{J}} \stackrel{\text{def}}{=} (\langle \mu_j, \alpha^\vee \rangle, 0)_{j \in \mathcal{J}} \in X^*(\underline{T})$. We also define

$$LG^{\text{bd},(v+p)v^{\hat{\mu}}}(R) \stackrel{\text{def}}{=} \left\{ (A_j) \in (LG(R))^{\mathcal{J}}, \text{ such that : } \begin{array}{l} \det A_j \in (R[v]^{\wedge(v(v+p))})^\times (v+p) v^{\langle \mu_j, \alpha^\vee \rangle} \\ A_j \in \text{Mat}_2(R[v]^{\wedge(v(v+p))}) \end{array} \right\}.$$

In particular we have a natural inclusion $LG^\tau \hookrightarrow LG^{\text{bd},(v+p)v^{\hat{\mu}}}$.

Lemma 3.1.2. (1) $L_1^+ \mathbf{G}$ is a normal subgroup of both of $L^+ \mathbf{G}$ and $L^+ \mathcal{G}$, and

$$\begin{aligned} L^+ \mathbf{G} &= L_1^+ \mathbf{G} \rtimes \text{GL}_2; \\ L^+ \mathcal{G} &= L_1^+ \mathbf{G} \rtimes B. \end{aligned}$$

(2) The multiplication maps $L_1^+ \mathbf{G} \times L^- \mathbf{G} \rightarrow LG$, $L^+ \mathcal{G} \times L^{--} \mathcal{G} \rightarrow LG$ are formally étale monomorphisms after p -adic completion.

(3) $\mathcal{A}(\eta)^{\wedge p}$ has an affine open cover $\mathcal{A}^{\wedge p}(\tilde{w})$ where \tilde{w} runs over $\text{Adm}(\eta)$ (cf. Table 1).

Proof. Item (1) is clear.

Item (2) for the map $L^+ \mathcal{G} \times L^{--} \mathcal{G} \rightarrow LG$ follows from [LLHLM23, Lemmas 3.2.2, 3.2.6] (using that $R[v]^{v(v+p)} = R[[v]]$ on rings where p is nilpotent), which implies the statement for $L_1^+ \mathbf{G} \times L^- \mathbf{G} \rightarrow LG$.

We prove item (3). Write by $LG^{--} \tilde{w}$ for the right translation of LG^{--} by $\tilde{w} \in \text{Adm}(\eta)$. By the previous item, $\mathcal{A}^{\wedge p}(\tilde{w}) \stackrel{\text{def}}{=} (L^+ \mathcal{G} \cdot LG^{--} \tilde{w}) \cap \mathcal{A}(\eta)^{\wedge p} = L^+ \mathcal{G} \cdot (LG^{--} \tilde{w} \cap \mathcal{A}(\eta)^{\wedge p})$ is an open subfunctor (since $L^+ \mathcal{G} \setminus \mathcal{A}(\eta)$ is finite type, cf. the proof of [LLHLM23, Corollary 3.2.10]). Consideration on $\overline{\mathbb{F}}$ -points shows that $\mathcal{A}^{\wedge p}(\tilde{w})$ for $\tilde{w} \in \text{Adm}(\eta)$ form an open cover.

Finally, if R is an \mathcal{O} -algebra where p is nilpotent, $L^- \mathcal{G}(R) \tilde{w}$ consists of matrices A of the form

$$A = \begin{pmatrix} 1 + \frac{1}{v}a & \frac{1}{v}b \\ c & 1 + \frac{1}{v}d \end{pmatrix} \tilde{w}$$

for $a, b, c, d \in R[\frac{1}{v}]$, and imposing on it the conditions (1), (2) from the definition of $\mathcal{A}(\eta)$ gives the explicit description in table 1. \square

TABLE 1. The affine cover of $L^+ \mathcal{G} \backslash \mathcal{A}(\eta)^{\wedge p}$.

\tilde{w}	t_η	$w_0 t_\eta$	$t_{w_0(\eta)}$
$L^+ \mathcal{G} \backslash \mathcal{A}^{\wedge p}(\tilde{w})$	$\begin{pmatrix} (v+p) & 0 \\ vx & 1 \end{pmatrix}$	$\begin{pmatrix} X & 1 \\ v & Y \end{pmatrix}$ $XY + p = 0$	$\begin{pmatrix} 1 & y \\ 0 & v+p \end{pmatrix}$

3.2. Loop groups and moduli of Breuil–Kisin modules. We now explain the description of $Y^{\eta, \tau}$ as a quotient stack (cf. [LLHLM23, 5.2.1]).

The p -adic completion of $\prod_{j \in \mathcal{J}} LG$ is endowed with the φ -twisted conjugation action defined by

$$(X_j)_{j \in \mathcal{J}} \cdot (A^{(j)})_{j \in \mathcal{J}} \stackrel{\text{def}}{=} (X_j A^{(j)} \varphi(X_{j-1})^{-1})_{j \in \mathcal{J}}$$

for $X_j, A^{(j)} \in LG^{\wedge p}$. We observe that (the p -adic completions of) $\prod_{j \in \mathcal{J}} L^+G$ and $\prod_{j \in \mathcal{J}} L^+ \mathcal{G}$ preserve $LG^{\text{bd}, (v+p)v^{\hat{\mu}}}$ and LG^τ under the φ -twisted conjugation action, respectively. Hence we can define the p -adic formal \mathcal{O} -quotient stacks $\left[LG^{\text{bd}, (v+p)v^{\hat{\mu}}} /_\varphi \prod_{j \in \mathcal{J}} L^+G \right]^{\wedge p}$ and $\left[LG^\tau /_\varphi \prod_{j \in \mathcal{J}} L^+ \mathcal{G} \right]^{\wedge p}$.

Lemma 3.2.1. *Let τ be a tame inertial type with small presentation $(s, \mu) \in \underline{W} \times X^*(\underline{T})$. Then any $\mathfrak{M} \in Y^{\eta, \tau}$ has an eigenbasis Zariski locally on $Y^{\eta, \tau}$ and the assignment $\mathfrak{M} \mapsto (A_{\mathfrak{M}, \beta}^{(j)})_{j \in \mathcal{J}} s_j^{-1} v^{\hat{\mu}}$ defines an isomorphism of p -adic formal algebraic stacks*

$$(3.2.2) \quad Y^{\eta, \tau} \xrightarrow{\sim} \left[LG^\tau /_\varphi \prod_{j \in \mathcal{J}} L^+ \mathcal{G} \right]^{\wedge p}$$

and hence a morphism

$$(3.2.3) \quad Y^{\eta, \tau} \rightarrow \left[LG^{\text{bd}, (v+p)v^{\hat{\mu}}} /_\varphi \prod_{j \in \mathcal{J}} L^+G \right]^{\wedge p}.$$

We have a morphism

$$\iota : \left[LG^{\text{bd}, (v+p)v^{\hat{\mu}}} /_\varphi \prod_{j \in \mathcal{J}} L^+G \right]^{\wedge p} \longrightarrow \Phi\text{-Mod}_K^{\text{ét}, 2}$$

sending the class of $A \stackrel{\text{def}}{=} (A^{(j)})_{j \in \mathcal{J}}$ to the étale φ -module $\iota(A)$ which is free of rank 2 and such that $\phi_{\iota(A)}^{(j)} : \iota(A)^{(j-1)} \rightarrow \iota(A)^{(j)}$ has matrix $A^{(j)}$ in the standard basis.

Proposition 3.2.4. *Let τ be a tame inertial type with a small presentation $(s, \mu) \in \underline{W} \times X^*(\underline{T})$. Assume $p - 2 > \max_j \langle \mu_j, \alpha^\vee \rangle$ (this is in particular satisfied if $p \geq 7$). We have a commutative diagram of p -adic formal algebraic stacks over $\mathrm{Spf}\mathcal{O}$:*

$$(3.2.5) \quad \begin{array}{ccc} Y^{\eta, \tau} & \xrightarrow{3.2.3} & \left[LG^{\mathrm{bd}, (v+p)v^{\hat{\mu}}} / \varphi \prod_{j \in \mathcal{J}} L^+G \right]^{\wedge p} \\ & \searrow \varepsilon_\tau & \nearrow \exists \\ & \mathcal{Z}^\tau & \\ & \downarrow & \nearrow \iota \\ & \Phi\text{-Mod}_K^{\acute{\mathrm{e}}\mathrm{t}, 2} & \end{array}$$

where the hooked diagonal arrow is a closed immersion. In particular, the dotted arrow exists and makes the diagram commute.

Proof. Denote by $Y^{\mathrm{bd}, \tau}$ the groupoid in the upper right vertex of the diagram (3.2.5).

The external triangle is commutative by Proposition 2.2.3 and Lemma 3.2.1. The factorization of ε_τ is by definition of \mathcal{Z}^τ .

Hence, the existence of the dotted arrow will follow once we prove that the diagonal hooked arrow is a closed immersion. Since ι is proper (as LG/L^+G is ind-proper and $Y^{\mathrm{bd}, \tau}$ is a finite type p -adic formal algebraic stack), it suffices to show that it is a monomorphism.

We prove that for any Noetherian \mathcal{O}/ϖ^a -algebra R , and any pair $A_1, A_2 \in Y^{\mathrm{bd}, \tau}(R)$, the morphism ι induces a bijection

$$\mathrm{Hom}_{Y^{\mathrm{bd}, \tau}}(A_1, A_2) \xrightarrow{\sim} \mathrm{Hom}_{\Phi\text{-Mod}}(\iota(A_1), \iota(A_2)).$$

The induced map is clearly injective, and we thus prove its surjectivity. Assume that there exists $X = (X_j)_{\mathcal{J}} \in LG(R)^{\mathcal{J}}$ such that

$$(3.2.6) \quad A_1^{(j)} = X_j A_2^{(j)} \left(\varphi(X_{j-1}) \right)^{-1}$$

or, equivalently,

$$(3.2.7) \quad \left(\varphi(X_{j-1}) \right) = (A_1^{(j)})^{-1} X_j A_2^{(j)}$$

for all $j \in \mathcal{J}$.

We show by induction on a that (3.2.7) forces $X_j \in L^+G(R)$ for all $j \in \mathcal{J}$.

Assume that $a = 1$. Thus $p = 0$ in R and $L^+G(R) = \mathrm{GL}_2(R[[v]])$. We first show that $X_j \in \mathrm{Mat}_2(R[[v]])$ for all $j \in \mathcal{J}$. Let $\kappa_j \in \mathbb{Z}$ be the pole order of X_j at v , i.e. $\kappa_j \in \mathbb{Z}$ is minimal such that $v^{\kappa_j} X_j \in \mathrm{Mat}_2(R[[v]])$. As $\det(A_1^{(j)}) \in R^\times v^{1 + \langle \mu_j, \alpha^\vee \rangle}$ we deduce from (3.2.7) that

$$1 + \kappa_j + \langle \mu_j, \alpha^\vee \rangle \geq p\kappa_{j-1}$$

which forces $\kappa_j \leq \max_i \left\{ \frac{\langle \mu_i, \alpha^\vee \rangle + 1}{p-1} \right\} < 1$. Reversing the roles of A_1 and A_2 , the same argument shows that $X_j^{-1} \in \mathrm{Mat}_2(R[[v]])$ for all $j \in \mathcal{J}$. Hence $X_j \in \mathrm{GL}_2(R[[v]])$ for all $j \in \mathcal{J}$ when $a = 1$, as wanted.

Assume the assertion up to $a - 1$. We can thus write $X_j = \tilde{X}_j(1 + \varepsilon_j)$ where $\tilde{X}_j \in L^+G(R)$, $\varepsilon_j \in \mathrm{Lie} LG((\varpi^{a-1}))$, for all $j \in \mathcal{J}$. (We denoted by $\mathrm{Lie} LG((\varpi^{a-1}))$ the monoid of 2 by 2 matrices with coefficients in the ideal of $R[v]^{\wedge (v+(v+p))} \left[\frac{1}{v(v+p)} \right]$ generated by ϖ^{a-1} . Similar comment applies for $\mathrm{Lie} L^+G((\varpi^{a-1}))$ below.) Replacing $A_1^{(j)}$ by $(\tilde{X}_j)^{-1} A_1^{(j)} \left(\varphi(\tilde{X}_{j-1}) \right)$, we can assume that (3.2.6) is

true with $X_j = 1 + \varepsilon_j$. In particular $A_2^{(j)} = (1 + \delta_j)A_1^{(j)}$ for some $\delta_j \in \text{Lie } LG((\omega^{a-1}))$. We now prove that actually $\varepsilon_j \in \text{Lie } L^+G((\omega^{a-1}))$. Then (3.2.6) becomes $1 = (1 + \varepsilon_j)(1 + \delta_j) \text{Ad}(A_1^{(j)})(1 - \varphi(\varepsilon_{j-1}))$ so that

$$(3.2.8) \quad (A_1^{(j)})^{-1} \varepsilon_j A_1^{(j)} + (A_1^{(j)})^{-1} (A_2^{(j)} - A_1^{(j)}) = \varphi(\varepsilon_{j-1}).$$

Let κ_j be the pole order of ε_j . Observe that

- $v^{\kappa_j + \langle \mu_j, \alpha^\vee \rangle} (v + p) \left((A_1^{(j)})^{-1} \varepsilon_j A_1^{(j)} \right) \in \text{Mat}_2(R[v]^{\wedge_{v(v+p)}})$
- $v^{\langle \mu_j, \alpha^\vee \rangle} (v + p) \left((A_1^{(j)})^{-1} (A_2^{(j)} - A_1^{(j)}) \right) \in \text{Mat}_2(R[v]^{\wedge_{v(v+p)}}).$

Since $(v + p)\varepsilon_j = v\varepsilon_j$ we deduce that

$$p\kappa_{j-1} \leq \max\{\kappa_j + \langle \mu_j, \alpha^\vee \rangle + 1, \langle \mu_j, \alpha^\vee \rangle + 1\}.$$

This forces $\kappa_j \leq \max_j \left\{ \frac{\langle \mu_i, \alpha^\vee \rangle + 1}{p-1} \right\} < 1$. So that indeed $\varepsilon_j \in \text{Lie } L^+G((\omega^{a-1}))$. \square

By taking fiber product, we thus obtain a commutative diagram:

$$(3.2.9) \quad \begin{array}{ccccc} \tilde{Y}^{\eta, \tau} & \xrightarrow{\pi} & \tilde{\mathcal{Z}}^\tau & \hookrightarrow & \left[LG^{\text{bd}, (v+p)v^{\hat{\mu}}} / \varphi \prod_{j \in \mathcal{J}} L_1^+ G \right]^{\wedge_p} \\ \downarrow \prod_{\mathcal{J}} \text{GL}_2 & \square & \downarrow \prod_{\mathcal{J}} \text{GL}_2 & \square & \downarrow \prod_{\mathcal{J}} \text{GL}_2 \\ Y^{\eta, \tau} & \longrightarrow & \mathcal{Z}^\tau & \hookrightarrow & \left[LG^{\text{bd}, (v+p)v^{\hat{\mu}}} / \varphi \prod_{j \in \mathcal{J}} L^+ G \right]^{\wedge_p} \\ & \searrow \varepsilon_\tau & \downarrow & \swarrow & \\ & & \Phi\text{-Mod}_K^{\text{ét}, 2} & & \end{array}$$

where the hooked arrows are closed immersions, the arrows decorated with $\prod_{\mathcal{J}} \text{GL}_2$ are $\prod_{\mathcal{J}} \text{GL}_2$ -torsors, and the central squares are cartesian, which defines the stacks $\tilde{Y}^{\eta, \tau}$, $\tilde{\mathcal{Z}}^\tau$.

Proposition 3.2.10. *Assume that*

$$2 \leq \langle \mu_j, \alpha^\vee \rangle \leq p - 2$$

and that if $\langle \mu_j, \alpha^\vee \rangle = p - 2$ for all j then some $s_j \neq \text{id}$ (this is in particular the case if $p \geq 5$ and (s, μ) is a small presentation with $\langle \mu_j, \alpha^\vee \rangle \geq 2$). Then $\varepsilon_\tau : Y^{\eta, \tau} \rightarrow \Phi\text{-Mod}_K^{\text{ét}, 2}$ is a closed immersion and thus $Y^{\eta, \tau} = \mathcal{Z}^\tau$ and $\tilde{Y}^{\eta, \tau} = \tilde{\mathcal{Z}}^\tau$.

Proof. The statement follows from a slight adaption of the proof of [LLHLM23, Proposition 5.4.3]. The essential two facts are:

- If R is an \mathbb{F} -algebra and $A, B \in \mathcal{A}(\eta)^{\mathcal{J}}(R)$, and $X \in LG^{\mathcal{J}}(R)$ such that

$$A_j = X_j B_j \text{Ad}(s_j^{-1} v^{\mu_j})(\varphi(X_{j-1}^{-1}))$$

then $X_j \in L^+ \mathcal{G}^{\mathcal{J}}(R)$.

- If R is an \mathbb{F} -algebra, J an R -module, $A \in \mathcal{A}(\eta)^{\mathcal{J}}(R)$, and $Y \in \text{Mat}_n^{\mathcal{J}}(J((v)))$ such that

$$\text{Ad}(A_j^{-1}) Y_j - \text{Ad}(s_j^{-1} v^{\mu_j})(\varphi(Y_{j-1})) \in \frac{1}{v} \text{Mat}_n(J[v])$$

then $Y_j \in \text{Lie } L^+ \mathcal{G}(J)$.

For the first item, we modify the proof of [LLHLM23, Lemma 5.4.4]. Set k_j to be the minimal integer such that $v^{k_j}X_j \in \text{Mat}_n(R[[v]])$. Once we can show $k_j \leq 0$, the argument in *loc.cit.* carries over verbatim. Now [LLHLM23, Equation (5.11)] gives $k_j \geq p(k_{j-1} - 1) + 1$, and thus $k_j \leq 1$ for all j . Furthermore, either $k_j \leq 0$ for all j or $k_j = 1$ for all j and the latter can only happen when

$$\langle \mu_j, \alpha^\vee \rangle = p - 2$$

for all j . Suppose $k_j = 1$ for all j . We get

$$v^{p-2}(vA_j^{-1})(vX_j)B_j = \text{Ad}(s_j^{-1}v^{\mu_j})(\varphi(vX_{j-1})).$$

Set $vX_j \bmod v = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in \text{Mat}_n(R)$. Then the above equation implies $a_j = c_j = d_j = 0$ for all j since the left-hand side is divisible by v^{p-2} . In turn, this implies that the $(2, 1)$ -entry of the left-hand side is divisible by v^{p-1} . But our hypothesis implies $s_j \neq \text{id}$ for some j , so that $b_{j-1} = 0$. But this contradicts the fact that $k_{j-1} > 0$.

A similar argument also proves the second item. □

3.3. Models for moduli of Breuil–Kisin modules. Throughout this section τ denotes a tame inertial type and $(s, \mu) \in \underline{W} \times X^*(T)$ a small presentation for it.

Let $\text{Gr}_1^\tau \hookrightarrow \text{Gr}_1^{\text{bd}, (v+p)v^{\hat{\mu}}}$ (resp. Gr_1) be the quotients $\prod_{j \in \mathcal{J}} L_1^+ G \backslash LG^\tau \hookrightarrow \prod_{j \in \mathcal{J}} L_1^+ G \backslash LG^{\text{bd}, (v+p)v^{\hat{\mu}}}$ (resp. $L_1^+ G \backslash LG$). We define $Y^{\text{mod}, \eta, \tau}$ as the quotient

$$(3.3.1) \quad \left[\text{Gr}_1^\tau / \prod_{\mathcal{J}} B\text{-sh.cnj} \right]^{\wedge_p}$$

by the shifted conjugation action $(b_j) \cdot (g_j) = (b_j g_j b_{j-1}^{-1})$. Define also $\tilde{Y}^{\text{mod}, \eta, \tau}$ as the fiber product

$$\begin{array}{ccc} \tilde{Y}^{\text{mod}, \eta, \tau} & \longrightarrow & \left(\text{Gr}_1^{\text{bd}, (v+p)v^{\hat{\mu}}} \right)^{\wedge_p} \\ \downarrow & \square & \downarrow \text{GL}_2^{\mathcal{J}} \\ Y^{\text{mod}, \eta, \tau} & \longrightarrow & \left[\text{Gr}_1^{\text{bd}, (v+p)v^{\hat{\mu}}} / \prod_{\mathcal{J}} \text{GL}_2\text{-sh.cnj} \right]^{\wedge_p} \end{array}$$

Proposition 3.3.2. *The p -adic formal scheme $\tilde{Y}^{\text{mod}, \eta, \tau}$ identifies with the (closed formal) subscheme of $\left(\prod_{\mathcal{J}} (\text{Gr}_1 \times (B \backslash \text{GL}_2, \mathcal{O})) \right)^{\wedge_p}$ consisting of pairs (X, g) such that $(g_j X_j g_{j-1}^{-1})_{j \in \mathcal{J}} \in \text{Gr}_1^\tau$.*

The natural $\text{GL}_2^{\mathcal{J}}$ action on $\tilde{Y}^{\text{mod}, \eta, \tau}$ is given by

$$(\kappa_j) \cdot ((X_j), (g_j)) = ((\kappa_j X_j \kappa_{j-1}^{-1}), (g_j \kappa_j^{-1})).$$

Proof. It follows from the definition that $\tilde{Y}^{\text{mod}, \eta, \tau}$ is the quotient of the space of triples $((X_j), (Y_j), (g_j)) \in \text{Gr}_1^{\text{bd}, (v+p)v^{\hat{\mu}}} \times \text{Gr}_1^\tau \times \text{GL}_2^{\mathcal{J}}$ satisfying $Y_j = g_j X_j (g_{j-1})^{-1}$ by the action of $B^{\mathcal{J}}$ given by

$$(b_j) \cdot ((X_j), (Y_j), (g_j)) = ((X_j), (b_j Y_j b_{j-1}^{-1}), (b_j g_j)).$$

This finishes the proof because the (class of the) triple $((X_j), (Y_j), (g_j))$ is uniquely determined by (the class of) $((X_j), (g_j))$. □

We define $\tilde{\mathcal{Z}}^{\text{mod},\tau}$ as the scheme theoretic image of projection map $\text{pr} : \tilde{Y}^{\text{mod},\eta,\tau} \rightarrow \text{Gr}_1^{\text{bd},(v+p)v^{\hat{\mu}}}$ sending (X, g) to X . This gives a factorization

$$(3.3.3) \quad \begin{array}{ccc} \tilde{Y}^{\text{mod},\eta,\tau} & \xrightarrow{\pi^{\text{mod}}} & \tilde{\mathcal{Z}}^{\text{mod},\tau} \\ & \searrow \text{pr} & \downarrow \iota \\ & & \text{Gr}_1^{\text{bd},(v+p)v^{\hat{\mu}}} \end{array}$$

For any $\tilde{z} \in \widetilde{W}^\vee$ by Lemma 3.1.2(2) we have a formally étale monomorphism

$$(3.3.4) \quad \text{Gr}_1^{\text{bd},(v+p)v^{\hat{\mu}}}(\tilde{z}) \stackrel{\text{def}}{=} [(L^-G)^{\mathcal{J}}\tilde{z}] \cap \text{Gr}_1^{\text{bd},(v+p)v^{\hat{\mu}}} \hookrightarrow \text{Gr}_1^{\text{bd},(v+p)v^{\hat{\mu}}}$$

after p -adic completion, which is an open immersion because the target is of finite type. Define $\tilde{U}(\tilde{z})$ to be the p -adic completion of the LHS of (3.3.4). Note that $\tilde{U}(\tilde{z}) = \prod_j \tilde{U}(\tilde{z}_j)$ has an obvious product structure.

Lemma 3.3.5. $\{\tilde{U}(\tilde{z})\}_{\tilde{z} \in \widetilde{W}^\vee}$ is a Zariski open covering for the p -adic completion of $\text{Gr}_1^{\text{bd},(v+p)v^{\hat{\mu}}}$.

Proof. Since $\text{Gr}_1^{\text{bd},(v+p)v^{\hat{\mu}}}$ is finite type, the $\tilde{U}(\tilde{z})$ are actually Zariski open formal subschemes of its p -adic completion. Looking at $\overline{\mathbb{F}}$ -points shows that they cover. \square

In particular, by diagram (3.3.3), $\tilde{U}(\tilde{z})$ induce open substacks $\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z}) \subseteq \tilde{Y}^{\text{mod},\eta,\tau}$, $\tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z}) \subseteq \tilde{\mathcal{Z}}^{\text{mod},\tau}$.

Lemma 3.3.6. The $\text{GL}_2^{\mathcal{J}}$ -shifted conjugates $g \cdot \tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})$ (resp. $g \cdot \tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z})$) for $\tilde{z} \in \text{Adm}(\eta)^{\mathcal{J}} s^{-1} v^{\hat{\mu}}$ and $g \in \text{GL}_2^{\mathcal{J}}(\overline{\mathbb{F}})$ form a Zariski open cover of $\tilde{Y}^{\text{mod},\eta,\tau}$ (resp. $\tilde{\mathcal{Z}}^{\text{mod},\eta,\tau}$).

Proof. This follows from Lemma 3.1.2(3). \square

Remark 3.3.7. The open charts $\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})$, $\tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z})$ in Lemma 3.3.6 are not preserved by the $\text{GL}_2^{\mathcal{J}}$ -shifted conjugation action in general. However, in what follows, we will be interested in various statements about $\tilde{Y}^{\text{mod},\eta,\tau}$, $\tilde{\mathcal{Z}}^{\text{mod},\tau}$ which are invariant for the $\text{GL}_2^{\mathcal{J}}$ -shifted conjugation action, and which can be checked Zariski locally. For such statements, it suffices to work with $\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})$, $\tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z})$ (instead of all their conjugates).

Lemma 3.3.8. $\tilde{Y}^{\text{mod},\eta,\tau}$ is an \mathcal{O} -flat local complete intersection of dimension $5f$ over \mathcal{O} .

Proof. As $\tilde{Y}^{\text{mod},\eta,\tau}$ is a $\text{GL}_2^{\mathcal{J}}$ -torsor over $Y^{\text{mod},\eta,\tau}$, it suffices to show that the latter is a \mathcal{O} -flat local complete intersection of dimension f over \mathcal{O} . Since Gr_1^{τ} is a $B^{\mathcal{J}}$ -torsor over $Y^{\text{mod},\eta,\tau}$ and over $L^+\mathcal{G} \backslash \mathcal{A}(\eta)$ (Lemma 3.1.2(1)) the result follows from Table 1. \square

The following lemma shows that on suitable p -power torsion, one can often exchange the φ -twisted conjugation action of a positive loop group with the left translation action:

Lemma 3.3.9. Assume that $p - 2 - \langle \mu_j, \alpha^\vee \rangle \geq N$ for all $j \in \mathcal{J}$. We have a commutative diagram

$$\begin{array}{ccc} \tilde{Y}^{\eta,\tau} \otimes_{\mathcal{O}} \mathcal{O}/p^N & \longrightarrow & \left[LG^{\text{bd},(v+p)v^{\hat{\mu}}} /_{\varphi} \prod_{j \in \mathcal{J}} L_1^+ G \right] \otimes_{\mathcal{O}} \mathcal{O}/p^N \\ \downarrow \cong & & \downarrow \cong \\ \tilde{Y}^{\text{mod},\eta,\tau} \otimes_{\mathcal{O}} \mathcal{O}/p^N & \longrightarrow & \text{Gr}_1^{\text{bd},(v+p)v^{\hat{\mu}}} \otimes_{\mathcal{O}} \mathcal{O}/p^N \end{array}$$

Proof. We claim that the two actions of $\prod_{j \in \mathcal{J}} L_1^+ \mathbf{G}$ on $LG^{\text{bd}, (v+p)v^{\widehat{\mu}}} \otimes_{\mathcal{O}} \mathcal{O}/p^N$ (resp. $LG^\tau \otimes_{\mathcal{O}} \mathcal{O}/p^N$) by φ -twisted conjugation and by left translation are equivalent, which would establish the existence of a unique right (resp. left) vertical isomorphism comptabile with the quotient maps from $LG^{\text{bd}, (v+p)v^{\widehat{\mu}}} \otimes_{\mathcal{O}} \mathcal{O}/p^N$ (resp $LG^\tau \otimes_{\mathcal{O}} \mathcal{O}/p^N$) and the commutativity of the diagram. The proof of the claim is similar to the proof of [LLHLM23, Lemma 5.2.2]: the result would follow from the fact that for any \mathcal{O}/p^N -algebra R and $A \in LG^{\text{bd}, (v+p)v^{\widehat{\mu}}}(R)$ the map

$$(X_j) \mapsto (X_j \text{Ad}(A_j) \varphi(X_{j-1}^{-1}))$$

is an automorphism of $\prod_{j \in \mathcal{J}} L_1^+ \mathbf{G}(R)$. In turn this follows from the fact that

$$v^p A_j^{-1} \in \frac{v^p}{(v+p)v^{\langle \mu_j, \alpha^\vee \rangle}} \text{Mat}_2(R[[v]]) \subset v^{p-N-\langle \mu_j, \alpha^\vee \rangle} \text{Mat}_2(R[[v]]) \subset v^2 \text{Mat}_2(R[[v]]),$$

where we use that $\frac{1}{(v+p)} R[[v]] \subset \frac{1}{v^N} R[[v]]$. □

Remark 3.3.10. The two actions $\prod_{j \in \mathcal{J}} L_1^+ \mathbf{G}$ on $LG^\tau \otimes_{\mathcal{O}} \mathcal{O}/p^N$ by φ -twisted conjugation and by left translation are equivalent under the weaker hypothesis that

- $N \leq p - 1 - \langle \mu_j, \alpha^\vee \rangle$ for all j ; and
- If $N = p - 1 - \langle \mu_j, \alpha^\vee \rangle$ for all j then some $s_j \neq \text{id}$.

In particular, this implies $Y^{\text{mod}, \eta, \tau} \otimes_{\mathcal{O}} \mathcal{O}/p^N \cong Y^{\eta, \tau} \otimes_{\mathcal{O}} \mathcal{O}/p^N$.

We explain this when $N = 1$, the general case follows in the same way. As in the above proof, we need to show that for an \mathbb{F} -algebra R and $A \in \mathcal{A}(\eta)^{\mathcal{J}}(R)$, the map

$$(X_j) \mapsto (X_j \text{Ad}(A_j s_j^{-1} v^{\widehat{\mu}_j}) \varphi(X_{j-1}^{-1}))$$

is an automorphism of $L_1^+ \mathbf{G}(R)^{\mathcal{J}}$. In turn, writing the Taylor expansion $X_j = 1 + vY_j + O(v^2)$, it suffices to show

$$(Y_j) \mapsto (Y_j - v^{p-2} A_j \text{Ad}(s_j^{-1} v^{\widehat{\mu}_j}) (Y_{j-1}) (vA_j^{-1}) \text{ mod } v)$$

is an automorphism of $\text{Mat}_n(R)^{\mathcal{J}}$. This will follow if we can show that the operator

$$\Psi : (Y_j) \mapsto (v^{p-2} A_j \text{Ad}(s_j^{-1} v^{\widehat{\mu}_j}) (Y_{j-1}) (vA_j^{-1}) \text{ mod } v)$$

is a nilpotent operator on $\text{Mat}_n(R)^{\mathcal{J}}$. Note that $\Psi(Y)_j$ is a linear operator that depends only on $c_{j-1} = (Y_{j-1})_{21}$, and that $(\Psi(Y)_j)_{21}$ is 0 if $s_j \neq \text{id}$ and is a multiple of c_{j-1} otherwise. It is clear from this description that $\Psi^{f+1} = 0$.

As a consequence of Lemma 3.3.9 and diagram 3.2.9, $\widetilde{U}(\widetilde{z})$ also induces open substacks $\widetilde{Y}^{\eta, \tau}(\widetilde{z})$, $\widetilde{Z}^\tau(\widetilde{z})$. The following Theorem, which makes Theorem 1.1.4(2) precise, is the main result of the paper.

Theorem 3.3.11. *Fix a small presentation (s, μ) of τ . Assume either $p > \max\{16f + 3, 16f - 12 + \max_j \langle \mu_j, \alpha^\vee \rangle\}$ or $p \geq 17$ and $K = \mathbb{Q}_p$. Then we have an isomorphism*

$$\widetilde{Z}^{\text{mod}, \tau}(\widetilde{z}) \cong \widetilde{Z}^\tau(\widetilde{z}).$$

Remark 3.3.12. (1) Since (s, μ) is small, $\max_j \langle \mu_j, \alpha^\vee \rangle \leq \frac{p+1}{2}$. In particular, for $f > 1$ the hypothesis on p is satisfied for all τ when $p > 32f - 23$.

(2) Theorem 3.3.11 is proven under the first hypothesis in section 4.5. The proof under the improved bound $p \geq 17$ for \mathbb{Q}_p is completed in section 5.5.1, cf. Remark 5.5.1 for the source of the improvements.

The proof of Theorem 3.3.11 will be performed in two steps:

- (1) We first show there is an isomorphism $\tilde{\mathcal{Z}}^\tau(\tilde{z}) \otimes_{\mathcal{O}} \mathcal{O}/p^N \xrightarrow{\sim} \tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z}) \otimes_{\mathcal{O}} \mathcal{O}/p^N$ for sufficiently large N .
- (2) We bound the power of p that belongs to the ideal of singularity of $\tilde{\mathcal{Z}}^{\text{mod},\eta,\tau}$ over \mathcal{O} . This allows us to lift the isomorphism mod p^N in the above step to an isomorphism over \mathcal{O} .

In the remainder of this section we will carry out the first step of the above strategy modulo some geometric facts about $\tilde{Y}^{\text{mod},\eta,\tau} \rightarrow \tilde{\mathcal{Z}}^{\text{mod},\tau}$ which will be established in the later sections. The second step of the strategy is carried out in Section 4.5.

3.3.1. *Modeling mod p^N .* Set $N \stackrel{\text{def}}{=} p - 2 - \max_j \langle \mu_j, \alpha^\vee \rangle \geq \frac{p-5}{2}$ and $\delta = \max\{0, |\{j \mid \langle \mu_j, \alpha^\vee \rangle \leq 1\}| - f + 1\}$.

Define $\tilde{\mathcal{Z}}_N^{\text{apx}}$ as the scheme theoretic image of the map

$$\tilde{Y}^{\eta,\tau} \otimes_{\mathcal{O}} \mathcal{O}/p^N \rightarrow \left[LG^{\text{bd},(v+p)v^{\hat{\mu}}} /_{\varphi} \prod_{j \in \mathcal{J}} L_1^+ G \right] \otimes_{\mathcal{O}} \mathcal{O}/p^N = \text{Gr}_1^{\text{bd},(v+p)v^{\hat{\mu}}} \otimes_{\mathcal{O}} \mathcal{O}/p^N.$$

We have the commutative diagram (where the morphisms \heartsuit and $\heartsuit\heartsuit$ exist by definition of scheme-theoretic image):

$$(3.3.13) \quad \begin{array}{ccccc} \tilde{Y}^{\eta,\tau} \otimes_{\mathcal{O}} \mathcal{O}/p^N & \xrightarrow{\sim \text{Lemma 3.3.9}} & \tilde{Y}^{\text{mod},\eta,\tau} \otimes_{\mathcal{O}} \mathcal{O}/p^N & & \\ \downarrow \pi & \searrow & \downarrow \pi^{\text{mod}} & & \\ \tilde{\mathcal{Z}}^\tau \otimes_{\mathcal{O}} \mathcal{O}/p^N & \xleftarrow{\heartsuit} & \tilde{\mathcal{Z}}_N^{\text{apx}} & \xrightarrow{\heartsuit\heartsuit} & \tilde{\mathcal{Z}}^{\text{mod},\tau} \otimes_{\mathcal{O}} \mathcal{O}/p^N \\ & \searrow (3.2.9) & \downarrow & & \downarrow \iota \\ & & \left[LG^{\text{bd},(v+p)v^{\hat{\mu}}} /_{\varphi} \prod_{\mathcal{J}} L_1^+ G \right] \otimes_{\mathcal{O}} \mathcal{O}/p^N & \xrightarrow{\sim} & \text{Gr}_1^{\text{bd},(v+p)v^{\hat{\mu}}} \otimes_{\mathcal{O}} \mathcal{O}/p^N \end{array}$$

Proposition 3.3.14. *The morphisms \heartsuit , $\heartsuit\heartsuit$ have a factorization*

$$\begin{array}{ccccc} \tilde{\mathcal{Z}}^\tau \otimes_{\mathcal{O}} \mathcal{O}/p^N & \xleftarrow{\heartsuit} & \tilde{\mathcal{Z}}_N^{\text{apx}} & \xrightarrow{\heartsuit\heartsuit} & \tilde{\mathcal{Z}}^{\text{mod},\tau} \otimes_{\mathcal{O}} \mathcal{O}/p^N \\ \uparrow & & \uparrow & & \uparrow \\ \tilde{\mathcal{Z}}^\tau \otimes_{\mathcal{O}} \mathcal{O}/p^{N-\delta} & \xrightarrow{\sim} & \tilde{\mathcal{Z}}_N^{\text{apx}} \otimes_{\mathcal{O}} \mathcal{O}/p^{N-\delta} & \xrightarrow{\sim} & \tilde{\mathcal{Z}}^{\text{mod},\tau} \otimes_{\mathcal{O}} \mathcal{O}/p^{N-\delta} \end{array}$$

In particular, the natural morphisms \heartsuit , $\heartsuit\heartsuit$ induce an isomorphism

$$\tilde{\mathcal{Z}}^\tau \otimes_{\mathcal{O}} \mathcal{O}/p^{N-\delta} \xrightarrow{\sim} \tilde{\mathcal{Z}}^{\text{mod},\tau} \otimes_{\mathcal{O}} \mathcal{O}/p^{N-\delta}.$$

Corollary 3.3.15. *Assume that $p \geq 7$ or $K = \mathbb{Q}_5$. Then*

$$\tilde{\mathcal{Z}}^\tau \otimes_{\mathcal{O}} \mathcal{O}/p \xrightarrow{\sim} \tilde{\mathcal{Z}}^{\text{mod},\tau} \otimes_{\mathcal{O}} \mathcal{O}/p.$$

Proof. First assume $p \geq 7$. If $\delta = 0$ then $N - \delta \geq \frac{p-5}{2} \geq 1$, otherwise $N - \delta \geq p - 4 > 1$. In either case, the result follows from Proposition 3.3.14.

Now consider the case $K = \mathbb{Q}_5$. If $\langle \mu, \alpha^\vee \rangle = 1$ then $N - \delta = 5 - 2 - 1 - 1 = 1$ and the result follows from Proposition 3.3.14. Otherwise, $\langle \mu, \alpha^\vee \rangle \geq 2$, thus $\tilde{\mathcal{Z}}^\tau = \tilde{Y}^{\eta,\tau}$ and $\tilde{\mathcal{Z}}^{\text{mod},\tau} = \tilde{Y}^{\text{mod},\eta,\tau}$ by Proposition 3.2.10. Then by Remark 3.3.10 $\tilde{Y}^{\text{mod},\eta,\tau}/p \cong \tilde{Y}^{\eta,\tau}/p$ and another invocation of Proposition 3.2.10 shows that $\tilde{Y}^{\text{mod},\eta,\tau}/p \cong \tilde{Y}^{\eta,\tau}/p$. \square

The proof of Proposition 3.3.14 crucially relies on the following two propositions, whose proofs will be postponed until the end of section 4.2.

Proposition 3.3.16. *Recall $\delta = \max\{0, |\{j \mid \langle \mu_j, \alpha^\vee \rangle \leq 1\}| - f + 1\}$. Then*

$$(3.3.17) \quad p^\delta \operatorname{coker} \left(\mathcal{O}_{\tilde{\mathcal{Z}}^{\text{mod}, \tau}} \rightarrow \pi_*^{\text{mod}} \left(\mathcal{O}_{\tilde{Y}^{\text{mod}, \eta, \tau}} \right) \right) = 0.$$

Proposition 3.3.18. *For $\ell > 0$*

$$R^\ell \pi_*^{\text{mod}} \mathcal{O}_{\tilde{Y}^{\text{mod}, \eta, \tau}} = 0.$$

Proof of Proposition 3.3.14. The statement is local on $\text{Gr}_1^{\text{bd}, (v+p)v^{\hat{\mu}}} \otimes_{\mathcal{O}} \mathcal{O}/p^N$, so it suffices to prove it after intersecting everything with $\tilde{U}(\tilde{z})$. Write $\tilde{\mathcal{Z}}^\tau(\tilde{z}) = \text{Spf } R$, $\tilde{\mathcal{Z}}^{\text{mod}, \tau}(\tilde{z}) = \text{Spf } R^{\text{mod}}$, and $\tilde{\mathcal{Z}}_N^{\text{apx}} \cap \tilde{U}(\tilde{z}) = \text{Spf } R_N^{\text{apx}}$. Also let $S = \pi_* \left(\mathcal{O}_{\tilde{Y}^{\text{mod}, \eta, \tau}(\tilde{z})} \right)$ and $S^{\text{mod}} = \pi_*^{\text{mod}} \left(\mathcal{O}_{\tilde{Y}^{\text{mod}, \eta, \tau}(\tilde{z})} \right)$ and $S_N = \pi_*^{\text{mod}} \left(\mathcal{O}_{\tilde{Y}^{\text{mod}, \eta, \tau}(\tilde{z})}/p^N \right)$. Note that S and S^{mod} are p -torsion free and, $S/p^N \hookrightarrow S_N = S^{\text{mod}}/p^N$ by Proposition 3.3.18.

Since $\tilde{\mathcal{Z}}^{\text{mod}, \tau}(\tilde{z})$ is the scheme theoretic image of $\tilde{Y}^{\text{mod}, \eta, \tau}(\tilde{z}) \rightarrow \tilde{U}(\tilde{z})$ we have $R^{\text{mod}} \subset S^{\text{mod}}$. Similarly $R \subset S$. Finally equation (3.3.17) shows that $S^{\text{mod}}/R^{\text{mod}}$ is p -torsion. We thus have the following commutative diagram

$$(3.3.19) \quad \begin{array}{ccccccc} S & \twoheadrightarrow & S/p^N & \hookrightarrow & S_N = S^{\text{mod}}/p^N & \longleftarrow & S^{\text{mod}} \\ \pi \uparrow & & \uparrow & & \uparrow & & \uparrow \pi^{\text{mod}} \\ R & \twoheadrightarrow & R/p^N & \xrightarrow{\heartsuit} & R_N^{\text{apx}} \xrightarrow{\heartsuit\heartsuit} & R^{\text{mod}}/p^N & \longleftarrow & R^{\text{mod}} \end{array}$$

where the hooked arrows are injective. Let C , C^{mod} denote the cokernel of the maps π , π^{mod} respectively. As S , S^{mod} are both p -flat, we deduce from (3.3.19) the following commutative diagram with exact rows (this defines C_N):

$$(3.3.20) \quad \begin{array}{ccccccc} & & \ker(\heartsuit\heartsuit) & & & & \\ & & \downarrow & & & & \\ 0 & \rightarrow & C^{\text{mod}}[p^N] & \rightarrow & R^{\text{mod}}/p^N & \xrightarrow{\pi^{\text{mod}}} & S^{\text{mod}}/p^N \rightarrow C^{\text{mod}}/p^N \rightarrow 0 \\ & & \downarrow & & \downarrow \heartsuit\heartsuit & & \parallel & & \parallel \\ 0 & \rightarrow & 0 & \rightarrow & R_N^{\text{apx}} & \hookrightarrow & S_N & \rightarrow & C_N \rightarrow 0 \\ & & \uparrow & & \uparrow \heartsuit & & \uparrow & & \uparrow \\ 0 & \rightarrow & C[p^N] & \rightarrow & R/p^N & \xrightarrow{\pi} & S/p^N & \rightarrow & C/p^N \rightarrow 0 \\ & & \parallel & & \uparrow & & & & \\ & & \ker(\heartsuit) & & & & & & \end{array}$$

Proposition 3.3.16 implies that $p^\delta C^{\text{mod}} = 0$ and hence $C^{\text{mod}}[p^N] = C^{\text{mod}}[p^\delta]$. Thus $\ker(\heartsuit\heartsuit)$ is annihilated by p^δ . We conclude that $p^\delta \ker(\heartsuit\heartsuit) \subseteq p^N R^{\text{mod}}$ and, as R^{mod} is p -flat, that $\ker(\heartsuit\heartsuit) \subseteq$

$p^{N-\delta}R^{\text{mod}}$. This implies the factorization

$$\begin{array}{ccc} R^{\text{mod}}/p^N & \xrightarrow{\heartsuit\heartsuit} & R_N^{\text{apx}} \\ \downarrow & \searrow \exists & \\ R^{\text{mod}}/p^{N-\delta} & & \end{array}$$

and hence an isomorphism $R^{\text{mod}}/p^{N-\delta} \cong R_N^{\text{apx}}/p^{N-\delta}$.

Now $p^\delta C_N = 0$ so $p^\delta(C/p^N) = 0$. Hence $p^\delta C \subset p^N C \subset p^{1+\delta} C$ and as C is p -adically separated we learn that $p^\delta C = 0$. We repeat the argument in the previous paragraph to obtain the factorization

$$\begin{array}{ccc} R/p^N & \xrightarrow{\heartsuit} & R_N^{\text{apx}} \\ \downarrow & \searrow \exists & \\ R/p^{N-\delta} & & \end{array}$$

and hence an isomorphism $R/p^{N-\delta} \cong R_N^{\text{apx}}/p^{N-\delta}$. \square

4. GEOMETRY OF LOCAL MODELS

4.1. Equations for $\tilde{Y}^{\text{mod},\eta,\tau}$. Let τ be a tame inertial type with small presentation (s, μ) . Everything we do depends on this choice but we usually suppress this dependence from the notation. Let $\tilde{w} \in \text{Adm}^\vee(\eta)^\mathcal{J}$ and set $\tilde{z} = \tilde{w}s^{-1}v^{\hat{\mu}}$. We consider the morphism $\text{GL}_{2,\mathcal{O}} \rightarrow B \backslash \text{GL}_{2,\mathcal{O}} \xrightarrow{\sim} \mathbb{P}_{\mathcal{O}}^1$ sending $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ to $[\gamma : \delta] \in \mathbb{P}_{\mathcal{O}}^1$. Given a Noetherian \mathcal{O} -algebra R where p is nilpotent, by Proposition 3.3.2 and the definition of $\mathcal{A}(\eta)$, we see that $\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})(R)$ is the groupoid of tuples $(l_j, \kappa_j, X_j)_{j \in \mathcal{J}} \in (\mathbb{P}^1 \times \text{GL}_2 \times L_1^- \text{G})^\mathcal{J}$ subject to the following conditions that for some (equivalently, any) lift $\tilde{l}_j \in \text{GL}_2(R)$ of l_j :

- (1) $X_j \tilde{w}_j \text{Ad}(s_j^{-1}v^{\hat{\mu}_j})(\tilde{l}_{j-1}^{-1}) \in \text{Mat}_2(R[[v]])$
- (2) $\det(X_j \tilde{w}_j) \in R[[v]]^\times(v+p)$
- (3) we have $l_j \kappa_j \cdot (X_j \tilde{w}_j \text{Ad}(s_j^{-1}v^{\hat{\mu}_j})(\tilde{l}_{j-1}^{-1}))|_{v=0} = [0 : 1]$.

The correspondence is explicitly given by sending the triplet (l_j, κ_j, X_j) to the pair $(\kappa_j X_j \tilde{z}_j, l_j)$. Note that $\text{Ad}(s_j^{-1}v^{\hat{\mu}_j})(B(R)) \in B(R[[v]])$ since μ is dominant, which justifies the independence of the choice of the lift $\tilde{l}_{j-1} \in \text{GL}_2(R)$ in items (1),(2),(3) above.

In item (3) we have used the following

Convention 4.1.1. Given $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{Mat}_2(R)$ and $x, y, z, t \in R$, the equality

$$[x : y] \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = [z : t]$$

is interpreted as $t(\alpha x + \gamma y) - z(\beta x + \delta y) = 0$.

Equivalently, this can be written in matrix form

$$(x \ y) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t \\ -z \end{pmatrix} = 0.$$

We will always adopt the above convention in the tables below, note that this allows us to interpret equations as $[x : y] = [z : t]$ even if neither side are actual elements of \mathbb{P}^1 . We will also

consider the following auxiliary space $\widetilde{Ba}(\tilde{z})$ of tuples $(l_j, \kappa_j, X_j, r_j)_{j \in \mathcal{J}} \in (\mathbb{P}^1 \times \mathrm{GL}_2 \times L_1^- \mathrm{G} \times \mathbb{P}^1)^{\mathcal{J}}$ satisfying the following variant of the conditions above:

- (1) $X_j \tilde{w}_j \mathrm{Ad}(s_j^{-1} v^{\hat{\mu}_j})(\tilde{r}_j^{-1}) \in \mathrm{Mat}_2(R[[v]])$
- (2) $\det(X_j \tilde{w}_j) \in R[[v]]^\times (v + p)$
- (3) we have $l_j \kappa_j \cdot (X_j \tilde{w}_j \mathrm{Ad}(s_j^{-1} v^{\hat{\mu}_j})(\tilde{r}_j^{-1}))|_{v=0} = [0 : 1]$.

Clearly, we have a decomposition $\widetilde{Ba}(\tilde{z}) = \prod_{j \in \mathcal{J}} \widetilde{Ba}_j(\tilde{z}_j)$ where the j -th factor classifies the quadruples $(l_j, \kappa_j, X_j, r_j)$. Furthermore $\widetilde{Y}^{\mathrm{mod}, \eta, \tau}(\tilde{z})$ is exactly the subspace of $\widetilde{Ba}(\tilde{z})$ where we impose that $r_j = l_{j-1}$ for all j .

Lemma 4.1.2. *The f -tuples $(X_j, r_j)_{j \in \mathcal{J}} \in (L_1^- \mathrm{G} \times \mathbb{P}^1)^{\mathcal{J}}(R)$ satisfying condition (1) are precisely those given in Table 2, according to $\tilde{w}_j \in \mathrm{Adm}^\vee(\eta_j)$, $s_j \in W$ and $k_j \stackrel{\mathrm{def}}{=} \langle \mu_j, \alpha^\vee \rangle$.*

TABLE 2. Description of $X_j \in L_1^- \mathrm{G}$

(\tilde{w}_j, s_j)	$(t_\eta, (12)), (w_0 t_\eta, (12)), (t_{w_0(\eta)}, \mathrm{id})$	$(t_\eta, \mathrm{id}), (w_0 t_\eta, \mathrm{id}), (t_{w_0(\eta)}, (12))$
	$s_j w_j^{-1} X_j w_j s_j^{-1} = \begin{pmatrix} 1 + \frac{A}{v^{k_j}} & \frac{B}{v} \\ \frac{C}{v^{k_j}} + \frac{C'}{v^{k_j-1}} & 1 + \frac{D}{v} \end{pmatrix}$ $A, B, C, C', D \in R,$	$s_j w_j^{-1} X_j w_j s_j^{-1} = \begin{pmatrix} 1 + \frac{A}{v} & 0 \\ \frac{C}{v} + \frac{C'}{v^{k_j+1}} & 1 \end{pmatrix}$ $A, C, C' \in R$
$k_j = 0$	$A = C = C' = 0$	$C' = 0$
$k_j = 1$	$[A : B] = [C : D] = r_j, C' = 0$	$[C' : 1] = r_j$
$k_j > 1$	$[A : B] = [C : D] = [C' : 1] = r_j$	$[C' : 1] = r_j$

Proof. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R)$ so that $r_j = [c : d]$. Also, let $w_j t_{v_j} \stackrel{\mathrm{def}}{=} \tilde{w}_j$ and write $s_j w_j^{-1} X_j w_j s_j^{-1} = \begin{pmatrix} 1 + \alpha & \beta \\ \gamma & 1 + \delta \end{pmatrix}$ so that $\alpha, \beta, \gamma, \delta \in \frac{1}{v} R[\frac{1}{v}]$. We abbreviate $k = k_j$.

Condition (1) is equivalent to

$$(4.1.3) \quad \begin{pmatrix} (1 + \alpha)v^{k+\varepsilon} & \beta v^{1-\varepsilon} \\ \gamma v^{k+\varepsilon} & (1 + \delta)v^{1-\varepsilon} \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \begin{pmatrix} v^k R[[v]] & R[[v]] \\ v^k R[[v]] & R[[v]] \end{pmatrix}$$

where $\varepsilon = 0$ (resp. $\varepsilon = 1$) if (\tilde{w}_j, s_j) is as in the first (resp. second) column of Table 2. From this we learn that $v^{1-\varepsilon}\beta, v^{1-\varepsilon}(1 + \delta), (1 + \alpha)v^{k+\varepsilon}, \gamma v^{k+\varepsilon} \in R[[v]]$. The first two conditions show that the second column of $\begin{pmatrix} 1 + \alpha & \beta \\ \gamma & 1 + \delta \end{pmatrix}$ has the form specified in the table.

We now show that α, γ also have the form specified in the table.

Case $\varepsilon = 0$. Looking at the first column of (4.1.3) we get

$$(4.1.4) \quad v^k(1 + \alpha)d - v\beta c \in v^k R[[v]];$$

$$(4.1.5) \quad v^k \gamma d - v(1 + \delta)c \in v^k R[[v]].$$

From a casewise check (on k , and recalling that $\alpha, \beta, \gamma, \delta \in \frac{1}{v} R[\frac{1}{v}]$) equations (4.1.4), (4.1.5) imply that αd and γd have the form specified in the table.

Since we already know $(1 + \alpha)v^k, \gamma v^k \in R[[v]]$ we only have to consider the case $k > 1$. But then equation (4.1.5) shows that $c \in R$ hence $d \in R^\times$ because $(c, d) = R$. But this implies α and γ are of the desired form.

Case $\varepsilon = 1$. Looking at the first column of (4.1.3) we get (using that $\beta = \delta = 0$)

$$(4.1.6) \quad v^{k+1}(1 + \alpha)d \in v^k R[[v]]$$

$$(4.1.7) \quad v^{k+1}\gamma d - c \in v^k R[[v]].$$

As above, note that these two equations imply that αd and γd have the form specified in the table.

Since we already know $(1 + \alpha)v^{k+1}, \gamma v^{k+1} \in R[[v]]$ we only have to consider the case $k > 0$. But then equation (4.1.7) shows that $c = 0$ hence $d \in R^\times$. Again this implies α and γ are of the desired form.

We have now shown that $\alpha, \beta, \gamma, \delta$ have the desired form. Plugging this information into equations (4.1.4), (4.1.5), (4.1.6), (4.1.7) immediately yields the remaining equations in the table. \square

Lemma 4.1.8. *For each j the spaces $\widetilde{Ba}_j(\widetilde{z}_j)$ have explicit presentations given by Table 3, in terms of the corresponding spaces in Table 2.*

TABLE 3. Presentations of $\widetilde{Ba}_j(\widetilde{z}_j)$.

$\langle \mu_j, \alpha^V \rangle$	\widetilde{w}_j	t_η	$w_0 t_\eta$	$t_{w_0(\eta)}$
	s_j			
> 1	(12)	Variables: A, B, C, C', D $A = BC', C = pC', D = p$ $r_j = [C' : 1]$ $l_j \kappa_j = [B : -p]$	Variables: A, B, C, C', D $A = BC', C = pC', D = p$ $r_j = [C' : 1]$ $l_j \kappa_j = [-p : B]$	Variables: A, C, C' $A = p$ $r_j = [C' : 1]$ $l_j \kappa_j = [0 : 1]$
	id	Variables: A, C, C' $A = p$ $r_j = [C' : 1]$ $l_j \kappa_j = [C : -p]$	Variables: A, C, C' $A = p$ $r_j = [C' : 1]$ $l_j \kappa_j = [-p : C]$	Variables: A, B, C, C', D $A = BC', C = pC', D = p$ $r_j = [C' : 1]$ $l_j \kappa_j = [0 : 1]$
$= 1$	(12)	Variables: A, B, C, C', D $A = p - D, C' = 0$ $r_j = [C : D] = [p - D : B]$ $l_j \kappa_j = [D - p : C] = [-B : D]$	Variables: A, B, C, C', D $A = p - D, C' = 0$ $r_j = [C : D] = [p - D : B]$ $l_j \kappa_j = [C : D - p] = [-D : B]$	Variables: A, C, C' $A = p$ $r_j = [C' : 1]$ $l_j \kappa_j = [0 : 1]$
	id	Variables: A, C, C' $A = p$ $r_j = [C' : 1]$ $l_j \kappa_j = [C : -p]$	Variables: A, C, C' $A = p$ $r_j = [C' : 1]$ $l_j \kappa_j = [-p : C]$	Variables: A, B, C, C', D $A = p - D, C' = 0$ $r_j = [C : D] = [p - D : B]$ $l_j \kappa_j = r_j$
$= 0$	id	Variables: A, C, C' $A = p, C' = 0$ $l_j \kappa_j = r_j \begin{pmatrix} 1 & 0 \\ -C & p \end{pmatrix}$	Variables: A, C, C' $A = p, C' = 0$ $l_j \kappa_j = r_j \begin{pmatrix} 0 & -1 \\ -p & C \end{pmatrix}$	Variables: A, B, C, C', D $A = 0, C = 0, C' = 0, D = p$ $l_j \kappa_j = r_j \begin{pmatrix} p & -B \\ 0 & 1 \end{pmatrix}$

The meaning of the variables A, B, C, C', D is in terms of the $X_j \in L_1^- G$ extracted from the corresponding entries in Table 2.

Proof. The equations involving A, B, C, C', D are exactly obtained from Table 2 by imposing the condition $\det(X_j) \in R[[v]]^\times \frac{(v+p)}{v}$. The formula for r_j also follows from Table 2.

Thus the only thing we need to verify is the equation involving $l_j \kappa_j$. We give a sample computation in the case when $(\widetilde{w}_j, \langle \mu_j, \alpha^V \rangle, s_j) = (t_{w_0(\eta)}, 1, \text{id})$, the others being similar and simpler. In this case let $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \text{GL}_2(R)$ (resp. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(R)$) be a lift of $l_j \kappa_j$ (resp. r_j). Thus condition

(3) (together with condition (1)) is the condition that the $(2, 1)$ -entry of $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} d & aB - Ab \\ -c & aD - Cb \end{pmatrix}$ is zero, which exactly means that $l_j \kappa_j = r_j$. \square

Remark 4.1.9. So far we only considered $\widetilde{Ba}_j(\tilde{z}_j), \widetilde{U}(\tilde{z}_j)$ as p -adic formal schemes. However Table 3 gives obvious de completions and we will work with the scheme version of these spaces in what follows.

4.2. Cohomological properties of $\widetilde{Y}^{\text{mod},\eta,\tau} \xrightarrow{\text{pr}} \text{Gr}_1^{\text{bd},(v+p)v^{\widehat{\mu}}}$. In this section we will use the explicit presentations from the previous section to study $R\pi_*^{\text{mod}}\mathcal{O}_{\widetilde{Y}^{\text{mod},\eta,\tau}}$ and hence prove Propositions 3.3.16, 3.3.18.

Since ι is a closed immersion, we have $\iota_* \circ R\pi_*^{\text{mod}}\mathcal{O}_{\widetilde{Y}^{\text{mod},\eta,\tau}} = R\text{pr}_*\mathcal{O}_{\widetilde{Y}^{\text{mod},\eta,\tau}}$. In particular Proposition 3.3.16 is equivalent to

$$p \text{ coker} \left(\mathcal{O} \left(\text{Gr}_1^{\text{bd},(v+p)v^{\widehat{\mu}}} \right) \rightarrow (\text{pr})_* \left(\mathcal{O} \left(\widetilde{Y}^{\text{mod},\eta,\tau} \right) \right) \right) = 0$$

and Proposition 3.3.18 is equivalent to $R\text{pr}_*\mathcal{O}_{\widetilde{Y}^{\text{mod},\eta,\tau}}$ concentrating in degree 0. In other words we can replace π^{mod} by pr in the statements of interest. Furthermore, these statements are local on the target so it suffices to analyze the situation after intersecting with $\widetilde{U}(\tilde{z})$ for $\tilde{z} \in \text{Adm}^\vee(\eta)^{\mathcal{J}} s^{-1}v^{\widehat{\mu}}$. Recall that diagram (3.3.3) gives a factorization

$$(4.2.1) \quad \widetilde{Y}^{\text{mod},\eta,\tau}(\tilde{z}) \xrightarrow{\pi^{\text{mod}}} \widetilde{Z}^{\text{mod},\tau}(\tilde{z}) \xrightarrow{\iota} \widetilde{U}(\tilde{z}).$$

To analyze pr we factorize

$$\widetilde{Y}^{\text{mod},\eta,\tau}(\tilde{z}) \hookrightarrow \widetilde{Ba}(\tilde{z}) = \prod_j \widetilde{Ba}_j(\tilde{z}_j) \xrightarrow{\text{pr}_{\widetilde{B}} \stackrel{\text{def}}{=} \prod \text{pr}_j} \widetilde{U}(\tilde{z}) = \prod_j \widetilde{U}(\tilde{z}_j)$$

where $\text{pr}_j : \widetilde{Ba}_j(\tilde{z}_j) \rightarrow \widetilde{U}(\tilde{z}_j)$ is the map

$$(l_j, \kappa_j, X_j, r_j) \mapsto \kappa_j X_j \tilde{z}_j.$$

Observe that we have an isomorphism $\widetilde{Ba}_j(\tilde{z}_j) \cong \text{GL}_2 \times Ba_j(\tilde{z}_j)$ where $Ba_j(\tilde{z}_j) = \widetilde{Ba}_j(\tilde{z}_j) \times_{\text{GL}_2} \{1\}$, given by $(l_j, \kappa_j, X_j, r_j) \mapsto (\kappa_j, (l_j \kappa_j, X_j, r_j))$. Let p_j (resp. q_j) be the obvious projections from $Ba_j(\tilde{z}_j) \subset \mathbb{P}^1 \times L_1^- \text{G} \times \mathbb{P}^1$ to the left (resp. right) \mathbb{P}^1 factor. We continue to denote $\text{pr}_j : Ba_j(\tilde{z}_j) \rightarrow \widetilde{U}(\tilde{z}_j)$ the map given by the projection from $Ba_j(\tilde{z}_j)$ to the $L_1^- \text{G}$ factor followed by right multiplication by \tilde{z}_j (this is compatible with $\text{pr}_j : \widetilde{Ba}_j(\tilde{z}_j) \rightarrow \widetilde{U}(\tilde{z}_j)$).

The following is immediate from Table 3.

Lemma 4.2.2. *Up to isomorphism, there are the following possibilities for $Ba_j(\tilde{z}_j)$:*

- (1) $Ba_j(\tilde{z}_j) = \mathbb{A}^2$ where p_j is the constant map $(C, C') \mapsto [0 : 1] \in \mathbb{P}^1$, q_j is the map $(C, C') \mapsto [C' : 1] \in \mathbb{P}^1$ and pr_j is the natural map extracting C and C' . This covers the cases $(\tilde{w}_j, s_j, k_j) = (t_{w_0(\eta)}, \text{id}, > 1), (t_{w_0(\eta)}, (12), \geq 1)$.
- (2) $Ba_j(\tilde{z}_j) = \text{Bl}_{(p,0)} \mathbb{A}^1 \times \mathbb{A}^1$ where $\text{Bl}_{(p,0)} \mathbb{A}^1 = \{([x : y], C) \mid Cx = py\} \subset \mathbb{P}^1 \times \mathbb{A}^1$ is the blowup of $\mathbb{A}^1_{\mathcal{O}}$ at the origin in its special fiber. Then p_j is the natural projection map from $\text{Bl}_{(p,0)} \mathbb{A}^1 \rightarrow \mathbb{P}^1$ and q_j is the natural inclusion $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ given by $C' \mapsto [C' : 1]$, and pr_j is the natural map to \mathbb{A}^2 extracting (according to the triple (\tilde{w}_j, s_j, k_j)) C, C' or B, C' . This covers $(\tilde{w}_j, s_j, k_j) = (t_\eta, (12), > 1), (t_\eta, \text{id}, \geq 1), (w_0 t_\eta, (12), > 1), (w_0 t_\eta, \text{id}, \geq 1)$.
- (3) $Ba_j(\tilde{z}_j) \subset \mathbb{P}^1 \times \mathbb{A}^1 \times \mathbb{P}^1$ consists of $([x : y], C, [x' : y'])$ such that $pxy' - yx' = 0$. Then p_j, q_j are the projections to the left, resp. right \mathbb{P}^1 factor, and pr_j is the natural map to \mathbb{A}^1 extracting C . This covers the cases when $k_j = 0$.
- (4) $Ba_j(\tilde{z}_j) \subset \mathbb{P}^1 \times \text{Mat}_2 \times \mathbb{P}^1$ consists of $([x : y], \begin{pmatrix} A & B \\ C & D \end{pmatrix}, [x' : y'])$ such that

- $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ has determinant 0 and trace p ;
- $(x' \ y') \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} = 0$;
- $(x \ y) \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} = 0$.

Then p_j, q_j are the projections to the left, resp. right \mathbb{P}^1 factor, and pr_j is the map extracting A, B, C, D . This covers $(\tilde{w}_j, s_j, k_j) = (t_\eta, (12), 1), (w_0 t_\eta, (12), 1)$.

- (5) It is the same as the previous case, except that instead of the third item we impose $[x : y] = [x' : y']$. This correspond to the case $(\tilde{w}_j, s_j, k_j) = (t_{w_0(\eta)}, \text{id}, 1)$.

Lemma 4.2.3. (1) pr_j is proper and when $k_j > 0$, pr_j becomes a closed immersion after inverting p ;

- (2) $Ba_j(\tilde{z}_j)$ is a \mathcal{O} -flat local complete intersection of relative dimension 2 over \mathcal{O} . The relative *dualizing sheaf* of $Ba_j(\tilde{z}_j)/\mathcal{O}$ is $q_j^* \mathcal{O}_{\mathbb{P}^1}(-1) \otimes p_j^* \mathcal{O}_{\mathbb{P}^1}(-1)$, except when $(\tilde{w}_j, s_j, k_j) = (t_{w_0(\eta)}, \text{id}, 1)$ where it is isomorphic to the structure sheaf.

- (3) $\mathcal{O}_{\tilde{U}(\tilde{z}_j)} \rightarrow \text{pr}_{j*} \mathcal{O}_{Ba_j(\tilde{z}_j)}$ is surjective.

The same assertions hold for $\widetilde{Ba}_j(\tilde{z}_j)$ and $\widetilde{Ba}(\tilde{z})$, but with relative dimensions 6 and 6f.

Proof. The first assertion is clear from Table 3 once we observe that $(l_j \kappa_j, r_j)$ can be uniquely solved for when p is invertible and $k_j > 0$.

For the second assertion, we inspect the five cases in Lemma 4.2.2. The result is obvious for case (1) and follows from Lemma 4.2.4 below for case (2).

For case (3), the result follows from the fact that $Ba_j(\tilde{z}_j) \hookrightarrow \mathbb{P}^1 \times \mathbb{A}^1 \times \mathbb{P}^1$ is a regular immersion cut out by an equation of bidegree $(1, 1)$, which has normal bundle $q_j^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes p_j^* \mathcal{O}_{\mathbb{P}^1}(1)$. (Note that in this case the relative dualizing sheaf is $q_j^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes p_j^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes q_j^* \mathcal{O}_{\mathbb{P}^1}(-2) \otimes p_j^* \mathcal{O}_{\mathbb{P}^1}(-2) = q_j^* \mathcal{O}_{\mathbb{P}^1}(-1) \otimes p_j^* \mathcal{O}_{\mathbb{P}^1}(-1)$.)

We turn to case (4). Set $t = x/y, s = x'/y'$. The $Ba_j(\tilde{z}_j)$ has an open cover given by

- $\text{Spec } \mathcal{O}[A, B, C, D, s, t]/(A - stD, B - tD, C - sD, p - D - stD)$;
- $\text{Spec } \mathcal{O}[A, B, C, D, s^{-1}, t]/(A - tC, B - ts^{-1}C, D - s^{-1}C, p - s^{-1}C - tC)$;
- $\text{Spec } \mathcal{O}[AB, C, D, s, t^{-1}]/(A - sB, C - st^{-1}B, D - tB, p - t^{-1}B - sB)$;
- $\text{Spec } \mathcal{O}[A, B, C, D, s, t^{-1}]/(B - s^{-1}A, C - t^{-1}A, D - s^{-1}t^{-1}A, p - A - s^{-1}t^{-1}A)$.

Thus we obtain locally a regular immersion from $Ba_j(\tilde{z}_j)$ to \mathbb{A}^6 . A simple computation shows that the determinant of the normal bundle of the regular immersion $Ba_j(\tilde{z}_j) \hookrightarrow \mathbb{P}^1 \times \mathbb{A}^4 \times \mathbb{P}^1$ is $q_j^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes p_j^* \mathcal{O}_{\mathbb{P}^1}(1)$ (hence again the relative dualizing sheaf is $q_j^* \mathcal{O}_{\mathbb{P}^1}(-1) \otimes p_j^* \mathcal{O}_{\mathbb{P}^1}(-1)$).

We finally deal with case (5). Then $Ba_j(\tilde{z}_j) \hookrightarrow \mathbb{P}^1 \times \mathbb{A}^4$ is a regular immersion cut out by the equations

$$A + D = p, AD - BC = 0, (x \ y) \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} = 0.$$

A similar computation as in the previous case yields the desired result. (The fact that the dualizing sheaf in this case is the structure sheaf has been communicated to us by Dat Pham.)

The third assertion follows from the fact that pr_j factors through the subscheme $Z \subset \tilde{U}(\tilde{z}_j)$ such that

- Z is normal;
- $\text{pr}_{j*} \mathcal{O}_{Ba_j(\tilde{z}_j)} = \mathcal{O}_Z$ after inverting p .

The existence of such a subscheme Z follows immediately from inspecting the cases in Lemma 4.2.2: in fact $Z = \mathbb{A}^2$ with coordinate C, C' or $Z = \text{Spec } \mathcal{O}[B, C, D]/((p - D)D - BC)$ (if $k_j > 0$) or $Z = \mathbb{A}^1$ with coordinate C if $k_j = 0$. \square

We make use of the following elementary computation.

Lemma 4.2.4. *Let $Bl_{(0,p)}\mathbb{A}^1 = \{([x : y], t) \mid xp = ty\} \subset \mathbb{P}^1 \times \mathbb{A}^1$ be the blowup of $\mathbb{A}^1_{/\mathcal{O}}$ at the ideal $(0, p)$. Let $\mathcal{O}(-k)$ be the pull back of $\mathcal{O}_{\mathbb{P}^1}(-k)$ by the projection to \mathbb{P}^1 . Then*

(1) For $k \geq 0$

$$H^n(Bl_{(0,p)}\mathbb{A}^1, \mathcal{O}(-k)) = \begin{cases} p^k \mathcal{O}[t] & \text{if } n = 0; \\ \text{annihilated by } p^{k-1} & \text{if } n = 1; \\ 0 & \text{if } n > 1. \end{cases}$$

(2) The relative dualizing sheaf of $Bl_{(0,p)}\mathbb{A}^1/\mathcal{O}$ is $\mathcal{O}(-1)$.

Proof. (1) $R\Gamma(Bl_{(0,p)}\mathbb{A}^1, \mathcal{O}(-k))$ is computed by the Čech complex:

$$\mathcal{O}[t, \frac{t}{p}] \oplus \left(\frac{p}{t}\right)^k \mathcal{O}[t, \frac{p}{t}] \rightarrow \mathcal{O}[t, \left(\frac{p}{t}\right)^{\pm 1}]$$

where all the terms are viewed as $\mathcal{O}[t]$ -submodules of $E[t^{\pm 1}]$ and the differential is given by $(f, g) \mapsto f - g$. The result now follows from an explicit computation.

(2) Since $Bl_{(0,p)}\mathbb{A}^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{A}^1$ is a regular immersion cut out by a degree 1 equation in the \mathbb{P}^1 coordinates, the normal bundle is $\mathcal{O}(1)$, hence the dualizing complex is $\mathcal{O}(-2) \otimes \mathcal{O}(1) = \mathcal{O}(-1)$. \square

The following Lemma will be the key to our analysis:

Lemma 4.2.5. *Let $j \in \mathcal{J}$ and let $\varepsilon_j, \delta_j \in \{0, 1\}$.*

(1) $\text{pr}_{j*} \left(q_j^*(\mathcal{O}_{\mathbb{P}^1}(-1))^{\varepsilon_j} \otimes_{\mathcal{O}_{Ba(\tilde{z}_j)}} p_j^*(\mathcal{O}_{\mathbb{P}^1}(-1))^{\delta_j} \right)$ is p -torsion free.

(2) The complex $R\text{pr}_{j*} \left(q_j^*(\mathcal{O}_{\mathbb{P}^1}(-1))^{\varepsilon_j} \otimes_{\mathcal{O}_{Ba(\tilde{z}_j)}} p_j^*(\mathcal{O}_{\mathbb{P}^1}(-1))^{\delta_j} \right)$ is concentrated in degree 0 if $(\varepsilon_j, \delta_j) \neq (1, 1)$, and is concentrated in degrees 0 and 1 if $(\varepsilon_j, \delta_j) = (1, 1)$.

(3) If $(\varepsilon_j, \delta_j) = (1, 1)$ then $R^1\text{pr}_{j*} \left(q_j^*(\mathcal{O}_{\mathbb{P}^1}(-1))^{\varepsilon_j} \otimes_{\mathcal{O}_{Ba(\tilde{z}_j)}} p_j^*(\mathcal{O}_{\mathbb{P}^1}(-1))^{\delta_j} \right)$ is

$$\begin{cases} \mathcal{O}\text{-torsion free} & \text{if } \langle \mu_j, \alpha^\vee \rangle = 0; \\ \text{isomorphic to } \mathcal{O}/p & \text{if } \langle \mu_j, \alpha^\vee \rangle = 1 \text{ and } (s_j, \tilde{w}_j) = (\text{id}, t_{w_0(\eta)}); \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The first item is obvious because $Ba_j(\tilde{z}_j)$ is flat over \mathcal{O} by Lemma 4.2.32. We now explain the cohomological computations.

For the remainder of the proof we set $\varepsilon \stackrel{\text{def}}{=} \varepsilon_j$, $\delta \stackrel{\text{def}}{=} \delta_j$ and $\mathcal{F} \stackrel{\text{def}}{=} q_j^*(\mathcal{O}_{\mathbb{P}^1}(-1))^\varepsilon \otimes_{\mathcal{O}_{Ba(\tilde{z}_j)}} p_j^*(\mathcal{O}_{\mathbb{P}^1}(-1))^\delta$. We work with the five cases in Lemma 4.2.2.

The computation is trivial for case (1), and follows from Lemma 4.2.4 for case (2).

We now turn to case (4). We have an open cover $U_1 = \{y \neq 0\}$, $U_2 = \{x \neq 0\}$. Then setting $t = x/y$ we see that U_1 is the space of $(t, C, [x' : y']) \subset \mathbb{A}^2 \times \mathbb{P}^1$ such that $x'(p - Ct) - Cy' = 0$.

Hence U_1 is isomorphic to $Bl_{(p,0)}\mathbb{A}^1 \times \mathbb{A}^1$ and $\mathcal{F}|_{U_1}$ is the pull-back of $\mathcal{O}(-1)^\varepsilon$ from the obvious map to \mathbb{P}^1 . Thus, from Lemma 4.2.4(1), we see that

- $R\Gamma(U_1, \mathcal{F}) = \mathcal{O}[C, t]p^\varepsilon$,
- Similarly $R\Gamma(U_2, \mathcal{F}) = \mathcal{O}[B, t^{-1}]p^\varepsilon$,
- $R\Gamma(U_1 \cap U_2, \mathcal{F}) = \mathcal{O}[C, t, t^{-1}]p^\varepsilon$

are all concentrated in degree zero. Thus the Čech cohomology spectral sequence computing $R\Gamma(Ba_j(\tilde{z}_j), \mathcal{F})$ degenerates at E_1 and $R\Gamma(Ba_j(\tilde{z}_j), \mathcal{F})$ is computed by

$$(4.2.6) \quad \mathcal{O}[t, C] \oplus t^{-\delta}\mathcal{O}[t^{-1}, B] \xrightarrow{d} \mathcal{O}[t^{\pm 1}, C]$$

where the differential is the obvious inclusion induced by the relation $B = t(p - tC)$. In turn, this complex is quasi isomorphic to the complex

$$(4.2.7) \quad t^{-\delta}\mathcal{O}[t^{-1}, B] \xrightarrow{d} \mathcal{O}[t^{\pm 1}, C]/\mathcal{O}[t, C] = \bigoplus_{k \geq 1} t^{-k}\mathcal{O}[C].$$

Since $\delta \in \{0, 1\}$, $t^{-k} \in \text{Im}(d)$ for $k \geq 1$. Now for $\ell > 0$ and $k \geq 0$

$$(4.2.8) \quad t^{-2k-\ell}(t(p-tC))^k = (-1)^k \frac{C^k}{t^\ell} + \dots \in \text{Im}(d)$$

where \dots is an \mathcal{O} -linear combination of $\frac{C^n}{t^m}$ where $0 \leq n < k$. Hence $\frac{C^k}{t^\ell} \in \text{Im}(d)$ by induction on k .

We now deal with case (5). Similar to the previous case, we have an open cover $U_2 = \{y \neq 0\}$, $U_1 = \{x \neq 0\}$ and set $t = y/x$.

We have, after choosing a trivialization of \mathcal{F} on U_1

$$\begin{aligned} \mathcal{F}(U_1) &= \frac{\mathcal{O}[t, B, C, D]}{(D - tC, p - D - Bt)} \cong \mathcal{O}[t, C] \\ \mathcal{F}(U_2) &= t^{-(\varepsilon+\delta)} \frac{\mathcal{O}[t^{-1}, B, C, D]}{((p-D) - Bt^{-1}, C - Dt^{-1})} \cong t^{-(\varepsilon+\delta)} \mathcal{O}[t^{-1}, B]. \end{aligned}$$

Thus $R\Gamma(Ba_j(\tilde{z}_j), \mathcal{F})$ is computed by the Čech complex

$$(4.2.9) \quad \mathcal{O}[t, C] \oplus t^{-\varepsilon-\delta}\mathcal{O}[t^{-1}, B] \xrightarrow{d} \mathcal{O}[t^\pm, C]$$

where the differential is the obvious inclusion induced by the relation $B = t(p - tC)$. As before this complex is quasi isomorphic to

$$t^{-\varepsilon-\delta}\mathcal{O}[t^{-1}, B] \xrightarrow{d} \mathcal{O}[t^\pm, C]/\mathcal{O}[t, C] = \bigoplus_{k \geq 1} t^{-k}\mathcal{O}[C].$$

Compared to case (4) the only new computation we have to make is when $(\varepsilon, \delta) = (1, 1)$. In this case, we have $t^{-k} \in \text{Im}(d)$ for $k \geq 2$ but only $pt^{-1} \in \text{Im}(d)$ (and $t^{-1} \notin \text{Im}(d)$). Equation (4.2.8) for $k, \ell \geq 1$ then shows that $\frac{C^k}{t^\ell} \in \text{Im}(d)$ for all $k, \ell \geq 1$. This means that the H^1 of the Čech complex is isomorphic to \mathcal{O}/p .

Finally we deal with case (3). Consider the open cover $U_1 = \{x \neq 0\}$, $U_2 = \{y \neq 0\}$. Write $t = y/x$, $s = x'/y'$. Then $U_2 \cong \mathbb{A}^2 = \text{Spec}(\mathcal{O}[s, C, t^{-1}]/(s-pt^{-1}))$, while $U_1 \cong Bl_{(0,p)}\mathbb{A}^1 \times \mathbb{A}^1$, where the coordinate on the blowup \mathbb{A}^1 is t . Now $\mathcal{F}|_{U_1} \cong \mathcal{O}(-1)^\varepsilon$ so $R\Gamma(U_1, \mathcal{F}) = p^\varepsilon\mathcal{O}[C, t]$ is concentrated in degree 0. On the other hand, U_2 is affine so $R\Gamma(U_2, \mathcal{F}) = s^\varepsilon t^{-\delta}\mathcal{O}[s, C, t^{-1}]/(s-pt^{-1})$. Moreover $R\Gamma(U_1 \cap U_2, \mathcal{F}) = p^\varepsilon\mathcal{O}[s, C, t^\pm]/(s-pt^{-1})$. Thus the Čech complex computing $R\Gamma(Ba_j(\tilde{z}_j), \mathcal{F})$ is given by

$$p^\varepsilon\mathcal{O}[C, t] \oplus s^\varepsilon t^{-\delta}\mathcal{O}[s, C, t^{-1}]/(s-pt^{-1}) \rightarrow p^\varepsilon\mathcal{O}[s, C, t^\pm]/(s-pt^{-1}).$$

This is quasi-isomorphic to the complex

$$t^{-\varepsilon-\delta}\mathcal{O}[C,t^{-1}] \rightarrow \mathcal{O}[C,t^{\pm 1}]/\mathcal{O}[C,t]$$

with differential induced by the natural inclusion. This has no H^1 if $(\varepsilon, \delta) \neq (1, 1)$, and has $H^1 \cong \mathcal{O}[C]$ if $(\varepsilon, \delta) = (1, 1)$. We are thus done with the cohomological computations. This finishes the proof. \square

Corollary 4.2.10. *Consider the commutative diagram:*

$$\begin{array}{ccc} \tilde{Y}^{\text{mod}, \eta, \tau}(\tilde{z}) & \xrightarrow{\Delta} & \widetilde{Ba}(\tilde{z}) \\ & \searrow \text{pr} & \downarrow \text{pr}_{\tilde{B}} \\ & & \tilde{U}(\tilde{z}) \end{array}$$

Then $R^i \text{pr}_{\tilde{B}*} \mathcal{O}_{\widetilde{Ba}(\tilde{z})} = 0$ if $i > 0$ and $\mathcal{O}_{\tilde{U}(\tilde{z})} \rightarrow \text{pr}_{\tilde{B}*} \mathcal{O}_{\widetilde{Ba}(\tilde{z})}$.

In particular, by letting $\mathcal{I}(\tilde{z})$ be the ideal sheaf defining the closed immersion Δ

- $\text{coker} \left(\mathcal{O}_{\tilde{U}(\tilde{z})} \rightarrow \text{pr}_{\tilde{B}*} \mathcal{O}_{\tilde{Y}^{\text{mod}, (\eta, \tau)}(\tilde{z})} \right) = R^1 \text{pr}_{\tilde{B}*} \mathcal{I}(\tilde{z})$
- $R^i \text{pr}_{\tilde{B}*} \mathcal{O}_{\tilde{Y}^{\text{mod}, (\eta, \tau)}(\tilde{z})} = R^{i+1} \text{pr}_{\tilde{B}*} \mathcal{I}(\tilde{z})$ for $i \geq 1$.

Proof. The first part follows from the fact that $R\text{pr}_{\tilde{B}*} \mathcal{O}_{\widetilde{Ba}(\tilde{z})} = \boxtimes_j R\text{pr}_{j*} \mathcal{O}_{\widetilde{Ba}_j(\tilde{z}_j)}$ and hence is concentrated in degree 0 by Lemma 4.2.5 (with $\varepsilon_j = \delta_j = 0$).

For the second part, we have the exact triangle

$$R\text{pr}_{\tilde{B}*} \mathcal{I}(\tilde{z}) \rightarrow R\text{pr}_{\tilde{B}*} \mathcal{O}_{\widetilde{Ba}(\tilde{z})} \rightarrow R\text{pr}_{\tilde{B}*} \mathcal{O}_{\tilde{Y}^{\text{mod}, (\eta, \tau)}(\tilde{z})} \rightarrow$$

which immediately gives the second item. For the first item, observe that the exact triangle implies $\text{coker} \left(\text{pr}_{\tilde{B}*} \mathcal{O}_{\widetilde{Ba}(\tilde{z})} \rightarrow \text{pr}_{\tilde{B}*} \mathcal{O}_{\tilde{Y}^{\text{mod}, (\eta, \tau)}(\tilde{z})} \right) = R^1 \text{pr}_{\tilde{B}*} \mathcal{I}(\tilde{z})$ but also

$$\text{coker} \left(\text{pr}_{\tilde{B}*} \mathcal{O}_{\widetilde{Ba}(\tilde{z})} \rightarrow \text{pr}_{\tilde{B}*} \mathcal{O}_{\tilde{Y}^{\text{mod}, (\eta, \tau)}(\tilde{z})} \right) = \text{coker} \left(\mathcal{O}_{\tilde{U}(\tilde{z})} \rightarrow \text{pr}_{\tilde{B}*} \mathcal{O}_{\tilde{Y}^{\text{mod}, (\eta, \tau)}(\tilde{z})} \right)$$

because $\mathcal{O}_{\tilde{U}(\tilde{z})} \rightarrow \text{pr}_{\tilde{B}*} \mathcal{O}_{\widetilde{Ba}(\tilde{z})}$. \square

Lemma 4.2.11. *Under the composite*

$$\tilde{Y}^{\text{mod}, \eta, \tau}(\tilde{z}) \xrightarrow{\Delta} \widetilde{Ba}(\tilde{z}) = \prod_j \widetilde{Ba}_j(\tilde{z}_j) = \prod_j \text{GL}_2 \times Ba_j(\tilde{z}_j),$$

$\tilde{Y}^{\text{mod}, \eta, \tau}(\tilde{z})$ is a complete intersection defined by the zero locus of maps $\mathfrak{s}_j : \mathcal{L}_j \stackrel{\text{def}}{=} q_j^*(\mathcal{O}_{\mathbb{P}^1}(-1)) \otimes p_{j-1}^*(\mathcal{O}_{\mathbb{P}^1}(-1)) \rightarrow \mathcal{O}_{\prod_j \text{GL}_2 \times Ba_j(\tilde{z}_j)}$. In particular, the ideal sheaf $\mathcal{I}(\tilde{z})$ defining Δ identifies with the “stupid” truncation

$$\tau_{<0} \left(\text{Kos}_{\bullet} \left(\bigoplus_{j \in \mathcal{J}} \mathcal{L}_j, (\mathfrak{s}_j) \right) \right).$$

Proof. It follows from the definitions that we have a diagram

$$\begin{array}{ccccc}
\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z}) & \hookrightarrow & \prod_j \widetilde{Ba}_j(\tilde{z}_j) & \longrightarrow & \prod_j \text{GL}_2 \times Ba_j(\tilde{z}_j) \\
\downarrow & & \downarrow & & \downarrow \text{id} \times p_j \times q_j \\
\prod_{\mathcal{J}} \text{GL}_2 \times \mathbb{P}^1 & \hookrightarrow & \prod_{\mathcal{J}} \text{GL}_2 \times \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\cong} & \prod_{\mathcal{J}} \text{GL}_2 \times \mathbb{P}^1 \times \mathbb{P}^1
\end{array}$$

where the bottom left map is $(\kappa_j, l_j)_j \mapsto (\kappa_j, l_j, l_{j-1})_j$ and the bottom right map is $(\kappa_j, l_j, r_j) \mapsto (\kappa_j, l_j, r_j)$.

The bottom right isomorphism only commutes with projection to the right \mathbb{P}^1 factor, but not to the left \mathbb{P}^1 factor. Nevertheless, the GL_2 -equivariance of $\mathcal{O}_{\mathbb{P}^1}(-1)$ shows that its pullback via projection to the j -th left (respectively, j -th right) \mathbb{P}^1 are compatible with the bottom isomorphism.

It follows that $\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})$ is the zero common locus of f maps $\mathfrak{s}_j : \mathcal{L}_j \rightarrow \mathcal{O}_{\widetilde{Ba}(\tilde{z})}$. By Lemma 3.3.8 and Lemma 4.2.3, $\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z}) \hookrightarrow \widetilde{Ba}(\tilde{z})$ has codimension f and $\widetilde{Ba}(\tilde{z})$ is local complete intersection. This implies that $\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})$ is the global complete intersection in $\widetilde{Ba}(\tilde{z})$ cut out by the \mathfrak{s}_j . \square

Remark 4.2.12. We note that if $(\tilde{w}_j, s_j, k_j) \neq (t_{w_0(\eta)}, \text{id}, 1)$ for all $j \in \mathcal{J}$ then the relative dualizing sheaf of $\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})/\mathcal{O}$ is trivial. Indeed by Lemma 4.2.11, $\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z}) \hookrightarrow \widetilde{Ba}(\tilde{z})$ is a regular immersion with normal bundle

$$\bigoplus q_j^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes p_{j-1}^* \mathcal{O}_{\mathbb{P}^1}(1)$$

which thus has determinant $\bigotimes_{\mathcal{J}} q_j^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes p_{j-1}^* \mathcal{O}_{\mathbb{P}^1}(1)$ and the desired triviality follows now from Lemma 4.2.3(2).

We compute $R\Gamma(\mathcal{I}(\tilde{z}))$ using the resolution from Lemma 4.2.11 and the Künneth formula. Let $\ell \in \{1, \dots, f\}$ and consider f -tuples $\underline{\varepsilon} = (\varepsilon_j)_{j \in \mathcal{J}}, \underline{\delta} = (\delta_j)_{j \in \mathcal{J}} \in \{0, 1\}^{\mathcal{J}}$ satisfying $\delta_j = \varepsilon_{j+1}$ and $\#\{j \in \mathcal{J}, \varepsilon_j = 1\} = \ell$. Then by Lemma 4.2.11 we have

$$\bigwedge^{\ell} \left(\bigoplus_{j \in \mathcal{J}} \mathcal{L}_j \right) = \bigoplus_{\underline{\varepsilon}, \underline{\delta}} \bigotimes_{j \in \mathcal{J}, \mathcal{O}_{\widetilde{Ba}}} \left(q_j^* (\mathcal{O}_{\mathbb{P}^1}(-1))^{\varepsilon_j} \otimes p_j^* (\mathcal{O}_{\mathbb{P}^1}(-1))^{\delta_j} \right)$$

where the direct sum runs over f -tuples $\underline{\varepsilon}, \underline{\delta}$ as above. As the $\widetilde{Ba}(\tilde{z}_j)$ are Tor-independent over \mathcal{O} , we deduce from the Künneth formula [Sta18, Tag 0FLQ] that

$$\begin{aligned}
R\Gamma \left(\bigotimes_{j \in \mathcal{J}, \mathcal{O}_{\widetilde{Ba}}} \left(q_j^* (\mathcal{O}_{\mathbb{P}^1}(-1))^{\varepsilon_j} \otimes p_j^* (\mathcal{O}_{\mathbb{P}^1}(-1))^{\delta_j} \right) \right) \\
\cong \bigotimes_{j \in \mathcal{J}, \mathcal{O}}^{\mathbb{L}} R\Gamma \left(q_j^* (\mathcal{O}_{\mathbb{P}^1}(-1))^{\varepsilon_j} \otimes_{\mathcal{O}_{\widetilde{Ba}(\tilde{z}_j)}} p_j^* (\mathcal{O}_{\mathbb{P}^1}(-1))^{\delta_j} \right)
\end{aligned}$$

From Lemma 4.2.5 we obtain the following corollaries:

Corollary 4.2.13. \bullet For $0 < \ell < f$, $R^{\ell} \text{pr}_{\tilde{B}}^* \left(\bigwedge^{\ell} \left(\bigoplus_{j \in \mathcal{J}} \mathcal{L}_j \right) \right) = 0$.

- $pR^f \text{pr}_{\tilde{B}*} \left(\bigotimes_{j \in \mathcal{J}} \mathcal{L}_j \right) = 0$; and
- $R^f \text{pr}_{\tilde{B}*} \left(\bigotimes_{j \in \mathcal{J}} \mathcal{L}_j \right) \neq 0$ if and only if for each $j \in \mathcal{J}$, either $\langle \mu_j, \alpha^\vee \rangle = 0$ or $\langle \mu_j, \alpha^\vee \rangle = 1$ and $(s_j, \tilde{w}_j) = (\text{id}, t_{w_0(\eta)})$.

Proof. Since the image of pr is affine, it suffices to check the statements after taking global sections, so we can replace all occurrences of $R^j \text{pr}_{\tilde{B}*}$ with $R^j \Gamma$.

Given our above discussion, it suffices to analyze

$$\mathrm{H}^\ell \left(\bigotimes_{j \in \mathcal{J}, \mathcal{O}}^{\mathbb{L}} R\Gamma \left(q_j^*(\mathcal{O}_{\mathbf{P}^1}(-1))^{\varepsilon_j} \otimes_{\mathcal{O}_{\overline{Ba}(\tilde{z}_j)}} p_j^*(\mathcal{O}_{\mathbf{P}^1}(-1))^{\delta_j} \right) \right)$$

for f -tuples $\underline{\varepsilon} = (\varepsilon_j)_{j \in \mathcal{J}}, \underline{\delta} = (\delta_j)_{j \in \mathcal{J}} \in \{0, 1\}^{\mathcal{J}}$ satisfying $\varepsilon_j = \delta_{j+1}$ and $\#\{j \in \mathcal{J}, \varepsilon_j = 1\} = \ell$. Note that these conditions imply that there are at most ℓ indices j such that $(\varepsilon_j, \delta_j) = (1, 1)$, with strict inequality if $\ell < f$. Thus the amplitude bound of Lemma 4.2.5 shows the above cohomology group vanishes for $\ell < f$, while

$$\begin{aligned} \mathrm{H}^f \left(\bigotimes_{j \in \mathcal{J}, \mathcal{O}}^{\mathbb{L}} R\Gamma \left(q_j^*(\mathcal{O}_{\mathbf{P}^1}(-1))^{\varepsilon_j} \otimes_{\mathcal{O}_{\overline{Ba}(\tilde{z}_j)}} p_j^*(\mathcal{O}_{\mathbf{P}^1}(-1))^{\delta_j} \right) \right) &\cong \\ &\cong \bigotimes_{j \in \mathcal{J}, \mathcal{O}} R^1 \Gamma \left(q_j^*(\mathcal{O}_{\mathbf{P}^1}(-1)) \otimes_{\mathcal{O}_{\overline{Ba}(\tilde{z}_j)}} p_j^*(\mathcal{O}_{\mathbf{P}^1}(-1)) \right). \end{aligned}$$

The result now follows from Lemma 4.2.5. \square

We note that the proof of the above corollary also shows

Corollary 4.2.14. *For any $\ell \in \{0, \dots, f\}$ and $k > \ell$*

$$R^k \text{pr}_{\tilde{B}*} \left(\bigwedge^{\ell} \left(\bigoplus_{j \in \mathcal{J}} \mathcal{L}_j \right) \right) = 0.$$

Proof of Propositions 3.3.16, 3.3.18. By Lemma 4.2.11 $R\text{pr}_* \mathcal{I}(\tilde{z})$ is filtered (in the derived sense) by $R\text{pr}_{\tilde{B}*} \left(\bigwedge^{\ell} \left(\bigoplus_{j \in \mathcal{J}} \mathcal{L}_j \right) \right) [\ell - 1]$ for $1 \leq \ell \leq f$. Then by Corollary 4.2.10, Proposition 3.3.16 follows from Corollary 4.2.13 and Proposition 3.3.18 from Corollary 4.2.14. \square

We also record the following, which will be used in subsection 4.6.2.

4.3. The naive models. Recall that $\tilde{\mathcal{Z}}^{\text{mod}, \tau}(\tilde{z})$ is the scheme theoretic image of $\tilde{Y}^{\text{mod}, \eta, \tau}(\tilde{z})$ under the map π^{mod} which forgets the elements $r_j = l_{j-1} \in \mathbb{P}^1$. In other words, the equations for $\tilde{\mathcal{Z}}^{\text{mod}, \tau}(\tilde{z})$ are obtained by eliminating the $r_j = l_{j-1}$ from the defining equations of $\tilde{Y}^{\text{mod}, \eta, \tau}(\tilde{z})$ which are extracted from Table 3. The goal of this section is to construct a slight enlargement $\tilde{\mathcal{Z}}^{\text{nv}, \tau}(\tilde{z})$ of $\tilde{\mathcal{Z}}^{\text{mod}, \tau}(\tilde{z})$, which has the advantage of being given by explicit equations.

We first introduce some auxiliary notation. We view $\mathcal{J} = \mathbb{Z}/f\mathbb{Z}$ as an oriented graph with edges going from j to $j - 1$.

Definition 4.3.1. Given the data (s, μ, \tilde{w}) , and letting $\tilde{z} \stackrel{\text{def}}{=} \tilde{w}s^{-1}v\hat{\mu}$:

- (1) Define $M_j(\tilde{z}_j)$ to be the scheme theoretic image of $Ba_j(\tilde{z}_j) \rightarrow \tilde{U}(\tilde{z}_j)$.
- (2) Let $j \in \mathcal{J}$. We say
 - j is of type *II* if either $k_j > 1$ or $(\tilde{w}_j, s_j, k_j) = (t_\eta, \text{id}, 1), (w_0 t_\eta, \text{id}, 1), (t_{w_0(\eta)}, (12), 1)$;
 - j is of type *I* if $(\tilde{w}_j, s_j, k_j) = (t_\eta, (12), 1), (w_0 t_\eta, (12), 1), (t_{w_0(\eta)}, \text{id}, 1)$;
 - j is of type 0 if $k_j = 0$.
- (3) A *fragmentation* of \mathcal{J} is the decomposition $\mathcal{J} = \bigcup_{k \in \mathcal{K}} \mathcal{J}_k$ into subsets \mathcal{J}_k such that:
 - \mathcal{J}_k is an oriented path in \mathcal{J} , i.e an ordered subset of the form $\{j, j-1, \dots, j-\ell\}$ for some $\ell \leq f$.
 - The endpoints of the path \mathcal{J}_k are not of type 0.
 - The interior points of the the path \mathcal{J}_k are of type 0.
 - \mathcal{J}_k is not a singleton unless $f = 1$.
Each \mathcal{J}_k is called a fragment of \mathcal{J} .

- Remark 4.3.2.* (1) The scheme theoretic image of $\widetilde{Ba}_j(\tilde{z}_j) \rightarrow \tilde{U}(\tilde{z}_j)$ is $\text{GL}_2 \times M_j(\tilde{z}_j)$.
- (2) Under our running assumption that τ is regular, \mathcal{J} must have a vertex not of type 0. This implies \mathcal{J} has a unique fragmentation, obtained by the minimal paths joining the vertices not of type 0. We also note that each fragment \mathcal{J}_k has a well-defined starting point and ending point (which may coincide, in which case the fragment is all of \mathcal{J}).
 - (3) It follows from Table 3 that
 - If j is type *II*: $M_j(\tilde{z}_j) \cong \mathbb{A}^2$.
 - If j is type *I*: $M_j(\tilde{z}_j) \cong M$, the space of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ which has determinant 0 and trace p .
 - If j is type 0: $M_j(\tilde{z}_j) \cong \mathbb{A}^1$.

We now define a subspace of $\prod_{\mathcal{J}} \text{GL}_2 \times M_j(\tilde{z}_j)$ using the fragmentation $\mathcal{J} = \bigcup \mathcal{J}_k$.

Definition 4.3.3. Let $\mathcal{J} = \bigcup \mathcal{J}_k$ be the fragmentation of \mathcal{J} . We define $\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})$ to be the closed subscheme of $\prod_{\mathcal{J}} \text{GL}_2 \times M_j(\tilde{z}_j) \hookrightarrow \text{GL}_2^{\mathcal{J}} \times \tilde{U}(\tilde{z})$ cut out by the matrix equations

$$M_{\text{out},o} \left(\prod_{\ell=o+1}^{i-1} T_\ell \right) M_{\text{in},i} = 0$$

for each fragment $\mathcal{J}_k = \{i, \dots, o\}$, where

- $M_{\text{in},i}$ is the initial matrix for $j = i$ in Table 4
- $M_{\text{out},o}$ is the final matrix for $j = o$ in Table 4
- T_ℓ is the transition matrix for $j = \ell$ (which are of type 0) in Table 4.

We also define $\mathcal{Z}^{\text{nv},\tau}(\tilde{z})$ and $\mathcal{Z}^{\text{mod},\tau}(\tilde{z})$ to be the fiber of $\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})$ and $\tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z})$ respectively, above $1 \in \text{GL}_2^{\mathcal{J}}$.

- Proposition 4.3.4.** (1) *The inclusion $\tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z}) \hookrightarrow \tilde{U}(\tilde{z})$ factors through $\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})$.*
- (2) *We have $\tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z})[\frac{1}{p}] = \tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})[\frac{1}{p}] = \tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})[\frac{1}{p}]$.*
- In particular, $\tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z})$ is the p -saturation (synonymously, the \mathcal{O} -flat part) of $\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})$.*

Proof. The first assertion follows from the fact that $\tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z})$ obeys the defining equations of $\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})$, which is a consequence of the relations in Table 3 and the relation $r_j = l_{j-1}$ in $\tilde{Y}^{\text{mod},\eta,\tau}$. Indeed, these defining equations were obtained by repeatedly substituting the relations between l_j and r_j when j is type 0 and $r_j = l_{j-1}$ until it becomes a relation between r_a and l_b where a, b are

TABLE 4.

$\langle \mu_j, \alpha^\vee \rangle$	\tilde{w}_j	t_η	$w_0 t_\eta$	$t_{w_0(\eta)}$
	s_j			
> 1	(12)	Type II Initial Matrix: $\begin{pmatrix} 1 \\ -C' \end{pmatrix}$ Final Matrix: $(B \ -p) \kappa_j^{-1}$	Type II Initial Matrix: $\begin{pmatrix} 1 \\ -C' \end{pmatrix}$ Final Matrix: $(-p \ B) \kappa_j^{-1}$	Type II Initial Matrix: $\begin{pmatrix} 1 \\ -C' \end{pmatrix}$ Final Matrix: $(0 \ 1) \kappa_j^{-1}$
	id	Type II Initial Matrix: $\begin{pmatrix} 1 \\ -C' \end{pmatrix}$ Final Matrix: $(C \ -p) \kappa_j^{-1}$	Type II Initial Matrix: $\begin{pmatrix} 1 \\ -C' \end{pmatrix}$ Final Matrix: $(-p \ C) \kappa_j^{-1}$	Type II Initial Matrix: $\begin{pmatrix} 1 \\ -C' \end{pmatrix}$ Final Matrix: $(0 \ 1) \kappa_j^{-1}$
$= 1$	(12)	Type I Initial Matrix: $\begin{pmatrix} D & -B \\ -C & p-D \end{pmatrix}$ Final Matrix: $\begin{pmatrix} p-D & -C \\ -B & D \end{pmatrix} \kappa_j^{-1}$	Type I Initial Matrix: $\begin{pmatrix} D & -B \\ -C & p-D \end{pmatrix}$ Final Matrix: $\begin{pmatrix} D & -B \\ -C & p-D \end{pmatrix} \kappa_j^{-1}$	Type II Initial Matrix: $\begin{pmatrix} 1 \\ -C' \end{pmatrix}$ Final Matrix: $(0 \ 1) \kappa_j^{-1}$
	id	Type II Initial Matrix: $\begin{pmatrix} 1 \\ -C' \end{pmatrix}$ Final Matrix: $(C \ -p) \kappa_j^{-1}$	Type II Initial Matrix: $\begin{pmatrix} 1 \\ -C' \end{pmatrix}$ Final Matrix: $(-p \ C) \kappa_j^{-1}$	Type I Initial Matrix: $\begin{pmatrix} D & -B \\ -C & p-D \end{pmatrix}$ Final Matrix: $\begin{pmatrix} p-D & B \\ C & D \end{pmatrix} \kappa_j^{-1}$
$= 0$	id	Type 0 Transition Matrix: $\begin{pmatrix} 1 & 0 \\ -C & p \end{pmatrix} \kappa_j^{-1}$	Type 0 Transition Matrix: $\begin{pmatrix} 0 & -1 \\ -p & C \end{pmatrix} \kappa_j^{-1}$	Type 0 Transition Matrix: $\begin{pmatrix} p & -B \\ 0 & 1 \end{pmatrix} \kappa_j^{-1}$

The meaning of the variables in this table are the same as that of the corresponding entry in Table 3.

not type 0, in which case one substitutes for r_a, l_b an expression in the variables on $\mathrm{GL}_2 \times M_a(\tilde{z}_a), \mathrm{GL}_2 \times M_b(\tilde{z}_b)$.

We now establish the second assertion. First, we show that the map $\tilde{Y}^{\mathrm{mod}, \eta, \tau}(\tilde{z}) \rightarrow \tilde{U}(\tilde{z})$ is a closed immersion after inverting p , i.e. we need to show r_j, l_j are determined by the remaining variables. For each j not of type 0, we can solve for r_j, l_j when p is invertible. Using the relation $r_j = l_{j-1}$ and the relations in Table 3, we can solve for the $r_{j'}, l_{j'}$ where j' is of type 0. Thus $\tilde{Z}^{\mathrm{mod}, \tau}(\tilde{z})[\frac{1}{p}] = \tilde{Y}^{\mathrm{mod}, \eta, \tau}(\tilde{z})[\frac{1}{p}]$.

To finish the proof, we need to show that $\tilde{Y}^{\mathrm{mod}, \eta, \tau}(\tilde{z})[\frac{1}{p}]$ surjects onto $\tilde{Z}^{\mathrm{nv}, \tau}(\tilde{z})[\frac{1}{p}]$, i.e. we need to produce r_j, l_j satisfying all requisite relations. We use the same procedure to define r_j, l_j as in the previous paragraph. The only potential issue is that for a fragment $\mathcal{J}_k = \{i, \dots, o\}$, the procedure gives two definitions of r_i : one by recursion in terms of l_o (and some variables at $j \in \{i-1, \dots, o+1\}$), the other by directly solving in terms of the variables at i . However the defining equations of $\tilde{Z}^{\mathrm{nv}, \tau}(\tilde{z})$ exactly guarantee that these two definitions coincide. \square

4.4. Obstruction bounds for naive models. We wish to establish a quantitative bound for the singular ideal of $\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})/\mathcal{O}$. This will be used in the next subsection to finish the proof of Theorem 3.3.11.

To save notation, in this section we abbreviate $\mathcal{Z} = \tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})$, $M_j = M_j(\tilde{z}_j)$ and $\mathcal{M} = \prod_{\mathcal{J}} \text{GL}_2 \times M_j$. We also define affine spaces \mathcal{A}_j with closed immersions $M_j \hookrightarrow \mathcal{A}_j$ as follows:

- $\mathcal{A}_j = M_j$ if j is not type I .
- If j is type I , then M_j is the space M of matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with determinant 0 and trace p , and we define $\mathcal{A}_j = \mathbb{A}^3$ and $M_j \hookrightarrow \mathcal{A}_j$ to be the map

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto (B, C, D)$$

so that M_j identifies with the hypersurface $D(p - D) = BC$ in \mathcal{A}_j .

We thus have an inclusion $\iota : \mathcal{Z} \hookrightarrow \mathcal{A} \stackrel{\text{def}}{=} \prod_{\mathcal{J}} \text{GL}_2 \times \mathcal{A}_j$. Let \mathcal{I} be the ideal of \mathcal{A} defining ι .

We first analyze a generating set of \mathcal{I} . To do this, recall the fragmentation $\mathcal{J} = \bigcup_{k \in \mathcal{K}} \mathcal{J}_k$. We also denote by $\mathcal{J}_I \subset \mathcal{J}$ the subset of j that are type of I . For each j of type I , we also let A_j, B_j, C_j, D_j denote the natural coordinates on M_j , so that $A_j + D_j = p$ and $A_j D_j = B_j C_j$.

It follows from Definition 4.3.3 that we have a decomposition

$$(4.4.1) \quad \mathcal{I} = \sum_k \mathcal{I}_{\mathcal{J}_k} + \sum_{j \in \mathcal{J}_I} (D_j(p - D_j) - B_j C_j)$$

where $\mathcal{I}_{\mathcal{J}_k}$ is the ideal generated by the entries of the matrix equation associated to \mathcal{J}_k in Definition 4.3.3. This gives a presentation of \mathcal{Z} in terms of \mathcal{A} . Note that $\mathcal{Z}[\frac{1}{p}]$ has codimension $c \stackrel{\text{def}}{=} |\mathcal{K}| + |\mathcal{J}_I|$ in $\mathcal{A}[\frac{1}{p}]$.

We write J for the Jacobian matrix of our chosen presentation of \mathcal{Z} . Our convention is that the columns are named by the variables and rows are labeled by (the chosen) generators of \mathcal{I} . We denote by H_c the ideal generated by

$$H_c = \sum_{\alpha} K_{(\alpha)} \Delta_{(\alpha)}$$

where

- α runs over choices of subsets of c rows of J , or equivalently a choice of c elements among our generating set for \mathcal{I} .
- $\Delta_{(\alpha)}$ is the ideal generated by the $c \times c$ minor of J whose rows belong α .
- $K_{(\alpha)}$ is the conductor ideal $\{x \in \mathcal{A} \mid x\mathcal{I} \subset \mathcal{I}_{(\alpha)}\}$ where $\mathcal{I}_{(\alpha)} \subset \mathcal{I}$ is the ideal generated by the subset of \mathcal{I} corresponding to the rows of α .

Note that H_c is contained in the ideal denoted by H_B in [Elk73, §0.2] corresponding to our fixed presentation.

Proposition 4.4.2. $p^{7|\mathcal{J}_I|+f} \in H_c$. In particular $p^{8f} \in H_c$.

Remark 4.4.3. Proposition 4.4.2 implies $\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})[\frac{1}{p}]$ is smooth over E . Of course this can also be seen directly from the explicit description of $\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})[\frac{1}{p}]$.

We observe the following structural properties of $\mathcal{I}_{\mathcal{J}_k}$:

Lemma 4.4.4. *Suppose the fragment $\mathcal{J}_k = (i, \dots, o)$. Let $\kappa_o = \begin{pmatrix} a_o & b_o \\ c_o & d_o \end{pmatrix}$ record the o -th GL_2 factor of \mathcal{A} .*

There exists $\alpha, \beta, \gamma, \delta, u$ only involving coordinates on the $(i-1)$ -th to $(o+1)$ -th factor of \mathcal{A} with u invertible, such that

$$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = up^{|\mathcal{J}_k \setminus \{i, o\}|}$$

such that:

(1) *If i, o are type II then $\mathcal{I}_{\mathcal{J}_k}$ is principal, generated by an element of the form*

$$(Y_o \quad -p) \kappa_o^{-1} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 \\ -X_i \end{pmatrix}$$

where Y_o, X_i are coordinates of $\mathcal{A}_o, \mathcal{A}_i$.

(2) *If i, o are type I then $\mathcal{I}_{\mathcal{J}_k}$ is generated by the entries of a matrix either of the form*

$$\begin{pmatrix} D_o & -B_o \\ -C_o & A_o \end{pmatrix} \kappa_o^{-1} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} D_i & -B_i \\ -C_i & A_i \end{pmatrix};$$

or

$$\begin{pmatrix} D_o & C_o \\ B_o & A_o \end{pmatrix} \kappa_o^{-1} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} D_i & -B_i \\ -C_i & A_i \end{pmatrix}.$$

(3) *If o is type II and i is type I then $\mathcal{I}_{\mathcal{J}_k}$ is generated by the entries of a matrix of the form*

$$(-p \quad Y_o) \kappa_o^{-1} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} D_i & -B_i \\ -C_i & A_i \end{pmatrix}$$

where Y_o is a coordinate in \mathcal{A}_o .

(4) *If o is type I and i is type II then $\mathcal{I}_{\mathcal{J}_k}$ is generated by the entries of a matrix either of the form*

$$\begin{pmatrix} D_o & -B_o \\ -C_o & A_o \end{pmatrix} \kappa_o^{-1} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 \\ -X_i \end{pmatrix},$$

or

$$\begin{pmatrix} D_o & C_o \\ B_o & A_o \end{pmatrix} \kappa_o^{-1} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 \\ -X_i \end{pmatrix}$$

where X_i is a coordinate on \mathcal{A}_i .

Proof. The form of $\mathcal{I}_{\mathcal{J}_k}$ follows from Table 4, after possibly rearranging the entries of the matrix equations. □

Proof of Proposition 4.4.2. We will show that for any choices $G_j, H_j \in \{A_j, D_j\}$ with $j \in \mathcal{J}_I$, the element $p^{f+|\mathcal{J}_I|} \prod_{j \in \mathcal{J}_I} G_j^2 H_j^2$ belongs to \mathbf{H}_c . This finishes the proof, since this implies \mathbf{H}_c contains

$$p^{f+|\mathcal{J}_I|} \prod_{j \in \mathcal{J}_I} (A_j^2, D_j^2)^2 \ni p^{f+7|\mathcal{J}_I|}.$$

Since the role of A_j, D_j is essentially symmetric in our argument, we will deal with the case $G_j = H_j = D_j$ for all $j \in \mathcal{J}$. We choose α be the subset consist of c generators of \mathcal{I} given by:

- For $k \in \mathcal{K}$ giving the fragment $\mathcal{J}_k = \{i, \dots, o\}$, α contains g_k , the generator of $\mathcal{I}_{\mathcal{J}_k}$ that correspond to the $(1,1)$ -th entry of the matrix equation described in 4.4.4 (this choice correspond to our choice of $G_o = D_o$ when $o \in \mathcal{J}_I$ and $H_i = D_i$ when $i \in \mathcal{J}_I$).
- α contains the generator $(p - D_j)D_j - B_j C_j$ for $j \in \mathcal{J}_I$.

We claim that with this choice of α

- (1) $K_{(\alpha)} \ni \prod_{j \in \mathcal{J}_I} G_j H_j$; and
- (2) $\Delta_{(\alpha)} \ni p^{f+|\mathcal{J}_I|} \prod_{j \in \mathcal{J}_I} G_j H_j$

which will finish the proof since $K_{(\alpha)} \Delta_{(\alpha)} \in \mathbb{H}_c$.

For each $k \in \mathcal{K}$, giving the fragment $\mathcal{J}_k = \{i, \dots, o\}$, we write the matrix whose entries generate $\mathcal{I}_{\mathcal{J}_k}$ in factorized form

$$M_k = X \begin{pmatrix} a_o & b_o \\ c_o & d_o \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} Y$$

as prescribed by Lemma 4.4.4. Note that the generators of \mathcal{I} not in α are exactly the entries $(M_k)_{rs}$ with $(r, s) \neq (1, 1)$. Since X is annihilated by left multiplication by either $(C_o \ D_o)$ or $(-B_o \ D_o)$ if $o \in \mathcal{J}_I$; and Y is annihilated by right multiplication by $\begin{pmatrix} B_i \\ D_i \end{pmatrix}$ if $i \in \mathcal{J}_I$, we see that any entry of M_k becomes divisible by $g_k = (M_k)_{11}$ upon multiplication by one of $1, D_o, D_i$, or $D_o D_i$ depending on the set $\{i, o\} \cap \mathcal{J}_I$. This implies the first of the above claims:

$$K_{(\alpha)} \ni \prod_{j \in \mathcal{J}_I} G_j H_j.$$

It remains to check the claim about $\Delta_{(\alpha)}$. Note that $\Delta_{(\alpha)}$ contains the $c \times c$ minors with row set α of any matrix of the form JU . Thus, it suffices to find, after modifying J by column operations, a $c \times c$ minor with row set α dividing $p^{f+|\mathcal{J}_I|} \prod_{j \in \mathcal{J}_I} G_j H_j$.

The entries of \mathcal{J}_k corresponding to row g_k and column a_o, b_o, c_o, d_o are exactly the $(1, 1)$ -th entry of the matrices

$$X \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} Y, \quad X \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} Y, \quad X \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} Y, \quad X \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} Y$$

respectively. Furthermore, no other row of J has a non-zero entry at the columns a_o, b_o, c_o, d_o . Hence, given $x, y, z, t \in \mathcal{O}(\mathcal{A})$, by modifying the columns J to JU where we take linear combinations of columns a_o, b_o, c_o, d_o but do not modify the remaining columns, we can guarantee that the resulting matrix has a column which has vanishing entry for any row other than g_k , and at row g_k the entry is the $(1, 1)$ -th entry of

$$X \begin{pmatrix} x & y \\ z & t \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} Y.$$

Now $\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = up^{|\mathcal{J}_k \setminus \{i, o\}|}$ with u a unit, so choosing x, y, z, t appropriately, we can make $\begin{pmatrix} x & y \\ z & t \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ become any matrix among

$$p^{|\mathcal{J}_k \setminus \{i, o\}|} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Using the explicit form of X, Y given in Lemma 4.4.4, we thus learn that after taking a linear combination of columns a_o, b_o, c_o, d_o , we can make the entry in row g_k become $p^{|\mathcal{J}_k \setminus \{i, o\}|} p, p^{|\mathcal{J}_k \setminus \{i, o\}|} D_o D_i, p^{|\mathcal{J}_k \setminus \{i, o\}|} p D_i$ or $p^{|\mathcal{J}_k \setminus \{i, o\}|} D_o$ (corresponding to the four cases in that Lemma, respectively).

Finally, we consider $j \in \mathcal{J}_I$, giving a generator $(p - D_j)D_j - B_j C_j$ belonging to α . The part of J corresponding to row $(p - D_j)D_j - B_j C_j$ and column B_j, C_j, D_j is

$$\begin{pmatrix} -C_j & -B_j & p - 2D_j \end{pmatrix}.$$

Since

$$\begin{pmatrix} -C_j & -B_j & p-2D_j \end{pmatrix} \begin{pmatrix} -4B_j \\ 0 \\ p-2D_j \end{pmatrix} = (p-2D_j)^2 + 4B_jC_j = p^2$$

we see that taking a linear combination of column B_j, C_j, D_j produces the entry p^2 in row $(p-D_j)D_j - B_jC_j$. Moreover that note that the entries of this linear combination at rows $(p-D_{j'})D_{j'} - B_{j'}C_{j'}$ are 0, where $j' \in \mathcal{J}_I$ but $j' \neq j$ (note however that we have no control on the entries on the remaining rows).

To summarize, by taking appropriate linear combination among columns a_o, b_o, c_o, d_o (for each $\mathcal{J}_k = \{i, \dots, o\}$) and among columns B_j, C_j, D_j (for each $j \in \mathcal{J}_I$), we can find a $c \times c$ submatrix with row set α which has block triangular form

$$\begin{pmatrix} P & * \\ 0 & Q \end{pmatrix}$$

where

- P is diagonal of size $|\mathcal{K}| \times |\mathcal{K}|$ whose entries belong to $\{p^{|\mathcal{J}_k \setminus \{i, o\}|}, p^{|\mathcal{J}_k \setminus \{i, o\}|} D_o D_i, p^{|\mathcal{J}_k \setminus \{i, o\}|} p D_i, p^{|\mathcal{J}_k \setminus \{i, o\}|} D_o\}$;
- Q is p^2 the identity matrix of size $|\mathcal{J}_I|$.

The determinant of this $c \times c$ matrix clearly divides

$$p^{f+|\mathcal{J}_I|} \prod_{j \in \mathcal{J}_I} G_j H_j.$$

□

4.5. Proof of Theorem 3.3.11. Set $N = p - 2 - \max_j \langle \mu_j, \alpha^\vee \rangle \geq \frac{p-5}{2}$ and $\delta = \max\{0, |\{j \mid \langle \mu_j, \alpha^\vee \rangle \leq 1\}| - f + 1\} \leq 1$.

Lemma 4.5.1. *Assume $p > \max\{16f + 3, 16f - 12 + \max_j \langle \mu_j, \alpha^\vee \rangle\}$. Then $N - \delta > 2f + 14|\mathcal{J}_I|$*

Proof. Suppose $\delta = 1$. Then

$$N - \delta = p - 3 > 16f \geq 2f + 14|\mathcal{J}_I|.$$

Otherwise, $\delta = 0$ and hence $|\mathcal{J}_I| < f$. Then

$$N - \delta > 16f - 14 \geq 2f + 14|\mathcal{J}_I|.$$

□

Recall from Proposition 3.3.14 that we have a diagram

$$\begin{array}{ccc} \tilde{\mathcal{Z}}^\tau \otimes_{\mathcal{O}} \mathcal{O}/p^{N-\delta} & \xrightarrow{\sim} & \tilde{\mathcal{Z}}^{\text{mod}, \tau} \otimes_{\mathcal{O}} \mathcal{O}/p^{N-\delta} \\ \downarrow & & \downarrow \\ \left[LG^{\text{bd}, (v+p)v^{\hat{\mu}}} / \varphi \prod_{\mathcal{J}} L_1^+ G \right] \otimes_{\mathcal{O}} \mathcal{O}/p^{N-\delta} & \xrightarrow{\sim} & \text{Gr}_1^{\text{bd}, (v+p)v^{\hat{\mu}}} \otimes_{\mathcal{O}} \mathcal{O}/p^{N-\delta} \end{array}$$

In particular, from Proposition 4.3.4 we get a closed immersion

$$\iota : \tilde{\mathcal{Z}}^\tau(\tilde{z}) \otimes_{\mathcal{O}} \mathcal{O}/p^{N-\delta} \xrightarrow{\sim} \tilde{\mathcal{Z}}^{\text{mod}, \tau}(\tilde{z}) \otimes_{\mathcal{O}} \mathcal{O}/p^{N-\delta} \hookrightarrow \tilde{\mathcal{Z}}^{\text{nv}, \tau}(\tilde{z}) \otimes_{\mathcal{O}} \mathcal{O}/p^{N-\delta}.$$

We now invoke [Elk73, Lemme 1] as in the proof of [LLHLM23, Proposition 3.3.9]: By Proposition 4.4.2, the integer h in *loc.cit.* can taken to be $f + 7|\mathcal{J}_I|$ while the integer k is 0 since $\tilde{\mathcal{Z}}^\tau(\tilde{z})$ is p -torsion free. By Lemma 4.5.1, $N - \delta > 2f + 14|\mathcal{J}_I|$, so we can produce a map $\tilde{\iota} : \tilde{\mathcal{Z}}^\tau(\tilde{z}) \rightarrow \tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})^{\wedge p}$ which agrees with ι modulo $p^{N-\delta-f-7|\mathcal{J}_I|}$. In particular, this implies $\tilde{\iota}$ is also a closed immersion. Since $\tilde{\mathcal{Z}}^\tau(\tilde{z})$ is \mathcal{O} -flat and the \mathcal{O} -flat part of $\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})^{\wedge p}$ is exactly $\tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z})$, we can factorize $\tilde{\iota}$

$$\tilde{\mathcal{Z}}^\tau(\tilde{z}) \hookrightarrow \tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z}) \hookrightarrow \tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})^{\wedge p}$$

such that the first map is an isomorphism modulo $p^{N-\delta-f-7|\mathcal{J}_I|}$. The following Lemma then implies that the inclusion $\tilde{\mathcal{Z}}^\tau(\tilde{z}) \hookrightarrow \tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z})$ is in fact an isomorphism, thus finishing the proof of Theorem 3.3.11.

Lemma 4.5.2. *Suppose we are given a surjection $\pi : R \rightarrow S$ of Noetherian p -adically complete \mathcal{O} -algebras, where S is \mathcal{O} -flat. Assume that π induces an isomorphism $\pi : R/\varpi \cong S/\varpi$. Then π is an isomorphism.*

Proof. Let $I = \ker \pi$, then since S is \mathcal{O} -flat we get a short exact sequence

$$0 \longrightarrow I/\varpi I \longrightarrow R/\varpi \longrightarrow S/\varpi \longrightarrow 0$$

Since π induces an isomorphism modulo ϖ , we learn that $I/\varpi I = 0$. But I is p -adically separated, so $I = 0$. \square

The following is immediate from our discussion

Corollary 4.5.3. *Assume that either $p > \max\{16f + 3, 16f - 12 + \max_j \langle \mu_j, \alpha^\vee \rangle\}$ or $p \geq 17$ and $K = \mathbb{Q}_p$. Then $\tilde{\mathcal{Z}}^\tau(\tilde{z})$ is isomorphic to the p -adic completion of the p -saturation of $\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})$.*

In particular, this realizes $\tilde{\mathcal{Z}}^\tau(\tilde{z})$ as the p -saturation of an explicitly presented affine p -adic formal scheme.

4.6. Applications.

4.6.1. *Galois deformation rings.* Recall that we have a shifted conjugation action map

$$\text{GL}_2^{\mathcal{J}} \times \text{Gr}_1^{\mathcal{J}} \rightarrow \text{Gr}_1^{\mathcal{J}}$$

given by the formula

$$(g_j, A_j)_{j \in \mathcal{J}} \mapsto (g_j A_j g_{j-1}^{-1})_{j \in \mathcal{J}}.$$

This action of $\text{GL}_2^{\mathcal{J}}$ clearly factors through the quotient $\text{GL}_2^{\mathcal{J}}/\Delta Z$ by the diagonally embedded copy of the center Z of GL_2 . Furthermore since

$$\prod_{j \in \mathcal{J}} \det g_j A_j g_{j-1}^{-1} = \prod_{j \in \mathcal{J}} \det A_j$$

we see that the quantity $\prod \det A_j$ modulo $\det L_1^+ G$ is invariant along orbits.

Lemma 4.6.1. *Let $\tilde{z} = (\tilde{z}_j) = (z_j v^{\nu_j}) \in \widetilde{W}^\vee$ such that $\langle \nu_j, \alpha^\vee \rangle \neq 0$ for some j .*

(1) *Suppose $\prod z_j = (12)$. Let $\bar{x} = ((1, \dots, 1), (\tilde{z}_j)) \in \text{GL}_2^{\mathcal{J}}(\mathbb{F}) \times \prod_{\mathcal{J}} L_1^- G(\mathbb{F}) \tilde{z}_j$ and \bar{y} its image under the shifted conjugation action. The shifted conjugation action map induces an isomorphism on completions at \bar{x} :*

$$\left(\text{GL}_2^{\mathcal{J}}/\Delta Z \times \prod_{\mathcal{J}} L_1^- G \tilde{z}_j \right)_{\bar{x}}^{\wedge} \cong \left(\text{GL}_2^{\mathcal{J}, \det=1} \times \prod_{\mathcal{J}} L_1^- G \tilde{z}_j \right)_{\bar{y}}^{\wedge}$$

(Here $\mathrm{GL}_2^{\mathcal{J}, \det=1}$ denotes the kernel of the product of determinant map $\mathrm{GL}_2^{\mathcal{J}} \rightarrow \mathbb{G}_m$, ΔZ denotes the image of the diagonal embedding $Z \hookrightarrow \mathrm{GL}_2^{\mathcal{J}}$ and the right-hand side of the displayed equation is viewed as a sub-functor of $(\prod_{\mathcal{J}} \mathrm{Gr}_1)_{\bar{y}}^{\wedge}$.)

(2) Suppose $\prod z_j = 1$. Choose a transversal slice V to T^{\vee} near 1 in GL_2 such that the tangent space $T_1 V = \mathfrak{n} \oplus \bar{\mathfrak{n}} = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \right\} \subset \mathfrak{gl}_2$. Let $\bar{x} \in \mathrm{GL}_2^{\mathcal{J}}(\mathbb{F}) \times T^{\vee}(\mathbb{F})\tilde{z}_0 \times \prod_{j \neq 0} L_1^- G(\mathbb{F})\tilde{z}_j$ be the tuple $((1, 1, \dots, 1), (\bar{t}\tilde{z}_0, (\tilde{z}_j)_{j \neq 0}))$, and \bar{y} its image under the shifted conjugation action. Then the shifted conjugation action map induces an isomorphism on completion at \bar{x}

$$\left(V \times \mathrm{GL}_2^{\mathcal{J} \setminus \{0\}} \times (T^{\vee} \times L_1^- G\tilde{z}_0) \times \prod_{j \neq 0} L_1^- G\tilde{z}_j \right)_{\bar{x}}^{\wedge} \cong \left(\mathrm{GL}_2^{\mathcal{J}} \times \prod_{\mathcal{J}} L_1^- G\tilde{z}_j \right)_{\bar{y}}^{\wedge}$$

Remark 4.6.2. The case $\langle \nu_j, \alpha^{\vee} \rangle = 0$ for all j happens exactly when $\tilde{z}_j \in W^{\vee} Z(\mathbb{F}((v)))$ for all j . In this case, the same method of proof shows that for $\bar{x} = ((1, 1, \dots, 1), (\kappa_0 \tilde{z}_0, (\tilde{z}_j)_{j \neq 0}))$ with image \bar{y} under the shifted conjugation action

$$\left(\mathrm{GL}_2^{\mathcal{J} \setminus \{0\}} \times (\mathrm{GL}_2 \times L_1^- G\tilde{z}_0) \times \prod_{j \neq 0} L_1^- G\tilde{z}_j \right)_{\bar{x}}^{\wedge} \cong \left(\mathrm{GL}_2^{\mathcal{J}} \times \prod_{\mathcal{J}} L_1^- G\tilde{z}_j \right)_{\bar{y}}^{\wedge}$$

Proof. (1) On a tangent vector $(1 + \varepsilon K_j, (1 + \varepsilon L_j)\tilde{z}_j)$ at \bar{x} , the formula for the action is

$$(1 + \varepsilon K_j)(1 + \varepsilon L_j)\tilde{z}_j(1 - \varepsilon K_{j-1}) = (1 + \varepsilon K_j - \varepsilon \mathrm{Ad}(\tilde{z}_j)(K_{j-1}) + \varepsilon L_j)\tilde{z}_j$$

where $K_j \in \mathrm{LieGL}_2 = \mathfrak{gl}_2$, $L_j \in \mathrm{Lie}L_1^- G = \frac{1}{v}\mathfrak{gl}_2[\frac{1}{v}]$.

Hence we need to show the map

$$S : (\mathfrak{gl}_2 \times \frac{1}{v}\mathfrak{gl}_2[\frac{1}{v}])^{\mathcal{J}} \rightarrow (v\mathfrak{gl}_2[[v]] \setminus \mathfrak{gl}_2((v)))^{\mathcal{J}} \cong (\mathfrak{gl}_2 \oplus \frac{1}{v}\mathfrak{gl}_2[\frac{1}{v}])^{\mathcal{J}}$$

given by

$$(K_j, L_j) \mapsto K_j - \mathrm{Ad}(\tilde{z}_j)(K_{j-1}) + L_j$$

is an isomorphism onto the subspace of the target consisting of tuples (Y_j) whose projection to $\mathfrak{gl}_2^{\mathcal{J}}$ factors through the subspace $\mathfrak{gl}_2^{\mathcal{J}, \mathrm{tr}=0}$ with sum of tuples whose sum of traces is 0. (We used the notation $\mathfrak{gl}_2((v))$ to denote the lie algebra of the loop group of GL_2 with loop variable v , i.e. the space of matrices with coefficients in Laurent power series in the variable v .)

Define $\Pi_j : \mathfrak{gl}_2 \rightarrow \mathfrak{gl}_2$ to be the linear endomorphism given by sending X to the projection of $\mathrm{Ad}(\tilde{z}_j)X \in \mathfrak{gl}_2((v))$ onto the \mathfrak{gl}_2 summand, i.e. extracting the v -degree 0 part of $\mathrm{Ad}(\tilde{z}_j)X$. Since $\tilde{z}_j = z_j v^{\nu_j}$ we see that

- $\Pi_j \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \mathrm{Ad}(z_j) \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ if $\langle \nu_j, \alpha^{\vee} \rangle = 0$; and
- $\Pi_j \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \mathrm{Ad}(z_j) \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ if $\langle \nu_j, \alpha^{\vee} \rangle \neq 0$

We first show that S is injective. Let $((K_j)_j, (L_j)_j) \in \ker S$ then

$$K_0 = \Pi_0 \Pi_{-1} \cdots \Pi_{-f+1}(K_0)$$

Set $K_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then our hypothesis on \tilde{z}_j shows that the composite $\Pi_0 \Pi_{-1} \cdots \Pi_{-f+1}$ sends K_0 to $\begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}$. Thus we get $K_0 \in \mathrm{Lie}(Z)$, and hence $K_j = K_0 \in \mathrm{Lie}(Z)$ for all j , and then $L_j = 0$. This shows the injectivity of S .

For surjectivity, it suffices to prove the surjectivity after projecting to the $\mathfrak{gl}_2^{\mathcal{J}}$ summand. Let $(Y_j) \in \mathfrak{gl}_2^{\mathcal{J}, \text{tr}=0}$. By repeated substitution in the system

$$K_j - \Pi_j(K_{j-1}) = Y_j$$

we see that the system has a solution if and only if

$$K_0 - \Pi_0 \Pi_{-1} \cdots \Pi_{-f+1}(K_0) = Y_0 + \Pi_0(Y_{-1}) + \Pi_0 \Pi_{-1}(Y_{-2}) + \cdots + \Pi_{-f+2}(Y_{-f+1})$$

has a solution $K_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Note that the right-hand side has trace 0. But

$$K_0 - \Pi_0 \Pi_{-1} \cdots \Pi_{-f+1}(K_0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a-d & b \\ c & d-a \end{pmatrix}$$

hence $K_0 - \Pi_0 \Pi_{-1} \cdots \Pi_{-f+1}(K_0)$ can become any trace 0 matrix. This gives surjectivity of S .

- (2) The argument is similar to the previous case. The only difference is that at $j = 0, 1$, the tangent vector equations (on the v -degree 0 part) we get are modified to

$$V_0 + T_0 - \Pi_0(K_{-1}) = Y_0$$

$$K_1 - \Pi_1(V_0) = Y_1$$

where $V_0 \in T_1 V = \mathfrak{n} \oplus \bar{\mathfrak{n}}$ and $T_0 \in \mathfrak{t}^\vee = \text{Lie} T^\vee$.

The existence and uniqueness of solutions to the system now boils down to solvability (in terms of V_0, T_0) of

$$V_0 + T_0 - \Pi_0 \Pi_{-1} \cdots \Pi_{-f+1}(V_0) = \Pi_0(Y_{-1}) + \Pi_0 \Pi_{-1}(Y_{-2}) + \cdots + \Pi_{-f+1}(Y_{-f}),$$

which now always has a unique solution since the left-hand side reduces to $V_0 + T_0$. \square

Theorem 4.6.3. *Let τ be a regular tame type with small presentation (s, μ) and assume that either $p > \max\{16f + 3, 16f - 12 + \max_j \langle \mu_j, \alpha^\vee \rangle\}$ or $p \geq 17$ and $K = \mathbb{Q}_p$. Suppose $\tilde{z}_j = \tilde{w}_j s_j^{-1} v^{\hat{\mu}_j}$ with $\tilde{w} = (\tilde{w}_j) \in \text{Adm}^\vee(\eta)^{\mathcal{J}}$. Assume that for at least one j , $\tilde{z}_j \notin W^\vee Z(\mathbb{F}(\!(v)\!))$. Let $\bar{t} \in T^\vee(\mathbb{F})$ and $\bar{\rho}$ be the unique semisimple Galois representation such that the matrix of the associated φ^f -module is $\bar{t} \tilde{z}_0 \varphi(\tilde{z}_{f-1}) \cdots \varphi^{f-1}(\tilde{z}_1)$ (see §2.2.2).*

- (1) *Suppose $\bar{\rho}$ is absolutely irreducible, and let $R_{\bar{\rho}}^{\eta, \tau}$ be the potentially Barsotti-Tate deformation ring of type τ . Then, up to enlarging \mathbb{F} , $R_{\bar{\rho}}^{\eta, \tau}$ is isomorphic to the completion of the p -saturation of $\mathcal{Z}^{\text{nv}, \tau}(\tilde{z})$ at \tilde{z} (see Definition 4.3.3).*
- (2) *Suppose $\bar{\rho}$ is reducible, and let $R_{\bar{\rho}}^{\eta, \tau}$ be the framed potentially Barsotti-Tate deformation ring of type τ . Then up to adding formal variables, $R_{\bar{\rho}}^{\eta, \tau}$ is isomorphic to the completion of the p -saturation of $\tilde{\mathcal{Z}}^{\text{mod}, \tau}(\tilde{z}) \times_{\text{GL}_2^{\mathcal{J}}} (T \times \{1\}^{\mathcal{J} \setminus \{0\}})$ at $(\bar{t} \tilde{z}_0, \tilde{z}_1, \dots, \tilde{z}_{f-1})$.*

Proof. We give the argument for the first case, the second case being similar. After twisting, we reduce to the case $\bar{t} = 1$. Let R be an Artinian local ring with residue field \mathbb{F} and $X = (\kappa_j A_j \tilde{z}_j)_j \in \prod_j \text{GL}_2(R) \times L_1^- G(A) \tilde{z}_j \subset \text{Gr}_1(R)^{\mathcal{J}}$ lifting \tilde{z} . Then by the first item of Lemma 4.6.1, there is a unique factorization

$$\kappa_j A_j = g_j B_j \tilde{z}_j g_{j-1}^{-1} Z_j$$

with $g = (g_j)_j \in (\text{GL}_2^{\mathcal{J}} / \Delta Z)(A)$, $B_j \in L_1^- G(A)$ lifting the identities over \mathbb{F} , $Z_0 = \begin{pmatrix} \det(\prod \kappa_j)^{1/2} & 0 \\ 0 & \det(\prod \kappa_j)^{1/2} \end{pmatrix}$ and $Z_j = 1 \in Z(A)$ for $j \neq 0$. Since $\tilde{\mathcal{Z}}^{\text{mod}, \tau}$ is invariant under the shifted conjugation action (and

translation by $Z^{\mathcal{J}}$, the condition that $X \in \tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z})(R)$ is equivalent to $(B_j \tilde{z}_j)_j \in \mathcal{Z}^{\text{mod},\tau}(\tilde{z})(R)$. Thus we learn that there is an isomorphisms after completions at \tilde{z}

$$\tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z})_{\tilde{z}}^{\wedge} \cong (\text{GL}_2^{\mathcal{J}} / \Delta Z \times Z)_1^{\wedge} \times \mathcal{Z}^{\text{mod},\tau}(\tilde{z})_{\tilde{z}}^{\wedge}.$$

In particular, this shows that $\mathcal{Z}^{\text{mod},\tau}(\tilde{z})_{\tilde{z}}^{\wedge}$ is \mathcal{O} -flat. Since $\tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z})$ is the p -saturation of $\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})$, and $\mathcal{Z}^{\text{mod},\tau}(\tilde{z})$, $\mathcal{Z}^{\text{nv},\tau}(\tilde{z})$ are the fibers above $1 \in \text{GL}_2^{\mathcal{J}}$ of $\tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z})$ and $\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})$ respectively, we see that $\mathcal{Z}^{\text{nv},\tau}(\tilde{z})$ and $\mathcal{Z}^{\text{mod},\tau}(\tilde{z})$ have the same p -saturation. Thus $\mathcal{Z}^{\text{mod},\tau}(\tilde{z})_{\tilde{z}}^{\wedge}$ is the p -saturation of $\mathcal{Z}^{\text{nv},\tau}(\tilde{z})_{\tilde{z}}^{\wedge}$.

Now, by Corollary 4.5.3, $R_{\tilde{z}}^{\eta,\tau}$ is a versal ring to $\tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z})$ at \tilde{z} , so $R_{\tilde{z}}^{\eta,\tau}$ has the desired description up to adding formal power series variables. To remove the extra power series variables, we use the following fact [Ham75, Theorem 5]: if R, S are complete local rings such that $R[[X]] \cong S[[X]]$ then $R \cong S$. \square

Remark 4.6.4. Definition 4.3.3 gives presentations of $\mathcal{Z}^{\text{nv},\tau}(\tilde{z})$ according to the entries of Table 4. However, it turns out that several different entries of Table 4 give isomorphic $\mathcal{Z}^{\text{nv},\tau}(\tilde{z})$ via obvious change of variables, and we can record the possible isomorphism type of $\mathcal{Z}^{\text{nv},\tau}(\tilde{z})$ in the following simpler way:

Given $(\tilde{w}_j)_{j \in \mathcal{J}} \in \text{Adm}^{\vee}(\eta)^{\mathcal{J}}$, the fragmentation $\mathcal{J} = \bigcup_{k \in \mathcal{K}} \mathcal{J}_k$, and the type of the endpoints of each \mathcal{J}_k , $\mathcal{Z}^{\text{nv},\tau}(\tilde{z})$ is the spectrum of $\otimes_{j \in \mathcal{J}} R_j / \sum_k \mathcal{I}_{\mathcal{J}_k}$ where for each fragment $\mathcal{J}_k = (i, i-1, \dots, o+1, o)$, $\mathcal{I}_{\mathcal{J}_k}$ is the ideal of $\otimes_{j \in \mathcal{J}_k} R_j$ generated by the equations

$$(4.6.5) \quad M_{\text{out},o} \cdot \left(\prod_{\ell=o+1}^{i-1} T_{\ell} \right) \cdot M_{\text{in},i} = 0,$$

and $M_{\text{out},o}$ (resp. $M_{\text{in},i}$, resp. T_{ℓ}) is the final matrix (resp. initial matrix, resp. transition matrix) appearing in the corresponding entry of Table 5 according to the type of o (resp. i).

4.6.2. Rational smoothness.

Theorem 4.6.6. (1) Let $\tilde{\mathcal{Z}}^{\text{mod},\text{nm}}$ be the normalization of $\tilde{\mathcal{Z}}^{\text{mod},\tau}$. Then $\tilde{\mathcal{Z}}^{\text{mod},\text{nm}}$ is resolution-rational (cf. [Kov, Definition 9.1]).

(2) Assume $p \geq 7$ or $K = \mathbb{Q}_5$. Then the same statements hold for the normalization \mathcal{Z}^{nm} of \mathcal{Z}^{τ} .

Proof. (1) The statement is local so it suffices to check it for $\tilde{\mathcal{Z}}^{\text{mod},\text{nm}}(\tilde{z})$.

Since

- $\tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z})$ is the scheme theoretic image of the proper map $\pi^{\text{mod}} : \tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z}) \rightarrow \tilde{U}(\tilde{z})$,
- π^{mod} is a closed immersion after inverting p (cf. Proposition 4.3.4),
- $\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})$ is normal,

it follows that $\pi_*^{\text{mod}} \mathcal{O}_{\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})} = \mathcal{O}_{\tilde{\mathcal{Z}}^{\text{mod},\text{nm}}(\tilde{z})}$. But Proposition 3.3.18 shows that in fact

$$R\pi_*^{\text{mod}} \mathcal{O}_{\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})} = \mathcal{O}_{\tilde{\mathcal{Z}}^{\text{mod},\text{nm}}(\tilde{z})}.$$

By Grothendieck duality and the properness of π^{mod} we have

$$(4.6.7) \quad R\pi_*^{\text{mod}} \omega_{\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})/\mathcal{O}} = \omega_{\tilde{\mathcal{Z}}^{\text{mod},\text{nm}}(\tilde{z})/\mathcal{O}}.$$

Since $\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})$ is easily seen to be resolution-rational (it is locally isomorphic to a product of $\text{Spec } \mathcal{O}[X, Y]/(XY - p)$), $\tilde{\mathcal{Z}}^{\text{mod},\text{nm}}(\tilde{z})$ is also resolution-rational.

TABLE 5. Equations for $\mathcal{Z}^{\text{nv},\tau}(\tilde{z})$

type \ \tilde{w}_j	t_η	$w_0 t_\eta$	$t_{w_0(\eta)}$
II	$R_j = \mathcal{O}[X_j, Y_j]$ Initial matrix: $\begin{pmatrix} 1 \\ -X_j \end{pmatrix}$ Final matrix: $\begin{pmatrix} Y_j & -p \end{pmatrix}$	$R_j = \mathcal{O}[X_j, Y_j]$ Initial matrix: $\begin{pmatrix} 1 \\ -X_j \end{pmatrix}$ Final matrix: $\begin{pmatrix} -p & Y_j \end{pmatrix}$	$R_j = \mathcal{O}[X_j, Y_j]$ Initial matrix: $\begin{pmatrix} 1 \\ -X_j \end{pmatrix}$ Final matrix: $\begin{pmatrix} 0 & 1 \end{pmatrix}$
I	$R_j = \frac{\mathcal{O}[X_j, Y_j, Z_j]}{((p - Y_j)Y_j - X_j Z_j)}$ Initial matrix: $\begin{pmatrix} Y_j & -X_j \\ -Z_j & p - Y_j \end{pmatrix}$ Final matrix: $\begin{pmatrix} p - Y_j & -Z_j \\ -X_j & Y_j \end{pmatrix}$	$R_j = \frac{\mathcal{O}[X_j, Y_j, Z_j]}{((p - Y_j)Y_j - X_j Z_j)}$ Initial matrix: $\begin{pmatrix} Y_j & -X_j \\ -Z_j & p - Y_j \end{pmatrix}$ Final matrix: $\begin{pmatrix} Y_j & -X_j \\ -Z_j & p - Y_j \end{pmatrix}$	$R_j = \frac{\mathcal{O}[X_j, Y_j, Z_j]}{((p - Y_j)Y_j - X_j Z_j)}$ Initial matrix: $\begin{pmatrix} Y_j & -X_j \\ -Z_j & p - Y_j \end{pmatrix}$ Final matrix: $\begin{pmatrix} p - Y_j & X_j \\ Z_j & Y_j \end{pmatrix}$
0	$R_j = \mathcal{O}[X_j]$ Transition Matrix: $\begin{pmatrix} 1 & 0 \\ -X_j & p \end{pmatrix}$	$R_j = \mathcal{O}[X_j]$ Transition Matrix: $\begin{pmatrix} 0 & -1 \\ -p & X_j \end{pmatrix}$	$R_j = \mathcal{O}[X_j]$ Transition Matrix: $\begin{pmatrix} p & -X_j \\ 0 & 1 \end{pmatrix}$

(2) It suffices to check the statement for $\tilde{\mathcal{Z}}^\tau$. Let $\tilde{\mathcal{Z}}^{\text{nm}}(\tilde{z})$ be the normalization of $\tilde{\mathcal{Z}}^\tau(\tilde{z})$. The same argument as above shows that

$$\pi_* \mathcal{O}_{\tilde{Y}^{\eta,\tau}(\tilde{z})} = \mathcal{O}_{\tilde{\mathcal{Z}}^{\text{nm}}(\tilde{z})}.$$

Note that by Proposition 3.2.10, in the $K = \mathbb{Q}_5$ case we only need to deal with the situation when $\langle \mu_j, \alpha^\vee \rangle = 1$. By Lemma 3.3.9

$$R\pi_* \mathcal{O}_{\tilde{Y}^{\eta,\tau}(\tilde{z})} \otimes_{\mathcal{O}}^{\mathbb{L}} \mathbb{F} = R\pi_* (\mathcal{O}_{\tilde{Y}^{\eta,\tau}(\tilde{z})} / \varpi) = R\pi_*^{\text{mod}} (\mathcal{O}_{\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})} / \varpi) = R\pi_*^{\text{mod}} \mathcal{O}_{\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})} \otimes_{\mathcal{O}}^{\mathbb{L}} \mathbb{F}$$

which concentrates in degree 0 by Proposition 3.3.18. Since π is proper, it follows that

$$(4.6.8) \quad R\pi_* \mathcal{O}_{\tilde{Y}^{\eta,\tau}(\tilde{z})} = \mathcal{O}_{\tilde{\mathcal{Z}}^{\text{nm}}(\tilde{z})}$$

$$(4.6.9) \quad R\pi_* (\mathcal{O}_{\tilde{Y}^{\eta,\tau}(\tilde{z})} / \varpi) = \mathcal{O}_{\tilde{\mathcal{Z}}^{\text{nm}}(\tilde{z})} / \varpi = \mathcal{O}_{\tilde{\mathcal{Z}}^{\text{mod},\text{nm}}(\tilde{z})} / \varpi$$

Applying Grothendieck duality to (4.6.9) as in the previous part, we conclude that $R\pi_* \omega_{(\tilde{Y}^{\eta,\tau}(\tilde{z})/\varpi)/\mathbb{F}} = \omega_{(\tilde{\mathcal{Z}}^{\text{nm}}(\tilde{z})/\varpi)/\mathbb{F}}$. Since $\tilde{\mathcal{Z}}^{\text{nm}}(\tilde{z})$ is an affine p -adic formal scheme, we also get $R\pi_* \omega_{\tilde{Y}^{\eta,\tau}(\tilde{z})/\mathcal{O}} = \omega_{\tilde{\mathcal{Z}}^{\text{nm}}(\tilde{z})/\mathcal{O}}$. This fact, (4.6.8) and the fact that $\tilde{Y}^{\eta,\tau}$ is resolution-rational now implies $\tilde{\mathcal{Z}}^{\text{nm}}(\tilde{z})$ is resolution rational. \square

We can also completely classify the non-normal locus of \mathcal{Z}^τ :

Theorem 4.6.10. *Suppose $p \geq 7$ or $K = \mathbb{Q}_5$. Let τ have small presentation (s, μ) . A Galois representation $\bar{\rho}$ gives a non-normal point of \mathcal{Z}^τ exactly when both of the following holds:*

- (1) For each j , $s_j = \text{id}$ and $\langle \mu_j, \alpha^\vee \rangle \in \{0, 1\}$.
- (2) $\bar{\rho}$ is Fontaine-Laffaille with inertial Hodge-Tate weights at embedding $j \in \mathcal{J}$ given by
 - $(1, 0)$ if $\langle \mu_j, \alpha^\vee \rangle = 0$.

- $(1, 1)$ if $\langle \mu_j, \alpha^\vee \rangle = 1$.

Proof. It follows from the proof of Theorem 4.6.6 that the non-normal locus of $\tilde{\mathcal{Z}}^\tau$ is exactly the support of

$$\text{coker}(\mathcal{O}_{\tilde{\mathcal{Z}}^\tau} \rightarrow \pi_* \mathcal{O}_{\tilde{Y}_{\eta, \tau}})$$

and when intersecting with $\tilde{U}(\tilde{z})$, it equals the support of

$$\text{coker}(\mathcal{O}_{\tilde{U}(\tilde{z})/\varpi} \rightarrow \pi_* \mathcal{O}_{\tilde{Y}_{\eta, \tau}/\varpi}) = \text{coker}(\mathcal{O}_{\tilde{U}(\tilde{z})/\varpi} \rightarrow \pi_*^{\text{mod}} \mathcal{O}_{\tilde{Y}_{\text{mod}, \eta, \tau}/\varpi})$$

Thus we reduce to investigating the non-normal locus of $\tilde{\mathcal{Z}}^{\text{mod}, \tau}(\tilde{z})$. It follows from the proof of Proposition 3.3.16 and Theorem 4.6.6 that the non-normal locus is exactly the support of $R^1 \text{pr}_{\tilde{B}^*} \mathcal{I}(\tilde{z})$ in Corollary 4.2.10. But Corollary 4.2.13 shows this support is non-empty exactly when for each j , either $\langle \mu_j, \alpha^\vee \rangle = 0$ or $(\langle \mu_j, \alpha^\vee \rangle, s_j, \tilde{w}_j) = (1, \text{id}, t_{w_0(\eta)})$. This shows that (s, μ) has the form described in (1).

We now suppose (s, μ) has the requisite form. Decompose $\mathcal{J} = \mathcal{J}_0 \amalg \mathcal{J}_1$ where $\mathcal{J}_0 = \{j | \langle \mu_j, \alpha^\vee \rangle = 0\}$, $\mathcal{J}_1 = \{j | \langle \mu_j, \alpha^\vee \rangle = 1\}$. Furthermore, if (s, μ) is of the right form, the proof of Lemma 4.2.5 furthermore identifies the support of $R^1 \text{pr}_{\tilde{B}^*} \mathcal{I}(\tilde{z})$ as the locus of tuples $(A_j) \in \tilde{U}(\tilde{z})(\mathbb{F})$ such that

- $A_j = g_j \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}$ if $j \in \mathcal{J}_1$.
- $A_j \in \{g_j \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}, g_j \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}\}$ if $j \in \mathcal{J}_0$;

for some $g_j \in \text{GL}_2(\mathbb{F})$ (and $B, C \in \mathbb{F}$). After modifying (A_j) by the shifted conjugation action of $\text{GL}_2^{\mathcal{J}}$, we can arrange so that $A_j \in \text{GL}_2(\mathbb{F}) \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}$ for $j \in \mathcal{J}_0$.

Now set $\lambda \in X_*(T)^\vee$ be such that $\lambda_j = (1, 0)$ for $j \in \mathcal{J}_0$ and $\lambda_j = (1, 1)$ for $j \in \mathcal{J}_1$. Analogous to [LLHM⁺, Proposition 2.2.6], the moduli space FL_λ of mod p Fontaine-Laffaille modules of weight λ has the following description: Let $H \cong \text{GL}_2^{|\mathcal{J}_1|} \times B^{|\mathcal{J}_0|} \subset \text{GL}_2^{\mathcal{J}}$ be the subgroup of tuples (X_j) such that $X_{j-1} \in \text{GL}_2$ if $j \in \mathcal{J}_1$ and $X_{j-1} \in B$ if $j \in \mathcal{J}_0$. Then FL_λ is the quotient $[\text{GL}_2^{\mathcal{J}}/H]$ given by the action

$$(X_j) \cdot (g_j) \mapsto (X_j g_j \overline{X_{j-1}}^{-1})$$

where the overline denotes the projection $B \rightarrow T$ when $j \in \mathcal{J}_0$, and is the identity map when $j \in \mathcal{J}_1$.

But then, analogous to [LLHM⁺, Proposition 8.2.4], the natural embedding $\text{FL}_\lambda \hookrightarrow \Phi\text{-Mod}_K^{\text{ét}, 2}$ identifies with

$$(g_j) \mapsto \left((g_j \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix})_{j \in \mathcal{J}_1}, (g_j \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix})_{j \in \mathcal{J}_0} \right).$$

This shows that the support of $R^1 \text{pr}_{\tilde{B}^*} \mathcal{I}(\tilde{z})$ consists of exactly the points in $\tilde{U}(\tilde{z})(\overline{\mathbb{F}})$ that induces points in the image of FL_λ in \mathcal{Z}^τ . \square

5. EQUATIONS OF THE DEFORMATION RING

In this section we apply the main results of Sections 3 and 4 (namely Theorems 3.3.11 and 4.6.3) to prove the conjectures of the series of papers [CDM18a, CDM18b, CDM23b], in particular that tamely potentially crystalline Barsotti-Tate deformation rings $R_{\bar{\rho}}^{\eta, \tau}$ only depend on the combinatorial gene $\mathbb{X}(\tau, \bar{\rho}|_{I_K})$ (Theorem 5.4.16). Throughout this section, except for §5.5, we assume that $K \neq \mathbb{Q}_p$.

5.1. Genetics. We recall and formalize into an abstract setup the notion of genes as introduced in [CDM18b, CDM23b], and recall the main conjectures of *loc. cit.*

5.1.1. Combinatorial genes. Inspired by the terminology of [CDM18b, CDM23b] we now define the notion of *combinatorial gene* associated to a pair $(\gamma, h) \in \mathbb{Z}/(p^f - 1)\mathbb{Z} \times \mathbb{Z}/(p^{2f} - 1)\mathbb{Z}$.

Let $\gamma \in \mathbb{Z}/(p^f - 1)\mathbb{Z}$ and $h \in \mathbb{Z}/(p^{2f} - 1)\mathbb{Z}$ such that $h \not\equiv 0$ modulo $p^f + 1$ and $h - 2\gamma - (\sum_{j=0}^{f-1} p^j) \not\equiv 0$ modulo $p^f - 1$. Consider the p -expansions

$$(5.1.1) \quad h - (p^f + 1) \left(h - \gamma - \sum_{j=0}^{f-1} p^j \right) \equiv p^{2f-1}v_0 + p^{2f-2}v_1 + \cdots + pv_{2f-2} + v_{2f-1} \pmod{p^{2f} - 1}$$

with $v_{j'} \in \{0, \dots, p-1\}$ for all $j' \in \mathcal{J}'$. A *combinatorial gene associated to (γ, h)* is a \mathcal{J}' -tuple $\mathbb{X} = \mathbb{X}(\gamma, h) \in \{\mathbf{A}, \mathbf{B}, \mathbf{AB}, \mathbf{0}\}^{\mathcal{J}'}$ which satisfies the following properties (see [CDM23b, Lemma B.1.3]):

- ♠1 if $v_{j'} = 0$ and $\mathbb{X}_{j'+1} = \mathbf{0}$, then $\mathbb{X}_{j'} = \mathbf{AB}$;
- ♠2 if $v_{j'} = 0$ and $\mathbb{X}_{j'+1} \neq \mathbf{0}$, then $\mathbb{X}_{j'} = \mathbf{A}$;
- ♠3 if $v_{j'} = 1$ and $\mathbb{X}_{j'+1} = \mathbf{0}$, then $\mathbb{X}_{j'} = \mathbf{0}$;
- ♠4 if $v_{j'} = 1$ and $\mathbb{X}_{j'+1} \neq \mathbf{0}$, then $\mathbb{X}_{j'} = \mathbf{B}$;
- ♠5 if $v_{j'} \geq 2$, then $\mathbb{X}_{j'} = \mathbf{0}$.

By [CDM23b, Lemma 1.3.3, Lemma B.1.7] a combinatorial gene \mathbb{X} associated to (γ, h) is well defined, and is unique by the proof of [CDM23b, Proposition 1.4.4]. Moreover, by [CDM23b, Proposition 1.3.2, Corollary 1.3.4], the \mathcal{J}' -tuple $\mathbb{X} = \mathbb{X}(\gamma, h)$ satisfies the following conditions

- ♣1 if $\mathbb{X}_{j'+1} = \mathbf{0}$, then $\mathbb{X}_{j'} \in \{\mathbf{AB}, \mathbf{0}\}$;
- ♣2 if $\mathbb{X}_{j'+1} \neq \mathbf{0}$, then $\mathbb{X}_{j'} \in \{\mathbf{A}, \mathbf{B}, \mathbf{0}\}$;
- ♣3 there exists an integer $j' \in \mathcal{J}'$ such that $\mathbb{X}_{j'} = \mathbf{0}$ or $\mathbb{X}_{j'} \neq \mathbb{X}_{j'+1}$.

The discussion after [CDM18b, Lemme 2.1.7] shows that:

Lemma 5.1.2. *Assume that $\gamma + \gamma' + (\sum_{j=0}^{f-1} p^j) \equiv h$ modulo $p^f - 1$, $h \not\equiv 0 \pmod{p^f + 1}$. Then for all $j' \in \mathcal{J}'$ we have:*

$$\mathbb{X}(\gamma', h)_{j'} = \begin{cases} \mathbf{A} & \text{if } \mathbb{X}(\gamma, h)_{j'} = \mathbf{B}, \\ \mathbf{B} & \text{if } \mathbb{X}(\gamma, h)_{j'} = \mathbf{A}, \\ \mathbb{X}(\gamma, h)_{j'} & \text{otherwise,} \end{cases}$$

and

$$\mathbb{X}(\gamma, p^f h)_{j'} = \mathbb{X}(\gamma, h)_{j'+f}.$$

Following the conditions (♣1)–(♣3) and Lemma 5.1.2 we define an *abstract combinatorial gene* as follows:

Definition 5.1.3. An *abstract combinatorial gene* is an equivalence class of a \mathcal{J}' -tuple $\mathbb{X} \in \{\mathbf{A}, \mathbf{B}, \mathbf{AB}, \mathbf{0}\}^{\mathcal{J}'}$ satisfying conditions (♣1), (♣2), (♣3), by the equivalence relation generated by

$$(5.1.4) \quad \mathbb{X}' \sim \mathbb{X} \quad \text{if for all } j' \in \mathcal{J}' \text{ we have } \begin{cases} \mathbb{X}'_{j'} = \{\mathbf{A}, \mathbf{B}\} \setminus \{\mathbb{X}_{j'}\} & \text{when } \mathbb{X}_{j'} \in \{\mathbf{A}, \mathbf{B}\} \\ \mathbb{X}'_{j'} = \mathbb{X}_{j'} & \text{otherwise.} \end{cases}$$

$$(5.1.5) \quad \mathbb{X}' \sim \mathbb{X} \quad \text{if for all } j' \in \mathcal{J}' \text{ we have } \begin{cases} \mathbb{X}'_{j'} & = \mathbb{X}_{j'+1} \\ \mathbb{X}'_{j'+f} & = \mathbb{X}_{j'+f+1}. \end{cases}$$

Note that the relation (5.1.5) implies that $(\mathbb{X}_{j'})_{j' \in \mathcal{J}'} \sim (\mathbb{X}_{j'+f})_{j' \in \mathcal{J}'}$, and that the notion of abstract combinatorial gene is independent of p .

Let $(\tau, \bar{\tau}')$ be a pair of tame inertial types satisfying the following determinant condition:

$$\text{(det)} \quad \det(\tau) \otimes_{\mathcal{O}} \mathbb{F} \equiv \det(\bar{\tau}') \otimes_{\mathbb{F}} \omega.$$

If $\tau = \tau(s, \mu) = \omega_f^\gamma \oplus \omega_f^{\gamma'}$ and $\bar{\tau}' = \bar{\tau}(\sigma, \nu) = \bar{\omega}_{2f}^h \oplus \bar{\omega}_{2f}^{p^f h}$ are a tame inertial type of niveau f and a tame inertial \mathbb{F} -type of level $2f$ as in §2.1.1, then condition (det) translates into the condition of Lemma 5.1.2. We thus define the gene $\mathbb{X}(\tau, \bar{\tau}')$ of the pair $(\tau, \bar{\tau}')$ as the abstract combinatorial gene associated to the \mathcal{J}' -tuple $\mathbb{X}(\gamma, h)$. This gives the motivation behind Definition 5.1.3: the relation (5.1.4) is imposed by the isomorphism $\omega_f^\gamma \oplus \omega_f^{\gamma'} \cong \tau \cong \omega_f^{\gamma'} \oplus \omega_f^\gamma$ and the relation (5.1.5) by the fact that the isomorphism class of $\tau, \bar{\tau}'$ does not depend on the choice of the embedding σ_0 . In particular, $\mathbb{X}(\tau, \bar{\tau}')$ only depends on the isomorphism class of τ and $\bar{\tau}'$, and is insensitive to twist by characters $\chi : I_K \rightarrow \mathcal{O}^\times$.

5.1.2. Genetic conjectures. The computations of [CDM18b, CDM23b] showed that the combinatorial genes contains non-trivial information on the generic and special fiber of Galois deformation rings with p -adic Hodge theory conditions.

Let $\bar{\rho} : G_K \rightarrow \text{GL}_2(\mathbb{F})$ be irreducible and τ a tame inertial type of niveau f . Then $\bar{\rho}|_{I_K}$ defines a tame inertial \mathbb{F} -type, so that the combinatorial gene $\mathbb{X}(\tau, \bar{\rho}|_{I_K})$ is defined. The authors of [CDM18b, CDM23b] propose the following conjecture (which is an integral version of [CDM18b, Conjecture 5.1.5])

Conjecture 5.1.6 (Conjecture 2 in [CDM23b]). *The deformation ring $R_\rho^{\eta, \tau}$ is determined by $\mathbb{X}(\tau, \bar{\rho}|_{I_K})$.*

They furthermore refine Conjecture 5.1.6 into the following

Conjecture 5.1.7 (Conjecture 5.2.7 [CDM18b], Conjecture 3.1.2 [CDM23b]). *There exists a decomposition $\mathbb{X}(\tau, \bar{\rho}|_{I_K}) = \cup_{i=0}^r (\mathbb{X}_j, \mathbb{X}_{j+f})_{j_i \leq j \leq j_{i+1}}$ such that*

$$R_\rho^{\eta, \tau} \cong \widehat{\otimes}_{i=0}^r R_i$$

where R_i is a complete local Noetherian \mathcal{O} -algebra depending only on $(\mathbb{X}_j, \mathbb{X}_{j+f})_{j_i \leq j \leq j_{i+1}}$.

Moreover, even if not stated as a conjecture, they suggest that

Conjecture 5.1.8 ([CDM23b, CDM23a]). *The deformation ring $R_\rho^{\eta, \tau}$ is “independent of p ”.*

Conjectures 5.1.6, 5.1.7 and 5.1.8 are proven in Theorem 5.4.16 for p sufficiently large “with respect to τ ”.

Prior to this work, Conjecture 5.1.6 was known when either τ has a presentation (s, μ) with $2 < \langle \mu_j, \alpha^\vee \rangle < p - 2$ for all $j \in \mathcal{J}$ or if $\binom{\mathbb{X}(\tau, \bar{\rho}|_{I_K})_{j+f}}{\mathbb{X}(\tau, \bar{\rho}|_{I_K})_j} = \binom{0}{0}$ for some $j \in \mathcal{J}$. In the first case, we have $0 \in \{\mathbb{X}_j, \mathbb{X}_{j+f}\}$ for all $j \in \mathcal{J}$ which in turn implies that implies that the Kisin variety of type (η, τ) attached to $\bar{\rho}$ is either empty or a single point by [CDM18b, Théorème 2.2.1(A)]. Conjecture 5.1.6 is true by Theorem 2.1.8 in this case follows from Theorem 2.1.8 and [CDM23b, Theorem 3]. In the second case, the Kisin variety is empty and the deformation ring is zero.

We now elaborate on Conjecture 5.1.8. Given an abstract combinatorial gene \mathbb{X} we prove the conjecture by constructing rings $R_{\mathbb{X}}$ which are (possibly zero) quotients of polynomial rings over $\mathbb{Z}[t]$ modulo an ideal $I_{\mathbb{X}} \subset R_{\mathbb{X}}$. These rings are independent of p (since abstract combinatorial genes are) and the Conjecture 5.1.8 would be proven by explicitly showing that $R_\rho^{\eta, \tau}$ is isomorphic to the completion of $R_{\mathbb{X}}/(t - p) \otimes \mathcal{O}$ at the ideal generated by t and the variables of $R_{\mathbb{X}}$, where

$\mathbb{X} = \mathbb{X}(\tau, \bar{\rho}|_{I_K})$ and the tensor product is along the map $\mathbb{Z}[t] \rightarrow \mathcal{O}$ defined by $t \mapsto p$. Again, by Theorem 2.1.8, the conjecture is known to be true when τ has a 2-generic lowest alcove presentation.

We conclude with the following observation. By [CDM18b, Proposition 4.1.3], we have $R_{\bar{\rho}}^{\eta, \tau} = 0$ as soon as $(\mathbb{X}(\tau, \bar{\rho}|_{I_K})_{j+f}, \mathbb{X}(\tau, \bar{\rho}|_{I_K})_j) = (0, 0)$ for some $j \in \mathcal{J}$. Hence, in what follows, we will be interested in combinatorial genes \mathbb{X} satisfying the further condition

$$\clubsuit 4 \quad (\mathbb{X}_{j+f}, \mathbb{X}_j) \neq (0, 0) \text{ for all } j \in \mathcal{J}.$$

5.2. Genetic translation. We fix our setup as in Theorem 4.6.3. Hence, let $\tau : I_K \rightarrow \mathrm{GL}_2(\mathcal{O})$ be a regular tame inertial type of niveau f with a small presentation (s, μ) and let $\tilde{w} = (w_j t_{\nu_j})_{j \in \mathcal{J}} \in \mathrm{Adm}^{\vee}(\eta)^{\mathcal{J}}$. Up to twist, we can furthermore assume that $\mu_{j,2} = 0$ for all $j \in \mathcal{J}$. We abbreviate $k_j \stackrel{\mathrm{def}}{=} \langle \mu_j, \alpha^{\vee} \rangle$ and set $\tilde{z} \stackrel{\mathrm{def}}{=} \tilde{w} s^{-1} v^{\hat{\mu}}$ in what follows (in particular, z_i equals $w_0^{-1} s_i^{-1}$ if $\tilde{w}_i = w_0 t_{\eta}$ and s_i^{-1} else).

In order to analyze genetic data associated to τ and \tilde{w} define $\lambda \in X^*(\underline{T})$ by the condition

$$(5.2.1) \quad (z_0 z_{f-1} \cdots z_1) v^{(\sum_{j \in \mathcal{J}} p^j \lambda_j)} = \tilde{z}_0 \varphi(\tilde{z}_{f-1}) \cdots \varphi^{f-1}(\tilde{z}_1)$$

and the $2f$ -tuple $(v'_{j'})_{j' \in \mathcal{J}'}$ by

$$\begin{pmatrix} v'_{2f-1-j} \\ v'_{f-1-j} \end{pmatrix} \stackrel{\mathrm{def}}{=} \lambda_j - \begin{pmatrix} k_{f-j} \\ k_{f-j} \end{pmatrix} \delta_{s_{\mathrm{or}, f-1-j} \neq \mathrm{id}}$$

where $j \in \{0, \dots, f-1\}$ and $s_{\mathrm{or}, f-1-j} = \prod_{i=0}^{f-1-j} s_i$ (Lemma 2.1.7). By smallness of (s, μ) , we have $v'_j, v'_{f+j} \in \{-\frac{p+1}{2}, \dots, \frac{p+3}{2}\}$ for all $j \in \{0, \dots, f-1\}$. Note that the $2f$ -tuple $(v'_{j'})_{j' \in \mathcal{J}'}$ depends on the triple (\tilde{w}, s, μ) , but we omit this dependence for sake of readability.

We assume from now on that $\prod_{j \in \mathcal{J}} z_j = w_0$, $\sum_{j \in \mathcal{J}} p^j (\mu_{j,1} + \mu_{j,2}) \equiv \sum_{j \in \mathcal{J}} p^j (\lambda_{j,1} + 1 + (\lambda_{j,2} + 1))$ modulo $p^f - 1$ and $\sum_{j \in \mathcal{J}} p^j (\lambda_{j,1} + p^f \lambda_{j,2}) \not\equiv 0$ modulo $p^f + 1$ (these hypotheses imply that $\bar{\rho}$ is irreducible, and its determinant is a cyclotomic shift of the determinant of $\bar{\tau}$). By construction, the $2f$ -tuple $(v'_{j'})_{j' \in \mathcal{J}'}$ extracts precisely the LHS of equation (5.1.1) (using equations (2.1.2) and (2.1.3)), and hence produces a gene $\mathbb{X}(v')$, satisfying items ($\clubsuit 1$), ($\clubsuit 2$), ($\clubsuit 3$) thanks to the assumptions in the previous sentence.

Tables 6 and 7 give respectively the explicit description of \tilde{z}_i and $\begin{pmatrix} v'_{f+j} \\ v'_j \end{pmatrix}$ according to (s, μ) and \tilde{w} and are directly obtained from the definitions.

TABLE 6. Genetic Translation-I

$s_i \backslash \tilde{w}_i$	t_{η}	$w_0 t_{\eta}$	$t_{w_0(\eta)}$
id	$\begin{pmatrix} k_i + 1 \\ 0 \end{pmatrix}$	$(12) \begin{pmatrix} k_i + 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} k_i \\ 1 \end{pmatrix}$
(12)	$(12) \begin{pmatrix} k_i \\ 1 \end{pmatrix}$	$\begin{pmatrix} k_i \\ 1 \end{pmatrix}$	$(12) \begin{pmatrix} k_i + 1 \\ 0 \end{pmatrix}$

This table records $\tilde{z}_i \stackrel{\mathrm{def}}{=} \tilde{w}_i s_i^{-1} t_{\mu_i}$.

TABLE 7. **Genetic Translation-II**

\tilde{w}_{j+1}	s_{j+1} \diagup $s_{\text{or},j}$	id	(12)
$t_\eta, w_0 t_\eta$	id	$\Sigma_j \begin{pmatrix} k_{j+1} + 1 \\ 0 \end{pmatrix}$	$\Sigma_j \begin{pmatrix} 1 \\ -k_{j+1} \end{pmatrix}$
	(12)	$\Sigma_j \begin{pmatrix} k_{j+1} \\ 1 \end{pmatrix}$	$\Sigma_j \begin{pmatrix} 0 \\ 1 - k_{j+1} \end{pmatrix}$
$t_{w_0 \eta}$	id	$\Sigma_j \begin{pmatrix} k_{j+1} \\ 1 \end{pmatrix}$	$\Sigma_j \begin{pmatrix} 0 \\ 1 - k_{j+1} \end{pmatrix}$
	(12)	$\Sigma_j \begin{pmatrix} k_{j+1} + 1 \\ 0 \end{pmatrix}$	$\Sigma_j \begin{pmatrix} 1 \\ -k_{j+1} \end{pmatrix}$

The entries of the table record $\begin{pmatrix} v'_{j+f} \\ v'_j \end{pmatrix}$ for $j \in \{0, \dots, f-1\}$, where

$\Sigma_j \stackrel{\text{def}}{=} \prod_{i=1}^j z_i^{-1} = (\prod_{i=1}^j s_i)(w_0)^{N_j}$ and $N_j \stackrel{\text{def}}{=} \#\{i \in \{1, \dots, j\}, \tilde{w}_i = w_0 t_\eta\}$. Note that $\Sigma_0 = \text{id}$ by definition.

5.3. Fibers of the map $(\tilde{w}, s, \mu) \mapsto \mathbb{X}(v')$. Given a triple (\tilde{w}, s, μ) we have produced a tuple $((s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type at } j), (v'_{f+j}, v'_j))_{j \in \mathcal{J}}$ and a gene $\mathbb{X}(v')$ attached to $(v'_{f+j}, v'_j)_{j \in \mathcal{J}}$, hence a map

$$(5.3.1) \quad (\tilde{w}, s, \mu) \mapsto ((s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type at } j), (v'_{f+j}, v'_j))_{j \in \mathcal{J}} \mapsto (v'_{f+j}, v'_j)_{j \in \mathcal{J}} \mapsto \mathbb{X}(v').$$

In this and the following section we analyze the shapes and types appearing in the fiber of the map (5.3.1). *From now onwards, we furthermore assume that $\mathbb{X}(v')$ satisfies condition $(\clubsuit 4)$.*

As a preliminary step, we record in Table 8 the fiber of the map

$$(5.3.2) \quad ((s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type at } j), (v'_{f+j}, v'_j))_{j \in \mathcal{J}} \mapsto (v'_{f+j}, v'_j)_{j \in \mathcal{J}}$$

Table 8 is obtained directly from Table 7, and the only relevant property to obtain the Table 8 is whether $v'_j \geq 2$, $v'_j = 1$, $v'_j = 0$ or $v'_j < 0$.

Recall that we identify \mathcal{J}' with $\mathbb{Z}/2f$, and in what follows objects such as $\Sigma_j, \mathbb{X}_j, v'_j$ are indexed by \mathcal{J}' . For objects that are previously indexed by \mathcal{J} we extend by f periodicity, with the exception of Σ_j where the extension is given by $\Sigma_{j+f} = w_0 \Sigma_j$.

5.3.1. Step one: fiber of the map $(v'_{f+j}, v'_j)_{j \in \mathcal{J}} \mapsto \mathbb{X}(v')$. Let $\mathbb{X} \in \{\mathbf{A}, \mathbf{B}, \mathbf{AB}, \mathbf{0}\}^{\mathcal{J}'}$ satisfy conditions $(\clubsuit 1), (\clubsuit 2), (\clubsuit 3)$ and $(\clubsuit 4)$. In this subsection we determine the fiber above \mathbb{X} of the map $(v'_{f+j}, v'_j)_{j \in \mathcal{J}} \mapsto \mathbb{X}(v')$.

TABLE 8. **Fibre of the map**

$$(s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type}_j) \mapsto \left\{ \begin{pmatrix} 0 \\ \geq 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ < 0 \end{pmatrix} \right\}$$

s_{j+1}	$s_{\text{or},j}$	Σ_j	$\begin{pmatrix} v'_{j+f} \\ v'_j \end{pmatrix} = \begin{pmatrix} 0 \\ \geq 2 \end{pmatrix}$	$\begin{pmatrix} v'_{j+f} \\ v'_j \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} v'_{j+f} \\ v'_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} v'_{j+f} \\ v'_j \end{pmatrix} = \begin{pmatrix} 0 \\ < 0 \end{pmatrix}$
id	id	id		(id, $t_{w_0(\eta)}$, 0)		
id	id	w_0	(id, t_η , II) (w_0 , $w_0 t_\eta$, II)	(id, t_η , 0) (w_0 , $w_0 t_\eta$, 0)		
id	w_0	id		(id, $t_{w_0(\eta)}$, 0)	(id, $t_{w_0(\eta)}$, I)	(id, $t_{w_0(\eta)}$, II)
id	w_0	w_0		(id, t_η , 0) (w_0 , $w_0 t_\eta$, 0)	(id, $t_{w_0(\eta)}$, I)	
w_0	id	id				
w_0	id	w_0	(w_0 , $t_{w_0(\eta)}$, II)			
w_0	w_0	id			(id, $w_0 t_\eta$, I) (w_0 , t_η , I)	(id, $w_0 t_\eta$, II) (w_0 , t_η , II)
w_0	w_0	w_0			(id, $w_0 t_\eta$, I) (w_0 , t_η , I)	

Given (v'_{f+j}, v'_j) and $(s_{j+1}, s_{\text{or},j}, \Sigma_j)$, the table entries record the triples $(z_{j+1}, \tilde{w}_j, \text{type at } j)$ such that $((s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type at } j), (v'_{f+j}, v'_j))_{j \in \mathcal{J}}$ is a 7-tuple associated to some (\tilde{w}, s, μ) . (Here $j \in \mathbb{Z}/f\mathbb{Z}$.) The table records the cases where $v'_{f+j} = 0$, for $j \in \mathbb{Z}/f\mathbb{Z}$, but we note that we can replace (v'_{f+j}, v'_j) with (v'_j, v'_{f+j}) at the cost of replacing Σ_j with $w_0 \Sigma_j$, and, similarly, we can replace (v'_{j+f}, v'_j) with $(1 - v'_{j+f}, 1 - v'_j)$ at the cost of replacing Σ_j and $s_{\text{or},j}$ with $w_0 \Sigma_j$ and $w_0 s_{\text{or},j}$. The blank boxes in the table correspond to configurations of $(s_{j+1}, s_{\text{or},j}, \Sigma_j, (v'_{j+f}, v'_j))$ which can not arise.

Lemma 5.3.3. *Assume that $\mathbb{X} = \mathbb{X}(v')$ for some $(v'_{f+j}, v'_j)_{j \in \mathcal{J}}$ associated to a triple (\tilde{w}, s, μ) as above (in particular \mathbb{X} satisfies conditions $(\clubsuit 1)$, $(\clubsuit 2)$, $(\clubsuit 3)$ and $(\clubsuit 4)$).*

Then, for each $j' \in \mathcal{J}'$, the values of v'_j are constraint by the third row of Table 9 according to the pair $\mathbb{X}_{f+j'}$, $\mathbb{X}_{f+j'+1}$.

Proof. Let $(v_{j'})_{j' \in \mathcal{J}'} \in \{0, \dots, p-1\}^{\mathcal{J}'}$ be the tuple defined by

$$\sum_{j'=0}^{2f-1} v_{j'} p^{2f-1-j'} \equiv \sum_{j'=0}^{2f-1} v'_{j'} p^{2f-1-j'} \pmod{p^{2f}-1}.$$

We will show that:

- if $\mathbb{X}_i \in \{\mathbf{A}, \mathbf{B}\}$ then $v_i = v'_i$;

TABLE 9. Genetic Translation-II

$\mathbb{X}_{f+j'}$	A	B	AB	0	0
$\mathbb{X}_{f+j'+1}$	$\neq 0$	$\neq 0$	0	$\neq 0$	0
$v'_{j'}$	0	1	$\{1, 0\}$	$\{\geq 2, < 0\}$	Any

- if $\mathbb{X}_i = \mathbf{AB}$ then $v_i \in \{v'_i, v'_i - 1\}$;
- if $\mathbb{X}_i = \mathbf{0}$ and $\mathbb{X}_{i+1} \neq \mathbf{0}$ then $v_i \in \{v'_i, v'_i + p\}$; and
- if $\mathbb{X}_i = \mathbf{0}$ and $\mathbb{X}_{i+1} = \mathbf{0}$ then $v_i \in \{v'_i, v'_i - 1, v'_i + p, v'_i + p - 1\}$.

By definition of combinatorial gene (equations ($\spadesuit 1$)-($\spadesuit 5$)) we deduce that:

- if $\mathbb{X}_i \in \{\mathbf{AB}, \mathbf{A}\}$ then $v_i = 0$;
- if $\mathbb{X}_i = \mathbf{B}$ then $v_i = 1$;
- if $\mathbb{X}_i = \mathbb{X}_{i+1} = \mathbf{0}$ then $v_i \in [1, p - 1]$; and
- if $\mathbb{X}_i = \mathbf{0}$ and $\mathbb{X}_{i+1} \neq \mathbf{0}$ then $v_i \in [2, p - 1]$.

By ($\clubsuit 3$) there exists $\ell \in \{0, \dots, 2f - 1\}$ such that $v_\ell \neq 0$. Let $\varepsilon \in \mathbb{Z}$ be such that

$$\sum_{i=0}^{2f-1} v_i p^{2f-1-i} = \sum_{i=0}^{2f-1} v'_i p^{2f-1-i} + \varepsilon(p^{2f} - 1)$$

As $v'_i \in \{-(p+1)/2, \dots, (p+3)/2\}$ we have

$$-p^{2f} + 1 < \sum_{i=0}^{2f-1} v'_i p^{2f-1-i} < p^{2f} - 1$$

hence $\varepsilon \in \{0, 1\}$.

Let $i_0 \stackrel{\text{def}}{=} \min\{i \geq 0, v'_i \neq 0\}$ (this is well defined by ($\clubsuit 3$)). As

$$-p^{2f-i_0-1} < \sum_{j=i_0+1}^{2f-1} v'_j p^{2f-1-j} < p^{2f-i_0-1}$$

we have

$$(5.3.4) \quad \varepsilon = \begin{cases} 0 & \text{if } v'_{i_0} > 0, \\ 1 & \text{if } v'_{i_0} < 0. \end{cases}$$

If $\varepsilon = 1$ then $v_i = p - 1$ for $0 \leq i \leq i_0 - 1$ and moreover $v_{i_0} = p + v'_{i_0} \geq 2$ if $i_0 < 2f - 1$ and $v_{i_0} = p - 1 + v'_{i_0} \geq 2$ if $i_0 = 2f - 1$. We conclude that $\mathbb{X}_i = \mathbf{0}$ for all $0 \leq i \leq i_0$ if $\varepsilon = 1$.

We have the following relation between v_{2f-1} and v'_{2f-1} , and we define $\varepsilon_{2f-1} \in \{0, 1\}$ as follows:

- if $v'_{2f-1} < 0$ and $\varepsilon = 0$ then $v_{2f-1} = p + v'_{2f-1} \geq 2$, $\mathbb{X}_{2f-1} = \mathbf{0}$ and we define $\varepsilon_{2f-1} \stackrel{\text{def}}{=} 1$,
- if $v'_{2f-1} \geq 0$ and $\varepsilon = 0$ then $v_{2f-1} = v'_{2f-1}$, and we define $\varepsilon_{2f-1} \stackrel{\text{def}}{=} 0$,
- if $v'_{2f-1} < 1$ and $\varepsilon = 1$ then $v_{2f-1} = p + v'_{2f-1} - 1 \geq 1$, $\mathbb{X}_{2f-1} = \mathbf{0}$ and we define $\varepsilon_{2f-1} = 1$,
- if $v'_{2f-1} \geq 1$ and $\varepsilon = 1$ then $v_{2f-1} = v'_{2f-1} - 1$ and we define $\varepsilon_{2f-1} \stackrel{\text{def}}{=} 0$.

By decreasing induction, we deduce for $i \in \{2f - 2, \dots, 0\}$ the following relations between v_i and v'_i , and define $\varepsilon_i \in \{0, 1\}$ as follows:

- (a) if $v'_i < 0$ and $\varepsilon_{i+1} = 0$ then $v_i = p + v'_i \geq 2$, $\mathbb{X}_i = 0$ and we define $\varepsilon_i \stackrel{\text{def}}{=} 1$,
- (b) if $v'_i \geq 0$ and $\varepsilon_{i+1} = 0$ then $v_i = v'_i$ and we define $\varepsilon_i \stackrel{\text{def}}{=} 0$,
- (c) if $v'_i < 1$ and $\varepsilon_{i+1} = 1$ then $v_i = p + v'_i - 1 \geq 1$, $\mathbb{X}_i = 0$ and we define $\varepsilon_i \stackrel{\text{def}}{=} 1$;
- (d) if $v'_i \geq 1$ and $\varepsilon_{i+1} = 1$ then $v_i = v'_i - 1$ and we define $\varepsilon_i \stackrel{\text{def}}{=} 0$.

By (5.3.4) we have $\varepsilon = \varepsilon_0$ and hence the relations (a)–(d) between v_i and v'_i hold for *any* $i \in \mathcal{J}'$, and define an element $(\varepsilon_i)_{\mathcal{J}'} \in \{0, 1\}^{\mathcal{J}'}$.

For any $i \in \{0, \dots, 2f - 1\}$ such that $\varepsilon_{i+1} = 1$, we have $\mathbb{X}_{i+1} = 0$. Hence $v_i = v'_i - 1$ only occurs for $\mathbb{X}_i \in \{0, \text{AB}\}$ by (1). The conclusion follows now from a direct application of conditions (1)–(5) (for instance if $\mathbb{X}_i = \text{AB}$ then $v_i = 0$ by 1, and from (b), (d) above we conclude that $v'_i \in \{0, 1\}$).

□

The following lemma improves Lemma 5.3.3.

Lemma 5.3.5. *Keep the assumptions of Lemma 5.3.3 let $i \in \{0, \dots, 2f - 1\}$.*

- (a) *If $\mathbb{X}_i = 0$ and $v'_i = 0$, then there exists $j \geq 0$ such that $v'_{j+i} < 0$ and $v'_{i+\ell} = 0$ for all $0 \leq \ell < j$.*
- (b) *If $\mathbb{X}_i = 0$ and $v'_i = 1$, then there exists $j \geq 0$ such that $v'_{j+i} \geq 2$ and $v'_{i+\ell} = 1$ for all $0 \leq \ell < j$.*
- (c) *If $\mathbb{X}_i = \text{AB}$ and $v'_i = 1$, then there exists $j \geq 0$ such that $v'_{j+i} < 0$ and $v'_{i+\ell} = 0$ for all $0 \leq \ell < j$.*
- (d) *If $\mathbb{X}_i = \text{AB}$ and $v'_i = 0$, then $v'_{i+1} \geq 1$.*

Proof. In the notation of the proof of Lemma 5.3.3 we have:

- (a) If $\mathbb{X}_i = 0$ and $v'_i = 0$, then $\varepsilon_{i+1} = 1$ and $\mathbb{X}_{i+1} = 0$ and $v'_{i+1} < 1$. The claimed result follows now by induction.
- (b) This is similar to (a).
- (c) If $\mathbb{X}_i = \text{AB}$ and $v'_i = 1$ then $v_i = v'_i - 1$, $\varepsilon_{i+1} = 1$ and $\mathbb{X}_{i+1} = 0$. Thus $v'_{i+1} \leq 0$. We conclude by (a).
- (d) If $\mathbb{X}_i = \text{AB}$ and $v'_i = 0$, then $v_i = v'_i$, $\varepsilon_{i+1} = 0$ and $\mathbb{X}_{i+1} = 0$. Thus $v'_{i+1} \geq 1$.

□

5.3.2. Step 2: Types and shapes in the fibers. In the following we assume that $\mathbb{X} \in \{\text{A}, \text{B}, \text{AB}, 0\}^{\mathcal{J}'}$ satisfies conditions (1)–(4). The following proposition analyzes the fiber of the composite map

$$\left((s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type at } j), (v'_{f+j}, v'_j) \right)_{j \in \mathcal{J}} \mapsto (v'_{f+j}, v'_j)_{j \in \mathcal{J}} \mapsto \mathbb{X}(v')$$

where the domain is the set of f -tuples attached to triples (\tilde{w}, s, μ) as in the end of Section 5.2.

Proposition 5.3.6. *Let $\mathbb{X} \in \{\text{A}, \text{B}, \text{AB}, 0\}^{\mathcal{J}'}$ satisfy conditions (1)–(4). Assume that $\mathbb{X} = \mathbb{X}(v')$ for some $(s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type of } j, (v'_{j+f}, v'_j))$ associated to a triple (\tilde{w}, s, μ) .*

- (1) *If $\binom{\mathbb{X}_{j+f}}{\mathbb{X}_j} = \binom{\text{A}}{0}$ then $(s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type of } j, (v'_{j+f}, v'_j))$ belongs to the set*

$$\left\{ \begin{array}{l} (-, w_0, \text{id}, -, -, II, (0, < 0)), (\text{id}, w_0, \text{id}, \text{id}, t_{w_0(\eta)}, I, (0, 0)), \\ (\text{id}, \text{id}, w_0, \text{id}, t_\eta, 0, (0, 1)), (-, \text{id}, w_0, -, -, II, (0, \geq 2)) \end{array} \right\}.$$

If $\binom{\mathbb{X}_{j+f}}{\mathbb{X}_j} = \binom{\mathbf{B}}{\mathbf{0}}$ then $(s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type of } j, (v'_{j+f}, v'_j))$ belongs to the set

$$\left\{ \begin{array}{l} (-, w_0, \text{id}, -, -, II, (1, < 0)), (\text{id}, w_0, \text{id}, \text{id}, t_\eta, 0, (1, 0)), \\ (\text{id}, \text{id}, w_0, \text{id}, t_{w_0(\eta)}, I, (1, 1)), (-, \text{id}, w_0, -, -, II, (1, \geq 2)) \end{array} \right\}.$$

(2) If $\binom{\mathbb{X}_{j+f}}{\mathbb{X}_j} = \binom{\mathbf{0}}{\mathbf{AB}}$ then $(s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type of } j, (v'_{j+f}, v'_j))$ belongs to the set

$$\left\{ \begin{array}{l} (\text{id}, \text{id}, \text{id}, w_0, w_0 t_\eta, II, (\geq 2, 0)), (w_0, \text{id}, \text{id}, \text{id}, w_0 t_\eta, II, (\geq 2, 1)), \\ (w_0, w_0, w_0, \text{id}, w_0 t_\eta, II, (< 0, 0)), (\text{id}, w_0, w_0, w_0, w_0 t_\eta, II, (< 0, 1)) \end{array} \right\}.$$

(3) If $\binom{\mathbb{X}_{j+f}}{\mathbb{X}_j} = \binom{\mathbf{A}}{\mathbf{AB}}$ then $(s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type of } j, (v'_{j+f}, v'_j))$ belongs to the set

$$\left\{ \begin{array}{l} (w_0, w_0, \text{id}, w_0, t_\eta, I, (0, 0)), (\text{id}, w_0, \text{id}, \text{id}, t_{w_0(\eta)}, 0, (0, 1)), \\ (w_0, w_0, w_0, \text{id}, w_0 t_\eta, I, (0, 0)), (\text{id}, w_0, w_0, w_0, w_0 t_\eta, 0, (0, 1)) \end{array} \right\}.$$

If $\binom{\mathbb{X}_{j+f}}{\mathbb{X}_j} = \binom{\mathbf{B}}{\mathbf{AB}}$, then $(s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}, \tilde{w}_j, \text{type of } j, (v'_{j+f}, v'_j))$ belongs to the set

$$\left\{ \begin{array}{l} (w_0, \text{id}, w_0, w_0, t_\eta, I, (1, 1)), (\text{id}, \text{id}, w_0, \text{id}, t_{w_0(\eta)}, 0, (1, 0)), \\ (w_0, \text{id}, \text{id}, \text{id}, w_0 t_\eta, I, (1, 1)), (\text{id}, \text{id}, \text{id}, w_0, w_0 t_\eta, 0, (1, 0)) \end{array} \right\}.$$

(4) If $\binom{\mathbb{X}_{j+f}}{\mathbb{X}_j} \in \left\{ \binom{\mathbf{A}}{\mathbf{A}}, \binom{\mathbf{A}}{\mathbf{B}}, \binom{\mathbf{B}}{\mathbf{A}}, \binom{\mathbf{B}}{\mathbf{B}} \right\}$ then $s_{j+1}, s_{\text{or},j}, (v'_{j+f}, v'_j)$ and type of j are determined by \mathbb{X} .
Moreover if $\binom{\mathbb{X}_{j+f}}{\mathbb{X}_j} \in \left\{ \binom{\mathbf{A}}{\mathbf{A}}, \binom{\mathbf{B}}{\mathbf{B}} \right\}$, then \mathbb{X} determines if either $\tilde{w}_j = t_{w_0(\eta)}$ or $\tilde{w}_j \in \{w_0 t_\eta, t_\eta\}$.

The other cases are deduced by the transformation $\binom{\mathbb{X}_{j+f}}{\mathbb{X}_j} \mapsto \binom{\mathbb{X}_j}{\mathbb{X}_{j+f}}$ (see the caption of Table 8).

Proof. Proof of (1). Assume $\mathbb{X}_j = \mathbf{0}$ and $v'_j = 1$. Let $j_1 = \max\{j' \geq j \text{ such that } v'_\ell = 1, j_0 \leq \ell \leq j'\}$. Then $v'_{j_1+1} \geq 2$ and for all $j \leq j' \leq j_1$ we have $\binom{\mathbb{X}_{f+j'}}{\mathbb{X}_{j'}} \in \left\{ \binom{\mathbf{A}}{\mathbf{0}}, \binom{\mathbf{B}}{\mathbf{0}} \right\}$, $(s_{j'+1}, s_{\text{or},j'}, \Sigma_{j'}, z_{j'+1}) = (\text{id}, \text{id}, w_0, \text{id})$ and

$$(\tilde{w}_{j'}, \text{type of } j', (v'_{f+j'}, v'_{j'})) = \begin{cases} (t_\eta, 0, (0, 1)) & \text{if } \mathbb{X}_{f+j'} = \mathbf{A}, \\ (t_{w_0(\eta)}, I, (1, 1)) & \text{if } \mathbb{X}_{f+j'} = \mathbf{B}. \end{cases}$$

Indeed by $\clubsuit 4$ and Lemma 5.3.5, j_1 is well defined, $\mathbb{X}_{j'} = \mathbf{0}$, for all $j \leq j' \leq j_1 + 1$ and $v'_{j_1+1} \geq 2$. Hence $\binom{v'_{j_1+1+f}}{v'_{j_1+1}} \in \left\{ \binom{\mathbf{0}}{\geq 2}, \binom{\mathbf{1}}{\geq 2} \right\}$.

By Table 8 and symmetry, the index $j_1 + 1$ is of type II and $s_{\text{or},j_1+1} = \text{id}$, $\Sigma_{j_1+1} = w_0$. By $\clubsuit 1$ and $\clubsuit 4$, for all $j \leq j' \leq j_1$ we have $\mathbb{X}_{j'+f} \in \{\mathbf{A}, \mathbf{B}\}$, hence $v'_{f+j'} \in \{0, 1\}$ is determined by $\mathbb{X}_{f+j'}$. That is, for $j \leq j' \leq j_1$, $\binom{v'_{j'+f}}{v'_{j'}} \in \left\{ \binom{\mathbf{0}}{\mathbf{1}}, \binom{\mathbf{1}}{\mathbf{1}} \right\}$.

By decreasing induction and Table 8, since $s_{\text{or},j_1+1} = \text{id}$ and $\Sigma_{j_1+1} = w_0$, we have for $j \leq j' \leq j_1$

- if $\binom{v'_{j'+f}}{v'_{j'}} = \binom{\mathbf{0}}{\mathbf{1}}$, $(s_{j'+1}, s_{\text{or},j'}, \Sigma_{j'}, z_{j'+1}, \tilde{w}_{j'}, \text{type of } j') = (\text{id}, \text{id}, w_0, \text{id}, t_\eta, 0)$,
- if $\binom{v'_{j'+f}}{v'_{j'}} = \binom{\mathbf{1}}{\mathbf{1}}$, $(s_{j'+1}, s_{\text{or},j'}, \Sigma_{j'}, z_{j'+1}, \tilde{w}_{j'}, \text{type of } j') = (\text{id}, \text{id}, w_0, \text{id}, t_{w_0(\eta)}, I)$.

The proof for $v'_j = 0$ can be deduced directly by symmetry. The cases $v'_j < 0$ and $v'_j \geq 2$ follow from Table 8.

Proof of (2). Assume $\binom{\mathbb{X}_{j+f}}{\mathbb{X}_j} = \binom{\mathbf{0}}{\mathbf{AB}}$. By $\clubsuit 1$ and $\clubsuit 4$, we have $\mathbb{X}_{j+1+f} \neq \mathbf{0}$. Hence $v'_{f+j} \geq 2$ or < 0 . By Table 8 and (1) we have $(s_{\text{or},j+1}, \Sigma_{j+1}) \in \{(w_0, \text{id}), (\text{id}, w_0)\}$.

Then

$$(s_{j+1}, s_{\text{or},j}, \Sigma_j, z_{j+1}) \in \{(w_0, \text{id}, \text{id}, \text{id}), (\text{id}, \text{id}, \text{id}, w_0), (\text{id}, w_0, w_0, w_0), (w_0, w_0, w_0, \text{id})\}$$

and $\tilde{w}_j = w_0 t_\eta$ and the type of j is *II*.

Proof of (3). If $\left(\frac{\mathbb{X}_{j+f}}{\mathbb{X}_j}\right) \in \left\{\left(\begin{smallmatrix} \mathbf{A} \\ \mathbf{AB} \end{smallmatrix}\right), \left(\begin{smallmatrix} \mathbf{B} \\ \mathbf{AB} \end{smallmatrix}\right)\right\}$, then $\left(v'_{j+f}, v'_j\right) = \left\{\left(\begin{smallmatrix} 0 \\ (0,1) \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ (0,1) \end{smallmatrix}\right)\right\}$, $\mathbb{X}_{j+1} = \mathbf{0}$ and

$$v'_{j+1} = \begin{cases} \geq 1 & \text{if } v'_j = 0, & \text{hence } (s_{\text{or},j+1}, \Sigma_{j+1}) = (\text{id}, w_0), \\ \leq 0 & \text{if } v'_j = 1, & \text{hence } (s_{\text{or},j+1}, \Sigma_{j+1}) = (w_0, \text{id}). \end{cases}$$

We conclude from Table 8.

Proof of (4). Assume $\mathbb{X}_j, \mathbb{X}_{j+f} \in \{\mathbf{A}, \mathbf{B}\}$.

- Assume that there exists j_1 such that

$$\begin{cases} \mathbb{X}_{j'}, \mathbb{X}_{j'+f} \in \{\mathbf{A}, \mathbf{B}\}, j \leq j' \leq j_1 - 1, \\ \mathbf{AB} \in \{\mathbb{X}_{j_1}, \mathbb{X}_{j_1+f}\}. \end{cases}$$

By (3) we see that s_{or,j_1} is determined by $\left(\frac{\mathbb{X}_{j_1+f}}{\mathbb{X}_{j_1}}\right)$. Since $\mathbb{X}_{j'}, \mathbb{X}_{j'+f} \in \{\mathbf{A}, \mathbf{B}\}$, for $j \leq j' \leq j_1 - 1$ we deduce by symmetry and decreasing induction from Table 8 that $s_{j'+1}, s_{\text{or},j'}$ and the type at j' are determined by $\left(\frac{\mathbb{X}_{j'+f}}{\mathbb{X}_{j'}}\right)$ for $j \leq j' \leq j_1 - 1$.

- Assume that such j_1 does not exist, i.e. that $\mathbb{X}_{j'} \in \{\mathbf{A}, \mathbf{B}\}$ for all $j' \in \mathcal{J}'$. Then for all $j' \in \mathcal{J}'$ either $s_{j'+1}$ or $s_{\text{or},j'}$ is determined by $\left(\frac{\mathbb{X}_{j'+f}}{\mathbb{X}_{j'}}\right)$ (cf. Table 8). If there exists $i \in \mathcal{J}$ such that $\left(\frac{\mathbb{X}_{i+f}}{\mathbb{X}_i}\right) \in \left\{\left(\begin{smallmatrix} \mathbf{A} \\ \mathbf{B} \end{smallmatrix}\right), \left(\begin{smallmatrix} \mathbf{B} \\ \mathbf{A} \end{smallmatrix}\right)\right\}$, there is a unique choice of $s_{\text{or},i}$ such that $s_{\text{or},f-1} = \text{id}$. If $\left(\frac{\mathbb{X}_{i+f}}{\mathbb{X}_i}\right) \in \left\{\left(\begin{smallmatrix} \mathbf{A} \\ \mathbf{A} \end{smallmatrix}\right), \left(\begin{smallmatrix} \mathbf{B} \\ \mathbf{B} \end{smallmatrix}\right)\right\}$ for all $i \in \mathbb{Z}/f\mathbb{Z}$ then the \mathcal{J} -tuple $(s_{\text{or},i})$ is determined by \mathbb{X} . By $(\clubsuit 3)$ there exists $j' \in \mathcal{J}'$ with $\mathbb{X}_{j'} \neq \mathbb{X}_{j'+1}$. By Tables 8 and symmetry we obtain $s_{j'+1} = w_0$. By induction we conclude that $(s_{j+1}, s_{\text{or},j})_{j \in \mathcal{J}}$ and the type of j are determined by \mathbb{X} .

Moreover, for $\left(\frac{\mathbb{X}_{j+f}}{\mathbb{X}_j}\right) \in \left\{\left(\begin{smallmatrix} \mathbf{A} \\ \mathbf{A} \end{smallmatrix}\right), \left(\begin{smallmatrix} \mathbf{B} \\ \mathbf{B} \end{smallmatrix}\right)\right\}$, we deduce from Table 8 (and symmetry) that the data of $(s_{j+1}, s_{\text{or},j})$ determines whether $\tilde{w}_j = t_{w_0(\eta)}$ or $\tilde{w}_j \in \{w_0 t_\eta, t_\eta\}$ \square

5.4. Naive equations associated to a gene. Let $\mathbb{X} \in \{\mathbf{A}, \mathbf{B}, \mathbf{AB}, \mathbf{0}\}^{\mathcal{J}'}$ be a gene satisfying conditions $(\clubsuit 1)$ – $(\clubsuit 4)$. Assume that $\mathbb{X} = \mathbb{X}(v')$ for some $(v'_{f+j}, v'_j)_{j \in \mathcal{J}}$ associated to a triple (\tilde{w}, s, μ) .

Definition 5.4.1. Let $\mathbb{X} = (\mathbb{X}_j)_{j \in \mathcal{J}'}$ be a gene satisfying $(\clubsuit 1)$ – $(\clubsuit 4)$. A *cluster* for \mathbb{X} is a sequence $(\mathbb{X}_{j+f}, \mathbb{X}_j)_{j_0 \leq j \leq j_1}$ such that there exists $\ell \in \{j_0, \dots, j_1\}$ satisfying

- $\mathbb{X}_j = \mathbf{0}$ for all $j \in \{\ell, \dots, j_1\}$, $\mathbb{X}_{j_1+1} \neq \mathbf{0}$ and $\mathbb{X}_{\ell-1} = \mathbf{AB}$,
- $\mathbf{0} \notin \{\mathbb{X}_j, \mathbb{X}_{j+f}\}$, $j_0 + 1 \leq j \leq \ell - 1$,
- $\mathbf{0} \in \{\mathbb{X}_{j_0}, \mathbb{X}_{j_0+f}\}$.

We remark that $\mathbb{X} = (\mathbb{X}_j)_{j \in \mathcal{J}'}$ has either a *unique* decomposition $\mathbb{X} = \cup_{i=0}^r (\mathbb{X}_{j_i}, \mathbb{X}_{j_i+f})_{j_i \leq j \leq j_{i+1}}$ into clusters (with the convention $j_{r+1} = j_0$), or does not have *any* cluster (in which case $\mathbb{X}_j \in \{\mathbf{A}, \mathbf{B}\}$ for all $j \in \mathcal{J}'$). Note finally that j_i are of type *II* for all $i = 0, \dots, r$ by Table 9.

Let $\cup_{k \in \mathcal{K}} \mathcal{J}_k$ be the fragmentation associated to the triple (\tilde{w}, s, μ) (Definition 4.3.12). Assume \mathbb{X} admits a decomposition into clusters $\mathbb{X} = \cup_{i=0}^r (\mathbb{X}_{j_i}, \mathbb{X}_{j_i+f})_{j_i \leq j \leq j_{i+1}}$. Note that given a cluster $(\mathbb{X}_{j+f}, \mathbb{X}_j)_{j_i \leq j \leq j_{i+1}}$ the sequence (j_i, \dots, j_{i+1}) is a union of fragments of \mathcal{K} , as both j_i and j_{i+1} are of type *II*. For $i \in \{0, \dots, r\}$, we denote

$$R_{[j_i, j_{i+1}]} \stackrel{\text{def}}{=} \mathcal{O}[Y_{j_i}] \otimes \left(\otimes_{i+1 \leq j \leq j_{i+1}-1} R_j\right) \otimes \mathcal{O}[X_{j_{i+1}}] \text{ and } \mathcal{I}_{[j_i, j_{i+1}]} \stackrel{\text{def}}{=} \sum_{k \in \mathcal{K} \cap [j_i, j_{i+1}]} \mathcal{I}_k$$

where for all $k \in \mathcal{K} \cap [j_i, j_{i+1}]$, the ideal $\mathcal{I}_k \subset R_{[j_i, j_{i+1}]}$ is generated by the entries of the matrix equation (4.6.5) associated to \mathcal{J}_k .

Proposition 5.4.2. *Let $\mathbb{X} = (\mathbb{X}_j, \mathbb{X}_{j+f})_{j \in \mathcal{J}}$ be a gene satisfying $(\clubsuit 1)$ – $(\clubsuit 3)$.*

- (1) *Assume that \mathbb{X} has a decomposition into clusters $\mathbb{X} = \cup_{i=0}^r (\mathbb{X}_{j_i}, \mathbb{X}_{j_i+f})_{j_i \leq j \leq j_{i+1}}$. Let (\tilde{w}, s, μ) be a triple in the fiber above \mathbb{X} of the map (5.3.1). Then*

$$\mathcal{Z}^{\text{nv}, \tau}(\tilde{z}) = \text{Spec} \left(\otimes_{i=0}^r R_{[j_i, j_{i+1}]} / \mathcal{I}_{[j_i, j_{i+1}]} \right).$$

Moreover, the p -saturation of $\mathcal{Z}^{\text{nv}, \tau}(\tilde{z})$ depends only on \mathbb{X} .

- (2) *Assume \mathbb{X} does not have clusters (i.e. $\mathbb{X} \in \{\mathbf{A}, \mathbf{B}\}^{\mathcal{J}'}$). Then $\mathcal{Z}^{\text{nv}, \tau}(\tilde{z})$ depends only on \mathbb{X} . In particular, the p -saturation of $\mathcal{Z}^{\text{nv}, \tau}(\tilde{z})$ depends only on \mathbb{X} .*

The proof of Proposition 5.4.2 relies on the analysis of each of the ideals $\mathcal{I}_{[j_i, j_{i+1}]}$. This analysis is preformed, for a fixed cluster, in Lemmas 5.4.3, 5.4.9 and 5.4.13 below, which deal with the $\mathbf{0}$, the \mathbf{AB} , and the (\mathbf{A}, \mathbf{B}) -part of the cluster.

Thus, we fix once and for all a cluster $(\mathbb{X}_j, \mathbb{X}_{j+f})_{j_0 \leq j \leq j_1}$ of \mathbb{X} . We let $\{k_0, \dots, k_s\} \stackrel{\text{def}}{=} \mathcal{K} \cap [j_0, j_1]$ so that

- for all $j = 0, \dots, s-1$, $\mathcal{J}_{k_j} = (i_{k_j}, i_{k_j} - 1, \dots, o_{k_j} + 1, o_{k_j}) \subset (j_1, j_1 - 1, \dots, j_0)$ and $o_{k_j} = i_{k_{j+1}}$
- $i_{k_0} = j_1$, $o_{k_s} = j_0$.

Let $s' \in [0, s]$ such that $\ell - 1 \in (o_{k_{s'}}, \dots, i_{k_{s'}} - 1)$.

Given $j \in [0, s]$, we define $\bar{\mathcal{I}}_{k_j}$ as the image of \mathcal{I}_{k_j} in $R_{[j_0, j_1]} / \sum_{j'=0}^{j-1} \mathcal{I}_{k_{j'}}$. We still denote by X_i, Y_i, Z_i etc. the variables of $R_{[j_0, j_1]}$ in the quotient $R_{[j_0, j_1]} / \sum_{j'=0}^{j-1} \mathcal{I}_{k_{j'}}$.

Lemma 5.4.3 (the $\mathbf{0}$ -part of a cluster). *For any $0 \leq i \leq s' - 1$, the ideal $\bar{\mathcal{I}}_{k_i}$ satisfies the following property: there exists $r_{k_i} \in \mathbb{N}$ such that*

- (1) $\bar{\mathcal{I}}_{k_i} = (Y_{o_{k_i}})$ if the type of both o_{k_i} and i_{k_i} is II ,
- (2) $\bar{\mathcal{I}}_{k_i} = (f_{k_i}, g_{k_i})$ if the type of o_{k_i} is I and the type of i_{k_i} is II ,
- (3) $p^{r_{k_i}} \bar{\mathcal{I}}_{k_i} = p^{r_{k_i}} (f_{k_i}, g_{k_i})(p, X_{i_{k_i}})$ if the type of both o_{k_i} and i_{k_i} is I ,
- (4) $p^{r_{k_i}} \bar{\mathcal{I}}_{k_i} = p^{r_{k_i}} (Y_{o_{k_i}})(p, X_{i_{k_i}})$ if the type of o_{k_i} is II and the type of i_{k_i} is I ,

where

$$\begin{cases} f_{k_i} \stackrel{\text{def}}{=} p - Y_{o_{k_i}} - X_{o_{k_i}+1} X_{o_{k_i}}, \\ g_{k_i} \stackrel{\text{def}}{=} Z_{o_{k_i}} - X_{o_{k_i}+1} (p - X_{o_{k_i}+1} X_{o_{k_i}}). \end{cases}$$

Proof. If $s' = 0$, there is nothing to prove. We assume now $s' \geq 1$.

By definition of ℓ , $\mathbb{X}_j = \mathbf{0}$ for all $j \in \{\ell, \dots, j_1\}$ and $\mathbb{X}_{j+f} \in \{\mathbf{A}, \mathbf{B}\}$ for all $j \in \{\ell, \dots, j_1 - 1\}$. Thus by Proposition 5.3.6(1) and an induction (using Table 8 to deal when j is of type II) we have for $j \in \{\ell, \dots, j_1 - 1\}$:

$$(5.4.4) \quad (\tilde{w}_j, \text{type of } j) \in \{(t_\eta, 0), (t_{w_0(\eta)}, I), (t_\eta, II), (t_{w_0(\eta)}, II)\}.$$

Recall that for any $k \in \mathcal{K}$ and fragment $(i_k, i_k + 1, \dots, o_k - 1, o_k)$, we know that i_k and o_k are of type either I or II , and $i_k - 1, \dots, o_k + 1$ are of type $\mathbf{0}$. Thus for any $i \in \{0, \dots, s' - 1\}$ we get from (5.4.4) and Table 5:

$$\prod_{l=o_{k_i}+1}^{i_{k_i}-1} T_l = \prod_{l=o_{k_i}+1}^{i_{k_i}-1} \begin{pmatrix} 1 & 0 \\ -X_l & p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \sum_{l=0}^{i_{k_i}-o_{k_i}-2} -p^l X_{o_{k_i}+1+l} & p^{i_{k_i}-o_{k_i}-1} \end{pmatrix}.$$

To prove the lemma we proceed by induction on $i \in \{0, \dots, s' - 1\}$ analyzing the matrices $M_{\text{out}, o_{k_i}}$ and $M_{\text{in}, i_{k_i}}$ on the fragments \mathcal{J}_{k_i} . We abbreviate $k \stackrel{\text{def}}{=} k_i$ in what follows. For $i = 0$, only the items (1) and (2) can happen (as i_{k_0} is of type *II*).

Proof of item (1). Assume o_k and i_k are of type *II*. Then by (5.4.4) and Table 5:

$$M_{\text{out}, o_k} = \begin{cases} (Y_{o_k}, -p) & \text{if } \tilde{w}_{o_k} = t_\eta, \\ (0, 1) & \text{if } \tilde{w}_{o_k} = t_{w_0(\eta)}, \end{cases} \quad \text{and } M_{\text{in}, i_k} = \begin{pmatrix} 1 \\ -X_{i_k} \end{pmatrix}.$$

After the change of variable $X_{o_{k+1}} \mapsto \sum_{l=0}^{i_k - o_k - 1} p^l X_{o_{k+1}+l}$ in $R_{[j_0, j_1]}$ we have

$$(5.4.5) \quad \left(\prod_{l=o_k+1}^{i_k-1} T_l \right) M_{\text{in}, i_k} = \begin{pmatrix} 1 \\ -X_{o_{k+1}} \end{pmatrix}.$$

(In particular, note that the “inner” variables of $R_{[j_0, j_1]}$ in the outcome of the Lemma are not the same as those used in Table 5. This change of variable can be checked not to be relevant in the gluing of Proposition 5.4.2, which uses equations from Table 5 involving only the “outer” variables of $R_{[j_0, j_1]}$ for each cluster.) Then

$$\mathcal{I}_k = \begin{cases} (Y_{o_k} - pX_{o_{k+1}}) & \text{if } \tilde{w}_{o_k} = t_\eta, \\ (X_{o_{k+1}}) & \text{if } \tilde{w}_{o_k} = t_{w_0(\eta)}. \end{cases}$$

Thus, up to the change of variable $Y_{o_k} \mapsto Y_{o_k} - pX_{o_{k+1}}$ if $\tilde{w}_{o_k} = t_\eta$ or $Y_{o_k} \mapsto X_{o_{k+1}}$, $X_{o_{k+1}} \mapsto Y_{o_k}$ if $\tilde{w}_{o_k} = t_{w_0(\eta)}$, we have $\mathcal{I}_k = (Y_{o_k})$ and item (1) holds.

Proof of item (2). Assume o_k is of type *I* and i_k is of type *II*. As before we have (5.4.5) after replacing $X_{o_{k+1}}$ by $\sum_{l=0}^{i_k - o_k - 1} p^l X_{o_{k+1}+l}$ and hence, using again (5.4.4) and Table 5

$$(5.4.6) \quad M_{\text{out}, o_k} \left(\prod_{l=o_k+1}^{i_k-1} T_l \right) M_{\text{in}, i_k} = \begin{pmatrix} p - Y_{o_k} & X_{o_k} \\ Z_{o_k} & Y_{o_k} \end{pmatrix} \begin{pmatrix} 1 \\ -X_{o_{k+1}} \end{pmatrix} = \begin{pmatrix} p - Y_{o_k} - X_{o_{k+1}}X_{o_k} \\ Z_{o_k} - Y_{o_k}X_{o_{k+1}} \end{pmatrix}$$

Thus $\mathcal{I}_k = (p - Y_{o_k} - X_{o_{k+1}}X_{o_k}, Z_{o_k} - X_{o_{k+1}}(p - X_{o_{k+1}}X_{o_k}))$ and item (2) holds.

Proof of item (3). Assume o_k and i_k are both of type *I*. Since j_1 is of type *II* and $i > 0$, there exist $i' \in [0, i - 1]$ such that $o_{k_l} = i_{k_{l+1}}$ is of type *I* for $i' \leq l \leq i - 1$ and $i_{k_{i'}}$ is of type *II*. By item (2) applied to $k_{i'}$ and noting that $o_{k_{i'}} = i_{k_{i'+1}}$ we see that $M_{\text{in}, i_{k_{i'+1}}}$ equals:

$$(5.4.7) \quad \begin{pmatrix} Y_{i_{k_{i'+1}}} & -X_{i_{k_{i'+1}}} \\ -Z_{i_{k_{i'+1}}} & p - Y_{i_{k_{i'+1}}} \end{pmatrix} \equiv \begin{pmatrix} 1 \\ -X_{i_{k_{i'+1}+1}} \end{pmatrix} (p - X_{i_{k_{i'+1}+1}}X_{i_{k_{i'+1}}}, -X_{i_{k_{i'+1}}}) \quad \text{modulo } \mathcal{I}_{k_{i'}}.$$

We conclude by (5.4.4) and Table 5) that the image $\bar{\mathcal{I}}_{k_{i'+1}}$ of $\mathcal{I}_{k_{i'+1}}$ in $R_{[j_0, j_1]} / (\sum_{l=0}^{i'} \mathcal{I}_{k_l})$ is generated by the equations

$$(5.4.8) \quad \begin{pmatrix} p - Y_{o_{k_{i'+1}}} & X_{o_{k_{i'+1}}} \\ Z_{o_{k_{i'+1}}} & Y_{o_{k_{i'+1}}} \end{pmatrix} \begin{pmatrix} 1 \\ -(\sum_{l=0}^{i_{k_{i'+1}} - o_{k_{i'+1}} - 2} p^l X_{o_{k_{i'+1}+1+l}} - p^{o_{k_{i'+1}} - o_{k_{i'+1}} - 1} X_{i_{k_{i'+1}}}) \end{pmatrix} (p - X_{i_{k_{i'+1}+1}}X_{i_{k_{i'+1}}}, -X_{i_{k_{i'+1}}}) = 0.$$

Hence, after the change of variable $X_{o_{k_{i'+1}+1}} \mapsto \sum_{l=0}^{i_{k_{i'+1}} - o_{k_{i'+1}} - 2} p^l X_{o_{k_{i'+1}+1+l}} - p^{o_{k_{i'+1}} - o_{k_{i'+1}} - 1} X_{i_{k_{i'+1}}}$,

the image $\bar{\mathcal{I}}_{k_{i'+1}}$ of $\mathcal{I}_{k_{i'+1}}$ in $R_{[j_0, j_1]} / (\sum_{i=0}^{i'} \mathcal{I}_{k_i})$ satisfies $\bar{\mathcal{I}}_{k_{i'+1}} = (f_{k_{i'+1}}, g_{k_{i'+1}})(p, X_{i_{k_{i'+1}}})$ where $f_{k_{i'+1}} = p - Y_{o_{k_{i'+1}}} - X_{o_{k_{i'+1}}}X_{o_{k_{i'+1}+1}}$ and $g_{k_{i'+1}} = Z_{o_{k_{i'+1}}} - X_{o_{k_{i'+1}+1}}(p - X_{o_{k_{i'+1}}}X_{o_{k_{i'+1}+1}})$. We now induct on $l \in \{0, \dots, i - i' - 1\}$ the case $l = 0$ being covered above. Indeed,

assuming by induction that $p^{l-1}\bar{\mathcal{I}}_{k_{i'+l-1}} = p^{l-1}(f_{k_{i'+l-1}}, g_{k_{i'+l-1}})(p, X_{i_{k_{i'+l-1}}})$ (note that the term $(p, X_{i_{k_{i'+l-1}}})$ only appears for $l > 1$) we can repeat the same computations above (multiplying bot side of equations (5.4.7), (5.4.8) by p^{l-1}) to show that the image $\bar{\mathcal{I}}_{k_{i'+l}}$ of $\mathcal{I}_{k_{i'+l}}$ in $R_{[j_0, j_1]}/\sum_{l'=0}^{l-1} \mathcal{I}_{k_{i'+l'}}$ satisfies $p^l \bar{\mathcal{I}}_{k_{i'+l}} = p^l(f_{k_{i'+l}}, g_{k_{i'+l}})(p, X_{i_{k_{i'+l}}})$, where $f_{k_{i'+l}} = p - Y_{o_{k_{i'+l}}} - X_{o_{k_{i'+l}}} X_{o_{k_{i'+l}+1}}$ and $g_{k_{i'+l}} = Z_{o_{k_{i'+l}}} - X_{o_{k_{i'+l}+1}}(p - X_{o_{k_{i'+l}}} X_{o_{k_{i'+l}+1}})$.

Proof of item (4). Assume i_k is of type *I* and o_k is of type *II*. Then k_{i-1} satisfies the hypotheses of (2) or (3) Similar computations as in (3) show now that the image of $p^{r_k} \mathcal{I}_k$ in $R_{[j_0, j_1]}/(\sum_{l=0}^{i-1} \mathcal{I}_{\mathcal{J}_l})$ is generated by the equations

$$p^{r_k} M_{\text{out}, o_k} \left(\begin{array}{c} 1 \\ -(\sum_{l=0}^{i_k - o_k - 2} p^l X_{o_k + 1 + l} - p^{o_{k_{i-1}} - o_k - 1} X_{i_{k_{i-1}}}) \end{array} \right) (p - X_{i_{k+1}} X_{i_k}, -X_{i_k}) = 0$$

where

$$M_{\text{out}, o_k} = \begin{cases} (Y_{o_k}, -p) & \text{if } \tilde{w}_{o_k} = t_\eta, \\ (0, 1) & \text{if } \tilde{w}_{o_k} = t_{w_0(\eta)}, \end{cases}$$

Then (after exchanging and/or replacing the variables in the same way as we did for (1)), we have $r_k \in \mathbb{N}$ such that $p^{r_k} \bar{\mathcal{I}}_k = p^{r_k}(Y_{o_k})(p, X_{i_k})$. □

Lemma 5.4.9 (the AB-part of a cluster). *Let $k \stackrel{\text{def}}{=} k_{s'}$.*

(1) *If the type of $\ell - 1$ is *I* then the image $\bar{\mathcal{I}}_k$ of \mathcal{I}_k in $R_{[j_0, j_1]}/\sum_{i=0}^{s'-1} \mathcal{I}_{k_i}$ satisfies the following property: there exists $r_k \in \mathbb{N}$ such that*

$$\begin{cases} \bar{\mathcal{I}}_k = (f_k, g_k) & \text{if } i_k \text{ is of type II,} \\ p^{r_k} \bar{\mathcal{I}}_k = p^{r_k}(f_k, g_k)(p, X_{i_k}) & \text{if } i_k \text{ is of type I,} \end{cases}$$

with $f_k \stackrel{\text{def}}{=} p - Y_{o_k} - X_{o_k+1} X_{o_k}$, $g_k \stackrel{\text{def}}{=} Z_{o_k} - X_{o_k+1}(p - X_{o_k+1} X_{o_k})$. Moreover there exists $r_{k_{s'+1}} \in \mathbb{N}$ such that $p^{r_{k_{s'+1}}} \bar{\mathcal{I}}_{k_{s'+1}}$ is generated by the equations

$$p^{r_{k_{s'+1}}} M_{\text{out}, o_k} \left(\prod_{l=o_k+1}^{\ell-2} T_l \right) M_{\ell-1} = 0$$

where,

$$M_{\ell-1} = \begin{cases} \Sigma_{\ell-1} \left(p - \frac{-X_{\ell-1}}{X_\ell X_{\ell-1}} \right) & \text{if } \mathbb{X}_{\ell-1+f} = \mathbf{A}, \\ w_0 \Sigma_{\ell-1} \left(p - \frac{-X_{\ell-1}}{X_\ell X_{\ell-1}} \right) & \text{if } \mathbb{X}_{\ell-1+f} = \mathbf{B}. \end{cases}$$

(2) *If the type $\ell - 1$ is 0, then the image $\bar{\mathcal{I}}_k$ of \mathcal{I}_k in $R_{[j_0, j_1]}/\sum_{i=0}^{s'-1} \mathcal{I}_{k_i}$ satisfies the following property: there exists $r_k \in \mathbb{N}$ such that $p^{r_k} \bar{\mathcal{I}}_k$ is generated by the equations*

$$p^{r_k} M_{\text{out}, o_k} \left(\prod_{l=o_k+1}^{\ell-2} T_l \right) M_{\ell-1} = 0$$

where

$$M_{\ell-1} \stackrel{\text{def}}{=} \begin{cases} (w_0)^{\delta_{\mathbb{X}_{f+\ell-1}=\mathbf{B}}} \Sigma_{\ell-1} \left(p - \frac{-X_{\ell-1}}{X_{\ell-1} X_\ell} \right) & \text{if } i_k \text{ is of type II,} \\ (w_0)^{\delta_{\mathbb{X}_{f+\ell-1}=\mathbf{B}}} \Sigma_{\ell-1} \left(p - \frac{-X_{\ell-1}}{X_{\ell-1} X_\ell} \right) (X_{\ell+1}, p) & \text{if } i_k \text{ is of type I,} \end{cases}$$

(3) If the type of $\ell - 1$ is *II*, then the image $\bar{\mathcal{I}}_k$ of \mathcal{I}_k in $R_{[j_0, j_1]} / \sum_{i=0}^{s'-1} \mathcal{I}_{k_i}$ satisfies the following property: there exists $r_k \in \mathbb{N}$ such that

$$\begin{cases} \bar{\mathcal{I}}_k = (h_k) & \text{if } i_k \text{ is of type II,} \\ p^{r_k} \bar{\mathcal{I}}_k = p^{r_k}(h_k)(p, X_{i_k}) & \text{if } i_k \text{ is of type I,} \end{cases}$$

where $h_k = (p + X_{o_k+1} Y_{o_k})$.

Proof. If the type of $\ell - 1$ is *I* or *0*, then $\ell - 1 > j_0$ (as j_0 is of type *II*) and $(\mathbb{X}_{\ell-1}, \mathbb{X}_{\ell-1+f}) \in \{(\mathbf{AB}, \mathbf{A}), (\mathbf{AB}, \mathbf{B})\}$. By Proposition 5.3.63,

$$(\tilde{w}_{\ell-1}, \text{ type of } \ell - 1) \in \{(t_\eta, I), (w_0 t_\eta, I), (w_0 t_\eta, 0), (t_{w_0(\eta)}, 0)\}.$$

Proof of Item (1). If $\ell - 1$ is of type *I*, then $o_k = \ell - 1$. Using 5.4.3 (2)-(3) to describe $R_{[j_0, j_1]} / \sum_{l=0}^{s'-1} \mathcal{I}_{k_l}$ and performing the reasoning of the proof of *loc. cit.* (replacing the final matrix at o_k in equations (5.4.6), (5.4.8) with the final matrix associated to either (t_η, I) or $(w_0 t_\eta, I)$ according to \tilde{w}_{o_k}) there exists $r_k \in \mathbb{N}$ such that the image $\bar{\mathcal{I}}_k$ of \mathcal{I}_k in $R_{[j_0, j_1]} / \sum_{l=0}^{s'-1} \mathcal{I}_{k_l}$ satisfies

$$\begin{cases} \bar{\mathcal{I}}_k = (f_k, g_k) & \text{if } i_k \text{ is of type II,} \\ p^{r_k} \bar{\mathcal{I}}_k = p^{r_k}(f_k, g_k)(p, X_{i_k}) & \text{if } i_k \text{ is of type I,} \end{cases}$$

where

$$(5.4.10) \quad \begin{cases} f_k \stackrel{\text{def}}{=} p - Y_{o_k} - X_{o_k} X_{o_k+1}, \\ g_k \stackrel{\text{def}}{=} Z_{o_k} - X_{o_k+1}(p - X_{o_k} X_{o_k+1}) \end{cases}$$

Again, note that the above equations are obtained by replacing X_{o_k+1} by $-X_{o_k+1}$ and, more importantly in what follows, exchanging X_{o_k} and Z_{o_k} (resp. replacing Y_{o_k} by $p - Y_{o_k}$) when $\tilde{w}_{\ell-1} = t_\eta$ (resp. $\tilde{w}_{\ell-1} = w_0 t_\eta$).

Letting $r_{k_{s'+1}}$ be $r_{k_{s'}} + 1$ if i_k is of type *I* and be 0 if i_k is of type *II*, we furthermore claim that $p^{r_{k_{s'+1}}} \bar{\mathcal{I}}_{k_{s'+1}}$ is generated by the equations

$$p^{r_{k_{s'+1}}} M_{\text{out}, o_{k_{s'+1}}} \left(\prod_{l=o_{k_{s'+1}}+1}^{\ell-2} T_l \right) M_{\ell-1} = 0$$

where,

$$M_{\ell-1} \stackrel{\text{def}}{=} (w_0)^{\delta_{(\mathbb{X}_f + \ell - 1 = \mathbf{B})}} \Sigma_{\ell-1} \begin{pmatrix} -X_{\ell-1} \\ p - X_\ell X_{\ell-1} \end{pmatrix}.$$

Indeed, keeping in mind the change of variables $X_{o_k} \leftrightarrow Z_{o_k}$ (resp. $Y_{o_k} \leftrightarrow p - Y_{o_k}$) when $\tilde{w}_{\ell-1} = t_\eta$ (resp. $\tilde{w}_{\ell-1} = w_0 t_\eta$) we obtain from row *I* of Table 5 (recall $i_{k_{s'+1}} = o_{k_{s'}} = \ell - 1$ is of type *I*) and (5.4.10)

$$\begin{aligned} p^{r_{k_{s'+1}}} M_{\text{in}, i_{k_{s'+1}}} &\equiv p^{r_{k_{s'+1}}} w_{\ell-1} \begin{pmatrix} p - X_{\ell-1} X_\ell & -X_\ell (p - X_{\ell-1} X_\ell) \\ -X_{\ell-1} & X_{\ell-1} X_\ell \end{pmatrix} w_{\ell-1}^{-1} \\ &= p^{r_{k_{s'+1}}} \begin{pmatrix} p - X_{\ell-1} X_\ell \\ -X_{\ell-1} \end{pmatrix} (1, -X_\ell) w_{\ell-1}^{-1} \end{aligned}$$

in $R_{[j_0, j_1]} / \sum_{l=0}^{s'} \mathcal{I}_{k_l}$, where $w_{\ell-1}$ is the permutation part of $\tilde{w}_{\ell-1}$. As $i_{k_{s'+1}}$ is of type *I* we deduce from Proposition 5.3.6(3) that $w_{\ell-1} = (w_0)^{\delta_{\mathbb{X}_f + \ell - 1 = \mathbf{B}}} \Sigma_{\ell-1}$ and the claim follows noting that the term $(1, -X_\ell) w_{\ell-1}^{-1}$ can be ignored when imposing condition (4.6.5).

Proof of item (2). A similar reasoning as in the proof of Lemma 5.4.3 shows that there exists $r_k \in \mathbb{N}$ and a change of variable for X_ℓ such that we have the following equality in $R_{[j_0, j_1]} / \sum_{i=0}^{s'-1} \mathcal{I}_{k_i}$:

$$(5.4.11) \quad p^{r_k} \left(\prod_{l=\ell}^{i_k-1} T_l \right)_{\text{in}, i_k} = \begin{cases} p^{r_k} \begin{pmatrix} 1 \\ -X_\ell \end{pmatrix} & \text{if } i_k \text{ is of type II,} \\ p^{r_k} \begin{pmatrix} 1 \\ X_\ell \end{pmatrix} (p - X_{i_k+1} X_{i_k}, -X_{i_k}) & \text{if } i_k \text{ is of type I.} \end{cases}$$

Moreover if $\ell - 1$ of type 0, then $(\tilde{w}_{\ell-1}, \text{ type of } \ell - 1) \in \{(w_0 t_\eta, 0), (t_{w_0(\eta)}, 0)\}$ and by Table 5 we have

$$(5.4.12) \quad T_{\ell-1} = w_{\ell-1} \begin{pmatrix} p & -X_{\ell-1} \\ 0 & 1 \end{pmatrix}$$

where $w_{\ell-1}$ is the permutation part of $\tilde{w}_{\ell-1}$.

As in the previous item, we now claim that there exists $r_k \in \mathbb{N}$ such that $p^{r_k} \bar{\mathcal{I}}_k$ is generated by the equations

$$p^{r_k} M_{\text{out}, o_k} \left(\prod_{l=o_k+1}^{\ell-2} T_l \right) M_{\ell-1} = 0$$

where

$$M_{\ell-1} = \begin{cases} (w_0)^{\delta_{\mathbb{X}^{f+\ell-1}=\mathbb{B}} \Sigma_{\ell-1}} \begin{pmatrix} p - X_{\ell-1} X_\ell \\ -X_{\ell-1} \end{pmatrix} & \text{if } i_k \text{ is of type II,} \\ (w_0)^{\delta_{\mathbb{X}^{f+\ell-1}=\mathbb{B}} \Sigma_{\ell-1}} \begin{pmatrix} p - X_{\ell-1} X_\ell \\ -X_{\ell-1} \end{pmatrix} (p, X_{\ell+1}) & \text{if } i_k \text{ is of type I.} \end{cases}$$

But this is clear from (5.4.11), (5.4.12) noting that from Proposition 5.3.6(3) we have $w_{\ell-1} = (w_0)^{\delta_{\mathbb{X}^{f+\ell-1}=\mathbb{B}} \Sigma_{\ell-1}}$ and that, when i_k is of type I, we can replace $(p - X_{i_k+1} X_{i_k}, -X_{i_k})$ by $(p, -X_{i_k})$ to obtain a system of equations equivalent to (4.6.5).

Proof of item (3). If $\ell - 1$ is of type II then $\ell - 1 = j_0 = o_k$ and $\begin{pmatrix} \mathbb{X}_{o_k+f} \\ \mathbb{X}_{o_k} \end{pmatrix} = \begin{pmatrix} 0 \\ \text{AB} \end{pmatrix}$. The argument in the proof of Lemma 5.4.3(1) and (4) shows that there exists $r_k \in \mathbb{N}$ such that $p^{r_k} \bar{\mathcal{I}}_k$ is generated by the equations (after an adequate change of variables)

$$\begin{cases} p^{r_k} M_{\text{out}, o_k} \begin{pmatrix} 1 \\ -X_{o_k+1} \end{pmatrix} = 0, r_k = 0 & \text{if } i_k \text{ is of type II,} \\ p^{r_k} M_{\text{out}, o_k} \begin{pmatrix} 1 \\ -X_{o_k+1} \end{pmatrix} (p - X_{i_k+1} X_{i_k}, -X_{i_k}) = 0 & \text{if } i_k \text{ is of type I,} \end{cases}$$

where $M_{\text{out}, o_k} = (-p, Y_{o_k})$ (since $\tilde{w}_{o_k} = w_0 t_\eta$ by Proposition 5.3.6(2)). The result follows. \square

Lemma 5.4.13 (the A, B-part of a cluster). *Assume $j_0 \neq \ell - 1$.*

(1) *There exists $r_{k_{s'}} \in \mathbb{N}$ such that*

$$p^{r_{k_{s'}}} \bar{\mathcal{I}}_{k_{s'}} = \begin{cases} p^{r_{k_{s'}}} \bar{\mathcal{I}}'_{k_{s'}} & \text{if type of } i_{k_{s'}} \text{ is II,} \\ p^{r_{k_{s'}}} \bar{\mathcal{I}}'_{k_{s'}} (p, X_{\ell+1}) & \text{if type of } i_{k_{s'}} \text{ is I,} \end{cases}$$

where $\bar{\mathcal{I}}'_{k_{s'}}$ is an ideal of $R_{[j_0, j_1]} / \sum_{i=0}^{s'-1} \mathcal{I}_{k_i}$ generated by elements which do not depend (up to an automorphism induced by an explicit change of variables at $o_{k_{s'}}$) on the choice of (\tilde{w}, s, μ) in the fiber above \mathbb{X} of the map (5.3.1).

(2) *For all $s'' \in \{s' + 1, \dots, s\}$, $\bar{\mathcal{I}}_{k_{s''}}$ is an ideal of $R_{[j_0, j_1]} / \sum_{l=0}^{s''-1} \mathcal{I}_{k_l}$ generated by elements which do not depend on the choice of (\tilde{w}, s, μ) in the fiber above \mathbb{X} of the map (5.3.1).*

Proof. Since $j_0 \neq \ell - 1$, $\mathbb{X}_{\ell-1} = \mathbf{AB}$, $\mathbb{X}_{\ell-1+f} \in \{\mathbf{A}, \mathbf{B}\}$ and $\mathbb{X}_j, \mathbb{X}_{j+f} \in \{\mathbf{A}, \mathbf{B}\}$ for all $j \in \{j_0 + 1, \dots, \ell - 2\}$. By Proposition 5.3.6 the triple $(s_{j+1}, s_{\text{or},j}, \text{type of } j)$ are determined by \mathbb{X} , for all $j \in \{j_0 + 1, \dots, \ell - 2\}$.

Proof of item (1). The case $o_{k_{s'}} = \ell - 1$ follows from Lemma 5.4.9 (1).

Assume $o_{k_{s'}} < \ell - 1$. By Lemma 5.4.9 (2), there exists $r_{k_{s'}} \in \mathbb{N}$ such that $p^{r_{k_{s'}}} \bar{\mathcal{I}}_{k_{s'}}$ is generated by the equations

$$(5.4.14) \quad p^{r_{k_{s'}}} M_{\text{out}, o_{k_{s'}}} \left(\prod_{l=o_{k_{s'}}+1}^{\ell-2} T_l \right) M_{\ell-1} = 0$$

where

$$M_{\ell-1} \stackrel{\text{def}}{=} \begin{cases} (w_0)^{\delta_{\mathbb{X}_{f+\ell-1}=\mathbf{B}}} \Sigma_{\ell-1} \begin{pmatrix} -X_{\ell-1} \\ p-X_{\ell-1}X_{\ell} \end{pmatrix} & \text{if } i_{k_{s'}} \text{ is of type } II, \\ (w_0)^{\delta_{\mathbb{X}_{f+\ell-1}=\mathbf{B}}} \Sigma_{\ell-1} \begin{pmatrix} -X_{\ell-1} \\ p-X_{\ell-1}X_{\ell} \end{pmatrix} (X_{\ell+1}, p) & \text{if } i_{k_{s'}} \text{ is of type } I. \end{cases}$$

We first prove, by decreasing induction on $j \in \{o_{k_{s'}} + 1, \dots, \ell - 1\}$, that in (5.4.14) we can replace

$$\left(\prod_{l=j}^{\ell-2} T_l \right) M_{\ell-1}$$

$$\begin{cases} \Sigma_j M'_j(p, X_{\ell+1}) & \text{if } i_{k_{s'}} \text{ is of type } I, \\ \Sigma_j M'_j & \text{if } i_{k_{s'}} \text{ is of type } II, \end{cases}$$

where M'_j depends only on \mathbb{X} .

The result is true for $\ell - 1$. For the inductive step, using the relation $M'_j = \Sigma_j^{-1} T_j \Sigma_{j+1} M'_{j+1}$, it is enough to prove that up to sign the matrix $\Sigma_j^{-1} T_j \Sigma_{j+1} = \Sigma_j^{-1} T_j \Sigma_j z_{j+1}$ only depends on \mathbb{X} for $j \in (o_{k_{s'}} + 1, \dots, \ell - 2)$. This is a casewise check using Table 8 and Table 5. Indeed, as j is of type 0 for all $j \in (o_{k_{s'}} + 1, \dots, \ell - 2)$ we have from Table 8 that

$$((w_0)^{\delta_{\mathbb{X}_{f+j}=\mathbf{B}}} \Sigma_j, \tilde{w}_j, z_{j+1}) \in \{(\text{id}, t_{w_0(\eta)}, \text{id}), (w_0, t_\eta, w_0), (w_0, t_\eta, \text{id})\}$$

(The factor $(w_0)^{\delta_{\mathbb{X}_{f+j}=\mathbf{B}}}$ is justified by Table 9) and an elementary computation from Table 5 shows that

$$\Sigma_j^{-1} T_j \Sigma_j z_{j+1} = (\pm) (w_0)^{\delta_{\mathbb{X}_{f+j}=\mathbf{B}}} \begin{pmatrix} p & -X_j \\ 0 & 1 \end{pmatrix} (w_0)^{\delta_{\mathbb{X}_{f+j}=\mathbf{B}}}.$$

We conclude that $p^{r_{k_{s'}}} \bar{\mathcal{I}}_{k_{s'}}$ is generated by the equations

$$(5.4.15) \quad p^{r_{k_{s'}}} M_{\text{out}, o_{k_{s'}}} M_{o_{k_{s'}}} = 0$$

where

$$M_{o_{k_{s'}}} = \begin{cases} \Sigma_{o_{k_{s'}}+1} M'_{o_{k_{s'}}+1}(p, X_{\ell+1}) & \text{if } i_{k_{s'}} \text{ is of type } I, \\ \Sigma_{o_{k_{s'}}+1} M'_{o_{k_{s'}}+1} & \text{if } i_{k_{s'}} \text{ is of type } II, \end{cases}$$

and $M'_{o_{k_{s'}}+1}$ depends only on \mathbb{X} .

We now perform a casewise analysis according to the type of $j \stackrel{\text{def}}{=} o_{k_{s'}}$.

- ♣1 Assume j is of type I . Then by Tables 9 and 8 we have $\begin{pmatrix} \mathbb{X}_{j+f} \\ \mathbb{X}_j \end{pmatrix} \in \left\{ \begin{pmatrix} \mathbf{A} \\ \mathbf{A} \end{pmatrix}, \begin{pmatrix} \mathbf{B} \\ \mathbf{B} \end{pmatrix} \right\}$. The system of equations (5.4.15) is equivalent to

$$p^{r_{k_{s'}}} \Sigma_j M_{\text{out}, j} \Sigma_{j+1} M'_{j+1}(p, X_{\ell+1})^{\delta_{i_{k_{s'}}=I}} = 0.$$

A direct check on Table 8 and Table 5 shows that $\Sigma_j M_{\text{out}, j} \Sigma_j z_{j+1}$ only depends on whether $\tilde{w}_j = t_{w_0\eta}$ or $\tilde{w}_j \in \{t_\eta, w_0 t_\eta\}$ up to the change of variables $X_j \longleftrightarrow Z_j$, $Y_j \longleftrightarrow p - Y_j$ (change of variables which happens exactly when $\Sigma_j = w_0$). By Proposition 5.3.6(4) we conclude

that the system of equation (5.4.15) only depends on \mathbb{X} up to the the change of variables $X_j \longleftrightarrow Z_j, Y_j \longleftrightarrow p - Y_j$.

- ♠2 Assume $j = o_{k_{s'}}$ is of type *II*. By definition of cluster and the fact that $o_{k_{s'}} \leq \ell - 1$, we conclude that j is of type *II*. A direct check on Table 8 (which provides the possible choices for $(\tilde{w}_j, \Sigma_j, z_j)$) and Table 5 shows that $M_{\text{out},j} \Sigma_j z_{j+1}$ only depends on whether $\tilde{w}_j = t_{w_0\eta}$ or $\tilde{w}_j \in \{t_\eta, w_0 t_\eta\}$. We conclude from Proposition 5.3.6(4) that the system of equation (5.4.15) only depends on \mathbb{X} .

Proof of item (2). In this case we necessarily have $s' < s$. Starting from case (♠1) above, we inductively analyze, for $s \geq s'' > s'$, the system of equations

$$M_{\text{out},o_{k_{s''}}} \left(\prod_{l=o_{k_{s''}}+1}^{i_{k_{s''}}-1} T_l \right) \Sigma_{o_{k_{s''}-1}} M_{\text{in},o_{k_{s''}-1}} \Sigma_{o_{k_{s''}-1}} = 0$$

where the presence of the $\Sigma_{o_{k_{s''}-1}}$ -conjugation on the matrix $M_{\text{in},o_{k_{s''}-1}}$ (defined in Table 5) is explained by the change of variables $X_{o_{k_{s''}-1}} \longleftrightarrow Z_{o_{k_{s''}-1}}, Y_{o_{k_{s''}-1}} \longleftrightarrow p - Y_{o_{k_{s''}-1}}$ when $\Sigma_{o_{k_{s''}-1}} = w_0$. Since the expression $M_{\text{in},o_{k_{s''}-1}}$ is independent of \mathbb{X} , we can now perform the same argument appearing in the proof of item (1) (where we replace $M'_{\ell-1}$ by $M_{\text{in},o_{k_{s''}-1}}$ in the initial inductive argument there). \square

Proof of Proposition 5.4.2. . Let $(\mathbb{X}_j, \mathbb{X}_{j+f})_{j \in \mathcal{J}}$ be a gene satisfying (♣1)–(♣3). If there exists $i \in \mathcal{J}$ such that $(\mathbb{X}_i, \mathbb{X}_{i+f}) = (0, 0)$ then the deformation ring is zero and the result is obvious. In what follows we assume that $(\mathbb{X}_j, \mathbb{X}_{j+f})_{j \in \mathcal{J}}$ satisfies (♣1)–(♣4).

Proof of item (1). Assume that there exists $i \in \mathcal{J}'$ such that $\mathbb{X}_i = 0$ and let $\mathbb{X} = \cup_{i=0}^r (\mathbb{X}_j, \mathbb{X}_{j+f})_{j_i \leq j \leq j_{i+1}}$ be the decomposition of \mathbb{X} into clusters. By Lemmas 5.4.3–5.4.13 applied on each fragments of \mathbb{X} we have

$$\mathcal{Z}^{\text{nv},\tau}(\tilde{z}) = \text{Spec} \left(\otimes_{i=0}^r R_{[j_i, j_{i+1}]} / \mathcal{I}_{[j_i, j_{i+1}]} \right).$$

and each $\mathcal{I}_{[j_i, j_{i+1}]}$ is a sum of the ideals described in

- Lemma 5.4.3, so that after p -saturation we can solve the variables Y_i, Z_i , while the variable X_i is free,
- Lemma 5.4.9 and Lemma 5.4.13 (1) so that after p -saturation these ideals produce equations which do not depend on the choice of (\tilde{w}, s, μ) in the fiber of the map (5.3.1) at \mathbb{X} .
- Lemma 5.4.13 (2), and these ideals admits generators which do not depend on the choice of (\tilde{w}, s, μ) in the fiber of the map (5.3.1) at \mathbb{X} .

We conclude that the p -saturation of $\mathcal{Z}^{\text{nv},\tau}(\tilde{z})$ depends only on \mathbb{X} .

Proof of item (2). The proof is similar to the proof of Proposition 5.3.6(4). If $\mathbb{X}_{j'} \in \{\mathbf{A}, \mathbf{B}\}$ for all $j' \in \mathcal{J}'$ then either s_{j+1} or $s_{\text{or},i}$ is determined by $\mathbb{X}_{j'}$, for all $j' \in \mathcal{J}$. If there exists $j_0 \in \mathcal{J}$ such that $\begin{pmatrix} \mathbb{X}_{j_0} \\ \mathbb{X}_{j_0+f} \end{pmatrix} \in \left\{ \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}, \begin{pmatrix} \mathbf{B} \\ \mathbf{A} \end{pmatrix} \right\}$ then s_{j_0+1} is uniquely determined (cf. Table 8). As $s_{\text{or},f-1} = \text{id}$ by Lemma 2.1.7 we conclude that $(s_{j+1}, s_{\text{or},j})_{j \in \mathcal{J}}$ is uniquely determined by $(\mathbb{X}_j, \mathbb{X}_{j+f})_{j \in \mathcal{J}}$.

If $\begin{pmatrix} \mathbb{X}_j \\ \mathbb{X}_{j+f} \end{pmatrix} \in \left\{ \begin{pmatrix} \mathbf{A} \\ \mathbf{A} \end{pmatrix}, \begin{pmatrix} \mathbf{B} \\ \mathbf{B} \end{pmatrix} \right\}$ for all $j \in \mathcal{J}$ then $(s_{\text{or},j})_{j \in \mathcal{J}}$ is determined by \mathbb{X} . By (♣3) there exists $j' \in \mathcal{J}'$ with $\mathbb{X}_{j'} \neq \mathbb{X}_{j'+1}$. By Table 8, we obtain $s_{j'+1} = (12)$ and by induction $(s_{j+1}, s_{\text{or},j})_{j \in \mathcal{J}}$ are determined.

As $(s_{j+1}, s_{\text{or},j}, \text{type of } j)$ are determined for all $j \in \mathcal{J}$, $\mathcal{Z}^{\text{nv},\tau}(\tilde{z})$ is determined by \mathbb{X} . \square

Theorem 5.4.16. *Let τ be a regular tame inertial type of niveau f with small presentation (s, μ) and assume that either $p > \max\{16f + 3, 16f - 12 + \max_j \langle \mu_j, \alpha^\vee \rangle\}$ or $p \geq 17$ and $K = \mathbb{Q}_p$.*

Let $\bar{\rho} : G_K \rightarrow \mathrm{GL}_2(\mathbb{F})$ be absolutely irreducible and such that $\det(\bar{\rho}) \otimes_{\mathbb{F}} \omega = \det(\tau) \otimes_{\mathcal{O}} \mathbb{F}$. Then $R_{\bar{\rho}}^{\eta, \tau}$ depends only on $\mathbb{X}(\tau, \bar{\rho}|_{I_K})$. Moreover, there exists an integer $r \geq 0$ and a decomposition $\mathbb{X}(\tau, \bar{\rho}|_{I_K}) = \cup_{i=0}^r (\mathbb{X}_{j_i}, \mathbb{X}_{j_i+f})_{j_i \leq j \leq j_{i+1}}$ such that

$$R_{\bar{\rho}}^{\eta, \tau} \cong \widehat{\otimes}_{i=0}^r R_i$$

where R_i is a complete local Noetherian \mathcal{O} -algebra depending only on $(\mathbb{X}_{j_i}, \mathbb{X}_{j_i+f})_{j_i \leq j \leq j_{i+1}}$.

Proof. This follows from Theorem 4.6.3 and Proposition 5.4.2. \square

5.5. Examples. We collect some examples computing potentially Barsotti–Tate deformation rings using the techniques of this article.

In what follows, given $\bar{x} \in \mathbb{F}^\times$ we denote by $\mathrm{un}_{\bar{x}}$ the unramified character of G_K sending p to \bar{x} .

5.5.1. Examples when $f=1$. We assume $p \geq 5$. By Proposition 3.2.10, we see immediately that if $\langle \mu, \alpha^\vee \rangle \geq 2$ then $R_{\bar{\rho}}^{\eta, \tau}$ is formally smooth over \mathcal{O} or $\mathcal{O}[[X, Y]]/(XY - p)$. This leaves the following cases

- (1) Case 1: $\tau = \tau((12), (1, 0))$ and $\tilde{w} \in \{t_\eta, w_0 t_\eta\}$ (thus $\tau = \omega_2 \oplus \omega_2^p$, $\tilde{z} \in \{w_0 t_{(1,1)}, t_{(1,1)}\}$ and $(\bar{\rho}|_{I_{\mathbb{Q}_p}})^{\mathrm{ss}} \in \{\bar{\omega}_2^{-1+p} \oplus \bar{\omega}_2^{1+p}, \bar{\omega} \oplus \bar{\omega}\}$).
- (2) Case 2: $\tau = \tau(\mathrm{id}, (1, 0))$ and $\tilde{w} = t_{w_0(\eta)}$ (thus $\tau = 1 \oplus \omega$, $\tilde{z} = t_{(1,1)}$ and $(\bar{\rho}|_{I_{\mathbb{Q}_p}})^{\mathrm{ss}} = \bar{\omega} \oplus \bar{\omega}$).

We analyze the p -saturation of $\tilde{\mathcal{Z}}^{\mathrm{nv}, \tau}(\tilde{z})$ in those cases.

We start with case 1. Since the two subcases give isomorphic spaces, we will work with $\tilde{w} = w_0 t_\eta$, so that $\tilde{z} = t_{(1,1)}$. From Table 4, $\tilde{\mathcal{Z}}^{\mathrm{nv}, \tau}(\tilde{z})$ is presented as the quotient of $\mathcal{O}[B, C, D, \alpha, \beta, \gamma, \delta]$ subject to $D(p - D) = BC$, $\alpha\delta - \beta\gamma$ invertible, and the relation

$$\begin{pmatrix} D & -B \\ -C & (p - D) \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} D & -B \\ -C & (p - D) \end{pmatrix} = 0$$

Using $D(p - D) = BC$, a simple manipulation shows that the above matrix equation is equivalent to $DF = BF = CF = (p - D)F = 0$ where $F = \gamma B + \beta C - \alpha D - \delta(p - D)$. It follows that the p -saturation $\tilde{\mathcal{Z}}^{\mathrm{mod}, \tau}(\tilde{z})$ is given by the equations

$$D(p - D) = BC, \gamma B + \beta C = \alpha D + \delta(p - D)$$

We check on Macaulay 2 that the ideal H_2 for the above relations together with the relations themselves contains p^2 , and has radical (p, B, C, D) . Thus the non-smooth locus of $\tilde{\mathcal{Z}}^{\mathrm{mod}, \tau}(\tilde{z})/\mathcal{O}$ is $p = B = C = D = 0$, which correspond to $\bar{\rho} \otimes \bar{\varepsilon}^{-1}$ unramified. Running through the proof of Theorem 3.3.11, we see its conclusion holds if $p - 2 - 1 - 1 > 4$. Thus for $p \geq 11$, after twisting $\bar{\rho}$, we see that $R_{\bar{\rho}}^{\eta, \tau}$ is isomorphic to the completion of

$$\mathcal{O}[B, C, D, \alpha, \beta, \gamma, \delta]/(D(p - D) - BC, \gamma B + \beta C - \alpha D - \delta(p - D))$$

at either $(p, B, C, D, \alpha - 1, \delta - 1, \beta, \gamma)$, $(p, B, C, D, \alpha - s, \delta - t, \beta, \gamma)$ (with $s \neq t \in \mathbb{F}^\times$) or $(p, B, C, D, \alpha - 1, \delta - 1, \beta - 1, \gamma)$.

We note that the second ideal correspond to the point $\bar{\rho} = \mathrm{un}_s \bar{\varepsilon} \oplus \mathrm{un}_t \bar{\varepsilon}$ with $s \neq t$, and after eliminating D using the fact that $\alpha - \delta$ is unit, and making a change of variable on B, C , we see that the completion of $\tilde{\mathcal{Z}}^{\mathrm{mod}, \tau}(\tilde{z})$ at $\bar{\rho}$ is a power series ring over $\mathcal{O}[[X, Y]]/(XY - p^2)$.

We now move to case 2. Reading off from Table 4 like in case 1, $\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})$ is presented as the quotient of $\mathcal{O}[B, C, D, \alpha, \beta, \gamma, \delta]$ subject to $D(p - D) = BC$, $\alpha\delta - \beta\gamma$ invertible, and the relations

$$\begin{aligned} BC\alpha + DC\beta + DB\gamma - BC\delta - pC\beta &= 0, \\ DC\alpha - C^2\beta - BC\gamma - DC\delta + pD\gamma &= 0 \\ DB\alpha - BC\beta - B^2\gamma - DB\delta - pB\alpha - pD\beta + pB\delta + p^2\beta &= 0. \end{aligned}$$

One can check that $\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})$ is already p -saturated, hence agrees with $\tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z})$. We check on Macaulay 2 that the ideal H_2 for the above relations together with the relations themselves contains p^5 , and has radical (p, B, C, D) . Thus the non-smooth locus of $\tilde{\mathcal{Z}}^{\text{mod},\tau}(\tilde{z})/\mathcal{O}$ is $p = B = C = D = 0$, which correspond to $\bar{\rho} \otimes \bar{\varepsilon}^{-1}$ unramified.

Running through the proof of Theorem 3.3.11, we see its conclusion hold if $p - 2 - 1 - 1 > 10$. Thus, if $p \geq 17$, after twisting $\bar{\rho}$, $R_{\bar{\rho}}^{\eta,\tau}$ is isomorphic to the completion of

$$\mathcal{O}[B, C, D, \alpha, \beta, \gamma, \delta] \left/ \left(\begin{array}{l} D(p - D) - BC, BC\alpha + DC\beta + DB\gamma - BC\delta - pC\beta, \\ DC\alpha - C^2\beta - BC\gamma - DC\delta + pD\gamma, \\ DB\alpha - BC\beta - B^2\gamma - DB\delta - pB\alpha - pD\beta + pB\delta + p^2\beta \end{array} \right) \right.$$

at either $(p, B, C, D, \alpha - 1, \delta - 1, \beta, \gamma)$, $(p, B, C, D, \alpha - s, \delta - t, \beta, \gamma)$ (with $s \neq t \in \mathbb{F}^\times$) or $(p, B, C, D, \alpha - 1, \delta - 1, \beta - 1, \gamma)$.

Remark 5.5.1. (Bounds on p for $f = 1$) Our explicit computations in this section allows us to slightly relax the requirement on p in Theorem 3.3.11. Specifically, the improvement in the proof of Theorem 3.3.11 comes from two sources

- We can get a map out of $\mathcal{O}[[X, Y]]/(XY - p^k)$ as soon as we have a map modulo p^{k+1} , instead of using the general Elkik bound which would have required a map modulo p^{2k+1} . This shows that our above models are valid for $p \geq 7$, unless (up to twists) $\bar{\rho} \otimes \bar{\varepsilon}^{-1}$ is unramified and has scalar semisimplification and $\tau = \omega_2 \oplus \omega_2^p$, or $\bar{\rho} \otimes \bar{\varepsilon}^{-1}$ is unramified and $\tau = 1 \oplus \omega$. We remark that these computations justify the claims made in [EGH, §7.5.13].
- In the remaining cases, we get a small saving in the bound required to apply Elkik's approximation theorem compared to the general bound of Proposition 4.4.2. This in particular shows that our models for these cases are valid for $p \geq 17$.

5.5.2. *Examples with $f = 3$.* Tables 11, 10 record several examples of deformation rings corresponding to the examples of the left column (resp. right column) of [CDM18b, Table 4] (by completion at the ideal generated by the variables $X_0, X_1, X_2, Y_0 \dots$).

TABLE 10. **Examples from [CDM18b, §5.3] for $f = 3$**

$\begin{array}{ c c c } \hline \uparrow & 0 & B & B \\ \hline & A & A & AB \\ \hline \downarrow & & & \end{array}$	$(II, w_0 t_\eta), (0, t_{w_0(\eta)}), (0, t_{w_0(\eta)})$	$R = \mathcal{O}[X_0, Y_0, X_1, X_2]$ $I^{\text{nv}} = I^{\text{p-sat}} = (X_0 Y_0 + p^3)$
$\begin{array}{ c c c } \hline \uparrow & 0 & A & B \\ \hline & B & A & AB \\ \hline \downarrow & & & \end{array}$	$(II, w_0 t_\eta), (I, w_0 t_\eta), (0, w_0 t_\eta)$	$R = \mathcal{O}[X_0, Y_0, X_1, Y_1, Z_1, X_2]$ $I^{\text{nv}} = \begin{pmatrix} Y_1(p - Y_1) - X_1 Z_1, & pY_1 + Y_0 Z_1, & pX_1 + Y_0(p - Y_1), \\ Y_1 X_0 + X_1(p + X_0 X_2), & Z_1 X_0 + (p - Y_1)(p + X_0 X_2) \end{pmatrix}$ $I^{\text{p-sat}} = \begin{pmatrix} Y_0 Z_1 + pY_1, & Y_1^2 + X_1 Z_1 - pY_1, & Y_0 Y_1 - pY_0 - pX_1, \\ X_0 X_1 X_2 + X_0 Y_1 + pX_1, & X_0 Y_1 X_2 - X_0 Z_1 - pX_0 X_2 + pY_1 - p^2 \end{pmatrix}$
$\begin{array}{ c c c } \hline \uparrow & A & A & B \\ \hline & A & B & B \\ \hline \downarrow & & & \end{array}$	$(I, w_0 t_\eta), (0, t_{w_0(\eta)}), (I, t_\eta)$	$R = \mathcal{O}[X_0, Y_0, Z_0, X_1, X_2, Y_2, Z_2]$ $I^{\text{nv}} = I^{\text{p-sat}} = \begin{pmatrix} (p - Y_2)Y_0 + Z_2 Z_0, & (p - Y_2)X_0 + Z_2(p - Y_0) \\ X_2 Y_0 + Z_0 Y_2, & X_2 X_0 + Y_2(p - Y_0), \\ Y_0(pY_2 + X_1 Z_2) + X_0 Z_2, & Z_0(pY_2 + X_1 Z_2) + Z_2(p - Y_0), \\ Y_0(pX_2 + X_1(p - Y_2)) + X_0(p - Y_2), & Z_0(pX_2 + X_1(p - Y_2)) + (p - Y_0)(p - Y_2), \\ Y_2(p - Y_2) - X_2 Z_2, & Y_0(p - Y_0) - X_0 Z_0 \end{pmatrix}$

TABLE 11. **Examples from [CDM18b, §5.3] for $f = 3$**

Gene	(type of j, \tilde{w}_j) $_{j=0,1,2}$	Equations for $\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})$ and $(\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z}))^{p\text{-sat}}$
$\begin{array}{ c c c } \hline \circ & B & A \\ \hline A & AB & \circ \\ \hline \end{array}$	$(II, t_\eta), (0, w_0 t_\eta), (II, t_\eta)$	$R = \mathcal{O}[X_0, X_1, X_2, Y_0]$ $I^{\text{nv}} = I^{p\text{-sat}} = (X_2 Y_0 + p^2)$
$\begin{array}{ c c c } \hline \circ & B & B \\ \hline A & AB & \circ \\ \hline \end{array}$	$(II, t_\eta), (0, w_0 t_\eta), (II, t_{w_0 t_\eta})$	$R = \mathcal{O}[X_0, X_1, X_2, Y_0]$ $I^{\text{nv}} = I^{p\text{-sat}} = (X_2 Y_0 + p^2)$
$\begin{array}{ c c c } \hline \circ & B & B \\ \hline B & A & AB \\ \hline \end{array}$	$(II, t_{w_0 t_\eta}), (0, t_\eta), (0, w_0 t_\eta)$	$R = \mathcal{O}[X_0, X_1, X_2, Y_0]$ $I^{\text{nv}} = I^{p\text{-sat}} = (X_0 Y_0 + p^2)$
$\begin{array}{ c c c } \hline A & B & B \\ \hline A & A & A \\ \hline \end{array}$	$(I, t_{w_0 t_\eta}), (0, t_\eta), (0, w_0 t_\eta)$	$R = \mathcal{O}[X_0, X_1, X_2, Y_0, Z_0]$ $I^{\text{nv}} = \begin{pmatrix} -X_0 Z_0 X_1 - p X_0 Z_0 X_2 - p^2 X_0 Y_0 - Y_0 Z_0 + p Z_0, & \\ -Y_0 Z_0 X_1 - p Y_0 Z_0 X_2 - p^2 Y_0^2 + Z_0^2, & \\ -X_0 Y_0 X_1 - p X_0 Y_0 X_2 + p^2 X_0^2 - Y_0^2 + p X_0 X_1 + p^2 X_0 X_2 + 2p Y_0 - p^2, & \\ -Y_0^2 X_1 - p Y_0^2 X_2 + p^2 X_0 Y_0 + Y_0 Z_0 + p Y_0 X_1 + p^2 Y_0 X_2 - p Z_0 & \end{pmatrix}$ $I^{p\text{-sat}} = \begin{pmatrix} Y_0^2 + X_0 Z_0 - p Y_0, & X_0 Z_0 X_1 + p X_0 Z_0 X_2 + p^2 X_0 Y_0 + Y_0 Z_0 - p Z_0, & \\ X_0 Y_0 X_1 + p X_0 Y_0 X_2 - p^2 X_0^2 - X_0 Z_0 - p X_0 X_1 - p^2 X_0 X_2 - p Y_0 + p^2, & \\ Y_0 Z_0 X_1 + p Y_0 Z_0 X_2 - p^2 X_0 Z_0 - Z_0^2 + p^3 Y_0 & \end{pmatrix}$
$\begin{array}{ c c c } \hline A & B & A \\ \hline A & A & B \\ \hline \end{array}$	$(I, t_{w_0 t_\eta}), (0, t_{w_0 t_\eta}), (0, w_0 t_\eta)$	$R = \mathcal{O}[X_0, X_1, X_2, Y_0, Z_0]$ $I^{\text{nv}} = \begin{pmatrix} p Z_0(p - Y_0) + (X_0 - X_1(p - Y_0))(-p Y_0 - Z_0 X_2), & -p(p - Y_0)^2 + (X_0 - X_1(p - Y_0))(p X_0 + X_2(p - Y_0)), \\ p Z_0^2 + (-Z_0 X_1 + Y_0)(-p Y_0 - Z_0 X_2), & -p Z_0(p - Y_0) + (-Z_0 X_1 + Y_0)(p X_0 + X_2(p - Y_0)), \\ & (p - Y_0) Y_0 - X_0 Z_0 \end{pmatrix}$ $I^{p\text{-sat}} = \begin{pmatrix} Y_0^2 + X_0 Z_0 - p Y_0, & X_1 X_2 Z_0^2 + p X_1 Y_0 Z_0 - X_2 Y_0 Z_0 + p X_0 Z_0 + p Z_0^2 - p^2 Y_0, & \\ X_1 X_2 Y_0 Z_0 - p X_0 X_1 Z_0 + X_0 X_2 Z_0 - p X_1 X_2 Z_0 + p X_0 Y_0 + p Y_0 Z_0 - p^2 Z_0, & \\ X_0 X_1 X_2 Z_0 + p X_0 X_1 Y_0 - X_0 X_2 Y_0 + p X_1 X_2 Y_0 + p X_0^2 - p^2 X_0 X_1 + p X_0 X_2 - p^2 X_1 X_2 + p X_0 Z_0 + p^2 Y_0 - p^3 & \end{pmatrix}$
$\begin{array}{ c c c } \hline \circ & B & B \\ \hline A & B & AB \\ \hline \end{array}$	$(II, t_\eta), (I, t_{w_0 t_\eta}), (0, w_0 t_\eta)$	$R = \mathcal{O}[X_0, Y_0, X_1, Y_1, Z_1, X_2]$ $I^{\text{nv}} = \begin{pmatrix} Y_0 Y_1 + p Z_1, & Y_0 X_1 + p(p - Y_1), & X_1(p + X_0 X_2) - X_0(p - Y_1), \\ Y_1(p + X_0 X_2) - X_0 Z_1, & (p - Y_1) Y_1 - X_1 Z_1 & \end{pmatrix}$ $I^{p\text{-sat}} = \begin{pmatrix} Y_1^2 + X_1 Z_1 - p Y_1, & Y_0 Y_1 + p Z_1, & Y_0 X_1 - p Y_1 + p^2 \\ X_0 Y_0 + p X_0 X_2 + p^2, & X_0 Y_1 X_2 - X_0 Z_1 + p Y_1, & X_0 X_1 X_2 + X_0 Y_1 - p X_0 + p X_1 \end{pmatrix}$
$\begin{array}{ c c c } \hline \circ & A & B \\ \hline AB & \circ & AB \\ \hline \end{array}$	$(II, w_0 t_\eta), (II, t_\eta), (0, t_{w_0 t_\eta})$	$R = \mathcal{O}[X_0, Y_0, X_1, Y_1, X_2]$ $I^{\text{nv}} = I^{p\text{-sat}} = (p + Y_0 X_1, X_0 X_2 Y_1 + p(X_0 + Y_1))$
$\begin{array}{ c c c } \hline \circ & A & B \\ \hline A & B & AB \\ \hline \end{array}$	$(II, t_\eta), (0, t_{w_0 t_\eta}), (0, w_0 t_\eta)$	$R = \mathcal{O}[X_0, Y_0, X_1, X_2]$ $I^{\text{nv}} = I^{p\text{-sat}} = (p X_0 Y_0 + (p + X_1 Y_0)(p + X_0 X_2))$

In the third column we express $\tilde{\mathcal{Z}}^{\text{nv},\tau}(\tilde{z})$ as $\text{Spec}(R/I^{\text{nv}})$, where $R = \bigotimes_{j=0}^2 R_j$ and $I^{\text{nv}}, (R_j)_{j=0,1,2}$ are extracted from the data of the second column and Table 5

APPENDIX A. COHEN–MACAULAYNESS OF THE NORMALIZATION OF \mathcal{Z}^τ

by DAT PHAM

The goal of this appendix is to show, under a very mild assumption, that the normalization of the tame Barsotti–Tate stack \mathcal{Z}^τ is always Cohen–Macaulay; see Theorem A.9 for a precise statement.

A key ingredient in the proof is the corresponding result for the model $\tilde{\mathcal{Z}}^{\text{mod},\tau}$:

Theorem A.1. *The normalization of $\tilde{\mathcal{Z}}^{\text{mod},\tau}$ (in the sense of [Sta18, Tag 035E]) is Cohen–Macaulay.*

Proof. As the statement is local, it suffices to work with the affine open $\tilde{\mathcal{Z}}^{\text{mod}}(\tilde{z})$ (where $\tilde{z} \in \text{Adm}(\eta)^{\mathcal{J}} s^{-1} v^{\hat{\mu}}$ as before). Recall from the proof of Theorem 4.6.6 (1) that the normalization $\tilde{\mathcal{Z}}^{\text{mod},\text{nm}}(\tilde{z})$ of $\tilde{\mathcal{Z}}^{\text{mod}}(\tilde{z})$ fits into the Stein factorization

$$\begin{array}{ccc} \tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z}) & \xrightarrow{\pi^{\text{mod},\text{nm}}} & \tilde{\mathcal{Z}}^{\text{mod},\text{nm}}(\tilde{z}) \\ & \searrow \pi^{\text{mod}} & \downarrow \\ & & \tilde{\mathcal{Z}}^{\text{mod}}(\tilde{z}), \end{array}$$

and that

$$(A.2) \quad \omega_{\tilde{\mathcal{Z}}^{\text{mod},\text{nm}}(\tilde{z})/\mathcal{O}} \simeq R\pi_*^{\text{mod},\text{nm}} \omega_{\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})/\mathcal{O}}.$$

We need to show that the RHS of (A.2) is concentrated in degree 0. As the map $\tilde{\mathcal{Z}}^{\text{mod},\text{nm}}(\tilde{z}) \rightarrow \tilde{\mathcal{Z}}^{\text{mod}}(\tilde{z})$ is affine, it suffices to show that $R^{>0} \text{pr}_* \omega_{\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})/\mathcal{O}} = 0$.

For this we need the following explicit description of the relative dualizing sheaf $\omega_{\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})/\mathcal{O}}$.

To this end, recall from Lemma 4.2.11 that the closed immersion $\Delta : \tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z}) \hookrightarrow \widetilde{Ba}(\tilde{z})$ is a global complete intersection of codimension f given by the vanishing locus of some f maps $\mathcal{L}_j := q_j^* \mathcal{O}_{\mathbb{P}^1}(-1) \otimes p_{j-1}^* \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \mathcal{O}_{\widetilde{Ba}(\tilde{z})}$ for $j \in \mathcal{J}$. Thus, the normal bundle of Δ has determinant $\bigotimes_{j \in \mathcal{J}} q_j^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes p_j^* \mathcal{O}_{\mathbb{P}^1}(1)$, and so combining with Lemma 4.2.3(2) we find that $\omega_{\tilde{Y}^{\text{mod}}(\tilde{z})/\mathcal{O}} \simeq \Delta^* \mathcal{L}$ where

$$\mathcal{L} := \bigotimes_{j \in \mathcal{J}_0} q_j^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes p_j^* \mathcal{O}_{\mathbb{P}^1}(1).$$

Here $\mathcal{J}_0 \subseteq \mathcal{J}$ is the subset of $j \in \mathcal{J}$ such that $(\tilde{w}_j, s_j, k_j) = (t_{w_0(\eta)}, \text{id}, 1)$.

Consider now the commutative diagram

$$\begin{array}{ccc} \tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z}) & \xleftarrow{\Delta} & \widetilde{Ba}(\tilde{z}) \\ & \searrow \text{pr} & \downarrow \text{pr}_{\widetilde{B}} \\ & & \tilde{U}(\tilde{z}). \end{array}$$

Recall that we wanted to show that $R^{>0} \text{pr}_*(\Delta^* \mathcal{L}) = 0$. We have $R\text{pr}_* \Delta^* \mathcal{L} \simeq R(\text{pr}_{\widetilde{B}})_*(\Delta_* \Delta^* \mathcal{L})$ and $\Delta_* \Delta^* \mathcal{L} \simeq \mathcal{L} \otimes \Delta_* \mathcal{O}_{\tilde{Y}^{\text{mod},\eta,\tau}(\tilde{z})}$ by the projection formula. Using the exact sequence (note that \mathcal{L} is flat)

$$0 \rightarrow \mathcal{L} \otimes \mathcal{I}(\tilde{z}) \rightarrow \mathcal{L} \rightarrow \mathcal{L} \otimes \Delta_* \mathcal{O}_{\tilde{Y}^{\text{mod}}(\tilde{z})} \rightarrow 0$$

(where as before $\mathcal{I}(\tilde{z})$ denotes the ideal sheaf of Δ), it suffices to show that $R^{>0}(\text{pr}_{\widetilde{B}})_* \mathcal{L} = 0$ and $R^{>1}(\text{pr}_{\widetilde{B}})_*(\mathcal{L} \otimes \mathcal{I}(\tilde{z})) = 0$. In turn as $\tilde{U}(\tilde{z})$ is affine, this is equivalent to showing that $H^{>0}(\widetilde{Ba}(\tilde{z}), \mathcal{L}) = 0$ and $H^{>1}(\widetilde{Ba}(\tilde{z}), \mathcal{L} \otimes \mathcal{I}(\tilde{z})) = 0$.

By Künneth formula we have

$$R\Gamma(\widetilde{Ba}(\widetilde{z}), \mathcal{L}) \simeq \bigotimes_{j \in \mathcal{J}_0}^{\mathbb{L}} R\Gamma(\widetilde{Ba}_j(\widetilde{z}_j), \underbrace{q_j^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes p_j^* \mathcal{O}_{\mathbb{P}^1}(1)}_{\mathcal{O}_{\mathbb{P}^1}(2)}) \otimes \bigotimes_{j \in \mathcal{J} \setminus \mathcal{J}_0}^{\mathbb{L}} R\Gamma(\widetilde{Ba}_j(\widetilde{z}_j), \mathcal{O}).$$

By Lemma 4.2.5 and Lemma A.4 below, each term appearing in the tensor product above is given by a flat module sitting in degree 0. Thus $R\Gamma(\widetilde{Ba}(\widetilde{z}), \mathcal{L})$ is indeed concentrated in degree 0.

We are left to show that $R^{>1}\Gamma(\widetilde{Ba}(\widetilde{z}), \mathcal{L} \otimes \mathcal{I}(\widetilde{z})) = 0$. As before $\mathcal{I}(\widetilde{z})$ admits a filtration with graded pieces $\bigwedge^l(\bigoplus_{j \in \mathcal{J}} \mathcal{L}_j)[l-1]$ for $1 \leq l \leq f$. Thus it suffices to show that

$$R^k\Gamma\left(\widetilde{Ba}(\widetilde{z}), \mathcal{L} \otimes \bigwedge^l\left(\bigoplus_{j \in \mathcal{J}} \mathcal{L}_j\right)\right) = 0$$

for all $1 \leq l \leq f$ and $k > l$. This amounts to showing that

$$(A.3) \quad R^k\Gamma\left(\widetilde{Ba}(\widetilde{z}), \mathcal{L} \otimes \bigotimes_{j \in \mathcal{J}} q_j^* \mathcal{O}_{\mathbb{P}^1}(-1)^{\epsilon_j} \otimes p_j^* \mathcal{O}_{\mathbb{P}^1}(-1)^{\delta_j}\right) = 0$$

for each tuple $(\epsilon_j) \in \{0, 1\}^{\mathcal{J}}$ with $\sum_j \epsilon_j = l$ (where $\delta_j := \epsilon_{j+1}$ as before). Using Künneth formula and the definition of \mathcal{L} we have

$$\begin{aligned} R\Gamma\left(\widetilde{Ba}(\widetilde{z}), \mathcal{L} \otimes \bigotimes_{j \in \mathcal{J}} q_j^* \mathcal{O}_{\mathbb{P}^1}(-1)^{\epsilon_j} \otimes p_j^* \mathcal{O}_{\mathbb{P}^1}(-1)^{\delta_j}\right) &\simeq \\ &\simeq \bigotimes_{j \in \mathcal{J}_0}^{\mathbb{L}} R\Gamma(\widetilde{Ba}_j(\widetilde{z}_j), \mathcal{O}_{\mathbb{P}^1}(2 - (\epsilon_j + \delta_j))) \otimes^{\mathbb{L}} \bigotimes_{j \in \mathcal{J} \setminus \mathcal{J}_0}^{\mathbb{L}} R\Gamma(\widetilde{Ba}_j(\widetilde{z}_j), q_j^* \mathcal{O}_{\mathbb{P}^1}(-1)^{\epsilon_j} \otimes p_j^* \mathcal{O}_{\mathbb{P}^1}(-1)^{\delta_j}). \end{aligned}$$

As $2 - (\epsilon_j + \delta_j) \geq 0 > -1$, by Lemma A.4 below the first tensor product is given by a flat \mathcal{O} -module sitting in degree 0. Now for each $j \in \mathcal{J} \setminus \mathcal{J}_0$ (even any $j \in \mathcal{J}$), we know that $R\Gamma(\widetilde{Ba}_j(\widetilde{z}_j), q_j^* \mathcal{O}(-1)^{\epsilon_j} \otimes p_j^* \mathcal{O}(-1)^{\delta_j})$ is represented by a two-term complex of flat \mathcal{O} -modules sitting in degree $[0, 1]$, and moreover has no H^1 unless $(\epsilon_j, \delta_j) = (1, 1)$. As $\sum_j \epsilon_j = l$, there are $\leq l$ indexes $j \in \mathcal{J} \setminus \mathcal{J}_0$ such that $(\epsilon_j, \delta_j) = (1, 1)$, and hence we conclude that the second tensor product has no H^k for $k > l$. This finishes the proof. \square

The following lemma was used above.

Lemma A.4. *For $j \in \mathcal{J}_0$ and $d \geq -1$, the complex $R\Gamma(Ba_j(\widetilde{z}_j), \mathcal{O}_{\mathbb{P}^1}(d))$ is given by a flat \mathcal{O} -module sitting in degree 0.*

Proof. Recall from Lemma 4.2.2 that if $j \in \mathcal{J}_0$, then $Ba_j(\widetilde{z}_j)$ identifies with the closed subscheme of $\mathbb{P}^1 \times \mathbb{A}^3$ parametrizing tuples $([x : y]; (A, B, C))$ such that $x(p - A) - yC = 0$ and $xB - yA = 0$. By the proof of Lemma 4.2.5, $R\Gamma(Ba_j(\widetilde{z}_j), \mathcal{O}_{\mathbb{P}^1}(d))$ is computed by the complex $\mathcal{O}[t, C] \oplus t^d \mathcal{O}[t^{-1}, B] \rightarrow \mathcal{O}[t^{\pm 1}, C]$ where the differential is defined by the relation $B = t(p - tC)$. Now it is shown in *loc.cit.* that this has no H^1 for $d = -1$ and hence the same holds for any $d \geq -1$. \square

We now turn to the analogue of Theorem A.1 for the Barsotti–Tate stack \mathcal{Z}^T itself; see Theorem A.9. Before doing so, we need to clarify what we mean by normalization in this setting. As far as we are aware, there is no general theory of normalization for formal algebraic stacks in the literature. For our purposes, it will be sufficient to use the following working definition.

We first recall the property of being normal, introduced in [Eme] for a large class of formal algebraic stacks. All stacks considered below will be p -adic formal algebraic stacks of finite type over $\mathrm{Spf}(\mathcal{O})$. Any such stack \mathcal{X} will be automatically residually Jacobson in the sense of [Eme, Definition 8.8] (since $\mathcal{X}_{\mathrm{red}}$ is a finite type algebraic stack over $\mathrm{Spec}(\mathbb{F})$). Accordingly, we say simply that \mathcal{X} is normal if for any smooth chart $\mathrm{Spf}(A) \rightarrow \mathcal{X}$, the ring A is normal. (By a smooth chart, we mean a morphism which is representable by algebraic spaces and smooth.) See [Eme, Definition 8.22] and the surrounding discussion.

Our construction of normalization will rest on following additional assumption:

(*) \mathcal{X} is p -flat, and the versal rings $\mathrm{Spf}(C) \rightarrow \mathcal{X}$ at any finite type points of \mathcal{X} satisfy $C[1/p]$ is normal.

We begin with the sanity check that it suffices to require (*) for *one* versal ring at each finite type point. Since any two versal rings at a same point agree up to adding formal variables, this follows from the following lemma. In particular, by [CEGSb, Corollary 5.2.19] and [Kis08, Theorem 3.3.8] we see that (*) is satisfied for the stack \mathcal{Z}^τ .

Lemma A.5. *Let C be a complete Noetherian local \mathcal{O} -algebra with finite residue field of characteristic $p > 0$. Then $C[1/p]$ is normal if and only if $C[[X]][1/p]$ is.*

Proof. The “if” part follows from [Sta18, Tag 033G] as $C \rightarrow C[[X]]$ is faithfully flat. Assume now that $C[1/p]$ is normal. It is standard that the residue field of $C[[X]][1/p]$ at any maximal ideal is a finite extension E' of E . Let \mathfrak{m} be such an ideal and let $C[[X]][1/p] \rightarrow E'$ be the resulting E' -algebra map. We want to show that $(C[[X]][1/p])^{\wedge \mathfrak{m}}$ is normal. Clearly if \mathfrak{m}' is the kernel of the induced map $(C \otimes_{\mathcal{O}} \mathcal{O}_{E'})[[X]][1/p] \rightarrow E'$, then $\mathfrak{m} \subseteq \mathfrak{m}'$ and the map $(C[[X]][1/p])^{\wedge \mathfrak{m}} \rightarrow ((C \otimes_{\mathcal{O}} \mathcal{O}_{E'})[[X]][1/p])^{\wedge \mathfrak{m}'}$ is faithfully flat. Moreover since $E \rightarrow E'$ is smooth (even étale), $(C \otimes_{\mathcal{O}} \mathcal{O}_{E'})[1/p] = C[1/p] \otimes_E E'$ is still normal by [Sta18, Tag 033C]. Thus by replacing C with $C \otimes_{\mathcal{O}} \mathcal{O}_{E'}$, we may assume that $E' = E$. Now, any such map $C[[X]][1/p] \rightarrow E$ necessarily sends X to an element $\lambda \in \mathfrak{m}_E$. If $\mathfrak{m}_C := \mathfrak{m} \cap C[1/p]$, then \mathfrak{m}_C is a maximal ideal of $C[1/p]$ and $\mathfrak{m} = (\mathfrak{m}_C, X - \lambda)$. It follows that

$$(C[[X]][1/p])^{\wedge \mathfrak{m}} = (C[[X]][1/p])^{\wedge (\mathfrak{m}_C, X - \lambda)} \simeq (C[1/p]^{\wedge \mathfrak{m}_C})[[X - \lambda]]$$

is normal by e.g. [Sta18, Tag 0BI0], as wanted. \square

Return to our discussion. Assuming (*) is satisfied, we will construct a finite morphism

$$\mathcal{X}^{\mathrm{nm}} \rightarrow \mathcal{X},$$

characterized by the property that for any smooth chart $\mathrm{Spf}(A) \rightarrow \mathcal{X}$, the pullback $\mathcal{X}^{\mathrm{nm}} \times_{\mathcal{X}} \mathrm{Spf}(A) \rightarrow \mathrm{Spf}(A)$ identifies with $\mathrm{Spf}(A^{\mathrm{nm}}) \rightarrow \mathrm{Spf}(A)$ where $\mathrm{Spec}(A^{\mathrm{nm}}) \rightarrow \mathrm{Spec}(A)$ is the normalization of $\mathrm{Spec}(A)$ in the sense of [Sta18, Tag 035E].

Lemma A.6. *Assume \mathcal{X} satisfies (*). If $\mathrm{Spf}(A) \rightarrow \mathcal{X}$ is a smooth chart, then $A[1/p]$ is normal.*

Proof. Given a prime ideal \mathfrak{q} of $A[1/p]$, choose a maximal ideal \mathfrak{m} of A containing \mathfrak{q} . Then $A^{\wedge \mathfrak{m}}$ is a versal ring to \mathcal{X} at the finite type point $\mathrm{Spec}(A/\mathfrak{m}) \rightarrow \mathcal{X}$, and thus $A^{\wedge \mathfrak{m}}[1/p]$ is normal by (*). Since $A_{\mathfrak{m}}[1/p] \rightarrow A^{\wedge \mathfrak{m}}[1/p]$ is faithfully flat, $A_{\mathfrak{m}}[1/p]$ is also normal. Now $A_{\mathfrak{q}}$ is a localization of $A_{\mathfrak{m}}[1/p]$, hence also normal. \square

Construction A.7. *If A is a p -flat Noetherian ring with $A[1/p]$ normal, we denote by A^{nm} the integral closure of A in $A[1/p]$. Concretely, if we let $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ be the minimal primes of A , then $A[1/p] \simeq \prod_i (A/\mathfrak{q}_i)[1/p]$ is a product of normal domains, and A^{nm} is the product over i of the*

integral closure of A/\mathfrak{q}_i in $(A/\mathfrak{q}_i)[1/p] \subseteq \text{Frac}(A/\mathfrak{q}_i)$. In particular, A^{nm} is normal and agrees with the normalization of A in the sense of [Sta18] by e.g. [Sta18, Tag 035P].

Remark A.8. Note that if A as above is quasi-excellent, then A^{nm} is finite over A (by e.g. [Sta18, Tag 07QV, Tag 035S]), hence also p -adically complete. This applies in the following two cases that are relevant to us:

- there is a smooth chart $\text{Spf}(A) \rightarrow \mathcal{X}$; or
- $A = C$ is a versal ring to \mathcal{X} at a finite type point.

Indeed, in the first case, A is quasi-excellent since $\text{Spf}(A) \rightarrow \text{Spf}(\mathcal{O})$ is in particular a finite type adic map; see the proof of [Eme, Corollary 8.25]. The second case follows from [Sta18, Tag 07QW] since A is a complete local Noetherian ring.

We claim that the formation of $\text{Spf}(A^{\text{nm}})$ is compatible with smooth base change in the smooth chart $\text{Spf}(A) \rightarrow \mathcal{X}$. Let $\text{Spf}(B) \rightarrow \text{Spf}(A)$ be a smooth chart. By [Eme, Lemma 8.18] the ring map $A \rightarrow B$ is flat. In particular, we have $B \subseteq A^{\text{nm}} \otimes_A B \subseteq B[1/p]$. We need to show that $A^{\text{nm}} \otimes_A B = B^{\text{nm}}$ in $B[1/p]$, or equivalently, $A^{\text{nm}} \otimes_A B$ is normal. Since the map $\text{Spf}(A^{\text{nm}} \otimes_A B) \rightarrow \text{Spf}(A^{\text{nm}})$ is smooth, and A^{nm} is normal and quasi-excellent (being finite over A , as already remarked), hence analytically normal, it follows from [Eme, Lemma 8.20] that $A^{\text{nm}} \otimes_A B$ is (analytically) normal, as wanted.

It follows that there is a finite morphism $\mathcal{X}^{\text{nm}} \rightarrow \mathcal{X}$ of p -adic formal algebraic stacks, uniquely characterized by the property that for any smooth chart $\text{Spf}(A) \rightarrow \mathcal{X}$, it pulls back to the natural map $\text{Spf}(A^{\text{nm}}) \rightarrow \text{Spf}(A)$. We call $\mathcal{X}^{\text{nm}} \rightarrow \mathcal{X}$ the normalization of \mathcal{X} . Clearly $\mathcal{X}^{\text{nm}} \simeq \mathcal{X}$ if and only if \mathcal{X} is normal in the sense recalled above. Furthermore, if $\mathcal{Y} \rightarrow \mathcal{X}$ is a smooth chart, then \mathcal{Y} also satisfies (*) and there is a natural cartesian diagram

$$\begin{array}{ccc} \mathcal{Y}^{\text{nm}} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{X}^{\text{nm}} & \longrightarrow & \mathcal{X}. \end{array}$$

We can finally state our main result on the singularity of the Barsotti–Tate stack \mathcal{Z}^τ .

Theorem A.9. *Assume either $p \geq 7$ or $K = \mathbb{Q}_5$. Then the normalization (in the sense above) of \mathcal{Z}^τ is Cohen–Macaulay.*

Proof. We would like to say that the normalization of \mathcal{Z}^τ identifies with the relative normalization of \mathcal{Z}^τ in $Y^{\eta,\tau}$. As in the case of the model $\tilde{\mathcal{Z}}^{\text{mod},\tau}$, the underlying reason for this is: $Y^{\eta,\tau}$ is normal and the map $Y^{\eta,\tau} \rightarrow \mathcal{Z}^\tau$ becomes an isomorphism after inverting p . However, since we are working in the context of formal stacks, we have to be careful on what we mean by “inverting p ”. Instead, we will algebraize the situation so as to work with honest schemes.

Consider a smooth chart $\text{Spf}(A) \rightarrow \mathcal{Z}^\tau$. We want to show that A^{nm} is Cohen–Macaulay. Note that for this we are free to localize $\text{Spf}(A)$ further in the smooth topology. We claim that we can localize $\text{Spf}(A)$ so that the proper p -adic formal algebraic space $Y^\wedge := Y^{\eta,\tau} \times_{\mathcal{Z}^\tau} \text{Spf}(A) \rightarrow \text{Spf}(A)$ can be algebraized to a proper scheme $Y \rightarrow \text{Spec}(A)$. By definition, the map $\text{Spf}(A) \rightarrow \mathcal{Z}^\tau$ corresponds to a rank 2 projective étale φ -module with A -coefficients, or equivalently a compatible system $\{\mathcal{M}_i\}$ of étale φ -module with (A/p^i) -coefficients for varying $i \geq 1$. By [EG21, Lemma 5.1.23], there is a smooth (even Nisnevich) cover $\text{Spec}(A_1) \rightarrow \text{Spec}(A/p)$ so that the base change $(\mathcal{M}_1)_{A_1}$ is free over $(W(k) \otimes_{\mathbb{Z}_p} A_1)((v^n))$ for some n . By [Sta18, Tag 0CKI], we can then inductively

construct a cartesian diagram

$$\begin{array}{ccccc} \mathrm{Spec}(A_1) & \hookrightarrow & \mathrm{Spec}(A_2) & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \\ \mathrm{Spec}(A/p) & \hookrightarrow & \mathrm{Spec}(A/p^2) & \longrightarrow & \dots \end{array}$$

with smooth vertical maps. The colimit $\varinjlim \mathrm{Spec}(A_i)$ thus defines a smooth cover $\mathrm{Spf}(A') \rightarrow \mathrm{Spf}(A)$ of p -adic affine formal schemes. Moreover, for each $i \geq 1$ the base change $(\mathcal{M}_i)_{A_i}$ is free over $(W(k) \otimes_{\mathbb{Z}_p} A_i)((v^n))$ (as this can be checked after extending scalars along the thickening $A_i \rightarrow A_1$) and we can inductively choose a compatible system of bases for these. Thus after replacing A by A' , we see by the proof of [EG21, Lemma 5.4.10] that the system $Y^\wedge := Y^{\eta, \tau} \times_{\mathcal{Z}^\tau} \mathrm{Spf}(A) = \{Y^{\eta, \tau} \times_{\mathcal{Z}^\tau} \mathrm{Spec}(A/p^i)\}_{i \geq 1}$ can be identified as a compatible system of projective closed subschemes inside an affine Grassmannian over $\mathrm{Spec}(A)$. It follows that $Y^\wedge := Y^{\eta, \tau} \times_{\mathcal{Z}^\tau} \mathrm{Spf}(A) \rightarrow \mathrm{Spf}(A)$ can be algebraized to a proper scheme $Y \rightarrow \mathrm{Spec}(A)$ by the Grothendieck existence theorem (cf. the proof of [Kis09, Prop. 2.1.10]).

Since the map $Y^{\eta, \tau} \rightarrow \mathcal{Z}^\tau$ is scheme-theoretically dominant, the same holds true for the flat base change $Y^\wedge = Y^{\eta, \tau} \times_{\mathcal{Z}^\tau} \mathrm{Spf}(A) \rightarrow \mathrm{Spf}(A)$. The theorem on formal functions then implies the same for the map $Y \rightarrow \mathrm{Spec}(A)$ of schemes.

We claim that the map $Y \rightarrow \mathrm{Spec}(A)$ becomes a closed immersion (hence an isomorphism) after inverting p . Note that since A is a p -adically complete topologically of finite type \mathcal{O} -algebra, $A[1/p]$ is Jacobson and its residue field at any maximal ideal is a finite extension of E . Thus, exactly as in the proof of [Kis08, Proposition 1.6.4 (1)], it suffices to show that the map $Y \rightarrow \mathrm{Spec}(A)$ is injective on $\mathrm{Spec}(C)$ -valued points for any finite flat \mathcal{O} -algebra C . By e.g. [Sta18, Tag 0A42], it suffices to show the analogous statement for $\mathrm{Spf}(C)$ -valued points of the completed map $\widehat{Y} \rightarrow \mathrm{Spf}(A)$ (note that since C is finite over \mathcal{O} , any map $\mathrm{Spec}(C) \rightarrow \mathrm{Spec}(A)$ is in particular proper, and so the assumption in the cited reference is indeed satisfied). In turn it suffices to show this for $\mathrm{Spf}(C)$ -valued points of the map $Y^{\eta, \tau} \rightarrow \mathcal{Z}^\tau$, but this follows from uniqueness of Breuil–Kisin lattices in finite height representations.

By [CEGSb, Corollary 4.5.3], Y^\wedge is p -flat and (analytically) normal. We claim that the scheme Y is also p -flat and normal. We give the proof for normality, the flatness case being identical. It suffices to show that the local ring $\mathcal{O}_{Y, y}$ at any closed point $y \in Y$ is normal. Choose an open affine $\mathrm{Spec}(B)$ of Y around y ; let $\mathfrak{m}_y \subseteq B$ be the corresponding maximal ideal. Since $Y \rightarrow \mathrm{Spec}(A)$ is proper and A is p -adically complete, $\mathfrak{m}_y \ni p$. In particular \mathfrak{m}_y gives rise to a maximal ideal, still denoted \mathfrak{m}_y , of $B^{\wedge p}$. Since $\mathrm{Spf}(B^{\wedge p}) \subseteq Y^\wedge$ is an open, $B^{\wedge p}$ is (analytically) normal. It follows that $B^{\wedge \mathfrak{m}_y} \simeq (B^{\wedge p})^{\wedge \mathfrak{m}_y}$ is normal, as claimed.

Since Y is a normal scheme, the ring $\mathcal{O}(Y)$ is normal (e.g. by [Sta18, Tag 0358]) and hence $\mathcal{O}(Y) = A^{\mathrm{nm}}$, the integral closure of A in $A[1/p] \simeq \mathcal{O}(Y)[1/p]$. Thus the normalization $\mathrm{Spec}(A^{\mathrm{nm}}) \rightarrow \mathrm{Spec}(A)$ fits into the Stein factorization

$$\begin{array}{ccc} Y & \xrightarrow{\pi^{\mathrm{nm}}} & \mathrm{Spec}(A^{\mathrm{nm}}) \\ & \searrow \pi & \downarrow \\ & & \mathrm{Spec}(A); \end{array}$$

this is the desired factorization alluded to at the beginning of the proof (but involves only schemes rather than formal stacks!). Recall that we are free to localize $\mathrm{Spf}(A)$ further in the smooth topology; in particular we may assume that $\mathrm{Spf}(A)$ lies over the smooth cover $\widetilde{\mathcal{Z}}^\tau \rightarrow \mathcal{Z}^\tau$. By

Lemma 3.3.9 and Corollary 3.3.15, the maps $\tilde{Y}^{\eta,\tau} \rightarrow \tilde{Z}^\tau$ and $\tilde{Y}^{\text{mod},\eta,\tau} \rightarrow \tilde{Z}^{\text{mod},\tau}$ identify after applying $(-)/p := (-) \otimes_{\mathcal{O}} \mathcal{O}/p$. In particular, we also have a smooth map $\text{Spec}(A/p) \rightarrow \tilde{Z}^{\text{mod},\tau}/p$, and the map $\pi : Y/p \rightarrow \text{Spec}(A/p)$ then identifies with the pullback $\pi^{\text{mod}} : \tilde{Y}^{\text{mod},\eta,\tau}/p \times_{\tilde{Z}^{\text{mod},\tau}/p} \text{Spec}(A/p) \rightarrow \text{Spec}(A/p)$.

By Proposition 3.3.18 and flat base change we have

$$R\pi_* \mathcal{O}_Y \otimes_{\mathcal{O}}^{\mathbb{L}} \mathcal{O}/p \simeq R\pi_*(\mathcal{O}_{Y/p}) \simeq R\pi_*^{\text{mod}}(\mathcal{O}_{\tilde{Y}^{\text{mod},\eta,\tau}/p \times_{\tilde{Z}^{\text{mod},\tau}/p} \text{Spec}(A/p)})$$

is concentrated in degree 0. It follows that for each $j > 0$, the finite A -module $R^j \pi_* \mathcal{O}_Y$ is infinitely divisible by p , and hence must be 0 (as A is p -adically complete and Noetherian). Thus $R\pi_* \mathcal{O}_Y$ is concentrated degree 0 whence $R\pi_*^{\text{nm}} \mathcal{O}_Y = \mathcal{O}_{\text{Spec}(A^{\text{nm}})}$. It follows that $\mathcal{O}_{\text{Spec}(A^{\text{nm}}/p)} = R\pi_*^{\text{nm}} \mathcal{O}_{Y/p} = \pi_*^{\text{nm}} \mathcal{O}_{Y/p}$ and so we obtain an isomorphism

$$\text{Spec}(A^{\text{nm}}/p) \simeq \tilde{Z}^{\text{mod,nm}}/p \times_{\tilde{Z}^{\text{mod},\tau}/p} \text{Spec}(A/p)$$

(as both sides identify with the normalization of the same map). Since the RHS is smooth over $\tilde{Z}^{\text{mod,nm}}/p$, it is Cohen–Macaulay by Theorem A.1. Thus A^{nm}/p is Cohen–Macaulay. Since A^{nm} is also p -adically complete (being finite over A), it follows that it is Cohen–Macaulay. This finishes the proof. \square

For a continuous representation $\bar{\rho} : G_K \rightarrow \text{GL}_2(\mathbb{F})$, let $R_{\bar{\rho}}^{\eta,\tau}$ be the universal framed deformation ring classifying lifts of $\bar{\rho}$ which are potentially Barsotti–Tate and of type τ . Recall from [CEGSb, Corollary 5.2.19] that $R_{\bar{\rho}}^{\eta,\tau}$ is a versal ring to \mathcal{Z}^τ at the finite type point given by $\bar{\rho}$.

To deduce the analogue of Theorem A.9 for the ring $R_{\bar{\rho}}^{\eta,\tau}$, we need the following lemma.

Lemma A.10. *Assume \mathcal{X} satisfies $(*)$ and \mathcal{X}^{nm} is Cohen–Macaulay. Then for any versal ring C to \mathcal{X} at a finite type point, the normalization C^{nm} is Cohen–Macaulay.*

Proof. We need to show that

- (i) for any smooth chart $\text{Spf}(A) \rightarrow \mathcal{X}$ and any maximal ideal \mathfrak{m} of A , if A^\wedge denotes the \mathfrak{m} -adic completion of A , then $(A^\wedge)^{\text{nm}}$ is Cohen–Macaulay; and
- (ii) if C is a versal ring X at a finite type point (e.g. $C = A^\wedge$ from (i)), then C^{nm} is Cohen–Macaulay if and only if $(C[[X]])^{\text{nm}}$ is.

We begin with (i). By assumption A^{nm} is Cohen–Macaulay. Since $A \rightarrow A^\wedge$ is flat, we have $A^\wedge \subseteq A^{\text{nm}} \otimes_A A^\wedge \subseteq A^\wedge[1/p]$. Since A^{nm} is finite over A , $A^{\text{nm}} \otimes_A A^\wedge$ is the \mathfrak{m} -adic completion of A^{nm} . As A^{nm} is quasi-excellent (being finite over A), its completion $A^{\text{nm}} \otimes_A A^\wedge$ is also normal (hence identifies with $(A^\wedge)^{\text{nm}}$) and Cohen–Macaulay. Now for (ii), since $C \rightarrow C[[X]]$ is flat, we have $C[[X]] \subseteq C^{\text{nm}} \otimes_C C[[X]] \subseteq C[[X]][1/p]$. Since C^{nm} is finite over C , $C^{\text{nm}} \otimes_C C[[X]] = C^{\text{nm}}[[X]]$ is also normal and thus identifies with $(C[[X]])^{\text{nm}}$. We are now done since C^{nm} is Cohen–Macaulay if and only if $C^{\text{nm}}[[X]]$ is. \square

The following is now immediate.

Corollary A.11. *Assume either $p \geq 7$ or $K = \mathbb{Q}_5$. Then for any $\bar{\rho} : G_K \rightarrow \text{GL}_2(\mathbb{F})$, the normalization of $R_{\bar{\rho}}^{\eta,\tau}$ in $R_{\bar{\rho}}^{\eta,\tau}[1/p]$ is Cohen–Macaulay.*

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DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, 2033 SHERIDAN ROAD, EVANSTON, ILLINOIS 60208, USA

Email address: lhvietbao@googlemail.com

INSTITUT DE MATHÉMATIQUES DE JUSSIEU- PARIS RIVE GAUCHE, SORBONNE UNIVERSITÉ, 4 PLACE JUSSIEU, 75005 PARIS, FRANCE

Email address: ariane.mezard@imj-prg.fr, dat.pham@imj-prg.fr

UNIVERSITÉ PARIS 8, LABORATOIRE D'ANALYSE, GÉOMÉTRIE ET APPLICATIONS, LAGA, UNIVERSITÉ SORBONNE PARIS NORD, CNRS, UMR 7539, F-93430, VILLETANEUSE, FRANCE

Email address: morra@math.univ-paris13.fr