Multiplicity theorems modulo p for $\mathbf{GL}_2(\mathbf{Q}_p)$

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Abstract

Let F be a non-archimedean local field, π an admissible irreducible $\mathbf{GL}_2(F)$-representation with complex coefficients. For a quadratic extension L/F and an $L^{\times}$-character χ a classical result of Tunnell and Saito establish a precise connection between the dimension of the Hom-space $\mathrm{Hom}_{L^{\times}}(\pi|_{L^{\times}}, \chi)$ and the normalized local factor of the pair (π, χ). The study of analogous Hom-spaces for complex valued representations has recently been generalized to $\mathbf{GL}_n$ in [AGRS] and their connections with local factors have been established by work of Waldspurger and Moeglin ([MW]).

In this paper we approach the analogous problem in the context of the p-modular Langlands correspondence for $\mathbf{GL}_2(\mathbf{Q}_p)$. We describe the restriction to Cartan subgroups of an irreducible p-modular representation π of $\mathbf{GL}_2(\mathbf{Q}_p)$ and deduce generalized multiplicity results on the dimension of the Ext-spaces $\mathrm{Ext}^i_{\mathscr{O}_L^{\times}}(\pi|_{\mathscr{O}_L^{\times}}, \chi)$ where $\mathscr{O}_L^{\times}$ is the ring of integers of a quadratic extension of $\mathbf{Q}_p$ and χ a smooth character of $\mathscr{O}_L^{\times}$.

Contents

1. Introduction, Notations and Preliminaries

Let F be a non-archimedean local field of characteristic 0, V a finite dimensional F-vector space endowed with a non degenerated quadratic form q. If E is a non isotropic line and W is the q-orthogonal of E in V we write G, H for the special orthogonal group of V and W respectively. Let π, ρ be irreducible admissible complex representations of $G(F)$, $H(F)$. The Gross-Prasad conjectures (cf. [GP1], [GP2]) predict a precise relation between $\dim(\mathrm{Hom}_{H(F)}(\pi|_{H(F)}, \rho))$ and the epsilon factor

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of the pair associated to (π, ρ). In this direction, Aizenbud, Gourevitch, Rallis and Schiffmann ([AGRS]) proved a "multiplicity one" result:

Theorem 1.1 ([AGRS], Theorem 1 and Theorem 1'). *In the previous hypothesis we have*

$$\dim(\mathrm{Hom}_{H(F)}(\pi|_{H(F)}, \rho)) \leqslant 1.$$

The multiplicity result of Theorem 1.1 recently allowed Waldspurger and Moeglin ([W1], [W2], [W3], [MW]) to prove the Gross-Prasad conjecture in a large number of cases.

Let us consider the situation for $\mathbf{GL}_2$. The Gross-Prasad conjecture is then a particular case of a result of Tunnell and Saito (cf. [Tun], [Sai]). More precisely, let π be an infinite dimensional irreducible admissible representation of $\mathbf{GL}_2(F)$ and σ_π the associated representation of the Weil group W_F (via the local Langlands correspondence). For a quadratic extension L/F we fix an $L^\times$-character χ which extends the central character ω_π of π (as usual, χ will be considered as a character of the Weil group W_L by local class field theory). Finally, we fix an embedding $L^\times \hookrightarrow \mathbf{GL}_2(F)$ and an additive character ψ of F, letting $\psi_L \stackrel{\mathrm{def}}{=} \psi \circ \mathrm{Tr}_{L/F}$.

The theorem is then the following (see also [Pra2], Theorem 1.1 and remark) :

Theorem 1.2 (Tunnell, Saito). *In the previous hypothesis, the conditions*

i) $\dim(\mathrm{Hom}_{L^\times}(\pi|_{L^\times}, \chi)) \neq 0$

ii) $\varepsilon(\sigma_\pi|_{W_L} \otimes \chi^{-1}, \psi_L)\omega_\pi(-1) = 1$

are equivalent.

Indeed, the problem of looking for multiplicities of $L^\times$-characters in $\pi|_{L^\times}$ goes back to a work of Silberger ([Sil]) and has then been approached in the work of Tunnell [Tun] and Prasad [Pra1]. In particular, the Tunnell-Saito theorem appears again in [Rag], where Raghuram gives explicit sufficient conditions for an $L^\times$-character to appear in $\pi|_{L^\times}$ for π supercuspidal.

In this paper we approach the p-modular analogue of such problems in the case $F = \mathbf{Q}_p$, giving a detailed description of the $L^\times$-structure for p-modular, absolutely irreducible and admissible representations of $\mathbf{GL}_2(\mathbf{Q}_p)$ and deducing certain mod-p multiplicity statements.

We rely on the works [Mo1], [Mo2], where we established the Iwahori and $\mathbf{GL}_2(\mathbf{Z}_p)$-structure for irreducible admissible $\mathbf{GL}_2(\mathbf{Q}_p)$-representations when $p \geqslant 3$. Hence, from now on we assume that p is an odd rational prime.

The results when π is supersingular (§3 and §4) are summed up in the following theorem. Recall that supersingular representations for $\mathbf{GL}_2(\mathbf{Q}_p)$ are parametrized, up to twist, by the universal representations $\pi(r, 0)$ where $r \in \{0, \dots, p-1\}$ (see §2 for the precise definition of the representations $\pi(r, 0)$).

Theorem 1.3 (Corollary 3.10 and Proposition 4.2). *Let $L/\mathbf{Q}_p$ be a quadratic extension, π be a supersingular representation and write ω_π for its central character.*

i) *Assume that $L/\mathbf{Q}_p$ is unramified and write $\{\eta_i\}_{i=0}^p$ for the $L^\times$-characters extending ω_π. There is an isomorphism*

$$\pi|_{L^\times} \cong \oplus_{i=0}^p \mathscr{F}_{\pi,0}(\eta_i) \bigoplus \oplus_{i=0}^p \mathscr{F}_{\pi,1}(\eta_i)$$

where each $\mathscr{F}_{\pi,\bullet}(\eta_i)$, for $\bullet \in \{0, 1\}$, is an infinite length uniserial representation of $L^\times$, with a scalar action of $p \in L^\times$ and Jordan-Hölder factors all isomorphic to η_i.

ii) *Assume that $L/\mathbf{Q}_p$ is totally ramified. Let $\mathscr{O}_L$ be its ring of integers and $\varpi \in L$ be an uniformizer. Then there is an $\mathscr{O}_L^\times$-equivariant exact sequence*

$$0 \to W^2 \to \left(\mathscr{U}_{\infty,0}^- \oplus \mathscr{U}_{\infty,1}^-\right)^2 \to \pi|_{\mathscr{O}_L^\times} \to 0$$

where, for $\bullet \in \{0,1\}$, the $\mathscr{O}_L^\times$-representations $\mathscr{U}_{\infty,\bullet}^-$ is uniserial, with Jordan-Hölder factors all isomorphic to $\omega_\pi|_{\mathbf{Z}_p^\times}$ and W is a 1-dimensional submodule of $\mathscr{U}_{\infty,0}^- \oplus \mathscr{U}_{\infty,1}^-$. Moreover, the ϖ-action on π is induced from the involution on $\mathscr{U}_{\infty,\bullet}^- \oplus \mathscr{U}_{\infty,\bullet}^-$ defined by $(x,y) \mapsto (y,x)$.

The techniques we used in the supersingular case can be adapted to deduce the $L^\times$-restriction of principal and special series (§5). The behavior is very similar:

THEOREM 1.4 (Corollary 5.4, Proposition 5.6). *Let $L/\mathbf{Q}_p$ be a quadratic extension, π be a principal or a special series and write ω_π for its central character.*

i) Assume that $L/\mathbf{Q}_p$ is unramified and write $\{\eta_i\}_{i=0}^p$ for the $L^\times$-characters extending ω_π. There is an isomorphism

$$\pi|_{L^\times} \cong \oplus_{i=0}^p \mathscr{F}_\pi(\eta_i)$$

where each $\mathscr{F}_\pi(\eta_i)$ is an infinite length uniserial representation of $L^\times$, with a scalar action of $p \in L^\times$ and Jordan-Hölder factors all isomorphic to η_i.

ii) Assume that $L/\mathbf{Q}_p$ is totally ramified. Let $\mathscr{O}_L$ be its ring of integers and $\varpi \in L$ be an uniformizer. If π is a principal series we have an $\mathscr{O}_L^\times$-equivariant isomorphism

$$\pi|_{\mathscr{O}_L^\times} \xrightarrow{\sim} \left(\mathscr{U}_\infty^-\right)^2 \tag{1}$$

where the $\mathscr{O}_L^\times$-representation $\mathscr{U}_\infty^-$ is uniserial, with Jordan-Hölder factors all isomorphic to $\omega_\pi|_{\mathbf{Z}_p^\times}$. Moreover the ϖ-action on π induces the natural involution $(x,y) \mapsto (y,x)$ on the RHS of (1). If π is a special series we have an $\mathscr{O}_L^\times$-equivariant exact sequence

$$0 \to W \to \left(\mathscr{U}_\infty^-\right)^2 \to \pi|_{\mathscr{O}_L^\times} \to 0$$

where, moreover, the $\mathscr{O}_L^\times$-representation W is 1-dimensional and stable under the natural involution on $\left(\mathscr{U}_\infty^-\right)^2$.

Although Theorem 1.3 and 1.4 give strong constraints on the $L^\times$-structure of an irreducible $\mathbf{GL}_2(\mathbf{Q}_p)$-representations π, we remark that it is *not* clear a priori if the $L^\times$-restriction of π depends *only* on its central character (at least, when π is supersingular).

The structure theorems 1.3 and 1.4 can be used to obtain generalized multiplicity statements. More precisely, we deduce that there can not be a naïve p-modular analogue of the Tunnell-Saito Theorem, simply because in our setting the Hom-spaces $\mathrm{Hom}_{\mathscr{O}_L^\times}(\pi,\chi)$ are *always* zero. This behavior has recently been established also in the p-adic setting by work of Dospinescu ([Dos], Théorème 6.3.1): if Π is a p-adic supersingular representation of $\mathbf{GL}_2(\mathbf{Q}_p)$ and δ is a p-adic continuous $L^\times$-character, then $\mathrm{Hom}_{L^\times}(\Pi^{\mathrm{an}},\delta) = 0$, where Π^{an} denotes the space of analytic vectors of Π.

Nevertheless, if we pursue the analysis to the *derived* bifunctors $\mathrm{Ext}^i_{\mathscr{O}_L^\times}(\circ,\circ)$ we find a new phenomenology: the Ext-spaces are now non-zero and, moreover, let us distinguish between the supersingular and non-supersingular case. The possibility of a connection with local constants, in the spirit of a Tunnell-Saito result, remains at present mysterious.

THEOREM 1.5 (Proposition 6.12 and 6.13). *Let $L/\mathbf{Q}_p$ be a quadratic extension, $\mathscr{O}_L$ its ring of integers and let e be the ramification degree. Let π be an infinite dimensional, admissible and absolutely irreducible $\mathbf{GL}_2(\mathbf{Q}_p)$-representation and let χ be a $L^\times$-character which extends the central character of π.*

The dimension of the Ext-spaces $\mathrm{Ext}^i_{\mathscr{O}_L^\times}(\pi|_{\mathscr{O}_L^\times},\chi|_{\mathscr{O}_L^\times})$ is given by:

$$\dim\big(\mathrm{Ext}^i_{\mathscr{O}_L^\times}(\pi|_{\mathscr{O}_L^\times},\chi|_{\mathscr{O}_L^\times})\big) = \begin{cases} 0 & \textit{if either } i=0 \textit{ or } i \geqslant 3 \\ 2e & \textit{if } i \in \{1,2\} \textit{ and } \pi \textit{ is supersingular.} \\ e & \textit{if } i \in \{1,2\} \textit{ and } \pi \textit{ is a principal or a special series.} \end{cases}$$

An immediate corollary is

Corollary 1.6 (Corollary 6.16). *Let $L/\mathbf{Q}_p$ be a quadratic extension and π be an admissible $\mathbf{GL}_2(\mathbf{Q}_p)$-representation of finite length, whose Jordan-Hölder factors are infinite dimensional and absolutely irreducible. Then*

$$\mathrm{Hom}_{L^\times}\big(\pi|_{L^\times},\chi\big)=0$$

for any $L^\times$-character χ.

As a byproduct of the proof of Theorem 1.5 (cf. §6.3.1) we detected the Ext-spaces in the "opposite order". More precisely, we have

Theorem 1.7 (Remark 6.17). *Let $L/\mathbf{Q}_p$ be a quadratic extension, $\mathscr{O}_L$ its ring of integers and let e be the ramification degree. Let π be an infinite dimensional, admissible and absolutely irreducible $\mathbf{GL}_2(\mathbf{Q}_p)$-representation and let χ be a $L^\times$-character which extends the central character of π.*

The dimension of the Ext-spaces $\mathrm{Ext}^i_{\mathscr{O}_L^\times}\big(\chi|_{\mathscr{O}_L^\times},\pi|_{\mathscr{O}_L^\times}\big)$ is given by:

$$\dim\big(\mathrm{Ext}^i_{\mathscr{O}_L^\times}\big(\chi|_{\mathscr{O}_L^\times},\pi|_{\mathscr{O}_L^\times}\big)\big)=\begin{cases}0 & \text{if } i\geqslant 2\\ 2e & \text{if } i\in\{0,1\} \text{ and } \pi \text{ is supersingular.}\\ e & \text{if } i\in\{0,1\} \text{ and } \pi \text{ is a principal or a special series.}\end{cases}$$

The relation between the Ext-spaces of Theorem 1.5 and Theorem 1.7 and the recent developments in the Gross-Prasad conjectures remains at present largely open ([Har]).

We give a more detailed account on the organization of the paper. After fixing the notations (§1.1), we introduce in §2 a self contained overview on supersingular representations of $\mathbf{GL}_2(\mathbf{Q}_p)$, recalling the structure theorems of [Mo2] (cf. Proposition 2.1) and the behavior of a certain uniserial representation $R^-_{\infty,0}$ of the Iwahori subgroup of $\mathbf{GL}_2(\mathbf{Z}_p)$ (cf. Lemma 2.3).

Sections 3, 4 and 5 are devoted to the realization of the structure theorems for the $L^\times$-restriction of absolutely irreducible admissible $\mathbf{GL}_2(\mathbf{Q}_p)$-representations.

We first deal with the supersingular, unramified case in §3. After recalling some elementary results on the $\mathbf{F}^\times_{p^2}$-restriction of a finite parabolic induction (§3.2), we detect a crucial decomposition result on the $\mathbf{GL}_2(\mathbf{Z}_p)$-representation induced from $R^-_{\infty,0}$ (Lemma 3.3). More precisely, we use the explicit description of $R^-_{\infty,0}$ (giving the extensions of the Iwahori-characters appearing in its socle filtration) to detect the extensions of the $\mathscr{O}_L^\times$-characters in the $\mathbf{GL}_2(\mathbf{Z}_p)$ induction of $R^-_{\infty,0}$.

Once Lemma 3.3 is established, we detect the structure theorem (Corollary 3.10) by standard arguments of homological algebra (cf. Lemma 3.4, Proposition 3.5).

The ramified, supersingular case is considered in 4. The structure result (Proposition 4.2) is in this situation an immediate consequence of the general theorems of §2. Finally, the behavior of principal and special series is dealt in §5 and the proofs are similar to those of the supersingular case.

The cohomological methods to detect the generalized multiplicity statements are developed in §6. After a section of standard preliminaries on the cohomological functors $\mathrm{Ext}^i_H\big(\circ,\circ\big)$ (where H is a p-adic analytic group), we determine some key dualities between Ext-spaces in §6.2.1. The first statement (Lemma 6.6) recalls that $\mathscr{O}_L^\times$ is a Poincaré duality group, while the second duality result (Proposition 6.6) is a consequence of a work of Schraen ([Sch], §1.3) on Iwasawa modules over a complete local Noetherian regular ring.

In §6.3 we compute the Ext-space for certain uniserial $\mathscr{O}_L^\times$-representations. The key result is Proposition 6.10, where we detect the dimension of the Ext-spaces by their explicit reslisation as co-limits of finite dimensional linear spaces.

The generalized multiplicity results (section 6.3.1) are now easy corollaries to the previously established structure theorems (§3, 4 and 5) and the cohomology of certain Iwasawa modules developed in section 6.3.

1.1 Notations

Let p be an odd prime. For a p-adic field F, with ring of integers $\mathscr{O}_F$ and (finite) residue field k_F, we write $x \mapsto \overline{x}$ for the reduction morphism $\mathscr{O}_F \to k_F$ and $\overline{x} \mapsto [\overline{x}]$ for the Teichmüller lift $k_F^\times \to \mathscr{O}_F^\times$ (we set $[0] \stackrel{\text{def}}{=} 0$).

We consider the general linear group $\mathbf{GL}_2$ and we write $\mathbf{B}$ for the Borel subgroup of upper triangular matrices and $\mathbf{T}$ for the maximal torus of diagonal matrices. This paper is focused on certain properties of p-modular representations of the p-adic group $G \stackrel{\text{def}}{=} \mathbf{GL}_2(\mathbf{Q}_p)$. We write $Z \stackrel{\text{def}}{=} Z(G)$, $K \stackrel{\text{def}}{=} \mathbf{GL}_2(\mathbf{Z}_p)$, respectively for the center and the maximal compact subgroup of G, and $\mathscr{T}$ for the Bruhat-Tits tree associated to G (cf. [Ser77]). We recall that the Iwahori subgroup of K, which we will denote by $K_0(p)$, is defined as the inverse image of the finite Borel $\mathbf{B}(\mathbf{F}_p)$ via the reduction morphism $K \to \mathbf{GL}_2(\mathbf{F}_p)$. The pro-$p$ Iwahori, i.e. the pro-p Sylow of $K_0(p)$, will be denoted by $K_1(p)$.

For notational convenience, we introduce the following elements:

$$s \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in G, \qquad \Pi \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix} \in G.$$

Let E be a p-adic field, with ring of integers $\mathscr{O}$ and finite residue field k (the "coefficient field"). A representation σ of a closed subgroup H of G is always understood to be smooth, with coefficients in k. If $h \in H$ we will sometimes write $\sigma(h)$ to mean the k-linear automorphism induced by the action of h on the underlying vector space of σ. Similarly, an irreducible G-representation will always understood to be *admissible* (i.e. the space of fixed vectors under any compact open subgroup of G is finite dimensional).

Let $H_2 \leqslant H_1$ be closed subgroups of G. For a smooth representation σ of H_2 we write $\mathrm{ind}_{H_2}^{H_1}\sigma$ to denote the compact induction of σ from H_2 to H_1. If $v \in \sigma$ and $h \in H_1$ we write $[h, v]$ for the unique element of $\mathrm{ind}_{H_2}^{H_1}\sigma$ supported in $H_2 h^{-1}$ and sending h^{-1} to v. We deduce in particular the following equalities:

$$h' \cdot [h, v] = [h'h, v], \qquad [hk, v] = [h, \sigma(k)v] \tag{2}$$

for any $h' \in H_1$, $k \in H_2$. The previous constructions will be mainly applied when $H_1 = G$ and $H_2 = KZ$ (cf. [Bre], §2.3) or when $H_1 = K$ and $H_2 = \mathbf{B}(\mathbf{Z}_p)$.

A *Serre weight* is an absolutely irreducible representation of K. Up to isomorphism they are of the form

$$\sigma_{r,t} \stackrel{\text{def}}{=} \det{}^t \otimes \mathrm{Sym}^r k^2 \tag{3}$$

where $r \in \{0, \ldots, p-1\}$ and $t \in \{0, \ldots, p-2\}$ (this gives a bijective parametrization of isomorphism classes of Serre weights by couples $(r, t) \in \{0, \ldots, p-1\} \times \{0, \ldots, p-2\}$). Recall that the K-representations $\mathrm{Sym}^r k^2$ can be identified with $k[X, Y]_r^h$, the linear subspace of $k_F[X, Y]$ described by the homogeneous polynomials of degree r, endowed with the K-action defined by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot X^{r-i}Y^i \stackrel{\text{def}}{=} (\overline{a}X + \overline{c}Y)^{r-i}(\overline{b}X + \overline{d}Y)^i$$

for $0 \leqslant i \leqslant r$.

We usually extend the action of K on a Serre weight to the group KZ, by imposing the scalar matrix $p \in Z$ to act trivially.

A k-valued character χ of the torus $\mathbf{T}(\mathbf{F}_p)$ will be considered, by inflation, as a smooth character of any subgroup of $K_0(p)$. We will write χ^s to denote the conjugate character of χ, defined by

$$\chi^s(t) \stackrel{\mathrm{def}}{=} \chi(sts)$$

for any $t \in \mathbf{T}(\mathbf{F}_p)$.

Similarly, if τ is any representation of $K_0(p)$, we will write τ^s to denote the conjugate representation, defined by

$$\tau^s(h) = \tau(\Pi h \Pi)$$

for any $h \in K_0(p)$.

Let $r \in \{0, \dots, p-1\}$. The following characters of $\mathbf{T}(\mathbf{F}_p)$ will play a central role in this paper:

$$\chi_r\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) \stackrel{\mathrm{def}}{=} a^r, \qquad \mathfrak{a}\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) \stackrel{\mathrm{def}}{=} ad^{-1}.$$

For any $l \in \mathbf{Z}$ we define $(\cdot)^l$ to be the $\mathbf{F}_p^\times$-character (resp. $\mathbf{F}_{p^2}^\times$-character) described by $\lambda \mapsto \lambda^l$ and $\omega : \mathbf{Q}_p^\times \to \mathbf{F}_p^\times$ to be the mod-p cyclotomic character.

If $H \leqslant K$ is a closed subgroup and τ is an H-representation we write $\left\{\mathrm{soc}^i(\tau)\right\}_{i \in \mathbf{N}}$ for its socle filtration (we set $\mathrm{soc}^0(\tau) \stackrel{\mathrm{def}}{=} \mathrm{soc}(\tau)$). We will use the notation

$$\mathrm{soc}^0(\tau) \text{---} \mathrm{soc}^1(\tau)/\mathrm{soc}^0(\tau) \text{---} \dots \text{---} \mathrm{soc}^{n+1}(\tau)/\mathrm{soc}^n(\tau) \text{---} \dots$$

to denote the sequence of consecutive graded pieces of the socle filtration for τ. In particular, each $\mathrm{soc}^{i+1}(\tau)/\mathrm{soc}^i(\tau) \text{---} \mathrm{soc}^{i+2}(\tau)/\mathrm{soc}^{i+1}(\tau)$ is a non-split extension.

Let L be a quadratic extension of $\mathbf{Q}_p$. The choice of a $\mathbf{Z}_p$-base of $\mathscr{O}_L$ gives an embedding of groups:

$$\begin{tikzcd} L^\times \cong \mathrm{Aut}_L(L) \arrow[r, hook, "\iota"] & \mathbf{GL}_2(\mathbf{Q}_p) \\ \mathscr{O}_L^\times \cong \mathrm{Aut}_{\mathscr{O}_L}(\mathscr{O}_L) \arrow[r, hook] \arrow[u, hook] & \mathbf{GL}_2(\mathbf{Z}_p) \arrow[u, hook] \end{tikzcd}$$

The aim of this paper is to describe the $L^\times$-restriction of absolutely irreducible admissible G-representations. The results presented here are independent on the choice of the $\mathbf{Z}_p$-base of $\mathscr{O}_L$, since the subgroups obtained by different choices are all conjugate in G.

We finally recall the Kronecker delta: if S is any set, and $s_1, s_2 \in S$ we define

$$\delta_{s_1,s_2} \stackrel{\mathrm{def}}{=} \begin{cases} 0 & \text{if} \quad s_1 \neq s_2 \\ 1 & \text{if} \quad s_1 = s_2. \end{cases}$$

2. Reminders on universal $\mathbf{GL}_2(\mathbf{Q}_p)$-representations

The aim of this section is to recall the structure theorems for universal $\mathbf{GL}_2(\mathbf{Q}_p)$-representations ([Mo2], Corollary 3.5 and Proposition 3.8). The main statement is recalled in Proposition 2.1.

Since these result will be of crucial importance in the rest of this work, and in order to make the present paper as self-contained as possible, we will shortly describe the construction of universal $\mathbf{GL}_2(\mathbf{Q}_p)$-representations and the realization of the structure theorems. The reader is invited to refer to [Mo2], §2 and §3 (or, [Mo4]) for the omitted details.

Let $r \in \{0, \dots, p-1\}$ and write $\sigma = \sigma_{r,0}$ for the associated Serre weight described in (3). In particular, the highest weight space of σ affords the character χ_r. We recall ([Ba-Li], [Her]) that the Hecke algebra $\mathscr{H}_{KZ}(\sigma) \stackrel{\text{def}}{=} \mathrm{End}_G(\mathrm{ind}_{KZ}^G \sigma)$ is commutative and isomorphic to the algebra of polynomials in one variable over k:

$$\mathscr{H}_{KZ}(\sigma) \xrightarrow{\sim} k[T].$$

The Hecke operator T is supported on the double coset $K\Pi KZ$ and completely determined as a suitable linear projection on σ (cf. [Her], Theorem 1.2); it admits an explicit description in terms of the Bruhat-Tits tree of $\mathbf{GL}_2(\mathbf{Q}_p)$ (cf. [Bre], §2.5).

The supersingular representation $\pi(r, 0)$ for $\mathbf{GL}_2(\mathbf{Q}_p)$ is then defined by the exact sequence

$$0 \to \mathrm{ind}_{KZ}^G \sigma \xrightarrow{T} \mathrm{ind}_{KZ}^G \sigma \to \pi(r, 0) \to 0.$$

Let $n \in \mathbf{N}$. We consider the element $\lambda_n(p) \in G$ defined by

$$\lambda_n(p) = \begin{bmatrix} 1 & 0 \\ 0 & p^n \end{bmatrix}$$

and we introduce the subgroup

$$K_0(p^n) \stackrel{\text{def}}{=} \big(\lambda_n(p) K \lambda_n^{-1}(p)\big) \cap K = \left\{ \begin{bmatrix} a & b \\ p^n c & d \end{bmatrix} \in K,\ c \in \mathbf{Z}_p \right\}.$$

The element $\begin{bmatrix} 0 & 1 \\ p^n & 0 \end{bmatrix}$ normalizes $K_0(p^n)$ and we define the $K_0(p^n)$-representation $\sigma^{(n)}$ as the $K_0(p^n)$ restriction of σ endowed with the twisted action of $K_0(p^n)$ by the element $\begin{bmatrix} 0 & 1 \\ p^n & 0 \end{bmatrix}$. Explicitly,

$$\sigma^{(n)}\left(\begin{bmatrix} a & b \\ p^n c & d \end{bmatrix} \right) \cdot X^{r-j} Y^j \stackrel{\text{def}}{=} \sigma\left(\begin{bmatrix} d & c \\ p^n b & a \end{bmatrix} \right) \cdot X^{r-j} Y^j.$$

Finally, for $n \geqslant 1$ we write

$$R_n^-(\sigma) \stackrel{\text{def}}{=} \mathrm{ind}_{K_0(p^n)}^{K_0(p)} \big(\sigma^{(n)}\big), \qquad R_0^- \stackrel{\text{def}}{=} \mathrm{cosoc}_{K_0(p)} \big(\sigma^{(1)}\big).$$

If the Serre weight σ is clear from the context, we set $R_n^- = R_n^-(\sigma)$. For notational convenience, we will write Y^r for a linear basis of R_0^-.

We have a K-equivariant isomorphism (deduced from Frobenius reciprocity)

$$\begin{aligned} \mathrm{ind}_{K_0(p)}^K R_n^- &\xrightarrow{\sim} k[K\lambda_n(p)KZ] \otimes_{k[KZ]} \sigma \\ [1, v] &\longmapsto \lambda_n(p) \otimes s \cdot v \end{aligned} \tag{4}$$

which lets us realize the Mackey decomposition for $\mathrm{ind}_{KZ}^G \sigma$:

$$\big(\mathrm{ind}_{KZ}^G \sigma\big)|_{KZ} \xrightarrow{\sim} \sigma^{(0)} \oplus \bigoplus_{n \geqslant 1} \mathrm{ind}_{K_0(p)}^K R_n$$

Here, $k[K\lambda_n(p)KZ]$ is the k-linear space on the double coset $K\lambda_n(p)KZ$, endowed with its natural structure of $(k[K], k[KZ])$-bimodule.

The interpretation in terms of the tree of $\mathbf{GL}_2(\mathbf{Q}_p)$ is clear: the $k[K]$-module R_n^- maps isomorphically onto the space of elements of $\mathrm{ind}_{KZ}^G \sigma$ having support on the double coset $K_0(p)\lambda_n(p)KZ$. In particular, if σ is the trivial weight, a linear basis for R_n^- is parametrized by the vertices of $\mathscr{T}$ lying at distance n from the central vertex and lying in the negative part of the tree.

The Hecke morphism T induces, by transport of structure, a family of $K_0(p)$-equivariant morphisms $\big\{T_n\big\}_{n \geqslant 1}$ defined on the $k[K_0(p)]$-modules R_n^- by the condition $T_n \stackrel{\text{def}}{=} T|_{R_n^-}$.

More precisely, one shows (cf. [Mo2], §2.1) that for any $n \geqslant 1$ the Hecke operator T_n admits a decomposition $T_n = T_n^+ \oplus T_n^-$ where[1] the morphisms $T_n^\pm : R_n^- \to R_{n\pm1}^-$ are obtained by compact induction (from $K_0(p^n)$ to K) from the following morphisms:

$$\begin{aligned} t_n^+ : \sigma^{(n)} &\hookrightarrow \mathrm{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \sigma^{(n+1)} \\ X^{r-j}Y^j &\mapsto \sum_{\lambda_n \in \mathbf{F}_p} (-\lambda_n)^j \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n] & 1 \end{bmatrix} [1, X^r]; \end{aligned}$$

$$\begin{aligned} t_{n+1}^- : \mathrm{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \sigma^{(n+1)} &\twoheadrightarrow \sigma^{(n)} \\ [1, X^{r-j}Y^j] &\mapsto \delta_{j,r} Y^r \end{aligned}$$

and, for $n = 0$, we have the natural epimorphism

$$\begin{aligned} T_1^- : R_1^- &\twoheadrightarrow R_0^- \\ X^{r-j}Y^j &\mapsto \delta_{j,r} Y^r \end{aligned}$$

(this shows that T_n^+ are monomorphisms and T_n^- epimorphisms for all $n \geqslant 1$).

The Hecke operators $T_n^\pm$ can be used to construct a family of amalgamated sums, in the following way. We define $R_0^- \oplus_{R_1^-} R_2^-$ as the push-out:

$$\begin{tikzcd} R_1^- \arrow[r, hook, "T_1^+"] \arrow[d, two heads, "-T_1^-"'] & R_2^- \arrow[d, two heads, dotted, "pr_2"] \\ R_0^- \arrow[r, hook, dotted] & R_0^- \oplus_{R_1^-} R_2^- \end{tikzcd}$$

and, assuming that we have inductively constructed $pr_{n-1} : R_{n-1}^- \twoheadrightarrow R_0^- \oplus_{R_1^-} \cdots \oplus_{R_{n-2}^-} R_{n-1}^-$ (where $n \geqslant 3$ is odd), we define the amalgamated sum $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-$ by the following co-cartesian diagram:

$$\begin{tikzcd} R_n^- \arrow[r, hook, "T_n^+"] \arrow[d, two heads, "-pr_{n-1} \circ T_n^-"'] & R_{n+1}^- \arrow[d, two heads, dotted, "pr_{n+1}"] \\ R_0^- \oplus_{R_1^-} R_2^- \oplus_{R_3^-} \cdots \oplus_{R_{n-2}^-} R_{n-1}^- \arrow[r, hook, dotted] & R_0^- \oplus_{R_1^-} R_2^- \oplus_{R_3^-} \cdots \oplus_{R_n^-} R_{n+1}^-. \end{tikzcd}$$

The amalgamated sums $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-$ (for n odd) form in an evident manner an inductive system and we define

$$R_{\infty,0}^- \overset{\text{def}}{=} \varinjlim_{n \in 2\mathbf{N}+1} R_0^- \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-.$$

We can repeat the previous construction for n even, defining an inductive system of $K_0(p)$-representations $R_1^- \oplus_{R_2^-} \cdots \oplus_{R_n^-} R_{n+1}^-$ and we write

$$R_{\infty,1}^- \overset{\text{def}}{=} \varinjlim_{n \in 2\mathbf{N}+2} R_1^- \oplus_{R_2^-} \cdots \oplus_{R_n^-} R_{n+1}^-.$$

[1] According to [Mo2], the morphisms $T_n^\pm$ should be written as $(T_n^\pm)^{\mathrm{neg}}$. We decided to use here the lighter notation $T_n^\pm$.

The relation between the representations $R^-_{\infty,\bullet}$ and the universal representation $\pi(\sigma,0)$ is described by the following structure theorem:

THEOREM 2.1 ([Mo2], Theorem 1.1 and 1.2). *The KZ restriction of the universal representation $\pi(r,0)$ decomposes as $\pi(r,0)|_{KZ} = R_{\infty,0}\oplus R_{\infty,1}$ and we have short exact sequences of K-representations*

$$0\to \mathrm{Sym}^{p-1-r}k^2\otimes\det{}^r\to \mathrm{ind}_{K_0(p)}^K\big(R^-_{\infty,0}\big)\to R_{\infty,0}\to 0$$
$$0\to \mathrm{Sym}^{r}k^2\to \mathrm{ind}_{K_0(p)}^K\big(R^-_{\infty,1}\big)\to R_{\infty,-1}\to 0.$$

Moreover, we have $K_0(p)$-equivariant exact sequences

$$0\to W_0\to\big(R^-_{\infty,1}\big)^s\oplus R^-_{\infty,0}\to R_{\infty,0}\to 0$$
$$0\to W_1\to\big(R^-_{\infty,0}\big)^s\oplus R^-_{\infty,1}\to R_{\infty,1}\to 0$$

where W_0, W_1 are appropriate 1-dimensional spaces affording the $K_0(p)$-characters χ_r and χ_r^s respectively.

Finally, the action of Π on $\pi(r,0)$ induces a k-linear isomorphism $R_{\infty,0}\to R_{\infty,1}$ which extends to the natural involution $\big(R^-_{\infty,1}\big)^s\oplus R^-_{\infty,0}\to\big(R^-_{\infty,0}\big)^s\oplus R^-_{\infty,1}$.

Proof. Omissis. This is Corollary 3.5, Proposition 3.7 and 3.8 in [Mo2]. $\square$

Thanks to Theorem 2.1, the precise understanding of the $K_0(p)$-modules $R^-_{\infty,0}$, $R^-_{\infty,1}$ gives us a complete control on the supersingular representation $\pi(r,0)$. The following result is the key phenomenon which lets us describe the phenomenology of the representations $R^-_{\infty,0}$, $R^-_{\infty,1}$. It relies *crucially* on the fact that we are working with the $\mathbf{Q}_p$-points of $\mathbf{GL}_2$:

PROPOSITION 2.2. *Let $n\geqslant 0$. The $k[K_0(p)]$-module R^-_{n+1} is uniserial, of dimension $(r+1)p^n$, and its socle filtration is described by*

$$\chi_{\underline{r}}^s\text{—}\chi_{\underline{r}}^s\mathfrak{a}\text{—}\chi_{\underline{r}}^s\mathfrak{a}^2\text{—}\dots\text{—}\chi_{\underline{r}}^s\mathfrak{a}^{(r+1)p^n-1}.$$

The $k[K_0(p)]$-modules $R^-_{\infty,0}$ and $R^-_{\infty,1}$ are uniserial, of infinite length and their socle filtration is described by

$$\begin{array}{ll} R^-_{\infty,0}: & \chi_r^s\mathfrak{a}^r\text{—}\chi_r^s\mathfrak{a}^{r+1}\text{—}\chi_r^s\mathfrak{a}^{r+2}\text{—}\dots\\ R^-_{\infty,1}: & \chi_r^s\text{—}\chi_r^s\mathfrak{a}\text{—}\chi_r^s\mathfrak{a}^2\text{—}\dots\end{array}$$

respectively.

Proof. The first statement concerning the structure of R^-_{n+1} is well known: see for instance [Mo2], Proposition 3.9 or [Pas], Propositions 4.7 and 5.9.

The second statement is obtained by passing to co-limits. More precisely, from [Mo2], Proposition 3.10, one has for $n\geqslant 1$

$$\dim\big(R^-_\bullet\oplus_{R^-_{\bullet+1}}\cdots\oplus_{R^-_n}R^-_{n+1}\big)=\begin{cases}1+(r+1)\frac{p^{n+1}-1}{p+1} & \text{if } \bullet=0\\ (r+1)\frac{p^{n+1}+1}{p+1} & \text{if } \bullet=1\end{cases}\tag{5}$$

which uniquely characterizes $R^-_\bullet\oplus_{R^-_{\bullet+1}}\cdots\oplus_{R^-_n}R^-_{n+1}$ as a quotient module of R^-_{n+1} (since the latter is uniserial).

One deduces that the socle filtration for $R^-_\bullet\oplus_{R^-_{\bullet+1}}\cdots\oplus_{R^-_n}R^-_{n+1}$ is described by

$$R^-_\bullet\oplus_{R^-_{\bullet+1}}\cdots\oplus_{R^-_n}R^-_{n+1}:\ \begin{cases}\chi_r^s\mathfrak{a}^r\text{—}\chi_r^s\mathfrak{a}^{r+1}\text{—}\chi_r^s\mathfrak{a}^{r+2}\text{—}\dots\text{—}\chi_r^s\mathfrak{a}^r & \text{if } \bullet=0\\ \chi_r^s\text{—}\chi_r^s\mathfrak{a}\text{—}\chi_r^s\mathfrak{a}^2\text{—}\dots\text{—}\chi_r^s\mathfrak{a}^r & \text{if } \bullet=1.\end{cases}$$

The structure of $R^-_{\infty,0}$, $R^-_{\infty,1}$ is now obtained by passing to co-limits (noticing that the transition morphisms are injective). $\square$

For $\bullet\in\{0,1\}$ let $\{\mathscr{F}_{j,\bullet}\}_{j\in\mathbf{N}}$ be the $K_0(p)$-socle filtration for $R^-_{\infty,\bullet}$. For notational convenience we set $\mathscr{F}_{-1,\bullet}\stackrel{\text{def}}{=}0$.

From [Mo3], Lemma 2.6 one deduces a linear basis $\mathscr{B}^-_{n+1}$ for R^-_{n+1}, endowed with a total order. Explicitly, we have a bijection

$$\begin{aligned}\{0,\dots,p-1\}^n\times\{0,\dots,r\}&\stackrel{\sim}{\longrightarrow}\mathscr{B}^-_{n+1}\\ (l_1,\dots,l_{n+1})&\longmapsto F^{(1,n)}_{(l_1,\dots,l_n)}(l_{n+1})\end{aligned}\tag{6}$$

where the element $F^{(1,n)}_{(l_1,\dots,l_n)}(l_{n+1})\in R^-_{n+1}$ is defined by

$$F^{(1,n)}_{(l_1,\dots,l_n)}(l_{n+1})\stackrel{\text{def}}{=}\sum_{\lambda_1\in\mathbf{F}_p}\lambda_1^{l_1}\begin{bmatrix}1&0\\p[\lambda_1]&1\end{bmatrix}\dots\sum_{\lambda_n\in\mathbf{F}_p}\lambda_n^{l_n}\begin{bmatrix}1&0\\p^n[\lambda_n]&1\end{bmatrix}[1,X^{r-l_{n+1}}Y^{l_{n+1}}]\in R^-_{n+1}.$$

The total ordering on $\mathscr{B}^-_{n+1}$ is then induced from the order of $\mathbf{N}$ via the *injective* map

$$\begin{aligned}\mathscr{B}^-_{n+1}&\hookrightarrow\mathbf{N}\\ F^{(1,n)}_{(l_1,\dots,l_n)}(l_{n+1})&\mapsto\sum_{j=0}^{n}p^jl_{j+1}.\end{aligned}$$

One checks (cf. [Mo3], Proposition 4.10, or an elementary computation) that the linear filtration on R^-_{n+1} induced from the linear order on $\mathscr{B}^-_{n+1}$ *is* the $K_0(p)$-socle filtration for R^-_{n+1}.

Lemma 2.3. *There exists a linear basis $\mathscr{B}_\infty=\{e_j,\ j\in\mathbf{N}\}$ for $R^-_{\infty,0}$ such that $e_0\in\mathscr{F}_{0,0}$ and such that for all $m\geqslant 1$ we have*

$$\left(\begin{bmatrix}a&b\\pc&d\end{bmatrix}-1\right)\cdot e_m\in\kappa_m\overline{c}e_{m-1}+\langle e_j\in\mathscr{B}_\infty,\ j<m-1\rangle_k\tag{7}$$

where $\begin{bmatrix}a&b\\pc&d\end{bmatrix}\in K_1(p)$ and $\kappa_m\in k^\times$ is an appropriate nonzero scalar depending only on e_m. In particular the linear basis $\mathscr{B}_\infty$ is compatible with the $K_0(p)$-socle filtration on $R^-_{\infty,0}$.

Similarly the $k[K_0(p)]$-module $R^-_{\infty,1}$ admits a linear basis $\mathscr{B}^1_\infty=\{f_j,\ j\in\mathbf{N}\}$ compatible with its socle filtration and verifying the analogous property to (7).

Proof. Let $n\geqslant 0$. An immediate manipulation (or [Mo3], Proposition 4.10) shows that we have a linear basis $\{e_j,\ 0\leqslant j<(r+1)p^n\}$ for R^-_{n+1} which is compatible with the socle filtration for R^-_{n+1} and which verifies (7): indeed, it suffices to define e_m as the unique element $F^{(1,n)}_{(l_1,\dots,l_n)}(l_{n+1})\in\mathscr{B}^-_{n+1}$ such that $\sum_{j=0}^n p^jl_{j+1}=m$.

The statement can now be deduced by passing to the quotients $\dots\oplus_{R^-_n}R^-_{n+1}$ and, then, to co-limits. $\square$

3. The unramified case

Throughout this section, we assume that $L/\mathbf{Q}_p$ is unramified. The main result is Proposition 3.10, which gives the $L^\times$-structure for the representation $\pi(r,0)$.

After fixing the notations and conventions (§3.1), we deal in §3.2 with the finite situation, i.e. with the $\mathbf{F}^\times_{p^2}$-restriction of finite parabolic inductions (cf. §3.2).

This will be needed in §3.3, where we study in detail the $\mathscr{O}_L^\times$-restriction for the induced representation $\mathrm{ind}_{K_0(p)}^K R_{\infty,0}^-$. The $k[K_0(p)]$-filtration $\{\mathscr{F}_{j,0}\}_{j\in\mathbf{N}}$ on $R_{\infty,0}^-$ induces a K-filtration on $\mathrm{ind}_{K_0(p)}^K R_{\infty,0}^-$ (whose graded pieces are finite parabolic induction) and we give the key technical lemma (Lemma 3.3) which describes the $\mathscr{O}_L^\times$-extensions between two consecutive pieces in the induced filtration on $\mathrm{ind}_{K_0(p)}^K R_{\infty,0}^-$.

We are then able to describe the $\mathscr{O}_L^\times$-socle filtration for $\mathrm{ind}_{K_0(p)}^K R_{\infty,0}^-$ (Corollary 3.6) hence the $L^\times$-structure for supersingular representations (Corollary 3.10).

3.1 Preliminaries and notations

Let α be a generator for the cyclic group $\mathbf{F}_{p^2}^\times$. In particular, α is a primitive element of $\mathbf{F}_{p^2}$ over $\mathbf{F}_p$ and write $X^2 - X\mathrm{Tr}(\alpha) + \mathrm{N}(\alpha)$ for its minimal polynomial (where Tr, N denote respectively the trace and norm of $\mathbf{F}_{p^2}$ over $\mathbf{F}_p$).

Hence, we have a $\mathbf{Z}_p$-linear isomorphism

$$\mathscr{O}_L \cong \mathbf{Z}_p \oplus \mathbf{Z}_p[\alpha]$$

and, since $\iota(p) = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$ acts trivially on $\pi(r,0)$, we are left to study the restriction $\pi(r,0)|_{\mathscr{O}_L^\times}$.

If $x, y \in \mathbf{Z}_p$ are such that

$$[\alpha^2] = [-\mathrm{N}(\alpha)] + [\alpha][\mathrm{Tr}(\alpha)] + p(x + [\alpha]y)$$

we have

$$\iota(a + [\alpha]b) = \begin{bmatrix} a & b[-\mathrm{N}(\alpha)] + pxb \\ b & a + b[\mathrm{Tr}(\alpha)] + pby \end{bmatrix}.$$

for any $a, b \in \mathbf{Z}_p$ verifying $a + [\alpha]b \in \mathscr{O}_L^\times$.

3.2 The finite case

For $l, m \in \mathbf{Z}$ we define the $\mathbf{GL}_2(\mathbf{F}_p)$-representation $V_{l,m}$ by

$$V_{l,m} \stackrel{\text{def}}{=} \mathrm{ind}_{\mathbf{B}(\mathbf{F}_p)}^{\mathbf{GL}_2(\mathbf{F}_p)} \chi_l^s \otimes \det{}^m.$$

The object of this section is to give a detailed description of the $\mathbf{F}_{p^2}^\times$-restriction of $V_{l,m}$, where $\mathbf{F}_{p^2}^\times$ is considered as a subgroup of $\mathbf{GL}_2(\mathbf{F}_p)$ via the embedding induced from the choice of the primitive element α. This is of course equivalent to the study of the $\mathscr{O}_L^\times$-restriction of $\mathrm{ind}_{K_0(p)}^K \chi_l^s \otimes \det^m$.

By Mackey decomposition, we have an isomorphism of $\mathbf{F}_{p^2}^\times$-representations

$$V_{l,m}|_{\mathbf{F}_{p^2}^\times} \xrightarrow{\sim} \mathrm{ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_{p^2}^\times} (\cdot)^{-l} \otimes \mathrm{N}^{m+l}. \tag{8}$$

The following explicit realization of the isomorphism (8) will be useful.

Define the permutation τ on $\{0, \ldots, p-1, \infty\}$ as follows. For $\lambda_0 \in \{0, \ldots, p-1\}$ we set

$$\tau(\lambda_0) \stackrel{\text{def}}{=} -\frac{\mathrm{N}(\alpha)}{\lambda_0 + \mathrm{Tr}(\alpha)} \quad \text{if } \lambda_0 \neq -\mathrm{Tr}(\alpha),$$
$$\tau(\lambda_0) \stackrel{\text{def}}{=} \infty \quad \text{if } \lambda_0 = -\mathrm{Tr}(\alpha)$$

and

$$\tau(\infty) \stackrel{\text{def}}{=} 0$$

In other words, we are considering the projective transformation on $\mathbf{P}^1(\mathbf{F}_p)$ associated to the matrix

$\begin{bmatrix} \mathrm{Tr}(\alpha) & 1 \\ -\mathrm{N}(\alpha) & 0 \end{bmatrix}$. We moreover define a map $x(\cdot) : \{0, \dots, p-1, \infty\} \to \mathbf{F}_p$ by

$$\begin{aligned} x(\lambda_0) &\overset{\mathrm{def}}{=} \lambda_0 + \mathrm{Tr}(\alpha) \quad \text{if } \lambda_0 \notin \{-\mathrm{Tr}(\alpha), \infty\}; \\ x(-\mathrm{Tr}(\alpha)) &\overset{\mathrm{def}}{=} -\mathrm{N}(\alpha); \\ x(\infty) &\overset{\mathrm{def}}{=} 1 \end{aligned}$$

and recall that a $\overline{\mathbf{F}}_p$-basis for $\mathrm{ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_{p^2}^\times}(\cdot)^{-l} \otimes \mathrm{N}^{m+l}$ is described by

$$\mathscr{B} = \Big\{ [\lambda_0 + \alpha, e] \quad \text{for } \lambda_0 \in \mathbf{F}_p; \quad [1, e] \Big\}.$$

The next lemma is elementary

Lemma 3.1. *We have an $\mathbf{F}_{p^2}^\times$-equivariant isomorphism defined by:*

$$\begin{aligned} V_{l,m}|_{\mathbf{F}_{p^2}^\times} &\overset{\sim}{\to} \mathrm{ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_{p^2}^\times}(\cdot)^{-l} \otimes \mathrm{N}^{m+l} \\ \left[\begin{bmatrix} \lambda_0 & 1 \\ 1 & 0 \end{bmatrix}, e \right] &\mapsto [\lambda_0 + \alpha, e]; \\ \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, e \right] &\mapsto [1, e](-1)^l. \end{aligned}$$

Proof. The $\overline{\mathbf{F}}_p$-linear morphism of the statement is clearly an isomorphism and we claim it is $\mathbf{F}_{p^2}^\times$-equivariant. It is enough to check the compatibility of the isomorphism with the α-action on a fixed base of $V_{l,m}|_{\mathbf{F}_{p^2}^\times}$. A direct computation gives

$$\begin{bmatrix} 0 & -\mathrm{N}(\alpha) \\ 1 & \mathrm{Tr}(\alpha) \end{bmatrix} \left[\begin{bmatrix} \lambda_0 & 1 \\ 1 & 0 \end{bmatrix}, e \right] = \begin{cases} (\mathrm{N}(\alpha))^m (-\tau(\lambda_0))^l \left[\begin{bmatrix} \tau(\lambda_0) & 1 \\ 1 & 0 \end{bmatrix}, e \right] & \text{if } \quad \lambda_0 \neq -\mathrm{Tr}(\alpha); \\ (\mathrm{N}(\alpha))^m [1, e] & \text{if } \quad \lambda_0 = -\mathrm{Tr}(\alpha); \end{cases}$$

$$\begin{bmatrix} 0 & -\mathrm{N}(\alpha) \\ 1 & \mathrm{Tr}(\alpha) \end{bmatrix} [1, e] = (\mathrm{N}(\alpha))^m (-\mathrm{N}(\alpha))^l \left[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, e \right]$$

and

$$\alpha[\lambda_0 + \alpha, e] = (\mathrm{N}(\alpha))^{m+l} (x(\lambda_0))^{-l} [\tau(\lambda_0) + \alpha, e].$$

The conclusion follows. $\square$

3.2.1 ***Study of*** $\mathrm{ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_{p^2}^\times}(\cdot)^l$

Let $l \in \{0, \dots, p-1\}$. As the order of the abelian group $\mathbf{F}_{p^2}^\times$ is prime to p the representation $\mathrm{ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_{p^2}^\times}(\cdot)^l$ decomposes into a direct sums of $(p+1)$-characters, which are precisely the $(p+1)$ possible extensions to $\mathbf{F}_{p^2}^\times$ of the $\mathbf{F}_p^\times$-character $\lambda \mapsto \lambda^l$.

Let $(s_0, s_1) \in \{0, \dots, p-1\}^2$ be such that $(s_0, s_1) \neq (p-1, p-1)$. The $\mathbf{F}_{p^2}^\times$-character defined by

$$\alpha \mapsto \alpha^{s_0 + p s_1}$$

extends the $\mathbf{F}_p^\times$-character $\lambda \mapsto \lambda^l$ if and only if

$$(s_0 + s_1) + p(s_0 + s_1) \equiv l + pl \bmod (p^2 - 1)$$

i.e. if and only if the couple (s_0, s_1) verifies one of the following relations:

$$s_0 + s_1 = l, \qquad s_0 + s_1 = p - 1 + l.$$

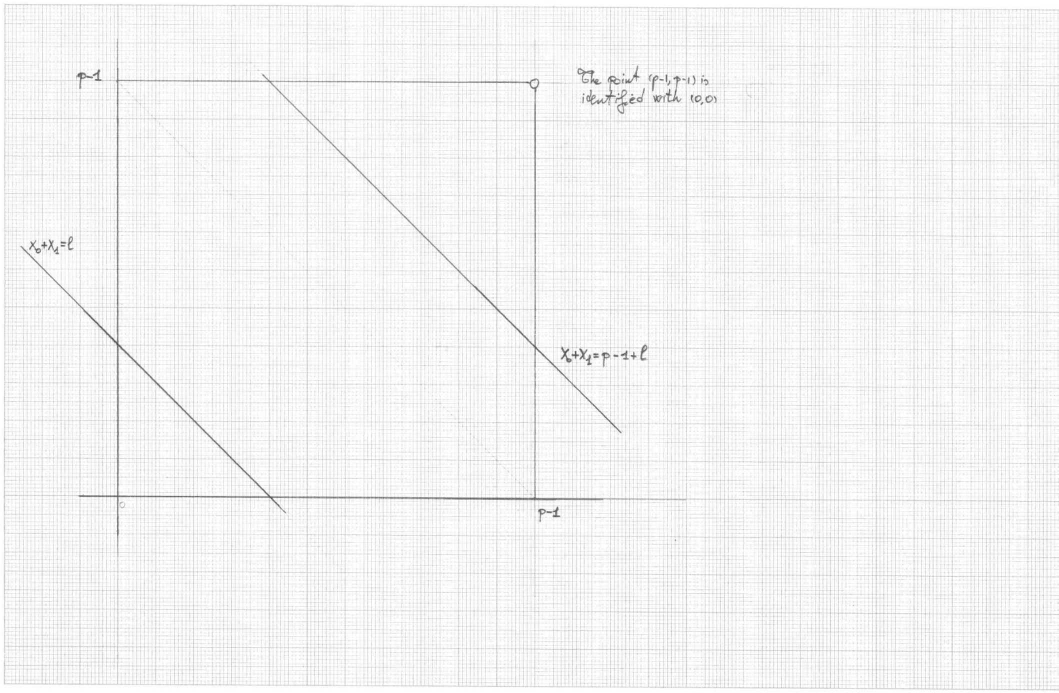

FIGURE 1: The combinatoric of admissible couples for l. The integer points in the square correspond to characters of $\mathbf{F}_{p^2}^{\times}$ and the admissible couples for l are then the integer points on the lines $X_0+X_1 = l$, $X_0+X_1=(p-1)+l$.

We will say that (s_0, s_1) is an *admissible couple for* l. The admissible couples for l can be visualized as in Figure III.1.

Let (s_0, s_1) be an admissible couple for l. We define the element $v^{(s_0,s_1)} \in \mathrm{ind}_{\mathbf{F}_p^{\times}}^{\mathbf{F}_{p^2}^{\times}}(\cdot)^l$ to be a linear generator of the $\mathbf{F}_{p^2}^{\times}$-character $\alpha \mapsto \alpha^{s_0+ps_1}$ appearing in the semisimplification of $\mathrm{ind}_{\mathbf{F}_p^{\times}}^{\mathbf{F}_{p^2}^{\times}}(\cdot)^l$. Hence, if we let e be a linear generator for $(\cdot)^l$, we have

$$v^{(s_0,s_1)} = \sum_{\lambda_0 \in \mathbf{F}_p} \mu_{\lambda_0}^{(s_0,s_1)}[\lambda_0+\alpha, e] + \mu_{\infty}^{(s_0,s_1)}[1,e]$$

for a $(p+1)$-tuple $\big((\mu_{\lambda_0}^{(s_0,s_1)})_{\lambda_0\in\mathbf{F}_p}, \mu_{\infty}^{(s_0,s_1)}\big) \in k^{p+1}$, which is moreover uniquely defined modulo $k^{\times}$. We will write $v^{(s_0,s_1)}(e)$ instead of $v^{(s_0,s_1)}$ if we need to emphasize the choice of the linear basis e.

The following lemma describes the scalars $\mu_{\lambda_0}^{(s_0,s_1)}$, $\mu_{\infty}^{(s_0,s_1)}$:

LEMMA 3.2. *Let (s_0, s_1) be an admissible couple for l and let $n \in \{0, \dots, p\}$. Then*

$$\mu_{\tau^{-n}(0)}^{(s_0,s_1)} = \mu_0^{(s_0,s_1)} \alpha^{n(s_0+ps_1)} \big(x(\tau^{-1}(0)) \cdot \dots \cdot x(\tau^{-n}(0))\big)^{-l}.$$

Proof. It is enough to study the action of α on $v^{(s_0,s_1)}$. A computation gives:

$$\alpha\cdot v^{(s_0,s_1)}=\sum_{\lambda_0\notin\{-\mathrm{Tr}(\alpha),\infty\}}\mu_{\lambda_0}^{(s_0,s_1)}\big(x(\lambda_0)\big)^l[\tau(\lambda_0)+\alpha,e]+\\ +\mu_{-\mathrm{Tr}(\alpha)}^{(s_0,s_1)}\big(x(-\mathrm{Tr}(\alpha))\big)^l[1,e]+\mu_\infty^{(s_0,s_1)}[\alpha,e]$$

and since $v^{(s_0,s_1)}$ is an eigenvector of associated eigencaracter $(\cdot)^{s_0+ps_1}$ we get the equations:

$$\begin{aligned}\mu_{\lambda_0}^{(s_0,s_1)}\big(x(\lambda_0)\big)^l&=\alpha^{s_0+ps_1}\mu_{\tau(\lambda_0)}^{(s_0,s_1)}\qquad\text{if }\lambda_0\notin\{-\mathrm{Tr}(\alpha),\infty\}\\ \mu_{-\mathrm{Tr}(\alpha)}^{(s_0,s_1)}\big(x(-\mathrm{Tr}(\alpha))\big)^l&=\alpha^{s_0+ps_1}\mu_\infty^{(s_0,s_1)}\\ \mu_\infty^{(s_0,s_1)}&=\alpha^{s_0+ps_1}\mu_0.\end{aligned}$$

The result is now clear for $n=0$ and it follows immediately for $n=1$ (as $\tau^{-1}(0)=\infty$, and $x(\infty)=1$). The general case follows by induction. □

3.3 Extensions inside irreducible representations

We are now able to describe the $\mathscr{O}_L^\times$-structure for the supersingular representation $\pi(r,0)$. Thanks to the intertwining $\pi(r,0)\overset{\sim}{\to}\pi(p-1-r,0)\otimes\omega^r$ it will be enough to study the summand $R_{\infty,0}$ appearing in the decomposition of Theorem 2.1 (cf. [Mo2], Proposition 4.1) and hence the $\mathscr{O}_L^\times$-socle filtration for the induced representation $\mathrm{ind}_{K_0(p)}^K R_{\infty,0}^-$.

The main result will be Proposition 3.5, from which one easily deduces the $\mathscr{O}_L^\times$-socle filtration for the supersingular representation $\pi(r,0)$ (Corollary 3.10).

Recall (cf. Proposition 2.2) that the $k[K_0(p)]$-module $R_{\infty,0}^-$ is uniserial with socle filtration $\big\{\mathscr{F}_j\big\}_{j\in\mathbf{N}}\overset{\mathrm{def}}{=}\big\{\mathscr{F}_{j,0}\big\}_{j\in\mathbf{N}}$ described by

$$\chi_r^s\mathfrak{a}^r—\chi_r^s\mathfrak{a}^{r+1}—\chi_r^s\mathfrak{a}^{r+2}—\chi_r^s\mathfrak{a}^{r+3}—...$$

We get the induced filtration $\big\{\mathrm{ind}_{K_0(p)}^K\mathscr{F}_j\big\}_{j\in\mathbf{N}}$ on $\mathrm{ind}_{K_0(p)}^K R_{\infty,0}^-$ whose graded pieces are described by

$$\mathrm{ind}_{K_0(p)}^K\chi_r^s\mathfrak{a}^r—\mathrm{ind}_{K_0(p)}^K\chi_r^s\mathfrak{a}^{r+1}—\mathrm{ind}_{K_0(p)}^K\chi_r^s\mathfrak{a}^{r+2}—\mathrm{ind}_{K_0(p)}^K\chi_r^s\mathfrak{a}^{r+3}—...\tag{9}$$

Furthermore (Lemma 2.3) we have a linear basis $\mathscr{B}_\infty=\big\{e_j,\ j\in\mathbf{N}\big\}$ for $R_{\infty,0}^-$ which is compatible with the filtration $\mathscr{F}_j$, i.e. for any integers $h\geqslant i\geqslant -1$ the set $\mathscr{B}_{h,i}\overset{\mathrm{def}}{=}\big\{e_j\in\mathscr{B},\ i<j\leqslant h\big\}$ is mapped to a a linear basis of the $k[K_0(p)]$-subquotient $\mathscr{F}_h/\mathscr{F}_i$. We will commit the abuse to employ the same notation for the elements $e_j\in\mathscr{B}_\infty$ and their images in the subquotients of $R_{\infty,0}^-$; this should cause no confusion.

We deduce, by restriction from (9), a $\mathscr{O}_L^\times$-equivariant filtration on $\mathrm{ind}_{K_0(p)}^K R_{\infty,0}^-$ given by

$$\mathrm{ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_{p^2}^\times}(\cdot)^r—\mathrm{ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_{p^2}^\times}(\cdot)^{r+2}\otimes\mathrm{N}^{-1}—\mathrm{ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_{p^2}^\times}(\cdot)^{r+4}\otimes\mathrm{N}^{-2}—...—\mathrm{ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_{p^2}^\times}(\cdot)^{r+2j}\otimes\mathrm{N}^{-j}—...$$

Fix $j\geqslant 1$. The $\mathscr{O}_L^\times$-subquotient $\mathrm{ind}_{K_0(p)}^K\big(\mathscr{F}_j/\mathscr{F}_{j-2}\big)$ of $\mathrm{ind}_{K_0(p)}^K R_{\infty,0}^-$ determines a $\mathscr{O}_L^\times$-extension

$$\mathrm{ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_{p^2}^\times}(\cdot)^{r+2j-2}\otimes\mathrm{N}^{-j+1}—\mathrm{ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_{p^2}^\times}(\cdot)^{r+2j}\otimes\mathrm{N}^{-j}$$

induced by the extension of $K_0(p)$-characters

$$\mathscr{F}_j/\mathscr{F}_{j-2}:\qquad\chi_{-r-2j+2}^s\det{}^{r+j-1}—\chi_{-r-2j}^s\det{}^{r+j}.\tag{10}$$

Recall that the characters in (10) admit the elements e_{j-1}, e_j as linear generators; moreover, given an admissible couple (s_0,s_1) for $r+2j$, we defined the elements $v^{(s_0,s_1)}(e_j)\in \mathrm{ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_{p^2}^\times}(\cdot)^{r+2j}\otimes \mathrm{N}^{-j}$. We will commit the abuse to employ the same notation for $v^{(s_0,s_1)}(e_j)$ and its canonical lift in $\mathrm{ind}_{K_0(p)}^K\big(\mathscr{F}_j/\mathscr{F}_{j-2}\big)$.

The following lemma is the key technical tool to prove Proposition 3.5. It describes the action of the pro-p part of $\mathscr{O}_L^\times$ on the elements $v^{(s_0,s_1)}(e_j)$ in the subquotient $\mathrm{ind}_{K_0(p)}^K\big(\mathscr{F}_j/\mathscr{F}_{j-2}\big)$.

Lemma 3.3. *Let $j\geqslant 1$ and (s_0,s_1) an admissible couple for $r+2j$. If $a,b\in\mathbf{Z}_p$ then*

$$\iota\big(1+p(a+[\alpha]b)\big)v^{(s_0,s_1)}(e_j)=v^{(s_0,s_1)}(e_j)+\mathrm{N}(\alpha)\bar{b}\kappa_j v^{(s_0-1,s_1-1)}(e_{j-1}) \tag{11}$$

where $\kappa_j\in k^\times$ and for $\bullet\in\{0,1\}$ the integers $(s_\bullet-1)\in\{0,\dots,p-1\}$ are defined by the conditions $(s_0-1)+p(s_1-1)\equiv s_0+ps_1-(p+1)\bmod (p^2-1)$ and $(s_0-1)+p(s_1-1)\neq p^2-1$.

In particular, the subquotient $\mathrm{ind}_{K_0(p)}^K\big(\mathscr{F}_j/\mathscr{F}_{j-2}\big)$ decomposes into a direct sum of $(p+1)$ uniserial representations (one for each admissible couple (s_0,s_1) for $r+2j$).

Proof. We have, by Lemma 3.1,

$$v^{(s_0,s_1)}(e_j)=\sum_{\lambda_0\in\mathbf{F}_p}\mu_{\lambda_0}^{(s_0,s_1)}\left[\begin{bmatrix}[\lambda_0] & 1\\ 1 & 0\end{bmatrix},e_j\right]+(-1)^{-r-2j}\mu_\infty^{(s_0,s_1)}[1,e_j]$$

and an elementary computation gives

$$\begin{bmatrix}1+pa & pb[-\mathrm{N}(\alpha)]+p^2bx\\ pb & 1+pa+pb[\mathrm{Tr}(\alpha)]+p^2by\end{bmatrix}\begin{bmatrix}[\lambda_0] & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}[\lambda_0] & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ p[-\bar{b}\big(\lambda_0^2+\lambda_0\mathrm{Tr}(\alpha)+\mathrm{N}(\alpha)\big)] & 1\end{bmatrix}\mathfrak{k}$$

where $\mathfrak{k}$ is an appropriate element in the pro-p part of $K_0(p^2)$. Hence, a simple manipulation using Lemma 2.3 gives the following equality in $\mathrm{ind}_{K_0(p)}^K\big(\mathscr{F}_j/\mathscr{F}_{j-2}\big)$:

$$\begin{aligned}&\iota\big(1+p(a+[\alpha]b)\big)v^{(s_0,s_1)}(e_j)-v^{(s_0,s_1)}(e_j)=\\ &\qquad=\bar{b}\kappa_j\Bigg(\sum_{\lambda_0\in\mathbf{F}_p}\mu_{\lambda_0}^{(s_0,s_1)}P(\lambda_0)\left[\begin{bmatrix}[\lambda_0] & 1\\ 1 & 0\end{bmatrix},e_{j-1}\right]+(-1)^{-r-2j}\mu_\infty^{(s_0,s_1)}[1,e_{j-1}]\Bigg).\end{aligned}$$

where we have defined $P(X)\overset{\mathrm{def}}{=}X^2+\mathrm{Tr}(\alpha)X+\mathrm{N}(\alpha)$ and $\kappa_j\in k^\times$ depends only on j (notice that $P(\lambda_0)\neq 0$).

We have

$$\begin{gathered}P(0)=\mathrm{N}(\alpha);\\ (-1)^{-r-2j}\mu_\infty^{(s_0,s_1)}=(-1)^{-r-2j}\alpha^{s_0+ps_1}=(-1)^{-r-2j}\alpha^{s_0-1+p(s_1-1)}\mathrm{N}(\alpha)=(-1)^{-r-2j}\mu_\infty^{(s_0-1,s_1-1)}\mathrm{N}(\alpha)\end{gathered}$$

and by Lemma 3.2 we are left to prove that

$$\mu_{\tau^{-n}(0)}^{(s_0,s_1)}P(\tau^{-n}(0))=\mathrm{N}(\alpha)\mu_{\tau^{-n}(0)}^{(s_0-1,s_1-1)}$$

where $2\leqslant n\leqslant p$.

This will be done by induction on n, the case $n=1$ being proved; for notational convenience, we put $P(\infty)\overset{\mathrm{def}}{=}1$. Assume the result is true for $n-1$; letting $i\overset{\mathrm{def}}{=}\tau^{-(n-1)}(0)$, by Lemma 3.2 and the inductive hypothesis we have:

$$\begin{aligned}\mu^{(s_0,s_1)}_{\tau^{-n}(0)}P(\tau^{-n}(0)) &= \mu^{(s_0,s_1)}_{\tau^{-1}(i)}P(\tau^{-1}(i))\\ &= \mu^{(s_0,s_1)}_{i}\alpha^{(s_0+ps_1)}P(\tau^{-1}(i))\big(x(\tau^{-1}(i))\big)^{-r-2j}\\ &= \mu^{(s_0-1,s_1-1)}_{i}\mathrm{N}(\alpha)(P(i))^{-1}\alpha^{(s_0+ps_1)}P(\tau^{-1}(i))\big(x(\tau^{-1}(i))\big)^{-r-2j+2}\big(x(\tau^{-1}(i))\big)^{-2}\\ &= \mathrm{N}(\alpha)\underbrace{\mu^{(s_0-1,s_1-1)}_{i}x(\tau^{-1}(i))^{-r-2j+2}\alpha^{s_0-1+p(s_1-1)}}_{\mu^{(s_0-1,s_1-1)}_{\tau^{-1}(i)}}\mathrm{N}(\alpha)(P(i))^{-1}P(\tau^{-1}(i))\big(x(\tau^{-1}(i))\big)^{-2}.\end{aligned}$$

To conclude the induction is then enough to show that for any $i_0\in\{0,\dots,p-1,\infty\}$ the following equality holds true:

$$\mathrm{N}(\alpha)(x(i_0))^{-2}P(i_0)=P(\tau(i_0)).$$

The cases $\tau(i_0)\in\{0,\infty\}$ are formal and a direct computation gives, for $\tau(i_0)\notin\{0,\infty\}$:

$$P(\tau(i_0))=\Big(-\frac{\mathrm{N}(\alpha)}{i_0+\mathrm{Tr}(\alpha)}\Big)^2-\frac{\mathrm{Tr}(\alpha)\mathrm{N}(\alpha)}{i_0+\mathrm{Tr}(\alpha)}+\mathrm{N}(\alpha)=\mathrm{N}(\alpha)x(i_0)^{-2}P(i_0).$$

This ends the inductive step and the proof of the first statement.

The assertion on the direct sum decomposition of $\mathrm{ind}^K_{K_0(p)}\big(\mathscr{F}_j/\mathscr{F}_{j-2}\big)$ is now clear, since $1+p\mathscr{O}_L$ acts trivially on the subrepresentation $\mathrm{ind}^K_{K_0(p)}\big(\mathscr{F}_{j-1}/\mathscr{F}_{j-2}\big)$, hence the equality (11) does not depend on the choice of the lift for the element $v^{(s_0,s_1)}(e_j)\in\mathrm{ind}^K_{K_0(p)}\big(\mathscr{F}_j/\mathscr{F}_{j-1}\big)$.

$\square$

In terms of Figure 1 the meaning of Lemma 3.3 is clear and illustrated in Figure 2.

We can now use Lemma 3.3 to prove that the whole representation $\mathrm{ind}^K_{K_0(p)}R^-_{\infty,0}$ admits a direct sum decomposition into uniserial pieces. First, we need the following elementary result on the cohomological bifunctors $\mathrm{Ext}^i_{\mathscr{O}_L^\times}(\circ,\circ)$ (cf. §6.1.1).

Lemma 3.4. *Let $\chi_1\neq\chi_2$ be smooth $\mathscr{O}_L^\times$-characters and let T be an uniserial $\mathscr{O}_L^\times$-representation of finite length, having all its Jordan-Hölder factors isomorphic to χ_1. Then*

$$\mathrm{Ext}^n_{\mathscr{O}_L^\times}(\chi_2,T)=0=\mathrm{Ext}^n_{\mathscr{O}_L^\times}(T,\chi_2)$$

for all $n\in\mathbf{N}$.

Proof. For $i\in\{0,1\}$ let e_{χ_i} be a generator for the linear space underlying the character χ_i.

As the characters χ_1, χ_2 are distinct, it exists $g_0\in\mathscr{O}_L^\times$ such that $\chi_1(g_0)\neq\chi_2(g_0)$. As the group $\mathscr{O}_L^\times$ is commutative, the maps

$$\begin{aligned}f_1:\chi_1&\to\chi_1\\ e_{\chi_1}&\mapsto g_0\cdot e_{\chi_1}-\chi_1(g_0)e_{\chi_1}\end{aligned}$$

and

$$\begin{aligned}f_2:\chi_2&\to\chi_2\\ e_{\chi_2}&\mapsto g_0\cdot e_{\chi_2}-\chi_1(g_0)e_{\chi_2}\end{aligned}$$

are $\mathscr{O}_L^\times$-equivariant morphisms and it is immediate to see that f_1 is the zero morphism while f_2 is an isomorphism.

By functoriality, the maps f_1, f_2 induce natural morphisms

$$\mathrm{Ext}^n_{\mathscr{O}_L^\times}(\chi_1,\chi_2)\xrightarrow{(f_1)_*}\mathrm{Ext}^n_{\mathscr{O}_L^\times}(\chi_1,\chi_2)\qquad\text{and}\qquad\mathrm{Ext}^n_{\mathscr{O}_L^\times}(\chi_1,\chi_2)\xrightarrow{(f_2)^*}\mathrm{Ext}^n_{\mathscr{O}_L^\times}(\chi_1,\chi_2)$$

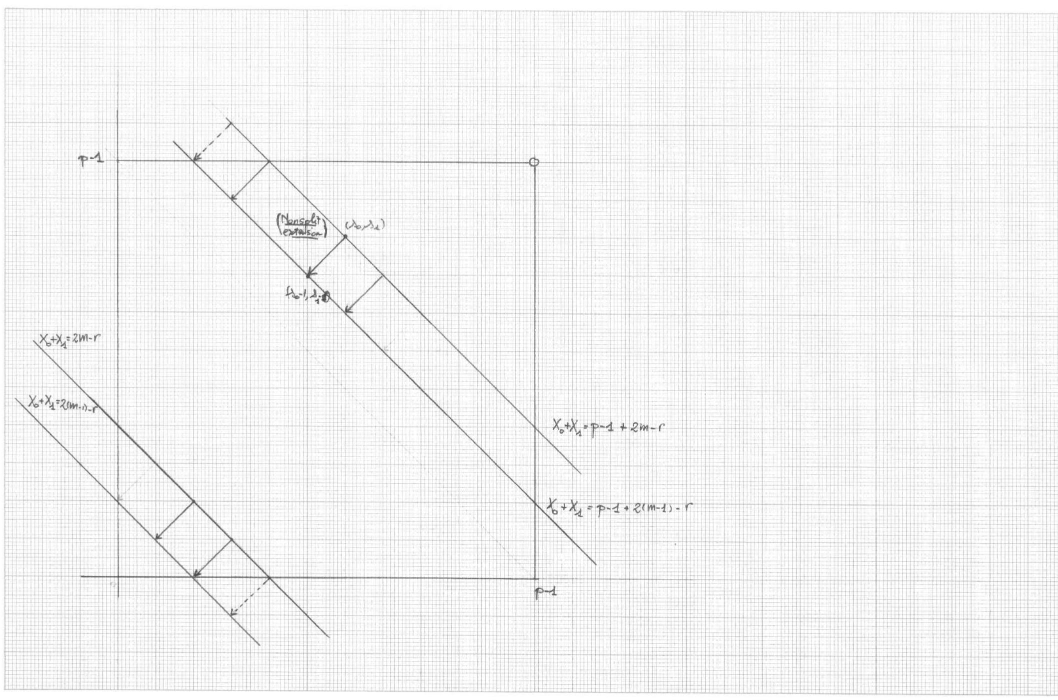

FIGURE 2: A graphic gloss of Lemma 3.3. Given the admissible couple (s_0, s_1) on the line $X_0 + X_1 = r + 2j$, we have a nonsplit extension with the admissible couple $(s_0 - 1, s_1 - 1)$ on the line $X_0 + X_1 = r + 2j - 2$

and we check that $(f_1)_* = (f_2)^*$. Thus $\mathrm{Ext}^n_{\mathscr{O}_L^\times}(\chi_1, \chi_2) = 0$. By dévissage on the length of T we get $\mathrm{Ext}^n_{\mathscr{O}_L^\times}(T, \chi_2) = 0$. The result on $\mathrm{Ext}^n_{\mathscr{O}_L^\times}(\chi_2, T)$ follows analogously. □

We deduce the desired result:

Proposition 3.5. *Let $j \geqslant 1$ and write $\{\eta_i\}_{i=0}^p$ for the set of the $(p+1)$-characters of $\mathbf{F}_{p^2}^\times$ extending the $\mathbf{F}_p^\times$-character $x \mapsto x^r$.*

The $\mathscr{O}_L^\times$-restriction of $\mathrm{ind}_{K_0(p)}^K \mathscr{F}_j$ admits a direct sum decomposition

$$\big(\mathrm{ind}_{K_0(p)}^K \mathscr{F}_j\big)|_{\mathscr{O}_L^\times} = \oplus_{i=0}^p \mathscr{F}_{j,0}(\eta_i)$$

where each representation $\mathscr{F}_{j,0}(\eta_i)$ is uniserial, of length j, having all its Jordan-Hölder factors isomorphic to η_i.

Proof. The proof is an induction on j. For $j = 0$ the result is clear since $\mathscr{F}_0 \cong \chi_r$ is a character.

Assume the result for $\mathrm{ind}_{K_0(p)}^K \mathscr{F}_{j-1}$ where $j \geqslant 1$. The exact sequence

$$0 \to \mathscr{F}_{j-1} \to \mathscr{F}_j \to \chi_r^s \mathfrak{a}^{r+j} \to 0$$

and the exactness of compact induction show that the $\mathscr{O}_L^\times$-restriction of $\mathrm{ind}_{K_0(p)}^K \mathscr{F}_j$ defines an element in $\mathrm{Ext}^1_{\mathscr{O}_L^\times}(\oplus_{i=0}^p \eta_i, \mathrm{ind}_{K_0(p)}^K \mathscr{F}_{j-1})$.

By the inductive hypothesis $\mathrm{ind}_{K_0(p)}^K \mathscr{F}_{j-1}$ decomposes into a direct sum of $\mathscr{O}_L^\times$-representations which are uniserial and with constant Jordan-Hölder factors. Hence by Lemma 3.4 we have

$$\mathrm{Ext}^1_{\mathscr{O}_L^\times}(\oplus_{i=0}^p \eta_i, \mathrm{ind}_{K_0(p)}^K \mathscr{F}_{j-1}) = \mathrm{Ext}^1_{\mathscr{O}_L^\times}(\oplus_{i=0}^p \eta_i, \oplus_{i=0}^p \mathscr{F}_{j-1,0}(\eta_i)) = \oplus_{i=0}^p \mathrm{Ext}^1_{\mathscr{O}_L^\times}(\eta_i, \mathscr{F}_{j-1,0}(\eta_i))$$

which shows precisely that $\mathrm{ind}_{K_0(p)}^K \mathscr{F}_j$ decomposes into a direct sum

$$\mathrm{ind}_{K_0(p)}^K \mathscr{F}_j = \oplus_{i=0}^p \mathscr{F}_{j,0}(\eta_i)$$

where each $\mathscr{F}_{j,0}(\eta_i)$ has all its Jordan-Hölder factors isomorphic to η_i.

This implies in particular that the natural injection $\mathrm{ind}_{K_0(p)}^K \mathscr{F}_{j-1} \hookrightarrow \mathrm{ind}_{K_0(p)}^K \mathscr{F}_j$ induces $(p+1)$ injections $\mathscr{F}_{j-1,0}(\eta_i) \hookrightarrow \mathscr{F}_{j,0}(\eta_i)$, one for each $i \in \{0, \dots, p\}$.

The fact that $\mathscr{F}_{j,0}(\eta_i)$ is uniserial is deduced immediately from Lemma 3.3. □

We deduce

Corollary 3.6. *The $\mathscr{O}_L^\times$-representation $\mathrm{ind}_{K_0(p)}^K R^-_{\infty,0}$ admits a direct sum decomposition*

$$\mathrm{ind}_{K_0(p)}^K R^-_{\infty,0} = \oplus_{i=0}^p \mathscr{F}_{\infty,0}(\eta_i)$$

where each representation $\mathscr{F}_{\infty,0}(\eta_i)$ is uniserial, of infinite length, having all its Jordan-Hölder factors isomorphic to η_i

Proof. We remarked, in the proof of Corollary 3.5, that for each $j \geqslant 1$ the natural injection $\mathrm{ind}_{K_0(p)}^K \mathscr{F}_{j-1} \hookrightarrow \mathrm{ind}_{K_0(p)}^K \mathscr{F}_j$ induces $(p+1)$ injections $\mathscr{F}_{j-1,0}(\eta_i) \hookrightarrow \mathscr{F}_{j,0}(\eta_i)$, one for each $i \in \{0, \dots, p\}$.

Since

$$\mathrm{ind}_{K_0(p)}^K R^-_{\infty,0} = \varinjlim_{j \in \mathbf{N}} \mathrm{ind}_{K_0(p)}^K \mathscr{F}_j$$

the statement follows from Corollary 3.5 by passing to co-limits. □

Remark 3.7. *The statement of Corollary 3.6 holds true if we replace $R^{-}_{\infty,0}$ with $R^{-}_{\infty,1}$, i.e. one has a decomposition*

$$\mathrm{ind}_{K_0(p)}^{K} R^{-}_{\infty,1} = \oplus_{i=0}^{p} \mathscr{F}_{\infty,1}(\eta_i)$$

where each representation $\mathscr{F}_{\infty,1}(\eta_i)$ is uniserial, of infinite length, having all its Jordan-Hölder factors isomorphic to η_i. This can deduced either from [Mo2], Proposition 4.2 if $p \geqslant 5$, or by a direct argument (which holds for $p \geqslant 3$), replacing the basis $\mathscr{B}_\infty$ with $\mathscr{B}^1_\infty$ (cf. Lemma 2.3).

3.4 Conclusion

It is now easy to deduce the decomposition result for the supersingular representation $\pi(r,0)$: from Corollary 3.6 and the structure Theorem 2.1 we are left to describe the $\mathbf{F}_{p^2}^{\times}$-restriction of a Serre weight.

This is worked out in the following lemma.

Lemma 3.8. *Let $r, m \in \{1,\dots,p-1\}$ and let $V \overset{\text{def}}{=} V_{-r,m} = \mathrm{ind}_{\mathbf{B}(\mathbf{F}_p)}^{\mathbf{GL}_2(\mathbf{F}_p)} \chi^s_{-r} \det^m$.*

i) The $\mathbf{F}_{p^2}^{\times}$-restriction of the socle $\mathrm{soc}(V)$ (resp. $\det^m|_{\mathbf{F}_{p^2}^{\times}}$ if $r = p-1$) decomposes as the direct sum of the characters $(\cdot)^{s_0+ps_1}$, where (s_0,s_1) are the admissible couples for r lying on the line $X_0 + X_1 = (p-1)+r$ (resp. $(s_0,s_1) = (0,0)$);

ii) the $\mathbf{F}_{p^2}^{\times}$-restriction of the cosocle $\mathrm{cosoc}(V)$ (resp. $\mathrm{St}\otimes\det^m|_{\mathbf{F}_{p^2}^{\times}}$ if $r = p-1$) decomposes as the direct sum of the characters $(\cdot)^{s_0+ps_1}$, where the (s_0,s_1) are the admissible couples for r lying on the line $X_0 + X_1 = r$.

Proof. Up to a twist we may assume $V = \mathrm{ind}_{\mathbf{B}(\mathbf{F}_p)}^{\mathbf{GL}_2(\mathbf{F}_p)} \chi^s_{p-1-r} \det^r$. It is now enough to show that

$$\mathrm{cosoc}(V)|_{\mathbf{F}_{p^2}^{\times}} \cong \mathrm{Sym}^r k^2$$

decomposes as the direct sum of the $\mathbf{F}_{p^2}^{\times}$-characters $(\cdot)^{s_0+ps_1}$ for the admissible couples (s_0,s_1) on the line $X_0 + X_1 = r$ (which implies that $\mathrm{soc}(V)|_{\mathbf{F}_{p^2}^{\times}}$ decomposes as the direct sum of the $\mathbf{F}_{p^2}^{\times}$-characters $(\cdot)^{s_0+ps_1}$ where $s_0 + ps_1 = (p-1)+r$).

For $r = 1$, the action of $\mathbf{GL}_2(\mathbf{F}_p)$ on $\mathrm{Sym}^1 k^2 \cong k^2$ is the natural one and the action of $\alpha \in \mathbf{F}_{p^2}^{\times}$ has spectrum $\mathscr{S} = \{\alpha, \alpha^p\}$ (indeed with the appropriate choice of a linear basis for k^2, the matrix associated to the α-action is $\begin{bmatrix} 0 & -N(\alpha) \\ 1 & Tr(\alpha) \end{bmatrix}$). This gives the case $r = 1$.

Let now $\{v_1, v_2\}$ be a linear basis of eigenvectors for the α-action on k^2. We define, for $j \in \{0,\dots,r\}$, the following element of $\mathrm{Sym}^r k^2$:

$$v_j \overset{\text{def}}{=} \underbrace{v_1 \vee v_1 \vee \cdots \vee v_1}_{j\text{-times}} \vee \underbrace{v_2 \vee v_2 \vee \cdots \vee v_2}_{(r-j)\text{-times}} \in \mathrm{Sym}^r k^2.$$

It follows by the definitions that $\{v_j,\ 0 \leqslant j \leqslant r\}$ is a linear basis for $\mathrm{Sym}^r k^2$ and each element v_j is an eigenvector for the α-action on $\mathrm{Sym}^r k^2$, having $\alpha^{j+p(r-j)}$ as an associated eigenvalue.

This implies the required result. □

Combining Corollary 3.6, Remark 3.7 and Lemma 3.8 we get the main result

Proposition 3.9. *For $\bullet \in \{0,1\}$ we have a direct sum decomposition*

$$R_{\infty,\bullet} \cong \oplus_{i=0}^{p} \overline{\mathscr{F}}_{\infty,\bullet}(\eta_i)$$

where each representation $\overline{\mathscr{F}}_{\infty,\bullet}(\eta_i)$ is uniserial, of infinite length, having Jordan-Hölder factors isomorphic to η_i.

Proof. This follows immediately from the exact sequence (Theorem 2.1)

$$0 \to \mathrm{Sym}^{p-1-r}k^2 \otimes \det^r \to \mathrm{ind}_{K_0(p)}^{K} R^{-}_{\infty,0} \to R_{\infty,0} \to 0$$

and the direct sum decomposition for the $\mathscr{O}_L^{\times}$-restriction of $\mathrm{Sym}^{p-1-r}k^2$ and $\mathrm{ind}_{K_0(p)}^{K} R^{-}_{\infty,0}$, given by Lemma 3.8 and Corollary 3.6.

The result on $R_{\infty,1}$ can be either deduced from the intertwining $\pi(r,0) \xrightarrow{\sim} \pi(p-1-r,0)\otimes\omega^r$ (which maps isomorphically $R_{\infty,1}$ in the source onto $R_{\infty,0}$ in the target, cf. [Mo2], Proposition 4.1), or from Remark 3.7.

Notice that if we define $\overline{\mathscr{F}_{j,0}}(\eta_i)$ to be the image of $\mathscr{F}_{j,0}(\eta_i)$ via the epimorphism $\mathrm{ind}_{K_0(p)}^{K} R^{-}_{\infty,0} \twoheadrightarrow R_{\infty,0}$ (cf. Proposition 3.5 for the subrepresentation $\mathscr{F}_{j,0}(\eta_i)$ of $\mathrm{ind}_{K_0(p)}^{K} R^{-}_{\infty,0}$), then

$$\overline{\mathscr{F}}_{\infty,\bullet}(\eta_i) = \varinjlim_{j\in\mathbf{N}} \overline{\mathscr{F}_{j,0}}(\eta_i)$$

and each $\overline{\mathscr{F}_{j,0}}(\eta_i)$ is uniserial, having Jordan-Hölder factors isomorphic to η_i and its length is $j+1$ (resp. j) if η_i corresponds to an admissible couple lying (resp. not lying) on the line $X_0+X_1=r$. □

In particular

Corollary 3.10. *Let π be a supersingular representation and write ω_π for its central character. The $L^{\times}$-restriction of π admits a splitting*

$$\pi|_{L^{\times}} \cong \oplus_{i=0}^{p}\mathscr{F}_{\pi,0}(\eta_i) \bigoplus \oplus_{i=0}^{p}\mathscr{F}_{\pi,1}(\eta_i)$$

where η_i are the $(p+1)$ smooth $L^{\times}$-characters extending the $\mathbf{Q}_p^{\times}$-character ω_π and, for $\bullet\in\{0,1\}$, each $\mathscr{F}_{\pi,\bullet}(\eta_i)$ is an infinite length uniserial representation of $L^{\times}$, with a scalar action of $p\in L^{\times}$, having all its Jordan-Hölder factors isomorphic to η_i.

Proof. Omissis. □

4. The ramified case

We assume now that $L/\mathbf{Q}_p$ is totally ramified. As $\mathscr{O}_L^{\times}$ injects into an Iwahori subgroup of $\mathbf{GL}_2(\mathbf{Q}_p)$, the structure of $\pi(r,0)|_{L^{\times}}$ is easily deduced from Theorem 2.1, as we outline in the following paragraphs. The main result is Proposition 4.2, giving the $L^{\times}$-structure for supersingular representations.

Let $\varpi\in\mathscr{O}_L$ be a uniformizer. With the choice of the $\mathbf{Z}_p$-base $\{\varpi,1\}$ for $\mathscr{O}_L$, we see that

$$\iota(a+\varpi b) = \begin{bmatrix} a & b \\ pb & a \end{bmatrix} \tag{12}$$

for any $a,b\in\mathbf{Z}_p$. In particular, $\iota(\mathscr{O}_L^{\times})$ is a subgroup of $K_0(p)$ and $\iota(\varpi)=\Pi$.

Thanks to Theorem 2.1, we deduce easily the $\mathscr{O}_L^{\times}$-restriction of the supersingular representation $\pi(r,0)$.

Proposition 4.1. *We have the following $\mathscr{O}_L^{\times}$-equivariant exact sequences:*

$$0 \to W_0 \to \mathscr{U}^{-}_{\infty,1}\oplus\mathscr{U}^{-}_{\infty,0} \to R_{\infty,0}|_{\mathscr{O}_L^{\times}} \to 0$$
$$0 \to W_1 \to \mathscr{U}^{-}_{\infty,0}\oplus\mathscr{U}^{-}_{\infty,1} \to R_{\infty,1}|_{\mathscr{O}_L^{\times}} \to 0$$

where $\mathscr{U}^{-}_{\infty,\bullet}$ are infinite length, uniserial representations of $\mathscr{O}_L^{\times}$ whose Jordan-Hölder factors are all isomorphic to the $\mathbf{F}_p^{\times}$-character $x\mapsto x^r$ and $W_\bullet$ are one dimensional.

Proof. We consider the first exact sequence (the other case being analogous) and use the notations of §2.

Recall that we have detected, in Lemma 2.3, natural linear basis $\mathscr{B}_\infty$, $\mathscr{B}^1_\infty$ for $R^-_{\infty,0}$, $R^-_{\infty,1}$ respectively such that for any $e_j \in \mathscr{B}_\infty$ (resp. $f_j \in \mathscr{B}^1_\infty$) and $x \in \mathscr{O}_L$ we have

$$\iota(1+\varpi x)\cdot e_j \in e_j - \overline{x}e_{j-1} + \mathscr{F}_{j-2,0} \qquad (\text{resp. } \iota(1+\varpi x)\cdot f_j \in f_j - \overline{x}f_{j-1} + \mathscr{F}_{j-2,1}). \tag{13}$$

Hence the $\mathscr{O}_L^\times$-representations $\mathscr{U}^-_{\infty,0} \overset{\text{def}}{=} R^-_{\infty,0}|_{\mathscr{O}_L^\times}$, $\mathscr{U}^-_{\infty,1} \overset{\text{def}}{=} R^-_{\infty,1}|_{\mathscr{O}_L^\times}$ are uniserial. Their Jordan-Hölder factors are described by the $\mathbf{F}_p^\times$-restriction of the characters $\chi_r^s\alpha^m$, for $m \in \mathbf{N}$, i.e. by the $\mathbf{F}_p^\times$-character $x \mapsto x^r$.

Moreover, by (12), we have $\left(R^-_{\infty,\bullet}\right)^s|_{\mathscr{O}_L^\times} = R^-_{\infty,\bullet}|_{\mathscr{O}_L^\times}$, which shows that the exact sequence of the statement are obtained, by $\mathscr{O}_L^\times$-restriction, from the exact sequences of Theorem 2.1. $\square$

We recall (cf. Theorem 2.1) that the Π-action on $\pi(r,0)$ induces a linear involution $R_{\infty,0} \to R_{\infty,1}$, which lifts to the natural involutions $R^-_{\infty,0} \to \left(R^-_{\infty,0}\right)^s$, $\left(R^-_{\infty,1}\right)^s \to R^-_{\infty,1}$.

If we define, for $m \in \mathbf{Z}$, the $L^\times$-representation U_m to be the two dimensional k-linear space on which $\mathscr{O}_L^\times$ acts by the character $x \mapsto \overline{x}^m$ and ϖ acts by a nontrivial involution, we therefore obtain the following structure result:

Proposition 4.2. *We have an $L^\times$-equivariant exact sequence*

$$0 \to U_r \to \mathscr{U}_0 \oplus \mathscr{U}_1 \to \pi(r,0)|_{L^\times} \to 0 \tag{14}$$

where the $k[L^\times]$-modules $\mathscr{U}_\bullet$ have infinite length, and the graded pieces of their socle filtration are all isomorphic to U_r.

Proof. For $\bullet \in \{0,1\}$ we define $\mathscr{U}_\bullet \overset{\text{def}}{=} \left(\mathscr{U}^-_{\infty,\bullet}\right)^2$ and we endow $\mathscr{U}_\bullet$ with the ϖ-action defined by the natural involution $(x,y) \mapsto (y,x)$.

The exact sequence (14) is immediately deduced from the description of the Π-action on $\pi(r,0)$ given in Theorem 2.1 and by Proposition 4.1 the graded pieces of the socle filtration on $\mathscr{U}_\bullet$ are all isomorphic to U_r. $\square$

5. The case of Principal and Special series

The aim of this section is to investigate the $L^\times$-restriction for principal and special series; the main results are the structure theorems given by Corollary 5.4 and Proposition 5.6. We first recall general structure theorems for tamely ramified parabolic inductions (cf. Proposition 5.2); this lets us reduce the investigation to a certain uniserial $k[K_0(p)]$-module (cf. Proposition 5.1) and therefore one can apply the techniques already seen in §3.3 and §4.

Prelimiraires on Principal and Special series. We recall (cf. [Ba-Li]) that, up to unramified twist, the irreducible principal series for $\mathbf{GL}_2(\mathbf{Q}_p)$ are described by the parabolic induction

$$\pi(r,\mu) \overset{\text{def}}{=} \mathrm{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)}(\mathrm{un}_\mu \otimes \omega^r \mathrm{un}_{\mu^{-1}})$$

where $\mu \in \overline{k}^\times$, un_μ is the unramified character of $\mathbf{Q}_p^\times$ verifying $\mathrm{un}_\mu(p) = \mu$, $r \in \{0,\dots,p-1\}$ and $(r,\mu) \notin \{(0,\pm 1), (p-1,\pm 1)\}$.

On the other hand, the special series are described (up to twist) by the short exact sequence

$$0 \to 1 \to \mathrm{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)} 1 \to \mathrm{St} \to 0. \tag{15}$$

We fix, once and for all, an integer $r \in \{0, \ldots, p-1\}$ and we define the following $k[K_0(p)]$-module

$$R_\infty^- \overset{\text{def}}{=} \varinjlim_{n \geqslant 1} \big(\mathrm{ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_r^s\big).$$

In analogy to the supersingular case, the module R_∞^- lets us control the representation theoretic properties of principal and special series, thanks to appropriate structure theorems. We remark that the structure of R_∞^- is particularly simple:

Proposition 5.1. *The $k[K_0(p)]$-module R_∞^- is uniserial, of infinite length and its socle filtration $\{\mathscr{F}_j\}_{j\in\mathbf{N}}$ is described by*

$$\chi_r^s — \chi_r^s\mathfrak{a} — \chi_r^s\mathfrak{a}^2 — \ldots.$$

Moreover, we have a linear basis $\mathscr{B} = \{e_j,\ j \in \mathbf{N}\}$ for R_∞^- such that $e_0 \in \mathrm{soc}\big(R_\infty^-\big)$ and for all $m \geqslant 1$ there exists a nonzero scalar $\kappa_m \in k^\times$ verifying

$$\left(\begin{bmatrix} a & b \\ pc & d \end{bmatrix} - 1\right) \cdot e_m \in \kappa_m \overline{c} e_{m-1} + \big\langle e_j,\ 0 \leqslant j \leqslant m-2\big\rangle_k$$

for any $\begin{bmatrix} a & b \\ pc & d \end{bmatrix} \in K_1(p)$. In particular the linear basis $\mathscr{B}$ is compatible with the $K_0(p)$-socle filtration on R_∞^-.

Proof. This is well known. Cf. for instance [Mo2], Proposition 3.9 or [Pas], Proposition 4.7 and 5.9. $\square$

The relation between the $k[K_0(p)]$-module R_∞^- and the tamely ramified principal series is described in the following

Proposition 5.2. *Let $\mu \in \overline{k}^\times$ and $r \in \{0, \ldots, p-1\}$. We have a K-equivariant isomorphism*

$$\big(\mathrm{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)}(\mathrm{un}_\mu \otimes \omega^r \mathrm{un}_{\mu^{-1}})\big)|_K \cong \mathrm{ind}_{K_0(p)}^K R_\infty^- \tag{16}$$

and a $K_0(p)$-equivariant isomorphism

$$\big(\mathrm{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)}(\mathrm{un}_\mu \otimes \omega^r \mathrm{un}_{\mu^{-1}})\big)|_{K_0(p)} \cong R_\infty^- \oplus \big(R_\infty^-\big)^s. \tag{17}$$

Moreover the Π-action on the LHS of (17) induces the natural involution

$$\begin{aligned} R_\infty^- &\longrightarrow \big(R_\infty^-\big)^s \\ v &\longmapsto \mu v. \end{aligned}$$

Proof. The K-isomorphism (16) is an easy consequence of Mackey decomposition theorem (cf. for instance [Mo2], §5, Lemma 5.1). Concerning the isomorphism (17), the Bruhat-Iwahori and the Mackey decompositions give a $k[K_0(p)]$-equivariant isomorphism

$$\big(\mathrm{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)}(\mathrm{un}_\mu \otimes \omega^r \mathrm{un}_{\mu^{-1}})\big)|_{K_0(p)} \cong k[K_0(p)] \otimes_{k[K_0(p^\infty)]} \eta \bigoplus k[K_0(p)sK_0(p)] \otimes_{k[K_0(p^\infty)]} \eta \tag{18}$$

where we set $\eta \overset{\text{def}}{=} \mathrm{un}_\mu \otimes \omega^r \mathrm{un}_{\mu^{-1}}$ and $K_0(p^\infty) \overset{\text{def}}{=} \mathbf{B}(\mathbf{Z}_p)$.

The second part of the statement follows by a direct manipulation (the action of Π on the LHS of (18) induces an involution which exchanges the two direct summands on the RHS). $\square$

5.1 The unramified case

We assume that $L/\mathbf{Q}_p$ is unramified and we adopt the notations already used in §3.

By Proposition 5.2 and 5.1 the tamely ramified principal series $\pi(r,\mu)$ is endowed with a K-equivariant filtration $\big\{\mathrm{ind}_{K_0(p)}^K\mathscr{F}_j\big\}_{j\in\mathbf{N}}$, described by

$$\mathrm{ind}_{K_0(p)}^K\chi_r^s\text{---}\mathrm{ind}_{K_0(p)}^K\chi_r^s\mathfrak{a}\text{---}\mathrm{ind}_{K_0(p)}^K\chi_r^s\mathfrak{a}^2\text{---}\mathrm{ind}_{K_0(p)}^K\chi_r^s\mathfrak{a}^3\text{---}\ldots$$

As we did in §3.3 we consider, for $j\geqslant 1$, the subquotient $\mathrm{ind}_{K_0(p)}^K\big(\mathscr{F}_j/\mathscr{F}_{j-2}\big)$. Recall that $\mathscr{F}_j/\mathscr{F}_{j-2}$ is linearly generated by the elements $e_{j-1},e_j\in\mathscr{B}$ and that we defined, for an admissible couple (s_0,s_1) for $2j$, the element $v^{(s_0,s_1)}(e_j)\in\mathrm{ind}_{K_0(p)}^K\big(\mathscr{F}_j/\mathscr{F}_{j-2}\big)$.

We then have:

LEMMA 5.3. *Let $j\geqslant 1$ and (s_0,s_1) be an admissible couple for $2j$. If $a,b\in\mathbf{Z}_p$ then*

$$\iota\big(1+p(a+[\alpha]b)\big)v^{(s_0,s_1)}(e_j)=v^{(s_0,s_1)}(e_j)+\mathrm{N}(\alpha)\overline{b}\kappa_j v^{(s_0-1,s_1-1)}(e_{j-1})\tag{19}$$

where $\kappa_j\in k^\times$ and $(s_i-1)\in\{0,\ldots,p-1\}$ are defined by the conditions $(s_0-1)+p(s_1-1)\equiv s_0+ps_1-(p+1)\bmod(p^2-1)$ and $(s_0-1)+p(s_1-1)\neq p^2-1$.

In particular, the subquotient $\mathrm{ind}_{K_0(p)}^K\big(\mathscr{F}_j/\mathscr{F}_{j-2}\big)$ decomposes into a direct sum of $(p+1)$ uniserial representations (one for each admissible couple (s_0,s_1) for $2j$).

Proof. The only property of $\mathscr{F}_j/\mathscr{F}_{j-2}$ which was used in the proof of Lemma 3.4 was the cocycle relation (7). Such property holds true also for the graded pieces of the socle filtration for R_∞^-, as stated in Proposition 5.1, and the argument of the proof of Lemma 3.4 applies line by line. □

Having Lemma 5.3, we see that the arguments of Corollaries 3.5 and 3.6 apply, hence:

COROLLARY 5.4. *Let π be a a special series or a principal series and write ω_π for its central character. The $L^\times$-restriction of π admits a splitting*

$$\pi|_{L^\times}\cong\oplus_{i=0}^p\mathscr{F}_\pi(\eta_i)$$

where η_i are the $(p+1)$ $L^\times$-characters extending the $\mathbf{Q}_p^\times$-character ω_π and each $\mathscr{F}_\pi(\eta_i)$ is an infinite length uniserial representation of $L^\times$, with a scalar action of p, having Jordan-Hölder factors isomorphic to η_i.

5.2 The ramified case

We assume now that $L/\mathbf{Q}_p$ is totally ramified. We use the notations of §4.

Thanks to Proposition 5.2 and 5.1 one deduces the $\mathscr{O}_L^\times$-restriction for the tamely ramified principal series $\pi(r,\mu)$:

PROPOSITION 5.5. *There is an $\mathscr{O}_L^\times$-equivariant decomposition for the tamely ramified principal series $\pi(r,\mu)$:*

$$\pi(r,\mu)|_{\mathscr{O}_L^\times}\xrightarrow{\sim}\big(\mathscr{U}_\infty^-\big)^2$$

where $\mathscr{U}_\infty^-$ is an infinite length, uniserial representations of $\mathscr{O}_L^\times$ whose Jordan-Hölder factors are all isomorphic to the $\mathbf{F}_p^\times$-character $x\mapsto x^r$.

Proof. Omissis. □

We recall that by Proposition 5.2 the Π action on $\pi(r,\mu)$ induces an involution on the direct sum decomposition (17). Defining the $L^\times$-representations U_r as in §4 we hence deduce

PROPOSITION 5.6. *The $L^\times$-restriction for the tamely ramified principal series $\pi(r,\mu)$ is described by:*

$$\pi(r,\mu)|_{L^\times}\xrightarrow{\sim}\mathscr{U}_\infty$$

where the $k[L^\times]$-module $\mathscr{U}_\infty$ has infinite length, and the graded pieces for its $k[L^\times]$-socle filtration are all isomorphic to U_r.

Proof. Omissis. $\square$

6. Cohomological methods

This section is devoted to prove the multiplicity statements on the dimension of the Ext-spaces $\mathrm{Ext}^i_{\mathscr{O}_L^\times}\big(\pi|_{\mathscr{O}_L^\times},\chi\big)$ for an irreducible $\mathbf{GL}_2(\mathbf{Q}_p)$-representation π and a smooth $\mathscr{O}_L^\times$-character χ. The main results are Proposition 6.12 and 6.13, which are deduced from the structure theorems of §3.4, §4 and §5 via the cohomological methods developed in this section.

More precisely, after recalling some generalities about the cohomological functors $\mathrm{Ext}^i_{\mathscr{O}_L^\times}$, $\mathrm{Tor}_i^{\mathscr{O}_L^\times}$ (cf. §6.1), we determine in §6.2 a key duality statement for the Ext-spaces of certain uniserial $\mathscr{O}_L^\times$-representations (Proposition 6.7).

The next crucial result is obtained in 6.3, where we determine the dimension of the Ext-spaces for certain uniserial $\mathscr{O}_L^\times$-representations (Proposition 6.10). At this point, the required multiplicities for absolutely irreducible $\mathbf{GL}_2(\mathbf{Q}_p)$-representations follow easily from the previously established structure theorems.

6.1 Preliminaries

The content of this section is essentially formal: we define precisely the categorical setting we work in, introducing the cohomological and homological functors which will be needed in §6.3. We recall the statement of Pontryagin duality and list some elementary properties of the previously introduced functors with respect to the duality. The main references will be [Bru] or [RZ], §5 (for a complete treatment of the categories of compact and discrete modules over profinite rings) or [S-W], §3 (whose results generalize line to line for mod p-Iwasawa algebras for compact p-adic analytic groups).

Let H be a compact p-adic analytic group (in the applications we will have $H\in\{\mathscr{O}_L^\times,\mathscr{O}_L,1\}$) and write $k[[H]]$ for the associated Iwasawa algebra. We recall (cf. [AB], Theorem 4.1) that $k[[H]]$ is a complete Noetherian semilocal ring (and indeed local if H is a pro-p-group). We consider the following categories:

−) the category $\mathrm{Mod}^{\mathrm{fl}}\big(k[[H]]\big)$ of discrete $k[[H]]$-modules of finite length;

−) the category $\mathrm{Mod}^{\mathrm{dis}}\big(k[[H]]\big)$ of discrete $k[[H]]$-modules;

−) the category $\mathrm{Mod}^{\mathrm{pro}}\big(k[[H]]\big)$ of profinite $k[[H]]$-modules (the homomorphisms being *continuous* and $k[[H]]$-linear).

Then the category $\mathrm{Mod}^{\mathrm{fl}}\big(k[[H]]\big)$ admits fully faithful embeddings both in $\mathrm{Mod}^{\mathrm{dis}}\big(k[[H]]\big)$ and $\mathrm{Mod}^{\mathrm{pro}}\big(k[[H]]\big)$ and it is well known (cf. [RZ] Proposition 5.4.2 and 5.4.4 or [Bru], §1) that $\mathrm{Mod}^{\mathrm{dis}}\big(k[[H]]\big)$ has enough injectives and exact co-limits while $\mathrm{Mod}^{\mathrm{pro}}\big(k[[H]]\big)$ enough projectives and exact limits.

We can embed the previous categories in the *Pontryagin category* $\mathrm{Pont}\big(k[[H]]\big)$, which is described as follow (cf. [S-W], §3). The objects of $\mathrm{Pont}\big(k[[H]]\big)$ are topological $k[[H]]$-modules which are either profinite or discrete; moreover, if $A,B\in\mathrm{Pont}\big(k[[H]]\big)$, we define the associated Hom-space as follows:

$$\mathrm{Hom}_H(A,B)\overset{\mathrm{def}}{=}\big\{A\overset{f}{\to}B,\ f\text{ is }k[[H]]\text{-linear, continuous and strict}\big\}.$$

We recall that a morphism $A\overset{f}{\to}B$ between topological $k[[H]]$-modules is *strict* if the quotient topology on $\mathrm{im}(f)$ coincides with the subspace topology induced from B. The k-linear space $\mathrm{Hom}_H(A,B)$ will be endowed with the compact-open topology.

It is clear from the definitions that the categories $\mathrm{Mod}^{\mathrm{dis}}\big(k[[H]]\big)$, $\mathrm{Mod}^{\mathrm{pro}}\big(k[[H]]\big)$ admit fully faithful embeddings in the Pontryagin category $\mathrm{Pont}\big(k[[H]]\big)$; moreover a projective object $P \in \mathrm{Mod}^{\mathrm{pro}}\big(k[[H]]\big)$ remains projective in $\mathrm{Pont}\big(k[[H]]\big)$: this follows from the fact that objects of $\mathrm{Mod}^{\mathrm{pro}}\big(k[[H]]\big)$ are filtrant co-limits of objects in $\mathrm{Mod}^{\mathrm{fl}}\big(k[[H]]\big)$, the transition morphisms being surjective (cf. [S-W] Proposition 3.4.1 or [Bru], Lemmas 2.2 and A.3). Similarly, an injective object in $\mathrm{Mod}^{\mathrm{dis}}\big(k[[H]]\big)$ remains injective in $\mathrm{Pont}\big(k[[H]]\big)$.

For an object $M \in \mathrm{Pont}\big(k[[H]]\big)$ we define

$$M^{\vee} \overset{\mathrm{def}}{=} \mathrm{Hom}_k(M, k)$$

where k is endowed with the discrete topology and the morphisms are understood to be *continuous* and *strict* (other than k-linear). We endow the module M with the compact-open topology and the H-action defined by

$$(h \cdot \phi)(t) \overset{\mathrm{def}}{=} \phi(h^{-1} \cdot t)$$

for any $h \in H$, $\phi \in M^{\vee}$, $t \in M$.

The Pontryagin duality asserts the following:

THEOREM 6.1 (Pontryagin duality). *The assignment $M \mapsto M^{\vee}$ defines an involutive, exact and contravariant functor on $\mathrm{Pont}\big(k[[H]]\big)$, which exchanges the subcategories $\mathrm{Mod}^{\mathrm{pro}}\big(k[[H]]\big)$ and $\mathrm{Mod}^{\mathrm{dis}}\big(k[[H]]\big)$ and preserves the length of the objects in $\mathrm{Mod}^{\mathrm{fl}}\big(k[[H]]\big)$. Moreover, we have a canonical isomorphism*

$$\mathrm{Hom}_H(M, N) \xrightarrow{\sim} \mathrm{Hom}_H(N^{\vee}, M^{\vee}). \tag{20}$$

Proof. This is well known. See for instance [Bru], Proposition 2.3, [S-W] Proposition 3.4.2 or [Eme], Lemma 2.2.7. □

6.1.1 ***Cohomological functors and their properties.*** In section 6.2 we will be interested in the left exact cohomological bifunctor $\mathrm{Hom}_H(\circ, \circ)$ defined on $\mathrm{Pont}\big(k[[H]]\big)$. It takes values in the category of k-linear topological spaces and since $\mathrm{Mod}^{\mathrm{dis}}\big(k[[H]]\big)$ (resp. $\mathrm{Mod}^{\mathrm{pro}}\big(k[[H]]\big)$) has enough injectives (resp. projectives) we can define its left derived bifunctors $\mathrm{Ext}^i_H\big(\circ, \circ\big)$ on the product categories[2] $\big(\mathrm{Mod}^{\mathrm{dis}}\big(k[[H]]\big)\big)^2$, $\big(\mathrm{Mod}^{\mathrm{pro}}\big(k[[H]]\big)\big)^2$ and $\mathrm{Mod}^{\mathrm{pro}}\big(k[[H]]\big) \times \mathrm{Mod}^{\mathrm{dis}}\big(k[[H]]\big)$. It is easy to see (cf. [Bru], §2) that $\mathrm{Ext}^i_H\big(A, B\big) \in \mathrm{Mod}^{\mathrm{disc}}(k)$ if $(A, B) \in \mathrm{Mod}^{\mathrm{pro}}\big(k[[H]]\big) \times \mathrm{Mod}^{\mathrm{dis}}\big(k[[H]]\big)$.

We notice that:

LEMMA 6.2. *Let $M, N \in \mathrm{Mod}^{\mathrm{fl}}\big(k[[H]]\big)$. Then $\mathrm{Ext}^i_H\big(M, N\big)$ is a discrete k-module of finite length.*

Proof. Since $k[[H]]$ is Noetherian, M admits a free resolution of $k[[H]]$-modules of finite type. It is therefore enough to prove that

$$\mathrm{Hom}_H(k[[H]], N)$$

is finite dimensional. Since N is of finite length its annihilator N is an open ideal of $k[[H]]$, and the conclusion follows as $k[[H]]$ is compact. □

In particular, we deduce

COROLLARY 6.3. *Let $M \in \mathrm{Mod}^{\mathrm{dis}}\big(k[[H]]\big)$, $N \in \mathrm{Mod}^{\mathrm{fl}}\big(k[[H]]\big)$. For all $i \geqslant 0$ there is a natural isomorphism*

$$\mathrm{Ext}^i_H\big(M, N\big) \xrightarrow{\sim} \varprojlim_{M_j \in J} \mathrm{Ext}^i_H\big(M_j, N\big)$$

[2] if we work with the categories of profinite $k[[H]]$-modules *of finite type* and discrete *admissible* $k[[H]]$-modules, the cohomological bifunctors $\mathrm{Ext}^i_H\big(\circ, \circ\big)$, $\mathrm{Tor}^H_i\big(\circ, \circ\big)$ can be defined on the whole associated Pontryagin category. See for instance [S-W], Theorems 3.7.2 and 3.7.4.

where J is the filtrant category of finite length submodules of M.

Similarly, if $M \in \mathrm{Mod}^{\mathrm{dis}}\big(k[[H]]\big)$, $N \in \mathrm{Mod}^{\mathrm{pro}}\big(k[[H]]\big)$, there is a natural isomorphism for any $i \geqslant 0$:

$$\mathrm{Ext}^i_H\big(N, M\big) \stackrel{\sim}{\longrightarrow} \varinjlim_{M_j \in J} \mathrm{Ext}^i_H\big(N, M_j\big)$$

where J is the filtrant category of finite length submodules of M.

Proof. Since each $\mathrm{Ext}^i_H\big(M_j, N\big)$ is finite by Lemma 6.2, the first statement follows from [Jen], Théorème 4.2 and 7.1.

Concerning the second isomorphism, we recall that if H is a compact p-adic analytic group then the functor $\mathrm{Ext}^i_H\big(N, \circ\big)$ is continuous (cf. [S-W] Theorem 3.7.2 or [RZ] Proposition 6.5.5). □

One easily checks, using the standard realization of a derived functor via projective/injective resolutions that the isomorphism (20) induces for any $i \geqslant 0$ an isomorphism

$$\mathrm{Ext}^i_H\big(N, M\big) \stackrel{\sim}{\longrightarrow} \mathrm{Ext}^i_H\big(M^\vee, N^\vee\big) \tag{21}$$

which is natural in $M \in \mathrm{Mod}^{\mathrm{dis}}\big(k[[H]]\big)$ and $N \in \mathrm{Mod}^{\mathrm{pro}}\big(k[[H]]\big)$.

Assume now that H is commutative. We can define the right exact bifunctor $\circ\widehat{\otimes}_{k[[H]]}\circ$ on the product category $\mathrm{Mod}^{\mathrm{pro}}\big(k[[H]]\big) \times \mathrm{Mod}^{\mathrm{pro}}\big(k[[H]]\big)$ (the *completed* tensor product of two profinite $k[[H]]$-modules). It takes values in the category profinite k-modules $\mathrm{Mod}^{\mathrm{pro}}\big(k[[k]]\big)$ and we write $\mathrm{Tor}^H_i\big(\circ, \circ\big)$ for its i-th left derived bifunctor.

By formal arguments of homological algebra (cf. [Bru], Corollary 2.6) we have, for any $i \geqslant 0$, a natural isomorphism of profinite k-modules

$$\mathrm{Tor}^H_i\big(\big(M^\vee\big)^{\mathrm{op}}, N\big) \stackrel{\sim}{\longrightarrow} \big(\mathrm{Ext}^i_H\big(N, M\big)\big)^\vee \tag{22}$$

where $M \in \mathrm{Mod}^{\mathrm{dis}}\big(k[[H]]\big)$, $N \in \mathrm{Mod}^{\mathrm{pro}}\big(k[[H]]\big)$ and $\big(M^\vee\big)^{\mathrm{op}}$ is the (right) $k[[H]]$-module obtained from $M^\vee$ via the anti-isomorphism of k-algebras $k[[H]] \stackrel{\sim}{\to} k[[H]]$ induced by the involution $h \mapsto h^{-1}$ on H.

Moreover, similarly to Lemma 6.2, we have

Lemma 6.4. *Let $M, N \in \mathrm{Mod}^{\mathrm{fl}}\big(k[[H]]\big)$. Then, for any $i \geqslant 0$, the k-module $\mathrm{Tor}^H_i\big(M, N\big)$ has finite length.*

Proof. Omissis. □

6.2 Reduction to local regular rings and dualities

The aim of this section is to specialize the constructions of §6.1 to the case $H \in \{\mathscr{O}_F^\times, \mathscr{O}_F\}$, for a p-adic field F. More precisely, we recall a canonical isomorphism between $\mathrm{Ext}^i_{\mathscr{O}_F^\times}\big(\circ, \circ\big)$ and $\mathrm{Ext}^i_{1+\varpi\mathscr{O}_F}\big(\circ, \circ\big)$ and we deduce in §6.2.1 a key duality result for the cohomological bifunctor $\mathrm{Ext}^i_{\mathscr{O}_F}\big(\circ, \circ\big)$, restricted to an appropriate subcategory of $\mathrm{Mod}^{\mathrm{dis}}\big(k[[\mathscr{O}_F^\times]]\big)$ (cf. Proposition 6.7). We crucially rely on the ring theoretic properties of the Iwasawa algebra of $\mathscr{O}_F^\times$, using previous work of Schraen [Sch].

Set $d \stackrel{\mathrm{def}}{=} [F : \mathbf{Q}_p]$ and let $\varpi \in F$ be an uniformizer. The following result is formal:

Lemma 6.5. *Let χ be an irreducible $k[[\mathscr{O}_F^\times]]$-module and $M \in \mathrm{Mod}^{\mathrm{dis}}\big(k[[\mathscr{O}_F^\times]]\big)$. Assume that the Jordan-Hölder factors of M are all isomorphic to χ.*

Then we have the following natural isomorphism:

$$\mathrm{Ext}^i_{\mathscr{O}_F^\times}\big(M, \chi\big) \stackrel{\sim}{\to} \mathrm{Ext}^i_{1+\varpi\mathscr{O}_F}\big(M|_{1+\varpi\mathscr{O}_F}, k\big)$$
$$\mathrm{Ext}^i_{\mathscr{O}_F^\times}\big(\chi, M\big) \stackrel{\sim}{\to} \mathrm{Ext}^i_{1+\varpi\mathscr{O}_F}\big(k, M|_{1+\varpi\mathscr{O}_F}\big)$$

Proof. By Corollary 6.3 it is enough to prove the statement when $M \in \mathrm{Mod}^{\mathrm{fl}}\big(k[[\mathscr{O}_F^\times]]\big)$.

We recall that the category $\mathrm{Mod}^{\mathrm{pro}}\big(k[k_F^\times]\big)$ is semisimple, in particular $\mathrm{Ext}^i_{k_F^\times}\big(T,\chi\big)=0$ for any $T\in\mathrm{Mod}^{\mathrm{pro}}\big(k[k_F^\times]\big)$ and any $i>0$.

Consider the first isomorphism. We write $A \overset{\mathrm{def}}{=} k[[1+\varpi\mathscr{O}_F]]$ (which is a local ring with residue field k, as $1+\varpi\mathscr{O}_F$ is pro-p) and we remark that for any $T\in\mathrm{Mod}^{\mathrm{pro}}\big(k[[\mathscr{O}_F^\times]]\big)$ one has a natural isomorphism:

$$\mathrm{Hom}_{\mathscr{O}_F^\times}(T,\chi)\xrightarrow{\sim}\mathrm{Hom}_{k_F^\times}(k\otimes_A T|_A,\chi).$$

We deduce the Grothendieck spectral sequence associated to the functors $k\otimes_A(\circ)|_A:\mathrm{Mod}^{\mathrm{pro}}\big(k[[\mathscr{O}_F^\times]]\big)\to\mathrm{Mod}^{\mathrm{pro}}\big(k[k_F^\times]\big)$ and $\mathrm{Hom}_{k_F^\times}(\circ,\chi):\mathrm{Mod}^{\mathrm{pro}}\big(k[k_F^\times]\big)\to\mathrm{Mod}^{\mathrm{pro}}\big(k\big)$, i.e.

$$\mathrm{Ext}^i_{k_F^\times}\big(\mathrm{Tor}_j^{1+\varpi\mathscr{O}_F}\big(k,M|_A\big),\chi\big)\Longrightarrow\mathrm{Ext}^{i+j}_{\mathscr{O}_F^\times}\big(M,\chi\big)\tag{23}$$

(notice that the restriction functor $(\,\circ\,)|_A$ is exact and sends projectives into projectives, hence the i-th left derived functor of $k\otimes_A(\,\circ\,)|_A$ coincides with $\mathrm{Tor}_i^{1+\varpi\mathscr{O}_F}\big(k,M|_A\big)$).

Using again the fact that $\mathrm{Ext}^i_{k_F^\times}\big(T,\chi\big)=0$ for $i\geqslant 1$, the spectral sequence (23) yields the isomorphisms

$$\mathrm{Hom}_{k_F^\times}\big(\mathrm{Tor}_j^{1+\varpi\mathscr{O}_F}\big(k,M|_A\big),\chi\big)\xrightarrow{\sim}\mathrm{Ext}^j_{\mathscr{O}_F^\times}\big(M,\chi\big).$$

We claim that the $k[k_F^\times]$-module $\mathrm{Tor}_j^{1+\varpi\mathscr{O}_F}\big(k,M|_A\big)$ has constant Jordan-Hölder factors, all isomorphic to χ. Indeed, we have a decomposition

$$k[[\mathscr{O}_F^\times]]\cong k[k_F^\times]\otimes_k A=\oplus_{e_l}e_l\cdot A$$

where the elements $e_l\in k[k_F^\times]$ are orthogonal idempotents parametrized by the set of irreducible $k[[\mathscr{O}_F^\times]]$-modules, and each e_lA is a projective $k[[\mathscr{O}_F^\times]]$-module with constant Jordan-Hölder factors.

Since M has constant Jordan-Hölder factors, we deduce that $M=e_\chi\cdot M$ (where $e_\chi\in k[[\mathscr{O}_F^\times]]$ is the idempotent corresponding to χ) and, similarly, we can refine a free resolution of M in $\mathrm{Mod}^{\mathrm{pro}}\big(k[[\mathscr{O}_F^\times]]\big)$ to a projective resolution formed by $k[[\mathscr{O}_F^\times]]$-modules with constant Jordan-Hölder factors, all isomorphic to χ.

This shows that $k[k_F^\times]$-module $\mathrm{Tor}_j^{1+\varpi\mathscr{O}_F}\big(k,M|_A\big)$ has Jordan-Holder factors isomorphic to χ, and, since the category $\mathrm{Mod}^{\mathrm{pro}}\big(k[k_F^\times]\big)$ is semisimple, one has

$$\mathrm{Hom}_{k_F^\times}\big(\mathrm{Tor}_j^{1+\varpi\mathscr{O}_F}\big(k,M|_A\big),\chi\big)=\mathrm{Hom}_k\big(\mathrm{Tor}_j^{1+\varpi\mathscr{O}_F}\big(k,M|_A\big),k\big)=\big(\mathrm{Tor}_j^{1+\varpi\mathscr{O}_F}\big(k,M|_A\big)\big)^\vee.$$

The conclusion follows now from the natural isomorphism (22).

The second isomorphism is proved in a completely analogous way: the details are left to the reader. $\square$

The statement of Lemma 6.5 will be particularly useful as the Iwasawa algebra $k[[1+\varpi\mathscr{O}]]$ is a complete Noetherian and regular local k-algebra, and its Krull dimension equals the dimension of $1+\varpi\mathscr{O}$ as a p-adic analytic group (cf. [AB], Theorem 3.6, 4.1 and 5.2).

6.2.1 ***Dualities.*** For a compact p-adic analytic group H, there are important duality results for the associated cohomology. In our situation, a classical duality statement specializes as follows:

LEMMA 6.6. *Let χ be an irreducible $k[[\mathscr{O}_F^\times]]$-module.*

For any $i\in\{0,\dots,d\}$ there is a natural isomorphism of $\mathrm{Mod}^{\mathrm{pro}}(k)$-valued functors defined on $\mathrm{Mod}^{\mathrm{dis}}\big(k[[\mathscr{O}_F^\times]]\big)$:

$$\mathrm{Ext}^i_{\mathscr{O}_F^\times}\big(\chi^\vee,(\,\circ\,)^\vee\big)\xrightarrow{\sim}\big(\mathrm{Ext}^{d-i}_{\mathscr{O}_F^\times}(\chi,\circ)\big)^\vee.\tag{24}$$

Proof. As the functor $\circ \otimes_k \chi$ defines an exact self-equivalence on $\mathrm{Pont}\big(k[[\mathscr{O}_F^\times]]\big)$ we can assume, without loss of generality, that χ is the trivial $k[[\mathscr{O}_F^\times]]$-module.

Since $\mathscr{O}_F^\times$ is a p-adic analytic group, which is p-torsion free, we deduce from [Laz] Théorème 2.5.8 (see also [S-W], Theorem 5.1.9) that $\mathscr{O}_F^\times$ is a Poincaré duality group. Therefore the isomorphism (24) follows once we have shown that the dimension of $\mathscr{O}_F^\times$ as a compact analytic group is $\dim(\mathscr{O}_F^\times) = d$ (see [S-W], Proposition 4.5.4 or [NSW] Chapter III §7).

This is clear: for any $n \geqslant 1$ the subgroup $1 + \varpi^n \mathscr{O}_F$ is an open pro-p subgroup of $\mathscr{O}_F^\times$ which is isomorphic, if n is large enough, to the additive group $\mathscr{O}_F$ (for instance, via the logarithm map). The conclusion follows from [DDMS], Theorem 8.36. $\square$

From the results of [Sch], we can moreover deduce another useful duality for the functor $\mathrm{Ext}^i_{\mathscr{O}_F^\times}\big(\circ, \chi\big)$, for an irreducible $k[[\mathscr{O}_F^\times]]$-module χ:

Proposition 6.7. *Let χ be an irreducible $k[[\mathscr{O}_F^\times]]$-module and $M \in \mathrm{Mod}^{\mathrm{dis}}\big(k[[\mathscr{O}_F^\times]]\big)$ be a discrete module with constant Jordan-Hölder factors, all isomorphic to χ.*

For any $i \in \{0, \dots, d\}$ we have a natural isomorphism of discrete k-modules:

$$\big(\mathrm{Ext}^i_{\mathscr{O}_F^\times}\big(M, \chi\big)\big)^\vee \xrightarrow{\sim} \mathrm{Ext}^{d-i}_{\mathscr{O}_F^\times}\big(\chi, M\big). \tag{25}$$

Proof. By Corollary 6.3 we can assume $M \in \mathrm{Mod}^{\mathrm{fl}}\big(k[[\mathscr{O}_F^\times]]\big)$, with constant Jordan-Hölder factors.

Hence, by Lemma 6.5, it will be enough to establish the natural isomorphism of discrete k-modules

$$\big(\mathrm{Ext}^i_{1+\varpi\mathscr{O}_F}\big(M, k\big)\big)^\vee \xrightarrow{\sim} \mathrm{Ext}^{d-i}_{1+\varpi\mathscr{O}_F}\big(k, M\big).$$

for $M \in \mathrm{Mod}^{\mathrm{fl}}\big(k[[1 + \varpi\mathscr{O}_F]]\big)$. Write A for the Iwasawa algebra associated to $1 + \varpi\mathscr{O}_F$. Since the latter is a compact uniform pro-p group of dimension d, we deduce that A is a complete regular Noetherian local k-algebra of dimension d. We write $\mathfrak{m}$ for its maximal ideal.

In this situation we have, by [Sch] Proposition 1.8, a natural isomorphism of discrete k-modules:

$$\big(\mathrm{Tor}_i^{1+\varpi\mathscr{O}_F}\big(k, M\big)\big)^\vee \xrightarrow{\sim} \mathrm{Tor}_{d-i}^{1+\varpi\mathscr{O}_F}\big(k, M^\vee\big). \tag{26}$$

Let $P^\bullet$ be a resolution for M by free A-modules of finite type. By exactness of Pontryagin duality we have

$$\begin{aligned}\big(\mathrm{Ext}^i_{1+\varpi\mathscr{O}_F}\big(M, k\big)\big)^\vee &= \mathrm{H}^{-i}\Big(\big(\mathrm{Hom}_A(P^\bullet, k)\big)^\vee\Big) \\ &= \mathrm{H}^{-i}\big(k \otimes_A P^\bullet\big) \\ &= \mathrm{Tor}_i^{1+\varpi\mathscr{O}_F}\big(k, M\big)\end{aligned} \tag{27}$$

where the second equality follows from $\big(\mathrm{Hom}_A(P^\bullet, k)\big)^\vee = \big(\mathrm{Hom}_k(k \otimes_A P^\bullet, k)\big)^\vee = k \otimes_A P^\bullet$ (notice that $k \otimes_A P^\bullet$ is, component wise, finite dimensional).

Combining (26), (27) and (21) we finally get

$$\Big(\mathrm{Ext}^i_{1+\varpi\mathscr{O}_F}\big(M, k\big)\Big)^\vee = \Big(\mathrm{Tor}_{d-i}^{1+\varpi\mathscr{O}_F}\big(k, M^\vee\big)\Big)^\vee = \mathrm{Ext}^{d-i}_{1+\varpi\mathscr{O}_F}\big(M^\vee, k\big) = \mathrm{Ext}^{d-i}_{1+\varpi\mathscr{O}_F}\big(k, M\big),$$

as required. $\square$

Remark 6.8. *We notice that we could have used Lemma 6.6 in order to obtain a generalized version of Proposition 6.7 and hence of [Sch], Proposition 1.8.*

More precisely, if χ is an irreducible $k[[\mathscr{O}_F^\times]]$-module, we have a natural isomorphism of $\mathrm{Mod}^{\mathrm{pro}}(k)$-valued functors defined on $\mathrm{Mod}^{\mathrm{dis}}\big(k[[\mathscr{O}_F^\times]]\big)$:

$$\mathrm{Ext}^i_{\mathscr{O}_F^\times}\big(\circ, \chi\big) \xrightarrow{\sim} \big(\mathrm{Ext}^{d-i}_{\mathscr{O}_F^\times}\big(\chi, \circ\big)\big)^\vee;$$

for this, it suffices to use Lemma 6.6 together with the continuity of the $\mathrm{Ext}^i_{\mathscr{O}_F^\times}(\chi^\vee,\circ)$*-functor (cf. [S-W], Theorem 3.7.2) and the isomorphism (21).*

In particular one deduces from (22) an isomorphism of $\mathrm{Mod}^{\mathrm{pro}}(k)$*-modules*

$$\left(\mathrm{Tor}_i^{\mathscr{O}_F^\times}(\chi,M)\right)^\vee \cong \mathrm{Tor}_{d-i}^{\mathscr{O}_F^\times}\left((M^\vee)^{op},\chi\right)$$

which is natural in $M \in \mathrm{Mod}^{\mathrm{dis}}(k[[\mathscr{O}_F^\times]])$*, generalizing [Sch], Proposition 1.8.*

6.3 Applications

If $F = L$ is a quadratic extension of $\mathbf{Q}_p$ and M is a uniserial $k[[\mathscr{O}_L^\times]]$-module with constant Jordan-Hölder factors, the results of §6.2 can be sharpened, in order to obtain the dimension of some Ext-spaces. The main result is Proposition 6.10, giving the dimension of the Ext-spaces $\mathrm{Ext}^i_{\mathscr{O}_L^\times}(\mathrm{soc}(M),M)$. Therefore the duality established in Proposition 6.7 and the structure theorems of sections 3.4, 4 and 5 let us determine the multiplicity of the Ext-spaces $\mathrm{Ext}^i_{\mathscr{O}_L^\times}(\pi|_{\mathscr{O}_L^\times},\chi)$ for an irreducible $\mathbf{GL}_2(\mathbf{Q}_p)$-representations π and a smooth $\mathscr{O}_L^\times$-character χ (§6.3.1).

We start from the case when M is an irreducible $k[[\mathscr{O}_L^\times]]$-module:

LEMMA 6.9. *Let* χ *be an irreducible* $k[[\mathscr{O}_L^\times]]$*-module. Then*

$$\dim\left(\mathrm{Ext}^i_{\mathscr{O}_L^\times}(\chi,\chi)\right) = \begin{cases} 1 & \text{if } \ i\in\{0,2\} \\ 2 & \text{if } \ i=1 \\ 0 & \text{if } \ i\geqslant 3. \end{cases}$$

Proof. The result is clear for $i \geqslant 3$ (since $\mathscr{O}_L^\times$ has cohomological dimension 2, cf. [Ser65], Corollaire (1)), for $i=0$, hence, by duality, for $i=2$.

Consider the case $i=1$. As for the proof of Lemma 6.6 we can assume that $\chi = k$ is the trivial $k[[\mathscr{O}_L^\times]]$-module.

$$\mathrm{Ext}^1_{\mathscr{O}_L^\times}(\chi,\chi) \cong H^1(\mathscr{O}_L^\times,k).$$

The result follows since $\dim(\mathscr{O}_L^\times)=2$ as a compact p-adic analytic group. $\square$

We are finally able to determine the Ext-spaces for uniserial discrete modules:

PROPOSITION 6.10. *Let* M *be a uniserial discrete* $k[[\mathscr{O}_L^\times]]$*-module of infinite length and* χ *an irreducible* $k[[\mathscr{O}_L^\times]]$*-module. Assume that the Jordan-Hölder factors of* M *are all isomorphic to* χ*.*

Then

$$\dim\left(\mathrm{Ext}^i_{\mathscr{O}_L^\times}(\chi,M)\right) = \begin{cases} 1 & \text{if } \ i\in\{0,1\} \\ 0 & \text{if } \ i\geqslant 2. \end{cases}$$

Proof. Let $\{M_j\}_{j\in\mathbf{N}}$ be the socle filtration for M and consider, for $j\in\mathbf{N}$, the exact sequence of discrete $k[[\mathscr{O}_L^\times]]$-modules

$$0\to M_j\to M_{j+1}\to\chi\to 0. \tag{28}$$

Since $\mathscr{O}_L^\times$ has cohomological dimension $\mathrm{cd}(\mathscr{O}_L^\times)=2$ we have the following exact sequence in cohomology

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\mathscr{O}_L^\times}(\chi,M_j)\to\mathrm{Hom}_{\mathscr{O}_L^\times}(\chi,M_{j+1})\to\mathrm{Hom}_{\mathscr{O}_L^\times}(\chi,\chi)\to\mathrm{Ext}^1_{\mathscr{O}_L^\times}(\chi,M_j)\to \\ &\to \mathrm{Ext}^1_{\mathscr{O}_L^\times}(\chi,M_{j+1})\to\mathrm{Ext}^1_{\mathscr{O}_L^\times}(\chi,\chi)\to\mathrm{Ext}^2_{\mathscr{O}_L^\times}(\chi,M_j)\to\mathrm{Ext}^2_{\mathscr{O}_L^\times}(\chi,M_{j+1})\to\mathrm{Ext}^2_{\mathscr{O}_L^\times}(\chi,\chi)\to 0. \end{aligned} \tag{29}$$

As M is unserial with Jordan-Hölder factors isomorphic to χ we have

$$\dim\left(\mathrm{Hom}_{\mathscr{O}_L^\times}(M_n,\chi)\right) = \dim\left(\mathrm{Hom}_{\mathscr{O}_L^\times}(\chi,M_n)\right) = 1$$

for any $n \in \mathbf{N}$ and hence, by Proposition 6.7, we have $\dim\big(\mathrm{Ext}^2_{\mathscr{O}_L^\times}(\chi, M_n)\big) = 1$ for any $n \in \mathbf{N}$.

It follows that $\mathrm{Hom}_{\mathscr{O}_L^\times}(\chi, M_j) \to \mathrm{Hom}_{\mathscr{O}_L^\times}(\chi, M_{j+1})$, $\mathrm{Ext}^2_{\mathscr{O}_L^\times}(\chi, M_{j+1}) \to \mathrm{Ext}^2_{\mathscr{O}_L^\times}(\chi, \chi)$ are respectively an isomorphism and the zero morphism and (29) simplifies to the following exact sequence

$$0 \to \mathrm{Hom}_{\mathscr{O}_L^\times}(\chi, \chi) \to \mathrm{Ext}^1_{\mathscr{O}_L^\times}(\chi, M_j) \stackrel{\theta_j}{\to} \mathrm{Ext}^1_{\mathscr{O}_L^\times}(\chi, M_{j+1}) \stackrel{\psi_j}{\to} \mathrm{Ext}^1_{\mathscr{O}_L^\times}(\chi, \chi) \to \mathrm{Ext}^2_{\mathscr{O}_L^\times}(\chi, M_j) \to 0. \quad (30)$$

We therefore obtain the required statement for $i \in \{0, 2\}$, from Corollary 6.3. Moreover, we deduce from (30), together with Lemma 6.9 and an immediate induction, that for any $n \in \mathbf{N}$ the linear space $\mathrm{Ext}^1_{\mathscr{O}_L^\times}(\chi, M_n)$ is 2-dimensional.

We claim that for all $n \in \mathbf{N}$ we have a linear space decomposition $\mathrm{Ext}^1_{\mathscr{O}_L^\times}(\chi, M_n) = \langle m_{n+1}\rangle \oplus \langle u_{n+1}\rangle$ such that for any $j \in \mathbf{N}$ the natural morphism θ_j decomposes into the direct sum of the zero morphism $\langle m_{j+1}\rangle \stackrel{0}{\to} \langle m_{j+2}\rangle$ and an isomorphism $\langle u_{j+1}\rangle \stackrel{\sim}{\to} \langle u_{j+2}\rangle$. By Corollary 6.3 this will imply the statement for $i = 1$.

Let $n \geqslant 1$. Using the Yoneda interpretation of Ext, the morphism ψ_{n-1} is defined by the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M_n \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & \chi \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \chi \arrow[r] & \psi_{n-1}(A) \stackrel{\mathrm{def}}{=} A/M_{n-1} \arrow[r] & \chi \arrow[r] & 0.
\end{tikzcd}$$

Since M_{n+1} (which defines an element in $\mathrm{Ext}^1_{\mathscr{O}_L^\times}(\chi, M_n)$) is uniserial, we deduce that $\psi_{n-1}(M_{n+1}) \neq 0$ and therefore, if u_{n+1} is a fixed linear generator for $\mathrm{im}(\theta_{n-1})$ the couple (M_{n+1}, u_{n+1}) defines a linear basis for $\mathrm{Ext}^1_{\mathscr{O}_L^\times}(\chi, M_n)$. For $n = 0$ we simply consider a linear decomposition $\mathrm{Ext}^1_{\mathscr{O}_L^\times}(\chi, \chi) = \langle M_1\rangle \oplus \langle u_1\rangle$ where u_1 is any nonzero element in a linear complement of $\langle M_1\rangle$.

Let $j \in \mathbf{N}$. Again by the Yoneda interpretation of Ext, we see that the image of the morphism $\mathrm{Hom}_{\mathscr{O}_L^\times}(\chi, \chi) \to \mathrm{Ext}^1_{\mathscr{O}_L^\times}(\chi, M_j)$ can be identified with the (class of the) element M_{j+1}. Hence the transition morphism θ_j (which has rank 1) is characterized by the conditions $\theta_j(M_{j+1}) = 0$, $\theta_j(u_{j+1}) = \kappa_{j+1} u_{j+2}$, for an appropriate $\kappa_{j+1} \in k^\times$.

This proves the claim and the result follows. ∎

By duality (Proposition 6.7) we obtain

Corollary 6.11. *Let M be a uniserial discrete $k[[\mathscr{O}_L^\times]]$-module of infinite length and χ an irreducible $k[[\mathscr{O}_L^\times]]$-module. Assume that the Jordan-Hölder factors of M are all isomorphic to χ.*

Then

$$\dim\big(\mathrm{Ext}^i_{\mathscr{O}_L^\times}(M, \chi)\big) = \begin{cases} 0 & \text{if either } \; i = 0 \text{ or } i \geqslant 3 \\ 1 & \text{if } \; i \in \{1, 2\}. \end{cases}$$

Proof. As noticed above, the result for $i \in \{0, 1, 2\}$ is deduced from Proposition 6.7 and 6.10. Concerning the case $i \geqslant 3$, the long exact sequence in cohomology associated to (28) and Lemma 6.9 yield a natural isomorphism

$$\mathrm{Ext}^i_{\mathscr{O}_L^\times}(M_j, \chi) \stackrel{\sim}{\longrightarrow} \mathrm{Ext}^i_{\mathscr{O}_L^\times}(M_{j+1}, \chi)$$

for any $j \in \mathbf{N}$. The result follows now from Corollary 6.3, noticing that $\mathrm{Ext}^i_{\mathscr{O}_L^\times}(M_0, \chi) = 0$ for $i \geqslant 3$ by Lemma 6.9. ∎

6.3.1 ***Multiplicity results.*** Let $L/\mathbf{Q}_p$ be a quadratic extension, π an irreducible, infinite dimensional $\mathbf{GL}_2(\mathbf{Q}_p)$-representation and χ a smooth $L^\times$-character. We are now able to determine the dimension of the Ext-spaces $\mathrm{Ext}^i_{\mathscr{O}_L^\times}\big(\pi|_{\mathscr{O}_L^\times},\chi\big)$, thanks to the structure theorems of sections 3, 4 and 5 and the cohomological methods developed in §6.3.

We start with the unramified situation:

Proposition 6.12. *Assume that the quadratic extension $L/\mathbf{Q}_p$ is unramified. Let π be an infinite dimensional, absolutely irreducible $\mathbf{GL}_2(\mathbf{Q}_p)$-representation and let χ be a smooth $L^\times$-character which extends the central character of π.*

Then

$$\dim\big(\mathrm{Ext}^i_{\mathscr{O}_L^\times}\big(\pi|_{\mathscr{O}_L^\times},\chi|_{\mathscr{O}_L^\times}\big)\big)=\begin{cases}0 & \text{if either } i=0 \text{ or } i\geqslant 3\\ 2 & \text{if } i\in\{1,2\} \text{ and } \pi \text{ is supersingular.}\\ 1 & \text{if } i\in\{1,2\} \text{ and } \pi \text{ is a principal or a special series.}\end{cases}$$

Proof. This is an immediate application of Corollary 3.10, Corollary 5.4 and Corollary 6.11. ☐

In the totally ramified setting the result is similar. Precisely, we have:

Proposition 6.13. *Assume that the quadratic extension $L/\mathbf{Q}_p$ is totally ramified. Let π be an infinite dimensional, absolutely irreducible $\mathbf{GL}_2(\mathbf{Q}_p)$-representation and let χ be a smooth $L^\times$-character which extends the central character of π.*

Then

$$\dim\big(\mathrm{Ext}^i_{\mathscr{O}_L^\times}\big(\pi|_{\mathscr{O}_L^\times},\chi|_{\mathscr{O}_L^\times}\big)\big)=\begin{cases}0 & \text{if either } i=0 \text{ or } i\geqslant 3\\ 4 & \text{if } i\in\{1,2\} \text{ and } \pi \text{ is supersingular.}\\ 2 & \text{if } i\in\{1,2\} \text{ and } \pi \text{ is a principal or special series.}\end{cases}$$

Before giving the proof, we nevertheless need two preliminary lemmas.

Lemma 6.14. *Assume that $L/\mathbf{Q}_p$ is totally ramified. For $\bullet\in\{0,1\}$ the $k[[\mathscr{O}_L^\times]]$-socle of $R_{\infty,\bullet}|_{\mathscr{O}_L^\times}$ is 2-dimensional. The $k[[\mathscr{O}_L^\times]]$-socle of $\mathrm{St}|_{\mathscr{O}_L^\times}$ is 2-dimensional.*

Proof. We consider the case of $R_{\infty,0}$ (the others being analogous) and use the notations of §4.

Recall (cf. (13) in the proof of Proposition 4.1) that the uniserial representations $\mathscr{U}^-_{\infty,0}$, $\mathscr{U}^-_{\infty,1}$ admits linear basis $\mathscr{B}_\infty$, $\mathscr{B}^1_\infty$ which are compatible, in the evident sense, with the socle filtration and verify in particular:

$$\iota(1+\varpi x)\cdot\big(\lambda e_1+\mu f_1\big)=\lambda e_1+\mu f_1-\overline{x}\big(\lambda e_0+\mu f_0\big)\tag{31}$$

for any $x\in\mathscr{O}_L$, $\lambda,\mu\in k$.

The short exact sequence of Proposition 4.1 easily implies the following exact sequence of discrete $k[[\mathscr{O}_L^\times]]$-modules:

$$0\to\langle\overline{e}_0,\overline{f}_0\rangle\to R_{\infty,0}\to\mathscr{U}^-_{\infty,0}/\langle e_0\rangle\oplus\mathscr{U}^-_{\infty,1}/\langle f_0\rangle\to 0$$

where we wrote $\overline{e}_0$, $\overline{f}_0$ to denote the image in $R_{\infty,0}$ of the elements $e_0\in\mathrm{soc}\big(\mathscr{U}^-_{\infty,0}\big)$, $f_0\in\mathrm{soc}\big(\mathscr{U}^-_{\infty,1}\big)$. Notice that $\langle\overline{e}_0,\overline{f}_0\rangle$ is a one dimensional subspace of $R_{\infty,0}$.

Hence, by taking the $(1+\varpi\mathscr{O}_L)$-invariants and recalling that $\mathscr{U}^-_{\infty,\bullet}$ are uniserial, we deduce a monomorphism:

$$\mathrm{soc}\big(R_{\infty,0}\big)\hookrightarrow\langle\overline{e}_0,\overline{f}_0,e_1,f_1\rangle.$$

It follows from (31) and Proposition 4.1 that the space of $(1+\varpi\mathscr{O}_L)$-fixed vectors of $R_{\infty,0}$ is 2-dimensional. ☐

Lemma 6.15. *Let $L/\mathbf{Q}_p$ be totally ramified and let $N, M \in \mathrm{Mod}^{\mathrm{dis}}\big(k[[\mathscr{O}_L^\times]]\big)$.*

Assume that M is uniserial with constant Jordan-Hölder factors, all isomorphic to an irreducible $k[[\mathscr{O}_L^\times]]$-module χ. Assume moreover that N has a 2-dimensional socle and fits in an exact sequence

$$0 \to \chi \to M \oplus M \to N \to 0. \tag{32}$$

Then $\mathrm{Ext}^i_{\mathscr{O}_L^\times}\big(\chi, N\big)$ is 2-dimensional if $i \in \{0,1\}$ and is zero dimensional otherwise.

Proof. It is an elementary computation of the dimension of the Ext-spaces appearing in the the long exact sequence in cohomology associated to (32), using Lemma 6.9, Proposition 6.10 and the hypothesis on the $k[[\mathscr{O}_L^\times]]$-socle of N. $\square$

We can now give the proof of Proposition 6.13:

Proof. We can assume that π is either supersingular or a special series: indeed for a principal series the statement follows immediately from Proposition 5.5 and Corollary 6.11.

Assume first that $i \in \{0,1,2\}$. In this case, by Proposition 6.7 it is equivalent to study the space $\mathrm{Ext}^i_{\mathscr{O}_L^\times}\big(\chi|_{\mathscr{O}_L^\times}, \pi|_{\mathscr{O}_L^\times}\big)$. Moreover, as the dimension of the Ext-spaces does not change by taking twists, we can assume that either $\pi = \pi(r,0,1)$ (supersingular case) or $\pi = \mathrm{St}$ (Steinberg case).

By Proposition 4.1 and Lemma 6.15 we deduce that for $\bullet \in \{0,1\}$ the space $\mathrm{Ext}^i_{\mathscr{O}_L^\times}\big(\chi|_{\mathscr{O}_L^\times}, R_{\infty,\bullet}|_{\mathscr{O}_L^\times}\big)$ is 2-dimensional if $i \in \{0,1\}$ and zero dimensional if $i \geqslant 2$ and we obtain an analogous statement in the Steinberg case by (15) and Proposition 5.5.

We are therefore left with the case $i \geqslant 3$ and this is now an elementary computation: it suffices to count the dimension of the Ext-spaces appearing in the long exact sequences in cohomology associated to the exact sequences of Proposition 4.1 and (15), using Lemma 6.9, Corollary 6.11 (and the known results for $i \leqslant 2$). $\square$

In particular, we see that

Corollary 6.16. *Let $L/\mathbf{Q}_p$ be a quadratic extension and π an admissible $\mathbf{GL}_2(\mathbf{Q}_p)$-representation of finite length, whose Jordan-Hölder factors are absolutely irreducible and infinite dimensional.*

For any smooth $L^\times$-character χ we have

$$\mathrm{Hom}_{L^\times}(\pi|_{L^\times}, \chi) = 0.$$

Proof. An immediate dévissage on the length of π, using Proposition 6.12, Proposition 6.13 and left exactness of the functor $\mathrm{Hom}_{\mathscr{O}_L^\times}\big(\circ, \chi|_{\mathscr{O}_L^\times}\big)$ shows that $\mathrm{Hom}_{\mathscr{O}_L^\times}(\pi|_{\mathscr{O}_L^\times}, \chi|_{\mathscr{O}_L^\times}) = 0$. Hence, *a fortiori* the required statement. $\square$

Remark 6.17. *We notice that, during the proof of Propositions 6.12, 6.13, we implicitly have proved the statement of Theorem 1.7, i.e. we have detected the Ext-spaces $\mathrm{Ext}^i_{\mathscr{O}_L^\times}\big(\chi|_{\mathscr{O}_L^\times}, \pi|_{\mathscr{O}_L^\times}\big)$ (the notations being the same as in the statement of Proposition 6.12, 6.13).*

More precisely, for $L/\mathbf{Q}_p$ unramified, the statement of Theorem 1.7 follows immediately from Corollary 3.10, Corollary 5.4 and Proposition 6.10. Similarly, if $L/\mathbf{Q}_p$ is totally ramified and π is a principal series, the statement follows from Proposition 5.5 and 6.10.

Finally, the case where $L/\mathbf{Q}_p$ is totally ramified and π is either a special series or a supersingular representation has been obtained in the proof of Proposition 6.13.

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