Lue Pan - II

Fix $K^p \leq GL(\mathbb{A}_f^p)$ ⇝ $\tilde{H} = \tilde{H}^1(K^p, \mathbb{Q}_p)$ completed cohomology

Thm. $\rho : G_{\mathbb{Q}} \to GL_2(\mathbb{Q}_p)$ $\mathcal{C}^0$, abs irr.

Assume (i) $\Pi_\rho := \mathrm{Hom}_{\mathbb{Q}_p[G_{\mathbb{Q}}]}(\rho, \tilde{H}) \neq 0$

(ii) $\rho|_{G_{\mathbb{Q}_p}}$ is de Rham with HT weights $0, k>0$.

Then ρ arises from an eigenform of weight $k+1$

De Rham rep^n of weights $0, k \geq 0$.

$G_{\mathbb{Q}_p} \curvearrowright V = \mathbb{Q}_p^{\oplus 2}$ Hodge-Tate of weights $0, k$: $\overbrace{V \otimes_{\mathbb{Q}_p} \mathbb{C}_p}^{W} = W_0 \oplus W_k$

$\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$

de Rham: $\dim_{\mathbb{Q}_p} (V \otimes B_{dR})^{G_{\mathbb{Q}_p}} = \dim V = 2$

Fontaine: $N : W_0 \to W_k(k)$ s.t. V de Rham $\Leftrightarrow N = 0$

Strategy

$(0, -k+1)$

Last time: $\tilde{\chi}_k = \overbrace{\widetilde{Z(\mathcal{U}(\mathfrak{gl}_2))}}^{=: Z} \longrightarrow \mathbb{Q}_p$

$$W := \tilde{H}^{la, \tilde{\chi}_k} \otimes_{\mathbb{Q}_p} \mathbb{C}_p = W_0 \oplus W_k$$

$\cup\!|$

ρ

has an action of $\mathbb{T} = \mathbb{Z}[T_\ell, S_\ell,$ ℓ s.t. $\underbrace{K^p \cap GL_2(\mathbb{Q}_\ell)}_{\cong GL_2(\mathbb{Z}_\ell)}$

Fontaine: $N_k : W_0 \to W_k(k)$ & ρ "appears" in $\operatorname{Ker} N_k$

[this thm implies the main Theorem]

Theorem: $\operatorname{Ker} N_k = \bigoplus_{\lambda : \mathbb{T} \to \overline{\mathbb{Q}}_p} (\operatorname{Ker} N_k)\widetilde{[\lambda]}$ generalized eigenspace

associated to eigenform of weight $k+1$

or λ appears in $\varinjlim_{K_p} H^0(Y_{K^pK_p}, \mathbb{Q}_p)$

appear only when $k = 1$

Rmk: $\lambda \overset{1:1}{\longleftrightarrow} \pi = \otimes \pi_\ell$

then: $\operatorname{Ker} N_k \widetilde{[\lambda]} = \operatorname{Ker} N_k[\lambda]$ if π is cuspidal

& π_p is either [supercuspidal or principal series]

ie. π_p non Steinberg

(ie. no need of taking generalized eigenspace)

__Recall__

$$\mathcal{X}_{K^p} \xrightarrow{\pi_{HT}} \mathbb{P}^1 = \mathcal{F}\ell$$

$$GL_2(\mathbb{Q}_p) \curvearrowright \mathcal{O} := \pi_{HT\,*}\, \mathcal{O}_{\mathcal{X}_{K^p}}$$

$$\mathcal{O}^{la} \subseteq \mathcal{O}$$

$GL_2(\mathbb{Q}_p)$ locally analytic sections

$$\theta_{\mathfrak{h}} \quad \circlearrowleft \quad \mathfrak{h}$$

encodes the action of Z & θ_{Sen}

Cartan subalgebra

$$\mathfrak{h} = \begin{pmatrix} * & \\ & * \end{pmatrix} \subseteq \begin{pmatrix} * & * \\ & * \end{pmatrix}$$

$$\downarrow$$

$$\mathfrak{h} \otimes \mathcal{O}_{\mathcal{F}\ell} \cong \mathfrak{b}^0/\mathfrak{n}^0 \curvearrowright \mathcal{O}^{la}$$

Get $\mathcal{O}^{la,\tilde{\chi}_k} = \mathcal{O}^{la,(1-k,0)} \oplus \mathcal{O}^{la,(1,-k)}$

HT wt : 0   HT wt : k

$$W_0 = H^1(\mathscr{Fl}, \mathcal{O}^{la,(1-k,0)}) \qquad (k \geq 2)$$

$$W_k = H^1(\mathscr{Fl}, \mathcal{O}^{la,(1,-k)})$$

<u>Construct</u>: $I_k : \mathcal{O}^{la,(1-k,0)} \longrightarrow \mathcal{O}^{la,(1,-k)}$ s.t. $H^1(I_k) = N_k$

"Complete now the picture":

$$\begin{tikzcd} & \mathcal{X}_{K^p} \arrow[dl, "\pi_{K_p}"'] \arrow[dr, "\pi_{HT}"] & \\ \mathcal{X}_{K^pK_p} & & \mathscr{Fl} \end{tikzcd}$$

$\mathcal{O}^{sm} \subseteq \mathcal{O}$ $GL_2(\mathbb{Q}_p)$-smooth sections

$$\mathcal{O}^{sm} = \varinjlim_{K_p} \pi_{HT*} \pi_{K_p}^{-1} \mathcal{O}_{\mathcal{X}_{K^pK_p}}$$

& <u>similarly</u>:

cusp

$$\Omega^{1,sm} := \varinjlim_{K_p} \pi_{HT*} \pi_{K_p}^{-1} \Omega^1_{\mathcal{X}_{K^pK_p}}(C)$$

Get $d : \mathcal{O}^{sm} \to \Omega^{1,sm}$

(Can repeat the "symmetric" situation & consider): $d_{\mathcal{J}\ell} : \mathcal{O}_{\mathcal{J}\ell} \to \Omega^1_{\mathcal{J}\ell}$

<u>Fact</u> $\mathcal{O}^{sm}, \mathcal{O}_{\mathcal{J}\ell} \subseteq \mathcal{O}^{la,(0,0)}$

same kind of completed tensor product] : one would like to extend $d \otimes 1 : \mathcal{O}^{sm} \otimes_{\mathbb{C}_p} \mathcal{O}_{\mathcal{J}\ell} \to \Omega^{1,sm} \otimes_{\mathbb{C}_p} \mathcal{O}_{\mathcal{J}\ell}$

$\leadsto$ $\mathcal{O}^{sm} \otimes_{\mathbb{C}_p} \mathcal{O}_{\mathcal{J}\ell} \longrightarrow \mathcal{O}^{la,(0,0)}$

<u>dense image</u>

<u>Theorem</u>: (1) $\exists !$ $\mathcal{C}^0$ map $d^1 : \mathcal{O}^{la,(0,0)} \longrightarrow \mathcal{O}^{la,(0,0)} \otimes_{\mathcal{O}^{sm}} \Omega^{1,sm}$ s.t.

(1.1) d^1 is $\mathcal{O}_{\mathcal{J}\ell}$-linear

(1.2) extends $d : \mathcal{O}^{sm} \to \Omega^{1,sm}$

"symmetric" situations

(2) $\exists !$ $\mathcal{C}^0$ map $\bar{d}^1 : \mathcal{O}^{la,(0,0)} \longrightarrow \mathcal{O}^{la,(0,0)} \otimes_{\mathcal{O}_{\mathcal{J}\ell}} \Omega^1_{\mathcal{J}\ell}$ s.t.

(2.1) $\bar{d}_1$ is $\mathcal{O}^{sm}$-linear

(2.2) extends $d_{\mathcal{J}\ell} : \mathcal{O}_{\mathcal{J}\ell} \to \Omega^1_{\mathcal{J}\ell}$

Bernstein-Belinson: d_{fl} comes from some Lie algebra action: $d_{fl} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\in \mathfrak{gl}_2(\mathbb{Q}_p)} dX$

if $X :=$ coordinate on $\mathbb{A}^1 \subseteq \mathbb{P}^1$

hence $\overline{d}^1 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} dX$

$\mathcal{E}$
$\downarrow$
Y

$\mathbb{D} = \mathcal{H}^1_{dR}(\mathcal{E}/Y)$ with:

(Fil, ∇) ⇝ $(\mathcal{O}^{la,(0,0)} \otimes \mathrm{Sym}^{k-1} \mathbb{D}) \longrightarrow \mathcal{O}^{la,(0,0)} \otimes \mathrm{Sym}^{k-1} \mathbb{D} \otimes \Omega^{1,sm}$

Kodaira-Spencer iso

$$\underbrace{\mathcal{O}^{la,(0,0)} \otimes \mathrm{Sym}^{k-1} \mathbb{D}}_{\tilde{\lambda} \in} \xrightarrow{d \otimes 1 + 1 \otimes \nabla} \mathcal{O}^{la,(0,0)} \otimes \mathrm{Sym}^{k-1} \otimes \Omega^{1,sm}$$

$\lambda \in \mathrm{gr}^0 \qquad \nabla(\tilde{\lambda}) \in \mathrm{Fil}^{k-1} \otimes \Omega^{1,sm}$

$\nabla : \mathrm{Fil}^1 \xrightarrow{\sim} (\ \ /\mathrm{Fil}^{k-1}) \otimes \Omega^{1,sm}$

Rmk BGG:

$d^k : \mathcal{O}^{la,(1-k,0)} \longrightarrow \mathcal{O}^{la,(1-k,0)} \otimes_{\mathcal{O}^{sm}} (\Omega^{1,sm})^{\otimes k}$

$\overline{d}^k : \mathcal{O}^{la,(1-k,0)} \longrightarrow \mathcal{O}^{la,(1-k,0)} \otimes_{\mathcal{O}_{fl}} (\Omega^1_{fl})^{\otimes k}$

⇝ $I_k := \overline{d}^k \circ d^k$

Consider the case $k=1$.

$$\mathcal{O}^{la,(0,0)} \xrightarrow{d^1} \mathcal{O}^{la,(0,0)} \otimes_{\mathcal{O}^{sm}} \Omega^{1,sm}$$

$$\downarrow \overline{d}^1 \otimes 1$$

$$\Omega^1_{fl} \otimes_{\mathcal{O}_{fl}} \mathcal{O}^{la,(0,0)} \otimes_{\mathcal{O}^{sm}} \Omega^{1,sm}$$

$$\cong (*)$$

$$\mathcal{O}^{la,(1,-1)}(1)$$

define the composite arrow as I_1

(*) Kodaira-Spencer isomorphism

(1) $\omega^{\otimes 2} \cong \Omega^1_{\mathcal{X}_{K^pK_p}}(C)$

(2) $\omega_{fl}^{\otimes(-2)} \cong \Omega^1_{fl}$

(3) $\pi_{HT}^* \omega_{fl} \cong \pi_{K_p}^* \omega(-1)$

Sen operator: $\theta_h \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \theta_{Sen}$

comes from the θ_h-action

$$\Rightarrow \Omega^1_{fl} \times \Omega^{1,Sen} \xrightarrow{\theta_h} \mathcal{O}^{la,(1,-1)}(1)$$

get a perfect pairing

<u>Rmk</u>: over $\mathbb{C}$:

$$\begin{tikzcd} & SL_2/SO_2 \arrow[dl] \arrow[dr] & \\ SL_2/B & & SL_2/\overline{B} \end{tikzcd}$$

anti-holomorphic Borel

analog of

$$\begin{tikzcd} & \mathcal{X}_{K^p} \arrow[dl, "\pi_{K_p}"'] \arrow[dr, "\pi_{HT}"] & \\ {} & & {} \end{tikzcd}$$

<u>Theorem</u> $\quad H^1(I_k) = c_k \cdot N_k \quad \exists\, c_k \in \mathbb{Q}^\times$

<u>Pf</u>: Uses $\mathbb{OB}_{dR}^+$ via Poincaré's lemma.

$\boxed{\underline{\text{Study}}: \operatorname{Ker} H^1(I_k)} = \operatorname{Ker}\left(H^1(\bar{d}^k) \circ H^1(d^k) \right) \qquad (\overline{\mathrm{DR}})$

$k=1$ case

$\boxed{\underline{\text{Lemma}}: H^1(\bar{d}^1) \text{ is injective}}$

In fact:

$$\underbrace{0 \to \mathcal{O}^{sm} \to \mathcal{O}^{la,(0,0)} \xrightarrow{\bar{d}^1} \mathcal{O}^{la,(0,0)} \otimes \Omega^1_{\mathcal{fl}} \to 0}$$

"$\bar{d}^1$-resolution of $\mathcal{O}^{sm}$"

"deRham C-lex" using the derivation on $\mathcal{fl}$

$$\Rightarrow H^1(\mathcal{fl}, \Omega^{1,sm}) \twoheadrightarrow \operatorname{Ker} H^1(\bar{d}^1)$$

$$\| $$

$$\varinjlim_{K_p} \underbrace{H^1(X_{K^pK_p}, \Omega^1(C))}_{=0}$$

Why we have such s.e.s.?

- $\bar{d} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} dx$, $\operatorname{Ker} \bar{d}$ is $GL_2(\mathbb{Q}_p)$-stable $\Rightarrow$ $\operatorname{Ker} \bar{d} = GL_2(\mathbb{Q}_p)$-smooth vectors $= \mathcal{O}^{sm}$
- $(\overline{\mathrm{DR}})_y$: the deRham C-lex of $\pi_{K_p}^{-1}(y) \cong K_p$ profinite set $=$ 0-dim^{l} $\Rightarrow$ <u>NO</u> H^1

By injectivity $\operatorname{Ker}(H^1(I_1)) = \operatorname{Ker}(H^1(d^1)) = \boxed{H^1(DR)}\big/ \mathrm{Fil}^1$ $\longrightarrow$ it suffices to study this!

$$H^0(\mathcal{O}^{la,(0,0)} \otimes \Omega^{1,sm}) \cong H^0(\Omega^{1,sm}) = M_2$$

DR: filtration
$$\left[\begin{array}{ccc} \mathcal{O}^{la,(0,0)} & \xrightarrow{d} & \mathcal{O}^{la,(0,0)} \otimes_{\mathcal{O}^{sm}} \Omega^{1,sm} \\ & & \cup \\ 0 & \longrightarrow & \mathcal{O}^{la,(0,0)} \otimes \Omega^{1,sm} \end{array}\right]$$

$y \in \mathbb{P}^1 = \mathbb{P}^1(\mathbb{Q}_p) \sqcup \Omega$

1. On $\pi_{HT}^{-1}(\Omega) \cong D^\times \backslash \mathcal{M}^{(0)}_{LT,\infty} \times (D \otimes \mathbb{A}_f^p) / K^p \cong \bigsqcup \mathcal{M}^{(0)}_{LT,\infty}$ have: $\mathcal{M}^{(0)}_{LT,\infty} \cong \mathcal{M}_{Dr,\infty}$

$$\begin{array}{ccc} & \mathcal{M}^{(0)}_{LT,\infty} & \\ {}^{GL_2(\mathbb{Q}_p)^0}\swarrow & & \searrow \\ \mathcal{H} & & \Omega \end{array}$$

$\mathcal{O}_D^\times$-covering

2. $\left[\begin{array}{l} \text{identify the Cartan actions} \\ \theta_{\mathfrak{h},LT} = \theta_{\mathfrak{h},Dr} \end{array}\right]$ $\longrightarrow$ $d_{LT} = d_{Dr}$ Hence $\mathcal{O}^{GL_2(\mathbb{Q}_p)-la} = \mathcal{O}^{D^\times-la}$ $\quad d_{LT} = d_{Dr}$

hence $d^1|_{\Omega}$ is surjective as $\operatorname{Ker} d^1$ is $GL_2(\mathbb{Q}_p)$-stable

$\leadsto \operatorname{Ker} d|_{\Omega} = \mathcal{O}_{D_p}^{\times}$-smooth vectors $\leadsto DR|_{\Omega} = \operatorname{Ker} d|_{\Omega}[0] = \varinjlim_n \left(\mathcal{A}^{K^p} \otimes \pi_n{}_* \mathcal{O}_{\mathcal{M}_n}\right)^{\mathcal{O}_{D_p}^{\times}}$

where $\mathcal{A} = \mathcal{C}^{sm}\left(D^{\times} \backslash D \otimes \mathbb{A}_f, \mathbb{C}_p\right)$

$= \bigoplus DR|_{\Omega}[\lambda]$ where $\lambda \leftrightarrow$ autom. rep^n of $(D\otimes\mathbb{A})^{\times}$

Moreover $DR|_{\mathbb{P}^1(\mathbb{Q}_p)}$

$$DR|_{\infty} = \mathcal{O}^{la,(0,0)}|_{\infty} \xrightarrow{d} \mathcal{O}^{la,(0,0)} \otimes \Omega^{1,sm}|_{\infty}$$

$$\begin{array}{ccc} \cong & & \\ \underbrace{M_0^{t} \hat{\otimes} \mathcal{O}_{\mathcal{F}\ell,\infty}}_{\text{"deperfection"}} & \xrightarrow{d\otimes 1} & M_2^{\dagger} \hat{\otimes} \mathcal{O}_{\mathcal{F}\ell,\infty} \end{array}$$

$\Rightarrow H^i(DR)_{\infty} = H^i_{rig}(I_g) \otimes \mathcal{O}_{\mathcal{F}\ell,\infty}$

$= \bigoplus H^1(DR)_{\infty}[\lambda]$