

Lue Pan - II

Fix $K \leq GL_n(\mathbb{A}_f^p)$ and $\tilde{H} = \tilde{H}^1(K, \mathbb{Q}_p)$ completed cohomology

Thm: $\rho: G_{\mathbb{Q}} \rightarrow GL_n(\mathbb{Q}_p)$ ℓ^0 , abs irr.

Assume (i) $\pi_{\rho} := \text{Hom}_{\mathbb{Q}_p}[G_{\mathbb{Q}}] (\rho, \tilde{H}) \neq 0$

(ii) $\rho|_{G_{\mathbb{Q}_p}}$ is de Rham with HT weights $0, k > 0$.

Then ρ arises from an eigenform of weight $k+1$

de Rham repⁿ of weights $0, k > 0$.

$G_{\mathbb{Q}_p} G V = \mathbb{Q}_p^{\oplus 2}$ Hodge-Tate of weights $0, k$: $\underbrace{V \otimes_{\mathbb{Q}_p} \mathbb{C}_p^{\widehat{W}}}_{\parallel} = W_0 \oplus W_k$

$\mathbb{C}_p = \widehat{\mathbb{Q}_p}$

de Rham: $\dim_{\mathbb{Q}_p} (V \otimes B_{dR})^{G_{\mathbb{Q}_p}} = \dim V = 2$

Frobenius: $N: W_0 \rightarrow W_k(k)$ s.t. V de Rham $\Leftrightarrow N=0$

Strategy

$$=: Z \quad (0, -k+1)$$

Last time:

$$\tilde{X}_k = \overline{Z(\mathcal{U}(\mathfrak{gl}_2))} \longrightarrow \mathbb{Q}_p$$

$$W := \tilde{H}^0(\tilde{X}_k) \otimes_{\mathbb{Q}_p} \mathbb{C}_p = W_0 \oplus W_k$$

U
p

has an action of $\Gamma = \mathbb{Z}[\Gamma, \text{Sl}]$
last: $K^* \cap \text{GL}_2(\mathbb{Q}_p)$
 $\cong \text{GL}_2(\mathbb{Z}_p)$

Fact: $N_k: W_0 \rightarrow W_k$

& p "appears" in $\text{Ker } N_k$

generalized eigenspace

This then implies the main theorem

Theorem: $\text{Ker } N_k = \bigoplus (\text{Ker } N_k)[\lambda]$

$\lambda: \Gamma \rightarrow \overline{\mathbb{Q}_p}$
associated to eigenform of weight $k+1$

appear only when $k=1$

or λ appears in $\varinjlim_{K_p} H^0(Y_{K_p}, \mathbb{Q}_p)$

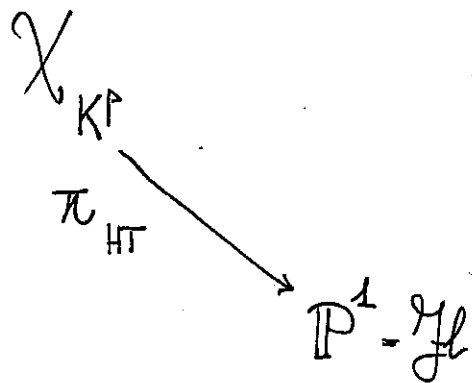
Prop: $\lambda \longleftrightarrow \pi = \otimes \pi_l$

then: $\text{Ker } N_k[\lambda] = \text{Ker } N_k[\lambda]$ if π is cuspidal

(ie. no need of taking generalized eigenspace)

& π_p is either supercuspidal or principal series
ie. π_p von Steenberg

Recall



$$G \mathcal{O} := \pi_{HT}^* \mathcal{O}_{X_{KP}}$$

$\mathcal{O}^{la} \subseteq \mathcal{O}$
 $\mathcal{O} \curvearrowright$ $GL_2(\mathbb{Q}_p)$ locally analytic sections

\mathcal{O}_h

Cartan subalgebra

encodes the action of $Z \times \mathcal{O}_{Sen}$

$$h = \begin{pmatrix} * & \\ & * \end{pmatrix} \subseteq \begin{pmatrix} * & * \\ & * \end{pmatrix}$$



$$h \otimes \mathcal{O}_{\mathbb{P}^1} \cong \mathcal{O}(0) / \mathcal{O}(n) \quad G\mathcal{O}^{la}$$

Get $\mathcal{O}^{la, \tilde{X}_k} = \mathcal{O}^{la, (1-k, 0)} \oplus \mathcal{O}^{la, (1, -k)}$

HT wt: 0

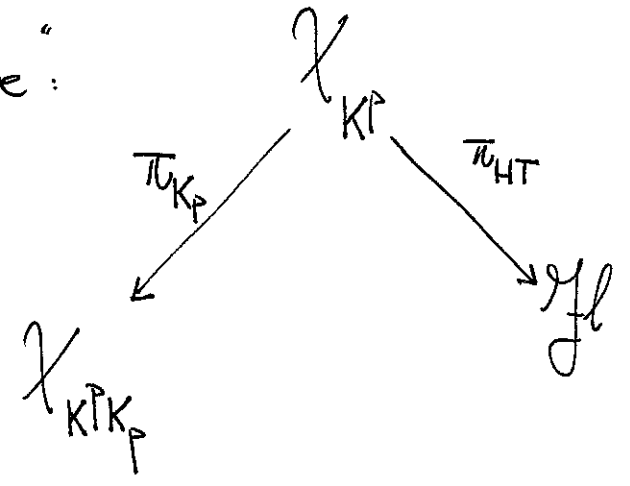
HT wt: k

$$W_0 = H^1(\mathcal{Y}^l, \mathcal{O}^{(a, (1-k, 0))}) \quad (k \geq 2)$$

$$W_k = H^1(\mathcal{Y}^l, \mathcal{O}^{(a, (1, -k))})$$

Construct: $I_k : \mathcal{O}^{(a, (1-k, 0))} \longrightarrow \mathcal{O}^{(a, (1, -k))}$ s.t. $H^1(I_k) = N_k$

"Complete using the picture":



$$\mathcal{O}^{sm} \subseteq \mathcal{O} \text{ } \mathcal{G}_2(\mathbb{Q})\text{-smooth sections}$$

$$\parallel \lim_{K_P} \pi_{HT*} \pi_{K_P}^{-1} \mathcal{O}_{X_{KPK_P}}$$

& similarly:

$$\Omega^{s, sm} := \lim_{K_P} \pi_{HT*} \pi_{K_P}^{-1} \Omega_{X_{KPK_P}}^1(\mathbb{C})$$

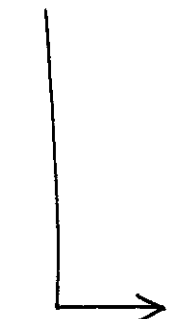
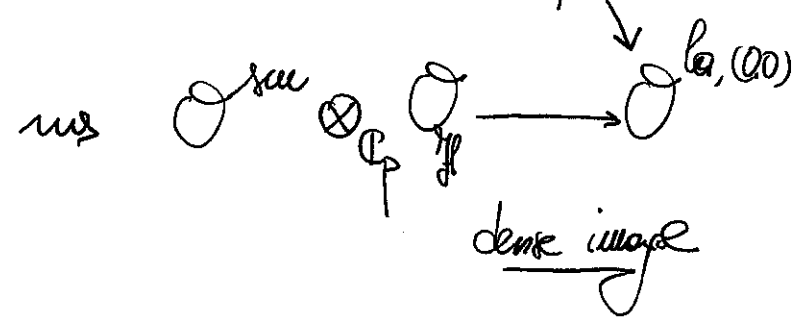
cusp
↓

Get $d: \mathcal{O}^{\text{sym}} \rightarrow \Omega^{1, \text{sym}}$

(can repeat the "symmetric" situation & consider): $d_{\text{eff}}: \mathcal{O}_{\text{eff}} \rightarrow \Omega_{\text{eff}}^1$

Fact $\mathcal{O}^{\text{sym}}, \mathcal{O}_{\text{eff}} \subseteq \mathcal{O}^{\text{la}, (0,0)}$

same kind of completed tensor product: one would like to extend $d \otimes 1: \mathcal{O}^{\text{sym}} \otimes_{\mathbb{C}_p} \mathcal{O}_{\text{eff}} \rightarrow \Omega^{1, \text{sym}} \otimes_{\mathbb{C}_p} \mathcal{O}_{\text{eff}}$



Theorem: (1) $\exists!$ $\mathcal{O}^{\text{la}, (0,0)}$ map $d^1: \mathcal{O}^{\text{la}, (0,0)} \rightarrow \mathcal{O}^{\text{la}, (0,0)} \otimes_{\mathcal{O}^{\text{sym}}} \Omega^{1, \text{sym}}$ s.t.

"symmetric" situation

- (1.1) d^1 is \mathcal{O}_{eff} -linear
- (1.2) extends $d: \mathcal{O}^{\text{sym}} \rightarrow \Omega^{1, \text{sym}}$

(2) $\exists!$ $\mathcal{O}^{\text{la}, (0,0)}$ map $d^1: \mathcal{O}^{\text{la}, (0,0)} \rightarrow \mathcal{O}^{\text{la}, (0,0)} \otimes_{\mathcal{O}_{\text{eff}}} \Omega_{\text{eff}}^1$ s.t.

- (2.1) d^1 is \mathcal{O}^{sym} -linear
- (2.2) extends $d_{\text{eff}}: \mathcal{O}_{\text{eff}} \rightarrow \Omega_{\text{eff}}^1$

Bernstein-Belinson: d_{diff} comes from some Lie algebra action:

$$d_{\text{diff}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} dX \in \mathfrak{gl}_2(\mathbb{C})$$

if $X := \text{coordinate on } \mathbb{A}^1 \subseteq \mathbb{P}^1$

have $d^1 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} dX$

$\text{Hom } Y \rightarrow D = \mathcal{H}'_{\text{diff}}(\mathbb{E}/Y)$ with:
 $(\text{Fil}, \nabla) \mapsto (\mathcal{O}^{b, (0,0)} \otimes \text{Sym}^{k-1} \nabla) \rightarrow \mathcal{O}^{b, (0,0)} \otimes \text{Sym}^{k-1} \nabla \otimes \Omega^{1,nu}$

Recall BGG: $d^k: \mathcal{O}^{b, (1-k, 0)} \rightarrow \mathcal{O}^{b, (1-k, 0)} \otimes_{\mathcal{O}_{\text{gr}}} (\Omega^{1,nu})^{\otimes k}$
 $d^k: \mathcal{O}^{b, (1-k, 0)} \rightarrow \mathcal{O}^{b, (1-k, 0)} \otimes_{\mathcal{O}_{\text{diff}}} (\Omega^1_{\text{diff}})^{\otimes k}$

Kodaira-Spencer iso

$$\mathcal{O}^{b, (0,0)} \otimes \text{Sym}^{k-1} \nabla \rightarrow \mathcal{O}^{b, (0,0)} \otimes \text{Sym}^{k-1} \Omega^{1,nu}$$

$\tilde{\lambda} \in \dots \downarrow \lambda \in \mathfrak{gr}^0 \quad \nabla(\tilde{\lambda}) \in \text{Fil}^{k-1} \otimes \Omega^{1,nu}$

$$\nabla: \text{Fil}^1 \xrightarrow{\sim} \left(\frac{\quad}{\text{Fil}^{k-1} \otimes \Omega^{1,nu}} \right)$$

Consider the case $k=1$.

$$\mathcal{O}^{b, (0,0)} \xrightarrow{d^1} \mathcal{O}^{b, (0,0)} \otimes_{\mathcal{O}_{\text{gr}}} \Omega^{1,nu}$$

$$\downarrow d^1 \otimes 1$$

$$\mathcal{O}'_{\text{diff}} \otimes \mathcal{O}^{b, (0,0)} \otimes_{\mathcal{O}_{\text{gr}}} \Omega^{1,nu}$$

$$\mathcal{O}^{b, (1,-1)} \otimes_{\mathbb{Z}(\ast)} (1)$$

define the composite maps as I_1

$$\text{we } I_k := d^k \circ d^k$$

(*) Kodaira-Spencer isomorphism

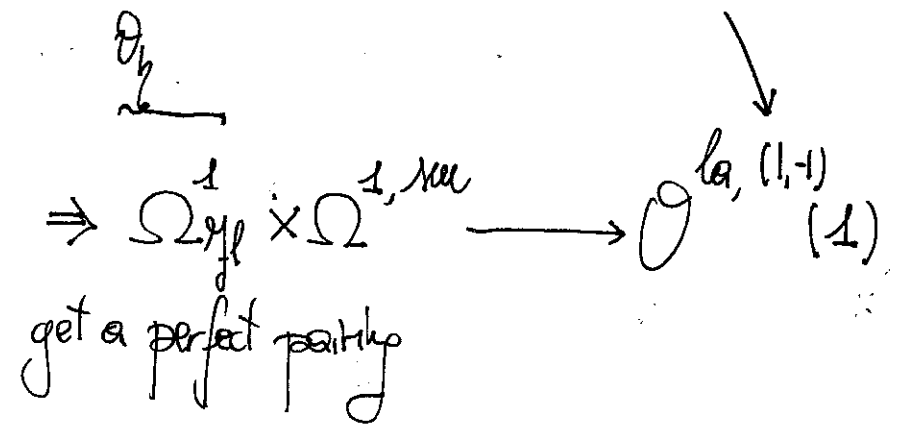
(1) $\omega^{\otimes 2} \cong \Omega_{X/K}^1 \otimes \Omega_{K/P}^1$ (C)

(2) $\omega_{Y/L}^{\otimes (-2)} \cong \Omega_{Y/L}^1$

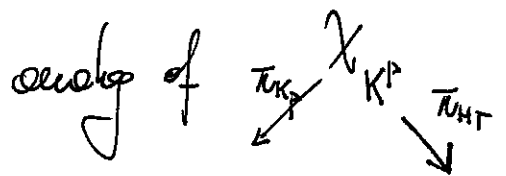
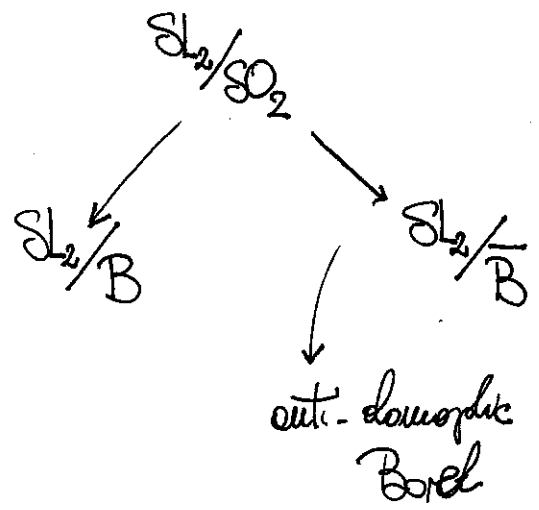
(3) $\pi_{HT}^* \omega_{Y/L} \cong \pi_{K/P}^* \omega(-1)$

See operator: $\mathcal{D}_Y \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathcal{D}_{See}$

comes from the \mathcal{D}_Y -action



Remark: over \mathbb{C} :



Theorem $H^1(I_k) = c_k \cdot N_k \quad \exists c_k \in \mathbb{Q}^*$

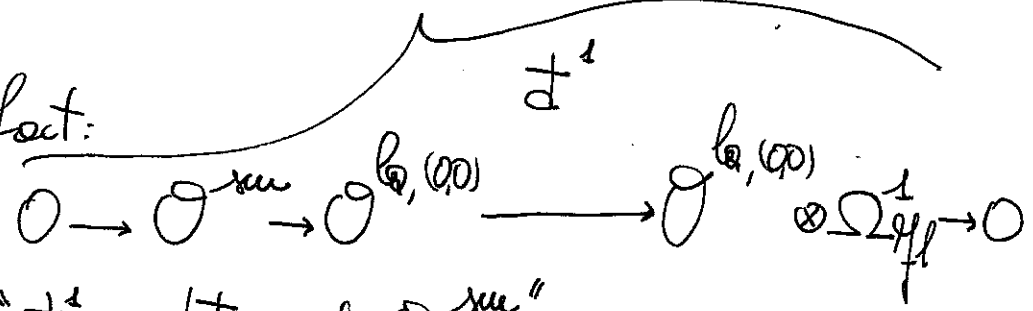
Pf: Uses $OB_{\mathbb{R}^+}$ via Poincaré's lemma.

Study: $\text{Ker } H^1(I_k) = \text{Ker} (H^1(d^k) \circ H^1(d^k)) \quad (\overline{\mathbb{R}})$

$k=1$ case

Lemma: $H^1(d^1)$ is injective

In fact:



" d^1 -resolution of \mathcal{O}_{sm} "

"de Rham C-lex using the derivation on \mathbb{R} "

$$\Rightarrow H^1(\mathcal{Y}_b, \Omega^{1, \text{sm}}) \longrightarrow \text{Ker } H^1(d^1)$$

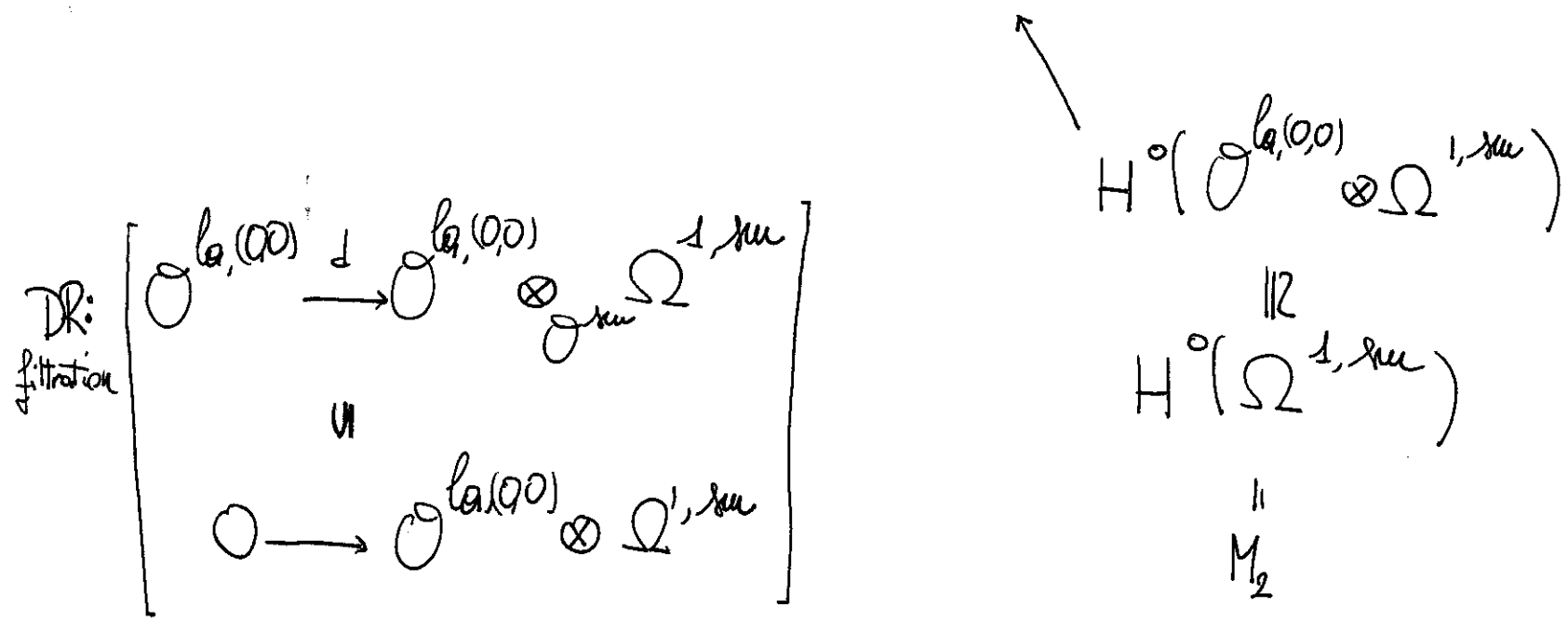
$$\lim_{\substack{\longrightarrow \\ K_p}} H^1(X_{K_p}, \Omega^1(\mathbb{C})) = 0$$

Why we have such s.e.s.?

• $d = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} dx$, $\text{Ker } d$ is $GL_2(\mathbb{Q}_p)$ -stable $\Rightarrow \text{Ker } d = GL_2(\mathbb{Q}_p)$ -smooth vectors = \mathcal{O}_{sm}

• $(\overline{\mathbb{R}})_y$: the de Rham C-lex of $\pi_{K_p}^{-1}(y) \cong K_p$ profinite set = 0-dim $\Rightarrow \underline{NO} H^1$

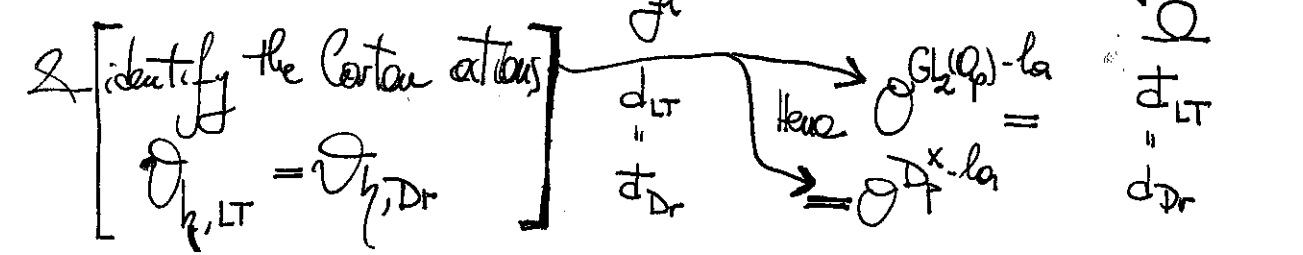
By injectivity, $\text{Ker}(H^1(I_1)) = \text{Ker}(H^1(d^1)) = \frac{[H^1(DR)]}{\text{Fil}^1}$ it suffices to study this!



$$y \in \mathbb{P}^1 = \mathbb{P}^1(\mathcal{O}_p) \sqcup \Omega$$

$$1. \mathcal{O}_y \pi_{HT}^{-1}(\Omega) \cong \mathbb{D}^x / \mathcal{M}_{LT, \infty}^{(0)} \times (\mathbb{D} \otimes \mathbb{A}_f^P) / K^P \cong \sqcup \mathcal{M}_{LT, \infty}^{(0)}$$

here: $\mathcal{M}_{LT, \infty}^{(0)} \cong \mathcal{M}_{Dr, \infty}^{(0)}$



hence $d^1|_{\Omega}$ is surjective as $\ker d^1$ is $GL_2(\mathbb{Q}_p)$ -stable

$\Rightarrow \ker d|_{\Omega} = \mathcal{O}_{D_p}^{\times}$ -module vectors $\Rightarrow DR|_{\Omega} = \ker d|_{\Omega} [0] = \varinjlim_n (A^{\otimes n} \otimes \pi_n^* \mathcal{O}_{\mathcal{M}_n})$

where $A = \mathcal{C}^{sm} \left(\frac{D \otimes A_f}{D^{\times}}, \mathbb{C}_p \right)$
 $= \bigoplus DR|_{\Omega}[\lambda]$ where $\lambda \leftrightarrow$ char. rep.^w of $(D \otimes A)^{\times}$

Moreover $DR|_{P(\mathbb{Q}_p)}$

$DR|_{\infty} = \mathcal{O}^{la, (0,0)}|_{\infty} \xrightarrow{d} \mathcal{O}^{la, (0,0)} \otimes \Omega^{1, sm}|_{\infty}$

$\xrightarrow{d \otimes 1} \mathcal{M}_2^t \hat{\otimes} \mathcal{O}_{\mathcal{H}_g, \infty}$
"deperfection"

$\Rightarrow H^i(DR)_{\infty} = H^i(\mathbb{I}_g) \otimes \mathcal{O}_{\mathcal{H}_g, \infty}$
 $= \bigoplus H^1(DR)_{\infty}[\lambda]$