Local Langlands correspondence for covering groups and geometrization

§ Motivation

p: prime number, E: p-adic loc. field. with res. field $\mathbb{F}_q$.

G: conn. reductive alg. gr $/E$

$WD_E = \mathbb{C} \rtimes W_E$: Weil-Deligne group of E. (An arithmetic Frob. element acts by q^{-1} on $\mathbb{C}$)

${}^L G = \hat{G} \rtimes W_E$: L-grp of G.

LLC conjecture

$\mathrm{Irr}(G(E)) = \{\text{irr. sm. rep of } G(E) \text{ over } \mathbb{C}\}/\sim \longrightarrow \Phi(G) = \{\text{L-parameters } WD_E \to {}^L G\}/\sim$

satisfying nice properties

Fargues-Scholze constructed a semi-simplified version of LLC using idea from geometric Langlands corr.

LLC is expected also for covering groups of conn. reductive groups.

Metaplectic group case (Gan-Savin)

$Sp_{2n}(E)$ has a non-linear central extension

$$1 \to \{\pm 1\} \longrightarrow Mp_{2n}(E) \to Sp_{2n}(E) \to 1 \qquad (\text{unique up to isom}).$$

An irr. sm rep of $Mp_{2n}(E)$ is called genuine if its restriction to $\{\pm 1\}$ is non-trivial.

$\exists$ construction of ${}^L Mp_{2n}$ s.t.

$$1 \to Sp_{2n}(\mathbb{C}) \longrightarrow {}^L Mp_{2n} \longrightarrow W_E \to 1$$

LLC for Mp_{2n}

$\mathrm{Irr}_{\mathrm{gen}}(Mp_{2n}(E)) = \{\text{genuine irr. sm. rep of } Mp_{2n}(E) \text{ over } \mathbb{C}\}/\sim \longrightarrow \Phi(Mp_{2n})$

This is constructed as follows:

Fix a non-trivial char $\psi : E \to \mathbb{C}^\times$.

The theta-corr gives

$$\Theta_\psi : \mathrm{Irr}_{\mathrm{gen}}(\mathrm{Mp}_{2n}(E)) \xrightarrow{\text{bij}} \mathrm{Irr}(SO(V_{2n+1}^+)) \amalg \mathrm{Irr}(SO(V_{2n+1}^-))$$

where V_{2n+1}^+ (resp. V_{2n+1}^-) is the split (resp. non-split) quadratic space of discriminant 1 and dimension $2n+1$ over E.

LLC for SO gives

$$\mathrm{Irr}(SO(V_{2n+1}^\pm)) \longrightarrow \{ L\text{-parameters} : WD_E \longrightarrow Sp_{2n}(\mathbb{C}) \times W_E \}$$

Further ψ determines an isom $Sp_{2n}(\mathbb{C}) \times W_E \cong {}^L\mathrm{Mp}_{2n}$.

Combining these maps we have

$$\mathrm{Irr}_{\mathrm{gen}}(\mathrm{Mp}_{2n}(E)) \longrightarrow \Phi(\mathrm{Mp}_{2n}) \quad (\text{independent of } \psi)$$

Gaitsgory-Lysenko formulated geometric Langlands corr. for covering groups of conn. reductive groups

<u>Hope</u>

Construct a semi-simplified version of LLC for covering grps using ideas from geometric Langlands corr.

(j.w. in progress with I. Gaisin, T. Koshikawa, Y. Zhao)

§ Objects in p-adic story (Fargues, Scholze)

$\mathrm{Perf}_{\overline{\mathbb{F}}_q}$: category of perfectoid spaces over $\overline{\mathbb{F}}_q$ equipped with v-topology.

Def A diamond over $\overline{\mathbb{F}}_q$ is a sheaf on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ written as X/R where $X, R \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$,

$R \subset X \times X$: equivalence relation s.t. $R \rightrightarrows X$ are pro-etale.

$\breve{E}$: completion of the max. unram. ext. of E.

Thm (Scholze)

$\exists$ diamond functor $\Diamond : \{\text{adic spaces}/\breve{E}\} \longrightarrow \{\text{diamonds}/\overline{\mathbb{F}}_q\}$

s.t. $X_{et} \cong (X^{\Diamond})_{et}$.

We write $\mathrm{Spd}\,\breve{E}$ for $\mathrm{Spa}(\breve{E}, \mathcal{O}_{\breve{E}})^{\Diamond}$.

Rem If X is a perfectoid sp over $\breve{E}$, $X^{\Diamond} = X^{\flat}$: tilting of X.

Rem For $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$,

$$\{S \to \mathrm{Spd}\,\breve{E}\} \overset{1:1}{\longleftrightarrow} \{\text{perfectoid sp } \underset{(\text{untilt})}{X} \text{ over } \breve{E} \text{ with } X^{\flat} \cong S\}/\sim$$

For an affinoid perfectoid sp $S = \mathrm{Spa}(R, R^+)$ over $\overline{\mathbb{F}}_q$,

we put $Y_S = \mathrm{Spa}(W_{\mathcal{O}_E}(R^+), W_{\mathcal{O}_E}(R^+)) \setminus V(p[\varpi])$

$\left(W_{\mathcal{O}_E}(R^+) = \mathcal{O}_E \otimes_{W(\overline{\mathbb{F}}_q)} W(R^+), \quad \varpi : \text{top. nilp. unit of } R \right)$

$X_S = Y_S / \varphi^{\mathbb{Z}}$ where φ acts via q-th power on R^+.

This glues together to give X_S for $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$,

called Fargues-Fontaine curve.

X_S is an adic sp over $\breve{E}$.

$S \to \mathrm{Spd}\,\breve{E} \iff$ perfectoid sp $S^\#$ over $\breve{E}$ with $(S^\#)^\flat \cong S$.

$\iff$ "Cartier divisor of deg 1" on Y_S (This is $S^\# \hookrightarrow Y_S$)

If $S = \mathrm{Spa}(R, R^+)$, Cartier divisor defined by $I \subset W(R^+)$ corresponds to $\mathrm{Spa}\left(W(R^+)/I \left[\frac{1}{[\varpi]}\right], W(R^+)/I \right)$.

This implies $S \to \mathrm{Spd}\,\breve{E}/\varphi^{\mathbb{Z}} \iff$ Cartier divisor of deg 1 on X_S.

$$\left(\text{This is } \begin{tikzcd} S^\# \arrow[r, hook] \arrow[dr, hook] & X_S \\ & Y_S \arrow[u] \end{tikzcd}\right)$$

$\mathrm{Div}^1_X := \mathrm{Spd}\,\breve{E}/\varphi^{\mathbb{Z}}$

Rem $\{\overline{\mathbb{Q}}_\ell\text{-loc. system on } \mathrm{Div}^1_X\} \iff \{\text{fin. dim. conti. rep. of } W_E\}$

Rem

$$X_S^\diamond \cong S \times \mathrm{Spd}\,\breve{E}/(\varphi \times \mathrm{id})^{\mathbb{Z}} \xleftrightarrow{\text{equiv of etale site}} S \times \mathrm{Spd}\,\breve{E}/(\mathrm{id} \times \varphi)^{\mathbb{Z}} = S \times \mathrm{Div}^1_X$$

For $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$, we put

$\mathrm{Bun}_G(S) =$ groupoids of G-bundles on X_S.

This defines a stack Bun_G on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$.

Bun_G contains $[*/G(E)]$ as an open substack corresponding the trivial G-bundle.

Hence smooth $\overline{\mathbb{Q}}_\ell$-shf on Bun_G gives a sm. rep of $G(E)$ via the restriction to $[*/G(E)]$.

We put $\mathrm{Div}_X^d = (\mathrm{Div}_X^1)^d / \Sigma_d$ (Σ_d: symmetric gp).

For $f_i : S \to \mathrm{Div}_X^1$ $(1 \leq i \leq d)$,
we have corresponding Cartier divisors $S_i^\# \hookrightarrow X_S$.

Those give Cartier divisors $S_i^\# \hookrightarrow X_S$

We ~~put~~ define $\mathrm{Gr}_{G, \mathrm{div}_X^d}$ by the sheafification of

$$S \longmapsto \left\{ \begin{array}{l} G\text{-bundles } \mathcal{G} \text{ on } X_S, \\ f_i : S \to \mathrm{Div}_X^1 \ (1 \leq i \leq d) \text{ up to permutation} \\ \text{trivialization of } \mathcal{G}|_{X_S \setminus \bigcup_{i=1}^d S_i^\#} \end{array} \right\}$$

We put $\mathrm{Gr}_G := \coprod_{d \geq 0} \mathrm{Gr}_{G, \mathrm{Div}_X^d}$

We have a natural morphism $\mathrm{Gr}_G \longrightarrow \mathrm{Bun}_G$.

This is a surjective morphism of stacks.

Gaitsgory–Lysenko's picture

X: proper sm. alg. curve over an alg. clos. field k

G: conn. reductive alg. gp over k.

$$\begin{tikzcd}
\left\{\begin{array}{l}\text{Brylinski-Deligne data}\\ \text{(extensions of } G \text{ by } K_2)\end{array}\right\} \arrow[r] \arrow[d] & \left\{\begin{array}{l}\text{etale parametrizations}\\ \text{using classifying spaces}\end{array}\right\} \arrow[d] & \\
\left\{\begin{array}{l}\text{factorization}\\ \text{line bundles on } Gr_G\end{array}\right\} \arrow[r] & \left\{\begin{array}{l}\text{factorization}\\ \text{gerbes on } Gr_G\end{array}\right\} \arrow[r] \arrow[dr, "\text{descent through } Gr_G \to Bun_G"'] & \left\{\begin{array}{l}\text{quadratic forms}\\ \text{on cocharacter lattices}\end{array}\right\} \\
 & & \{\text{gerbes on } Bun_G\}
\end{tikzcd}$$

study this using ω_X

	Geometric case	p-adic case
π_1 of	X: proper sm. curve	Div_X^1 (X: F-F curve)
canonical bundle	ω_X	?
structure map	$X_S := X \times S \to S$	no map $X_S \dashrightarrow S$ but $X_S \supseteq X_S^\diamond \xleftrightarrow{\text{equiv of etale site}} S \times \mathrm{Div}_X^1 \to S$

Motto Try to work with etale objects (rather than coherent objects)

p-adic picture

$$\begin{tikzcd}
 & \{\text{etale parametrization}\} \arrow[r] \arrow[d] & \{\text{quadratic forms}\} \\
\{\text{gerbes on } Gr_G\} & \{\text{gerbes on } Bun_G\} \arrow[l, "\text{via pull-back}"] &
\end{tikzcd}$$

§ Higher algebra (Lurie)

Set $\subset$ Groupoid $\subset \cdots \subset$ Spc : ∞-cat. of spaces

Spc_* : ∞-cat. of pointed spaces

$\Omega : \mathrm{Spc}_* \to \mathrm{Grp}(\mathrm{Spc}) : (* \to S) \mapsto * \times_S *$

For $i \geq 0$ and $(* \to S) \in \mathrm{Spc}_*$, we put

$\pi_i(* \to S) := \pi_0(\Omega^i(* \to S)) \in \mathrm{Set}$ ($\mathrm{Grp}(\mathrm{Set})$ if $i \geq 1$)

($\pi_0 : \mathrm{Spc} \to \mathrm{Sets}$: taking the set of connected components)

e.g.

If S is a groupoid $\{*_G\}$ with $\mathrm{Aut}(*_G) = G$ (G: grp).

then $\Omega(* \to S) = G \in \mathrm{Grp}(\mathrm{Set})$, $\Omega(* \to G) = *$.

Hence $\pi_0(* \to S) = *$, $\pi_1(* \to S) = G$, $\pi_i(* \to S) = *$ $(i \geq 2)$.

$\exists$ the left adjoint B of $\Omega : \mathrm{Spc}_* \to \mathrm{Grp}(\mathrm{Spc})$, called the functor of classifying space

$\exists$ concept of $\mathbb{E}_n$-structure $(n \geq 0)$

$$\begin{array}{ccccccc} \mathrm{Spc} \leftarrow & \mathbb{E}_0(\mathrm{Spc}) & \leftarrow & \mathbb{E}_1(\mathrm{Spc}) & \leftarrow \cdots & \leftarrow & \mathbb{E}_\infty(\mathrm{Spc}) \\ & \| & & \| & & & \| \\ & \mathrm{Spc}_* & & \mathrm{Monoid}(\mathrm{Spc}) & & & \mathrm{CommMonoid}(\mathrm{Spc}) \\ & & & (x \cdot y)\cdot z \simeq x \cdot (y \cdot z) & & & x \cdot y \simeq y \cdot x \end{array}$$

For $n \geq 1$, $\mathbb{E}_n^{grp}(\mathrm{Spc}) \subset \mathbb{E}_n(\mathrm{Spc})$ is the full subcat. of group like objects

def'd as the preimage of $\mathrm{Grp}(\mathrm{Spc}) \subset \mathbb{E}_1(\mathrm{Spc})$ under $\mathbb{E}_n(\mathrm{Spc}) \to \mathbb{E}_1(\mathrm{Spc})$.

B and Ω induce $\mathbb{E}_n^{grp}(\mathrm{Spc}) \underset{\Omega}{\overset{B}{\rightleftarrows}} \mathbb{E}_{n-1}^{grp}(\mathrm{Spc})$ for $n \geq 2$.

Def A Picard groupoid is a symmetric monoidal groupoid $\mathcal{A}$

$$\left(\exists \otimes,\ \exists \sigma_{a,b,c} : (a\otimes b)\otimes c \xrightarrow{\sim} a\otimes(b\otimes c),\ \exists \tau_{a,b} : a\otimes b \xrightarrow{\sim} b\otimes a \right)$$

s.t. $-\otimes a : \mathcal{A} \to \mathcal{A}$ is equivalence for $\forall a \in \mathcal{A}$.

It is called strictly commutative if $\tau_{a,a} = \mathrm{id}$ for $\forall a \in \mathcal{A}$.

Thm (SGA4, XVIII, §1.4)

$\exists$ equivalence of cat

$$D^{[-1,0]}(\mathbb{Z}) = \left\{ \begin{array}{l} \text{derived cat of cpx } K \\ \text{s.t. } H^i(K)=0 \ (i \notin [-1,0]) \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{small, strictly commutative} \\ \text{Picard groupoid} \end{array} \right\}$$

$$(K^{-1} \xrightarrow{d} K^0) \longmapsto \mathcal{A} \text{ s.t. } \left\{ \begin{array}{l} \mathrm{ob}(\mathcal{A}) = K^0 \quad \otimes := + \\ \mathrm{Mor}(a,b) = \{ m \in K^{-1} \mid dm = b-a \} \\ \sigma_{a,b,c} = \mathrm{id},\ \tau_{a,b} = \mathrm{id} \end{array} \right.$$

$$\begin{tikzcd}
Ch^{[0]}(\mathbb{Z}) \arrow[r, phantom, "\subset"] \arrow[ddd] & Ch^{[-1,0]}(\mathbb{Z}) \arrow[r, phantom, "\subset"] \arrow[d] & \cdots \arrow[r, phantom, "\subset"] & Ch^{\leq 0}(\mathbb{Z}) \arrow[d] & \\
& D^{[-1,0]}(\mathbb{Z}) \arrow[d, phantom, "\wr\wr"] & & D^{\leq 0}(\mathbb{Z}) \arrow[dr] & \\
& \left\{ \begin{array}{c} \text{sm. str. comm} \\ \text{Picard groupoid} \end{array} \right\} \arrow[d] & & & \mathbb{E}_\infty^{grp}(Spc) \arrow[d] \\
Set \arrow[r, phantom, "\subset"] & Groupoid \arrow[r, phantom, "\subset"] & \cdots \arrow[r, phantom, "\subset"] & Spc & Spc_* \arrow[l]
\end{tikzcd}$$

If $Ch^{\leq 0}(\mathbb{Z}) \to Spc_*$, then $H^{-n}(K^\bullet) \cong \pi_n(* \to S)$ for $n \geq 0$.

$K^\bullet \mapsto (* \to S)$

For $n \geq 0$, we have

$$\begin{tikzcd}
D^{\leq 0}(\mathbb{Z}) \arrow[r, "{[n]}"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & D^{\leq 0}(\mathbb{Z}) \arrow[d] \\
\mathbb{E}_\infty^{grp}(Spc) \arrow[r, "B^n"] & \mathbb{E}_\infty^{grp}(Spc)
\end{tikzcd}$$

$\mathcal{C}$: site.

$Shv(\mathcal{C})$: ∞-cat of sheaves of spaces on $\mathcal{C}$.

We can consider $*$ (pointed version), B, Ω, π_n, $\mathbb{E}_n$, $\cdots$ also for $Shv(\mathcal{C})$.

Prop A : a shf of ab. grp. on $\mathcal{C}$. $c \in \mathcal{C}$.

We have $\Gamma(c, B^n(A)) = \tau^{\leq 0} R\Gamma(c, A[n]) \in \mathbb{E}_\infty^{grp}(Spc)$,

$\pi_k(\Gamma(c, B^n(A))) \cong H^{n-k}(c, A)$ for $0 \leq k \leq n$.

$\left(B^n : \mathbb{E}_\infty^{grp}(Shv(\mathcal{C})) \to \mathbb{E}_\infty^{grp}(Shv(\mathcal{C})) \right)$

$Shv_*(\mathcal{C})_{\geq n} \subset Shv_*(\mathcal{C})$: the full subcat of $\mathcal{F} \in Shv_*(\mathcal{C})$ s.t. $\pi_k(\mathcal{F}) = *$ for $0 \leq k < n$.

<u>Prop</u> B^n induces the equivalence of ∞-cat $\mathbb{E}_n^{grp}(Shv(\mathcal{C})) \xrightarrow{\sim} Shv_*(\mathcal{C})_{\geq n}$.

§ <u>Etale parametrization</u> (Deligne, Gaitsgory-Lysenko, Zhao)

E : p-adic loc. field. $N \geq 1$. Assume $|\mu_N(E)| = N$

G : conn. reductive alg gp $/E$

Let $\mu : BG \to B^4(\mu_N^{\otimes 2})$ be a morphism of pointed sheaves on $(\text{Affine sch}/E)_{et}$
(etale parameter)

<u>Covering group</u>

By the equivalence $B : \mathbb{E}_1^{grp}(Shv(\mathcal{C})) \xrightarrow{\sim} Shv_*(\mathcal{C})_{\geq 1}$

μ gives an $\mathbb{E}_1$-morphism $G \to B^3(\mu_N^{\otimes 2})$.

So we have $G(E) \to \Gamma(E, B^3(\mu_N^{\otimes 2}))$

$\pi_0(\Gamma(E, B^3(\mu_N^{\otimes 2}))) \cong H^3(E, \mu_N^{\otimes 2}) = 0$ ($\because$ coh. dim of $E = 2$)

Hence $\Gamma(E, B^3(\mu_N^{\otimes 2})) \cong B(\Gamma(E, B^2(\mu_N^{\otimes 2})))$.

So we have an $\mathbb{E}_1$-morphism

$$G(E) \to B(\Gamma(E, B^2(\mu_N^{\otimes 2}))) \to B(H^2(E, \mu_N^{\otimes 2})) \underset{\text{Tate duality isom.}}{\cong} B(\mu_N(E))$$

This gives $1 \to \mu_N(E) \to \tilde{G}_\mu \to G(E) \to 1$.

$$\begin{tikzcd}
 & (a, a^{-1}) \arrow[d, phantom, "\cap"] & (ag, a^{-1}) \arrow[d, phantom, "\cap"] & & \\
1 \arrow[r] & \mu_n \times \mu_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \tilde{G}_\mu \times \tilde{G}_\mu \arrow[r] \arrow[d] & G \times G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mu_n \arrow[r] \arrow[d, equal] & \square \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & G \times G \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mu_n \arrow[r] & \tilde{G}_\mu \arrow[r] & G & \\
 & & g & &
\end{tikzcd}$$

Classification of etale parameters

Λ: fin. free $\mathbb{Z}$-mod.

$H^{(1)}(\Lambda) := \mathrm{Sym}^2(\Lambda) \oplus \Lambda$

with grp str $(s_1, a_1) + (s_2, a_2) = (s_1 + s_2 + a_1 \otimes a_2, a_1 + a_2)$ $\quad (s_i \in \mathrm{Sym}^2(\Lambda),\ a_i \in \Lambda)$

$H^{(2)}(\Lambda) = [\Lambda \otimes \Lambda \longrightarrow H^{(1)}(\Lambda)] \in D^{[-1,0]}(\mathbb{Z})$

given by $\Lambda \otimes \Lambda \twoheadrightarrow \mathrm{Sym}^2(\Lambda) \hookrightarrow H^{(1)}(\Lambda)$

$P^{(2)}(\Lambda)$: str. comm. Picard groupoid ass. to. $H^{(2)}(\Lambda)$

We view $\Lambda \times B(\overset{2}{\wedge} \Lambda)$ as str. comm Picard groupoid

by $\nabla_{\lambda_1, \lambda_2, \lambda_3} = \mathrm{id}$

$\tau_{\lambda_1, \lambda_2} := \lambda_1 \wedge \lambda_2 : \lambda_1 + \lambda_2 \cong \lambda_2 + \lambda_1$ $\quad (\lambda_1, \lambda_2, \lambda_3 \in \Lambda)$

Prop $P^{(2)}(\Lambda) \cong \Lambda \times B(\overset{2}{\wedge} \Lambda)$

(∵) Define $F : \Lambda \times B(\overset{2}{\wedge}\Lambda) \to P^{(2)}(\Lambda)$ by $\lambda \mapsto (0, \lambda)$, $\mathrm{Hom}(\lambda_1, \lambda_2) \xrightarrow{\mathrm{id}} \mathrm{Hom}(F(\lambda_1), F(\lambda_2))$

Give aditivity of F by $\lambda_1 \otimes \lambda_2$: $\underset{(0,\, \lambda_1 + \lambda_2)}{F(\lambda_1 + \lambda_2)} \xrightarrow{\sim} \underset{(\lambda_1 \otimes \lambda_2,\, \lambda_1 + \lambda_2)}{F(\lambda_1) + F(\lambda_2)}$.

Then we have

$$
\begin{tikzcd}
F(\lambda_1 + \lambda_2) \arrow[r, "\lambda_1 \otimes \lambda_2", "\sim"'] \arrow[d, "\lambda_1 \otimes \lambda_2 - \lambda_2 \otimes \lambda_1"'] \arrow[dr, phantom, "\circlearrowright"] & F(\lambda_1) + F(\lambda_2) \arrow[d, equal, "\tau_{F(\lambda_1), F(\lambda_2)} = \mathrm{id}"] \\
F(\lambda_2 + \lambda_1) \arrow[r, "\lambda_2 \otimes \lambda_1", "\sim"'] & F(\lambda_2) + F(\lambda_1)
\end{tikzcd}
$$

Hence F is an isom of Picard groupoids. $\square$

We have $(\overset{2}{\wedge} \Lambda)[1] \to H^{(2)}(\Lambda) \to \Lambda$ in $D^{[-1,0]}(\mathbb{Z})$.

In $\mathbb{E}_1^{grp}(\mathrm{Spc})$, we have $B(\overset{2}{\wedge}\Lambda) \to P^{(2)}(\Lambda) \rightleftarrows \Lambda$

$\mathbb{E}_1$-section, not $\mathbb{E}_\infty$-morphism.

$\mathrm{Quad}(\Lambda)$: the $\mathbb{Z}/N$-valued quad forms on Λ.

For $Q \in \mathrm{Quad}(\Lambda)$, we define $b_Q(\lambda_1, \lambda_2) := Q(\lambda_1 + \lambda_2) - Q(\lambda_1) - Q(\lambda_2)$.

η: μ_N-torsor over $\mathrm{Spec}\, E$ of N-th roots of (-1).

Since η^2 is trivial, η gives $\mathbb{Z}/2\mathbb{Z} \xrightarrow{\eta} \mu_N[1]$

$$\begin{tikzcd}
\Lambda \otimes \Lambda \arrow[r, "b_Q"] \arrow[d] & \mathbb{Z}/N\mathbb{Z} \arrow[d] & & \\
b_{Q,\eta} : \overset{2}{\wedge} \Lambda \arrow[r] & \mathbb{Z}/N\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \arrow[r, "\mathrm{id} \otimes \eta"] & \mathbb{Z}/N\mathbb{Z} \otimes_{\mathbb{Z}} \mu_N[1] \cong \mu_N[1]
\end{tikzcd}$$

<u>Def</u> Define $\theta^{(2)}(\Lambda)$ by the triples (Q, F, h)

s.t. (1) $Q \in \mathrm{Quad}(\Lambda)$

(2) $F : H^{(2)}(\Lambda) \longrightarrow \mu_N[2]$: morphism in $D^{\leq 0}(\mathbb{Z})$

(3) h : isom between $b_{Q,\eta} : \overset{2}{\wedge} \Lambda \longrightarrow \mu_N[1]$

and $[-1]$ of $\overset{2}{\wedge} \Lambda[1] \longrightarrow H^2(\Lambda) \xrightarrow{F} \mu_N[2]$

T : split torus over E.

$\Lambda_T := \mathrm{Hom}(T, \mathbb{G}_m)$, $\check{\Lambda}_T = \mathrm{Hom}(\mathbb{G}_m, T)$

<u>Prop</u> $\mathrm{Map}_*(BT, B^4 \mu_N^{\otimes 2}) \cong \theta^2(\check{\Lambda}_T)$

<u>Rem</u> $\mathbb{G}_m \longrightarrow \mathbb{G}_m : x \mapsto x^N$ gives $\Phi : \mathbb{G}_m \to B\mu_N$.

For $x \in \Lambda_T$, $x^*(B\Phi) \in \mathrm{Map}_*(BT, B^2\mu_N)$.

Any pointed morphism $BT \to B^4 \mu_N$ can be written as

$x_1^*(B\Phi) \cup x_2^*(B\Phi) + (B^2 Y_\tau)(x^*(B\Phi))$ for $x_1, x_2, x \in \Lambda_T$, $\tau \in \mu_N[2]$,

where $Y_\tau : \mathbb{Z}/N\mathbb{Z} \longrightarrow B^2\mu_N$ determined by τ.

Then the associated quad. form in $\mathrm{Sym}^2(\Lambda_T)/N$ is the image of $x_1 \otimes x_2$.

G = conn. reductive alg. gp $/E$

Assume that G is split.

Take $B \subset G$: Borel. T: the quot. of B by the unip. radical.

$(\Lambda_T, \Delta, \check{\Lambda}_T, \check{\Delta})$: based root datum of G for B.

$$\mathrm{Quad}(\check{\Lambda}_T)_{st} = \{ Q \in \mathrm{Quad}(\check{\Lambda}_T) \mid b_Q(\check{\alpha}, \lambda) = \langle \alpha, \lambda \rangle Q(\check{\alpha}) \text{ for } {}^{\forall}\alpha \in \Delta, {}^{\forall}\lambda \in \check{\Lambda}_T \}$$

(strict)

Define $\theta^2(\check{\Lambda}_T)_{st} \subset \theta^2(\check{\Lambda}_T)$ by the condition $Q \in \mathrm{Quad}(\check{\Lambda}_T)_{st}$.

$$\mathrm{Map}_*(BG, B^4\mu_N^{\otimes 2}) \to \mathrm{Map}_*(B(B), B^4\mu_N^{\otimes 2}) \cong \mathrm{Map}_*(BT, B^4\mu_N^{\otimes 2}) \cong \theta^2(\check{\Lambda}_T)$$

gives $R_B : \mathrm{Map}_*(BG, B^4\mu_N^{\otimes 2}) \to \theta^2(\check{\Lambda}_T)_{st}$.

G_{sc} : Simply connected cov. of the derived subgrp of G with B_{sc} and T_{sc} induced by B.

<u>Thm</u> (Zhao)

We have

$$\begin{tikzcd}
\mathrm{Map}_*(BG, B^4\mu_N^{\otimes 2}) \arrow[r, "R_B"] \arrow[d] \arrow[dr, phantom, "\square"] & \theta^2(\check{\Lambda}_T)_{st} \arrow[d] \\
\mathrm{Map}_*(BG^{sc}, B^4\mu_N^{\otimes 2}) \arrow[r, "R_{B^{sc}}"] \arrow[dr, "\sim"'] & \theta^2(\check{\Lambda}_{T^{sc}})_{st} \arrow[d] \\
& \mathrm{Quad}(\check{\Lambda}_{T^{sc}})_{st}
\end{tikzcd}$$

Therefore $\mathrm{Map}_*(BG, B^4\mu_N^{\otimes 2})$ is classified by (Q, F, h, φ)

where (1) $Q \in \mathrm{Quad}(\check{\Lambda}_T)_{st}$

(2) $F : H^{(2)}(\check{\Lambda}_T) \to \mu_N[2]$

(3) h: isom between two morphisms $\overset{2}{\wedge}\check{\Lambda}_T \to \mu_N[1]$ given by Q and F.

(4) φ: isom between the restriction $(F, h)|_{\check{\Lambda}_{T_{sc}}}$

and (F_{sc}, h_{sc}) determined by $Q|_{\check{\Lambda}_{T_{sc}}}$ and $R_{B_{sc}}$.

§ Construction of L-group

$\mu : BG \to B^4 \mu_N^{\otimes 2} \rightsquigarrow (Q, F, h, \varphi)$

$\check{\Lambda}_T^{\#} := \{ \lambda \in \check{\Lambda}_T \mid b_Q(\lambda, \lambda') = 0 \text{ for } \forall \lambda' \in \check{\Lambda}_T \}$

$\Lambda_T^{\#} \subset \Lambda_T \otimes \mathbb{Q}$: the dual of $\check{\Lambda}_T^{\#}$.

$\check{\Delta}^{\#} := \{ \mathrm{ord}(Q(\check{\alpha}))\, \check{\alpha} \mid \check{\alpha} \in \check{\Delta} \}$ (ord $(Q(\alpha))$ is the order of $Q(\alpha) \in \mathbb{Z}/N\mathbb{Z}$)

$\Delta^{\#} := \{ \mathrm{ord}(Q(\check{\alpha}))^{-1} \alpha \mid \alpha \in \Delta \}$.

Then $\Delta^{\#} \subset \Lambda_T^{\#}$ by $Q \in \mathrm{Quad}(\Lambda)_{st}$.

H : reductive grp over $\overline{\mathbb{Q}_\ell}$ ass. to $(\check{\Lambda}_T^{\#}, \check{\Delta}^{\#}, \Lambda_T^{\#}, \Delta^{\#})$.

e.g. $G = Sp_{2n}$. $N = 2$. $\check{\Delta} = \{ \check{\alpha}_1, \check{\alpha}_2, \cdots, \check{\alpha}_n \}$ ($\check{\alpha}_1$: short one)

The only non-trivial $Q \in \mathrm{Quad}(\check{\Lambda}_T)_{st}$ is given by

$Q(\check{\alpha}_1) = 1, \quad Q(\check{\alpha}_i) = 0 \quad (2 \le i \le n)$.

So $\check{\Delta}^{\#} = \{ 2\check{\alpha}_1, \check{\alpha}_2, \cdots, \check{\alpha}_n \}$: type C_n. $\quad H = Sp_{2n}$.

Consider the $\mathbb{E}_1$-morphism

$$F_1 : \check{\Lambda}_T \underset{\mathbb{E}_1\text{-section}}{\longrightarrow} \check{\Lambda}_T \times B(\overset{2}{\wedge} \check{\Lambda}_T) \cong P^{(2)}(\check{\Lambda}_T) \xrightarrow{F} B^2 \mu_N$$

The restriction $F_1|_{\check{\Lambda}_T^{\#}}$ naturaly has a str. of $\mathbb{E}_\infty$-morph

and gives $F_1^{\#} : \check{\Lambda}_T^{\#} \longrightarrow B^2 \mu_N$.

$\check{\Lambda}_T^{\#, r} = \mathbb{Z} \cdot \check{\Delta}^{\#} \subset \check{\Lambda}_T^{\#}$

Then Q_{sc} is trivial on $\check{\Lambda}_T^{\#, r}$. Hence (F_{sc}, h_{sc}) is trivial on $\check{\Lambda}_T^{\#, r}$.

By $\varphi : (F, h)|_{\check{\Lambda}_{T_{sc}}} \cong (F_{sc}, h_{sc})$, (F, h) is also triv. on $\check{\Lambda}_T^{\#, r}$.

This implies that $F_1^{\#}$ factors through $\overline{F_1^{\#}} : \check{\Lambda}_T^{\#}/\check{\Lambda}_T^{\#.r} \longrightarrow B^2\mu_N$.

We fix $\iota : \mu_N(E) \hookrightarrow \overline{\mathbb{Q}}_\ell^\times$.

Then we have an $\mathbb{E}_\infty$-morphism

$$F_\mu : \Lambda_{Z_H} \cong \check{\Lambda}_T^{\#}/\check{\Lambda}_T^{\#.r} \xrightarrow{\overline{F_1^{\#}}} B^2\mu_N \longrightarrow B^2\,\overline{\mathbb{Q}}_\ell^\times .$$

We have the exact seq

$$0 \to \mathrm{Hom}(\Lambda_{Z_H}, \overline{\mathbb{Q}}_\ell^\times[2]) \longrightarrow \mathrm{Map}_{\mathbb{E}_\infty}(\Lambda_{Z_H}, B^2\overline{\mathbb{Q}}_\ell^\times) \to \mathrm{Hom}(\Lambda_{Z_H}/2, \overline{\mathbb{Q}}_\ell^\times) \to 0$$

$$(x \mapsto \tau_{x,x})$$

$$f \in \mathrm{Map}_{\mathbb{E}_\infty}(\Lambda_{Z_H}, B^2\overline{\mathbb{Q}}_\ell^\times) \cong \left\{ \begin{array}{l} \text{étale stack of Picard groupoid } A \text{ over } \mathrm{Spec}\, E \\ \text{s.t. } \pi_0(A) \cong \Lambda_{Z_H},\ \pi_1(A) \cong \overline{\mathbb{Q}}_\ell^\times \end{array} \right\}$$

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & * \arrow[d] \\ \Lambda_{Z_H} \arrow[r, "f"] & B^2(\overline{\mathbb{Q}}_\ell^\times) \end{tikzcd}$$

A with comm. constraint $\tau_{x,y}$ for $x, y \in \Lambda_{Z_H}$

We have a splitting

$$\mathrm{Hom}(\Lambda_{Z_H}/2, \overline{\mathbb{Q}}_\ell^\times) \longrightarrow \mathrm{Map}_{\mathbb{E}_\infty}(\Lambda_{Z_H}, B^2(\overline{\mathbb{Q}}_\ell^\times))$$

by sending $x \mapsto (-1)^{\varepsilon(x)}$ $\quad (\varepsilon : \Lambda_{Z_H} \to \mathbb{Z}/2\mathbb{Z})$

to the stack of Picard groupoids $\Lambda_{Z_H} \times B(\overline{\mathbb{Q}}_\ell^\times)$

with the comm. constraint $\tau_{x,y} = (-1)^{\varepsilon(x)\varepsilon(y)}$ for $x, y \in \Lambda_{Z_H}$.

Therefore we can write F_μ as (F_2, F_{sgn})

where $F_2 \in \mathrm{Hom}(\Lambda_{Z_H}, \overline{\mathbb{Q}}_\ell^\times[2])$, $F_{\mathrm{sgn}} \in \mathrm{Hom}(\Lambda_{Z_H}/2, \overline{\mathbb{Q}}_\ell^\times)$.

$$\mathrm{Hom}(\Lambda_{Z_H}, \overline{\mathbb{Q}}_\ell^\times[2]) \cong Z_H(\overline{\mathbb{Q}}_\ell)[2]$$

F_2 gives a section of $B^2(Z_H(\overline{\mathbb{Q}}_\ell))(E)$

General facts

A: sheaf of ab. grps on E_{et}

$t \in B(A)(E)$ gives $\mathcal{T} = \operatorname{Spec} E \underset{\substack{\text{base pt} \\ \searrow \; B(A) \; \swarrow t}}{\times} \operatorname{Spec} E$ (A-torsor over E).

Then $^{\exists} E'/E$: etale $\mathcal{T}(E') \neq \phi$, $^{\forall} x \in \mathcal{T}(E')$, $\mathrm{Hom}(x, x) = id_x$

$^{\forall} x, y \in \mathcal{T}(E')$, $^{\exists!} a \in A(E')$, $\mathrm{Hom}(x, y) = \{a\}$

$g \in B^2(A)(E)$ gives $\mathcal{g} = \operatorname{Spec} E \underset{\substack{\text{base pt} \\ \searrow \; B^2(A) \; \swarrow g}}{\times} \operatorname{Spec} E$ (A-gerbe over E)

$^{\exists} E'/E$: etale $\mathcal{g}(E') \neq \phi$, $^{\forall} x \in \mathcal{g}(E')$. $\mathcal{H}om(x, x) \cong A_{E'}$

$^{\forall} x, y \in \mathcal{g}(E')$, $^{\exists!} t \in B(A)(E')$, $\mathcal{H}om(x, y) = t$

We apply this to $F_2 \in B^2(Z_H(\overline{\mathbb{Q}}_\ell))(E)$.

Take $x \in \mathcal{g}(\overline{E})$.

Then $Z_{H,2} := \coprod_{w \in W_E} \mathrm{Hom}(x, w^* x)$ has a grp str. by

$$\mathrm{Hom}(x, w_1^* x) \times \mathrm{Hom}(x, w_2^* x) \cong \mathrm{Hom}(w_2^* x, (w_1 w_2)^* x) \times \mathrm{Hom}(x, w_2^* x)$$

$$\to \mathrm{Hom}(x, (w_1 w_2)^* x).$$

This gives the extension

$$1 \to \underset{\substack{\rotatebox{90}{$\cong$} \\ Z_H(\overline{\mathbb{Q}}_\ell)}}{\mathrm{Hom}(x, x)} \to Z_{H,2} \to W_E \to 1$$

Define a grp str on $\{\pm 1\} \times E^\times$ by $(\varepsilon_1, a_1)(\varepsilon_2, a_2) = (\varepsilon_1 \varepsilon_2 (a_1, a_2), a_1 a_2)$

where (a_1, a_2) is the Hilbert symbol.

Then we have the exact seq. $1 \to \{\pm 1\} \to \{\pm 1\} \times E^\times \to E^\times \to 1$

Pulling back along $W_E \to W_E^{ab} \cong E^\times$

and pushing along $F_{sgn} \in \mathrm{Hom}(\Lambda Z_H/2, \overline{\mathbb{Q}}_\ell^\times) \cong \mathrm{Hom}(\{\pm1\}, Z_H(\overline{\mathbb{Q}}_\ell))$,

we obtain $1 \to Z_H(\overline{\mathbb{Q}}_\ell) \to Z_{H.sgn} \to W_E \to 1$.

Taking the Baer sum of $Z_{H.\mathcal{L}}$ and $Z_{H.sgn}$,

and pushing along $Z_H \hookrightarrow H$,

we obtain $1 \to H \to {}^L\widetilde{G}_\mu \to W_E \to 1$.

§ Geometric Satake

Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid over $\overline{\mathbb{F}}_p$.

For $S \to \mathrm{Div}_X^1$, we have a Cartier divisor $S^\# = \mathrm{Spa}(R^\#, R^{+\#}) \hookrightarrow X_S$

def'd by an ideal $\mathcal{I}_S = (\xi) \subset \mathcal{O}_{X_S}$.

We put

$B^+_{\mathrm{Div}_X^1}(S) :=$ the global section of $\mathcal{I}_S$-adic completion of $\mathcal{O}_{X_S}$.

$B_{\mathrm{Div}_X^1}(S) := B^+_{\mathrm{Div}_X^1}(S)[\frac{1}{\xi}]$

$L^+_{\mathrm{Div}_X^1} G(S) = G(B^+_{\mathrm{Div}_X^1}(S))$, $L_{\mathrm{Div}_X^1} G(S) = G(B_{\mathrm{Div}_X^1}(S))$.

We define $\mathrm{Hec}_{G.\mathrm{Div}_X^1}$ by $\left[L^+_{\mathrm{Div}_X^1} G \backslash L_{\mathrm{Div}_X^1} G / L^+_{\mathrm{Div}_X^1} G \right]$.

From $\mu : BG \to B^4 \mu_N^{\otimes 2}$, we can construct

μ_N-gerbe $\mathcal{G}_\mu$ on $\mathrm{Hec}_{G.\mathrm{Div}_X^1}$ as follows:

Let $S \to \mathrm{Div}^1_X$.

μ gives $G \to B^3(\mu_N^{\otimes 2})$.

Hence we have

$$L_{\mathrm{Div}^1_X}(G)(S) = \Gamma(\mathrm{Spec}(B_{\mathrm{Div}^1_X}(S)), G) \to \Gamma(\mathrm{Spec}(B_{\mathrm{Div}^1_X}(S)), \mu_N^{\otimes 2}[3])$$

$$\to \Gamma(\mathrm{Spec}\, R^\sharp, \hat{i}^! \mu_N^{\otimes 2}[4])$$

$$\begin{tikzcd}
\mathrm{Spec}\, R^\sharp \arrow[r, hook, "\hat{i}"] \arrow[d, equal] & \mathrm{Spec}\, B^+_{\mathrm{Div}^1_X}(S) \arrow[d] & \mathrm{Spec}\, B_{\mathrm{Div}^1_X}(S) \arrow[l, hook'] \arrow[d] \\
\mathrm{Spec}\, R^\sharp \arrow[r, hook, "i_{alg}"] & X_S^{alg} & X_S^{alg} \setminus \mathrm{Spec}\, R^\sharp \arrow[l, hook'] \\
S^\sharp \arrow[r, hook, "i"] \arrow[u] & X_S \arrow[u] & X_S \setminus S^\sharp \arrow[l, hook'] \arrow[u]
\end{tikzcd}$$

Further we have

$$\Gamma(\mathrm{Spec}\, R^\sharp, \hat{i}^! \mu_N^{\otimes 2}[4]) \cong \Gamma(\mathrm{Spec}\, R^\sharp, i_{alg}^! \mu_N^{\otimes 2}[4]) \to \Gamma(S^\sharp, i^! \mu_N^{\otimes 2}[4])$$

$$\to \Gamma(X_S, \mu_N^{\otimes 2}[4]) \cong \Gamma(X_S^\diamond, \mu_N^{\otimes}[4]) \cong \Gamma(S \times \mathrm{Div}^1_X, \mu_N^{\otimes}[4])$$

$$\to \Gamma(S, \mu_N[2]) \quad (\because \mathrm{Div}^1_X \to * \text{ is proper, coh. sm. of rel. dim } 1)$$

Hence we have $L_{\mathrm{Div}^1_X} G \to B^2 \mu_N$

This induces μ_N-gerbe $\mathcal{G}_\mu$ on $\mathrm{Hec}_{G, \mathrm{Div}^1_X}$.

We can define $\mathrm{Sat}_{G,\mu}$ as a twist of $\mathrm{Perv}^{ULA}(\mathrm{Hec}_{G,\mathrm{Div}^1_X})$ by $\mathcal{G}_\mu$.

<u>Expected geom Satake</u>

$\exists$ monoidal equiv : $\mathrm{Rep}_{{}^L G_\mu} \xrightarrow{\sim} \mathrm{Sat}_{G,\mu}$