On some representations of the Iwahori subgroup

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Abstract

Let $p \geqslant 5$ be a prime number. In [BL94] Barthel and Livné gave a classification for irreducible representations of $\mathrm{GL}_2(F)$ over $\overline{\mathbf{F}}_p$, for F a p-adic field, discovering some objects, referred to as "supersingular", which appear as subquotients of universal representations $\pi(\underline{r}, 0, 1)$. In this paper we give a detailed description of the Iwahori structure of such universal representations, in the case when F is an unramified extension of $\mathbf{Q}_p$. We determine a fractal structure which shows how and why the techniques used for $\mathbf{Q}_p$ fail and which lets us determine"natural" subrepresentations of the universal object $\pi(\underline{r}, 0, 1)$. As a corollary, we get the Iwahori structure of tamely ramified principal series.

Contents

1. Introduction

Let p be a prime number and F a p-adic field. In the papers [BL94], [BL95] Barthel and Livné studied a classification (recently generalized for general $\mathrm{GL}_n(F)$ by Herzig in [Her]) for the representations of $\mathrm{GL}_2(F)$ with coefficients in an algebraic closure of $\mathbf{F}_p$. Besides characters, principal unramified series and special series, they found a new class of irreducible objects referred to as "supersingular", which are defined, up to twist, as subquotients of a universal representation, which we will note $\pi(\underline{r}, 0, 1)$ for an f-tuple $\underline{r} = (r_0, \ldots, r_{f-1})$, where f is the residual degree of F. The existence of

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supersingular representations is assured by a Zorn-type argument (see [BL95], Proposition 11) and a complete exhaustive study for supersingular representations is a relevant open problem in the emerging p-adic Langlands program. Indeed, in a conjectural mod p-Langlands correspondence it is expected that the supersingular objects are intimately related to Galois representations arising from elliptic curves with supersingular reduction.

This is actually the case if $F = \mathbf{Q}_p$ (when the universal representations are indeed irreducible). Such result is due to Breuil [Bre03] where he reaches a complete classification of supersingular representations thanks to direct computations on the ring of Witt vectors of $\mathbf{F}_p$. If $F \neq \mathbf{Q}_p$ the situation is not clear. For the time being, the problem of classifying supersingular representations looks to be infinitely more involved compared to its Galois analogue (known from the works of Serre [Ser72]). The methods of Paskunas [Pas] and Breuil-Paskunas [Br-Pa], which associate an *infinite* family $\Pi(\rho)$ of supersingular representations to a single Galois object ρ, are a major progress in this direction, but it is not clear, especially after the work of Hu [Hu1], how to distingush in a canonical way a privileged supersingular representation inside $\Pi(\rho)$. We remark that the methods of [Pas] and [Br-Pa] have been improved by Hu's canonical diagrams in [Hu2]; unfortunately canonical diagrams are difficult to calculate explicitly.

Another approach to the problem has been treated by Schein in [Sch] where he studies the universal representations for a totally ramified extension $F/\mathbf{Q}_p$. He detects a natural quotient V_{e-1} of $\pi(\underline{r}, 0, 1)$ which enjoys an universal property with respect to supersingular representations whose $\mathrm{GL}_2(\mathscr{O}_F)$-socle respects a certain combinatorics conjecturally associated to suitable Galois representations arising from elliptic curves with supersingular reduction (the modular weights introduced in [BDJ] and generalised in [Sch1])

In this paper we describe the Iwahori structure for the universal representation $\pi(\underline{r}, 0, 1)$ in the case where $F/\mathbf{Q}_p$ is unramified, generalizing Breuil's method. In particular, our result give the irreducibility for $F = \mathbf{Q}_p$ and shows how and why the universal representations fail to be irreducible otherwise. With "Iwahori structure" we mean that we are able to detect the Iwahori-socle filtration for $\pi(\underline{r}, 0, 1)$ as well as the extension between two consecutive graded pieces. As a byproduct we will deduce the Iwahori structure of principal and special series and the presence of a natural injection $c\text{−}\mathrm{Ind}_{KZ}^{G} V \hookrightarrow \pi(\underline{r}, 0, 1)$. The reader will find out that, as soon as $F \neq \mathbf{Q}_p$, the Iwahori-socle filtration for the universal representation relies on an extremely complicated combinatorics.

The main result of this paper is to show that such combinatorics can be handled with the help of some simple Euclidean data; such a method -a far reaching generalisation of the techniques of [Bre03]- can be briefly described as follow. We detect a natural $\overline{\mathbf{F}}_p$-basis $\mathscr{B}$ of $\pi(\underline{r}, 0, 1)$ as well as an injection:

$$\mathscr{B} \hookrightarrow \mathbf{Z}^{[F:\mathbf{Q}_p]};$$

as we will show, its image $\mathfrak{R}$ is explicitely known. For $v \in \mathscr{B}$ we define the set of antecedents $\mathfrak{S}_v$ of v as the set of $v' \in \mathscr{B}$ such that $v' = v - e_s$ where e_s is the s-th element of the canonical base of $\mathbf{Z}^{[F:\mathbf{Q}_p]}$. When we claim that the Iwahori structure for $\pi(\underline{r}, 0, 1)$ is described by $\mathfrak{R}$ we mean the following facts (see Definition 1.7 for a precise formalism):

i) the Iwahori-socle filtration $\left\{\pi(\underline{r}, 0, 1)_J\right\}$ is obtained from $\mathfrak{R}$ by removing successively the points with empty antecedents;

ii) if $v_0, v_1 \in \mathscr{B}$ and $J \in \mathbf{N}$ are such that v_i is an eigenvector for the $(J-i)$-th graded piece $\pi(\underline{r}, 0, 1)_{J-i}/\pi(\underline{r}, 0, 1)_{J-i-1}$ of the socle filtration of the universal representation, then the linear space $\langle v_0, v_1\rangle$ gives a nontrivial equivariant extension of v_1 by v_0 *inside the quotient* $\pi(\underline{r}, 0, 1)_J/\pi(\underline{r}, 0, 1)_{J-2}$ if and only if v_0 is an antecedent of v_1.

According to this terminology the main result is the following (see Proposition 5.18):

Theorem 1.1. *The Iwahori structure of the universal representations is described by* $\mathfrak{R}$.

We give in Figure 1 the idea of such structure for the quadratic unramified extension of $\mathbf{Q}_p$.

As annonced, we get some other byproducts as

Theorem 1.2. *Let* π *be a tamely ramified principal series and write* $\pi|_I = \pi^+ \oplus \pi^-$ *for the Mackey decomposition deduced from the restriction of* π *to the Iwahori subgroup* I*. Then the Iwahori structure of* π^+ *(resp.* π^-*) is described by the lattice* $\mathbf{N}^{[F:\mathbf{Q}_p]}$*, naturally embedded in* $\mathbf{Z}^{[F:\mathbf{Q}_p]}$.

and

Theorem 1.3. *Let* $\underline{r} \notin \{(0,\dots,0),(p-1,\dots,p-1)\}$ *and let* χ^s *be the conjugate character of* $(\sigma_{\underline{r}})^{U(\mathbf{F}_q)}$*. There is a sub* KZ*-representation* $V \leqslant \pi(\underline{r},0,1)|_{KZ}$ *isomorphic to the kernel of the natural map*

$$\mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{GL}_2(\mathbf{F}_q)}\chi^s/\mathrm{soc}(\mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{GL}_2(\mathbf{F}_q)}\chi^s) \twoheadrightarrow \mathrm{cosoc}(\mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{GL}_2(\mathbf{F}_q)}\chi^s)$$

and such that the map (induced by Frobenius reciprocity)

$$c\mathrm{-Ind}_{KZ}^{G}V \to \pi(\underline{r},0,1)$$

is injective.

We remark that the existence of such an injective morphism has already been discovered by Paskunas in an unpublished draft.

Such results rely on an heavy formalism and they need preparation to be handled. In particular, from section §4 we start using the Euclidean dictionary as a key tool to manage the combinatorics of the representation under study. In order to guide the reader the statements and the proofs are preceded by a detailed translation in Euclidean terms (otherwise they would sound as empty exercises of combinatorics) and each section opens with an exhaustive description of the Euclidean strategy adopted to reach our aims.

The reasons which make our strategy work are essentially three:

i) we detect a suitable basis $\mathscr{B}$ of the universal representation which is well behaved with respect to the action of the Iwahori subgroup and the canonical Hecke operator $T \in \mathrm{End}_G(c\mathrm{-Ind}_{KZ}^{G}\sigma_{\underline{r}})$;

ii) the action of the Iwahori subgroup on the elements of $\mathscr{B}$ can be read through certain universal Witt polynomials whose homogeneous degree is known;

iii) the correspondence between the elements of the basis $\mathscr{B}$ and integer points in $\mathbf{R}^{[F:\mathbf{Q}_p]}$ is compatible with the homogeneous degree of the polynomials of *ii*).

We hope the tehcniques introduced in this paper can be the starting point for further developments in the conjectural mod p Langlands programm. In particular our approach in terms of "harmonic analysis", developed here by explicit methods, should be adapted in a more general setting, for instance for GL_n and for any finite extension of $\mathbf{Q}_p$. Nevertheless, the combinatoric for such generalisations should become soon extremely intricate and we think it can not reasonably be handled without the use of a more general and synthetic framework on harmonic analysis on the Bruhat-Tits building.

We believe that suitable comparisons between Breuil-Paskunas methods and the explicit description of $\pi(\underline{r},0,1)$ in terms of the basis $\mathscr{B}$ could shed new light in the research of good supersingular representaitons of $\mathrm{GL}_2(F)$. A natural questions, suggested by the referee, is the following: given a supersingular Breuil-Paskunas representation π and a Serre weight σ appearing as a subobject of $\pi|_{KZ}$, is it possible to describe the kernel of the natural map $\pi(\sigma,0,\mu) \twoheadrightarrow \pi$ in terms of the Euclidean

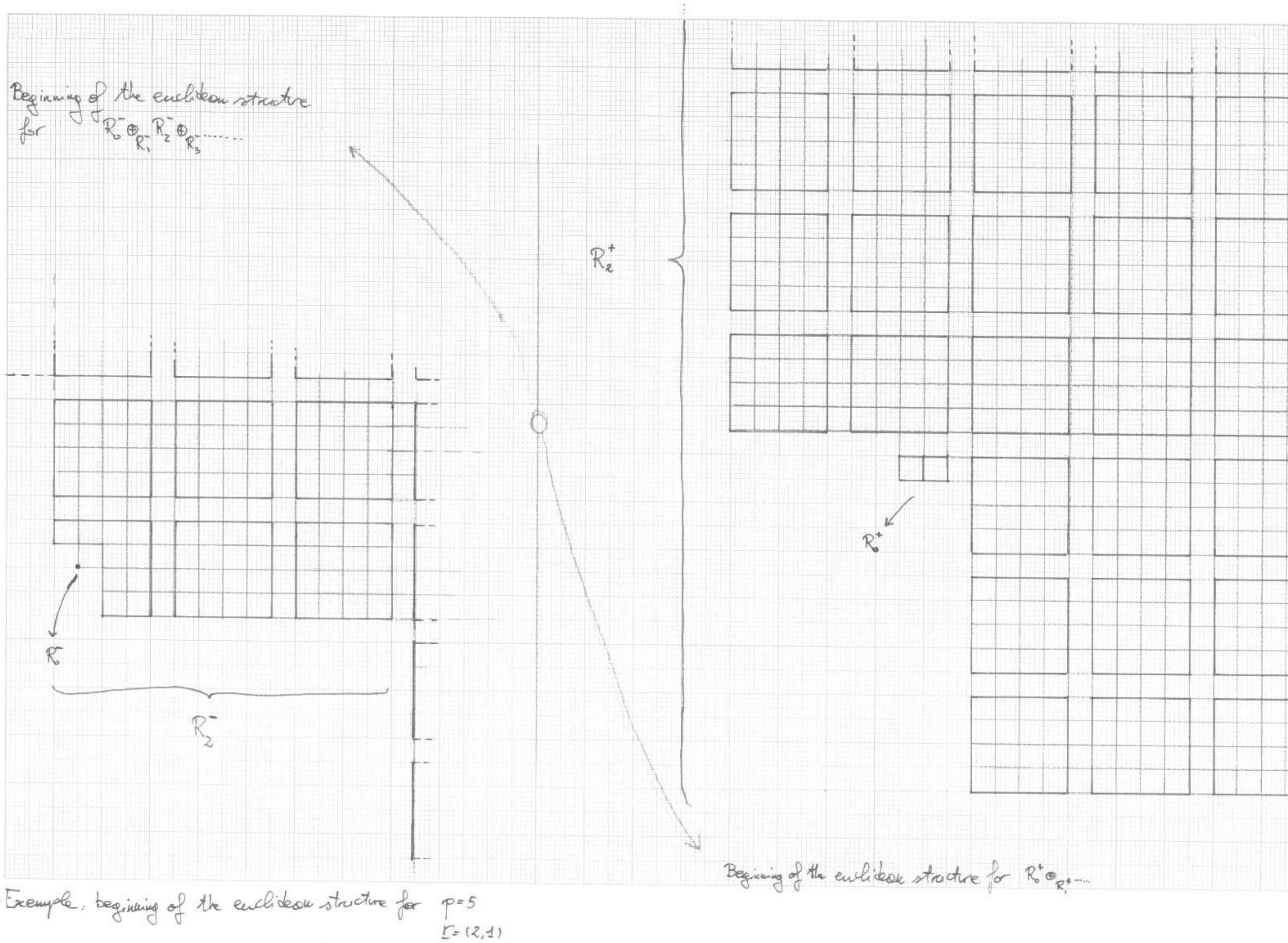


Figure 1: The picture represents part of the Euclidean structure associated to $\pi(\underline{r},0,1)$, in the particular case $p=5$, $f=2$, $\underline{r}=(2,1)$, according to the decomposition of $\pi(\underline{r},0,1)$ given by Propositions 2.9 and 3.5. The axes let us parametrise certains elements of the compact induction according to the immersions of $\mathbf{F}_{p^2}$ in $\overline{\mathbf{F}}_p$. The Iwahori-socle for $\pi(\underline{r},0,1)$ is deduced by the points having empty antecedent, according to Definition 1.7.

structure associated to $\pi(\sigma, 0, \mu)$? (here μ is a convenient unramified character of Z, choosen in such a way that the central character of $\pi(\sigma, 0, \mu)$ and π coincide).

The precise terms of such comparison are still unclear to the author. We suppose they should clarify the nature of the several parameters appearing in the constructions of [Br-Pa], giving a hierarchy which should be hopefully deduced from the Euclidean structure associated to $\pi(\underline{r}, 0, 1)$.

The structure of the paper is then the following.

The first two sections §2 and §3 are formal and do not need the hypothesis $F/\mathbf{Q}_p$ unramified. Section §2 is essentially a dictionary which let us detect a natural equivariant filtration on the KZ-restriction of the universal representation. We first introduce a family of KZ-representations $\{R_n\}_{n\in\mathbf{N}}$. Through some convenient Hecke operators $T_n^{\pm} : R_n \to R_{n\pm 1}$ we define inductively a direct system of amalgamated sums (each of them endowed with a natural filtration) which leads to an explicit isomorphism (Proposition 2.9):

$$\pi(\sigma_{\underline{r}}, 0, 1)|_{KZ} \xrightarrow{\sim} \varinjlim_{n\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}) \oplus \varinjlim_{n\,\mathrm{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}).$$

We remark that such isomorphism was already drafted by Breuil in [Bre].

In section 3 we start from an Iwahori-splitting $R_{n+1} = R_{n+1}^{+} \oplus R_{n+1}^{-}$ to deduce, in the same flavour of the preceeding section, an inductive system of amalgamated sums $\cdots \oplus_{R_n^{\pm}} R_{n+1}^{\pm}$. Such amalgamated sums are endowed with a natural Iwahori-filtration revealed by a short exact sequence

$$0 \to \cdots \oplus_{R_{n-2}^{\pm}} R_{n-1}^{\pm} \to \cdots \oplus_{R_n^{\pm}} R_{n+1}^{\pm} \to R_{n+1}^{\pm}/R_n^{\pm} \to 0. \tag{1}$$

The resulting inductive limits are related to the universal representation by the following

Proposition 1.4. *We have an exact Iwahori-equivariant sequence*

$$0 \to \langle (v_+, v_-)\rangle_{\overline{\mathbf{F}}_p} \to (\varinjlim_{n\,\mathrm{odd}} R_0^{+} \oplus_{R_1^{+}} \cdots \oplus_{R_n^{+}} R_{n+1}^{+}) \oplus (\varinjlim_{n\,\mathrm{odd}} R_0^{-} \oplus_{R_1^{-}} \cdots \oplus_{R_n^{-}} R_{n+1}^{-}) \to$$
$$\to (\varinjlim_{n\,\mathrm{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})|_{K_0(p)} \to 0$$

where $v_{\pm} \in \varinjlim_{n\,\mathrm{odd}} R_0^{\pm} \oplus_{R_1^{\pm}} \cdots \oplus_{R_n^{\pm}} R_{n+1}^{\pm}$ *(and are explicitly known).*

We have an analogous result in the even case.

It will therefore be enough to focus our attention on the inductive limits of section §3.
The Euclidean dictionary is developed in section 4. Thanks to the natural filtration on the inductive limits, we are primarly concerned with the Iwahori structure of the representations $R_{n+1}^{\pm}$. We detect a convenient $\overline{\mathbf{F}}_p$-basis $\mathscr{B}_{n+1}^{\pm}$ (Lemma 2.6) and determine a natural way to identify the elements of $\mathscr{B}_{n+1}^{\pm}$ with integer valued points of $\mathbf{R}^{[F:\mathbf{Q}_p]}$ (see section 4.1.1 for details). If we write $\mathscr{R}_{n+1}^{\pm}$ to denote the image of $\mathscr{B}_{n+1}^{\pm}$ in the $[F:\mathbf{Q}_p]$-dimensional real Euclidean space (such an image looks as a parallelepipoid of side $p^{n+\epsilon}(\underline{r}+\underline{1})$ for $\epsilon \in \{0,1\}$ according to the cases R_{n+1}^{+}, R_{n+1}^{-}) then

Proposition 1.5. *The structure* $\mathscr{R}_{n+1}^{\pm}$ *describes, according to Definition 1.7, the Iwahori structure of* $R_{n+1}^{\pm}$.

Because of the geometry of the polytope $\mathscr{R}_{n+1}^{\pm}$ we indeed see that the socle filtration can be detected by successive cuttings by suitable hyperplanes (parallel to the antidiagonal $X_0+\cdots+X_{f-1} = 0$).

We similarly deduce the structure of tamely ramified principal series (Proposition 1.2). Unfortunately, these results rely on a careful analysis of the behaviour of some universal Witt polynomials,

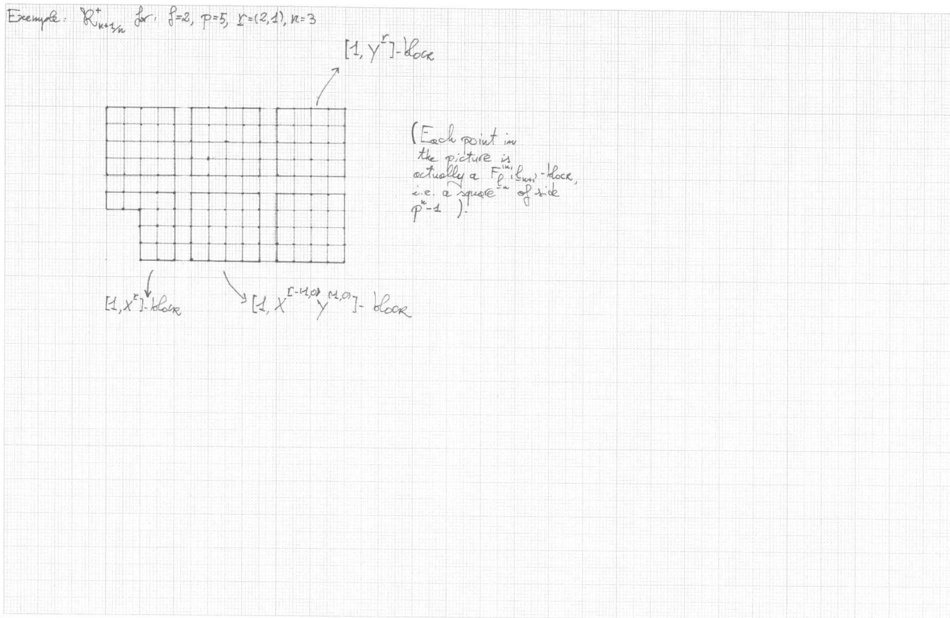


FIGURE 2: This is the structure associated to the quotient R^+_{n+1}/R^+_n, in the particular case $p=5$, $f=2$, $\underline{r}=(2,1)$. The structure of such quotients is more complicated compared to R^+_{n+1}; it is deduced from a delicate subdivision of R^+_{n+1}/R^+_n into increasing subspaces, suggested by the geometry of $\mathscr{R}^+_{n+1/n}$.

contained in the two appendices A and B.

Section §5 deals finally with the universal representation $\pi(\underline{r},0,1)$. We are first concerned with the graded pieces of the natural filtrations introduced in §3: it is the object of §5.1. Thanks to the behaviour of the canonical basis $\mathscr{B}^\pm_n$ with respect to the Hecke operators of §3 we easily determine a natural basis $\mathscr{B}^\pm_{n+1/n}$ for each $R^\pm_{n+1}/R^\pm_n$ and we associate an Euclidean structure $\mathscr{R}^\pm_{n+1/n}$ to it. Such a structure is more complicated than the previous $\mathscr{R}^\pm_{n+1}$ and can not be determined directely by Proposition 1.5 but a suitable decomposition of $\mathscr{R}^\pm_{n+1/n}$ as a union of inreasing polytopes enable us to state the

PROPOSITION 1.6. *The structure $\mathscr{R}^\pm_{n+1/n}$ describes, according to Definition 1.7, the Iwahori structure of $R^\pm_{n+1}/R^\pm_n$.*

An example, for $\underline{r}=(2,1)$, of the Euclidean image of $\mathscr{R}^+_{n+1/n}$ is given in Figure 2.

As a byproduct, the natural filtrations of section §3 and the previous description of the basis $\mathscr{B}^\pm_{n+1}$ let us deduce Proposition 1.3.

The conclusion is in section §5.2 where we study the amalgamated sums $\cdots \oplus_{R^\pm_n} R^\pm_{n+1}$. Again, the behaviour of the canonical base $\mathscr{B}^\pm_n$ with respect to the Hecke operators let us deduce, by induction on the exact sequence (1), an Euclidean structure, say $\mathfrak{R}^\pm_{\text{even,odd}}$. Such a structure has a regular fractal nature, due to a convenient glueing of the blocks $\mathscr{R}^\pm_{n+1/n}$. *Simple* remarks on the geometry of $\mathfrak{R}^\pm_{\text{even,odd}}$, as well as the fact that $\cdots \oplus_{R^\pm_{n-2}} R^\pm_{n-1}$ is a Iwahori-*subrepresentation* of $\cdots \oplus_{R^\pm_n} R^\pm_{n+1}$, let us deduce the main result of Theorem 1.1.

We introduce now the basic conventions and notations of the paper (we essentially use the formalism and notations of [Bre03]).

Fix a prime [1] $p \geqslant 5$ and let F be a finite unramified extension of $\mathbf{Q}_p$; let $f \overset{\text{def}}{=} [F:\mathbf{Q}_p]$ be the residue degree. We write $\mathscr{O}_F$ to denote the ring of integers of F and fix the uniformizer $p\in\mathscr{O}_F$: let k_F be the residue field; it is a finite field with $q \overset{\text{def}}{=} p^f$ elements. We fix an isomorphism $k_F\cong\mathbf{F}_q$; as F is unramified, we deduce an isomorphism $\mathscr{O}_F\cong W(\mathbf{F}_q)$ where $W(\mathbf{F}_q)$ denotes the ring of Witt vectors of $\mathbf{F}_q$. We will write $[\cdot]:\mathbf{F}_q^\times\rightarrow W(\mathbf{F}_q)^\times$ to denote the Teichmüller character (putting $[0]\overset{\text{def}}{=}0$). We finally fix an algebraic closure $\overline{\mathbf{F}}_p$ of $\mathbf{F}_q$.

For any $k\in\mathbf{N}$ the natural action of $\mathrm{GL}_2(\mathbf{F}_q)$ on $\mathbf{F}_q^2$ lets us determine, by functoriality of the k-th symmetric power, the $\mathrm{GL}_2(\mathbf{F}_q)$-representation $\mathrm{Sym}^k\mathbf{F}_q^2$. It is isomorphic (up to a choice of an $\mathbf{F}_q$-basis for $\mathbf{F}_q^2$) to $\mathbf{F}_q[X,Y]_k^h$, the homogeneous component of degree k of the ring $\mathbf{F}_q[X,Y]$, endowed with the usual modular action:

$$\begin{bmatrix} a & b\\ c & d\end{bmatrix} X^{k-i}Y^i=(aX+cY)^{k-i}(bX+dY)^i.$$

We recall that for $s\in\mathbf{N}$, $(\mathbf{F}_q[X,Y]_k^h)^{Frob^s}$ is the representation obtained by functoriality, in the evident way, from the field automorphism $x\mapsto x^{p^s}$ defined on $\mathbf{F}_q$.

For $\tau\in\mathrm{Gal}(\mathbf{F}_q/\mathbf{F}_p)$ and $r_\tau,t_\tau\in\{0,\dots,p-1\}$ we consider the $\mathrm{GL}_2(\mathbf{F}_q)$-representation

$$\sigma_{\{r_\tau\},\{t_\tau\}}\overset{\text{def}}{=}\bigotimes_{\tau\in\mathrm{Gal}(\mathbf{F}_q/\mathbf{F}_p)}(\det{}^{t_\tau}\otimes_{\mathbf{F}_q}\mathrm{Sym}^{r_\tau}\mathbf{F}_q^2)\otimes_\tau\overline{\mathbf{F}}_p;$$

such representations, called *Serre weights*, exhaust all irreducible $\mathrm{GL}_2(\mathbf{F}_q)$-representations with coefficients in $\overline{\mathbf{F}}_p$ (and they are pairwise non isomorphic if we impose $t_\tau<p-1$ for at least one element $\tau\in\mathrm{Gal}(\mathbf{F}_q/\mathbf{F}_p)$). A Serre weight is said to be regular if $1\leqslant r_i\leqslant p-3$ for all i (see [Gee], Definition 2.1.5).

We fix once for all an immersion $\tau:\mathbf{F}_q\hookrightarrow\overline{\mathbf{F}}_p$. Such a choice determines, up to twist, a manifest isomorphism

$$\sigma_{\{r_\tau\},\{t_\tau\}}\cong\sigma_{(r_0,\dots,r_{f-1})}\overset{\text{def}}{=}\bigotimes_{s=0}^{f-1}(\overline{\mathbf{F}}_p[X_s,Y_s]_{r_s}^h)^{Frob^s}$$

for a suitable $\underline{r}\overset{\text{def}}{=}(r_0,\dots,r_{f-1})\in\{0,\dots,p-1\}^f$; such an isomorphism will be assumed to be fixed once for all throughout the paper. We notice that the choice of another immersion acts on the right hand side by a cyclic permutation on the indices s in the obvious sense.

Write $G\overset{\text{def}}{=}\mathrm{GL}_2(F)$, $K\overset{\text{def}}{=}\mathrm{GL}_2(\mathscr{O}_F)$ and $Z\overset{\text{def}}{=}Z(G)$. We write $K_0(p)$ to denote the Iwahori subgroup of K. The $\mathrm{GL}_2(\mathbf{F}_q)$-representation $\sigma_{\underline{r}}$ will be seen, by the inflation map $K\twoheadrightarrow\mathrm{GL}_2(\mathbf{F}_q)$, as a smooth representation of K. By imposing $p\in Z$ to act trivially, the smooth K-action on $\sigma_{\underline{r}}$ extends to a smooth action of KZ: by abuse of notation we will write $\sigma_{\underline{r}}$ to denote either the $\mathrm{GL}_2(\mathbf{F}_q)$, the K or the KZ-representation obtained by this procedure (or, as usual, the underlying vector space of $\sigma_{\underline{r}}$).

Similarly, the character

$$\begin{aligned}\chi_{\underline{r}}:B(\mathbf{F}_q)&\rightarrow\overline{\mathbf{F}}_p^\times\\ \begin{bmatrix} a & b\\ 0 & d\end{bmatrix}&\mapsto a^{\sum_{s=0}^{f-1}p^sr_s}\end{aligned}$$

will be considered, by inflation as a character of any open subgroup of $K_0(p)$. We write then $\chi_{\underline{r}}^s$ to

[1]For a technical reason, the case $p=3$ is slightly more delicate: see the note in Proposition 4.7.

denote the conjugate character of $\chi_{\underline{r}}$. We denote by $\mathfrak{a}$ the character

$$\begin{aligned} B(\mathbf{F}_q) &\to \overline{\mathbf{F}}_p^{\times} \\ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} &\mapsto ad^{-1}. \end{aligned}$$

Recall the compact induction:

$$c\text{--}\mathrm{Ind}_{KZ}^{G}\sigma_{\underline{r}}$$

defined as the $\overline{\mathbf{F}}_p$-linear space of functions $f : G \to \sigma_{\underline{r}}$, compactly supported modulo Z, verifying $f(kg) = k \cdot f(g)$ for any $k \in K$, $g \in G$; it is endowed with the smooth left action of G defined by right translations.

For $g \in G$, $v \in \sigma_{\underline{r}}$ we define $\big[g, v\big] \in c\text{--}\mathrm{Ind}_{KZ}^{G}\sigma_{\underline{r}}$ as the unique function f supported in KZg^{-1} and such that $f(g^{-1}) = v$. Then we have

$$\begin{aligned} g' \cdot \big[g, v\big] &= \big[g'g, v\big] && \text{for } g' \in G \\ \big[gk, v\big] &= \big[g, k \cdot v\big] && \text{for } k \in KZ. \end{aligned}$$

Each function $f \in c\text{--}\mathrm{Ind}_{KZ}^{G}\sigma_{\underline{r}}$ can be written as a $\overline{\mathbf{F}}_p$-linear combination of a finite family of functions $\big[g, v\big]$; if g varies in a fixed system of coset representatives for G/KZ and v varies in a fixed $\overline{\mathbf{F}}_p$-basis of $\sigma_{\underline{r}}$ the aforementioned writing is then unique.

We leave to the reader the task to adapt the previous definitions and remarks to such objects as

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\tau$$

where $K_0(p^{n+1}) \overset{\circ}{\leqslant} K_0(p^m) \overset{\circ}{\leqslant} K$ are the open subgroups of K defined by (3) and τ is a smooth representation of $K_0(p^{n+1})$.

From [BL94], Proposition 8-(1) there exists a canonical Hecke operator (depending on $\underline{r}$) $T \in \mathrm{End}_G(c\text{--}\mathrm{Ind}_{KZ}^{G}\sigma_{\underline{r}})$. It realizes an isomorphism between the $\overline{\mathbf{F}}_p$-algebra of endomorphisms $\mathrm{End}_G(c\text{--}\mathrm{Ind}_{KZ}^{G}\sigma_{\underline{r}})$ and the ring of polynomials in one variable over $\overline{\mathbf{F}}_p$. We then define the universal representation of $\mathrm{GL}_2(F)$ as the cokernel of the canonical operator T:

$$\pi(\underline{r}, 0, 1) \overset{\text{def}}{=} \mathrm{coker}(T).$$

We recall some conventions on the multiindex notations. We write $\underline{\alpha} \overset{\text{def}}{=} (\alpha_0, \ldots, \alpha_{f-1})$ to denote an f-tuple $\underline{\alpha} \in \mathbf{N}^f$. If $\underline{\alpha}, \underline{\beta}$ are f-tuples we define

i) $\underline{\alpha} + \underline{\beta} \overset{\text{def}}{=} (\alpha_0 + \beta_0, \ldots, \alpha_{f-1} + \beta_{f-1})$;
ii) $\underline{\alpha} \geqslant \underline{\beta}$ if and only if $\alpha_s \geqslant \beta_s$ for all $s \in \{0, \ldots, f-1\}$;
iii) $\binom{\underline{\alpha}}{\underline{\beta}} \overset{\text{def}}{=} \prod_{s=0}^{f-1} \binom{\alpha_s}{\beta_s}$.

For $n \in \mathbf{N}$ we will write $\underline{n} \overset{\text{def}}{=} (n, \ldots, n) \in \mathbf{N}^f$.

If $\underline{\alpha} + \underline{\beta} = \underline{r}$ we define the following element of $\sigma_{\underline{r}}$:

$$X^{\underline{\alpha}} Y^{\underline{\beta}} \overset{\text{def}}{=} \otimes_{s=0}^{f-1} X_s^{\alpha_s} Y_s^{\beta_s};$$

for $\lambda \in \mathbf{F}_q$ and $\underline{\alpha} \in \{0, \ldots, p-1\}^f$ we put

$$\lambda^{\underline{\alpha}} \overset{\text{def}}{=} \lambda^{\sum_{s=0}^{f-1} p^s \alpha_s}.$$

For an integer $n \in \mathbf{N}$ we define $\lfloor n \rfloor \in \{0, \ldots, f-1\}$ as the unique integer $m \in \{0, \ldots, f-1\}$ congruent to n modulo f. Similarly, if $n \neq 0$ we define $\lceil n \rceil \in \{1, \ldots, q-1\}$ as the unique integer $m \in \{1, \ldots, q-1\}$ congruent to n modulo $q-1$; we set $\lceil 0 \rceil \overset{\text{def}}{=} 0$.

Let R be a smooth representation of $K_0(p)$ over $\overline{\mathbf{F}}_p$. We recall the definition of the *socle filtration* $\{\mathrm{soc}_N(R)\}_{N\in\mathbf{N}}$ on R: we set $\mathrm{soc}_0(R)=\mathrm{soc}(R)$ (the maximal semisimple subrepresentation of R) and, assuming $\mathrm{soc}_N(R)$ being defined, the submodule $\mathrm{soc}_{N+1}(R)$ is defined to be the inverse image of $\mathrm{soc}(R/\mathrm{soc}_N(R))$ via the natural projection $R\twoheadrightarrow R/\mathrm{soc}_N(R)$. We set formally $\mathrm{soc}_{-1}(R)=0$. We therefore get an increasing, exhausting and separate filtration on R, with semisimple layers.

Throughout the paper we describe the socle filtration of R by means of subset $\mathscr{R}\subseteq\mathbf{Z}^f$ suitably associated to R: this is a crucial formalism whose meaning we define precisely in the following Definition.

Definition 1.7. *Let $\mathscr{B}$ be an $\overline{\mathbf{F}}_p$-basis of R and P a bijection of $\mathscr{B}$ onto a subset $\mathscr{R}$ in $\mathbf{Z}^f$. Let $\mathscr{B}'\subseteq\mathscr{B}$ be a subset and $\mathscr{R}'$ denote its image through the bijection P; for $v\in\mathscr{B}'$ we define the set of antecedents of v in $\mathscr{R}'$ as:*

$$\mathfrak{S}_v(\mathscr{B}')\overset{\text{def}}{=}\big\{w\in\mathscr{B}'\ \text{s.t.}\ P(w)=P(v)-e_s\,\text{for}\,s\in\{0,\dots,f-1\}\big\}$$

(where $\{e_0,\dots,e_{f-1}\}$ is the canonical basis of $\mathbf{Z}^f$).

1) *We say that* the socle filtration $\{\mathrm{soc}_N(R)\}_{N\in\mathbf{N}}$ of R is described by $\mathscr{R}$ *if the following holds: there exists an increasing family $\{\mathscr{B}_N\}_{N\in\mathbf{N}}$ of subsets of $\mathscr{B}$ such that*
   i) for all $N\in\mathbf{N}$ the family $\mathscr{B}_N$ is an $\overline{\mathbf{F}}_p$-basis of $\mathrm{soc}_N(R)$;
   ii) for all $N\in\mathbf{N}$ an $\overline{\mathbf{F}}_p$-basis for $\mathrm{soc}(R/\mathrm{soc}_{N-1}(R))$ is described as
   $$\big\{v\in\mathscr{B}\setminus\mathscr{B}_{N-1},\quad \text{s.t.}\,\mathfrak{S}_v(\mathscr{B}\setminus\mathscr{B}_{N-1})=\emptyset\big\}.$$
2) *If the socle filtration of R is described by $\mathscr{R}$ we will say that* the extensions between two graded pieces are described by $\mathscr{R}$ *if the following holds true:*

   for all $N\in\mathbf{N}$ and $v\in\mathscr{B}_{N+1}$ the $\overline{\mathbf{F}}_p$-linear subspace $E_{v,N}$ of $R/\mathrm{soc}_{N-1}(R)$ generated by $v,\mathfrak{S}_v(\mathscr{B}\setminus\mathscr{B}_{N-1})$ is $K_0(p)$-stable and for each $w\in\mathfrak{S}_v(\mathscr{B}\setminus\mathscr{B}_{N-1})$ the induced extension
   $$0\to\overline{w}\to E_{v,N}/\langle\mathfrak{S}_v(\mathscr{B}\setminus\mathscr{B}_{N-1})\setminus\{w\}\rangle_{\overline{\mathbf{F}}_p}\to\overline{v}\to 0$$
   is nonsplit (with the obvious meaning of $\overline{w}$, $\overline{v}$).

In Euclidean terms, the meaning of Definition 1.7 is the following:

1) the socle filtration of R is obtained from $\mathscr{R}$ by removing successively the points having empty antecedent: a linear basis $\mathscr{B}_0$ for $\mathrm{soc}_0(R)$ is described by the points of $\mathscr{R}$ having empty antecedent; assuming we have a linear basis $\mathscr{B}_N$ for $\mathrm{soc}_N(R)$ then a linear basis for $\mathrm{soc}_{N+1}(R)$ is given by the disjoint union of $\mathscr{B}_N$ and the points of $P(\mathscr{B}\setminus\mathscr{B}_N)$ having empty antecedent;
2) the segments between v and the set of its antecedents let us detemines all the nonsplit extensions between two graded pieces of the socle filtration.

By abuse of terminology, we will call *lattice* of $\mathbf{R}^f$ a *subset* of $\mathbf{Z}^f$ (the latter being naturally embedded in $\mathbf{R}^f$) containing a linear base for $\mathbf{R}^f$. The subset $\mathscr{R}$ will be often callet the *associated lattice* for the representation R.

2. Preliminaries

As we outlined in the introduction, the main aim of this section is to describe the Iwahori-structure of the universal representations $\pi(\underline{r},0,1)$ of $\mathrm{GL}_2(F)$ over $\overline{\mathbf{F}}_p$.

Such representations have a completely explicit description in terms of the Bruhat-Tits tree and of the Hecke operator T given in [Bre03], §2 and their Iwahori structure can indeed be found by direct methods. Nevertheless, the extremely involved combinatorics of such results leads us to introduce an intermediary step -namely a suitable KZ-filtration- which lets us handle, in a reasonable way, the high amount of technical computations. Precisely, we start (cf. definition 2.3) by introducing the KZ-representations

$$R_{n+1} \stackrel{\text{def}}{=} \mathrm{Ind}_{K_0(p^{n+1})}^{K} \sigma_{\underline{r}}^{(n+1)}$$

(where $\sigma_{\underline{r}}^{(n+1)}$ is a $K_0(p^{n+1})$-representation obtained by twisting the action of $K_0(p^{n+1})$ on $\sigma_{\underline{r}}|_{K_0(p^{n+1})}$). Such objects are endowed with an action of suitable "Hecke" operators $T_n^{\pm}: R_n \to R_{n\pm 1}$ (cf. Lemma 2.7), with respect to which we are able to define (inductively) a direct system of amalgamated sums $\cdots \oplus_{R_n} R_{n+1}$ (cf. Proposition 2.8). Such amalgamated sums fit in a natural commutative diagram (see Proposition 2.8) which lets us deduce a natural filtration on the resulting inductive limits. The final result is then the isomorphism of Proposition 2.9, which relates the KZ-restriction of the universal representation $\pi(\underline{r}, 0, 1)|_{KZ}$ to the inductive limits constructed above; in particular, we have a natural KZ-equivariant filtration on the universal representation $\pi(\underline{r}, 0, 1)$.

In Lemma 2.6 we introduce a "canonical" basis for the representations R_{n+1}. Such basis is well behaved with respect to both the action of the Hecke operators and the action of the Iwahori subgroup: this will be the key observation which lead us to the description of the Iwahori structure for $\pi(\underline{r}, 0, 1)$.

We remark that the isomorphism of Proposition 2.9 does not rely on the fact that $F/\mathbf{Q}_p$ is unramified: the content of this section can be generalised in the evident manner for any finite extension F of $\mathbf{Q}_p$.

Reminders on the universal representations $\pi(\underline{r}, 0, 1)$**.** For $n \in \mathbf{N}_{\geqslant 1}$ we define

$$I_n \stackrel{\text{def}}{=} \{\sum_{j=0}^{n-1} p^j [\lambda_j] \quad \text{for } \lambda_j \in \mathbf{F}_q\}$$

and we put $I_0 \stackrel{\text{def}}{=} \{0\}$. The sets I_n let us describe the Bruhat-Tits tree in the following way: if $n, m \in \mathbf{N}$, $\lambda \in I_n$ and

$$g_{n,\lambda}^0 \stackrel{\text{def}}{=} \begin{bmatrix} p^n & \lambda \\ 0 & 1 \end{bmatrix}, \qquad g_{n,\lambda}^1 \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ p\lambda & p^{n+1} \end{bmatrix}$$

we get a decomposition

$$KZ\alpha^{-m}KZ = \coprod_{\lambda \in I_m} g_{m,\lambda}^0 KZ \coprod \coprod_{\lambda \in I_{m-1}} g_{m,\lambda}^1 KZ \tag{2}$$

thus describing the vertex of the tree having distance m from KZ (where we have written $\alpha \stackrel{\text{def}}{=} g_{0,0}^1$). The canonical Hecke operator $T \in \mathrm{End}_G(\mathrm{Ind}_{KZ}^G \sigma_{\underline{r}})$, defined in [Bre03] §2.7, is then characterized as follow:

Lemma 2.1. *For $n \in \mathbf{N}_{>}$, $\lambda \in I_n$ and $0 \leqslant \underline{j} \leqslant \underline{r}$ we have:*

$$T([g_{n,\lambda}^0, X^{\underline{r}-\underline{j}} Y^{\underline{j}}]) = \sum_{\lambda_n \in \mathbf{F}_q} [g_{n+1,\lambda+p^n[\lambda_n]}^0, (-\lambda_n)^{\underline{j}} X^{\underline{r}}] + [g_{n-1,[\lambda]_{n-1}}^0, \delta_{\underline{j},\underline{r}} (\lambda_{n-1} X + Y)^{\underline{r}}]$$

$$T([g_{n,\lambda}^1, X^{\underline{r}-\underline{j}} Y^{\underline{j}}]) = \sum_{\lambda_n \in \mathbf{F}_q} [g_{n+1,\lambda+p^n[\lambda_n]}^1, (-\lambda_n)^{\underline{r}-\underline{j}} Y^{\underline{r}}] + [g_{n-1,[\lambda]_{n-1}}^1, \delta_{\underline{j},\underline{0}} (X + \lambda_{n-1} Y)^{\underline{r}}]$$

where $[\cdot]_{n-1}: I_n \to I_{n-1}$ denotes the truncation of the $(n-1)$-th p-adic digit.

If $n=0$ we have

$$T([1_G, X^{\underline{r}-\underline{j}}Y^{\underline{j}}]) = \sum_{\lambda_0\in\mathbf{F}_q}[g^0_{1,[\lambda_0]}, (-\lambda_0)^{\underline{j}}X^{\underline{r}}] + [\alpha, \delta_{\underline{j},\underline{r}}Y^{\underline{r}}]$$
$$T([\alpha, X^{\underline{r}-\underline{j}}Y^{\underline{j}}]) = \sum_{\lambda_1\in\mathbf{F}_q}[g^1_{1,[\lambda_1]}, (-\lambda_1)^{\underline{r}-\underline{j}}Y^{\underline{r}}] + [1_G, \delta_{\underline{j},\underline{0}}X^{\underline{r}}]$$

.

Proof. A computation shows that the statement of Lemme 3.1.1 in [Bre03] has an obvious generalisation for $f>1$. The result follows then from *Ibid.*, §2.5. □

For $n\in\mathbf{N}$ we define the $\overline{\mathbf{F}}_p$-subspace of $\mathrm{Ind}_{KZ}^G\sigma_{\underline{r}}$:

$$W(n)\overset{\text{def}}{=}\{f\in \mathrm{Ind}_{KZ}^G\sigma_{\underline{r}},\quad \text{s.t. the support of } f \text{ is contained in } KZ\alpha^{-n}KZ\}.$$

By Cartan decomposition the subspaces $W(n)$ are KZ-stable for all $n\in\mathbf{N}$ and therefore

Lemma 2.2. *There is a natural KZ-equivariant isomorphism*

$$\mathrm{Ind}_{KZ}^G\sigma_{\underline{r}} \overset{\sim}{\to} \bigoplus_{n\in\mathbf{N}} W(n).$$

The representations R_n's and the dictionary. Let $n\in\mathbf{Z}_{\geqslant -1}$; we define the open subgroups of K:

$$K_0(p^{n+1})\overset{\text{def}}{=}\left\{g\in K,\ \text{s.t. } g=\begin{bmatrix} a & b\\ p^{n+1}c & d\end{bmatrix} \text{ for } a,b,c,d\in\mathscr{O}_F\right\}. \tag{3}$$

As $\begin{bmatrix} 0 & 1\\ p^{n+1} & 0\end{bmatrix}$ normalizes $K_0(p^{n+1})$, the representation $\sigma_{\underline{r}}|_{K_0(p^n)}$ induces, by conjugation, a $K_0(p^{n+1})$-representation which will be denoted as $\sigma_{\underline{r}}^{n+1}$ (or simply $\sigma_{\underline{r}}$ if there is no risk of confusion). Explicitly, we have

$$\sigma_{\underline{r}}^{(n+1)}\left(\begin{bmatrix} a & b\\ p^{n+1}c & d\end{bmatrix}\right) = \sigma_{\underline{r}}\left(\begin{bmatrix} d & c\\ p^{n+1}b & a\end{bmatrix}\right).$$

We can therefore introduce the representations R_{n+1}:

Definition 2.3. *Let $n\in\mathbf{Z}_{\geqslant -1}$. The K-representation R_{n+1} is defined as*

$$R_{n+1}\overset{\text{def}}{=}\mathrm{Ind}_{K_0(p^{n+1})}^K\sigma_{\underline{r}}^{n+1}.$$

We can extend the action of K on R_{n+1} to an action of KZ by letting $p\in Z$ act trivially; the resulting representation will be denoted again by R_{n+1} and we will pass from the one to the other without commentary.

Thanks to the decomposition (2) we get the following, elementary, description of the R_n:

Lemma 2.4. *Let $n\in\mathbf{Z}_{\geqslant -1}$ Then:*

i) right translation by $\alpha^{n+1}w$ induces a bijection

$$K/K_0(p^{n+1})\overset{\sim}{\to} KZ\alpha^{-n-1}KZ/KZ;$$

ii) we have a decomposition

$$K=\coprod_{\lambda\in I_{n+1}}\begin{bmatrix} \lambda & 1\\ 1 & 0\end{bmatrix}K_0(p^{n+1})\coprod\coprod_{\lambda'\in I_n}\begin{bmatrix} 1 & 0\\ p\lambda' & 1\end{bmatrix}K_0(p^{n+1});$$

Moreover, if $1 \leqslant m \leqslant n$ we have a decomposition

$$K_0(p^m) = \coprod_{\lambda' \in I_{n+1-m}} \begin{bmatrix} 1 & 0 \\ p^m \lambda' & 1 \end{bmatrix} K_0(p^{n+1});$$

iii) the family

$$\left\{ \Big[\begin{bmatrix} \lambda & 1 \\ 1 & 0 \end{bmatrix}, X^{\underline{r}-\underline{j}} Y^{\underline{j}} \Big], \Big[\begin{bmatrix} 1 & 0 \\ p\lambda' & 1 \end{bmatrix}, X^{\underline{r}-\underline{j}} Y^{\underline{j}} \Big] \text{ for } \lambda \in I_{n+1},\ \lambda' \in I_n, \underline{0} \leqslant \underline{j} \leqslant \underline{r} \right\}$$

defines an $\overline{\mathbf{F}}_p$-basis for the representation R_{n+1}. Moreover, if $1 \leqslant m \leqslant n$, the family

$$\{ \Big[\begin{bmatrix} 1 & 0 \\ p^m \lambda' & 1 \end{bmatrix}, X^{\underline{r}-\underline{j}} Y^{\underline{j}} \Big] \text{ for } \lambda \in I_{n+1-m},\ \underline{0} \leqslant \underline{j} \leqslant \underline{r} \}$$

defines an $\overline{\mathbf{F}}_p$-basis for the representation $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} \sigma_{\underline{r}}$.

Proof. Omissis. □

The relation between the representations R_n and the compact induction $\mathrm{Ind}_{KZ}^G \sigma_{\underline{r}}|_{KZ}$ is then described by the following

Proposition 2.5. *Let $n \in \mathbf{Z}_{\geqslant -1}$. We have a KZ-equivariant isomorphism*

$$\Phi_{n+1} : W(n+1) \xrightarrow{\sim} R_{n+1}$$

such that

$$\Phi_{n+1}(\big[g^0_{n+1,\lambda}, X^{\underline{r}-\underline{j}} Y^{\underline{j}}\big]) = \Big[\begin{bmatrix} \lambda & 1 \\ 1 & 0 \end{bmatrix}, X^{\underline{r}-\underline{j}} Y^{\underline{j}} \Big]$$

$$\Phi_{n+1}(\big[g^1_{n,\lambda'}, X^{\underline{r}-\underline{j}} Y^{\underline{j}}\big]) = \Big[\begin{bmatrix} 1 & 0 \\ p\lambda' & 1 \end{bmatrix}, X^{\underline{j}} Y^{\underline{r}-\underline{j}} \Big]$$

for $n \geqslant 0$ and

$$\Phi_0(\big[1_G, X^{\underline{r}-\underline{j}} Y^{\underline{j}}\big]) = X^{\underline{j}} Y^{\underline{r}-\underline{j}}$$

for $n = 0$.

In particular, we have a KZ-equivariant isomorphism

$$\mathrm{Ind}_{KZ}^G \sigma_{\underline{r}} \xrightarrow{\sim} \bigoplus_{n \in \mathbf{N}} R_n$$

Proof. Elementary (see for instance [Mo], Proposition 3.4, whose proof generalizes line by line). □

We introduce now a convenient $\overline{\mathbf{F}}_p$-basis for the representation R_{n+1}. Thanks to the transitivity

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} \sigma_{\underline{r}} \cong \mathrm{Ind}_{K_0(p^{m+1})}^{K_0(p^m)} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^{m+1})} \sigma_{\underline{r}}$$

(where $0 \leqslant m \leqslant n$) we see that a Vandermonde argument together with an immediate induction give us the following:

Lemma 2.6 (Definition). *Let $n \in \mathbf{N}$. An $\overline{\mathbf{F}}_p$ basis for the K-representation R_{n+1} is described by the elements*

$$F^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}(\underline{l}_{n+1}) \overset{\text{def}}{=} \sum_{i=1}^{n} \sum_{\lambda_i \in \mathbf{F}_q} (\lambda_i^{\frac{1}{p^i}})^{\underline{l}_i} \begin{bmatrix} 1 & 0 \\ p^i[\lambda_i^{\frac{1}{p^i}}] & 1 \end{bmatrix} \big[1, X^{\underline{r}-\underline{l}_{n+1}} Y^{\underline{l}_{n+1}}\big]$$

$$F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) \overset{\text{def}}{=} \sum_{\lambda_0 \in \mathbf{F}_q} \lambda_0^{\underline{l}_0} \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix} \big[1, F^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}(\underline{l}_{n+1})\big]$$

for $\underline{l}_i \in \{0,\dots,p-1\}^f$ *(where* $i \in \{0,\dots,n\}$*) and* $\underline{l}_{n+1} \leqslant \underline{r}$*, with the obvious conventions that if* $n=0$ *we have*

$$F_{\emptyset}^{(1,0)}(\underline{l}_1) \overset{\text{def}}{=} \big[1, X^{\underline{r}-\underline{l}_{n+1}} Y^{\underline{l}_n+1}\big].$$

For notational convenience we define

$$F_{\emptyset}^{(0,-1)}(\underline{l}_0) \overset{\text{def}}{=} (-1)^{\underline{l}_0} X^{\underline{l}_0} Y^{\underline{r}-\underline{l}_0}$$
$$F_{\emptyset}^{(1,-1)}(\emptyset) \overset{\text{def}}{=} Y^{\underline{r}}.$$

Such basis will be denoted by $\mathscr{B}_{n+1}$*.*

The subset $\mathscr{B}^+_{n+1} \subset \mathscr{B}_{n+1}$ *described by the elements of the form* $F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1})$ *will be referred to as the set of* positive elements *of* R_{n+1}*; the* $\overline{\mathbf{F}}_p$*-linear subspace generated by the positive elements will be denoted as* R^+_{n+1}*.*

Similarly the subset $\mathscr{B}^-_{n+1} \subset \mathscr{B}_{n+1}$ *described by elements of the form* $F^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}(\underline{l}_{n+1})$ *will be referred to as the set of* negative elements *of* R_{n+1}*; the* $\overline{\mathbf{F}}_p$*-linear subspace generated by the negative elements will be denoted as* R^-_{n+1}*.*

Hecke operators on the R_{n+1}***.*** Let $n \in \mathbf{N}$. Thanks to Lemma 2.1 the $W(n)$-restriction of the operator T gives the $\overline{\mathbf{F}}_p$-linear morphism

$$T|_{W(n)} : W(n) \to W(n-1) \oplus W(n+1).$$

Such restriction is KZ-equivariant (by Cartan decomposition) and composition by the natural projections gives us the KZ-equivariant operators

$$T^+_n : W(n) \to W(n+1) \qquad T^-_n : W(n) \to W(n-1).$$

By transport of structure (via the isomorphisms of Lemma 2.5) we get morphisms

$$T^+_n : R_n \to R_{n+1} \qquad T^-_n : R_n \to R_{n-1}$$

(where we used the same notations for the operators on $W(n)$ and R_n). Their description in terms of the canonical basis of R_{n+1} is immediate, following from Lemmas 2.1 and 2.5:

Lemma 2.7. *Let* $n > 0 \in \mathbf{N}$*. The* KZ*-equivariant operators* T^+_n, T^-_n *are characterized by*

$$T^+_n : R_n \to R_{n+1}$$
$$\big[1, X^{\underline{r}-\underline{l}_n} Y^{\underline{l}_n}\big] \mapsto (-1)^{\underline{l}_n} \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{\underline{l}_n} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} \big[1, X^{\underline{r}}\big]$$

$$T^-_n : R_n \to R_{n-1}$$
$$\big[1, X^{\underline{r}-\underline{l}_n} Y^{\underline{l}_n}\big] \mapsto \begin{cases} \delta_{\underline{r},\underline{l}_n}\big[1, Y^{\underline{r}}\big] & \text{if } n > 1 \\ \delta_{\underline{r},\underline{l}_n} Y^{\underline{r}} & \text{if } n = 1. \end{cases}$$

For $n = 0$ *we have*

$$R_0 \hookrightarrow R_1$$
$$X^{\underline{r}-\underline{l}_0} Y^{\underline{l}_0} \mapsto \sum_{\lambda_0 \in \mathbf{F}_q} (-1)^{\underline{r}-\underline{l}_0} \lambda_0^{\underline{r}-\underline{l}_0} \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix} \big[1, X^{\underline{r}}\big] + \delta_{\underline{l}_0,\underline{0}}\big[1, X^{\underline{r}}\big].$$

Moreover, the operators T^+_n *are monomorphisms for all* $n \in \mathbf{N}$ *and the operators* T^-_n *are epimorphisms for all* $n \in \mathbf{N}_{\geqslant 1}$*.*

Proof. The characterisation of the operators $T^{\pm}_n$ follows by the explicit descriptions given in Lemmas 2.1 and 2.5.

As T_n^+ maps the basis $\mathscr{B}_n$ into a subset of $\mathscr{B}_{n+1}$, the operator is injective for $n \geqslant 1$. As $\left[1, Y^{\underline{r}}\right]$ (resp. $Y^{\underline{r}}$) is a K-generator for R_{n-1} (resp. R_0) for $n \geqslant 2$ (resp. $n = 1$), the operator T_n^- is surjective. $\square$

We identify R_n with a K-subrepresentation of R_{n+1} via the monomorphism T_n^+ without any further commentary. For any odd integer $n \geqslant 1$ we use the Hecke operators $T_n^{\pm}$ to define (inductively) the amalgamated sum $R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_n} R_{n+1}$ via the following co-cartesian diagram

$$\begin{array}{ccc}
R_n & \overset{T_n^+}{\hookrightarrow} & R_{n+1} \\
\Big\downarrow {\scriptstyle -pr_{n-1}\circ T_n^-} & & \vdots\,{\scriptstyle pr_{n+1}} \\
R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_{n-2}} R_{n-1} & \cdots\cdots\!\!> & R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_n} R_{n+1}
\end{array}$$

(where we define pr_0 to be the identity map). Similarly we define the amalgamated sums $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}$ for any positive even integer $n \in \mathbf{N}_{>}$. The following result is then formal

Proposition 2.8. *For any odd integer $n \in \mathbf{N}$, $n \geqslant 1$ we have a natural commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R_n & \xrightarrow{T_n^+} & R_{n+1} & \longrightarrow & R_{n+1}/R_n & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle -pr_{n-1}\circ T_n^-} & & \Big\downarrow {\scriptstyle pr_{n+1}} & & \Big\| & & \\
0 & \longrightarrow & R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1} & \longrightarrow & R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1} & \xrightarrow{\pi} & R_{n+1}/R_n & \longrightarrow & 0
\end{array}$$

with exact lines.

We have an analogous result concerning the family

$$\{R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}\}_{n \in 2\mathbf{N}\setminus\{0\}}.$$

Proof. Formal. See for instance [Mo], Proposition 4.1. $\square$

The following result let us complete the dictionary

Proposition 2.9. *We have a KZ-equivariant isomorphism*

$$\pi(\sigma_{\underline{r}}, 0, 1)|_{KZ} \overset{\sim}{\to} \varinjlim_{n\,\mathrm{odd}} \left(R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}\right) \oplus \varinjlim_{n\,\mathrm{even}} \left(R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}\right). \tag{4}$$

Proof. The proof is formal and identical to [Mo], Proposition 3.9. $\square$

Remark 2.10. *We can give analogous (in the evident way) definitions in the case where F is any finite extension of $\mathbf{Q}_p$: we would then get a statement completely similar to Proposition 2.9.*

3. First description of the Iwahori structure

The goal of this section is to give a first, general description for the $K_0(p)$-representation $\pi(\underline{r}, 0, 1)|_{K_0(p)}$. The endpoint is Proposition 3.5, which is the "Iwahori analogue" of Proposition 2.9 of the preceeding section. More precisely, for each $n \in \mathbf{N}$ the block R_{n+1} has a natural $K_0(p)$-equivariant splitting (by Mackey decomposition)

$$R_{n+1} = R_{n+1}^+ \oplus R_{n+1}^-$$

which is compatible with the Hecke operators $T_n^{\pm}$ in the obvious sense (cf. Lemma/Definition 3.2). This will enable us to repeat the constructions of §2, i.e. the construction of the inductive family of amalgamated sums $\cdots \oplus_{R_n^{\pm}} R_{n+1}^{\pm}$, endowed with a natural filtration (cf. Lemma 3.4) .

Thanks to Proposition 3.5 we see that we can content ourselves to the study of the amalgamated sums $\cdots \oplus_{R_n^\pm} R_{n+1}^\pm$: actually we have a $K_0(p)$-equivariant surjection

$$\begin{array}{c}(\varinjlim_{n\,\mathrm{odd}} \cdots \oplus_{R_n^+} R_{n+1}^+) \oplus (\varinjlim_{n\,\mathrm{odd}} \cdots \oplus_{R_n^-} R_{n+1}^-) \oplus (\varinjlim_{n\,\mathrm{even}} \cdots \oplus_{R_n^+} R_{n+1}^+) \oplus (\varinjlim_{n\,\mathrm{even}} \cdots \oplus_{R_n^-} R_{n+1}^-)\\ \downarrow \\ \pi(\underline{r},0,1)|_{K_0(p)}\end{array}$$

whose kernel is "small" (and explicitly determined).

The following elementary result will be crucial.

Lemma 3.1. *Let $a \in \{0,\dots,q-1\}$. Then*

$$\sum_{\lambda\in\mathbf{F}_q} \lambda^a = \begin{cases} 0 & \text{if } a \neq q-1 \\ -1 & \text{if } a = q-1.\end{cases}$$

Proof. Omissis. □

The representations $R_{n+1}^\pm$ and the Hecke operators $(T_n^\pm)^{\mathrm{pos,\,neg}}$. Fix $n \in \mathbf{N}$; the $\overline{\mathbf{F}}_p$-linear decomposition given by Lemma 2.6

$$R_{n+1} \cong R_{n+1}^+ \oplus R_{n+1}^- \tag{5}$$

is easily checked to be $K_0(p)$-equivariant (realising the Mackey decomposition for $R_{n+1}|_{K_0(p)}$) and we clearly have a $K_0(p)$-equivariant isomorphism

$$R_{n+1}^- \xrightarrow{\sim} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} \sigma_{\underline{r}}^{n+1}.$$

We moreover define the following $K_0(p)$-representations:

$$R_0^+ \overset{\mathrm{def}}{=} R_0, \qquad R_0^- \overset{\mathrm{def}}{=} \langle Y^{\underline{r}}\rangle_{\overline{\mathbf{F}}_p}, \qquad (R_1/R_0)^+ \overset{\mathrm{def}}{=} \mathrm{Im}(R_1^+ \hookrightarrow R_1 \twoheadrightarrow R_1/R_0).$$

The decomposition given in (5) and the description of Lemma 2.7 lets us define [2] the Hecke operators $(T_n^\pm)^{\mathrm{pos,\,neg}}$ on the representations $R_{n+1}^\pm$:

Lemma 3.2 (Definition). *Let $n \in \mathbf{N}_{\geqslant 1}$.*

i) The restriction of the Hecke operator T_n^+ on the $K_0(p)$-subrepresentations R_n^+, R_n^- of R_n induces two $K_0(p)$-equivariant monomorphisms,

$$(T_n^+)^{\mathrm{pos}} : R_n^+ \hookrightarrow R_{n+1}^+$$
$$(T_n^+)^{\mathrm{neg}} : R_n^- \hookrightarrow R_{n+1}^-$$

ii) The restriction of Hecke operator T_n^- on the $K_0(p)$-subrepresentations R_n^+, R_n^- of R_n induces two $K_0(p)$-equivariant epimorphisms,

$$(T_n^-)^{\mathrm{pos}} : R_n^+ \twoheadrightarrow R_{n-1}^+$$
$$(T_n^-)^{\mathrm{neg}} : R_n^- \twoheadrightarrow R_{n-1}^-$$

Proof. Except for the operator $(T_1^-)^{\mathrm{pos}}$, the result follows immediately from the decomposition $R_n|_{K_0(p)} \cong R_n^+ \oplus R_n^-$ and the properties and characterisations of the Hecke operators $T_n^\pm$.

Concerning $(T_1^-)^{\mathrm{pos}} : R_1^+ \to R_0$ we notice that

$$(T_1^-)^{\mathrm{pos}}(F_{\underline{l}_0(\underline{r})}^{(0)}) = \sum_{\underline{i}\leqslant\underline{r}} X^{\underline{r}-\underline{i}} Y^{\underline{i}} \Big(\sum_{\lambda_0\in\mathbf{F}_q} \lambda_0^{\underline{l}_0+\underline{i}}\Big)$$

[2] We apologize to the reader if the notation $(T_n^\pm)^{\mathrm{pos,\,neg}}$ looks heavy. We believe it is convenient if we want to be precise and keep track of the various parameters on which depend the representations we deal with.

and the result follows from Lemma 3.1. $\square$

Corollary 3.3. *The natural $K_0(p)$-equivariant map*

$$R_2^+ \to (R_1/R_0)^+$$

is an epimorphism.

Proof. Omissis. $\square$

Amalgamated sums and first description of the Iwahori structure. Using the Hecke operators defined in Lemma 3.2 we can introduce the following amalgamated sums, analogously to the constructions of §2.

Let $n \in \mathbf{N}$ be odd and $\bullet \in \{+,-\}$. We can define inductively a natural $K_0(p)$-representation $R_0^\bullet \oplus_{R_1^\bullet} \cdots \oplus_{R_n^\bullet} R_{n+1}^\bullet$ together with canonical morphisms $pr_{n+1}^\bullet, \iota_{n-1}^\bullet$ via the co-cartesian diagram

$$\begin{array}{ccc} R_n^\bullet & \xhookrightarrow{(T_n^+)^\bullet} & R_{n+1}^\bullet \\ {\scriptstyle -(pr_{n-1})^\bullet \circ (T_n^-)^\bullet}\downarrow & & \vdots\, {\scriptstyle \exists!\ (pr_{n+1})^\bullet} \\ R_0^\bullet \oplus_{R_1^\bullet} \cdots \oplus_{R_{n-2}^\bullet} R_{n-1}^\bullet & \xrightarrow[\exists!]{\iota_{n-1}^\bullet} & R_0^\bullet \oplus_{R_1^\bullet} \cdots \oplus_{R_n^\bullet} R_{n+1}^\bullet. \end{array}$$

(with the convention that $(T_j^\pm)^+ \stackrel{\text{def}}{=} (T_j^\pm)^{\text{pos}}$ and $(T_j^\pm)^- \stackrel{\text{def}}{=} (T_j^\pm)^{\text{neg}}$).

For $n \in \mathbf{N}$ even and $\bullet \in \{+,-\}$ we can define the amalgamated sums $(R_1/R_0)^\bullet \oplus_{R_2^\bullet} \cdots \oplus_{R_n^\bullet} R_{n+1}^\bullet$, together with canonical morphisms $pr_{n+1}^\bullet, \iota_{n-1}^\bullet$ in the evident analogous way (with the convention that $(R_1/R_0)^- = R_1^-$.)

The following result is similar to Proposition 2.8:

Lemma 3.4. *Let $n \in \mathbf{N}$ be odd, $\bullet \in \{+,-\}$. Then $\iota_{n-1}^\bullet$ is a monomorphism, $pr_{n+1}^\bullet$ is an epimorphism and we have a $K_0(p)$-equivariant commutative diagram with exact lines:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R_n^\bullet & \xrightarrow{(T_n^+)^\bullet} & R_{n+1}^\bullet & \xrightarrow{\pi_{n+1}} & R_{n+1}^\bullet/R_n^\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle -(T_n^-)^\bullet} & & & & & & \\ & & R_{n-1}^\bullet & & \downarrow{\scriptstyle pr_{n+1}^\bullet} & & \| & & \\ & & \downarrow{\scriptstyle pr_{n-1}^\bullet} & & & & & & \\ 0 & \longrightarrow & R_0^\bullet \oplus_{R_1^\bullet} \cdots \oplus_{R_{n-2}^\bullet} R_{n-1}^\bullet & \xrightarrow{\iota_{n-1}} & R_0^\bullet \oplus_{R_1^\bullet} \cdots \oplus_{R_n^\bullet} R_{n+1}^\bullet & \xrightarrow{\pi_{n+1}} & R_{n+1}^\bullet/R_n^\bullet & \longrightarrow & 0. \end{array}$$

We have an analogous result when $n \in \mathbf{N}_>$ is even.

Proof. The proof is identical to Proposition 2.8, using that the maps $R_1^\bullet \stackrel{(T_1^-)^\bullet}{\twoheadrightarrow} R_0^\bullet$ and $R_2^\bullet \stackrel{(T_2^-)^\bullet}{\twoheadrightarrow} (R_1/R_0)^\bullet$ are epimorphisms. $\square$

In order to give a first description of the $K_0(p)$-representation $\pi(\underline{r}, 0, 1)|_{K_0(p)}$ we are now left to determine the relations between the amalgamated sums $\cdots \oplus_{R_n^\bullet} R_{n+1}^\bullet$ and the restriction $(\cdots \oplus_{R_n} R_{n+1})|_{K_0(p)}$.

We will treat in detail the analysis of the limit $(\varinjlim_{n,\text{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})|_{K_0(p)}$. The case n even is proved in a similar way and is left to the reader.

Proposition 3.5. *The decomposition $R_n|_{K_0(p)} \cong R_n^+ \oplus R_n^-$ induces the following $K_0(p)$-equivariant exact sequences:*

$$0 \to \langle (F_\emptyset^{(0,-1)}(\underline{0}), -F_\emptyset^{(1,-1)}(\emptyset)) \rangle_{\overline{\mathbf{F}}_p} \to (\varinjlim_{n\,\mathrm{odd}} R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_n^+} R_{n+1}^+) \oplus (\varinjlim_{n\,\mathrm{odd}} R_0^- \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-) \to$$
$$\to (\varinjlim_{n\,\mathrm{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})|_{K_0(p)} \to 0$$

and

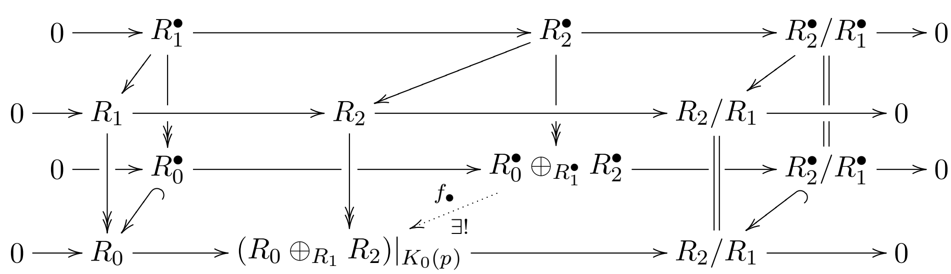
$$0 \to \langle (F_{\underline{r}}^{(0)}(\underline{0}), -F_\emptyset^{(1,0)}(\underline{0})) \rangle_{\overline{\mathbf{F}}_p} \to (\varinjlim_{n\,\mathrm{even}} (R_1/R_0)^+ \oplus_{R_2^+} \cdots \oplus_{R_n^+} R_{n+1}^+) \oplus (\varinjlim_{n\,\mathrm{even}} R_1^- \oplus_{R_2^-} \cdots \oplus_{R_n^-} R_{n+1}^-) \to$$
$$\to (\varinjlim_{n\,\mathrm{even}} (R_1/R_0) \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})|_{K_0(p)} \to 0.$$

Proof. Let us assume n odd, leaving the case n even to the reader (the proof being analogous). The functor $\varinjlim$ is exact if the index category is filtrant and the forgetful functor $For : \mathcal{R}ep_{K_0(p)} \to \mathcal{V}ect_{\overline{\mathbf{F}}_p}$ commutes with $\varinjlim$. It is therefore enough to show that we have an inductive system of exact sequences

$$0 \to \langle (F_\emptyset^{(0,-1)}(\underline{0}), -F_\emptyset^{(1,-1)}(\emptyset)) \rangle_{\overline{\mathbf{F}}_p} \to (R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_n^+} R_{n+1}^+) \oplus (R_0^- \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-) \to$$
$$\to (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})|_{K_0(p)} \to 0$$

with the natural morphisms $R_0^\bullet \oplus_{R_1^\bullet} \cdots \oplus_{R_n^\bullet} R_{n+1}^\bullet \to R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}$ being injective. The proof will be an induction on n.

Let $\bullet \in \{+, -\}$. By the universal property of the push out we deduce the following commutative diagram



and the morphism $f_\bullet$ is injective by the four Lemma applied to the "bottom" diagram: recall that $(T_0^+)^\bullet$ is injective and we check easily the injectivity of the morphism $R_2^\bullet/R_1^\bullet \to R_2/R_1$. We deduce the commutative diagram with exact lines

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R_0^+ \oplus R_0^- & \longrightarrow & (R_0^+ \oplus_{R_1^+} R_2^+) \oplus (R_0^- \oplus_{R_1^-} R_2^-) & \longrightarrow & (R_2^+/R_1^+) \oplus (R_2^-/R_1^-) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R_0 & \longrightarrow & (R_0 \oplus_{R_1} R_2)|_{K_0(p)} & \longrightarrow & R_2/R_1 & \longrightarrow & 0. \end{array} \tag{6}$$

The isomorphism $(R_2^+/R_1^+) \oplus (R_2^-/R_1^-) \xrightarrow{\sim} R_2/R_1$ and the exact sequence

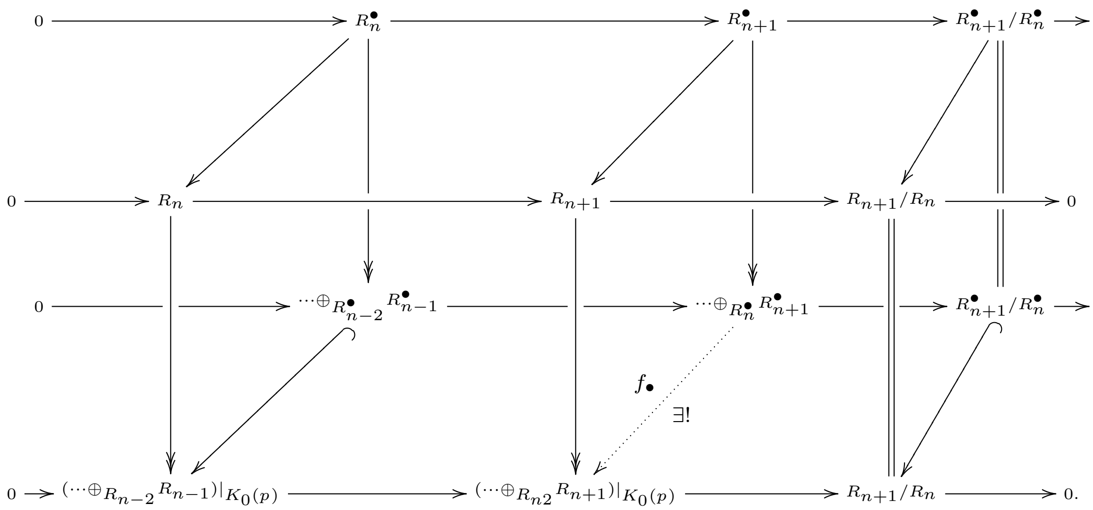
$$0 \to \langle (F_\emptyset^{(0,-1)}(\underline{0}), -F_\emptyset^{(1,-1)}(\emptyset)) \rangle \to R_0^+ \oplus R_0^- \to R_0 \to 0$$

give the result, via the snake Lemma applied to the diagram (6).

We treat now the inductive step. By the inductive hypothesis and the definition of the Hecke operators $(T_n^{\pm})^{\mathrm{pos,neg}}$, we dispose of the commutative diagrams

$$\begin{array}{ccc} R_n^\bullet & \hookrightarrow & R_n \\ \downarrow & & \downarrow \\ R_{n-1}^\bullet & \hookrightarrow & R_{n-1} \\ \downarrow & & \downarrow \\ R_0^\bullet \oplus_{R_1^\bullet} \cdots \oplus_{R_{n-2}^\bullet} R_{n-1}^\bullet & \hookrightarrow & R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1} \end{array}$$

(our inductive hypothesis giving the injectivity of the lower arrow) from which we deduce the following commutative diagram with exact rows



Again, the morphism $f_\bullet$ is injective by the four Lemma and we deduce as well the following commutative diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ (R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_{n-1}^+} R_{n-1}^+) \oplus (R_0^- \oplus_{R_1^-} \cdots \oplus_{R_{n-1}^-} R_{n-1}^-) & \longrightarrow & (R_0 \oplus_{R_1} \cdots \oplus_{R_{n-1}} R_{n-1})|_{K_0(p)} \\ \downarrow & & \downarrow \\ (R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_{n+1}^+} R_{n+1}^+) \oplus (R_0^- \oplus_{R_1^-} \cdots \oplus_{R_{n+1}^-} R_{n+1}^-) & \longrightarrow & (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})|_{K_0(p)} \\ \downarrow & & \downarrow \\ (R_{n+1}^+/R_n^+) \oplus (R_{n+1}^-/R_n^-) & \longrightarrow & R_{n+1}/R_n \\ \downarrow & & \downarrow \\ 0 & & 0. \end{array}$$

As the natural morphism $(R_{n+1}^+/R_n^+) \oplus (R_{n+1}^-/R_n^-) \to R_{n+1}/R_n$ is an isomorphism, the conclusion follows by applying the snake Lemma and using the exact sequence

$$0 \to \langle (F_\emptyset^{(0,-1)}(\underline{0}), -F_\emptyset^{(1,-1)}(\emptyset)) \rangle_{\overline{\mathbf{F}}_p} \to (R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_{n-2}^+} R_{n-1}^+) \oplus (R_0^- \oplus_{R_1^-} \cdots \oplus_{R_{n-2}^-} R_{n-1}^-) \to$$
$$\to (R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1})|_{K_0(p)} \to 0.$$

coming from the inductive hypothesis. □

4. Representations of the Iwahori subgroups

In this section we introduce the fundamental techniques which let us describe *easily* the Iwahori structure of the representations $R_{n+1}^{\pm}$, appeared in §3, in terms of simple Euclidean data. Appropriate refinements of such methods let us, later on, describe more complicate objects, such as the representations $R_{n+1}^{\pm}/R_n^{\pm}$ or the universal representations $\pi(\underline{r}, 0, 1)$ appearing in §5.

We hope that suitable improvements of the ideas and techniques presented here will eventually lead to the detection of the "good" supersingular representations which should appear in a mod p local Langlands correspondence (see also Remark 4.3).

We focus our attention on the representations $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} 1$: the description of $R_{n+1}^{\pm}$ can be obtained with identical techniques (cf. sections §4.1.3 or 4.2). The Iwahori structure of such objects -given by Proposition 4.2- may look complicated, but the key point is that its combinatorics can be controlled by an easy Euclidean method which we outline as follows.

First of all we detect a "canonical" $\overline{\mathbf{F}}_p$-basis $\mathscr{B}$ for the representation $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} 1$ (definition 4.1). We see that each element $F_{\underline{l}_1,\dots,\underline{l}_n}^{(1,n)} \in \mathscr{B}$ is parametrized by a family of f-tuples $\underline{l}_i \in \{0, \dots, p-1\}^f$, family which can be used to define a point (in the naïve sense) $(x_0, \dots, x_{f-1}) \in \mathbf{R}^{f-1}$. In this way, we can associate, bijectively, the elements of the basis $\mathscr{B}$ to the integer points of an f-hypercube $\mathscr{R}$, of side $p^n - 1$ embedded in $\mathbf{Z}^{f-1}$: this is detailed in paragraph 4.1.1.

The following step (§4.1.2) consists in veryfing that the Euclidean lattice $\mathscr{R}$ describes the Iwahori structure of $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} 1$, in the sense of Definition 1.7. As $\mathscr{R}$ is an f-hypercube, this means that the Iwahori socle filtration of $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} 1$ is deduced by successive intersections of the lattice $\mathscr{R}$ with the antidiagonals $X_0 + \cdots + X_{f-1} = \textit{constant}$ (see also Figure 3), i.e.:

i) a linear basis for the N-th composition factor of the socle filtration of $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} 1$ is given by the point lying below the hyperplane $X_0 + \cdots + X_{f-1} = N$

ii) a linear basis for the N-th layer of the socle filtration is given by the points lying on the hyperplane $X_0 + \cdots + X_{f-1} = N$.

This is the content of Proposition 4.2, the technical heart of the methods introduced in this paper. We verify, via the delicate estimates on Witt vectors of Appendices A and B, that the behaviour of the canonical elements $F_{\underline{l}_1,\dots,\underline{l}_{f-1}}^{(1,n)}$ fits the previous Euclidean picture.

The interested reader is invited to see the beginning of section 4.1.2 for further details concerning the general techniques and phenomena appearing in the proof of Proposition 4.2.

As annonced the same techniques let us detect the $K_0(p)$-structure for the representations $R_{n+1}^{\pm}$: the involved combinatorics can be handled with the help of a simple Euclidean picture (an f-parallelepipoid). The precise statements are Propositions 4.10 and 4.11 which deal with R_{n+1}^{-} and R_{n+1}^{+} respectively.

The constructions and computations of this section let us, as an application, determine the Iwahori structure for principal and special series: this is the object of §4.3. Again, in terms of Euclidean space, we see that the successive layers for the $K_0(p)$-socle filtration are detected by the intersections of $\mathbf{N}^f$ (the "hypercube" associated to such series) with the hyperplanes $X_0 + \cdots + X_{f-1} = \textit{constant}$.

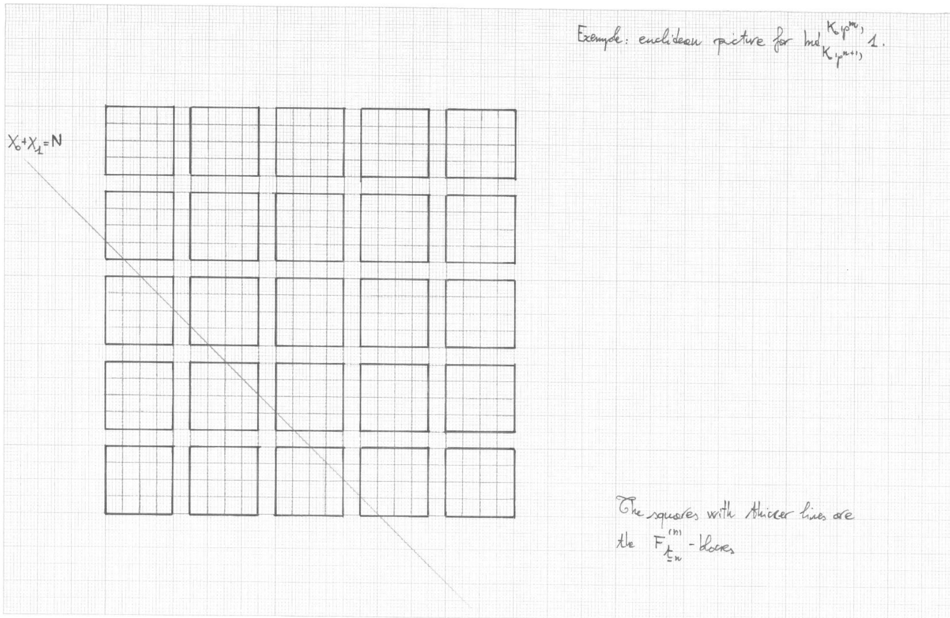


Figure 3: The Euclidean picture of $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1$ for $p=5$, $m\leqslant n-1$. The N-th composition factor for the socle filtration is described by the points of $\mathscr{R}$ lying below the line $X_0+X_1=N$. Each point should be interpreted as a $F^{(n-1,n)}_{\underline{l}_{n-1},\underline{l}_n}$-block; the square with thicker lines are then the $F^{(n)}_{\underline{t}_n}$-blocks.

4.1 The negative case.

Let $1\leqslant m\leqslant n$ be integers. In this section we examine the $K_0(p)$-socle filtration (and the extensions between two consecutive graded pieces) for the representations $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\chi$ where $\chi: K_0(p^{n+1})\to\overline{\mathbf{F}}_p^{\times}$ is a smooth character of $K_0(p^{n+1})$ (i.e. the inflation of a character of the finite Borel $B(\mathbf{F}_q)$ by the morphism $K_0(p^{n+1})\twoheadrightarrow B(\mathbf{F}_q)$). Thanks to the canonical isomorphism :

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\chi\cong(\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1)\otimes\chi$$

we can assume that $\chi=1$ is the trivial character. Finally, let $\{e\}$ be an $\overline{\mathbf{F}}_p$-basis for the underlying vector space associated to the character χ.

We introduce now the canonical base of $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1$ and its interpretation in terms of lattices of $\mathbf{R}^f$.

Definition 4.1. *For $j\in\{m,\dots,n\}$ let $\underline{l}_j=(l_j^{(0)},\dots,l_j^{(f-1)})\in\{0,\dots,p-1\}^f$ be a f-tuple. We define the element $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}\in\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1$ as*[3]

$$F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}\overset{\mathrm{def}}{=}\sum_{j=m}^{n}\sum_{\lambda_j\in\mathbf{F}_q}(\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j}\begin{bmatrix}1&0\\p^j[\lambda_j^{\frac{1}{p^j}}]&1\end{bmatrix}[1,e].$$

[3]As remarked by the Referee, if λ_j runs over $\mathbf{F}_q$ so does $\lambda_j^{\frac{1}{p^j}}$ and the expression for $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}$ may be simplifid. Nevertheless we find our writing well adapted when we need to manipulate the (pseudo-)homogeneous degree of universal Witt polynomials in a coherent way, see for instance the note in Proposition 4.4.

For a notational convenience, we define $F^{(n+1,n)}_{\underline{l}_{n+1},\dots,\underline{l}_n} \overset{\text{def}}{=} [1,e]$ *and* $\underline{l}_{n+1} \overset{\text{def}}{=} \underline{0}$.

The set

$$\mathscr{B} \overset{\text{def}}{=} \left\{ F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} \in \mathrm{Ind}^{K_0(p^m)}_{K_0(p^{n+1})} 1, \text{ for } (\underline{l}_m,\dots,\underline{l}_n) \in \left\{ \{0,\dots,p-1\}^f \right\}^{n+1-m} \right\}$$

is an $\overline{\mathbf{F}}_p$*-basis for* $\mathrm{Ind}^{K_0(p^m)}_{K_0(p^{n+1})} 1$.

The fact that $\mathscr{B}$ is an $\overline{\mathbf{F}}_p$ basis for $\mathrm{Ind}^{K_0(p^m)}_{K_0(p^{n+1})} 1$ is again an induction together with a Vandermonde argument as for Lemma 2.6.

4.1.1 ***Interpretation in terms of lattices.*** As anticipated in the introduction, each element of $\mathscr{B}$ can be seen as a "point" of a $\mathbf{Z}$-lattice in the standard Euclidean f-dimensional space $\mathbf{R}^f$: such correspondence is given by the injective map

$$\begin{aligned} \mathscr{B} &\overset{P}{\hookrightarrow} \mathbf{R}^f \\ F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} &\mapsto \left(\sum_{j=m}^{n} p^{j-m} l_j^{(\lfloor j-m \rfloor)}, \dots, \sum_{j=m}^{n} p^{j-m} l_j^{(\lfloor f-1+j-m \rfloor)} \right) \end{aligned} \tag{7}$$

whose image will be denoted by $\mathscr{R}$. We notice that $\mathscr{R}$ is a f-hypercube of side $p^{n-m+1}-1$, with a natural recursive structure in the following sense: for an f-tuple $\underline{t}_n \in \{0,\dots,p-1\}^f$, the subset of functions whose last p-adic digits are fixed to be equal to $\underline{t}_n$, i.e.

$$\left\{ F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_{n-1},\underline{t}_n} \in \mathscr{B} \quad \underline{l}_j \in \{0,\dots,p-1\}^f, \text{ for } m \leqslant j \leqslant n-1 \right\},$$

corresponds to an f-sub-hypercube of $\mathscr{R}$ of side $p^{n-m}-1$ via the bijection P. It will be referred to as the $F^{(n)}_{\underline{t}_n}$-*block* of $\mathscr{R}$. The hypercube $\mathscr{R}$ is then obtained as the juxtaposition of the $F^{(n)}_{\underline{t}_n}$-blocks for varying $\underline{t}_n \in \{0,\dots,p-1\}^f$. This is visualized, for instance, in Figure 3, where the squares with thicker lines correspond to the $F^{(n)}_{\underline{t}_n}$-blocks. The notion of block can be adapted in the evident way if considering the functions where the k last p-adic digits are fixed (with $1 \leqslant k \leqslant n-m+1$): we get this way a f-sub-hypercube of $\mathscr{R}$ of side $p^{n-m+1-k}-1$.

We are therefore allowed to apply the terminology of real Euclidean spaces to the elements of $\mathscr{B}$, meaning their image through the map P. In particular if $e_i \overset{\text{def}}{=} (\delta_{0,i},\dots,\delta_{f-1,i}) \in \{0,1\}^f$ we define $F^{m,n}_{(\underline{l}_m,\dots,\underline{l}_n)-e_i}$ by

$$F^{m,n}_{(\underline{l}_m,\dots,\underline{l}_n)-e_i} = \begin{cases} 0 \quad \text{if } P^{\leftarrow}(P(F^{m,n}_{\underline{l}_m,\dots,\underline{l}_n}) - e_i) = \emptyset \\ \text{the only element of } P^{\leftarrow}(P(F^{m,n}_{\underline{l}_m,\dots,\underline{l}_n}) - e_i) \text{ otherwise.} \end{cases}$$

In order to give the statement concerning the $K_0(p^m)$-structure of $\mathrm{Ind}^{K_0(p^m)}_{K_0(p^{n+1})}\chi$ we still need some notation. If $(\underline{l}_m,\dots,\underline{l}_n)$ is a $(n+1-m)f$-tuple, we define

$$\begin{aligned} N_{m,n}(\underline{l}_m,\dots,\underline{l}_n) &\overset{\text{def}}{=} \sum_{s=0}^{f-1} l_m^{(s)} + p(\sum_{s=0}^{f-1} l_{m+1}^{(s)}) + \dots + p^{n-m}(\sum_{s=0}^{f-1} l_n^{(s)}) \\ e(\underline{l}_m,\dots,\underline{l}_n) &\overset{\text{def}}{=} (\sum_{s=0}^{f-1} p^s l_m^{(s)}) + \dots + (\sum_{s=0}^{f-1} p^s l_n^{(s)}); \end{aligned}$$

in particular any $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}$ lies on the antidiagonal $X_0 + \dots + X_{f-1} = N_{m,n}(\underline{l}_m,\dots,\underline{l}_n)$.

Let $N\in\mathbf{N}$. We define the $\overline{\mathbf{F}}_p$-linear subspace

$$(\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1)_N\overset{\mathrm{def}}{=}\langle F^{m,n}_{\underline{l}_m,\dots,\underline{l}_n}\in\mathscr{B}\quad\text{s.t.}\quad N_{m,n}(\underline{l}_m,\dots,\underline{l}_n)<N\rangle_{\overline{\mathbf{F}}_p};$$

it is the subspace generated by the functions lying strictly below the antidiagonal $X_0+\dots X_{f-1}=N$.

We refer the reader to Figure 3 to have the Euclidean interpretation in the case $f=2$.

Let $(\underline{l}_m,\dots,\underline{l}_n)$ be a fixed -tuple. For $s\in\{0,\dots,f-1\}$, we define

$$\Xi_s\overset{\mathrm{def}}{=}\left\{a\in\{m,\dots,n\},\quad\text{s.t. } l_a^{\lfloor s+a-m\rfloor}\neq 0\right\}$$

and we set

$$a_0(s)\overset{\mathrm{def}}{=}\begin{cases}\min(\Xi_s) & \text{if } \Xi_s\neq\emptyset\\ n+1 & \text{otherwise.}\end{cases}$$

The Euclidean meaning of $a_0(s)$ is clear: if we consider the $F^{(a_0(s),n)}_{\underline{l}_{a_0(s)},\dots,\underline{l}_n}$-block then the function $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}$ lies on its s-th face (which is a $(f-1)$-hypercube of side $p^{a_0(s)-m}-1$).

The $K_0(p^m)$-structure of $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\chi$ is then given by the following

Proposition 4.2. *Let $\underline{r}\overset{\mathrm{def}}{=}(r_0,\dots,r_{f-1})\in\{0,\dots,p-1\}^{f-1}$ be a $f-$tuple, m,n be integers such that $1\leqslant m\leqslant n$ and let $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}\in\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\chi_{\underline{r}}^s$ be as in definition 4.1. If $a,b,c,d\in\mathscr{O}_F$ are integers such that $g\overset{\mathrm{def}}{=}\begin{bmatrix}a & b\\ p^mc & d\end{bmatrix}\in K_0(p^m)$ we have*

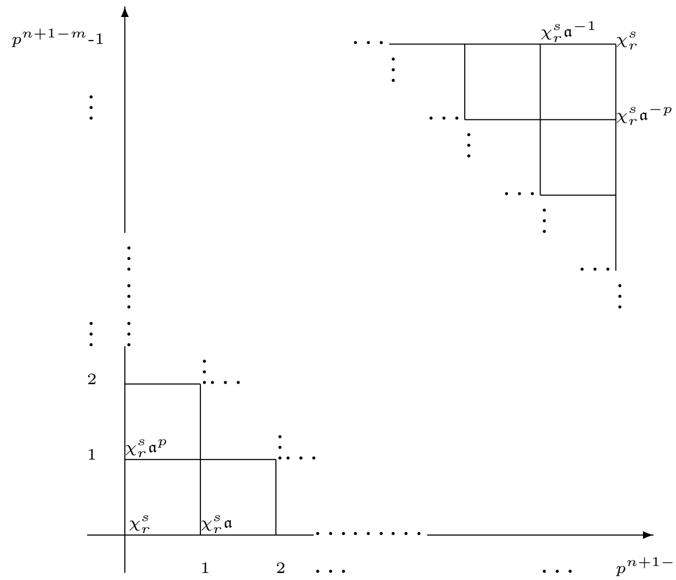
$$gF^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}=\mathfrak{a}^{e(\underline{l}_m,\dots,\underline{l}_n)}\chi_{\underline{r}}^s(g)(F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}-\sum_{s=0}^{f-1}(\overline{ca}^{-1})^{p^s}l_{a_0(s)}^{\lfloor s+a_0(s)-m\rfloor}F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n-e_s}+y)$$

where, putting $N\overset{\mathrm{def}}{=}N_{m,n}(\underline{l}_m,\dots,\underline{l}_n)$, we have $y\in(\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\chi_{\underline{r}}^s)_{N-1}$.

In particular, the $K_0(p)$-socle filtration, as well as the extensions between two consecutive graded pieces, of $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\chi_{\underline{r}}^s$ is described by the associated lattice $\mathscr{R}$.

We emphatise again the meaning of Proposition 4.2 in terms of lattices in $\mathbf{R}^f$: the socle filtration of $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\chi$ is given by cutting up the hypercube $\mathscr{R}$ by the antidiagonals $X_0+\cdots+X_{f-1}=N$ (precisely, soc_N is obtained from the cutting by the antidiagonal $X_0+\cdots+X_{f-1}=N$); the extensions between two consecutive graded pieces are visualized by the segments of length 1 obtained from the cutting of $\mathscr{R}$ by two consecutive antidiagonals $X_0+\cdots+X_{f-1}=N$, $X_0+\cdots+X_{f-1}=N-1$.

Here below an example for $f = 2$.



Here, each "point" in the lattice corresponds to a function $F^{m,n}_{\underline{l}_m,\dots,\underline{l}_n} \in \mathscr{B}$ according to the map P described in (7). The N-th composition factor $\mathrm{soc}_N(\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1)$ of the socle filtration can be read as the intersection of R with the semispace $X_0 + \cdots + X_{f-1} \leqslant N$, and the N-th graded piece $\mathrm{soc}_N(\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1)/\mathrm{soc}_{N-1}(\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1)$ as the intersection with the antidiagonal $X_0 + \cdots + X_{f-1} = N$. Finally, a "point" of coordinates $(\sum_{j=m}^n p^{j-m} l_j^{(\lfloor j-m \rfloor)}, \sum_{j=m}^n p^{j-m} l_j^{(\lfloor 1+j-m \rfloor)})$ should be understood as the character $\chi^s_{\underline{r}} \mathfrak{a}^{e(\underline{l}_m,\dots,\underline{l}_n)}$.

Remark 4.3. *We hope that suitable improvements of the techniques introduced here could lead to a better understanding of some important representations of the Iwahori subgroup ($R^{\pm}_{n+1}$, $R^{\pm}_{n+1}/R^{\pm}_n$, the universal representations...). For instance, our result shows that for a fixed point $P \in \mathscr{R}$ lying on the hyperplane $X_0 + \cdots + X_{f-1} = N$, the $K_0(p)$-subrepresentation generated by P lives inside the linear space generated by P, the elements $P - e_i$ for $i \in \{0, \dots, f-1\}$ and* some *elements lying strictly below the hyperplane $X_0 + \cdots + X_{f-1} = N - 1$. In particular, it is* not *clear (and probably* false*) that the $K_0(p)$-subrepresentation generated by $P = (x_0, \dots, x_{f-1})$ lives in the subspace*

$$X_0 \leqslant x_0 \cap \cdots \cap X_{f-1} \leqslant x_{f-1}.$$

An answer to this question would be of great importance in understanding supersingular representations for $\mathrm{GL}_2(F)$.

4.1.2 ***Proof of Proposition 4.2.*** The section is devoted to the proof of Proposition 4.2. It is the technical part of the paper and the methods rely on a careful analysis of suitable invariants associated to certain universal Witt polynomials. Such invariants, together with the choice of the "natural" linear basis $\mathscr{B}$, lead us to the following key phenomena:

i) the elements of the canonical basis $\mathscr{B}$ are "well behaved" with respect to the action of $g \in K_0(p)$, i.e. one can naturally describe $gF^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}$ as a linear combination of elements of $\mathscr{B}$;

ii) the parameters describing the elements appearing in the linear development of $gF^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}$ depend on some universal Witt polynomyals, whose (pseudo-)homogeneous degree is known (see section 6.4 for the precise definition of pseudo-homogeneity).

iii) the correspondence between the elements of $\mathscr{B}$ and the points in the associated hypercube is well behaved with respect to the homogeneous degree of the universal Witt polynomials.

It is here that we need the results of the Appendices A, B, which deal with certains invariants of some universal Witt polynomials; throughout the proofs of Propositions 4.4, 4.5, 4.7 we make use of some notations introduced in such appendices, in particular §6.2, 6.3 and 6.4.1 (we will give precise references in the proofs as well).

Thanks to the decomposition

$$K_0(p^m) = H \cdot \begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1+p\mathscr{O}_F & 0 \\ 0 & 1+p\mathscr{O}_F \end{bmatrix} \begin{bmatrix} 1 & 0 \\ p^m\mathscr{O}_F & 1 \end{bmatrix} \tag{8}$$

for $m \geqslant 1$ we are led to study separately the actions of lower unipotent, diagonal and upper unipotent matrices on the elements of the canonical basis $\mathscr{B}$: this will be the object of the next three paragraphs.

The action of lower unipotents matrices. We study here the action of the closed subgroup $\begin{bmatrix} 1 & 0 \\ p^m\mathscr{O}_F & 1 \end{bmatrix}$ of $K_0(p^m)$ on $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1$; we first need to introduce a family of $\overline{\mathbf{F}}_p$-subspaces of $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1$.

Let $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} \in \mathscr{B}$ and set $(x_0,\dots,x_{f-1}) \overset{\mathrm{def}}{=} P(F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}) \in \mathscr{R}$. We define the $\overline{\mathbf{F}}_p$-subspace $\mathfrak{W}_{(\underline{l}_m,\dots,\underline{l}_n)}$ of $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1$ as follows

$$P(\mathfrak{W}_{(\underline{l}_m,\dots,\underline{l}_n)}) \overset{\mathrm{def}}{=} \{(x'_0,\dots,x'_{f-1}) \in \mathscr{R} \quad \text{s.t. it exists}\, k \geqslant 0 \,\text{for which}$$
$$k(p-1) \leqslant \sum_{s=0}^{f-1}(x_s - x'_s) < (k+1)(p-1) \,\text{and}\, x'_j \leqslant x_j + k \,\text{for all}\, j = 0,\dots,f-1\}.$$

The image $P(\mathfrak{W}_{(\underline{l}_m,\dots,\underline{l}_n)}) \subseteq \mathbf{R}^f$ looks as a snowflake: in Figure 4 an example for $f=2$ (and $p=5$).

It is immediate to check that if $F^{(m,n)}_{\underline{l}'_m,\dots,\underline{l}'_n} \in \mathfrak{W}_{(\underline{l}_m,\dots,\underline{l}_n)}$ then $\mathfrak{W}_{(\underline{l}'_m,\dots,\underline{l}'_n)} \subseteq \mathfrak{W}_{(\underline{l}_m,\dots,\underline{l}_n)}$. The action of $\begin{bmatrix} 1 & 0 \\ p^m\mathscr{O}_F & 1 \end{bmatrix}$ is then described in the following

Proposition 4.4. *Let* $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} \in \mathscr{B}$, *and write* $N \overset{\mathrm{def}}{=} N_{m,n}(\underline{l}_m,\dots,\underline{l}_n)$. *Let* $g = \begin{bmatrix} 1 & 0 \\ p^m c & 1 \end{bmatrix} \in \begin{bmatrix} 1 & 0 \\ p^m\mathscr{O}_F & 1 \end{bmatrix}$ *for* $c \in \mathscr{O}_F$. *Then we have*

$$g \cdot F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} = F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} - \sum_{s=0}^{f-1} \overline{c}^{p^s} l_{a_0(s)}^{\lfloor s+a_0(s)-m \rfloor} F^{(m,n)}_{(\underline{l}_m,\dots,\underline{l}_n)-e_s} + y$$

for a suitable $y \in (\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1)_{N-1}$. *More precisely, via the projection*

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1 \overset{pr}{\twoheadrightarrow} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1/(\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1)_{N-(p^f+2)},$$

the image of the element y *is contained in the image of the subspace* $\mathfrak{W}_{(\underline{l}_m,\dots,\underline{l}_n)}$.

Proof. As the action of $\begin{bmatrix} 1 & 0 \\ p^m\mathscr{O}_F & 1 \end{bmatrix}$ is continuous, we can assume that c belongs to a set of topological generators (for the additive structure) of $\mathscr{O}_F$; in particular, we can assume $c = [\mu^{\frac{1}{p^m}}]$

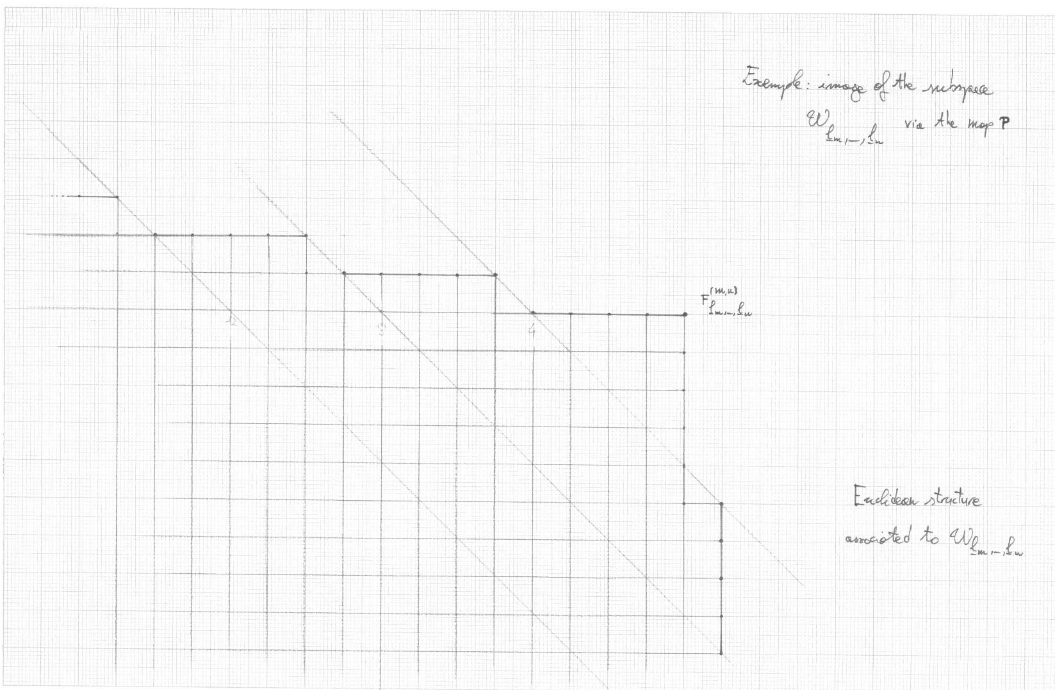


Figure 4: Euclidean interpretation of $\mathfrak{W}_{(\underline{l}_m,\dots,\underline{l}_n)}$ for $p = 5$. By Proposition 4.4 the representation generated by $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}$ under the action of lower unipotent matrices lives inside the linear space generated by $\mathfrak{W}_{(\underline{l}_m,\dots,\underline{l}_n)}$. Notice the fractal structure due to the behaviour of Witt polynomials.

for $\mu\in\mathbf{F}_q$.

Using the notations of §6.2, we can write the following equality in $p^m\mathscr{O}_F/p^{n+1}\mathscr{O}_F$:

$$p^m[\mu^{\frac{1}{p^m}}]+\sum_{j=m}^{n}p^j[\lambda_j^{\frac{1}{p^j}}]=\sum_{j=m}^{n}p^j[\lambda_j^{\frac{1}{p^j}}+(\widetilde{S}_{j-m}^{\frac{1}{p^j}})] \tag{9}$$

A direct computation describes[4] the action of g on the function $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}$:

$$\begin{bmatrix}1 & 0\\ p^m[\mu] & 1\end{bmatrix}F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}=$$
$$=\sum_{j=m}^{n-1}\sum_{\underline{i}_j\leqslant\underline{l}_j}\binom{\underline{l}_j}{\underline{i}_j}(-s_0(\widetilde{S}_0)^{\underline{i}_m})\sum_{\lambda_j\in\mathbf{F}_q}(\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j-\underline{i}_j}(-s_{j-m}(\widetilde{S}_{j-m+1})^{\frac{1}{p^{j+1}}})^{\underline{i}_{j+1}}\begin{bmatrix}1 & 0\\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1\end{bmatrix}\big[1,F^{(n)}_{\underline{l}_n-\underline{i}_n}\big].$$

As $deg(s_{j-1}(\widetilde{S}_j))\leqslant p^j$ for each $j\in\{1,\dots,n-m\}$ we can apply Proposition 7.3 (with $T_{m+j}=s_{j-1}(\widetilde{S}_j)$) to conclude that

$$g\cdot F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}=F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}+\sum_{s=0}^{f-1}\beta_sF^{(m,n)}_{(\underline{l}_m,\dots,\underline{l}_n)-e_s}+y$$

where $y\in\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1$ is the element described in the statement, for suitable elements $\beta_s\in\mathbf{F}_q$. We are now left to prove that $\beta_s=-(\mu^{\frac{1}{p^m}})^{p^s}l_{a_0(s)}^{\lfloor s+a_0(s)-m\rfloor}$.

We use the notations of Proposition 7.3 (in particular, we need the quantities $\kappa_a^{(b),s}$, $\kappa_a^{(b)}$ used in its proof) and we recall that, for $b=m+1,\dots,n$, a polynomial $-s_{b-m-1}(\widetilde{S}_{b-m}(\underline{X},Y))$ is homogeneous of degree p^{b-m} if X_a has degree p^a, Y degree p^0 (and $\widetilde{S}_0=Y$).

If we pick an element

$$x\overset{\mathrm{def}}{=}\sum_{\lambda_m\in\mathbf{F}_q}(\lambda_m^{\frac{1}{p^m}})^{\kappa_m}\begin{bmatrix}1 & 0\\ p^m[\lambda_m^{\frac{1}{p^m}}] & 1\end{bmatrix}\dots\sum_{\lambda_n\in\mathbf{F}_q}(\lambda_n^{\frac{1}{p^n}})^{\kappa_n}\begin{bmatrix}1 & 0\\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1\end{bmatrix}\big[1,e\big]$$

appearing in the development of $gF^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}$ we have, for $b\in\{m+1,\dots,n\}$,

$$\sum_{a=m}^{b-1}p^{a-m}\kappa_a^{(b),s}=i_b^{(s)}p^{b-m}-\alpha_b^{(s)}$$

where $i_b^{(s)}(p^{b-m}-1)\geqslant\alpha_b^{(s)}\geqslant i_b^{(s)}$ is the exponent of Y in the fixed monomial of $-s_{b-1-m}(\widetilde{S}_{b-m})^{i_b^{(s)}}$ (recall that any monomial $Y^c\prod_{i=0}^{b-1-m}X_i^{a_i}$ with $c=0$ or $\sum a_i=0$ appears in the development of

[4]It is in such situations that Definition 4.1 turns out to be useful, as it let us handle in a coherent way the exponents of the λ_j in the development of $(\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j-\underline{i}_j}(-s_{j-m}(\widetilde{S}_{j-m+1})^{\frac{1}{p^{j+1}}})^{\underline{i}_{j+1}}$.

$-s_{b-1-m}(\widetilde{S}_{b-m})$ with coefficient zero unless $b=m$). Considering that $p\geqslant 3$ the inequalities

$$\mathfrak{s}(\kappa_m)+p\,\mathfrak{s}(\kappa_{m+1})+\cdots+p^{n-m}\,\mathfrak{s}(\kappa_n)\leqslant$$

$$\leqslant(\mathfrak{s}(\underline{l}_m-\underline{i}_m)+\mathfrak{s}(p^{\lfloor -1\rfloor}\kappa_m^{(m+1)})+\cdots+\mathfrak{s}(p^{\lfloor -(n-m)\rfloor}\kappa_m^{(n)}))+$$
$$+p(\mathfrak{s}(\underline{l}_{m+1}-\underline{i}_{m+1})+\mathfrak{s}(p^{\lfloor -1\rfloor}\kappa_{m+1}^{(m+2)})+\cdots+\mathfrak{s}(p^{\lfloor -(n-m-1)\rfloor}\kappa_{m+1}^{(n)}))+\dots$$
$$\dots+p^{n-m-1}(\mathfrak{s}(\underline{l}_{n-1}-\underline{i}_{n-1})+\mathfrak{s}(p^{\lfloor -1\rfloor}\kappa_{n-1}^{(n)}))+p^{n-m}(\mathfrak{s}(\underline{l}_n-\underline{i}_n))\leqslant$$

$$\leqslant\mathfrak{s}(\underline{l}_m-\underline{i}_m)+\sum_{s=0}^{f-1}\mathfrak{s}(\kappa_m^{(m+1),s})+$$
$$p(\mathfrak{s}(\underline{l}_{m+1}-\underline{i}_{m-1}))+(\sum_{s=0}^{f-1}(\mathfrak{s}(\kappa_m^{(m+2),s}+p\,\mathfrak{s}(\kappa_{m+1}^{(m+2),s}))))+\dots$$
$$\dots+(\sum_{s=0}^{f-1}(\mathfrak{s}(\kappa_m^{(n),s})+p\,\mathfrak{s}(\kappa_{m+1}^{(n),s})+\cdots+p^{n-m-1}\,\mathfrak{s}(\kappa_{n-1}^{(n),s})))+p^{n-m}\,\mathfrak{s}(\underline{l}_n-\underline{i}_n)\leqslant$$

$$\leqslant\sum_{a=m}^{n}p^{a-m}(\mathfrak{s}(\underline{l}_a-\underline{i}_a))+\sum_{b=m+1}^{n}\left(p^{b-m}(\mathfrak{s}(\underline{i}_b))-\sum_{s=0}^{f-1}\alpha_b^{(s)}\right)$$

have to be equalities if we furthermore require our element to lie on the hyperplane $X_0+\cdots+X_{f-1}=N-1$; in particular we must have $i_b^{(s)}=0$ for all couples $(b,s)\in\{m,\dots,n\}\times\{0,\dots,f-1\}$ except one and only one, say (b_0,s_0), for which we must have $i_{b_0}^{(s_0)}=1$.

We notice that for $b_0\neq m$ we require furthermore that $\alpha_{b_0}=1$ i.e. the exponent of Y appearing in the fixed monomial of $-s_{b_0-m-1}(\widetilde{S}_{b_0-m})$ is 1. Thanks to Lemmas 6.3 and 6.4 we check that

$$x=-(\mu^{\frac{1}{p^m}})^{p^{s_0}}(l_{a_0(s)}^{\lfloor s+a_0(s)-m\rfloor})F^{(m,n)}_{\underline{l}_m,\dots,\underline{n}-e_{s_0}}$$

as required. □

The action of diagonal matrices. We are going to study the action of the subgroup

$$\begin{bmatrix}1+p\mathscr{O}_F & 0\\ 0 & 1+p\mathscr{O}_F\end{bmatrix}$$

on the elements of $\mathscr{B}$. If $z\in p^m\mathscr{O}_F/p^{n+1}\mathscr{O}_F$, an elementary computation shows that

$$\begin{bmatrix}1+pa & 0\\ 0 & 1+pd\end{bmatrix}\begin{bmatrix}1 & 0\\ z & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ z' & 1\end{bmatrix}\mathfrak{k}$$

where $\mathfrak{k}\in K_0(p^{n+1})$ is upper unipotent modulo p and $z'\in p^m\mathscr{O}_F/p^{n+1}\mathscr{O}_F$ is determined by the condition

$$z'\equiv(1+pa)^{-1}(1+pd)z\bmod p^{n+1}. \tag{10}$$

We can therefore content ourself studying the action of an element of the form $x\overset{\text{def}}{=}\begin{bmatrix}1 & 0\\ 0 & 1+p\alpha\end{bmatrix}$ for $\alpha\in\mathscr{O}_F$.

Proposition 4.5. *Let* $g\in\begin{bmatrix}1+p\mathscr{O}_F & 0\\ 0 & 1+p\mathscr{O}_F\end{bmatrix}$ *and fix* $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}\in\mathscr{B}$; *write* $N\overset{\text{def}}{=}N_{m,n}(\underline{l}_m,\dots,\underline{l}_n)$.

We then have the equality

$$g\cdot F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}=F^{m,n}_{\underline{l}_m,\dots,\underline{l}_n}+y$$

where $y\in \mathrm{Ind}^{K_0(p^m)}_{K_0(p^{n+1})}1)_{N-1}$.

More precisely, via the projection

$$\mathrm{Ind}^{K_0(p^m)}_{K_0(p^{n+1})}1\overset{pr}{\twoheadrightarrow}\mathrm{Ind}^{K_0(p^m)}_{K_0(p^{n+1})}1/(\mathrm{Ind}^{K_0(p^m)}_{K_0(p^{n+1})}1)_{N-(p^f+2)},$$

the image of y *is contained in the image of the subspace* $\mathfrak{W}_{(\underline{l}_m,\dots,\underline{l}_n)}$ *and writing*

$$y=\sum_{i\in I}\beta_i F^{(m,n)}_{\underline{l}_m(i),\dots,\underline{l}_n(i)}$$

(for a suitable set of indices I *and scalars* $\beta_i\in\overline{\mathbf{F}}_p^{\times}$*) we have that each function* $F^{(m,n)}_{\underline{l}_m(i),\dots,\underline{l}_n(i)}$ *which is not in the kernel* $\ker(pr)$ *lies on an hyperplane*

$$X_0+\dots+X_{f-1}=N-t(p-1)$$

for some $t\in\mathbf{N}_>$.

Proof. The proof is completely analogous to the proof of Proposition 4.4. As remarked above, it is enough to consider the case $x=\begin{bmatrix}1&0\\0&1+p\alpha\end{bmatrix}$ where $\alpha=\sum_{j=0}^{\infty}p^j[\alpha_j^{\frac{1}{p^j}}]$. Using the notations of §6.3 we see that

$$(1+p\alpha)(\sum_{j=m}^{n}p^j[\lambda_j^{\frac{1}{p^j}}])\equiv\sum_{j=m}^{n}p^j[\lambda_j^{\frac{1}{p^j}}+\widetilde{Q}_j^{\frac{1}{p^j}}]\bmod p^{n+1}$$

and we deduce

$$\begin{bmatrix}1&0\\0&1+p\alpha\end{bmatrix}F^{m,n}_{\underline{l}_m,\dots,\underline{l}_n}=$$
$$=\sum_{j=m}^{n-1}\sum_{\substack{\underline{i}_j\leqslant\underline{l}_j\\ \underline{i}_m=\underline{0}}}\binom{\underline{l}_j}{\underline{i}_j}\sum_{\lambda_j\in\mathbf{F}_q}(\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j-\underline{i}_j}(-q_{j-m}(\widetilde{Q}_{j+1-m}))^{\underline{i}_{j+1}}\begin{bmatrix}1&0\\p^j[\lambda_j^{\frac{1}{p^j}}]&1\end{bmatrix}\big[1,F^{(n)}_{\underline{l}_n-\underline{i}_n}\big]\qquad(11)$$

(with the obvious conventions if $n\in\{m,m+1\}$). As each polynomial $(-q_{j-1}(\widetilde{Q}_j))\in\mathbf{F}_p[\lambda_m,\dots,\lambda_{j-1-m}]$, for $1\leqslant j\leqslant n-m$ is homogeneous of degree p^j (in the shifted grading for which λ_{m+h} is homogeneous of degree p^h for $h\geqslant 0$) we can apply Proposition 7.3 with $T_{m+j}=(-q_{j-1}(\widetilde{Q}_j))$ to get the first part of the statement.

We are left to prove 2). Consider an integer $t\in\mathbf{N}$ and an hyperplane $\mathfrak{H}:X_0+\dots X_{f-1}=N-t$. Following the proof of Proposition 7.3, a necessary condition for an element

$$\sum_{\lambda_m\in\mathbf{F}_q}(\lambda_m^{\frac{1}{p^m}})^{\kappa_m}\begin{bmatrix}1&0\\p^m[\lambda_m^{\frac{1}{p^m}}]&1\end{bmatrix}\dots\sum_{\lambda_n\in\mathbf{F}_q}(\lambda_n^{\frac{1}{p^n}})^{\kappa_n}\begin{bmatrix}1&0\\p^n[\lambda_n^{\frac{1}{p^n}}]&1\end{bmatrix}\big[1,e\big]$$

appearing in the developement of (11) to lie in $\mathfrak{H}$ is then

$$\sum_{j=m}^{n}p^{j-m}\,\mathfrak{s}(\kappa_j)\equiv N-t\bmod p-1.$$

Again, as each polynomial $(-q_{j-1}(\widetilde{Q}_j))$, for $1\leqslant j\leqslant n-m$ is homogeneous of degree p^j, and $\mathfrak{s}(h)\equiv h\bmod p-1$ we deduce that inequalities (25), (26), (27) and (28) appearing in the proof of

Proposition 7.3 are actually *equalities* in $\mathbf{Z}/(p-1)$ so that we get

$$\sum_{j=m}^{n} p^{j-m}\,\mathfrak{s}(\kappa_j) \equiv N - \mathfrak{s}(\underline{i_m}) \bmod p-1 = N.$$

The conclusion follows. □

The action of upper unipotent matrices. We are left to study the action of the closed subgroup $\begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix}$ on the elements of $\mathscr{B}$. We recall that the action of $K_0(p^m)$ is continuous on $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1$ and the natural topology on $\begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix}$ coincides with the topology induced (via the natural immersion) by $K_0(p^m)$. Thanks to the isomorphisms of abelian topological groups

$$\begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix} \cong \mathscr{O}_F \cong (\mathbf{Z}_p)^f$$

where the latter isomorphism is determined by the choice of a primitive element $\alpha \in \mathbf{F}_q$ of $\mathbf{F}_q$ over $\mathbf{F}_p$ (cf. Serre [Ser], Proposition 16 Ch.I) it is enough to study the action of elements $g \in \begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix}$ of the form $g = \begin{bmatrix} 1 & [\mu] \\ 0 & 1 \end{bmatrix}$ for $\mu \in \mathbf{F}_q$.

We start with an elementary computation:

Lemma 4.6. *Let $z \in p^m\mathscr{O}_F/p^{n+1}\mathscr{O}_F$ and $\mu \in \mathbf{F}_q$. We have the following equality:*

$$\begin{bmatrix} 1 & [\mu] \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z' & 1 \end{bmatrix}\mathfrak{k}$$

where $\mathfrak{k} \in K_0(p^{n+1})$ is upper unipotent modulo p and $z' \in p^m\mathscr{O}_F/p^{n+1}\mathscr{O}_F$ is uniquely determined by the condition

$$z' \equiv z(1+z[\mu])^{-1} \bmod p^{n+1} \equiv \sum_{j=0}^{N}((-1)^j z^{j+1}[\mu^j]) \bmod p^{n+1}$$

for $N \stackrel{\text{def}}{=} \lfloor \frac{n+1}{m} \rfloor$.

Proof. Omissis. □

We are now left to use Lemma 4.6 and the results of §6.4 in order to describe the required action of $\begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix}$:

Proposition 4.7. *Let $g \in \begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix}$ and fix $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} \in \mathscr{B}$. Write* [5] $N \stackrel{\text{def}}{=} N_{m,n}(\underline{l}_m,\dots,\underline{l}_n)$. *In the quotient space* [6]

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1/(\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1)_{N-(p^m-2)+1}$$

[5] Of course, this N does not have anything to do with $N \stackrel{\text{def}}{=} \lfloor \frac{n+1}{m} \rfloor$. We believe this conflict of notations will not give rise to any confusion, as the meaning of N will be clear from the context.

[6] It is here that the assumption $p \neq 3$ is important: indeed, for $p = 3$, $m = 1$ the quotient space is $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1/(\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1)_N$ which is too small to deduce interesting information about the action of upper unipotent matrices.

we have the equality

$$g\cdot F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}=F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}.$$

Proof. As remarked at the begining of this paragraph, we can assume $g=\begin{bmatrix}1&[\mu]\\0&1\end{bmatrix}$ where $\mu\in\mathbf{F}_q$.

Using Lemma 4.6 and the results (and notations) of §6.4.1 we get the following equality in $\mathscr{O}_F/(p^{n+1})$:

$$\sum_{j=0}^{N}(-1)^j z^{j+1}[\mu^j]\equiv\sum_{j=m}^{n}p^j[\lambda_j^{\frac{1}{p^j}}+\widetilde{U}_j^{\frac{1}{p^j}}]\,\mathrm{mod}\,p^{n+1}$$

so that, inside $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1$, we have:

$$gF^{m,n}_{\underline{l}_m,\dots,\underline{l}_n}=\sum_{j=m}^{n-1}\sum_{\substack{\underline{i}_j\leqslant\underline{l}_j\\ \underline{i}_m=\underline{0}}}\binom{\underline{l}_j}{\underline{i}_j}\sum_{\lambda_j\in\mathbf{F}_q}(\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j-\underline{i}_j}(-u_j(\widetilde{U}_{j+1}^{\frac{1}{p^{j+1}}}))^{\underline{i}_{j+1}}\begin{bmatrix}1&0\\p^j[\lambda_j^{\frac{1}{p^j}}]&1\end{bmatrix}\left[1,F^{(n)}_{\underline{l}_n-\underline{i}_n}\right]$$

and we recall that $\widetilde{U}_j=0$ for $m\leqslant j\leqslant 2m-1$. As for each $2m\leqslant j\leqslant n$ the polynomial $-u_{j-1}(\widetilde{U}_j)$ is pseudo-homogeneous of degree $p^j-p^m(p^m-2)$ the conclusion follows from Proposition 7.4, with $V_j=-u_{j-1}(\widetilde{U}_j)$. □

Proof of Proposition 4.2. The last step in order to complete the proof of Proposition 4.2 is immediate:

Proposition 4.8. *Let $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}\in\mathscr{B}$ and let $a,d\in\mathbf{F}_q$. We then have the following equality in $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1$:*

$$\begin{bmatrix}[a]&0\\0&[d]\end{bmatrix}F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}=\mathfrak{a}^{e(\underline{l}_m,\dots,\underline{l}_n)}(\begin{bmatrix}[a]&0\\0&[d]\end{bmatrix})F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}.$$

In particular

$$\begin{bmatrix}[a]&0\\0&[d]\end{bmatrix}F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n-e_s}=\mathfrak{a}^{e(\underline{l}_m,\dots,\underline{l}_n)-p^s}(\begin{bmatrix}[a]&0\\0&[d]\end{bmatrix})F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n-e_s}.$$

Proof. We just remark that for $z=\sum_{j=m}^{n}p^j[\lambda_j]\in p^m\mathscr{O}_F/p^{n+1}\mathscr{O}_F$ we have

$$\begin{bmatrix}[a]&0\\0&[d]\end{bmatrix}\begin{bmatrix}1&0\\z&1\end{bmatrix}=\begin{bmatrix}1&0\\z[a^{-1}d]&1\end{bmatrix}\begin{bmatrix}[a]&0\\0&[d]\end{bmatrix}$$

and that

$$z[a^{-1}d]=\sum_{j=m}^{n}p^j[\lambda_j(a^{-1}d)].$$

□

Finally, for $a,b,c,d\in\mathscr{O}_F$ as in the statement of Proposition 4.2, we recall the matrix equality

$$\begin{bmatrix}a&b\\p^mc&d\end{bmatrix}=\begin{bmatrix}[a]&0\\0&[d]\end{bmatrix}\begin{bmatrix}1&0\\p^mz&1\end{bmatrix}\begin{bmatrix}1+px&0\\0&1+pw\end{bmatrix}\begin{bmatrix}1&y\\0&1\end{bmatrix}$$

where $x,y,z,w\in\mathscr{O}_F$ are suitable integers verifying $\overline{z}=\overline{cd^{-1}}$. The result follows now from Propositions 4.4, 4.5, 4.7 and Lemma 4.8. □

Remark 4.9. *We note that the bijection (7)* depends *on the immersion $\tau : \mathbf{F}_q \hookrightarrow \overline{\mathbf{F}}_p$ fixed in the introduction and should be noted as P_τ. As another immersion $\tau' : \mathbf{F}_q \hookrightarrow \overline{\mathbf{F}}_p$ is obtained by composing τ with a power ϕ^a of the frobenius on $\mathbf{F}_q$ we see that the map $P_{\tau'}$ is obtained by composing P_τ with a power Φ^a, where $\Phi \in \mathrm{End}(\mathbf{R}^f)$ is defined by $\Phi(e_s) = e_{\lfloor s+1 \rfloor}$. Hence, as the antidiagonal is fixed under Φ, Proposition 4.2 does not depend on τ.*

4.1.3 ***The structure of the representations R_n^-.*** Fix an integer $n \in \mathbf{N}$. We describe here the socle filtration (and the extensions between two consecutive graded pieces) for the $K_0(p)$-representations R_{n+1}^-. Again, we can identify the negative elements of R_{n+1}^- with the points of a lattice of $\mathbf{R}^f$ according to the following injective map

$$\begin{aligned} \mathscr{B}_{n+1}^- &\hookrightarrow \mathbf{R}^f \\ F^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}(\underline{l}_{n+1}) &\mapsto \Big(\sum_{a=1}^{n+1} p^{a-1} l_a^{\lfloor s+a-1 \rfloor}\Big)_{s\in\{0,\dots,f-1\}} \end{aligned}$$

whose image will be denoted by $\mathscr{R}_{n+1}^-$; we define in the evident way the subspaces $(R_{n+1}^-)_N$ for $N \in \mathbf{N}$.

The structure of R_{n+1}^- is then sumarized in the following

Proposition 4.10. *Let $n \in \mathbf{N}$, $F^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}(\underline{l}_{n+1}) \in \mathscr{B}_{n+1}^-$ and let $a, b, c, d \in \mathscr{O}_F$ be such that $g \overset{\text{def}}{=} \begin{bmatrix} a & b \\ pc & d \end{bmatrix} \in K_0(p)$. Define finally the integer $N \overset{\text{def}}{=} N_{1,n+1}(\underline{l}_1, \dots, \underline{l}_{n+1})$.*

We have the equality

$$gF^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}(\underline{l}_{n+1}) = \mathfrak{a}^{e(\underline{l}_1,\dots,\underline{l}_{n+1})}\chi^s_{\underline{r}}(g)\Big(F^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}(\underline{l}_{n+1}) - \sum_{s=0}^{f-1}(\overline{ca}^{-1})^{p^s} l_{a_0(s)}^{\lfloor s+a_0(s)-1 \rfloor}(-1)^{\delta_{a_0(s),n+1}} F^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}(\underline{l}_{n+1}) + y\Big)$$

where $y \in (R_{n+1}^-)_{N-1}$.

In particular, the $K_0(p)$-socle filtration of R_{n+1}^-, as well as the extensions between two consecutive graded pieces, are described by the associated lattice $\mathscr{R}_{n+1}^-$.

Proof. We notice that we have a $K_0(p^{n+1})$-equivariant monomorphism

$$\begin{aligned} \sigma_{\underline{r}}^{(n+1)} &\hookrightarrow \mathrm{Ind}_{K_0(p^{n+2})}^{K_0(p^{n+1})} \chi^s_{\underline{r}} \\ X^{\underline{r}-\underline{l}_{n+1}} Y^{\underline{l}_{n+1}} &\mapsto (-1)^{\underline{l}_{n+1}} \sum_{\lambda_{n+1}\in\mathbf{F}_q} (\lambda_{n+1}^{\frac{1}{p^{n+1}}})^{\underline{l}_{n+1}} \begin{bmatrix} 1 & 0 \\ p^{n+1}[\lambda_{n+1}^{\frac{1}{p^{n+1}}}] & 1 \end{bmatrix} [1, e]. \end{aligned}$$

By transitivity and exactness of the induction functor $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)}(\bullet)$ we get a $K_0(p)$-equivariant monomorphism

$$\begin{aligned} R_{n+1}^- &\hookrightarrow \mathrm{Ind}_{K_0(p^{n+2})}^{K_0(p)} \chi^s_{\underline{r}} \\ F^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}(\underline{l}_{n+1}) &\mapsto (-1)^{\underline{l}_{n+1}} F^{(1,n+1)}_{\underline{l}_1,\dots,\underline{l}_n,\underline{l}_{n+1}}. \end{aligned}$$

The conclusion is now immediate from Proposition 4.2. □

4.2 The positive case

This section is again divided into two parts. We begin with the study of the $K_0(p)$-representations R_{n+1}^+, for $n \in \mathbf{N}$: they are described in Proposition 4.11. We subsequently switch our attention introducing other $K_0(p)$ representations (the representations $(\mathrm{Ind}_{K_0(p^{n+1})}^K \chi^s)^+$ defined in §4.3) which will let us describe the $K_0(p)$-restriction of principal and special series (see §4.3).

The philosophy is completely analogous to the one of the previous paragraph: we verify by a direct computation on the ring of Witt vectors that the $K_0(p)$-structure of such objects can be described in terms of f-parallelepipoids in the Euclidean space $\mathbf{R}^f$.

Fix $n \in \mathbf{N}$. We introduce the injective map

$$\mathscr{B}^+_{n+1} \hookrightarrow \mathbf{R}^f$$
$$F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) \mapsto (\sum_{i=0}^{n+1} p^i l_i^{(\lfloor s+i \rfloor)})_{s\in\{0,\dots,f-1\}}$$

which lets us interpret the positive elements of R^+_{n+1} as points in a convenient lattice of $\mathbf{R}^f$. The image of such map (which is a parallelepipoid of side $p^{n+1}(r_s+1)-1$) will be denoted as $\mathscr{R}^+_{n+1}$. We still need the following notations (see also §4.1.1):

i) for a $(n+2)f$-tuple $(\underline{l}_0,\dots,\underline{l}_{n+1}) \in \left\{\{0,\dots,p-1\}^f\right\}^{n+2}$ define the integers

$$N_{0,n+1}(\underline{l}_0,\dots,\underline{l}_{n+1}) \overset{\text{def}}{=} \sum_{a=0}^{n+1} p^a \mathfrak{s}(\underline{l}_a)$$
$$e(\underline{l}_0,\dots,\underline{l}_{n+1}) \overset{\text{def}}{=} (\sum_{s=0}^{f-1} p^s l_0^{(s)}) + \dots + (\sum_{s=0}^{f-1} p^s l_{n+1}^{(s)});$$

ii) for $N \in \mathbf{N}$ we define the $\overline{\mathbf{F}}_p$-linear subspace

$$(R^+_{n+1})_N \overset{\text{def}}{=} \left\langle F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) \in \mathscr{B}^+_{n+1} \quad \text{s.t.} \quad N_{0,n+1}(\underline{l}_0,\dots,\underline{l}_{n+1}) < N \right\rangle_{\overline{\mathbf{F}}_p};$$

iii) for a fixed $(n+2)f$-tuple $(\underline{l}_0,\dots,\underline{l}_{n+1})$ and $s \in \{0,\dots,f-1\}$, we define

$$\Xi_s \overset{\text{def}}{=} \left\{a \in \{0,\dots,n+1\}, \quad \text{s.t. } l_a^{\lfloor s+a \rfloor} \neq 0\right\}$$

and we set

$$a_0(s) \overset{\text{def}}{=} \begin{cases} \min(\Xi_s) & \text{if } \Xi_s \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

For a given positive element $F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1})$ we define the subspace $\mathfrak{W}_{(\underline{l}_0,\dots,\underline{l}_{n+1})}$ in the evident, similar way.

The structure of R^+_{n+1} is then given by

Proposition 4.11. *Let $n \in \mathbf{N}$, $F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) \in \mathscr{B}^+_{n+1}$ and let $a,b,c,d \in \mathscr{O}_F$ be such that $g \overset{\text{def}}{=} \begin{bmatrix} a & b \\ pc & d \end{bmatrix} \in K_0(p)$. Define finally the integer $N \overset{\text{def}}{=} N_{0,n+1}(\underline{l}_0,\dots,\underline{l}_{n+1})$. We then have*

$$gF^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) = (\mathfrak{a}^{-1})^{e(\underline{l}_0,\dots,\underline{l}_{n+1})}\chi_{\underline{r}}(g)(F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) - \sum_{s=0}^{f-1}(\overline{bd}^{-1})^{p^s} l_{a_0(s)}^{\lfloor s+a_0(s)\rfloor}(-1)^{\delta_{a_0(s),n+1}} F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) + y)$$

where $y \in (R^+_{n+1})_{N-1}$.

In particular, the $K_0(p)$-filtration, as well as the extensions between two consecutive pieces, is described by the associated lattice $\mathscr{R}^+_{n+1}$.

Proof. The proof is analogous to the proof of Proposition 4.2, using this time Lemma 6.17 and Proposition 7.5. The details are left as an exercise to the reader. □

4.2.1 ***On some other $K_0(p)$-representations.*** As announced in the introduction, we define and study some $K_0(p)$-representations (denoted as $(\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi)^+$) which naturally appear when dealing with the Iwahori structure of principal and special series. The reader will realize soon that the behaviour of the representations $(\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi)^+$ can be treated with the same methods of §4.2 and 4.1; the proofs will be therefore omitted.

Fix an integer $n \in \mathbf{N}$, a smooth character $\chi : K_0(p^{n+1}) \to \overline{\mathbf{F}}_p^{\times}$ and an $\overline{\mathbf{F}}_p$-basis $\{e\}$ for the underlying vector space of χ. The $K_0(p)$-representation $(\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi)^+$ is defined as the $K_0(p)$-subrepresentation induced by $\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi$ on the $\overline{\mathbf{F}}_p$-subspace

$$\langle \Big[\begin{bmatrix} [z] & 1 \\ 1 & 0 \end{bmatrix}, e \Big] \in \mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi, \quad z \in I_{n+1} \rangle_{\overline{\mathbf{F}}_p}$$

(the $K_0(p)$-stability of such $\overline{\mathbf{F}}_p$-linear space is immediately verified). Again, we have the

Definition 4.12. *Let $j \in \{0, \dots, n\}$ and let $\underline{l}_j \in \{0, \dots, p-1\}^f$ be a f-tuple. We define the following element of $(\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi)^+$:*

$$F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n} \overset{\mathrm{def}}{=} \sum_{\lambda_0 \in \mathbf{F}_q} \lambda_0^{\underline{l}_0} \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix} \sum_{j=1}^{n} \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j} \begin{bmatrix} 1 & 0 \\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1 \end{bmatrix} [1, e].$$

The family

$$\mathscr{B}^+ \overset{\mathrm{def}}{=} \{F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n} \in (\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi)^+, \quad \underline{l}_j \in \{0, \dots, p-1\}^f \quad \text{for all } j \in \{0, \dots, n\}\}$$

is an $\overline{\mathbf{F}}_p$-basis for $(\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi)^+$.

Exactly as we did for R^+_{n+1}, each given element $F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}$ of $\mathscr{B}^+$ will be read as a point in a convenient lattice $\mathscr{R}$ of $\mathbf{R}^f$ and the integers $a_0(s)$ (for $s \in \{0, \dots, f-1\}$) can be assigned. Moreover, if $N \in \mathbf{N}$, the subspaces $((\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi)^+)_N$ are defined in the similar, evident way (see the introduction of §4.2 for details).

The structure of the representations $(\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi)^+$ is then described in the next

Proposition 4.13. *Let $\underline{r} \in \{0, \dots, p-1\}^f$ be an f-tuple, $n \in \mathbf{N}$ an integer and let $a, b, c, d \in \mathscr{O}_F$ be such that $g \overset{\mathrm{def}}{=} \begin{bmatrix} a & b \\ pc & d \end{bmatrix} \in K_0(p)$. Fix an element $F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n} \in \mathscr{B}^+$ and set $N \overset{\mathrm{def}}{=} N_{0,n}(\underline{l}_0, \dots, \underline{l}_n)$.*

Then

$$g \cdot F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n} = (\mathfrak{a}^{-1})^{e(\underline{l}_0,\dots,\underline{l}_n)} \chi_{\underline{r}}(g) (F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n} - \sum_{s=0}^{f-1} (\overline{bd}^{-1})^{p^s} l_{a_0(s)}^{\lfloor s+a_0(s) \rfloor} F^{0,n}_{(\underline{l}_0,\dots,\underline{l}_n)-e_s} + y)$$

where $y \in (\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_{\underline{r}}^{s^+})_{N-1}$.

In particular the $K_0(p)$-socle filtration of $(\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_{\underline{r}}^{s})^+$, as well as the extensions of two consecutive graded pieces, are described by the associated lattice $\mathscr{R}$.

Proof. Omissis. □

4.3 The Iwahori structure of Principal and Special Series

We are now able to describe easily the Iwahori-structure of principal and special series for $\mathrm{GL}_2(F)$. Such result is essentially a formal consequence of the previous sections §4.1 and §4.2.1.

For $\lambda \in \overline{\mathbf{F}}_p^{\times}$ and $\underline{r} \in \{0, \dots, p-1\}^f$ we consider the smooth parabolic induction

$$\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \mu_\lambda \otimes \omega^{\underline{r}} \mu_{\lambda^{-1}}$$

where ω denotes the mod p cyclotomic character and μ_λ the unramified character verifying $\mu_\lambda(p) = \lambda$. It is well known that for $(\underline{r}, \lambda) \notin \{(\underline{0}, \pm 1), (\underline{p-1}, \pm 1)\}$ such inductions are irreducible, while, if $(\underline{r}, \lambda) \in \{(\underline{0}, \pm 1), (\underline{p-1}, \pm 1)\}$ they have length 2 and a unique infinite dimensional factor, the Steinberg representation (see also [BL94]). Thanks to the Iwahori decomposition and Mackey theorem we have

$$\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \mu_\lambda \otimes \omega^{\underline{r}} \mu_{\lambda^{-1}}|_K \xrightarrow{\sim} \mathrm{Ind}_{B(\mathscr{O}_F)}^{\mathrm{GL}_2(\mathscr{O}_F)} \chi_{\underline{r}}^s$$

and, since the elements $f \in \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \mu_\lambda \otimes \omega^{\underline{r}} \mu_{\lambda^{-1}}$ are locally constant functions and $B(\mathscr{O}_F) \backslash \mathrm{GL}_2(\mathscr{O}_F)$ is compact we have a natural isomorphism

$$\mathrm{Ind}_{B(\mathscr{O}_F)}^{\mathrm{GL}_2(\mathscr{O}_F)} \chi_{\underline{r}}^s \xrightarrow{\sim} \varinjlim_{n \in \mathbf{N}} \mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_{\underline{r}}^s.$$

Again, we can use Mackey decomposition to deduce

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_{\underline{r}}^s|_{K_0(p)} \xrightarrow{\sim} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^s \oplus (\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_{\underline{r}}^s)^+$$

so that, by the exactness property of filtrant inductive limit, we get

$$\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \mu_\lambda \otimes \omega^{\underline{r}} \mu_{\lambda^{-1}}|_{K_0(p)} \xrightarrow{\sim} (\varinjlim_{n \in \mathbf{N}} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^s) \oplus (\varinjlim_{n \in \mathbf{N}} (\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_{\underline{r}}^s)^+). \tag{12}$$

The isomorphism (12) lets us reduce to the case of the finite inductions $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^s$, $(\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_{\underline{r}}^s)^+$, whose structure is completely described in Propositions 4.2 and 4.13. Therefore

Theorem 4.14. *Let $\lambda \in \overline{\mathbf{F}}_p^{\times}$ and $\underline{r} \in \{0, \dots, p-1\}^f$ an f-tuple. For any $m \in \mathbf{N}_>$ we write*

$$F_{\underline{0}, \dots, \underline{0}, \dots}^{(m, \infty)} \in \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \mu_\lambda \otimes \omega^{\underline{r}} \mu_{\lambda^{-1}}$$

to denote the characteristic function of $K_0(p^m)$.

The $K_0(p)$-restriction of the parabolic induction admits a natural splitting

$$\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \mu_\lambda \otimes \omega^{\underline{r}} \mu_{\lambda^{-1}}|_{K_0(p)} \xrightarrow{\sim} (\varinjlim_{n \in \mathbf{N}} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^s) \oplus (\varinjlim_{n \in \mathbf{N}} (\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_{\underline{r}}^s)^+).$$

Moreover an $\overline{\mathbf{F}}_p$-basis $\mathscr{B}^-$ for $\varinjlim_{n \in \mathbf{N}} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^s$ (risp. $\mathscr{B}^+$ for $\varinjlim_{n \in \mathbf{N}} (\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_{\underline{r}}^s)^+$) is described by the elements

$$F_{\underline{l}_1, \dots, \underline{l}_n, \dots}^{(1, \infty)} \overset{\mathrm{def}}{=} \sum_{\lambda_1 \in \mathbf{F}_q} (\lambda_1^{\frac{1}{p}})^{\underline{l}_1} \begin{bmatrix} 1 & 0 \\ p[\lambda_1^{\frac{1}{p}}] & 1 \end{bmatrix} \dots \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{\underline{l}_n} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} \dots$$

(risp. the elements

$$F_{\underline{l}_0, \dots, \underline{l}_n, \dots}^{(0, \infty)} \overset{\mathrm{def}}{=} \sum_{\lambda_0 \in \mathbf{F}_q} \lambda_0^{\underline{l}_0} \begin{bmatrix} [\lambda_1] & 1 \\ 1 & 0 \end{bmatrix} \dots \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{\underline{l}_1} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} \dots)$$

for a varying sequence $(\underline{l}_n)_{n \in \mathbf{N}_>} \in \{0, \dots, p-1\}^{(\mathbf{N}_>)}$ (resp. $(\underline{l}_n)_{n \in \mathbf{N}} \in \{0, \dots, p-1\}^{(\mathbf{N})}$).

If we associate the elements of such basis to points in $\mathbf{R}^f$ *according to the law*

$$F^{(1,\infty)}_{\underline{l}_1,\ldots,\underline{l}_n,\ldots,} \mapsto (\sum_{i=1}^{\infty} p^{i-1} l_i^{\lfloor s+i-1\rfloor})_{s\in\{0,\ldots,f-1\}}$$
$$F^{(0,\infty)}_{\underline{l},\ldots,\underline{l}_n,\ldots,} \mapsto (\sum_{i=0}^{\infty} p^{i} l_i^{\lfloor s+i\rfloor})_{s\in\{0,\ldots,f-1\}}$$

and write $\mathscr{R}^-$ *(resp* $\mathscr{R}^+$*) to denote the image of* $\mathscr{B}^-$ *(resp.* $\mathscr{B}^+$*) by this map, then the* $K_0(p)$*-socle filtration for* $\varinjlim_{n\in\mathbf{N}} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)}\chi_{\underline{r}}^s$ *(resp. for* $\varinjlim_{n\in\mathbf{N}}(\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_{\underline{r}}^s)^+$*), as well as the extentions between two graded pieces, is described by the associated lattice* $\mathscr{R}^-$ *(risp.* $\mathscr{R}^+$*).*

The Iwahori structure of irreducible principal series follows.

As far as the Steinberg representation is concerned, we just need to notice the following fact:

Lemma 4.15. *Assume* $\underline{r}\in\{\underline{0},\underline{p-1}\}$ *and let* $n\in\mathbf{N}$*. We have a* $K_0(p)$*-equivariant exact sequence*

$$0\to\langle(F^{(0)}_{\underline{0}},F^{(1,0)}_{\emptyset})\rangle\to \mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_{\underline{r}}^{s^+}\oplus\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)}\chi_{\underline{r}}^{s}\to(\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_{\underline{r}}^{s}/\langle 1\rangle)|_{K_0(p)}\to 0.$$

Proof. The proof is an induction on n, the case $n=0$ being well known (cf. [Br-Pa], Lemma 2.6).

For the general case, we leave to the reader the easy task to check that we have a natural commutative diagram with exact lines

$$\begin{array}{ccc}
0 & & 0\\
\downarrow & & \downarrow\\
\mathrm{Ind}_{K_0(p^{n})}^{K_0(p)}\chi_{\underline{r}}^{s}\oplus\mathrm{Ind}_{K_0(p^{n})}^{K}\chi_{\underline{r}}^{s^+} & \twoheadrightarrow & (\mathrm{Ind}_{K_0(p^{n})}^{K}\chi_{\underline{r}}^{s}/\langle 1\rangle)\\
\downarrow & & \downarrow\\
\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)}\chi_{\underline{r}}^{s}\oplus\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_{\underline{r}}^{s^+} & \twoheadrightarrow & (\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_{\underline{r}}^{s}/\langle 1\rangle)\\
\downarrow & & \downarrow\\
(\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)}\chi_{\underline{r}}^{s}/\mathrm{Ind}_{K_0(p^{n})}^{K_0(p)}\chi_{\underline{r}}^{s})\oplus\big((\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_{\underline{r}}^{s})^+/(\mathrm{Ind}_{K_0(p^{n})}^{K}\chi_{\underline{r}}^{s})^+\big) & \overset{\cong}{\twoheadrightarrow} & \mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_{\underline{r}}^{s}/\mathrm{Ind}_{K_0(p^{n})}^{K}\chi_{\underline{r}}^{s}\\
\downarrow & & \downarrow\\
0 & & 0
\end{array}$$

and the snake Lemma together with the inductive hypothesis gives us the exact sequence

$$0\to\langle(F^{(0)}_{\underline{0}},F^{(1,0)}_{\emptyset})\rangle\to \mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_{\underline{r}}^{s^+}\oplus\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)}\chi_{\underline{r}}^{s}\to(\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_{\underline{r}}^{s}/\langle 1\rangle)|_{K_0(p)}\to 0.$$

This ends the proof. □

5. The structure of the universal representation

In this section we show how the technical results of §4 concerning the representations $R^{\pm}_{n+1}$ and the formalism of §3 let us describe the Iwahori structure for the universal representation $\pi(\underline{r},0,1)$. Again, we develop an Euclidean dictionary which enable us to handle the involved combinatorics of $\pi(\underline{r},0,1)|_{K_0(p)}$: the conclusion is then Proposition 5.18, which loosely speaking shows that the required structure is obtained by a juxtaposition of the blocks $R^{\pm}_{n+1}$ in a fractal way. As a byproduct,

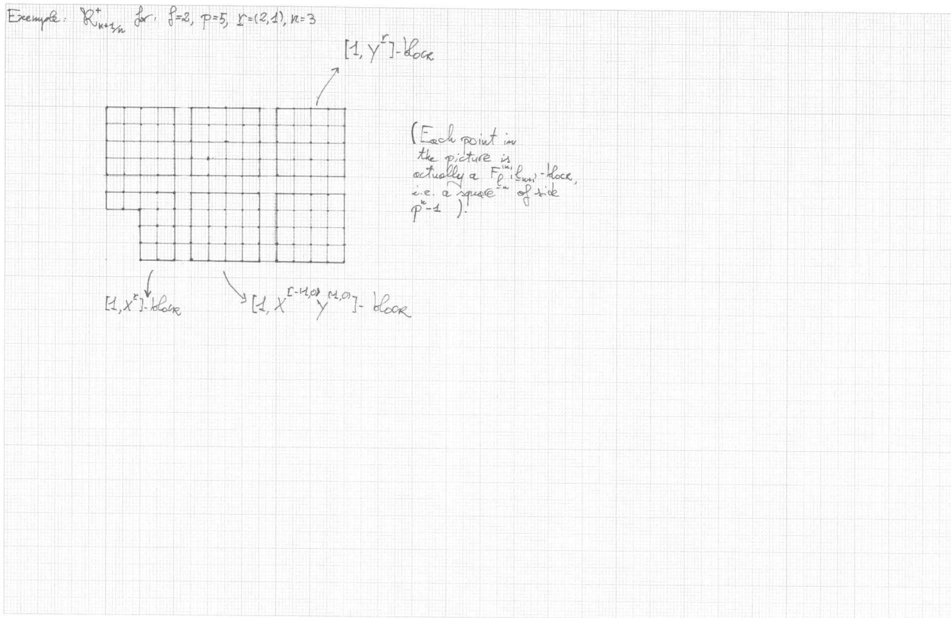


Figure 5: Euclidean structure for $R^{\pm}_{n+1}/R^{\pm}_{n}$ in the case $f = 2$, $p = 5$, $\underline{r} = (2, 1)$. It is obtained as the set theoretic difference of the stuctures associated to $R^{\pm}_{n+1}$ and $R^{\pm}_{n}$.

we will exhibit a natural injective map

$$c\text{−}\mathrm{Ind}_{KZ}^{G} V \hookrightarrow \pi(\underline{r}, 0, 1)$$

where $V \leqslant \pi(\underline{r}, 0, 1)|_{KZ}$ is a convenient KZ-subrepresentation of $\pi(\underline{r}, 0, 1)|_{KZ}$. We remark that a similar injective map has been detected independently by Paskunas in an unpublished draft.

We give here a more precise description of this section. Thanks to Proposition 3.5 we can content ourselves to the study of the representations $\varinjlim_{n\,\mathrm{odd}} R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_n^+} R_{n+1}^+$ and $\varinjlim_{n\,\mathrm{odd}} R_0^- \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-$. As seen in Proposition 3.4, such $K_0(p)$-representations have a natural filtration whose graded pieces are isomorphic to the quotients R^+_{n+1}/R^+_n, R^-_{n+1}/R^-_n respectively.

Such quotients are studied in §5.1. As we did in sections §4.1.3 and §4.2 -concerning the $K_0(p)$-structure of R^+_{n+1} and R^-_{n+1}- we introduce a natural correspondence between a “canonical” $\overline{\mathbf{F}}_p$-base $\mathscr{B}^{\pm}_{n+1/n}$ for $R^{\pm}_{n+1}/R^{\pm}_n$ and a convenient lattice (denoted as $\mathscr{R}^{\pm}_{n+1/n}$) in $\mathbf{R}^f$. Thanks to the behaviour of the canonical Hecke operator $(T_n^+)^{\mathrm{pos,neg}}$ with respect to the elements of $\mathscr{B}^{\pm}_{n+1/n}$ we see that such a lattice is in fact the set-theoretic difference of the lattices $\mathscr{R}^{\pm}_{n+1}$ and $\mathscr{R}^{\pm}_n$ (cf. Lemma 5.1): Figure 5 shows this phenomenon for $f = 2$.

As we did in §4, we need to check that the Euclidean structure $\mathscr{R}^{\pm}_{n+1/n}$ describes the Iwahori structure of the quotient $R^{\pm}_{n+1}/R^{\pm}_n$ in the sense of Definition 1.7. Unfortunately, we can not use directly the results of section 4 to concude that the $K_0(p)$-structure of $R^{\pm}_{n+1}/R^{\pm}_n$ is predicted by the lattice $\mathscr{R}^{\pm}_{n+1/n}$. For instance, if $v \in \mathscr{B}^+_{n+1}$ lies on the antidiagonal $X_0 + \cdots + X_{f-1} = N$, Proposition 4.11 describes the $K_0(p)$-representation generated by v *modulo the subspace* $X_0 + \cdots + X_{f-1} \leqslant N - 2$, while the combinatorics of the lattice $\mathscr{R}^+_{n+1/n}$ shows that we need of a much finer knowledge of $\langle K_0(p) \cdot v\rangle$: loosely speaking, the socle filtration of $R^{\pm}_{n+1}/R^{\pm}_n$ should be obtained from suitable simultaneous cuttings [7] by the f hyperplanes $X_0 + \cdots + X_{f-1} = p^n(r_i + 1) + constant$, for $i \in \{0, \ldots, f-1\}$.

It is therefore necessary to perfect the estimates made in the proofs of Propositions 4.10, 4.11: this is the object of §5.1.1 and Proposition 5.3, where we show that the socle filtration of R^+_{n+1}/R^+_n is described by the associated Euclidean datum $\mathscr{R}^+_{n+1/n}$. Again, we rely on some delicate arguments on Witt vectors contained in §6.4. We remark that the behaviour of $(R_1/R_0)^+$ (resp. $R^-_0 \oplus_{R^-_1} R^-_2$) is slighty different from that of R^+_{n+1}/R^+_n for $n \geqslant 1$ (resp. R^-_{n+1}/R^-_n for $n \geqslant 2$) if the Serre weight is non-regular (see §5.1.2).

In section §5.2 we determine the structure of the amalgamated sums $\cdots \oplus_{R^\pm_n} R^\pm_{n+1}$: their structure can be *easily* determined from the results concerning of $R^\pm_{n+1}/R^\pm_n$.

First of all, we detect a "natural" linear base $\mathscr{B}^\pm_{al,n}$ for the amalgamated sums $\cdots \oplus_{R^\pm_n} R^\pm_{n+1}$ (Lemmas 5.13, 5.14): this can be done thanks to the compatibility of the elements of $\mathscr{B}^\pm_n \subset R^\pm_n$ with respect to the Hecke operators $(T^-_n)^{\mathrm{pos,neg}}$. In particular, the natural projection on $R^\pm_{n+1}/R^\pm_n$ let us let us identify the elements of the canonical basis $\mathscr{B}^\pm_{n+1/n}$ of $R^\pm_{n+1}/R^\pm_n$ with suitable elements of $\mathscr{B}^\pm_{al,n}$.

Again, the elements of $\mathscr{B}^\pm_{al,n}$ admit a natural parametrisation in terms of a convenient lattice $\cdots \oplus_{\mathscr{R}^\pm_n} \mathscr{R}^\pm_{n+1}$ in $\mathbf{Z}^f$ (see the paragraph following Lemma 5.14 for a precise realisation of the Euclidean data associated to $\mathscr{B}^\pm_{al,n}$). As we will see (§5.2), each Euclidean datum $\cdots \oplus_{\mathscr{R}^\pm_n} \mathscr{R}^\pm_{n+1}$ is obtained by a convenient "glueing" of the Euclidean datum $\mathscr{R}^\pm_{n+1/n}$ of $R^\pm_{n+1}/R^\pm_n$ with the Euclidean datum of $\cdots \oplus_{R^\pm_{n-2}} R^\pm_{n-1}$: this give raise to a complicate fractal picture (see Figure 6).

The last step is then to prove that such fractal lattice describes the Iwahori structure of the amalgamated sum $\cdots \oplus_{R^\pm_n} R^\pm_{n+1}$ in the sense of Definition 1.7. This is the content of Theorem 5.18.

Let us consider for instancethe "positive sums". We see that Proposition 5.3 together with a simple Euclidean argument implies that the linear space V_J, generated by

i) a linear basis for the J-th composition factor of $\cdots \oplus_{R^+_{n-2}} R^+_{n-1}$,

ii) a linear basis $\mathscr{B}_J$ for the J-th composition factor of R^+_{n+1}/R^+_n (seen as a subset of $\mathscr{B}^+_{al,n}$ via the above identification),

is $K_0(p)$-stable and the filtration obtained this way has semisimple layers[8]. As $\cdots \oplus_{R^+_{n-2}} R^+_{n-1}$ is a subrepresentation, the $K_0(p)$-stability is verified once we check that the f cutting hyperplanes $X_0 + \cdots + X_{f-1} = p^n(r_i + 1) + J$ for R^+_{n+1}/R^+_n lie strictly below the cutting hyperplanes for the $(J-1)$-th composition factor of $\cdots \oplus_{R^+_{n-2}} R^+_{n-1}$: indeed the structure Theorem for R^+_{n+1}/R^+_n let us conclude that the $K_0(p)$-subrepresentation generated by an element $v \in \mathscr{B}_J$ lives in $V_{J-1} + \langle v \rangle$ (we invite the reader to the discussion after Remark 5.15 for more details). In particular Proposition 4.11 let us conclude that the linear space V_J, deduced from the Euclidean datum in the usual sense, is actually the J-th composition factor for the socle filtration of $\cdots \oplus_{R^+_n} R^+_{n+1}$.

In Figure 6 an example of the glueing of blocks [9] and their fractal stucture.

As annonced, we can combine Lemma 5.1 and Proposition 3.4 to exhibit a natural *injective*

[7] see Figure 5 and Figure 7 for an example or the discussion after Proposition 5.3 for a precise formalism about the simultaneous cuttings of the Euclidean data $\mathscr{R}^+_{n+1/n}$ by the f hyperplanes $X_0 + \cdots + X_{f-1} = p^n(r_i + 1) + constant$.

[8] Thus it is easier to treat the glueing of the quotients R^+_{n+1}/R^+_n than the quotients themselves.

[9] Strictly speaking, the figure gives the glueing of blocks R^+_{n-1}/R^+_{n-2} and R^+_{n+1}/R^+_n, i.e. the structure of $R^+_{n-1}/R^+_{n-2} \oplus_{R^+_n} R^+_{n+1}$. If we want to get the picture of the *whole* amalgamated sum $\cdots \oplus_{R^+_n} R^+_{n+1}$ we should insert a "even smaller" structure inside the point $(1, 2)$ of the rectangle drawn on the left in Figure 6.

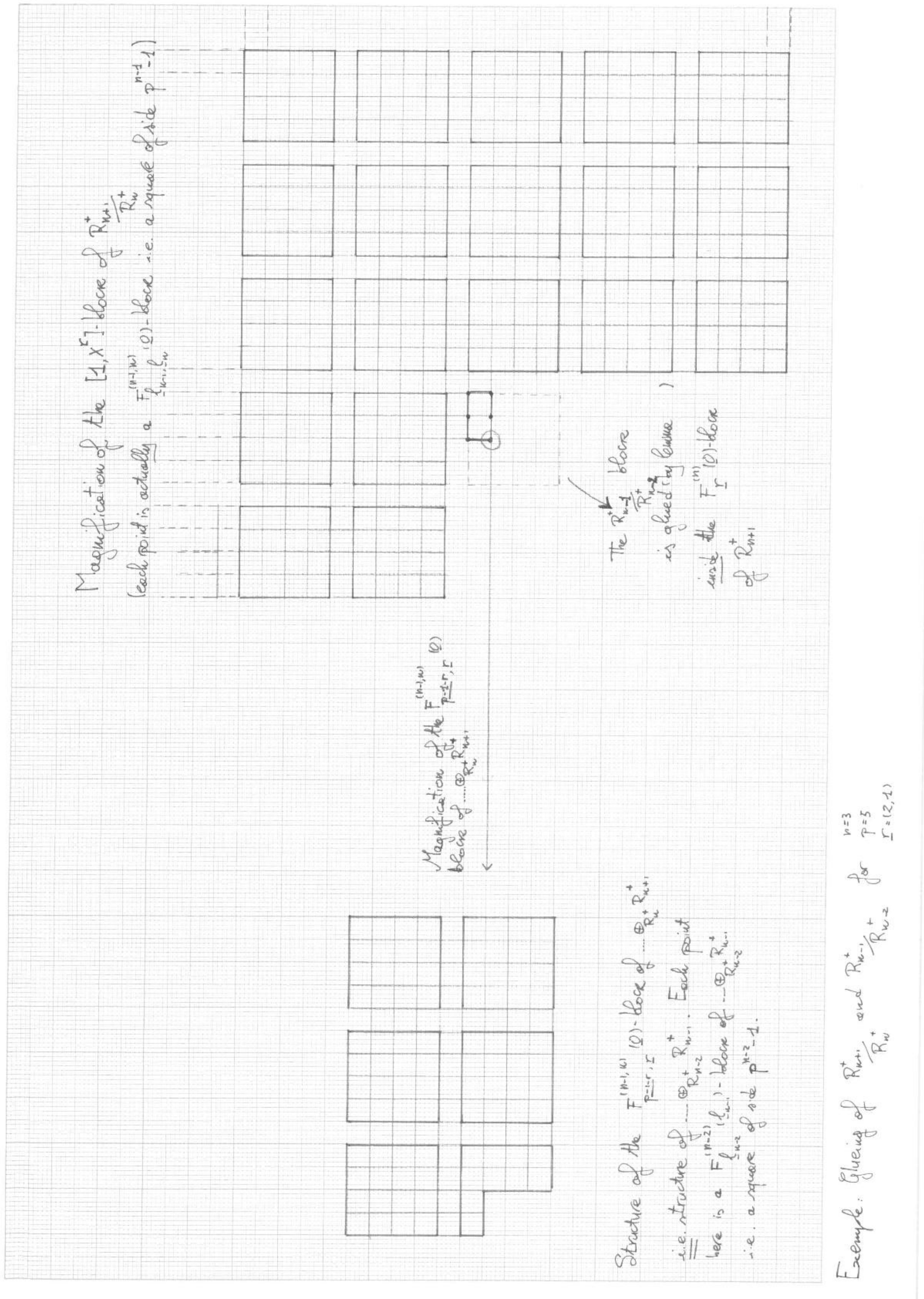


Figure 6: The glueing and the fractal structure in the case $f = 2$, $p = 5$, $\underline{r} = (2, 1)$. On the right side we have the particular of the $\left[1, X^{\underline{r}}\right]$-block for $R^+_{n-1} \oplus_{R^+_n} R^+_{n+1}$. Notice the glueing of $\mathscr{R}^+_{n-1}$ inside the $F^{(n)}_{\underline{r}}(\underline{0})$-block of R^+_{n+1}/R^+_n (Lemma 5.13). For the structure of $R^+_{n+1} \oplus_{R^+_n} R^+_{n-1}/R^+_{n-2}$ we should further consider the $F^{(n-1,n)}_{\underline{p-1-r},\underline{r}}(\underline{0})$-block and its magnification (left hand side). Repeating this process, we reveal the fractal nature of the structure associated to $R^+_0 \oplus_{R^+_1} \cdots \oplus_{R^+_n} R^+_{n+1}$.

morphism -whose existence was known by an unpublished work of Paskunas-

$$c\text{--}\mathrm{Ind}_{KZ}^{G} V \hookrightarrow \pi(\underline{r},0,1)|_{KZ}$$

where $V \leqslant \pi(\underline{r},0,1)|_{KZ}$ is a convenient KZ-subrepresentation of $\pi(\underline{r},0,1)|_{KZ}$: this is the object of Proposition 5.10.

As the cutting hyperplanes are fixed by the linear transformation $e_s \mapsto e_{\lfloor s+1 \rfloor}$ of $\mathbf{R}^f$ the results of §5.1 and §5.2 do not depend on the immersion $\tau : \mathbf{F}_q \hookrightarrow \overline{\mathbf{F}}_p$ (remark 4.9).

5.1 The structure of the quotients $R^{\bullet}_{n+1}/R^{\bullet}_n$

In the flavour of §4.1.3 and §4.2 we start by describing a suitable $\overline{\mathbf{F}}_p$-basis for the quotients $R^{\bullet}_{n+1}/R^{\bullet}_n$.

Lemma 5.1. *Let $n \in \mathbf{N}_{\geqslant 1}$.*

1) *An $\overline{\mathbf{F}}_p$-basis $\mathscr{B}^{+}_{n+1/n}$ for R^{+}_{n+1}/R^{+}_n is described as the homomorphic image (via the natural projection $R^{+}_{n+1} \twoheadrightarrow R^{+}_{n+1}/R^{+}_n$) of the elements*

$$F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(l_{n+1}) \in \mathscr{B}^{+}_{n+1}$$

such that $\underline{l}_n \not\leqslant \underline{r}$ if $\underline{l}_{n+1} = \underline{0}$.

2) *An $\overline{\mathbf{F}}_p$-basis $\mathscr{B}^{-}_{n+1/n}$ for R^{-}_{n+1}/R^{-}_n is described as the homomorphic image (via the natural projection $R^{-}_{n+1} \twoheadrightarrow R^{-}_{n+1}/R^{-}_n$) of the elements*

$$F^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}(l_{n+1}) \in \mathscr{B}^{-}_{n+1}$$

such that $\underline{l}_n \not\leqslant \underline{r}$ if $\underline{l}_{n+1} = \underline{0}$.

If $n = 0$ then an $\overline{\mathbf{F}}_p$-basis for $(R_1/R_0)^+$ is described as the homomorphic image (via the natural projection $R^{+}_1 \twoheadrightarrow (R_1/R_0)^+$) of the elements

$$F^{(0)}_{\underline{l}_0}(\underline{l}_1)$$

such that $\underline{l}_1 \not\leqslant \underline{r}$ if $\underline{l}_1 = \underline{0}$ and of the element $F^{(0)}_{\underline{r}}(\underline{0})$.

Proof. The result follows immediately from the definition of the operators $(T^{+}_n)^{\mathrm{pos,neg}}$. Indeed, for $n \geqslant 1$ we have (with the obvious conventions if $n = 1$):

$$(T^{+}_n)^{\mathrm{pos}}(F^{(0,n-1)}_{\underline{l}_0,\dots,\underline{l}_{n-1}}(\underline{l}_n)) = (-1)^{\underline{l}_n} F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{0});$$
$$(T^{+}_n)^{\mathrm{neg}}(F^{(1,n-1)}_{\underline{l}_1,\dots,\underline{l}_{n-1}}(\underline{l}_n)) = (-1)^{\underline{l}_n} F^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}(\underline{0})$$

while, for $n = 0$ we have

$$T_0(F^{(0,-1)}_{\emptyset}(\underline{l}_0)) = F^{(0)}_{\underline{l}_0}(\underline{0}) + (-1)^{\underline{r}} \delta_{\underline{l}_0,\underline{0}} F^{(1,0)}_{\emptyset}(\underline{0}).$$

□

As usual the elements of the basis $\mathscr{B}^{\pm}_{n+1/n}$ will be read as the elements of a convenient lattice $\mathscr{R}^{\pm}_{n+1/n}$ of $\mathbf{R}^f$.

Interpretation in terms of Euclidean data. Exactly as we did in sections §4.1.3 and §4.2 we have natural injections $\mathscr{B}^{\pm}_{n+1/n} \hookrightarrow \mathbf{R}^f$ which let us interpret the elements of $\mathscr{B}^{\pm}_{n+1/n}$ as points in a convenient lattice $\mathscr{R}^{\pm}_{n+1/n}$ of $\mathbf{R}^f$: the details can safely be left to the reader.

The Euclidean interpretation of Lemma 5.1 is therefore clear: for $n \geqslant 1$ the lattice $\mathscr{R}^{+}_{n+1/n}$ (resp.

$\mathscr{R}^-_{n+1/n}$) of $\mathbf{R}^f$, is obtained from the lattice of R^+_{n+1} (resp. R^-_{n+1}) by removing the simplex

$$\{(x_0,\dots,x_{f-1})\in\mathscr{R}^+_{n+1}\quad\text{s.t.}\,x_s<p^n(r_{\lfloor n+s\rfloor}+1)\,\text{for all}\,s=0,\dots,f-1\}$$

(resp.

$$\{(x_0,\dots,x_{f-1})\in\mathscr{R}^-_{n+1}\quad\text{s.t.}\,x_s<p^{n-1}(r_{\lfloor n+s-1\rfloor}+1)\,\text{for all}\,s=0,\dots,f-1\})$$

(i.e. $\mathscr{R}^{\pm}_{n+1/n}$ is obtained as the set-theoretical difference of $\mathscr{R}^{\pm}_{n+1}\setminus\mathscr{R}^{\pm}_{n}$).

As usual, we have to prove that the Euclidean datum $\mathscr{R}^{\pm}_{n+1/n}$ describes the Iwahori structure of $R^{\pm}_{n+1}/R^{\pm}_{n}$ in the sense of Definition 1.7. This is the content of Proposition 5.3.

We refer the reader to Figure 5 for an example in residual degree $f=2$.

The lattice $\mathscr{R}^+_{1/0}$ associated to $(R_1/R_0)^+$ is similarly obtained from the lattice associated to R^+_1, by removing the subset

$$\big\{(x_0,\dots,x_{f-1})\in\mathscr{R}^+_{n+1}\quad\text{s.t.}\,x_s<(r_{\lfloor n+s\rfloor}+1)\,\text{for all}s=0,\dots,f-1\big\}\setminus\big\{(r_0,\dots,r_{f-1})\big\}.$$

We will see that the lattice $\mathscr{R}^+_{1/0}$ (resp. the lattice naturally associated to $R^-_0\oplus_{R^-_1}R^-_2$) does *not* describe the $K_0(p)$-structure of $(R_1/R_0)^+$ (resp. $R^-_0\oplus_{R^-_1}R^-_2$) *sic et simpliciter*, unless the f-tuple $\underline{r}$ is sufficiently regular. A harmless modification of the formalism used for $\mathscr{R}^+_{n+1/n}$ when $n\geqslant 1$ (resp. $\mathscr{R}^-_{n+1/n}$ when $n\geqslant 2$) lets us detect their $K_0(p)$-socle filtration also for $(R_1/R_0)^+$ (resp. $R^-_0\oplus_{R^-_1}R^-_2$) in the non generic case: see section §5.1.2 and Propositions 5.6, 5.7 and 5.8 for details.

We will describe in detail the $K_0(p)$-structure of R^+_{n+1}/R^+_n for $n\geqslant 1$; as annonced, the negative case (for $n\geqslant 2$) will be left to the reader.

Preliminaries: partitioning the lattice. As annonced in the introduction to §5, the mere knowledge of the $K_0(p)$-socle filtration for R^+_{n+1} does *not* allow us determine the structure of the quotient R^+_{n+1}/R^+_n, as for $v\in\mathscr{B}^+_{n+1}$ lying on the antidiagonal $X_0+\dots+X_{f-1}=N$, Proposition 4.11 describes $\langle K_0(p)\cdot v\rangle$ *modulo the subspace* $X_0+\dots+X_{f-1}\leqslant N-2$.

For instance, if we pick two points $v_0,\,v_1\in\mathscr{R}^+_{n+1/n}$ with empty antecedent it is not clear that the $K_0(p)$-representation generated by $\{v_0,\,v_1\}$ is 2-dimensional and semisimple: consider $v_0=F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1})$, $v_1=F^{(0,n)}_{\underline{l}'_0,\dots,\underline{l}'_n}(\underline{l}'_{n+1})$ with $\underline{l}_j=\underline{l}'_j=\underline{0}$ for all $j\neq n$, $\underline{l}_n=(0,\dots,0,r_s+1,0,\dots,0)$, $\underline{l}_n=(0,\dots,0,r_{s'}+1,0,\dots,0)$ and $r_s>r_{s'}$; by Proposition 4.11 we only know that v_1 may lie in the $K_0(p)$ representation generated by v_0 (Figure 7).

Notice that this phenomena happens only if $F\neq\mathbf{Q}_p$: if $F=\mathbf{Q}_p$ the structure of the quotients is immediate from the structure of R^+_{n+1}.

We modify the strategy of section 4.2. We show that the $K_0(p)$-strucure of R^+_{n+1} is again predicted by $\mathscr{R}^+_{n+1}$ but each cutting antidiagonal $X_0+\dots+X_{f-1}=$ *constant* of section §4.2 is now replaced by f-antidiagonals of the form $X_0+\dots+X_{f-1}=p^n(r_{\lfloor n+s\rfloor}+1)+$ *constant*: we will say that $X_0+\dots+X_{f-1}=p^n(r_{\lfloor n+s\rfloor}+1)+$ *constant* is the *s-th cutting hyperplane* of R^+_{n+1}/R^+_n. This means that we divide the lattice $\mathscr{R}^+_{n+1/n}$ into sub-blocks $\mathfrak{V}_s$, for $s\in\{0,\dots,f-1\}$, of increasing size (cf. definition 5.2); the J-th composition factor for the $K_0(p)$-socle filtration of R^+_{n+1}/R^+_n is then obtained as the sum of the f subspaces determined by the intersection of the block $\mathfrak{V}_s$ with the antidiagonal $X_0+\dots+X_{f-1}=p^n(r_s+1)+$ *constant*, for varying $s\in\{0,\dots,f-1\}$: it is the content of Proposition 5.3. In Figure 8, an example of how the increasing blocks (and successive cuttings) look like.

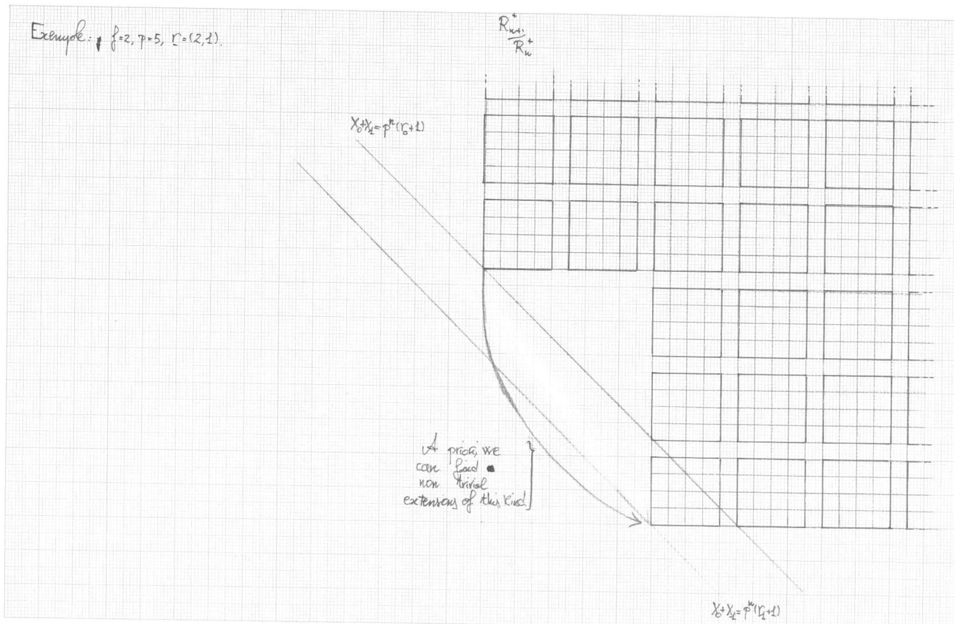


Figure 7: With the only Proposition 4.11 we can not exclude a priori some non trivial extensions, *inside the quotient* R^+_{n+1}/R^+_n, between elements lying on hyperplanes at a distance greater than 1. In the example of the picture (again, given for $f=2$, $p=5$, $\underline{r}=(2,1)$), we could have a non trivial extension between the elements $F^{0,n}_{\underline{0},\dots,\underline{0},(3,0)}(\underline{0})$ and $F^{0,n}_{\underline{0},\dots,\underline{0},(0,2)}(\underline{0})$: Proposition 4.11 tells only that the subrepresentation generated by $F^{0,n}_{\underline{0},\dots,\underline{0},(3,0)}(\underline{0})$ lives in a linear space generated by a family which may contain the element $F^{0,n}_{\underline{0},\dots,\underline{0},(0,2)}(\underline{0})$, as this element lies strictly below the hyperplane $X_0+\dots+X_{f-1}=3p^n-1$ as soon as $n>0$.

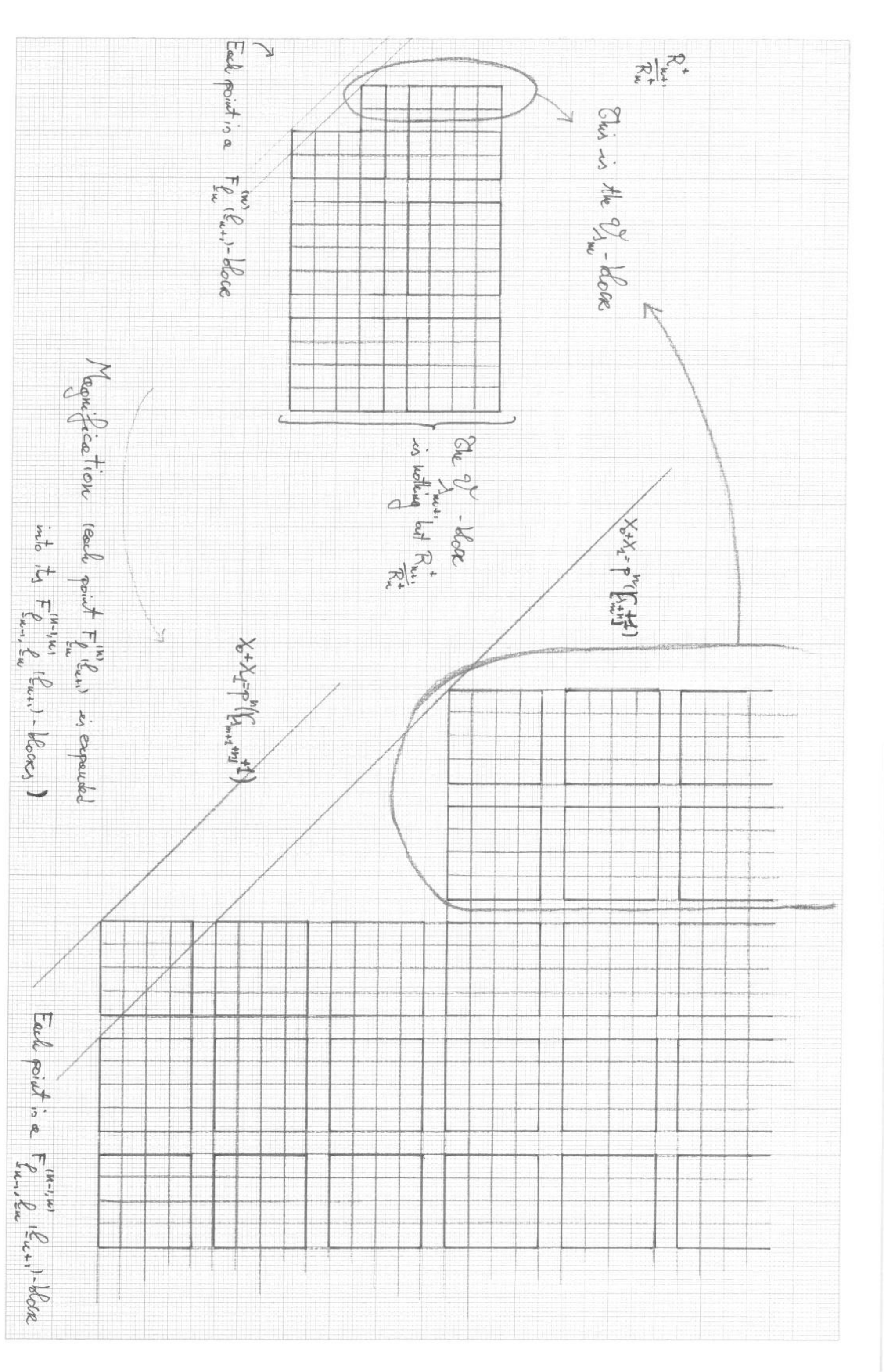


Figure 8: Euclidean interpretation of the filtration of R^+_{n+1}/R^+_n by means of the subspaces (or "blocks") $\mathfrak{V}_{s_m} \subseteq \mathfrak{V}_{\mathfrak{s}_{m+1}}$, in the case $f = 2$, $p = 5$, $\underline{r} = (2, 1)$. The socle of R^+_{n+1}/R^+_n is then obtained as the sum of the points of $\mathfrak{V}_{s_m}$ lying below $X_0 + X_1 = p^n(r_{\lfloor s_m+n \rfloor} + 1)$ and the points of $\mathfrak{V}_{s_{m+1}}$ lying below $X_0 + X_1 = p^n(r_{\lfloor s_{m+1}+n \rfloor} + 1)$ (by construction, $r_{\lfloor s_m+n \rfloor} \geqslant r_{\lfloor s_{m+1}+n \rfloor}$).

We determine the decomposition of $\mathscr{R}^+_{n+1/n}$ into increasing blocks. Fix $n \geqslant 0$ and define $s_m \in \{0, \dots, f-1\}$ by the condition

$$r_{\lfloor s_m+n \rfloor} = \max\left\{r_{\lfloor s+n \rfloor}, \quad s \in \{0, \dots, f-1\}\right\}.$$

We fix an ordering

$$p-1 \geqslant r_{\lfloor s_m+n \rfloor} \geqslant r_{\lfloor s_{m+1}+n \rfloor} \geqslant \dots \geqslant r_{\lfloor s_{m+f-1}+n \rfloor} \geqslant 0 \tag{13}$$

and define the following $\overline{\mathbf{F}}_p$-subspaces of R^+_{n+1}/R^+_n:

Definition 5.2. *For $k \in \{0, \dots, f-1\}$ define $\mathfrak{V}_{s_{m+k}}$ as the $\overline{\mathbf{F}}_p$-subspace of R^+_{n+1}/R^+_n generated by the elements $F^{(0,n)}_{\underline{l}_0, \dots, \underline{l}_n}(\underline{l}_{n+1}) \in \mathscr{B}^+_{n+1/n}$ verifying the properties:*

i) for $s \notin \{s_m, \dots, s_{m+k}\}$ we have

$$l_n^{\lfloor s+n \rfloor} \leqslant r_{\lfloor s+n \rfloor};$$

ii) for $s \notin \{s_m, \dots, s_{m+k}\}$ we have

$$l_{n+1}^{\lfloor s+n+1 \rfloor} = 0.$$

By abuse of notation, we will also write $\mathfrak{V}_{s_{m+k}}$ to denote the image of the canonical basis (in the obvious sense) of $\mathfrak{V}_{s_{m+k}}$ in the lattice $\mathscr{R}^+_{n+1/n}$. The geometric meaning of the previous definition is the following: the block $\mathfrak{V}_{s_{m+k}}$ is described as the intersection of the subset

$$\{X_{s_{m+k+1}} < p^n(r_{\lfloor s_{m+k+1}+n \rfloor} + 1)\} \cap \dots \cap \{X_{s_{m+f-1}} < p^n(r_{\lfloor s_{m+f-1}+n \rfloor} + 1)\}$$

with the lattice $\mathscr{R}^+_{n+1/n}$: in other words, we give restrictions on the coordinates $x_{s_{m+k+1}}, \dots, x_{s_{m+f-1}}$ of a point $(x_0, \dots, x_{f-1}) \in \mathscr{R}^+_{n+1/n}$ to lie in the block $\mathfrak{V}_{s_{m+k}}$.

Notice that in order to detect if a function $F^{(0,n)}_{\underline{l}_0, \dots, \underline{l}_n}(\underline{l}_{n+1}) \in \mathscr{B}^+_{n+1/n}$ belongs to the subspace $\mathfrak{V}_{s_{m+k}}$ we *only need to study the last two f-tuples $\underline{l}_n$, $\underline{l}_{n+1}$.*

Obviously, the subspaces $\mathfrak{V}_{s_{m+k}}$ describe (for $n \geqslant 1$) an exhaustive increasing filtration on R^+_{n+1}/R^+_n as a *$\overline{\mathbf{F}}_p$-vector space.*

The following crucial result shows that the lattice $\mathscr{R}^+_{n+1/n}$ lets us detect the required $K_0(p)$-structure for $n \geqslant 1$.

Proposition 5.3. *Assume $n \in \mathbf{N}_{\geqslant 1}$. Let $a, b, c, d \in \mathscr{O}_F$, $g \overset{\text{def}}{=} \begin{bmatrix} 1+pa & b \\ pc & 1+pd \end{bmatrix} \in K_0(p)$, fix an element $F^{(0,n)}_{\underline{l}_0, \dots, \underline{l}_n}(\underline{l}_{n+1}) \in \mathfrak{V}_{s_{m+k}}$ for some $k \in \{0, \dots, f-1\}$ and write $N_{0,n+1}(\underline{l}_0, \dots, \underline{l}_{n+1}) = p^n(r_{\lfloor s_{m+k}+n \rfloor} + 1) + J$ for some $J \in \mathbf{N}$. Finally, consider the linear development*

$$gF^{(0,n)}_{\underline{l}_0, \dots, \underline{l}_n}(\underline{l}_{n+1}) = \sum_{i \in I} \beta(i) F^{(0,n)}_{\underline{l}_0(i), \dots, \underline{l}_n(i)}(\underline{l}_{n+1}(i))$$

(where I is a suitable set of indices and $\beta(i) \in \overline{\mathbf{F}}_p^\times$ are scalars).

Fix an index $i_0 \in I$ and assume there exists $k' \in \{k+1, \dots, f-1\}$, minimal with respect to the property $F^{(0,n)}_{\underline{l}_0(i_0), \dots, \underline{l}_n(i_0)}(\underline{l}_{n+1}(i_0)) \in \mathfrak{V}_{s_{m+k'}} \setminus \mathfrak{V}_{s_{m+k}}$.

Then we have

$$N_{0,n+1}(\underline{l}_0(i_0), \dots, \underline{l}_{n+1}(i_0)) \leqslant p^n(r_{\lfloor s_{m+k'}+n \rfloor} + 1) + J - 2. \tag{14}$$

In particular, the lattice $\mathscr{R}^+_{n+1/n}$ describes the $K_0(p)$-socle filtration, as well as the extensions between two consecutive graded pieces, of R^+_{n+1}/R^+_n.

We explain here the geometric meaning of Proposition 5.3: we pick a function in the k-th block $F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1})\in\mathfrak{V}_{s_{m+k}}$, liyng on the antidiagnal $X_0+\cdots+X_{f-1}=p^n(r_{\lfloor s_{m+k}+n\rfloor}+1)+J$ and $F^{(0,n)}_{\underline{l}_0(i_0),\dots,\underline{l}_n(i_0)}(\underline{l}_{n+1}(i_0))$ a function appearing (with nonzero linear coefficient) in the linear development of $(g-1)F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1})$. A priori $F^{(0,n)}_{\underline{l}_0(i_0),\dots,\underline{l}_n(i_0)}(\underline{l}_{n+1}(i_0))$ belongs to the linear space generated by the points of $\mathscr{R}^+_{n+1/n}$ lying below the antidiagonal $X_0+\cdots+X_{f-1}=p^n(r_{\lfloor s_{m+k}+n\rfloor}+1)+J-1$.

But, if $F^{(0,n)}_{\underline{l}_0(i_0),\dots,\underline{l}_n(i_0)}(\underline{l}_{n+1}(i_0))$ happens to belong to a *strictly bigger block*, say $\mathfrak{V}_{s_{m+k'}}$ with $k'>k$ and minimal with respect to this property, then it lies *strictly below* the antidiagonal $X_0+\cdots+X_{f-1}=p^n(r_{\lfloor s_{m+k'}+n\rfloor}+1)+J-1$. In other words, the $K_0(p)$-subrepresentation generated by $F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1})$ lives in the linear space generated by

$$\big\{\underline{x}\in\mathfrak{V}_{s_{m+k}},\,\sum_i x_i\leqslant p^n(r_{\lfloor s_{m+k}+n\rfloor}+1)+J\big\}\bigsqcup\coprod_{j=k+1}^{f-1}\big\{\underline{x}\in\mathfrak{V}_{s_{m+j}},\,\sum_i x_i\leqslant p^n(r_{\lfloor s_{m+j}+n\rfloor}+1)+J-2\big\}.$$

As $r_{\lfloor s_{m+j}+n\rfloor}\leqslant r_{\lfloor s_{m+k}+n\rfloor}$ for $j\geqslant k$ (by the choosen ordering (13)) we see that this is a refinement of Proposition 4.11.

Thanks to this phenomenon, we can invoke Proposition 4.11 to deduce the $K_0(p)$-structure for R^+_{n+1}/R^+_n from the associated lattice $\mathscr{R}^+_{n+1/n}$. More precisely, we see that

i) the linear space V_J generated by the f-subspaces

$$\Big\langle\big\{\underline{x}\in\mathfrak{V}_{s_{m+k}},\,\sum_i x_i<p^n(r_{s_{m+k}}+1)+J\big\}\Big\rangle$$

(i.e. the points of the k-th block $\mathfrak{V}_{s_{m+k}}$ lying strictly below the antidiagonal $X_0+\cdots+X_{f-1}=p^n(r_{s_{m+k}}+1)+J$)) is stable under the action of $K_0(p)$ (Proposition 5.3);

ii) the points of a k-block $\mathfrak{V}_{s_{m+k}}$ lying on the antidiagonal $X_0+\cdots+X_{f-1}=p^n(r_{s_{m+k}}+1)+J$ are fixed under the action of the pro-p-Iwahori inside the quotient $(R^+_{n+1}/R^+_n)/V_J$ (Proposition 5.3; note that such points may be equal to zero in the quotient);

iii) if $x\in\mathfrak{V}_{s_{m+k}}$ lies on the antidiagonal $X_0+\cdots+X_{f-1}=p^n(r_{s_{m+k}}+1)+J$ then the $K_0(p)$-subrepresentation generated by x *inside the quotient* $(R^+_{n+1}/R^+_n)/V_{J-1}$ is either zero or generated by x and the $x-e_i$ for $i=0,\dots,f-1$ (Propositions 4.11, 5.3).

We deduce that $V_J=\mathrm{soc}_J(R^+_{n+1}/R^+_n)$, i.e. the Iwahori structure of the quotient R^+_{n+1}/R^+_n is obtained from the Euclidean datum $\mathscr{R}^+_{n+1/n}$ as well as the extensions between two consecutive graded pieces.

Notice moreover that the statement of Proposition 5.3 is empty if $f=1$: in the rest of §5.1 we will assume $f\geqslant 2$.

5.1.1 ***Proof of Proposition 5.3.*** The rest of this section is devoted to the proof of Proposition 5.3. Thanks to the decomposition (8) we can study separately the actions of lower unipotent, diagonal and upper unipotent matrices on the elements of R^+_{n+1}: this will be the object of the next three paragraphs. The proofs are similar to the proofs of Propositions 4.4, 4.5 and 4.7, but need a delicate extra argument due to the irregular structure of the lattice $\mathscr{R}^+_{n+1/n}$. In particular, in order to control the action of lower unipotent matrices, we will need the estimates of Appendix A, §**??**.

The action of upper unipotent matrices. We study here the case where $g \in \begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix}$, and again we assume $g = \begin{bmatrix} 1 & [\mu] \\ 0 & 1 \end{bmatrix}$ for $\mu \in \mathbf{F}_q$. As in Proposition 4.4 we write

$$gF^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) =$$
$$= \sum_{j=1}^{n+1} \sum_{\underline{i}_j \leqslant \underline{l}_j} \binom{\underline{l}_j}{\underline{i}_j} \sum_{\underline{i}_0 \leqslant \underline{l}_0} \binom{\underline{l}_0}{\underline{i}_0} (T_0)^{\underline{i}_0} \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j - \underline{i}_j} (T_{j+1}^{\frac{1}{p^{j+1}}})^{\underline{i}_{j+1}} \begin{bmatrix} 1 & 0 \\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1 \end{bmatrix} \left[1, f_{\underline{l}_{n+1}-\underline{i}_{n+1}}\right]$$

where for notational convenience, we commit the abuse of writing $\begin{bmatrix} 1 & 0 \\ p^0[\lambda_0] & 1 \end{bmatrix}$ instead of $\begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix}$ and where we have set

$$f_{\underline{l}_{n+1}-\underline{i}_{n+1}} \stackrel{\text{def}}{=} (-1)^{\underline{i}_{n+1}} X^{\underline{r}-(\underline{l}_{n+1}-\underline{i}_{n+1})} Y^{\underline{l}_{n+1}-\underline{i}_{n+1}},$$

$T_0 \stackrel{\text{def}}{=} -s_0(\widetilde{S}_0)$, $T_{j+1} \stackrel{\text{def}}{=} -s_j(\widetilde{S}_{j+1})$ for $j \in \{0,\dots,n\}$.

Developing the polynomials T_{j+1} we write

$$gF^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) = \sum_{i \in I} \beta(i) F^{(0,n)}_{\underline{l}_0(i),\dots,\underline{l}_n(i)}(\underline{l}_{n+1}(i))$$

for a suitable set of indices I and scalars $\beta(i) \in \overline{\mathbf{F}}_p^{\times}$ We pick a vector v appearing in the linear development of $gF^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1})$:

$$v \stackrel{\text{def}}{=} F^{(0,n)}_{\lceil \underline{\kappa}_0 \rceil,\dots,\lceil \underline{\kappa}_n \rceil}(\lceil \underline{\kappa}_{n+1} \rceil);$$

where, as in Proposition 7.3, we write for $0 \leqslant a \leqslant n+1$

$$\underline{\kappa}_a = \underline{l}_a - \underline{i}_a + \sum_{b=a+1}^{n+1} p^{\lfloor a-b \rfloor} \kappa_a^{(b)}$$

and, for $a+1 \leqslant b \leqslant n+1$ we have

$$\kappa_a^{(b)} = \sum_{s=0}^{f-1} p^s \kappa_a^{(b),s}$$

where $\kappa_a^{(b),s}$ is the exponent of λ_a in $(T_b)^{i_b^{(s)}}$. By the definition of the subspace $\mathfrak{V}_{s_{m+k}}$ we see that

$$\underline{\kappa}_n = \underline{l}_n - \underline{i}_n + p^{\lfloor -1 \rfloor} \kappa_n^{(n+1)} =$$
$$= \sum_{h=0}^{k} p^{\lfloor s_{m+h}+n \rfloor} \big(l_n^{(\lfloor s_{m+h}+n \rfloor)} - i_n^{(\lfloor s_{m+h}+n \rfloor)} + \kappa_n^{(n+1),\lfloor s_{m+h}+n+1 \rfloor}\big) + \sum_{h=k+1}^{f-1} p^{\lfloor s_{m+h}+n \rfloor} \big(l_n^{(\lfloor s_{m+h}+n \rfloor)} - i_n^{(\lfloor s_{m+h}+n \rfloor)}\big)$$

If $v \notin \mathfrak{V}_{s_{m+k}}$ then we define

$$k' \stackrel{\text{def}}{=} \min\left\{c \in \{k+1,\dots f-1\},\ \text{s.t.}\ \lceil \kappa_n^{(\lfloor s_{m+c}+n \rfloor)} \rceil > r_{\lfloor s_{m+c}+n \rfloor}\right\}$$

Observe that $k' > k$ by construction and we *necessarily* have $\underline{\kappa}_n \neq 0$ and the equality

$$\mathfrak{s}(\underline{l}_n - \underline{i}_n + p^{\lfloor -1 \rfloor} \kappa_n^{(n+1)}) = \sum_{s=0}^{f-1} l_n^{(s)} - i_n^{(s)} + \kappa_n^{(n+1),\lfloor s+1 \rfloor} - \widetilde{j}(p-1)$$

for a suitable $\widetilde{j} \geqslant 1$. Following the inequalities (26), (27), (28) of Proposition 7.3 (i.e. using the subadditivity of the function $\mathfrak{s}$ and the fact that the polynomials T_j are homogeneous of degree p^j

if λ_i is defined to have degree p^i) we get

$$\mathfrak{s}(\underline{\kappa}_0)+\cdots+p^{n+1}\mathfrak{s}(\underline{\kappa}_{n+1})\leqslant p^n(r_{\lfloor s_{m+k}+n\rfloor}+1)+J-\mathfrak{s}(\underline{i}_0)+p^n(p-1)\widetilde{j}.$$

As $n\geqslant 1$ the inequality

$$p^n(r_{\lfloor s_{m+k}+n\rfloor}-r_{\lfloor s_{m+k'}+n\rfloor})\leqslant \widetilde{j}p^n(p-1)+\mathfrak{s}(\underline{i}_0)-2$$

is then obvious if either $\widetilde{j}\geqslant 2$ or $r_{\lfloor s_{m+k'}+n\rfloor}>0$.

Assume finally $\widetilde{j}=1$ and $r_{\lfloor s_{m+k'}+n\rfloor}=0$. Therefore the p-adic development of $\lceil\kappa_n\rceil$ has the form

$$(l_n^{(0)}-i_n^{(0)}+\kappa_n^{(n+1),1},\dots,l_n^{(s)}-i_n^{(s)}+\kappa_n^{(n+1),s+1}-p,l_n^{(s+1)}-i_n^{(s+1)}+\kappa_n^{(n+1),s+2}+1,\dots)$$

for a unique $s\in\{s_m,\dots,s_{m+k}\}$. The condition $x\notin\mathfrak{V}_{s_{m+k}}$ imposes $\lfloor s+1\rfloor\notin\{s_m,\dots,s_{m+k}\}$ and the minimality condition on k' imposes $\lfloor s_{m+k'}+n\rfloor=\lfloor s+1\rfloor$, in particular $r_{\lfloor s+1\rfloor}=0$. As $\kappa_n^{(n+1),s+1}$ is the coefficient of $\lambda_n^{\frac{1}{p^n}}$ in the fixed monomial of $s(\widetilde{S}_{n+1})^{i_{n+1}^{\lfloor s+1\rfloor}}$ and $i_n^{\lfloor s+1\rfloor}\leqslant r_{\lfloor s+1\rfloor}$ we get a contradiction.

The action of diagonal matrices. The next step is to study the action of an element $g\in\begin{bmatrix}1+p\mathscr{O}_F & 0\\ 0 & 1+p\mathscr{O}_F\end{bmatrix}$; again we can assume $g=\begin{bmatrix}1+p\alpha & 0\\ 0 & 1\end{bmatrix}$. The arguments are completely analogous to those of the previous paragraph, in this case using the fact that the polynomials $q_{j-1}(\widetilde{Q}_j)$ of §6.3 are homogeneous of degree p^j. The details are left to the reader.

The action of lower unipotent matrices. In this section we deal with the action of an element $g\in\begin{bmatrix}1 & 0\\ p\mathscr{O}_F & 1\end{bmatrix}$; again, we assume $g=\begin{bmatrix}1 & 0\\ p[\mu] & 1\end{bmatrix}$. This case is more delicate than the previous and we need to recall and carry on the accurate estimates seen in the appendix A §6.4.2.

As for Proposition 4.7, we write

$$gF^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1})=\sum_{j=0}^{n}\sum_{\substack{\underline{i}_{j+1}\leqslant\underline{l}_{j+1}\\ \underline{i}_0=0}}\binom{\underline{l}_{j+1}}{\underline{i}_{j+1}}\sum_{\lambda_j\in\mathbf{F}_q}(\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j-\underline{i}_j}(V_{j+1}^{\frac{1}{p^{j+1}}})^{\underline{i}_{j+1}}\begin{bmatrix}1 & 0\\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1\end{bmatrix}[1,f_{\underline{l}_{n+1}-\underline{i}_{n+1}}]$$

where for notational convenience, we commit the abuse of writing $\begin{bmatrix}1 & 0\\ p^0[\lambda_0] & 1\end{bmatrix}$ instead of $\begin{bmatrix}[\lambda_0] & 1\\ 1 & 0\end{bmatrix}$ and where we have set

$$f_{\underline{l}_{n+1}-\underline{i}_{n+1}}\overset{\text{def}}{=}(-1)^{\underline{i}_{n+1}}X^{\underline{r}-(\underline{l}_{n+1}+\underline{i}_{n+1})}Y^{\underline{l}_{n+1}-\underline{i}_{n+1}}$$

and $V_{j+1}\overset{\text{def}}{=}-u_j(\widetilde{U}_{j+1})$ for $j\in\{0,\dots,n\}$. We develop the polynomials $V_{j+1}^{\underline{i}_{j+1}}$, recognizing again a sum of elements of the basis $\mathscr{B}^+_{n+1/n}$. We pick a vector

$$v\overset{\text{def}}{=}F^{(0,n)}_{\lceil\kappa_0\rceil,\dots,\lceil\kappa_n\rceil}(\lceil\kappa_{n+1}\rceil);$$

as in the previous paragraph we write for $0\leqslant a\leqslant n+1$

$$\underline{\kappa}_a=\underline{l}_a-\underline{i}_a+\sum_{b=a+1}^{n+1}p^{\lfloor a-b\rfloor}\kappa_a^{(b)}$$

and, for $a+1\leqslant b\leqslant n+1$ we have

$$\kappa_a^{(b)}=\sum_{s=0}^{f-1}p^s\kappa_a^{(b),s}$$

where $\kappa_a^{(b),s}$ is the exponent of λ_a in $(V_b)^{i_b^{(s)}}$. Again, using the notations of Lemmas 6.19 and 6.20, we focus our attention on

$$\begin{aligned}\underline{\kappa}_n &= \underline{l}_n - \underline{i}_n + p^{\lfloor -1\rfloor}\kappa_n^{(n+1)} = \\ &= \sum_{h=0}^{k} p^{\lfloor s_{m+h}+n\rfloor}(l_n^{(\lfloor s_{m+h}+n\rfloor)} - i_n^{(\lfloor s_{m+h}+n\rfloor)} + B_n^{\lfloor s_{m+h}+1+n\rfloor}(0) + pB_{n+1}^{\lfloor s_{m+h}+1+n\rfloor}(1)) + \\ &\quad + \sum_{h=k+1}^{f-1} p^{\lfloor s_{m+h}+n\rfloor}(l_n^{(\lfloor s_{m+h}+n\rfloor)} - i_n^{(\lfloor s_{m+h}+n\rfloor)})\end{aligned}$$

(where we can again assume $\underline{\kappa}_n \neq 0$) and we distinguish the following four possibilities.

I). Assume$\sum_{h=0}^{k} B_{n+1}^{\lfloor s_{m+h}+1+n\rfloor}(1) = 0$. The condition $v \notin \mathfrak{V}_{s_{m+k}}$ imposes that

$$\mathfrak{s}(\kappa_n) = \sum_{s=0}^{f-1} l_n^{(s)} - i_n^{(s)} + B_n^{\lfloor s+1\rfloor}(0) - \widetilde{j}(p-1)$$

for $\widetilde{j} \in \mathbf{N}$, $\widetilde{j} \geqslant 1$. We recall that for each $j \in \{0, \dots, n-1\}$ the polynomial V_j is pseudohomogeneous of degree $p^j - (p-2)$ (see Definition 6.11) so that the subadditivity of $\mathfrak{s}$ and Lemma 6.20 give

$$\sum_{j=0}^{n+1} p^j \mathfrak{s}(\underline{\kappa}_j) \leqslant \sum_{j=0}^{n+1} p^j \mathfrak{s}(\underline{l}_j) - (p-2)(\sum_{j=0}^{n+1} \mathfrak{s}(\underline{i}_j)) - p^n\widetilde{j}(p-1)$$

and the conclusion follows.

II). Assume $\sum_{h=0}^{k} B_{n+1}^{\lfloor s_{m+h}+1+n\rfloor}(1) \geqslant 2$. Then we have

$$\sum_{s=0}^{f-1}\sum_{j=0}^{n} p^j \mathfrak{s}(\kappa_j^{(n+1),s}) \leqslant p^{n+1}\mathfrak{s}(\underline{i}_{n+1}) - 2p^n(p-2).$$

The conclusion is now easy and left to the reader.

III). Assume $1 = \sum_{h=0}^{k} A_{n+1}^{\lfloor s_{m+h}+1+n\rfloor}(1) = \sum_{h=0}^{k} B_{n+1}^{\lfloor s_{m+h}+1+n\rfloor}(1) = 1$ (see Lemma 6.20 for the quantities $A_{n+1}^s(1)$). Let $h_1 \in \{0, \dots, k\}$ the unique integer such that $B_{n+1}^{\lfloor s_{m+h_1}+1+n\rfloor}(1) = 1$. We can again distinguish the following two subcases:

III)$_{\mathrm{A}}$ Assume

$$\mathfrak{s}(\kappa_n) = \sum_{s=0}^{f-1}(l_n^{(s)} - i_n^{(s)} + B_n^{(s+1)}(0) + B_{n+1}^{(s)}(1)) - \widetilde{j}(p-1)$$

for $\widetilde{j} \in \mathbf{N}$, $\widetilde{j} \geqslant 1$. In this case the reader can check that

$$\sum_{j=0}^{n+1} p^j \mathfrak{s}(\kappa_j) \leqslant \sum_{j=0}^{n+1} p^j \mathfrak{s}(\underline{l}_j) - (p-2)(\sum_{j=0}^{n} \mathfrak{s}(\underline{i}_j)) - p^n\widetilde{j}(p-1) - (p-2)p^n$$

and the conclusion follows.

III)$_{\mathrm{B}}$ Assume finally

$$\mathfrak{s}(\kappa_n) = \sum_{s=0}^{f-1}(l_n^{(s)} - i_n^{(s)} + B_n^{(s+1)}(0) + B_{n+1}^{(s)}(1)).$$

Such condition, together with $v \notin \mathfrak{V}_{s_{m+k}}$ imposes that $\lfloor s_{m+h_1} + 1\rfloor \notin \{s_m, \dots, s_{m+k}\}$; by minimality of k' we conclude that $\lfloor s_{m+h_1} + 1\rfloor = s_{m+k'}$; in particular $r_{\lfloor s_{m+k'}+n\rfloor} > 0$ (Lemma

6.20-3)). We deduce that the choosen monomial of $u_n(\widetilde{U}_{n+1}^{\frac{1}{p^{n+1}}})^{\underline{i}_{n+1}}$ is of the form

$$\lambda_0^{\alpha'_0}\cdot\ldots\cdot\lambda_n^{\alpha'_n}(\lambda_0\lambda_n^{\frac{1}{p^n}})^{p^{\lfloor s_{m+h_1}+1+n\rfloor}}$$

where the integers α'_j verify

$$\sum_{j=0}^{n}p^j\mathfrak{s}(\alpha'_j)\leqslant(p^{n+1}-(p-2))(\mathfrak{s}(\underline{i}_{n+1}-1)).$$

By subadditivity of the function $\mathfrak{s}$ we find finally

$$\sum_{j=0}^{n+1}p^j\mathfrak{s}(\underline{\kappa}_j)\leqslant\sum_{j=0}^{n+1}p^j\mathfrak{s}(\underline{l}_j)-(p-2)(\sum_{j=0}^{n}\mathfrak{s}(\underline{i}_j))+(p^{n+1}-(p-2))(\mathfrak{s}(\underline{i}_{n+1})-1)+$$
$$+(1+p^n)-p^{n+1}\mathfrak{s}(\underline{i}_{n+1})$$

(where the integer $1+p^n$ is deduced from the monomial $\lambda_0\lambda_n^{\frac{1}{p^n}}$) and the conclusion follows easily noticing that $\sum_{j=0}^{n+1}\mathfrak{s}(\underline{i}_j)\geqslant 1$.

The proof of Proposition 5.3 is therefore complete.

Remark 5.4. *The reader has noticed that if we assume $r_s\leqslant p-2$ for all $s\in\{0,\ldots,f-1\}$ then the inequality (14) in the statement can be replaced by the following, stronger, inequality*

$$N_{0,n+1}(\underline{l}_0(i_0),\ldots,\underline{l}_{n+1}(i_0))\leqslant p^n+J-2.$$

5.1.2 ***The case*** $n=0$**.** In this section we study the $K_0(p)$-structure of $(R_1/R_0)^+$; the negative counterpart, i.e. the $K_0(p)$-structure of $R_0^-\oplus_{R_1^-}R_2^-$ is left to the reader.

We see that if the Serre weight σ happens not to be regular, the associated lattice $\mathscr{R}_{1/0}^+$ needs not describe the Iwahori structure of $(R_1/R_0)^+$ in the sense of Definition 1.7 and we need a slight modification of our methods according to the combinatorics of $\underline{r}$ (see Proposition 5.8).

This is due to technical reasons: raughly speaking, for $n=0$ the f cutting hyperplanes $X_0+\cdots+X_{f-1}=(r_s+1)+J$ are "very close" to each other and, in the non regular case, we may get some extra extensions between functions lying on different hyperplanes.

Otherwise, in the regular case, we see that the lattice $\mathscr{R}_{1/0}^+$ describes the Iwahori structure of $(R_1/R_0)^+$ in the usual sense (Proposition 5.6, 5.7).

In what follows, we fix $k\in\{0,\ldots,f-1\}$ and an element $F_{\underline{l}_0}^{(0)}(\underline{l}_1)\in\mathfrak{V}_{s_{m+k}}\setminus\langle F_{\underline{r}}^{(0)}(\underline{0})\rangle_{\overline{\mathbf{F}}_p}$. Let $g\in K_0(p)$. We fix an element $v=F_{\lceil\underline{\kappa}_0\rceil}^{(0)}(\lceil\underline{\kappa}_1\rceil)$ appearing (with a nonzero linear coefficient) in the $\overline{\mathbf{F}}_p$-linear development of $gF_{\underline{l}_0}^{(0)}(\underline{l}_1)$, for suitable integers $\underline{\kappa}_0,\underline{\kappa}_1\in\mathbf{N}$.

We assume there exists an integer $k'\in\{k+1,\ldots,f-1\}$ such that $v\notin\mathfrak{V}_{s_{m+k'}}\setminus\mathfrak{V}_{s_{m+k}}$ and k' is minimal with respect to this property.

The next lemma can be verified by an easy computation on the ring $\mathbf{W}_1(\mathbf{F}_q)$:

Lemma 5.5. *In the previous hypothesis we have*

$$N_{0,1}(\underline{\kappa}_0,\underline{\kappa}_1)=N_{0,1}(\underline{l}_0,\underline{l}_1)-\epsilon$$

where

1) *if* $g\in\begin{bmatrix}1&\mathscr{O}_F\\0&1\end{bmatrix}$ *then* $\epsilon=\mathfrak{s}(\underline{i}_0)+\mathfrak{s}(\underline{i}_1)+\widetilde{j}(p-1)$ *where* $\widetilde{j}\geqslant 1$ *and* $\mathfrak{s}(\underline{i}_0)+\mathfrak{s}(\underline{i}_1)\geqslant 1$;

2) *if* $g \in \begin{bmatrix} 1+p\mathscr{O}_F & 0 \\ 0 & 1+p\mathscr{O}_F \end{bmatrix}$ *then* $\epsilon = \mathfrak{s}(\underline{i}_1)(p-1) + \widetilde{j}(p-1)$ *where* $\mathfrak{s}(\underline{i}_1) \geqslant 1$ *and* $\widetilde{j} \in \mathbf{N}$*;*

3) *if* $g \in \begin{bmatrix} 1 & 0 \\ p\mathscr{O}_F & 1 \end{bmatrix}$ *then* $\epsilon = \mathfrak{s}(\underline{i}_1)(p-2) + \widetilde{j}(p-1)$ *where* $\mathfrak{s}(\underline{i}_1) \geqslant 1$ *and* $\widetilde{j} \in \mathbf{N}$.

Moreover:

1_A) *if in case* 1) *we have* $\widetilde{j} = 1$ *then we necessarly have* $s_{m+k'} = \lfloor s+1 \rfloor$ *for an index* s *verifying* $s \in \{s_m, \dots, s_{m+k}\}$ *and* $\lfloor s+1 \rfloor \notin \{s_m, \dots, s_{m+k}\}$*; moreover* $r_{s_{m+k'}} > 0$*;*

2_B) *if in case* 2) *we have* $\widetilde{j} = 0$ *and* $\mathfrak{s}(\underline{i}_1) = 1$ *then we have*

$$\lceil \kappa_0 \rceil = (l_0^{(0)}, \dots, l_0^{(s)}, l_0^{\lfloor s+1 \rfloor} + 1, l_0^{\lfloor s+2 \rfloor}, \dots, l_0^{(f-1)})$$

where the index s *verify* $s \in \{s_m, \dots, s_{m+k}\}$ *and* $\lfloor s+1 \rfloor \notin \{s_m, \dots, s_{m+k}\}$*. Furthermore* $r_{\lfloor s+1 \rfloor} = r_{s_{m+k'}} > 0$.

3_B) *if in case* 3) *we have* $\widetilde{j} = 0$ *and* $\mathfrak{s}(\underline{i}_1) = 1$ *then we have*

$$\lceil \kappa_0 \rceil = (l_0^{(0)}, \dots, l_0^{(s)}, l_0^{\lfloor s+1 \rfloor} + 2, l_0^{\lfloor s+2 \rfloor}, \dots, l_0^{(f-1)})$$

where the index s *verify* $s \in \{s_m, \dots, s_{m+k}\}$ *and* $\lfloor s+1 \rfloor \notin \{s_m, \dots, s_{m+k}\}$*. Furthermore* $r_{\lfloor s+1 \rfloor} = r_{s_{m+k'}} > 0$.

Proof. The proof, a direct computation, is left to the reader. □

Thanks to its explicit nature, the description of the socle filtration for $(R_1/R_0)^+$ can be easily deduced from Lemma 5.5. We have to distinguish three cases, according to the combinatorics of the f-tuple $\underline{r}$; the proofs are left as an exercie to the reader (see [Mo1] for details).

Proposition 5.6. *Assume that the* f*-tuple verifies one of the following hypotheses:*

I_A). *For each* $s \in \{0, \dots, f-1\}$ *the condition*

$$\begin{cases} r_s \geqslant r_{\lfloor s+1 \rfloor} \geqslant 1 \\ r_s - r_{\lfloor s+1 \rfloor} \in \{p-2, p-3\} \end{cases}$$

is false.

I_B). *The* f*-tuple is of the form* $(0, \dots, 0, r_{s_m}, 0, \dots, 0)$.

Then the socle filtration of $(R_1/R_0)^+$*, together with the extensions between two consecutive graded pieces, is described by the associated lattice* $\mathscr{R}^+_{1/0}$.

Proof. Omissis. See [Mo1], Proposition 5.8. □

Proposition 5.7. *Assume that for all* $s \in \{0, \dots, f-1\}$ *we have* $\sum_{s=0}^{f-1}(r_s) \geqslant r_s + 1$ *and that the condition*

$$\begin{cases} r_s \geqslant r_{\lfloor s+1 \rfloor} \geqslant 1 \\ r_s - r_{\lfloor s+1 \rfloor} = p-2 \end{cases}$$

is false.

Then the socle filtration for $(R_1/R_0)^+$ *is described by the lattice* $\mathscr{R}^+_{1/0}$.

Proof. Omissis. See [Mo1], Proposition 5.9. □

We finally deal with the remaining case -the socle filtration is here slightly more complicated: in Euclidean terms, the blocks $\mathfrak{V}_{s_{m+k}}$ for $r_{s_{m+k}} = p-1$ should be cut by the hyperplanes $X_0 + \dots + X_{f-1} = (r_{s_{m+k}} + 1) + J$ or $X_0 + \dots + X_{f-1} = (r_{s_{m+k}} + 1) + J - 1$ according to a condition on $r_{s_{m+k}+1}$.

Proposition 5.8. *Assume there exist an index $s \in \{0, \dots, f-1\}$ such that $r_s = p-1$ and $r_{\lfloor s+1 \rfloor} = 1$. Up to reordering, we assume there exists integers $0 \leqslant k_1 \leqslant k_0$ such that $r_{s_{m+j}} = p-1$ for all $j \in \{0, \dots, k_0\}$ and*

$$\begin{cases} r_{\lfloor s_{m+j}+1 \rfloor} \neq 1 & \text{if} \quad 0 \leqslant j \leqslant k_1 - 1, \\ r_{\lfloor s_{m+j}+1 \rfloor} = 1 & \text{if} \quad k_1 \leqslant j \leqslant k_0. \end{cases}$$

Then the J-th factor for the socle filtration of $(R_1/R_0)^+$ is described by the subspace

$$\mathscr{V}_J \overset{\text{def}}{=} \langle F_{\underline{r}}^{(0)}(\underline{0}) \rangle_{\overline{\mathbf{F}}_p} + \sum_{k=0}^{f-1} \langle F_{\underline{l}_0}(\underline{l}_1) \in \mathfrak{V}_{s_{m+k}}, \quad N_{(0,1)}(\underline{l}_0, \underline{l}_1) \leqslant (r_{s_{m+k}} + 1) + J - \delta_{k_1 \leqslant k \leqslant k_0} \rangle_{\overline{\mathbf{F}}_p}.$$

In particular, the socle filtration is deduced from the lattice $\mathscr{R}^+_{1/0}$ by cutting the k-th block by the hyperplane $X_0 + \cdots + X_{f-1} = (r_{s_{m+k}} + 1) + J - \delta_{k_1 \leqslant k \leqslant k_0}$.

Proof. Omissis. See [Mo1], Proposition 5.10. □

5.1.3 ***Application: the universal representation contains infinitely many compact inductions.*** As annonced in the introduction of §5 we are able to describe a G-equivariant natural injection

$$c\text{–}\mathrm{Ind}_{KZ}^G V \hookrightarrow \pi(\underline{r}, 0, 1)$$

for $\underline{r} \notin \{\underline{0}, \underline{p-1}\}$ where V is a convenient KZ-subrepresentation of $\pi(\underline{r}, 0, 1)|_{KZ}$. An analogous result has been discovered by Paskunas in an unpublished draft.

The proof can be outlined as follow. Via the isomorphism of Proposition 2.9 we define the representation V as a suitable subrepresentation of R_1/R_0: by Frobenius reciprocity we get a morphism $\phi : c\text{–}\mathrm{Ind}_{KZ}^G V \to \pi(\underline{r}, 0, 1)$. From a basis of V we construct a convenient $\overline{\mathbf{F}}_p$-basis for the compact induction $c\text{–}\mathrm{Ind}_{KZ}^G V$ and therefore we only have to check that ϕ maps such basis into a linearly independent family of $\pi(\underline{r}, 0, 1)$.

This can be easily verified combining Proposition 3.4, Lemma 5.1 and Proposition 3.5.

We start from the following elementary fact:

Lemma 5.9. *The K subrepresentation $\mathrm{Fil}^{\underline{0}}(R_1)$ of R_1 generated by $\left[1, X^{\underline{r}}\right]$ is naturally isomorphic to the finite principal series $\mathrm{Ind}_{K_0(p)}^K \chi_{\underline{r}}^s$ and $\mathrm{soc}(\mathrm{Fil}^{\underline{0}}(R_1)) \cong R_0$ via the monomorphism $R_0 \hookrightarrow R_1$.*

Proof. Omissis. □

Let $\widetilde{V}$ denote the kernel of the natural map

$$\mathrm{Fil}^{\underline{0}}(R_1)/R_0 \twoheadrightarrow \mathrm{cosoc}(\mathrm{Fil}^{\underline{0}}(R_1));$$

we define $V \leqslant \pi(\underline{r}, 0, 1)|_{KZ}$ as the homomorphic image of $\widetilde{V}$ via the isomorphism given in Proposition 2.9. Therefore, by Frobenius reciprocity, we get a morphism

$$\phi : c\text{–}\mathrm{Ind}_{KZ}^G V \to \pi(\underline{r}, 0, 1).$$

We claim that

Theorem 5.10. *Assume $\underline{r} \notin \{\underline{0}, \underline{p-1}\}$. Then ϕ is a monomorphism.*

Proof. We show that the composite morphism of ϕ with the isomorphism (4)

$$c\text{–}\mathrm{Ind}_{KZ}^G V \overset{\phi}{\to} \pi(\underline{r}, 0, 1) \overset{\sim}{\to} \varinjlim_{n \,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}) \oplus \varinjlim_{n \,\mathrm{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})$$

maps an $\overline{\mathbf{F}}_p$-basis of $c-\mathrm{Ind}_{KZ}^G V$ onto a linearly independent family of the amalgamated sums on the right hand side.

By the well known results concerning the structure of finite principal series for $\mathrm{GL}_2(\mathbf{F}_q)$ we have

Lemma 5.11. *Assume* $\underline{r} \notin \{\underline{0}, \underline{p-1}\}$. *For an* f*-tuple* $\underline{t} \in \{0, \dots, p-1\}^f$ *such that* $\underline{t} \not\leq \underline{r}$ *and* $\underline{r} \not\leq \underline{t}$ *the element* $v_{\underline{t}} \in V$ *is defined as*

$$v_{\underline{t}} \overset{\mathrm{def}}{=} \sum_{\mu_0 \in \mathbf{F}_q} \mu_0^{\underline{t}} \begin{bmatrix} p & [\mu_0] \\ 0 & 1 \end{bmatrix} [1, X^{\underline{r}}].$$

An $\overline{\mathbf{F}}_p$*-basis* $\mathcal{V}$ *for the compact induction* $c-\mathrm{Ind}_{KZ}^G V$ *is described by the elements*

$$G_{\emptyset}^{(0,-1)}(\underline{t}) \overset{\mathrm{def}}{=} [1, v_{\underline{t}}]$$

$$G_{\underline{l}_0,\dots,\underline{l}_n}^{(1,n)}(\underline{t}) \overset{\mathrm{def}}{=} \sum_{\lambda_1 \in \mathbf{F}_q} (\lambda_1^{\frac{1}{p}})^{\underline{l}_1} \begin{bmatrix} 1 & 0 \\ p[\lambda_1^{\frac{1}{p}}] & 1 \end{bmatrix} \dots \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{\underline{l}_n} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ p^{n+1} & 0 \end{bmatrix} [1, v_{\underline{t}}]$$

$$G_{\underline{l}_0,\dots,\underline{l}_n}^{(0,n)}(\underline{t}) \overset{\mathrm{def}}{=} \sum_{\lambda_0 \in \mathbf{F}_q} \lambda_0^{\underline{l}_0} \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix} [1, G_{\underline{l}_0,\dots,\underline{l}_n}^{(1,n)}(\underline{t})]$$

where $n \in \mathbf{N}$, $\underline{l}_j \in \{0, \dots, p-1\}^f$ *for all* $j \in \{0, \dots, n\}$, *and* $\underline{t} \in \{0, \dots, p-1\}^f$ *verify the conditions* $\underline{t} \not\leq \underline{r}$ *and* $\underline{r} \not\leq \underline{t}$.

Proof. It is elementary and left to the reader. See [Mo1], Lemma 5.13 for details. □

We recall that the morphism ϕ is G-equivariant and the isomorphism (4) is KZ-equivariant. We deduce the equalities

$$\phi(G_{\underline{l}_0,\dots,\underline{l}_n}^{(0,n)}(\underline{t})) = pr(F_{\underline{l}_0,\dots,\underline{l}_n,\underline{t}}^{(0,n+1)}(\underline{0}))$$
$$\phi(G_{\underline{l}_1,\dots,\underline{l}_n}^{(1,n)}(\underline{t})) = pr(F_{\underline{l}_1,\dots,\underline{l}_n,\underline{t}}^{(1,n+1)}(\underline{0}))$$
$$\phi(G_{\emptyset}^{(0,-1)}(\underline{t})) = pr(F_{\underline{t}}^{(0)}(\underline{0}))$$

where we wrote pr to denote the natural epimorphisms of Proposition 3.5.

As the kernel of the epimorphism pr is known and we dispose of a suitable $\overline{\mathbf{F}}_p$-basis of the inductive limits $\varinjlim_{n\,\mathrm{odd}} R_0^{\pm} \oplus_{R_1^{\pm}} \dots \oplus_{R_n^{\pm}} R_{n+1}^{\pm}$, $\varinjlim_{n\,\mathrm{even}} (R_1/R_0)^{\pm} \oplus_{R_2^{\pm}} \dots \oplus_{R_n^{\pm}} R_{n+1}^{\pm}$ we check that the elements $pr(F_{\underline{l}_0,\dots,\underline{l}_n,\underline{t}}^{(0,n+1)}(\underline{0}))$, $pr(F_{\underline{l}_1,\dots,\underline{l}_n,\underline{t}}^{(1,n+1)}(\underline{0}))$ and $pr(F_{\underline{t}}^{(0)}(\underline{0}))$ of the inductive limits $\varinjlim_{n\,\mathrm{odd}} R_0 \oplus_{R_1} \dots \oplus_{R_n} R_{n+1}$, $\varinjlim_{n\,\mathrm{even}} (R_1/R_0) \oplus_{R_2} \dots \oplus_{R_n} R_{n+1}$ are linearly independent, as required. □

Remark 5.12. *Let* $\mathfrak{V}$ *the image of the composite map obtained by* ϕ *and the isomorphism (4). By the proof of Proposition 5.10 the reader can easily describe, in terms of the lattices* $\dots \oplus_{\mathscr{R}_n^{\pm}} \mathscr{R}_{n+1}^{\pm}$, *the inverse image of* $\mathfrak{V}$ *by the natural epimorphism* pr *of Proposition 3.5.*

5.2 The structure of the amalgamated sums

We are now ready to describe how two blocks $R_{n+1}^{\bullet}/R_n^{\bullet}$ and $R_{n-1}^{\bullet}/R_{n-2}^{\bullet}$ should be glued together. We will see that such glueing is more or less a formal consequence of the geometric interpretation of the amalgamated sums, as announced in the introduction of §5.

As for section 5.1 we will give the detailed proofs for the positive case: the negative part is deduced analogously.

First, we want to understand the image of an element $F^{(0,n)}_{\underline{l}_0,\ldots,\underline{l}_n}(\underline{l}_{n+1}) \in R^+_{n+1}$ (resp. $F^{(1,n)}_{\underline{l}_1,\ldots,\underline{l}_n}(\underline{l}_{n+1}) \in R^-_{n+1}$) via the projection $(pr_{n+1})^{\mathrm{pos}}$ (resp. $(pr_{n+1})^{\mathrm{neg}}$) of Lemma 3.4.

Lemma 5.13. *Let $n \in \mathbf{N}_{\geqslant 1}$. The image of the element $F^{(0,n)}_{\underline{l}_0,\ldots,\underline{l}_n}(\underline{l}_{n+1}) \in R^+_{n+1}$ via the projection pr^{pos}_{n+1} is described as follow:*

1) *If either $\underline{l}_{n+1} \neq \underline{0}$ or $\underline{l}_{n+1} = \underline{0}$ and $\underline{l}_n \nleqslant \underline{r}$ then*
$$\pi_{n+1}(pr_{n+1})^{\mathrm{pos}}(F^{(0,n)}_{\underline{l}_0,\ldots,\underline{l}_n}(\underline{l}_{n+1})) = \pi_{n+1}(F^{(0,n)}_{\underline{l}_0,\ldots,\underline{l}_n}(\underline{l}_{n+1}));$$
2) *If $\underline{l}_{n+1} = \underline{0}$, $\underline{l}_n = \underline{r}$ and $\underline{l}_{n-1} \geqslant \underline{p-1-r}$ then*
$$(-1)^{\underline{r}}(pr_{n+1})^{\mathrm{pos}}(F^{(0,n)}_{\underline{l}_0,\ldots,\underline{l}_n}(\underline{l}_{n+1})) = \iota^{\mathrm{pos}}_{n-1}(F^{(0,n-2)}_{\underline{l}_0,\ldots,\underline{l}_{n-2}}(\underline{l}_{n-1} - \underline{p-1-r})) + \delta_{\underline{r},\underline{p-1}}\delta_{\underline{l}_{n-1},\underline{p-1}}\iota^{\mathrm{pos}}_{n-1}(F^{(0,n-2)}_{\underline{l}_0,\ldots,\underline{l}_{n-2}}(\underline{0}));$$
3) *If either $\underline{l}_{n+1} = \underline{0}$, $\underline{l}_n = \underline{r}$ and $\underline{l}_{n-1} \ngeqslant \underline{p-1-r}$ or $\underline{l}_{n+1} = \underline{0}$ and $\underline{l}_n \lneqq \underline{r}$ then*
$$(pr_{n+1})^{\mathrm{pos}}(F^{(0,n)}_{\underline{l}_0,\ldots,\underline{l}_n}(\underline{l}_{n+1})) = 0.$$

Proof. Assertion 1) is clear by Lemma 5.1. We assume now that $\underline{l}_{n+1} = \underline{0}$ and $\underline{l}_n \leqslant \underline{r}$. Thus,
$$F^{(0,n)}_{\underline{l}_0,\ldots,\underline{l}_n}(\underline{0}) = (-1)^{\underline{l}_n}(T^+_n)^{\mathrm{pos}}(F^{(0,n-1)}_{\underline{l}_0,\ldots,\underline{l}_{n-1}}(\underline{l}_n))$$
so that we get the following equality in the amalgamated sum $\cdots \oplus_{R^+_n} R^+_{n+1}$:
$$(pr_{n+1})^{\mathrm{pos}}(F^{(0,n)}_{\underline{l}_0,\ldots,\underline{l}_n}(\underline{0})) = \iota^+_{n-1} \circ pr^+_{n-1} \circ (-T^-_n)^{\mathrm{pos}}((-1)^{\underline{l}_n}(F^{(0,n-1)}_{\underline{l}_0,\ldots,\underline{l}_{n-1}}(\underline{l}_n))).$$
In order to get the statement, we are now left to describe
$$(T^-_n)^{\mathrm{pos}}((F^{(0,n-1)}_{\underline{l}_0,\ldots,\underline{l}_{n-1}}(\underline{l}_n))).$$
Let assume $n \geqslant 2$ (the case $n = 1$ is treated in an analogous way and is left to the reader). By the characterisation of the operator T^-_n we have
$$(T^-_n)^{\mathrm{pos}}((F^{(0,n-1)}_{\underline{l}_0,\ldots,\underline{l}_{n-1}}(\underline{l}_n))) = 0$$
if $\underline{l}_n \neq \underline{r}$, while, for $\underline{l}_n = \underline{r}$, we have
$$\begin{aligned}
&(T^-_n)^{\mathrm{pos}}((F^{(0,n-1)}_{\underline{l}_0,\ldots,\underline{l}_{n-1}}(\underline{l}_n)))) = \\
&= \sum_{j=0}^{n-2}\sum_{\lambda_j \in \mathbf{F}_q}(\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j}\begin{bmatrix} 1 & 0 \\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1\end{bmatrix}\Big[1, \sum_{\lambda_{n-1} \in \mathbf{F}_q}(\lambda_{n-1}^{\frac{1}{p^{n-1}}})^{\underline{l}_{n-1}}(\lambda_{n-1}^{\frac{1}{p^{n-1}}}X + Y)^{\underline{r}}\Big] = \\
&= \sum_{\underline{i} \leqslant \underline{r}}\binom{\underline{r}}{\underline{i}}\sum_{j=0}^{n-2}\sum_{\lambda_j \in \mathbf{F}_q}(\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j}\begin{bmatrix} 1 & 0 \\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1\end{bmatrix}\Big[1, X^{\underline{r}-\underline{i}}Y^{\underline{i}}\sum_{\lambda_{n-1} \in \mathbf{F}_q}(\lambda_{n-1}^{\frac{1}{p^{n-1}}})^{\underline{l}_{n-1}+\underline{r}-\underline{i}}\Big].
\end{aligned}$$
By Lemma 3.1, the quantity
$$\sum_{\lambda_{n-1} \in \mathbf{F}_q}(\lambda_{n-1}^{\frac{1}{p^{n-1}}})^{\underline{l}_{n-1}+\underline{r}-\underline{i}}$$
is non zero (indeed assuming the value -1) if and only if $\underline{l}_{n+1} + \underline{r} - \underline{i} \equiv 0 \bmod q-1$ and $\underline{l}_{n+1} + \underline{r} - \underline{i} \neq 0$. The result follows. ☐

The result concerning the negative part is similar

Lemma 5.14. *Let $n \in \mathbf{N}_{\geqslant 1}$. The image of the element $F^{(1,n)}_{\underline{l}_1,\ldots,\underline{l}_n}(\underline{l}_{n+1}) \in R^-_{n+1}$ via the projection pr^{neg}_{n+1} is described as follow:*

1) *If either* $\underline{l}_{n+1} \neq \underline{0}$ *or* $\underline{l}_{n+1} = \underline{0}$ *and* $\underline{l}_n \not\leqslant \underline{r}$ *then*

$$\pi_{n+1}(pr_{n+1})^{\mathrm{neg}}(F^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}(\underline{l}_{n+1})) = \pi_{n+1}(F^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}(\underline{l}_{n+1}));$$

2) *If* $\underline{l}_{n+1} = \underline{0}$, $\underline{l}_n = \underline{r}$ *and* $\underline{l}_{n-1} \geqslant \underline{p-1-r}$ *(the latter condition being empty if $n=1$) then*

$$(-1)^{\underline{r}}(pr_{n+1})^{\mathrm{neg}}(F^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}(\underline{l}_{n+1})) = \iota^{\mathrm{neg}}_{n-1}(F^{(1,n-2)}_{\underline{l}_1,\dots,\underline{l}_{n-2}}(\underline{l}_{n-1}-\underline{p-1-r}))+\delta_{\underline{r},\underline{p-1}}\delta_{\underline{l}_{n-1},\underline{p-1}}\iota^{\mathrm{pos}}_{n-1}(F^{(1,n-2)}_{\underline{l}_1,\dots,\underline{l}_{n-2}}(\underline{0}));$$

3) *If either* $\underline{l}_{n+1} = \underline{0}$, $\underline{l}_n = \underline{r}$ *and* $\underline{l}_{n-1} \not\geqslant \underline{p-1-r}$ *(the latter condition being empty if $n=1$) or* $\underline{l}_{n+1} = \underline{0}$ *and* $\underline{l}_n \lneq \underline{r}$ *then*

$$(pr_{n+1})^{\mathrm{pos}}(F^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}(\underline{l}_{n+1})) = 0.$$

Proof. It is analogous to the proof of Proposition 5.13 and it is left to the reader. □

Interpretation in terms of Euclidean data. We dispose of a canonical $\overline{\mathbf{F}}_p$-basis for the representation $\cdots \oplus_{R_n^\pm} R_{n+1}^\pm$, which is obtained in the obvious way by an induction from Proposition 3.4 and Lemma 5.1.

Exactly as we did in §5.1 we have a natural way to associate an element of such canonical basis to a point in $\mathbf{R}^f$: again, we obtain a lattice, which we will denote by $\cdots \oplus_{\mathscr{R}_n^\pm} \mathscr{R}_{n+1}^\pm$.

In such Euclidean setting Proposition 5.13 is clear: it tells that lattice $\cdots \oplus_{\mathscr{R}_n^+} \mathscr{R}_{n+1}^+$ is obtained as the union of the lattice $\mathscr{R}_{n+1/n}^+$ associated to R_{n+1}^+/R_n^+ and the image of the lattice $\cdots \oplus_{\mathscr{R}_{n-2}^+} \mathscr{R}_{n-1}^+$ associated to the amalgamated sum $\cdots \oplus_{R_{n-2}^+} R_{n-1}^+$ (which, inductively, can be assumed to be known) under the translation

$$\begin{aligned} \mathbf{R}^f &\to \mathbf{R}^f \\ (x_i)_i &\mapsto (x_i + p^{n-1}(p-1-r_{\lfloor i+n-1 \rfloor}) + p^n r_{\lfloor i+n \rfloor}). \end{aligned} \tag{15}$$

Notice that in particular the lattice $\cdots \oplus_{\mathscr{R}_{n-2}^+} \mathscr{R}_{n-1}^+$ is glued inside the $F^n_{\underline{r}}(\underline{0})$-block of $\mathscr{R}_{n+1}^+$.

We stress again in Figure 9 the glueing and the fractal structure for $f = 2$ (noticing the glueing of $\cdots \oplus_{\mathscr{R}_{n-2}^+} \mathscr{R}_{n-1}^+$ inside the $F^{(n)}_{\underline{r}}(\underline{0})$-block of $\mathscr{R}_{n+1/n}^+$).

The evident analogous considerations for the negative part $\cdots \oplus_{\mathscr{R}_n^-} \mathscr{R}_{n+1}^-$ are left to the reader.

Remark 5.15. *Notice that if $f = 1$ then it follows directly from Propositions 5.13 and 5.14 that the $K_0(p)$-structure (and the extensions between two consecutive graded pieces) of the representations $\dots_{R_n^\bullet} R_{n+1}^\bullet$ are given by the associated lattices $\cdots \oplus_{\mathscr{R}_n^\bullet} \mathscr{R}_{n+1}^\bullet$. In particular, we deduce that each of these representations has a space of I_1 invariants of dimension 1, recovering [Bre03], Théorème 3.2.4.*

By remark 5.15 we can assume from now on that $f \geqslant 2$.

Structure of the Universal representation and Euclidean datum. We are now left to prove that the socle filtration (and the extension between two consecutive graded pieces) of the $K_0(p)$-representation $\cdots \oplus_{R_n^+} R_{n+1}^+$ is described by the associated Euclidean datum in the sense of Definition 1.7. As we have seen for R_{n+1}^+/R_n^+, the main task is to show that the "natural" linear filtration on $\cdots \oplus_{R_n^+} R_{n+1}^+$ deduced from the Euclidean structure $\cdots \oplus_{\mathscr{R}_n^+} \mathscr{R}_{n+1}^+$ is indeed $K_0(p)$-equivariant with semisimple layers. By proposition 4.11, it follows then that such natural linear filtration is the socle filtration.

We need therefore a precise control on the $K_0(p)$ representation generated by an element of the canonical base of $\cdots \oplus_{R_n^+} R_{n+1}^+$; as $\cdots \oplus_{R_{n-2}^+} R_{n-1}^+$ is a $K_0(p)$-*subrepresentation* we will see, by a simple Euclidean argument, that the statement of proposition 5.3 will be sufficient.

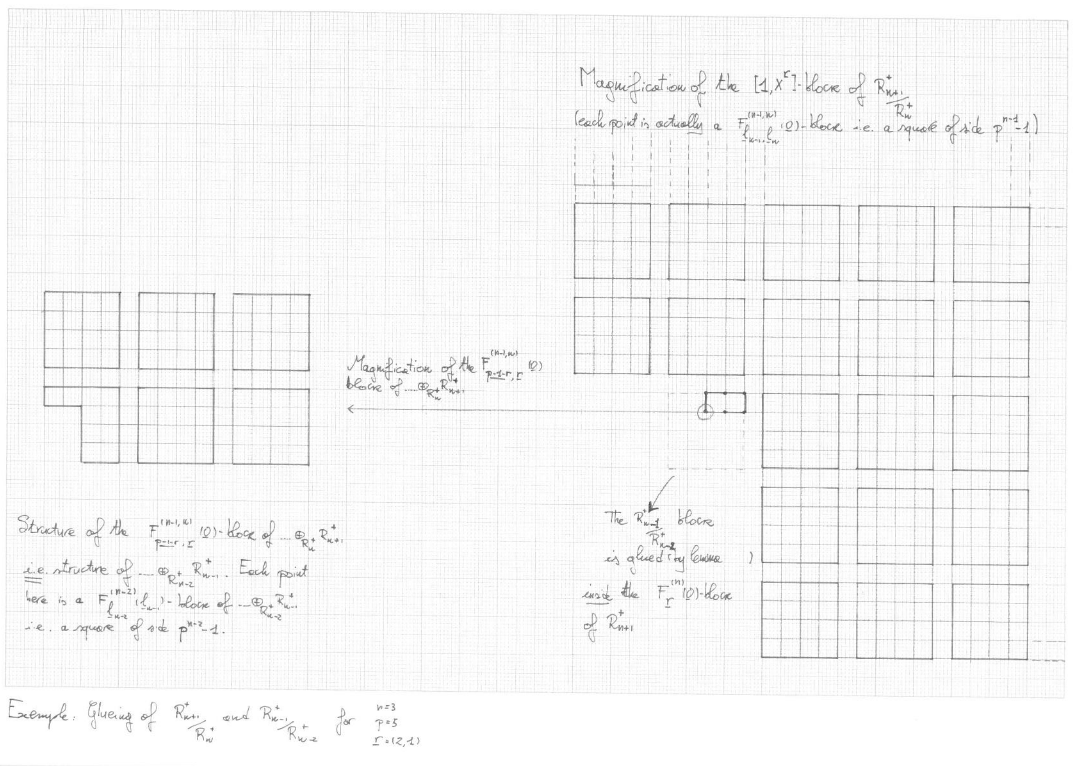


Figure 9: This picture shows how to glue together the datum of R_{n+1}^+/R_n^+ and R_{n-1}^+/R_{n-2}^+ (Lemma 5.13). Repeating this process for all R_{n+1-2i}^+, $i \in \mathbf{N}$, $i \leqslant \frac{n+1}{2}$ gives rise to a complicate fractal structure.

Let $\mathscr{B}^+_{al,n}$ be the canonical basis of $\cdots \oplus_{R^+_n} R^+_{n+1}$; its description in terms of the canonical basis of $\cdots \oplus_{R^+_{n-2}} R^+_{n-1}$ and R^+_{n+1}/R^+_n is clear, as well as the relations between $\mathscr{B}^+_{al,n}$ and the canonical basis of R^+_{n+1} (Lemma 5.13). In particular, the projection π_{n+1} of Lemma 3.4 let us identify the canonical base $\mathscr{B}^+_{n+1/n}$ of R^+_{n+1}/R^+_n with a convenient subset of $\mathscr{B}^+_{al,n}$; in Euclidean terms we are considering the Euclidean datum $\mathscr{R}^+_{n+1/n}$ embedded in $\cdots \oplus_{\mathscr{R}^+_n} \mathscr{R}^+_{n+1}$.

For $N \in \mathbf{N}$ we consider the following subsets of $\mathscr{B}^+_{al,n}$:

i) a linear basis $\mathscr{B}_{N,sub}$ for the N-th composition factor of the socle filtration for $\cdots \oplus_{R^+_{n-2}} R^+_{n-1}$;

ii) A linear basis $\mathscr{B}_{N,qt}$ for the N-th composition factor of R^+_{n+1}/R^+_n (such basis is seen as a subset of $\mathscr{B}^+_{al,n}$ via the previous identification).

The result below gives us the desired control of the action of $K_0(p)$ in Euclidean terms:

Proposition 5.16. *Let $N \in \mathbf{N}$, $v \in \mathscr{B}_{N,2}$ and $g \in K_0(p)$. Assume moreover that if $n = 1$ and $f = 2$ then $(r_0, r_1) \notin \{(p-2, 0), (0, p-2)\}$.*

Then the element $(g-1)\cdot v$ is contained in the linear space generated by $\mathscr{B}_{N-1,qt}$ and $\mathscr{B}_{N-2,sub}$.

In particular,

i) the linear space V_N generated by $\mathscr{B}_{N,sub}$, $\mathscr{B}_{N,qt}$ is $K_0(p)$ stable;

ii) the filtration $\{V_N\}_{N\in\mathbf{N}}$ has semisimple layers;

iii) modulo V_{N-2} there are no extensions between the elements of $\mathscr{B}_{N,qt}$ and $\mathscr{B}_{N-1,sub}$.

Proof. Define, for any $n \geqslant 1$,

$$M_n \overset{\text{def}}{=} \sum_{s=0}^{f-1} (p^{n-1}(p-1-r_{\lfloor s+n-1 \rfloor}) + p^n r_{\lfloor s+n \rfloor});$$

in particular the hyperplane $X_0 + \cdots + X_{f-1} = M_n$ contains the image of the point $\underline{0}$ via the translation (15). Except in the case where $f = 2$ and $(r_0, r_1) \in \{(p-1, 0), (0, p-1), (p-2, 0), (0, p-2)\}$, we have

$$M_n > p^n(r_{s_0} + 1) \tag{16}$$

for any $s_0 \in \{0, \ldots, f-1\}$ (and we actually have an equality if and only if $f = 2$, $(r_0, r_1) \in \{(p-2, 0), (0, p-2)\}$ and $r_{s_0} = p-2$).

By the Euclidean interpretation, Proposition 5.3 and an immediate induction [10] on n, the statement is proved if we show the following:

1) if $n \geqslant 3$, that an hyperplane of the form $X_0 + \cdots + X_{f-1} = p^n(r_{\lfloor n+s \rfloor} + 1) + N$ lies strictly below an hyperplane of the form $X_0 + \cdots + X_{f-1} = M_n + p^{n-2}(r_{\lfloor n+s \rfloor} + 1) + N$ for any choice of indices $s_0, s_1 \in \{0, \ldots, f-1\}$, i.e.

$$p^n(r_{s_0} + 1) < M_n + p^{n-2}(r_{s_1} + 1);$$

2) similarly, if $n = 2$, that for any choice of inices $s_0, s_1 \in \{0, \ldots, f-1\}$ we have

$$p^2(r_{s_0} + 1) < M_2 + (r_{s_1} + 1) - \delta$$

where $\delta \in \{0, 1\}$ is nonzero if and only if either the f-tuple $\underline{r}$ verifies the hypothesis I_B) of Proposition 5.6 and $s_1 = s_m$ (see the introduction of §5.1 for the definition of s_m) or the the f-tuple $\underline{r}$ verifies the hypothesis of Proposition 5.8 and $s_1 \in \{s_{m+k_1}, \ldots, s_{m+k_0}\}$.

[10]if $f = 2$ and $(r_0, r_1) \in \{(p-2, 0), (0, p-2)\}$ then induction works as well thanks to Remark 5.17.

3) if $n = 1$, that

$$p(r_{s_0} + 1) < M_1.$$

The three conditions follow from (16) if $f \geqslant 3$. If $f = 2$ and $(r_0, r_1) \in \{(p-1, 0), (0, p-1)\}$ we have $\mathfrak{V}_{s_m} = \{0\}$ so that the three conditions should be checked only for $r_{s_0} = 0$ (we have no cutting hyperplane of the form $X_0 + \cdots + X_{f-1} = p^n(\lfloor r_{s_m} + n \rfloor + 1)$ in this case!). $\square$

Remark 5.17. *Notice that for $n = 1$, $f = 2$ and $(r_0, r_1) \in \{(p-2, 0), (0, p-2)\}$ the statement of Proposition 5.16 holds true if we replace $\mathscr{B}_{N-2,sub}$ by $\mathscr{B}_{N-1,sub}$. Indeed, in this situation the cutting hyperplanes for R_0^+ and $\mathfrak{V}_{s_m}$ coincides.*

Theorem 5.18. *Let $n \geqslant 1$ and consider the $K_0(p)$-representation $\cdots \oplus_{R_n^+} R_{n+1}^+$.*

The socle filtration and the extensions between two consecutive graded pieces are described by the associated lattice $\cdots \oplus_{\mathscr{R}_n^+} \mathscr{R}_{n+1}^+$, with the conventions of section §5.1.2 and Propositions 5.6, 5.7 and 5.8 concerning the lattice associated to the $K_0(p)$-structure of $(R_1/R_0)^+$.

Proof. It is a formal consequence of Proposition 5.16, Remark 5.17 and Proposition 4.11. $\square$

6. Appendix A: Some remarks on Witt polynomials

The aim of this appendix is to collect some technical results concerning Witt polynomials. After a section of general reminders (§6.1), we will treat in detail the case of the universal polynomials for the sum and the product (§6.2 and §6.3). In section §6.4 we study the Witt polinomials of a certain power series in the ring $W(\mathbf{F}_q)$: in this situation it is more complicate to keep track of the exponents of such polynomials and we are therefore led to introduce the notion of "pseudo homogeneity" (definition 6.11).

6.1 Reminder on Witt polynomials

The description of the socle filtration for the aforementioned representations of $\mathrm{GL}_2(F)$ relies crucially on the behaviour of the universal Witt polynomials. After some generalities, we focus on specific situations related to the study of the action of lower unipotent, diagonal and upper unipotent matrices in $\mathrm{GL}_2(\mathscr{O}_F)$. The interested reader is referred to [Ser], [Bou] or [Bos] for more details concerning the formalism of Witt polynomials.

For $n \in \mathbf{N}$ the n-th Witt polynomial $W_n(\underline{X}) \in \mathbf{Z}[X_0, \ldots, X_n]$ is defined by

$$W_n(\underline{X}) \overset{\text{def}}{=} \sum_{i=0}^{n} X_i^{p^{n-i}} p^i.$$

As the ring endomorphism

$$\begin{aligned} \mathbf{Z}[\frac{1}{p}][X_0, \ldots, X_n] &\xrightarrow{\omega_n} \mathbf{Z}[\frac{1}{p}][X_0, \ldots, X_n] \\ X_j &\longmapsto W_j(X_0, \ldots, X_j) \end{aligned}$$

is bijective, we get a family of polynomials $M_0(X_0), \ldots, M_n(X_0, \ldots, X_n) \in \mathbf{Z}[\frac{1}{p}][X_0, \ldots, X_n]$ which are uniquely determined by the condition:

$$M_j(W_0(\underline{X}), \ldots, W_n(\underline{X})) = X_j.$$

They are of course described inductively by

$$M_n = \frac{1}{p^n}(X_n - p^{n-1} M_{n-1}(\underline{X})^p - \cdots - p M_1(X_0, X_1)^{p^{n-1}} - M_0(X_0)^{p^n}).$$

The following Lemma lets us deduce the universal Witt polynomials describing the ring structure of $W(\mathbf{F}_q)$:

Proposition 6.1. *Let $\Phi \in \mathbf{Z}[\zeta,\xi]$ be a polynomial in the variables ζ,ξ. For all $n \in \mathbf{N}$ there exist polynomials $\phi_n \in \mathbf{Z}[X_0,\dots,X_n,Y_0,\dots,Y_n]$, uniquely determied by the conditions*

$$W_n(\phi_0,\dots,\phi_n) = \Phi(W_n(X_0,\dots,X_n), W_n(Y_0,\dots,Y_n)).$$

Sketch of the proof. The proof is constructive: we considering the commutative diagram

$$\begin{array}{ccc} \mathbf{Z}[\frac{1}{p}][X_0,\dots,X_n] & \xrightarrow[\sim]{\omega_n} & \mathbf{Z}[\frac{1}{p}][X_0,\dots,X_n] \\ \downarrow f & & \vdots \\ \mathbf{Z}[\frac{1}{p}][X_0,\dots,X_n,Y_0,\dots,Y_n] & \xrightarrow[\sim]{\omega_n\otimes\omega_n} & \mathbf{Z}[\frac{1}{p}][X_0,\dots,X_n,Y_0,\dots,Y_n] \end{array}$$

where $f : \mathbf{Z}[\frac{1}{p}][\underline{X}] \to \mathbf{Z}[\frac{1}{p}][\underline{X},\underline{Y}]$ is defined by $f(X_j) \stackrel{\text{def}}{=} \Phi(X_j,Y_j)$ for any $j \in \{0,\dots,n\}$; the polynomial ϕ_n is then given by

$$\phi_n(\underline{X},\underline{Y}) \stackrel{\text{def}}{=} (\omega_n \otimes \omega_n) \circ f \circ \omega_n^{-1}(X_n).$$

The fact that such ϕ_n have integer coefficients is an induction on n. □

We apply Proposition 6.1 to the polynomials

$$\Phi(\zeta,\xi) = \zeta + \xi, \qquad \Phi(\zeta,\xi) = \zeta\xi$$

to get the universal polynomials for the sum and the product respectively. They will be denoted as $S_n, Prod_n \in \mathbf{Z}[X_0,\dots,X_n,Y_0,\dots,Y_n]$ and are described inductively by

$$S_n(\underline{X},\underline{Y}) = \frac{1}{p^n}(W_n(\underline{X}) + W_n(\underline{Y}) - p^{n-1}S_{n-1}(\underline{X},\underline{Y})^p - \dots - pS_1(\underline{X},\underline{Y})^{p^{n-1}} - S_0(\underline{X},\underline{Y})^{p^n})$$

$$Prod_n(\underline{X},\underline{Y}) = \frac{1}{p^n}(W_n(\underline{X})W_n(\underline{Y}) - p^{n-1}Prod_{n-1}(\underline{X},\underline{Y})^p - \dots - pProd_1(\underline{X},\underline{Y})^{p^{n-1}} - Prod_0(\underline{X},\underline{Y})^{p^n}).$$

In section 4 we are interested in such operations as rise to the N-th power or the alternate sum $\sum_{j=1}^{N}(-1)^{j+1}\underline{X}(j)$ of N elements. We can of course adapt the arguments of Proposition 6.1 (or, use an induction on N) to determine the universal Witt polynomials associated to such operations. We will write $Pot_n^N(\underline{X}) \in \mathbf{Z}[X_0,\dots,X_n]$, $S_n^N(\underline{X(1)},\dots,\underline{X(N)}) \in \mathbf{Z}[X(1)_0,\dots,X(1)_n,\dots,X(N)_0,\dots,X(N)_n]$ for the n-th Witt polinomial associated to the rise to the N-th power and the alternate sum of N elements respectively. We have then the recursive relations:

$$\begin{aligned} &Pot_n^N(\underline{X}) = \frac{1}{p^n}(W_n(\underline{X})^N - p^{n-1}Pot_{n-1}^N(\underline{X})^p - \\ &\dots - pPot_1^N(\underline{X})^{p^{n-1}} - Pot_0^N(\underline{X})^{p^n}) \\ &S_n^N(\underline{X(1)},\dots,\underline{X(N)}) = \frac{1}{p^n}(\sum_{j=1}^{N}(-1)^{j+1}W_n(\underline{X(j)}) - p^{n-1}S_{n-1}^N(\underline{X(1)},\dots,\underline{X(N)})^p - \\ &\dots - pS_1^N(\underline{X(1)},\dots,\underline{X(N)})^{p^{n-1}} - S_0^N(\underline{X(1)},\dots,\underline{X(N)})^{p^n}). \end{aligned}$$

6.2 Some special polynomials-I

In this paragraph we collect some thechnical results concerning some Witt polynomials which appear naturally in the study of the action of $\begin{bmatrix} 1 & 0 \\ p\mathscr{O}_F & 1 \end{bmatrix}$ (resp. $\begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix}$) for the representations of §4.1 (resp. of §4.2), see also the proof of Proposition 4.4.

For $n \in \mathbf{N}$ we define $S_n(\underline{X}, Y_0) \in \mathbf{Z}[X_0, \dots, X_n, Y_0]$ as the specialisation of $S_n(\underline{X}, \underline{Y})$ at $\underline{Y} = (Y_0, 0, \dots, 0, \dots)$. We recall

Lemma 6.2. *For $n \in \mathbf{N}$ the polynomial $S_n(\underline{X}, \underline{Y})$ is an homogeneous polynomial in $\underline{X}, \underline{Y}$, of degree p^n if we define the elemets X_j, Y_j to be homogeneous of degree p^j.*

Proof. Omissis. □

Thus, if we set

$$\widetilde{S}_n(\underline{X}, Y_0) \stackrel{\text{def}}{=} S_n(\underline{X}, Y_0) - X_n$$

we see that $\widetilde{S}_j(\underline{X}, Y_0)$ is a polynomial in $\mathbf{Z}[X_0, \dots, X_{n-1}, Y_0]$, homogeneous of degree p^n. Moreover, as $\widetilde{S}_n(\underline{X}, 0) = 0$ we see that $\widetilde{S}_n(\underline{X}, Y_0)$ belongs to the ideal generated by Y_0.

We define inductively the following family of automorphisms: we put

$$\begin{aligned} s_0 : \mathbf{Z}[X_0, Y_0] &\to \mathbf{Z}[X_0, Y_0] \\ X_0 &\mapsto X_0 - Y_0 \\ Y_0 &\mapsto Y_0 \end{aligned}$$

and, assuming $s_{j-1} : \mathbf{Z}[X_0, \dots, X_{j-1}, Y_0] \to \mathbf{Z}[X_0, \dots, X_{j-1}, Y_0]$ being constructed, we define

$$\begin{aligned} s_j : \mathbf{Z}[X_0, \dots, X_j, Y_0] &\to \mathbf{Z}[X_0, \dots, X_j, Y_0] \\ X_j &\mapsto X_j - s_{j-1}(\widetilde{S}_j) \end{aligned}$$

By their very construction, the s_j are graded homomorphisms; in particular $s_j(\widetilde{S}_j)$ is homogeneous of degree p^j, and belongs to the ideal (Y_0) inside $\mathbf{Z}[X_0, \dots, X_j, Y_0]$. We can actually prove the following result

Lemma 6.3. *For any $n \geqslant 1$ we have*

$$s_{n-1}(S_n(\underline{X}, Y_0) - X_n) = -(S_n(\underline{X}, -Y_0) - X_n).$$

Proof. The case $n = 1$ is elementary:

$$s_0(S_1(X_0, X_1, Y_0) - X_1) = s_0(\frac{1}{p}(X_0^p + Y_0^p - (X_0 + Y_0)^p)) = \frac{1}{p}((X_0 - Y_0)^p + Y_0^p - X_0^p) = -(S_1(X_0, X_1, Y_0) - X_1).$$

Concerning the general case, we write

$$\begin{aligned} S_n(X_0, \dots, X_n, Y_0) - X_n = \frac{1}{p^n}\big[X_0^{p^n} + Y_0^{p^n} - p^{n-1}(S_{n-1}(\underline{X}, Y_0)^p - X_{n-1}^p) - \dots \\ \dots - p(S_1(X_0, X_1, Y_0)^{p^{n-1}} - X_1^{p^{n-1}}) - (X_0 + Y_0)^{p^n}\big]. \end{aligned} \tag{17}$$

For $j \in \{1, \dots, n-1\}$ we have

$$\begin{aligned} s_j(S_j(X_0, \dots, X_j, Y_0)^{p^{n-j}} - X_j^{p^{n-1}}) &= (s_{j-1}(S_j(X_0, \dots, X_j, Y_0) - X_j) + s_j(X_j))^{p^{n-j}} - (s_j(X_j))^{p^{n-j}} \\ &= X_j^{p^{n-j}} - (X_j - s_{j-1}(S_j(X_0, \dots, X_j, Y_0) - X_j))^{p^{n-j}} \\ &= X_j^{p^{n-j}} - (X_j + S_j(X_0, \dots, X_j, -Y_0) - X_j))^{p^{n-j}} \\ &= -(S_j(X_0, \dots, X_j, -Y_0)^{p^{n-j}} - X_j^{p^{n-j}}). \end{aligned}$$

As $s_{n-1}(S_n(X_0,\dots,X_n,Y_0)-X_n)=s_n(S_n(X_0,\dots,X_n,Y_0)-X_n)$ we are left to compute

$$s_n\Bigg(\frac{1}{p^n}\Bigg(X_0^{p^n}+Y_0^{p^n}-p^{n-1}(S_{n-1}(\underline{X},Y_0)^p-X_{n-1}^p)-\dots$$

$$\dots-p(S_1(X_0,X_1,Y_0)^{p^{n-1}}-X_1^{p^{n-1}})-(X_0+Y_0)^{p^n}\Bigg)\Bigg)=$$

$$=\frac{1}{p^n}\Bigg((X_0-Y_0)^{p^n}+Y_0^{p^n}-p^{n-1}s_{n-1}(S_{n-1}(\underline{X},Y_0)^p-X_{n-1}^p)-\dots$$

$$\dots-ps_1(S_1(X_0,X_1,Y_0)^{p^{n-1}}-X_1^{p^{n-1}})-(X_0)^{p^n}\Bigg)$$

and the result follows as $s_j(S_j(X_0,\dots,X_j,Y_0)^{p^{n-j}}-X_j^{p^{n-1}})=-(S_j(X_0,\dots,X_j,-Y_0)^{p^{n-j}}-X_j^{p^{n-j}})$ for all $j\in\{1,\dots n-1\}$. □

We will also need a cleaner statement concerning the monomials of $S_n(X_0,\dots,X_n,Y_0)$:

Lemma 6.4. *For all $n\geqslant 1$ the coefficient of the monomial $X_0^{p-1}\dots X_{n-1}^{p-1}Y_0$ appearing in the development of the universal Witt polynomial $S_n(X_0,\dots,X_n,Y_0)$ is 1, up to sign.*

Proof. The proof is again an induction on n: the case $n=1$ is evident.

For the general case, consider

$$S_n(\underline{X},Y_0)=\frac{1}{p^n}(W_n(\underline{X})+Y_0^{p^n}-p^{n-1}S_{n-1}(\underline{X},Y_0)^p-\dots-pS_1(\underline{X},Y_0)^{p^{n-1}}-S_0(\underline{X},Y_0)^{p^n}).$$

A monomial of the form $X_0^{p-1}\dots X_{n-1}^{p-1}Y_0$ lies therefore inside

$$-\frac{1}{p}(S_{n-1}(X_0,\dots,X_{n-1},Y_0)^p-X_{n-1}^{p-1})$$

and the inductive hypothesis yields

$$S_{n-1}(X_0,\dots,X_{n-1},Y_0)=X_{n-1}+X_0^{p-1}\dots X_{n-2}^{p-1}Y_0+x(X_0,\dots,X_{n-2},Y_0)$$

where $x(X_0,\dots,X_{n-2},Y_0)\in\mathbf{Z}[X_0,\dots,X_{n-2},Y_0]$ doesn't contains the monomial $X_0^{p-1}\dots X_{n-2}^{p-1}Y_0$. Finally, we have

$$(S_{n-1}(X_0,\dots,X_{n-1},Y_0))^p=\sum_{\substack{i+j+k=p\\ 0\leqslant i,j,k}}\frac{p!}{i!j!k!}X_{n-1}^i(X_0^{p-1}\dots X_{n-2}^{p-1}Y_0)^j(x(X_0,\dots,X_{n-2},Y_0))^k$$

and the conclusion follows. □

6.3 Some special polynomials -II

In this section we deal with some Witt polynomials which appear naturally when we study the action of the diagonal matrices $\begin{bmatrix}1+p\mathscr{O}_F & 0\\ 0 & 1+\mathscr{O}_F\end{bmatrix}$, see in particular the proof of Proposition 4.5. Recall that

Lemma 6.5. *Let $n\in\mathbf{N}$. The n-th universal Witt polynomial of the product $Prod_n(\underline{X},\underline{Y})$ is an homogeneous element of $(\mathbf{Z}[\underline{Y}])[\underline{X}]$ (resp. $(\mathbf{Z}[\underline{X}])[\underline{Y}]$) provided that X_j (resp. Y_j) is homogeneous of degree p^j for any $0\leqslant j\leqslant n$.*

Proof. Elementary. □

Remark 6.6. *In the present paragraph, we will be concerned with the image in* $\mathbf{F}_p[\underline{X},\underline{Y}]$ *of the universal Witt polynomials* $S_n(\underline{X},\underline{Y}), Prod_n(\underline{X},\underline{Y})$*. Such images will be denoted again by* $S_n(\underline{X},\underline{Y}), Prod_n(\underline{X},\underline{Y})$*, in order not to overload notations. As* $p\cdot 1 = 0$ *multiplication by* p *is the composite of Frobenius and Verschiebung.*

For $N \in \mathbf{N}$, let $z' = (\lambda'_0, \dots, \lambda'_N, 0\dots, 0, \dots) \in W(\mathbf{F}_q)$ and let $\alpha = (\alpha_0, \alpha_1, \dots) \in W(\mathbf{F}_q)$; we need to describe

$$z' + p\alpha \cdot z' \mod p^{N+1} \tag{18}$$

in terms of the universal Witt polynomials.

Lemma 6.7. *For* $0 \leqslant j \leqslant N$*, the* j*-th Witt polynomial of the development of (18) is an homogeneous element* $Q_j(\underline{\lambda}', \underline{\alpha})$ *of degree* p^j *in* $(\mathbf{F}_p[\alpha_0, \dots, \alpha_{j-1}])[\lambda'_0, \dots, \lambda'_j]$ *if we define, for* $0 \leqslant s \leqslant j$*,* λ'_s *to be homogeneous of degree* p^s*.*

Proof. It is a strightforward consequence of Lemmas 6.2 and 6.5. More precisely, from 6.5 we see that

$$p\cdot z' \cdot \alpha = (0, Prod_0(\lambda'^p_0, \alpha_0^p), \dots, \underbrace{Prod_{j-1}(\lambda'^p_0, \dots, \lambda'^p_{j-1}, \alpha^p_0, \dots, \alpha^p_{j-1})}_{j\text{ th entry}} \dots)$$

where each $Prod_{j-1}(\underline{\lambda}', \underline{\alpha})^p$ is homogeneous of degree p^j (provided that λ'_s is homogeneous of degree p^s for $0 \leqslant s \leqslant j-1$). Furthermore, $Q_j(\underline{\lambda}', \underline{\alpha})$ is the specialisation of $S_j(\underline{X}, \underline{Y})$ at $\underline{X} = z', \underline{Y} = p\cdot z' \cdot \alpha$ and we use Lemma 6.2 to get the desired result. □

As we did in §6.2 we define (for $0 \leqslant j \leqslant N$)

$$\widetilde{Q}_j \stackrel{\text{def}}{=} Q_j(\underline{\lambda}', \underline{\alpha}) - \lambda'_j.$$

For $j \neq 0$ it is a polynomial in $(\mathbf{F}_p[\alpha_0, \dots, \alpha_{j-1}])[\lambda'_0, \dots, \lambda'_{j-1}]$, homogeneous of degree p^j.

We can finally define, inductively, a family of ring homomorphisms: we let

$$q_0 : \mathbf{F}_p[\lambda'_0] \to \mathbf{F}_p[\lambda'_0]$$

be the identity map, and, assuming q_{j-1} being constructed for $j \geqslant 1$, we define

$$q_j : \mathbf{F}_p[\lambda'_0, \dots, \lambda_j, \alpha_0, \dots, \alpha_{j-1}] \to \mathbf{F}_p[\lambda'_0, \dots, \lambda'_j, \alpha_0, \dots, \alpha_{j-1}]$$

by the condition

$$\begin{aligned} \lambda'_j &\mapsto \lambda'_j - q_{j-1}(\widetilde{Q}_j) \\ \alpha_{j-1} &\mapsto \alpha_{j-1} \\ q_j|_{\mathbf{F}_p[\lambda'_0, \dots, \lambda_{j-1}, \alpha_0, \dots, \alpha_{j-2}]} &= q_{j-1} \end{aligned}$$

(and the obvious formalism: if $j=1$ we just forget α_{j-2} from the formulas).

We deduce:

Lemma 6.8. *For* $0 \leqslant j \leqslant N$*, the polynomial* $q_{j-1}(\widetilde{Q}_j)$ *is homogeneous of degree* p^j *in* $\lambda'_0, \dots, \lambda'_{j-1}$*.*

Proof. The morphism q_{j-1} is a graded ring homomorphism. □

6.4 Some special Witt polynomials -III

In this paragraph we study some Witt polynomials giving the action of $\begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix}$ (resp. $\begin{bmatrix} 1 & 0 \\ p\mathscr{O}_F & 1 \end{bmatrix}$) for the representations of §4.1 (resp. of §4.2). A tipycal example is the proof of Proposition 4.7 Such study is more delicate than the previous sections (§6.2 and §6.3) and relies crucially on the fact that we deal with Witt vectors $x \in W(\mathbf{F}_q)$ which are NOT invertible.

We start with a general remark

Lemma 6.9. *Let $N, n \in \mathbf{N}$.*

i) The n-th universal Witt polynomial of the rise to the N-th power $Pot_n^N(\underline{X})$ is an homogeneous element of degree Np^n in $\mathbf{Z}[X_0, \dots, X_n]$ provided that X_j is homogeneous of degree p^j for any $0 \leqslant j \leqslant n$.

ii) The n-th universal Witt polynomial associated to the alternate sum of N elements $S_n^N(\underline{X(1)}, \dots, \underline{X(N)})$ is an homogeneous element of degree p^n in $\mathbf{Z}[X(1)_0, \dots, X(1)_n, \dots, X(N)_0, \dots, X(N)_n]$ if we define $X(l)_j$ to be homogeneous of degree p^j, for any $l \in \{1, \dots, N\}$.

Proof. The result is elementary once we notice that, for $p \geqslant 3$, the universal Witt polynomials $Inv_n(\underline{X})$ of the additive inverse of $\underline{X}$ is simply $Inv_n(\underline{X}) = -X_n$. □

As in §6.3 we have the following

Remark 6.10. *In the present paragraph, we will be concerned with polynomials with coefficients in $\mathbf{F}_p$ obtained by reducing modulo p the coefficients of the universal Witt polynomials $S_n^N(\underline{X}, \underline{Y})$, $Pot_n^N(\underline{X})$, $S_n(\underline{X}, \underline{Y})$, $Prod_n(\underline{X}, \underline{Y})$. In order not to overload notations, such images will be denoted again by $S_n^N(\underline{X}, \underline{Y}), \dots$. As $p \cdot 1 = 0$, the multiplication by p is the composite of Frobenius and Verschiebung.*

Fix $0 \leqslant m \leqslant n$ and consider the ring $\mathbf{F}_p[\lambda_m, \dots, \lambda_n]$.

Definition 6.11. *Let $M \in \mathbf{N}$. A monomial $\lambda_m^{\alpha_m} \dots \lambda_n^{\alpha_n} \in \mathbf{F}_p[\lambda_m, \dots, \lambda_n]$ is said to be* pseudo-homogeneous of degree M *if the following holds:*

there exist an integer $L \in \mathbf{N}$ and integers $\beta_l(j) \in \mathbf{N}$ for $j \in \{1, \dots, L\}$, $l \in \{m, \dots, n\}$ such that

i) for all $l \in \{m \dots, n\}$ we have

$$\alpha_l = \sum_{j=1}^{L} p^{j-1} \beta_l(j)$$

ii) we have

$$p^m(\sum_{j=1}^{L} \beta_m(j)) + \dots + p^n(\sum_{j=1}^{L} \beta_n(j)) \leqslant M.$$

A polynomial in $\mathbf{F}_p[\lambda_m, \dots, \lambda_n]$ is said to be pseudo-homogeneous of degree M *if it is a sum of pseudo-homogeneous monomials of degree M.*

Notice that a monomial $\lambda_m^{\alpha_m} \dots \lambda_n^{\alpha_n}$ can be pseudo-homogeneous of several degrees (for instance, λ_0^p): such notion let us consider any p-th power $\lambda_j^{p^k}$ as pseudo-homogeneous of degree $p^{j+k'}$, with $0 \leqslant k' \leqslant k$. Definition 6.11 is flexible enough to handle information on the exponents of some complicate Witt polynomials, yet strong enough to make these informations interesting for our aims[11].

The following result is imediate

Lemma 6.12. *Fix m, n as above. Then:*

[11] We suggest the reader to make some example of pseudo homogeneous polynomials of low degree ($p-1$, p, $p+1$, etc...).

i) *If $P_1, P_2 \in \mathbf{F}_p[\lambda_m, \dots, \lambda_n]$ are pseudo-homogeneous of degree M_1, M_2 respectively, then $P_1 P_2$ is pseudo-homogeneous of degree $M_1 + M_2$.*

ii) *if $P_1 \in \mathbf{F}_p[\lambda_m, \dots, \lambda_n]$ is pseudo-homogeneous of degree M_1 then P_1^p is again pseudohomogeneous of degree M_1.*

Proof. Omissis. □

Remark 6.13. *If $P \in \mathbf{F}_p[\lambda_m, \dots, \lambda_n]$ is pseudo-homogeneous and we specialise P on an element of $\mathbf{F}_q^{n-m+1}$, we see that the integer L in definition 6.11 can be assumed to verify $L \leqslant f$.*

We are now ready to focus our attention on some Witt vectors in $W(\mathbf{F}_q)$.

6.4.1 *The negative case.* For $1 \leqslant m \leqslant n$, let $z \stackrel{\text{def}}{=} (0, \dots, 0, \lambda_m, \dots, \lambda_n, 0, \dots)$ and $[\mu] \stackrel{\text{def}}{=} (\mu, 0, \dots)$ be elements of $W(\mathbf{F}_q)$. We are interested in the Witt development of

$$\sum_{j=0}^{N} (-1)^j z^{j+1} [\mu^j] \quad \text{mod} \quad p^{n+1} \tag{19}$$

where $N \stackrel{\text{def}}{=} \lfloor \frac{n+1}{m} \rfloor$. For $j \in \{m, \dots, n\}$ write finally $U_j(\underline{\lambda}, \mu) \in \mathbf{F}_p[\lambda_m, \dots, \lambda_j, \mu]$ for the j-th polynomial of the Witt development of (19) and put

$$\widetilde{U}_j(\underline{\lambda}, \mu) \stackrel{\text{def}}{=} U_j - \lambda_j.$$

We notice that $\widetilde{U}_j = 0$ if $m \leqslant j \leqslant 2m - 1$ and $\widetilde{U}_{2m} = -\lambda_m^{2p^m}$.

We have a rough estimate for the degree of the $\widetilde{U}_h$

Lemma 6.14. *Let $h \in \{2m, \dots, n\}$. Then $\widetilde{U}_h \in \mathbf{F}_p[\lambda_m, \dots, \lambda_{h-1}, \mu]$ and is pseudo homogeneous of degree $p^h - p^m(p^m - 2)$.*

Proof. If $\widetilde{z} \stackrel{\text{def}}{=} (\lambda_m^{\frac{1}{p^m}}, \dots, \lambda_n^{\frac{1}{p^m}}, 0, \dots)$ then we recall that $Pot_l^{j+1}(\widetilde{z})$ is homogeneous of degree $(j+1)p^l$ (if λ_s is homogeneous of degree p^s). Thus the Witt development of $z^{j+1}[\mu]^j$ has the form

$$z^{j+1}[\mu]^j = (0, \dots, 0, \underbrace{Pot_0^{j+1}(\lambda_m^{p^{mj}})(\mu^j)^{p^{m(j+1)}}}_{\text{position } m(j+1)}, \dots, \underbrace{Pot_l^{j+1}(\lambda_m^{p^{mj}}, \dots, \lambda_{m+l}^{p^{mj}})(\mu^j)^{p^{m(j+1)+l}}}_{\text{position } m(j+1)+l}, \dots)$$

and $Pot_l^{j+1}(\lambda_m^{p^{mj}}, \dots, \lambda_{m+l}^{p^{mj}})(\mu^j)^{p^{m(j+1)+l}}$ is homogeneous of degree $(j+1)p^{l+m(j+1)}$ and actually is pseudo-homogeneous of degree $(j+1)p^{l+m}$.

Thus, if $a_{(j+1)m}(j), \dots, a_h(j)$ is an $(h - (j+1)m + 1)$-tuple of integers, the polynomial

$$\prod_{l=0}^{h-(j+1)m} (Pot_l^{j+1}(\lambda_m^{p^{mj}}, \dots, \lambda_{m+l}^{p^{mj}})(\mu^j)^{p^{m(j+1)+l}})^{a_{(j+1)m+l}(j)}$$

is pseudo-homogeneous of degree

$$(j+1)(p^m a_{(j+1)m}(j) + \dots + p^{h-mj} a_h(j)).$$

By Lemma 6.9 we see that a monomial of $S_h^{N+1}(\underline{X}(1), \dots, \underline{X}(N+1))$ has the following form:

$$\mathfrak{X} \stackrel{\text{def}}{=} \prod_{l_0=0}^{h} X_{l_0}(1)^{a_{l_0}(0)} \cdots \prod_{l_N=0}^{h} X_{l_N}(N+1)^{a_{l_N}(N)}$$

where

$$\sum_{l_0=0}^{h} p^{l_0} a_{l_0}(0) + \cdots + \sum_{l_N=0}^{h} p^{l_N} a_{l_N}(N) = p^h.$$

As U_h is the specialisation of $S_h^{(N+1)}$ at

$$(\underline{X}(j+1))_{j\in\{0,\dots,N\}} = (z^{j+1}[\mu^j])_{j\in\{0,\dots,N\}}$$

we see in particular that $\widetilde{U_h} \in \mathbf{F}_p[\lambda_m, \dots, \lambda_{h-1}, \mu]$.

Assume now that

1) if j verifies $h \geqslant (j+1)m$ we have $a_{l_j}(j) = 0$ for all $l_j < (j+1)m$;
2) if j verifies $h < (j+1)m$ we have $a_{l_j}(j) = 0$.

Then Lemma 6.12 shows that the specialisation of $\mathfrak{X}$ is pseudo-homogeneous of degree

$$d \stackrel{\text{def}}{=} \sum_{j=0}^{N} (j+1)(\sum_{i=(j+1)m}^{h} p^{i-jm} a_i(j)).$$

Letting

$$x_{j+1} \stackrel{\text{def}}{=} \sum_{i=(j+1)m}^{h} p^{i-mj} a_i(j)$$

for $j \in \{0, \dots, h\}$ we get

$$d = p^h - \sum_{j=0}^{N} (p^{jm} - (j+1)) x_j$$

and the conclusion follows from Lemma 6.15 below. □

Lemma 6.15. *Let $j \in \{0, \dots, N\}$ and let*

$$\mathfrak{X} \stackrel{\text{def}}{=} \prod_{l_0=0}^{h} X_{l_0}(1)^{a_{l_0}(0)} \cdots \prod_{l_N=0}^{h} X_{l_N}(N+1)^{a_{l_N}(N)}$$

be a monomial of $S_h^{(N+1)}(\underline{X}(1), \dots, \underline{X}(N+1))$.

If $a_{l_i}(i) = 0$ for all $i \neq j$ and $l_i \in \{0, \dots, h\}$ then

$$\mathfrak{X} = (-1)^{j+1} X_h(j).$$

Proof. An immediate induction on h shows that if we specialise $S_h^{(N+1)}$ at

$$(X_0(i), \dots, X_h(i)) = (0, \dots, 0)$$

for $i \neq j$ we get

$$S_h^{(N+1)}(\underline{0}, \dots, \underline{0}, \underline{X}(j), \underline{0}, \dots, \underline{0}) = (-1)^{j+1} X_h(j)$$

and the claim follows. □

We finally introduce a family of ring homomorphisms, for $m \leqslant j \leqslant n$,

$$u_j : \mathbf{F}_p[\lambda_m, \dots, \lambda_j, \mu] \to \mathbf{F}_p[\lambda_m, \dots, \lambda_j, \mu]$$

defined inductively as follow: u_m is the identity map and, assuming u_{j-1} being constructed, we define u_j as the unique extension of u_{j-1} to $\mathbf{F}_p[\lambda_m, \dots, \lambda_j, \mu]$ such that

$$\lambda_j \mapsto \lambda_j - u_{j-1}(\widetilde{U}_j).$$

We have the

Lemma 6.16. *Let $h \in \{2m, \dots, n\}$. Then $u_h(\widetilde{U}_h)$ is pseudo-homogeneous of degree $p^h - p^m(p^m-2)$.*

Proof. Arguing by induction, we can assume that $u_l(\lambda_l)$ is pseudo-homogeneous of degree p^l for all $l \in \{m, \dots, h-1\}$. As $\widetilde{U}_h$ is pseudo-homogeneous of degree $p^h - p^m(p^m-2)$ by Lemma 6.14, the claim follows from Lemma 6.12. □

6.4.2 *The positive case* This section is essentially a re-edition of §6.4.1, where we take $m = 0$. The results presented here will be used in 4.2, precidely for the proofs of Propositions 4.11, 4.13 where we give a description of the $K_0(p)$-representations R^+_{n+1}.

Let $(\lambda_0, \dots, \lambda_n, 0, \dots) \in \mathbf{W}(\mathbf{F}_q)$.
We are interested in the Witt development $(U_0(\lambda_0,\mu), U_1(\lambda_0,\lambda_1,\mu), \dots, U_{n+1}(\lambda_0,\dots,\lambda_{n+1},\mu), 0, \dots)$ of

$$z(1+p[\mu]z)^{-1} \equiv \sum_{j=0}^{n+1} p^j[\mu](-1)^j z^{j+1} \operatorname{mod} p^{n+2}.$$

We check immediately that $U_0 = \lambda_0$ and $U_1 = \lambda_1 - \lambda_0^{2p}\mu$.

We define, for $h = 0, \dots, n+1$, $\widetilde{U}_h \stackrel{\text{def}}{=} U_h - \lambda_h$. The following result is the analogous of Lemma 6.14

Lemma 6.17. *Let $h \in \{1, \dots, n+1\}$. Then $\widetilde{U}_h \in \mathbf{F}_p[\lambda_0, \dots, \lambda_{h-1}, \mu]$ is pseudohomogeneous of degree $p^h - (p-2)$.*

Proof. The proof is completely analogous to the proof of Lemma 6.14 and left to the reader (see [Mo1], Lemma 6.17 for details). □

As in section §6.4.1 we define inductively, for $h = 0, \dots, n+1$, the ring morphisms

$$u_h : \mathbf{F}_p[\lambda_0, \dots, \lambda_h, \mu] \to \mathbf{F}_p[\lambda_0, \dots, \lambda_h, \mu]$$

by the condition $u_h(\lambda_h) \stackrel{\text{def}}{=} \lambda_h - u_{h-1}(\widetilde{U}_h)$ for $h \geqslant 1$ and $u_0 \stackrel{\text{def}}{=} id$. Then

Lemma 6.18. *Let $1 \leqslant h \leqslant n+1$. Then $u_h(\widetilde{U}_h)$ is pseudo-homogeneous of degree $p^h - (p-2)$.*

Proof. As for Lemma 6.16 it is a consequence of Lemma 6.12 and Lemma 6.17. □

Still others remarks on some universal Witt polynomials. In this paragraph we pursue the technical computations of §6.4.2; the results here will be used in §5.1, Proposition 5.3. Indeed, the structure of the quotients $R^\bullet_{n+1}/R^\bullet_n$ is more complicate than for $R^\bullet_{n+1}$, and it can not be deduced from Lemmas 6.16, 6.18; we therefore need to look more closely to the structure of the polynomial $\widetilde{U}_{n+1}$ and $u_n(\widetilde{U}_n+1)$ (the notations being the same as for §6.4.2).

The following description is deduced as in the proof of Lemma 6.14. Let $z = (\lambda_0, \dots, \lambda_n, 0) \in \mathbf{W}_{n+1}(\mathbf{F}_q)$ and write

$$\sum_{j=0}^{n+1} p^j[\mu](-1)^j z^{j+1} = (U_0, \dots, U_{n+1}).$$

for $U_j \in \mathbf{F}_p[\lambda_0, \dots, \lambda_j, \mu]$. We recall that U_h is obtained by specializing the universal polynomial $S_h^{n+2}(\underline{X}(1), \dots, \underline{X}(n+2))$ at

$$\underline{X}(j+1) = (0, \dots, 0, \underbrace{(Pot_0^{j+1}(\underline{\lambda}))^{p^j}(\mu^j)^{p^j}}_{\text{position } j}, \dots, \underbrace{(Pot_l^{j+1}(\underline{\lambda}))^{p^j}(\mu^j)^{p^{j+l}}}_{\text{position } j+l}, \dots).$$

We recall moreover that a monomial $\mathfrak{X}$ of $S_h^{n+2}(\underline{X}(1),\dots,\underline{X}(n+2))$ has the form

$$\mathfrak{X}=\prod_{l_0=0}^{h}X_{l_0}(1)^{a_{l_0}(0)}\cdots\prod_{l_{n+1}=0}^{h}X_{l_{n+1}}(n+2)^{a_{l_{n+1}}(n+1)} \tag{20}$$

where the integers $a_{l_i}(i)$ verify

$$\sum_{l_0=0}^{h}p^{l_0}a_{l_0}(0)+\cdots+\sum_{l_{n+1}}^{h}p^{l_{n+1}}a_{l_{n+1}}(n+1)=p^h;$$

Therefore a monomial $\lambda_0^{\alpha_0}\cdot\dots\cdot\lambda_h^{\alpha_h}$ issued from U_h verifies

$$\sum_{j=0}^{h}p^j\mathfrak{s}(\alpha_j)\leqslant\sum_{j=0}^{h}(j+1)(\sum_{i=j}^{h}p^{i-j}a_i(j))=p^h-\sum_{j=1}^{h}(p^j-(j+1))x_j$$

where we have set

$$x_j\overset{\text{def}}{=}\sum_{i=j}^{h}p^{i-j}a_i(j).$$

We focus our attention for the case $h=n+1$, obtaining thus the following

Lemma 6.19. *A monomial of $\widetilde{U}_{n+1}$ has the following form*

$$\lambda_n^{a_n(0)+pa_{n+1}(1)}\cdot\lambda_{n-1}^{\alpha_{n-1}}\cdot\dots\cdot\lambda_0^{\alpha_0}$$

where the exponents verify the following properties:

1) *we have $a_n(0)\in\{0,\dots,p-1\}$ and $a_{n+1}(1)\in\{0,1\}$,*
2) *letting $x_j\overset{\text{def}}{=}\sum_{i=j}^{n+1}p^{i-j}a_i(j)$ we have*

$$\sum_{j=0}^{n-1}p^j\mathfrak{s}(\alpha_j)+p^n(a_n(0)+a_{n+1}(1))\leqslant p^{n+1}-\sum_{j=1}^{n+1}(p^j-(j+1))x_j$$

3) *if $a_{n+1}(1)=1$ then the monomial has the form*

$$\lambda_0^{p^{n+1}}\lambda_n^p.$$

Proof. The fact that $a_n(0)\neq p$ follows from the fact that in the polynomial S_{n+1}^{n+2} the coefficient of $X_n(1)^p$ is zero (the proof is the usual one: see Lemma 6.15). Assertion 2) is deduced from 1) (and the fact that $f\geqslant 2$). Assertion 3) follows noticing that $(Pot_n^2(z))^p=2\lambda_0^{p^{n+1}}\lambda_n^p+x$ where $x\in\mathbf{F}_p[\lambda_0,\dots,\lambda_{n-1}]$. □

We recall the ring morphism $u_n:\mathbf{F}_p[\lambda_0,\dots,\lambda_n,\mu]\to\mathbf{F}_p[\lambda_0,\dots,\lambda_n,\mu]$ (cf. 6.4.2). If $i_{n+1}^{(s)}\in\mathbf{N}$ deduce the following

Lemma 6.20. *In the preceding notations, a monomial issued from $u_n(\widetilde{U}_{n+1})^{i_{n+1}^{(s)}}$ has the following form*

$$(\lambda_0^{p^{n+1}}\lambda_n^p)^{B_{n+1}^{(s)}(1)}\lambda_n^{B_n^{(s)}(0)}\lambda_{n-1}^{\beta_{n-1}}\cdot\dots\cdot\lambda_0^{\beta_0}$$

and there exists convenient integers $A_i(j)\geqslant 0$ (depending on the choosen monomial) such that

1) *we have*

$$\mathfrak{s}(\beta_0+p^{n+1}B_{n+1}^{(s)}(1))+\sum_{j=1}^{n-1}p^j\mathfrak{s}(\beta_j)+p^n(B_n^{(s)}(0)+B_{n+1}^{(s)}(1))\leqslant p^{n+1}i_{n+1}^{(s)}-\sum_{j=1}^{n+1}\sum_{i=j}^{n+1}(p^j-(j+1))p^{i-j}A_i(j);$$

2) *we have* $A_i(j)=0$ *for all couples* (i,j) *if and only if* $i^{(s)}_{n+1}=0$*;*

3) *we have* $0\leqslant B^{(s)}_{n+1}(1)\leqslant A^{(s)}_{n+1}(1)\leqslant i^{(s)}_{n+1}$.

Proof. Lemma 6.19 shows that a monomial $\underline{\lambda}^{\underline{\alpha}}\overset{\text{def}}{=}\lambda_n^{a_n(0)+pa_{n+1}(1)}\cdot\lambda_{n-1}^{\alpha_{n-1}}\cdot\dots\cdot\lambda_0^{\alpha_0}$ issued from $\widetilde{U}_{n+1}$ is pseudo homogeneous of degree

$$d_0\overset{\text{def}}{=}p^{n+1}-\sum_{j=1}^{n+1}\sum_{i=j}^{n+1}(p^j-(j+1))p^{i-j}a_i(j).$$

A monomial issued from $u_n(\underline{\lambda}^{\underline{\alpha}})$ is of the form

$$\underline{\lambda}^{\underline{\beta}'}\overset{\text{def}}{=}(\lambda_0^{p^{n+1}}\lambda_n^p)^{b^{(s)}_{n+1}(1)}\lambda_n^{b^{(s)}_n(0)}\lambda_{n-1}^{\beta'_{n-1}}\cdot\ldots\lambda_0^{\beta'_0}\tag{21}$$

where $0\leqslant b_n(0)\leqslant a_n(0)$ and $0\leqslant b_{n+1}(1)\leqslant a_{n+1}(1)$. Moreover, as $\underline{\lambda}^{\underline{\alpha}}$ is pseudo-homogeneous of degree d_0, so it is for $\underline{\lambda}^{\underline{\beta}}$ and in particular we have

$$\mathfrak{s}(\beta'_0+p^{n+1}b_{n+1}(1))+\sum_{j=1}^{n-1}p^j\mathfrak{s}(\beta'_j)+p^n(b_n(0)+b_{n+1}(1))\leqslant d_0.$$

(as $\mathfrak{s}(b_n(0)+pb_{n+1}(1))=b_n(0)+b_{n+1}(1)!$).

A monomial issued from $u_n(\widetilde{U}_{n+1})^{i^{(s)}_{n+1}}$ is the product of $i^{(s)}_{n+1}$ monomials of the form $\underline{\lambda}^{\underline{\beta}'}$, and thus of the form

$$(\lambda_0^{p^{n+1}}\lambda_n^p)^{B^{(s)}_{n+1}(1)}\lambda_n^{B^{(s)}_n(0)}\lambda_{n-1}^{\beta_{n-1}}\cdot\ldots\lambda_0^{\beta_0}$$

where $B^{(s)}_{n+1}(1)$, $B^{(s)}_n(0)$ is the sum of $i^{(s)}_{n+1}$ terms of the form $b^{(s)}_{n+1}(1)$, $b^{(s)}_n(0)$.

If each of the monomials $\underline{\lambda}^{\underline{\beta}'}$ comes from $u_n(\underline{\lambda}^{\underline{\alpha}})$, the statement follows easily from the subadditivity of the function $\mathfrak{s}$ and the additivity of the pseudo-homogeneous degree, once we define each integer $A_i(j)$ to be the sum of $i^{(s)}_{n+1}$ terms of the form $a_i(j)$, one for each monomial $\underline{\lambda}^{\underline{\alpha}}$ (the integers $a_i(j)$ being defined as for Lemma 6.19).

□

7. Appendix B: Two rough estimates

The aim of this appendix is to estimate the behaviour of some "discrete Fourier transforms" which appear naturally in the study of the socle filtration for the representations $R^{\pm}_{n+1}$, $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)}1$, etc... According to the Euclidean vocabulary developped in Sections 4 and 5 such behaviour is related to the reduction mod p^f-1 of the exponents of some (pseudo-)homogeneous polynomials.

The first tool is discussed in §7.1: it is an elementary description of the function $\mathfrak{s}$ giving the digit sum of the reduction modulo p^f-1 of a natural number. In §7.2 the properties of the function $\mathfrak{s}$ and the results on Witt polynomials stated in §6 will be used to describe in detail some explicit vectors of the aforementioned representations (Propositions 7.3, 7.4 and 7.5).

7.1 Remark on the proof of Stickelberger's Theorem

In this section we recall the construction and the properties of a certain function $s:\mathbf{Z}\to\mathbf{N}$ which appears in the proof of Stickelberger's theorem.

If $\mathfrak{p}$ is a prime of $\mathbf{Q}(\zeta_{q-1})$ lying above p, the reduction modulo $\mathfrak{p}$, $\mathbf{Z}[\zeta_{q-1}]\to\mathbf{F}_q$ admits a multiplicative section

$$\omega_{\mathfrak{p}}:\mathbf{F}_p^{\times}\to\mathbf{Z}[\zeta_{q-1}]$$

which induces an isomorphisms on the group μ_{q-1} of $q-1$-th roots of unity. If $\mathfrak{P}$ is the prime of $\mathbf{Q}(\zeta_{q-1},\zeta_p)$ lying above $\mathfrak{p}$, we define a function $s:\mathbf{Z}\rightarrow\mathbf{N}$ by

$$s(n)\overset{\text{def}}{=}val_{\mathfrak{P}}(g(\omega_{\mathfrak{p}}^{-n}))$$

where $val_{\mathfrak{P}}$ denotes the $\mathfrak{P}$-adic valuation and $g(\omega_{\mathfrak{p}}^{-n})$ denotes the Gauss sum of the character $\omega_{\mathfrak{p}}^{-n}:\mathbf{F}_q^{\times}\rightarrow\mu_{q-1}$ (see [Was], §6.1 for the definition of the Gauss sum $g(\omega_{\mathfrak{p}}^{-n})$).

We need to modify slightly this function as follow:

$$\begin{aligned}\mathfrak{s}:\mathbf{N}&\rightarrow\mathbf{N}\\ n&\mapsto\begin{cases}s(n)\,\text{if either}\,n\not\equiv 0\,\text{mod}\,q-1\text{ or }n=0\\ f(p-1)\quad\text{otherwise}\end{cases}\end{aligned}$$

The following lemma is then easily deduced from the well known properties of the function s (cf. [Was], §6.2):

Lemma 7.1. *Let $n,m\in\mathbf{N}$. Then:*

a) $\mathfrak{s}(0)=0$ *and* $\mathfrak{s}(1)=1$*;*

b) $0\leqslant\mathfrak{s}(m+n)\leqslant\mathfrak{s}(n)+\mathfrak{s}(m)$*;*

c) $\mathfrak{s}(pn)=\mathfrak{s}(n)$*;*

d) if $0\leqslant n\leqslant q-1$ *and* $(a_0,\dots,a_{f-1})$ *are the digits of the* p*-adic development of* n*, we have*

$$\mathfrak{s}(n)=a_0+a_1+\cdots+a_{f-1}.$$

In particular, $\mathfrak{s}(n)\leqslant n$ *for any* $n\in\mathbf{N}$*, with equality if and only if* $n\in\{0,\dots,p-1\}$*.*

We can improve the statement of *b)*:

Lemma 7.2. *Let $b_0,\dots,b_{f-1}\in\mathbf{N}$ be integers.*

Then there exist integers m_s,n_s, where $s\in\{0,\dots,f-1\}$ such that:

1) *for all* $s\in\{0,\dots,f-1\}$

$$c_s\overset{\text{def}}{=}b_s-pm_s+n_{\lfloor s-1\rfloor}\in\{0,\dots,p-1\};$$

2) *we have*

$$\widetilde{j}\overset{\text{def}}{=}\sum_{s=0}^{f-1}m_s=\sum_{s=0}^{f-1}n_s;$$

3) *we have*

$$\sum_{s=0}^{f-1}p^sb_s\equiv\sum_{s=0}^{f-1}p^sc_s\mod p^f-1;$$

4) *we have the equality*

$$\mathfrak{s}(\sum_{s=0}^{f-1}p^sb_s)=\sum_{s=0}^{f-1}b_s-\widetilde{j}(p-1).$$

Proof. Assume first that $b_s\in\{0,\dots,p-1\}$ for all $s\geqslant 1$ and $b_0\geqslant p$. There exist (unique) integers m_s, for $s=0,\dots,f-1$ such that

i) $b_s+m_{s-1}-pm_s\in\{0,\dots,p-1\}$ for all $s\geqslant 1$ and $b_0-pm_0\in\{0,\dots,p-1\}$;

ii) we have the equality

$$\sum_{s=0}^{f-1}b_sp^s=(b_0-pm_0)+\sum_{s=0}^{f-1}p^s(b_s+m_{s-1}-pm_s)+p^{f-1}m_{f-1}.\qquad(22)$$

As we work modulo $q-1$ the equality (22) reads

$$\sum_{s=0}^{f-1} b_s p^s \equiv (b_0 - pm_0 + m_{f-1}) + \sum_{s=0}^{f-1} p^s (b_s + m_{s-1} - pm_s) \bmod q-1.$$

If $b_0 - pm_0 + m_{f-1} \in \{0,\dots,p-1\}$ we get the result. If not, we only have to check that $0 \leqslant b_0 - pm_0 + m_{f-1} < b_0$ (so that the iteration of the preceding procedure eventually stops). As $-pm_1 + b_1 + m_0 \geqslant 0$ and $b_1 \leqslant p-1$ we get $m_1 \leqslant \frac{p-1+m_0}{p}$ and, inductively, $m_{s+1} \leqslant \frac{p^{s+1}-1+m_0}{p^{s+1}}$. Thus

$$-pm_0 + m_{f-1} \leqslant -pm_0 + \frac{p^{f-1}-1+m_0}{p^{f-1}} < 0$$

if $m_0 \geqslant 1$.

For the general case, we notice that there exists unique integers m'_s such that $b_s + m'_{s-1} - pm'_s \in \{0,\dots,p-1\}$ for all $s \geqslant 1$ and $b_0 - m'_0 \in \{0,\dots,p-1\}$. As we work modulo $q-1$ we get

$$\sum_{s=0}^{f-1} b_s p^s \equiv (b_0 - pm'_0 + m'_{f-1}) + \sum_{s=0}^{f-1} p^s (b_s + m'_{s-1} - pm'_s) \bmod q-1.$$

and we are in the previous case. □

7.2 Two rough estimates

In this section we study some elements of $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$ which appear naturally in the study of the socle filtration for $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$ (but the results adapt immediately for the representations $R_{n+1}^{\pm}$). In particular, we will be able to have a partial control of the action of $K_0(p^m)$ on $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$.

The following proposition holds for a fixed pair (m,n) of integers such that $0 \leqslant m \leqslant n$; if $m=0$ we just have to replace the matrix $\begin{bmatrix} 1 & 0 \\ p^m[\lambda_m] & 1 \end{bmatrix}$ with $\begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix}$ in the expressions (23) and (24). Finally we recall the definition of the $\overline{\mathbf{F}}_p$-linear subspace $\mathfrak{W}_{(\underline{l}_m,\dots,\underline{l}_n)}$ of $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$ for a given $(n+1-m)f$-tuple $(\underline{l}_m,\dots,\underline{l}_n) \in \big\{\{0,\dots,p-1\}^f\big\}^{n+1-m}$, given in §4.1.2.

Proposition 7.3. *Let $F^{m,n}_{\underline{l}_m,\dots,\underline{l}_n} \in \mathscr{B}$, and $N \overset{\mathrm{def}}{=} N_{m,n}(\underline{l}_m,\dots,\underline{l}_n)$. For $m \leqslant j \leqslant n$ let $T_j \in \mathbf{F}_p[\lambda_m,\dots,\lambda_{j-1}]$ be a polynomial of degree $\deg(T_j) \leqslant p^{j-m}$ (where, for $j \in \{0,\dots,n-1\}$, we define λ_{j+m} to be homogeneous of degree p^j), and $\underline{i}_j$ be a f-tuple such that $\underline{i}_j \leqslant \underline{l}_j$. Finally, fix $M < p^f - 1$ and define the element*

$$x \overset{\mathrm{def}}{=} \sum_{j=m}^{n-1} \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j - \underline{i}_j} (T_{j+1}^{\frac{1}{p^{j+1}}})^{\underline{i}_{j+1}} \begin{bmatrix} 1 & 0 \\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1 \end{bmatrix} \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{\underline{l}_n - \underline{i}_n} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} \big[1,e\big]. \quad (23)$$

Then the image of x under the projection

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1 \twoheadrightarrow \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1 / (\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1)_{N-M}$$

is contained in the image of the subspace $\mathfrak{W}_{(\underline{l}_m,\dots,\underline{l}_n)}$.

Proof. The technique of the proof is very simple: we fix $0 \leqslant t \leqslant M$ and $k \in \mathbf{N}$ such that $k(p-1) \leqslant t < (k+1)(p-1)$. If we write x as a suitable sum of elements $F^{(m,n)}_{\underline{l}'_m,\dots,\underline{l}'_n}$, the statement is proved if we check that any such element lying in the antidiagonal $X_0 + \dots + X_{f-1} = N - t$ verifies $x'_j \leqslant x_j + k$ for all $j = 0,\dots,f-1$ (where, as usual, $(x_0,\dots,x_{f-1})$, $(x'_0,\dots,x'_{f-1})$ are the coordinates of $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}$, $F^{(m,n)}_{\underline{l}'_m,\dots,\underline{l}'_n}$ via the map (7)).

This is a long computation. If we expand each of the polynomials $T_{m+1}^{\underline{i}_{m+1}}, \ldots, T_n^{\underline{i}_n}$, we obtain:

$$\sum_{\mathrm{i} \in \mathrm{I}} \beta_i \sum_{\lambda_m \in \mathbf{F}_q} (\lambda_m^{\frac{1}{p^m}})^{\kappa_m(i)} \begin{bmatrix} 1 & 0 \\ p^m[\lambda_m^{\frac{1}{p^m}}] & 1 \end{bmatrix} \cdots \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{\kappa_n(i)} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} [1, e] \tag{24}$$

where I is a suitable set of indices, $\beta_i \in \overline{\mathbf{F}}_p$, and the exponents $\kappa_j(i)$ (for $j \in \{m, \ldots, n\}$) admit the following explicit description: [12]

$$\kappa_a = p^{\lfloor -1 \rfloor} \kappa_a^{(a+1)} + \cdots + p^{\lfloor -(n-a) \rfloor} \kappa_a^{(n)} + \underline{l}_a - \underline{i}_a$$

and (for $a+1 \leqslant b \leqslant n$)

$$\kappa_a^{(b)} = \kappa_a^{(b),0} + p\kappa_a^{(b),1} + \cdots + p^{f-1}\kappa_a^{(b),f-1}$$

where each $\kappa_a^{(b),s}$ is the exponent of λ_a apperaring in a fixed monomial of $(T_b)^{i_b^{(s)}}$.

Recall that, by the hypothesis on the T_b, we have

$$\kappa_m^{(b),s} + p\kappa_{m+1}^{(b),s} + \cdots + p^{b-1-m}\kappa_{b-1}^{(b),s} \leqslant p^{b-m} i_b^{(s)}. \tag{25}$$

Thanks to Lemma 7.1, we have the following inequalities:

$$\mathfrak{s}(\kappa_m) + p\,\mathfrak{s}(\kappa_{m+1}) + \cdots + p^{n-m}\,\mathfrak{s}(\kappa_n) \leqslant \tag{26}$$

$$\begin{aligned} \leqslant & (\mathfrak{s}(\underline{l}_m - \underline{i}_m) + \mathfrak{s}(p^{\lfloor -1 \rfloor}\kappa_m^{(m+1)}) + \cdots + \mathfrak{s}(p^{\lfloor -(n-m) \rfloor}\kappa_m^{(n)})) + \\ & +p(\mathfrak{s}(\underline{l}_{m+1} - \underline{i}_{m+1}) + \mathfrak{s}(p^{\lfloor -1 \rfloor}\kappa_{m+1}^{(m+2)}) + \cdots + \mathfrak{s}(p^{\lfloor -(n-m-1) \rfloor}\kappa_{m+1}^{(n)})) + \ldots \\ & \cdots + p^{n-m-1}(\mathfrak{s}(\underline{l}_{n-1} - \underline{i}_{n-1}) + \mathfrak{s}(p^{\lfloor -1 \rfloor}\kappa_{n-1}^{(n)})) + p^{n-m}(\mathfrak{s}(\underline{l}_n - \underline{i}_n)) \leqslant \end{aligned} \tag{27}$$

$$\begin{aligned} & \leqslant \mathfrak{s}(\underline{l}_m - \underline{i}_m) + \sum_{s=0}^{f-1} \mathfrak{s}(\kappa_m^{(m+1),s}) + \\ & p(\mathfrak{s}(\underline{l}_{m+1} - \underline{i}_{m-1})) + (\sum_{s=0}^{f-1}(\mathfrak{s}(\kappa_m^{(m+2),s} + p\,\mathfrak{s}(\kappa_{m+1}^{(m+2),s})) + \ldots \\ & \cdots + (\sum_{s=0}^{f-1}(\mathfrak{s}(\kappa_m^{(n),s}) + p\,\mathfrak{s}(\kappa_{m+1}^{(n),s}) + \cdots + p^{n-m-1}\,\mathfrak{s}(\kappa_{n-1}^{(n),s}))) + p^{n-m}\,\mathfrak{s}(\underline{l}_n - \underline{i}_n) \leqslant \end{aligned} \tag{28}$$

$$\leqslant \mathfrak{s}(\underline{l}_m - \underline{i}_m) + p\,\mathfrak{s}(\underline{i}_{m+1}) + p\,\mathfrak{s}(\underline{l}_{m+1} - \underline{i}_{m+1}) + \cdots + p^{n-m}\,\mathfrak{s}(\underline{i}_n) + p^{n-m}\,\mathfrak{s}(\underline{l}_n - \underline{i}_n)$$

where the inequality (28) is deduced from (25) and Lemma 7.1-*d*).

If we impose our function to lie on the hyperplane $X_0 + \cdots + X_{f-1} = t$ we get a "control" on the exponents $\kappa_a^{(b),s}$. More precisely,

i) the inequality (26) give rise to the conditions:

$$\mathfrak{s}(\kappa_a) = \mathfrak{s}(\underline{l}_a - \underline{i}_a) + \mathfrak{s}(\kappa_a^{(a+1)}) + \cdots + \mathfrak{s}(\kappa_a^{(n)}) - u_a(p-1)$$

for $a \in \{m, \ldots, n-1\}$ and some $u_a \in \mathbf{N}$;

ii) the inequality (27) give rise to the conditions:

$$\mathfrak{s}(\kappa_a^{(b)}) = \mathfrak{s}(\kappa_a^{(b),0}) + \cdots + \mathfrak{s}(\kappa_a^{(b),f-1}) - w_a^{(b)}(p-1)$$

[12] From now on, we fix an index $i \in I$, and we put $\kappa_j \stackrel{\text{def}}{=} \kappa_j(i)$

where $a \in \{m, \dots, n-1\}$, $b \in \{a+1, \dots, n\}$ and some $w_a^{(b)} \in \mathbf{N}$;

iii) the inequality (28) give rise to the conditions

$$\mathfrak{s}(\kappa_a^{(b),s}) = \kappa_a^{(b),s} - v_a^{(b),c}(p-1)$$

where $a \in \{m, \dots, n-1\}$, $b \in \{a+1, \dots, n\}$, $s \in \{0, \dots, f-1\}$ and some $v_a^{(b),s} \in \mathbf{N}$;

iv) condition $t < (k+1)(p-1)$ imposes finally

$$\sum_{a=m}^{n-1} p^{a-m} u_a + \sum_{a=m}^{n-1} p^{a-m} (\sum_{b=a+1}^{n} w_a^{(b)}) + \sum_{a=m}^{n-1} p^{a-m} (\sum_{b=a+1}^{n} \sum_{s=0}^{f-1} v_a^{(b),s}) \leqslant k.$$

First, notice that the condition $k(p-1) < p^f - 1$ implies $\kappa_a^{(b),s} \leqslant p^f - 1$ for all possible choices of a, b, s (as $\mathfrak{s}(\kappa_a^{(b),s}) \leqslant \lceil \kappa_a^{(b),s} \rceil$). If $\kappa_a^{(b),s}(i)$, for $i \in \{0, \dots, f-1\}$, are the cyphers of the p-adic development of $\kappa_a^{(b),s}$, we then see that *iii*) gives the necessary condition

$$\sum_{i=1}^{f-1} \kappa_a^{(b),s}(i) \leqslant v_a^{(b),s}$$

(indeed, $v_a^{(b),s}$ can uniquely written as $v_a^{(b),s} = \alpha_{a,b,s}(1) + (p+1)\alpha_{a,b,s}(2) + \dots + \alpha_{a,b,s}(f-1)(1 + p + \dots + p^{f-1})$ for suitable integers $\alpha_{a,b,s}(j)$).

Fix now $a \in \{m, \dots, n-1\}$, $b \in \{a+1, \dots, n\}$. Working in $\mathbf{Z}/(p^f - 1)$, we see that

$$\kappa_a^{(b),0} + \dots + p^{f-1} \kappa_a^{(b),f-1} \equiv \sum_{j=0}^{f-1} p^j (\kappa_a^{(b),0}(j) + \kappa_a^{(b),1}(\lfloor j-1 \rfloor) + \dots + \kappa_a^{(b),f-1}(\lfloor j-(f-1) \rfloor)).$$

Using Lemma 7.2 we see that condition *ii*) lets us deduce the p-adic expansion of $\kappa_a^{(b)}$:

$$\begin{aligned} \kappa_a^{(b)}(j) &= \kappa_a^{(b),0}(j) + \dots + \kappa_a^{(b),f-1}(\lfloor j-(f-1) \rfloor) - p\alpha_a^{(b)}(j) + \beta_a^{(b)}(j) \\ &= \kappa_a^{(b),j}(0) + \rho_a^{(b)}(j) - p\alpha_a^{(b)}(j) \end{aligned} \tag{29}$$

where the integers $\alpha_a^{(b)}(j)$, $\beta_a^{(b)}(j)$ verify

$$\sum_{j=0}^{f-1} \alpha_a^{(b)}(j) = \sum_{j=0}^{f-1} \beta_a^{(b)}(j) = w_a^{(b)}$$

and

$$\rho_a^{(b)}(j) = \sum_{s \in \{0 \dots, f-1\} \setminus \{j\}} \kappa_a^{(b),s}(\lfloor j-s \rfloor) + \beta_a^{(b)}(j) \leqslant \sum_{s=0}^{f-1} v_a^{(b),s} + w_a^{(b)}.$$

Similarly, condition *i*) lets us deduce the p-adic development of κ_a:

$$\begin{aligned} \kappa_a(j) &= l_a^{(j)} - i_a^{(j)} + \sum_{b=a+1}^{n} \kappa_a^{(b)}(\lfloor j+b-a \rfloor) - pA_a(j) + B_a(j) \\ &= l_a^{(j)} - i_a^{(j)} + \sum_{b=a+1}^{n} \kappa_a^{(b),\lfloor j+b-a \rfloor}(0) + \mathfrak{R}_a(j) - p(\sum_{b=a+1}^{n} \alpha_a^{(b)}(\lfloor j+b-a \rfloor) + A_a(j)) \end{aligned}$$

where the integers $A_a(j)$, $B_a(j), \mathfrak{R}_a(j)$ verify

$$\sum_{j=0}^{f-1} A_a(j) = \sum_{j=0}^{f-1} B_a(j) = u_a$$

and

$$\mathfrak{R}_a(j) = \sum_{b=a+1}^{n} \rho_a^{(b)}(\lfloor j+b-a \rfloor) + B_a(j) \leqslant u_a + \sum_{b=a+1}^{n} (\sum_{s=0}^{f-1} v_a^{(b),s} + w_a^{(b)}).$$

We finally have all the ingredients to give the rough estimate of the statement. We fix a "coordinate" j. A strightforward but tedious computation gives

$$\sum_{a=m}^{n} p^{a-m}\kappa_a(j) = \sum_{a=m}^{n} p^{a-m}(l_a^{(j)} - i_a^{(j)} + \sum_{b=a+1}^{n} \kappa_a^{(b),\lfloor j+b-a \rfloor}(0) + \mathfrak{R}_a(j) - p\mathfrak{A}_a(j)) =$$

$$= x_j - \sum_{a=m}^{n} p^{a-m} i_a^{\lfloor j+a-m \rfloor} + \sum_{b=m+1}^{n}\sum_{a=m}^{b-1} p^{a-m}\kappa_a^{(b),\lfloor j+b-m \rfloor}(0) +$$

$$+ \sum_{a=m}^{n-1} p^{a-m}\mathfrak{R}_a(j) - p(\sum_{a=m}^{n} p^{a-m}\mathfrak{A}_a(j))$$

where $\mathfrak{A}_a(j) \in \mathbf{N}$ are convenient integers (and notice that $\mathfrak{R}_n(j) = 0!$). The conclusion follows as

$$\sum_{a=m}^{n-1} p^{a-m}\mathfrak{R}_a(j) \leqslant \sum_{a=m}^{n-1} p^{a-m}(u_a + \sum_{b=a+1}^{n} w_a^{(b)} + \sum_{b=a+1}^{n}\sum_{s=0}^{f-1} v_a^{(b),s}) \leqslant k$$

and

$$\sum_{a=m}^{b-1} \kappa_a^{(b),s}(0) \leqslant p^{b-m} i_b^{(s)}$$

for any $b \in \{m+1, \dots, n\}$ and $s \in \{0, \dots, f-1\}$. □

The following rough estimate will help us understand the action of $\begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix}$ (resp. of $\begin{bmatrix} 1 & 0 \\ p\mathscr{O}_F & 1 \end{bmatrix}$) on the representations in §4.1 (resp. §4.2) and it will be used in the proof of Proposition 4.7. Apparently, the result is unsatisfactory if we want to describe the K-socle filtration for the representations $\pi(\underline{r}, \lambda, 1)$, unless we impose some conditions, depending on p, on the residue degree f (we expect a condition of the form $f \leqslant \frac{p+1}{2}$, [Mo2]).

Proposition 7.4. *Let $1 \leqslant m \leqslant n$ be integers and consider $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} \in \mathscr{B}$; let $N \overset{\text{def}}{=} N_{m,n}(\underline{l}_m, \dots, \underline{l}_n)$. For $2m \leqslant j \leqslant n$ let $V_j \in \mathbf{F}_p[\lambda_m, \dots, \lambda_{j-1}]$ be a pseudo-homogeneous polynomial of degree $\deg(V_j) \leqslant p^j - p^m(p^m-2)$ and $\underline{i}_j$ be a f-tuple such that $\underline{i}_j \leqslant \underline{l}_j$. Finally, fix $M < p^m - 2$ and define $V_j \overset{\text{def}}{=} 1$, $\underline{i}_j = \underline{0}$ for $m \leqslant j \leqslant 2m-1$.*

The element x defined as

$$x \overset{\text{def}}{=} \sum_{j=m}^{n-1} \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j - \underline{i}_j} (V_{j+1}^{\frac{1}{p^{j+1}}})^{\underline{i}_{j+1}} \begin{bmatrix} 1 & 0 \\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1 \end{bmatrix} \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{\underline{l}_n - \underline{i}_n} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} [1, e]$$

and the element $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}$ have the same image under the projection

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1 \twoheadrightarrow \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1 / (\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1)_{N-M}.$$

Proof. The idea of the proof is completely analogous of that of Proposition 7.3 the main difference being that here we are *not* able to give an estimate of the coordinates of the points appearing in the development of x.

As in 7.3 we consider an element appearing in the development of x:

$$\sum_{\lambda_m\in\mathbf{F}_q}(\lambda_m^{\frac{1}{p^m}})^{\kappa_m(i)}\begin{bmatrix}1 & 0\\ p^m[\lambda_m^{\frac{1}{p^m}}] & 1\end{bmatrix}\dots\sum_{\lambda_n\in\mathbf{F}_q}(\lambda_n^{\frac{1}{p^n}})^{\kappa_n(i)}\begin{bmatrix}1 & 0\\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1\end{bmatrix}[1,e].$$

The exponents κ_a (for $a\in\{m,\dots,n\}$) admit the following explicit description:

$$\kappa_a=p^{\lfloor -1\rfloor}\kappa_a^{(a+1)}+\dots+p^{\lfloor n-a\rfloor}\kappa_a^{(n)}+\underline{l}_a-\underline{i}_a$$

and (for $a+1\leqslant b\leqslant n$)

$$\kappa_a^{(b)}=\kappa_a^{(b),0}+p\kappa_a^{(b),1}+\dots+p^{f-1}\kappa_a^{(b),f-1}$$

where each $\kappa_a^{(b),s}$ is the exponent of λ_a appeararing in a fixed monomial of $(V_b)^{i_b^{(s)}}$.

As each V_b is pseudo-homogeneous, for each triple (a,b,s) we have

$$\kappa_a^{(b),s}=\beta_a^{(b),s}(1)+\dots+p^{f-1}\beta_a^{(b),s}(f)$$

where the integers $\beta_a^{(b),s}(j)$ verify

$$\sum_{j=1}^f\beta_m^{(b),s}(j)+p(\sum_{j=1}^f\beta_{m+1}^{(b),s}(j))+\dots+p^{b-m-1}\sum_{j=1}^f(\beta_{b-1}^{(b),s}(j))\leqslant(p^{b-m}-(p^m-2))i_b^{(s)}.$$

As for the inequalities (26), (27), (28), we use Lemma 7.1 to obtain

$$\sum_{a=m}^n p^{a-m}\mathfrak{s}(\kappa_a)\leqslant N-(p^m-2)(\sum_{a=2m}^n\mathfrak{s}(\underline{i}_a))$$

and the conclusion follows. □

We state an analogous result in the case $m=0$.

Proposition 7.5. *Let $n\geqslant 0$ and $F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_{n+1}}\in\mathscr{B}^+_{n+1}$; let $N\overset{\text{def}}{=}N_{0,n+1}(\underline{l}_0,\dots,\underline{l}_{n+1})$. For $1\leqslant h\leqslant n+1$ let $V_h\in\mathbf{F}_p[\lambda_0,\dots,\lambda_{h-1}]$ be a pseudo homogeneous polynomial of degree $p^h-(p-2)$ and $\underline{i}_h\leqslant\underline{l}_h$ be an f-tuple. We finally fix $M\in\{0,\dots,p-3\}$ and put $\underline{i}_0\overset{\text{def}}{=}\underline{0}$, $V_{n+2}\overset{\text{def}}{=}1$.*

The element

$$x\overset{\text{def}}{=}\sum_{\lambda_0\in\mathbf{F}_q}\lambda_0^{\underline{l}_0-\underline{i}_0}(V_1^{\frac{1}{p}})^{\underline{i}_1}\begin{bmatrix}[\lambda_0] & 1\\ 1 & 0\end{bmatrix}\sum_{j=1}^{n+1}\sum_{\lambda_j\in\mathbf{F}_q}(\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j-\underline{i}_j}(V_{j+1}^{\frac{1}{p^{j+1}}})^{\underline{i}_{j+1}}\begin{bmatrix}1 & 0\\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1\end{bmatrix}[1,e]$$

and the element $F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_{n+1}}$ have the same image under the projection

$$\mathrm{Ind}_{K_0(p^{n+2})}^K1)^+\twoheadrightarrow(\mathrm{Ind}_{K_0(p^{n+2})}^K1)^+/((\mathrm{Ind}_{K_0(p^{n+2})}^K1)^+)_{N-M}$$

Proof. The proof is completely analogous to the proof of Proposition 7.4 and is left to the reader (see [Mo1], Proposition 7.5 for details). □

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