ON MOD p LOCAL-GLOBAL COMPATIBILITY FOR $\mathrm{GL}_3(\mathbb{Q}_p)$ IN THE NON-ORDINARY CASE

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ABSTRACT. Let $F/\mathbb{Q}$ be a CM field where p splits completely and $\bar{r} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$ a continuous modular Galois representation. Assume that $\bar{r}$ is non-ordinary and nonsplit reducible (niveau 2) at a place w above p. We show that the isomorphism class of $\bar{r}|_{\mathrm{Gal}(\overline{F}_w/F_w)}$ is determined by the $\mathrm{GL}_3(F_w)$-action on the space of mod p algebraic automorphic forms by using the refined Hecke action of [HLM]. We also give a nearly optimal weight elimination result for niveau two Galois representations compatible with the explicit conjectures of [Her09] and [GHS]. Moreover, we prove the modularity of certain Serre weights, in particular, when the Fontaine-Laffaille invariant takes special value ∞, our methods provide with the modularity of a certain shadow weight.

CONTENTS

1. INTRODUCTION

Let p be a prime. In this paper, we address a problem about local-global compatibility in the mod p Langlands program for $\mathrm{GL}_3(\mathbb{Q}_p)$. In [Ser87], J-P. Serre conjectured that if $\bar{r} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is a modular Galois representation, then the minimal weight of a modular form giving rise to $\bar{r}$ is determined (in an explicit way) from the local datum $\bar{r}|_{I_p}$, where I_p denotes the inertia group at p. From the explicit description, one easily sees that the conjectured minimal weight actually determines the isomorphism class of $\bar{r}|_{I_p}$. Serre interpreted this as evidence for compatible mod p local and global Langlands correspondences (cf. *loc. cit.*, Section 3.4). These correspondences were established along with their p-adic analogues in several works of many authors—Breuil, Berger, Colmez, Dospinescu, Emerton, Kisin, and Paskunas to name a few (see [Bre03, Col10, Eme]). In particular, $\bar{r}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ can be recovered from the minimal weight and the Hecke action on it.

One would hope for analogous correspondences in greater generality. For a CM extension F/F^+ in which p splits completely, fix a place $w|p$. For a modular Galois representation $\bar{r} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$, one could consider the $\mathrm{GL}_3(F_w)$-representation $\Pi(\bar{r})$ coming from the space of mod p automorphic forms on a definite unitary group. It is not known whether $\Pi(\bar{r})$ depends only on $\bar{r}|_{\mathrm{Gal}(\overline{F}_w/F_w)}$. It is expected that if $\bar{r}|_{\mathrm{Gal}(\overline{F}_w/F_w)}$ is tamely ramified, then it is determined by the set of modular Serre weights (the $\mathrm{GL}_3(\mathbb{Z}_p)$-socle of $\Pi(\bar{r})$) and the Hecke action on its constituents. However, this is not true if $\bar{r}|_{\mathrm{Gal}(\overline{F}_w/F_w)}$ is wildly ramified, and the question of determining $\bar{r}|_{\mathrm{Gal}(\overline{F}_w/F_w)}$ from $\Pi(\bar{r})$ lies deeper than the weight part of Serre's conjecture. Using a *refined* Hecke action, we show that the $\mathrm{GL}_3(\mathbb{Q}_p)$-action on $\Pi(\bar{r})$ determines $\bar{r}|_{\mathrm{Gal}(\overline{F}_w/F_w)}$ in many non-ordinary cases following the work in the ordinary cases of [HLM] for $\mathrm{GL}_3(\mathbb{Q}_p)$ and [BD14] for GL_2 over unramified extensions of $\mathbb{Q}_p$.

In order to present the main results in more detail we need to fix some notation. We let $E/\mathbb{Q}_p$ be a finite extension, $\mathcal{O}_E$ its ring of integers and $\mathbb{F}$ its residue field. These are the rings of coefficients of our representations and are always assumed to be sufficiently large. Let $\bar{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_3(\mathbb{F})$ be a continuous reducible indecomposable Galois representation. It is not hard to see from the results of [GG12] that the semisimplification of $\bar{\rho}$ is often determined by the modular Serre weights of $\bar{\rho}$ and the Hecke actions on them. If $\bar{\rho}$ is Fontaine-Laffaille, the extension class, and hence the isomorphism class of $\bar{\rho}$, is determined by an invariant $\mathrm{FL}(\bar{\rho}) \in \mathbf{P}^1(\mathbb{F})$ generalizing the one in [HLM] (cf. Definition 2.8).

One can also define a parameter on the automorphic side. Let I_1 denote the standard pro-p Iwahori subgroup. If π_p is a smooth $\mathbb{F}$-valued representation of $\mathrm{GL}_3(\mathbb{Q}_p)$, which verifies certain multiplicity one properties with respect to its $\mathrm{GL}_3(\mathbb{Z}_p)$-socle, then there is a natural action of certain group algebra operators S, S' on (a_2, a_1, a_0)-isotypic parts of $\pi_p^{I_1}$ (isotypic with respect to the residual action of the finite torus) and one can associate a non-zero parameter to the pair (S, S') (see Section 5 for the precise definition of the operators and their properties).

The main result of this paper is to prove that the two local parameters defined above coincide when the local representations are obtained from the cohomology of unitary arithmetic manifolds (cf. Theorem 6.13). Let $F/\mathbb{Q}$ be a CM field with F^+ its maximal totally real subfield and let $\bar{r} : G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$ be a continuous Galois representation. Assume that p is totally split in F and fix a place $w_0|v_0$ of F, F^+ respectively, above p. We assume that $\bar{r}$ is modular: for the purpose of this introduction this means that there exists a totally definite unitary group G defined over F^+ (outer form of GL_3 and split at places

above p), a tame level $U^p \leq G(\mathbf{A}_{F^+}^{\infty,p})$ away from p and a maximal ideal $\mathfrak{m}_{\bar{r}}$ associated to $\bar{r}$ in the anemic Hecke algebra acting on $S^{\mathrm{sm}}(U^p, \mathbb{F})$ (the space of algebraic automorphic forms with infinite level at p and coefficients in $\mathbb{F}$) such that $S^{\mathrm{sm}}(U^p, \mathbb{F})[\mathfrak{m}_{\bar{r}}] \neq 0$.

We write $W(\bar{r})$ for the irreducible smooth $\mathrm{GL}_3(\mathcal{O}_{F^+,p})$-representations V over $\mathbb{F}$ such that

$$\mathrm{Hom}_{\mathcal{G}(\mathcal{O}_{F^+,p})}\left(V^\vee, S^{\mathrm{sm}}(U^p, \mathbb{F})[\mathfrak{m}_{\bar{r}}]\right) \neq 0.$$

(the set of *Serre* weights of $\bar{r}$) and we fix a *Fontaine-Laffaille* set of weights V^{v_0} away from v_0 (i.e. V^{v_0} is an irreducible smooth representation of $\prod_{v|p,\ v \neq v_0} \mathrm{GL}_3(\mathcal{O}_{F_v^+})$ and there exists an irreducible smooth $\mathrm{GL}_3(\mathcal{O}_{F_{v_0}^+})$-representation V_{v_0} such that $V^{v_0} \otimes V_{v_0} \in W(\bar{r})$; see Definition 6.5 for details on the definition of V^{v_0}). In particular, we define the space $S^{\mathrm{sm}}(U^{v_0}, V^{v_0})[\mathfrak{m}_{\bar{r}}]$ of algebraic automorphic forms of infinite level at v_0 and coefficients in V^{v_0}; it is a $G(F_{v_0}^+)$-representation.

Theorem 1.1. *In the previous hypothesis and settings, let* $U = U_{v_0} \times U^{v_0} \leq G(\mathbf{A}_F^{\infty,p}) \times \mathcal{G}(\mathcal{O}_{F^+,p})$ *be a sufficiently small compact open, where* $U^{v_0} \subset G(\mathbf{A}_{F^+}^{\infty,v_0})$. *We make the following assumptions:*

(i) $\bar{r}|_{G_{F_{w_0}}}$ *is indecomposable of residual niveau 2 as in (2.1.1) with genericity condition (2.1.2);*

(ii) $\mathrm{FL}(\bar{r}|_{G_{F_{w_0}}}) \notin \{0, \infty\}$;

(iii) $\bar{r}$ *is Fontaine-Laffaille at all places dividing p;*

(iv) $\bar{r}$ *is unramified at places away from p;*

(v) $\bar{r}$ *has an image containing* $\mathrm{GL}_3(k)$ *for some* $k \subset \mathbb{F}$ *with* $\#k > 9$;

(vi) $\overline{F}^{\ker(\mathrm{ad}\bar{r})}$ *does not contain* $F(\zeta_p)$.

Let S, S' *be the group algebra operators defined in Section 5 (associated to the triple of integers* $(-a_0, -a_1, -a_2)$*). Then*

$$(1.0.1) \qquad S' \circ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & 0 & 0 \end{pmatrix} = (-1)^{a_2 - a_1} \cdot \frac{a_1 - a_0}{a_2 - a_1} \cdot \mathrm{FL}(\bar{r}|_{G_{F_{w_0}}}) \cdot S$$

on $S^{\mathrm{sm}}(U^{v_0}, V^{v_0})[\mathfrak{m}_{\bar{r}}]^{I,(-a_1,-a_0,-a_2)}[U_2]$, *where the notation* $(\bullet)^{I,(-a_1,-a_0,-a_2)}$ *denotes the* $(-a_1, -a_0, -a_2)$-*isotypic part, for the residual action of the finite torus, of the pro-p Iwahori fixed vectors of* $S^{\mathrm{sm}}(U^{v_0}, V^{v_0})[\mathfrak{m}_{\bar{r}}]$, *and* U_2 *is a carefully chosen* U_p-*operator.*

In the theorem above, the global assumption (iii)-(vi) are needed in order to obtain a freeness result for a Hecke algebra acting on $S^{\mathrm{sm}}(U^{v_0}, V^{v_0})[\mathfrak{m}_{\bar{r}}]$ (cf. Theorem 6.16).

As mentioned before, in order to obtain Theorem 1.1 one needs a certain multiplicity one condition on the $G(\mathcal{O}_{F_{v_0}^+})$-socle. This is obtained by a thorough *type elimination* in niveau 2, which highlights that the set of Serre weights for $\bar{r}$ depends on the associated Fontaine-Laffaille parameter.

When $\bar{r}|_{G_{F_{w_0}}}$ is semisimple, there is a conjectural description of the set $W_{w_0}^?(\bar{r})$ of irreducible smooth representations V_{v_0} of $G(\mathcal{O}_{F^+,p})$ such that $V^{v_0} \otimes V_{v_0} \in W(\bar{r})$ (cf. [Her09]). When $\bar{r}|_{G_{F_{w_0}}}$ is not semisimple, we define here an explicit set $W_{w_0}^?(\bar{r})$, which depends on the Fontaine-Laffaille parameter associated to $\bar{r}|_{G_{F_{w_0}}}$ (cf. Definition 6.3). We remark that in the set $W_{w_0}^?(\bar{r})$ we can distinguish an explicit subset $W_{w_0}^{?,obv}(\bar{r})$ of obvious weights (related to "obvious" crystalline lifts of $\bar{r}|_{G_{F_{w_0}}}$). Our main result on Serre weights for $\bar{r}$ is contained in the following theorem:

Theorem 1.2. *Assume that $\bar{r}$ verifies assumption (i) of Theorem 1.1. Then*

$$W_{w_0}(\bar{r}) \subseteq W_{w_0}^?(\bar{r}).$$

Moreover, the obvious *weights $F(a_2 - 1, a_1, a_0 + 1)$ and $F(a_2 - 1, a_0 + 1, a_1 - p + 1)$ are always modular, while, if the Fontaine-Laffaille parameter at w_0 verifies $\mathrm{FL}(\bar{r}|_{G_{F_{w_0}}}) = \infty$, the* shadow *weight $F(a_2, a_0, a_1 - (p - 1))$ is modular.*

Finally, assume that F is unramified at all finite places and that there is a RACSDC automorphic representation Π of $\mathrm{GL}_3(\mathbf{A}_F)$ of level prime to p such that

(i) $\bar{r} \simeq \bar{r}_{p,i}(\Pi)$;
(ii) *For each place $w | p$ of F, $r_{p,i}(\Pi)|_{G_{F_w}}$ is potentially diagonalizable;*
(iii) $\bar{r}(G_{F(\zeta_p)})$ *is adequate.*

Then we have the following inclusion:

$$W_{w_0}^{?,obv}(\bar{r}) \subseteq W_{w_0}(\bar{r}).$$

Remark 1.3. If $\bar{r}|_{G_{F_w}}$ is split, and $\bar{r}$ verifies items (i)-(iii) of Theorem 1.2 we can always prove that $W_{w_0}^{?,obv}(\bar{r}) \cap W_L \subseteq W_{w_0}(\bar{r})$ where $W_{w_0}^{?,obv}(\bar{r}) \cap W_L$ is the set of obvious lower weights of $\bar{r}$ at w_0 (cf. §6.3)

For the weight elimination results, we classify rank 2 simple Breuil modules with descent data of niveau 2 corresponding to the 2-dimensional irreducible quotient of $\bar{r}|_{G_{F_{w_0}}}$. The classification of the rank 2 simple Breuil modules is also heavily used to show the connection between the Fontaine-Laffaille parameter and a Frobenius eigenvalue of certain potentially crystalline lift of $\bar{r}|_{G_{F_{w_0}}}$ (cf. Proposition 4.3 and Theorem 4.5). The proof of weight existence is here performed by purely Galois cohomology arguments. We remark that along the proof of Theorem 1.2, we obtain a potential diagonalizability result, which lets us infer that representations satisfying the hypotheses of Theorem 1.1 do exist (cf. Theorem 6.17).

We conclude this introduction with an overview of the sections of this paper. In the remainder of this introduction, we introduce the notation that will be used throughout the paper. In Section 2, we analyze the local mod p Galois representation $\bar{\rho}_0$ in terms of Fontaine–Laffaille theory. We also classify rank 2 simple Breuil modules with tame descent data, and show the existence of crystalline lifts with certain Hodge–Tate weights of the representation $\bar{\rho}_0$. In Section 3, we perform elimination of Galois types, by determining the structure of possible Breuil modules with descent data corresponding to the representation $\bar{\rho}_0$. In Section 4 we completely determine the filtration of strongly divisible modules lifting the Breuil modules, with a carefully chosen descent datum, corresponding to the representation $\bar{\rho}_0$. The filtration on strongly divisible modules gives information of the eigenvalues of the Frobenius map of the corresponding weakly admissible filtered (ϕ, N)-modules, and we find an explicit relation between certain Frobenius eigenvalues and the Fontaine–Laffaille parameter. In Section 5, we quickly review certain group algebra operators and their properties, developed in [HLM]. Our main results are stated and proved in Section 6. We establish weight elimination result in Section 6.3, and prove mod p local-global compatability and modularity of certain weights in Section 6.4. A freeness result for a Hecke algebra acting on $S^{\mathrm{sm}}(U^{v_0}, V^{v_0})[\mathfrak{m}_{\bar{r}}]$ is proved in Section 6.5.

1.1. **Notation.** Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. All number fields $F/\mathbb{Q}$ will be considered as subfields in $\overline{\mathbb{Q}}$ and we write $G_F \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ to denote the absolute Galois group of F. For any rational prime $\ell \in \mathbb{Q}$, we fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$ and an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$ (and so an inclusion $G_{\mathbb{Q}_\ell} \hookrightarrow G_{\mathbb{Q}}$). In a similar fashion, we fix an algebraic closure

$\overline{\mathbb{F}}_\ell$ for the residue field $\mathbb{F}_\ell$ of $\mathbb{Q}_\ell$. As above, all algebraic extensions of $\mathbb{Q}_\ell$ (resp. $\mathbb{F}_\ell$) will be considered as subfields in the fixed algebraic closure $\overline{\mathbb{Q}}_\ell$ (resp. $\overline{\mathbb{F}}_\ell$).

Let $f \geq 1$. We let $K_0 \stackrel{\mathrm{def}}{=} W(k)[\frac{1}{p}]$ be the unramified extension of degree f of $\mathbb{Q}_p$. We consider the Eisenstein polynomial $E(u) \stackrel{\mathrm{def}}{=} u^e + p \in \mathbb{Z}_p[u]$ where $e = p^f - 1$. We fix a root $\varpi = \sqrt[e]{-p} \in \overline{\mathbb{Q}}_p$ and set $K \stackrel{\mathrm{def}}{=} K_0(\varpi)$. In particular, K/K_0 is a tamely, totally ramified extension of K_0 of degree e and a uniformizer ϖ.

Let E be a finite extension of $\mathbb{Q}_p$. We write $\mathcal{O}_E$ for its ring of integers, $\mathbb{F}$ for its residue field and $\varpi_E \in \mathcal{O}_E$ to denote an uniformizer. From now on, we fix an embedding $\widetilde{\sigma}_0 : K \hookrightarrow E$, hence an embedding $\sigma_0 : k \hookrightarrow \mathbb{F}$.

The choice of $\varpi \in K$ provides us with a map:

$$
\begin{aligned}
\widetilde{\omega}_\varpi : \mathrm{Gal}(K/\mathbb{Q}_p) &\longrightarrow W(\mathbb{F}_{p^f})^\times \\
g &\longmapsto \frac{g(\varpi)}{\varpi}
\end{aligned}
$$

whose reduction mod ϖ will be denoted as ω_ϖ. Note that the choice of the embedding $\sigma_0 : k \hookrightarrow \mathbb{F}$ provides us with a fundamental character of niveau f, namely $\omega_f \stackrel{\mathrm{def}}{=} \sigma_0 \circ \omega_\varpi|_{\mathrm{Gal}(K/K_0)}$.

Write φ for the absolute Frobenius on k. By extension of scalars, the ring $k \otimes_{\mathbb{F}_p} \mathbb{F}$ is equipped with a Frobenius endomorphism $\varphi \otimes 1$ and with a $\mathrm{Gal}(K/\mathbb{Q}_p)$-action via $\omega_\varpi \otimes 1$. In particular, we recall the standard idempotent elements $e_\sigma \in k \otimes_{\mathbb{F}_p} \mathbb{F}$ defined for $\sigma \in \mathrm{Hom}(k, \mathbb{F})$, which verify $\varphi(e_\sigma) = e_{\sigma \circ \varphi^{-1}}$ and $(\lambda \otimes 1)e_\sigma = (1 \otimes \sigma(\lambda))e_\sigma$. We write $\widehat{e}_\sigma \in W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ for the standard idempotent elements; they reduce to e_σ modulo p.

Given a p-adic Galois representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(E)$, we write $\rho^\vee$ to denote the linear dual representation. Given a potentially semistable representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(E)$, we write $\mathrm{WD}(\rho)$ to denote the associated Weil-Deligne representation as defined in [CDT99], Appendix B.1. We refer to $\mathrm{WD}(\rho)|_{I_{\mathbb{Q}_p}}$ as to the *inertial type* associated to ρ. Note that, in particular, $\mathrm{WD}(\rho)$ is defined via the (*covariant*) filtered (φ, N)-module $\mathrm{D}_{\mathrm{st}}^{\mathbb{Q}_p}(\rho) \stackrel{\mathrm{def}}{=} \varinjlim_{H/\mathbb{Q}_p} (\mathrm{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} \rho)^{G_H}$ (and $\mathrm{D}_{\mathrm{st}}^{*, \mathbb{Q}_p}$ denotes the *contravariant* filtered (φ, N)-module).

2. The local Galois side

In this section, we analyze the local mod p Galois representations we impose in terms of Fontaine–Laffaille theory. After recalling some integral p-adic Hodge theory, we classify rank 2 simple Breuil modules with tame descent data of niveau 1 and 2, which will be used in Sections 3 and 4. We also show the existence of crystalline lifts with certain Hodge–Tate weights of the local mod p representations, which will be useful later.

2.1. The Fontaine-Laffaille parameter.
Let $\overline{\rho}_0 : G_{\mathbb{Q}_p} \to \mathrm{GL}_3(\mathbb{F})$ be a continuous Galois representation. We assume that $\overline{\rho}_0$ is of niveau 2, i.e. an extension of a 2-dimensional irreducible representation by a character. More precisely, we may let

$$
(2.1.1) \qquad \overline{\rho}_0|_{I_{\mathbb{Q}_p}} \cong \begin{pmatrix} \omega^{a_2+1} & * & * \\ 0 & \omega_2^{(a_1+1)+p(a_0+1)} & 0 \\ 0 & 0 & \omega_2^{a_0+1+p(a_1+1)} \end{pmatrix}
$$

for some integers $a_0, a_1, a_2 \in \mathbb{N}$. It is obvious that it can be rewritten as follows:

$$\overline{\rho}_0|_{I_{\mathbb{Q}_p}} \cong \begin{pmatrix} \omega^{(a_2-a_0-1)+1} & * & * \\ 0 & \omega_2^{(a_1-a_0-1)+1} & 0 \\ 0 & 0 & \omega_2^{p((a_1-a_0-1)+1)} \end{pmatrix} \otimes \omega^{a_0+1}.$$

We let $\overline{\rho}_2$ be the one-dimensional subrepresentation such that $\overline{\rho}_2|_{I_{\mathbb{Q}_p}} \cong \omega^{a_2+1}$ and $\overline{\rho}_{10}$ the two-dimensional irreducible quotient such that $\overline{\rho}_{10}|_{I_{\mathbb{Q}_p}} \cong \omega_2^{a_0+1+p(a_1+1)} \oplus \omega_2^{(a_1+1)+p(a_0+1)}$.

2.1.1. *Preliminaries on Fontaine-Laffaille theory.* We briefly recall the theory of Fontaine-Laffaille modules with $\mathbb{F}$-coefficients and its relation with mod-p Galois represent-ations. The main reference will be [HLM], Section 2.1.

A *Fontaine-Laffaille module* $(M, \mathrm{Fil}^\bullet M, \phi_\bullet)$ over $k \otimes_{\mathbb{F}_p} \mathbb{F}$ is the datum of

   (i) a finite $k \otimes_{\mathbb{F}_p} \mathbb{F}$-module M, free over k;
   (ii) a separated, exhaustive and decreasing filtration $\{\mathrm{Fil}^j M\}_{j \in \mathbb{Z}}$ on M by $k \otimes_{\mathbb{F}_p} \mathbb{F}$ submodules (the *Hodge filtration*), which are k-direct summands;
   (iii) A φ-semilinear Frobenius isomorphism $\phi_\bullet : \mathrm{gr}^\bullet M \to M$

Note that, by property (iii), a Fontaine-Laffaille module is indeed free over $k \otimes_{\mathbb{F}_p} \mathbb{F}$.

Defining the morphisms in the obvious way, we obtain the abelian category $\mathbb{F}\text{-}\mathcal{FL}_k$ of Fontaine-Laffaille modules over $k \otimes_{\mathbb{F}_p} \mathbb{F}$. If the field k is clear from the context, we simply write $\mathbb{F}\text{-}\mathcal{FL}$ to lighten the notation.

Given a Fontaine-Laffaille module M, the set of its Hodge-Tate weights in the direction of $\sigma \in \mathrm{Gal}(k/\mathbb{F}_p)$ is defined as

$$\mathrm{HT}_\sigma \stackrel{\mathrm{def}}{=} \left\{ i \in \mathbb{N}, \ \dim_{\mathbb{F}} \left(\frac{e_\sigma \, \mathrm{Fil}^i M}{e_\sigma \, \mathrm{Fil}^{i+1} M} \right) \neq 0 \right\}.$$

In the remainder of this paper we will be focused on Fontaine-Laffaille modules in *parallel* Hodge-Tate weights, i.e. we will assume that for all $i \in \mathbb{N}$, the submodules $\mathrm{Fil}^i M$ are free over $k \otimes_{\mathbb{F}_p} \mathbb{F}$.

Definition 2.1. Let M be a Fontaine-Laffaille module in parallel Hodge-Tate weights. A $k \otimes_{\mathbb{F}_p} \mathbb{F}$ basis $\underline{f} = (f_1, \ldots, f_n)$ on M is *compatible with the filtration* if for all $i \in \mathbb{N}$ there exists $j_i \in \mathbb{N}$ such that $\mathrm{Fil}^i M = \sum_{j=j_i}^n k \otimes_{\mathbb{F}_p} \mathbb{F} \cdot f_j$. In particular, the principal symbols $(\mathrm{gr}(f_1), \ldots, \mathrm{gr}(f_n))$ provide a $k \otimes_{\mathbb{F}_p} \mathbb{F}$ basis for $\mathrm{gr}^\bullet M$.

Note that if the graded pieces of the Hodge filtration have rank at most one then any two compatible basis on M are related by a lower triangular matrix in $\mathrm{GL}_n(k \otimes_{\mathbb{F}_p} \mathbb{F})$. Given a Fontaine-Laffaille module and a compatible basis $\underline{f}$, it is convenient to describe the Frobenius action via a matrix $\mathrm{Mat}_{\underline{f}}(\phi_\bullet) \in \mathrm{GL}_3(k \otimes_{\mathbb{F}_p} \mathbb{F})$, defined in the obvious way using the principal symbols $(\mathrm{gr}(f_1), \ldots, \mathrm{gr}(f_n))$ as a basis on $\mathrm{gr}^\bullet M$.

It is customary to write $\mathbb{F}\text{-}\mathcal{FL}^{[0,p-2]}$ to denote the full subcategory of $\mathbb{F}\text{-}\mathcal{FL}$ formed by those modules M verifying $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{p-1} M = 0$ (it is again an abelian category). We have the following description of mod p Galois representations of G_{K_0} via Fontaine-Laffaille modules:

Theorem 2.2. *There is an exact fully faithful contravariant functor*

$$\mathrm{T}^*_{\mathrm{cris}, K_0} : \ \mathbb{F}\text{-}\mathcal{FL}_k^{[0,p-2]} \to \mathrm{Rep}_{\mathbb{F}}(G_{K_0})$$

which is moreover compatible with the restriction over unramified extensions: if K_0'/K_0 is unramified, with residue field k'/k, then

$$\mathrm{T}^*_{\mathrm{cris},K_0'}(k' \otimes_k M) \cong \mathrm{T}^*_{\mathrm{cris},K_0}(M)|_{G_{K_0'}}.$$

Proof. The statement with $\mathbb{F}_p$-coefficients is in [FL82], Théorème 6.1; its analogue with $\mathbb{F}$-coefficient is a formal argument which is left to the reader (cf. also [GL14], Theorem 2.2.1). $\square$

We will simply write $\mathrm{T}^*_{\mathrm{cris}}$ if the base field K_0 is clear from the context.

It is well known, (for instance [GG12], Lemma 3.1.5), that under mild condition on the inertial weights, $\overline{\rho}_0$ is Fontaine-Laffaille:

Proposition 2.3. *Let $\overline{\rho}_0 : G_{\mathbb{Q}_p} \to \mathrm{GL}_3(\mathbb{F})$ be as in (2.1.1). If the triple $(a_2, a_1, a_0) \in \mathbb{Z}^3$ verifies $p - 2 \geq (a_2 - a_0 - 1) \geq a_1 - a_0 \geq 2$ then $\overline{\rho}_0$ is Fontaine-Laffaille.*

In order to obtain results on local-global compatibility and to perform weight elimination (cf. Section 3), we shall assume a stronger genericity condition on the integers a_i,

Definition 2.4. We say that a niveau 2 Galois representation $\overline{\rho}_0 : G_{\mathbb{Q}_p} \to \mathrm{GL}_3(\mathbb{F})$ as in (2.1.1) is *generic* if the triple (a_2, a_1, a_0) satisfy the condition

$$(2.1.2) \qquad\qquad p - 3 > (a_2 - a_0 - 1) > (a_1 - a_0) > 3.$$

2.1.2. ***The Fontaine-Laffaille parameter.*** Let $\overline{\rho}_0$ be as in (2.1.1) and assume that the integers $a_i \in \mathbb{N}$ verify the generic condition (2.1.2). By Proposition 2.3 there is a Fontaine-Laffaille module M such that $\mathrm{T}^*_{\mathrm{cris}}(M) \cong \overline{\rho}_0 \otimes \omega^{-a_0 - 1}$ and which is moreover endowed with a filtration by Fontaine-Laffaille submodules $M_0 \subsetneq M_1 \subsetneq M_2 = M$ induced via $\mathrm{T}^*_{\mathrm{cris}}$ from the cosocle filtration on $\overline{\rho}_0$ (cf. Theorem 2.2).

Lemma 2.5. *Assume (2.1.2) and let $M \in \mathbb{F}\text{-}\mathcal{FL}$ be such that $\mathrm{T}^*_{\mathrm{cris}}(M) \cong \overline{\rho}_0 \otimes \omega^{-a_0 - 1}$. Then there exists a basis $\underline{f} = (f_0, f_1, f_2)$ on M which is compatible with the Hodge filtration $\mathrm{Fil}^\bullet M$ and with the filtration by Fontaine-Laffaille submodules on M, and such that*

$$(2.1.3) \qquad\qquad \mathrm{Mat}_{\underline{f}}(\phi_\bullet) = \begin{pmatrix} 0 & \mu_1^{-1} & x \\ \mu_0^{-1} & z & y \\ 0 & 0 & \mu_2^{-1} \end{pmatrix}$$

for some $\mu_i \in \mathbb{F}^\times$, $x, y, z \in \mathbb{F}$.

Proof. For the rest of this proof, we set $c \overset{\mathrm{def}}{=} a_2 - a_1 - 1$, $r \overset{\mathrm{def}}{=} a_1 - a_0 - 1$. In particular M has Hodge-Tate weights $\{0, r+1, c+1\}$.

Let N be the rank two irreducible Fontaine-Laffaille submodule of M corresponding to $\mathrm{T}^*_{\mathrm{cris}}(N) \cong \frac{\overline{\rho}_0}{\omega^{c+1}}$. Then we have $\mathrm{Fil}^i N = N \cap \mathrm{Fil}^i M$ for all $i \in \mathbb{N}$. As N is irreducible, we can find a basis (f_0, f_1) on N, such that $\mathrm{Fil}^1 N = \cdots = \mathrm{Fil}^{r+1} N = \langle f_1 \rangle$ and $\mathrm{Mat}_{(f_0, f_1)}(\phi_\bullet) = \begin{pmatrix} 0 & \mu_1^{-1} \\ \mu_0^{-1} & z \end{pmatrix}$. Let f_2 be a generator of $\mathrm{Fil}^{r+2} M$. As $\mathrm{Fil}^{r+2} N = 0$ and the Frobenius on N is induced from the Frobenius on M it is obvious that the element $\mathrm{Mat}_{(f_0, f_1, f_2)}(\phi_\bullet) \in \mathrm{GL}_3(\mathbb{F})$ has the desired shape (2.1.3). $\square$

Remark 2.6. Keep the notation in the proof of Lemma 2.5. As N is a rank two irreducible Fontaine-Laffaille module, it is easy to show that it is always possible to choose (f_0, f_1) so that $z = 0$.

The Fontaine-Laffaille invariant $\mathrm{FL}(\overline{\rho}_0)$ associated to $\overline{\rho}_0$ is defined in terms of $\mathrm{Mat}_{\underline{f}}(\phi_\bullet)$.

Lemma 2.7. *Keep the hypotheses and the notation of Lemma 2.5. Assume moreover that* x, y *are not both zero, so that* $[x : y] \in \mathbf{P}^1(\mathbb{F})$ *is well defined. Then the elements*

$$\left(\mu_0 \mu_1, \mu_2, \left[-x : \det \begin{pmatrix} \mu_1^{-1} & x \\ z & y \end{pmatrix} \right] \right)$$

deduced from $\mathrm{Mat}_{\underline{f}}(\phi_\bullet)$ *do not depend on the choice of a basis which is compatible with both the Hodge and the submodule filtration on* M.

Proof. The proof is an elementary computation in $\mathrm{GL}_3(\mathbb{F})$. Indeed, let $\underline{f}$ be a basis on M as in the statement of Lemma 2.5. Then the matrix $B \in \mathrm{GL}_3(\mathbb{F})$ associated to a change of basis (compatible with the Hodge filtration) on M is lower triangular and the requirement that the new basis is compatible with the submodule filtration on M provides us the following equation:

$$B \cdot \mathrm{Mat}_{\underline{f}}(\phi_\bullet) \cdot \mathrm{gr}(B)^{-1} = \begin{pmatrix} 0 & \lambda_1^{-1} & x' \\ \lambda_0^{-1} & z' & y' \\ 0 & 0 & \lambda_2^{-1} \end{pmatrix}$$

where the diagonal matrix $\mathrm{gr}(B)$ is defined by $\mathrm{gr}(B)_{i,i} = (B)_{i,i}$, and the left hand side is in an element in $\mathrm{GL}_3(\mathbb{F})$.

By letting $B = \begin{pmatrix} \alpha & 0 & 0 \\ \delta & \beta & 0 \\ \epsilon & \eta & \gamma \end{pmatrix}$, an easy computation provides us with

$$\begin{pmatrix} 0 & \lambda_1^{-1} & x' \\ \lambda_0^{-1} & z' & y' \\ 0 & 0 & \lambda_2^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \mu_1^{-1}\beta^{-1}\alpha & \alpha x \gamma^{-1} \\ \alpha^{-1}\mu_0^{-1}\beta & \mu_1^{-1}\beta^{-1}\delta + z & x\gamma^{-1}\delta + y\gamma^{-1}\beta \\ 0 & 0 & \mu_2^{-1} \end{pmatrix}.$$

We have

$$\left[-\alpha x \gamma^{-1} : \det \begin{pmatrix} \mu_1^{-1}\beta^{-1}\alpha & \alpha x \gamma^{-1} \\ \mu_1^{-1}\beta^{-1}\delta + z & x\gamma^{-1}\delta + y\gamma^{-1}\beta \end{pmatrix} \right] = \left[-x : \det \begin{pmatrix} \mu_1^{-1} & x \\ z & y \end{pmatrix} \right]$$

and the conclusion is now clear. $\qquad\square$

Definition 2.8. Keep the hypothesis and notation of Lemma 2.5, and let M be the Fontaine-Laffaille module associated to $\overline{\rho}_0 \otimes \omega^{-a_0-1}$, $\underline{f} = (f_0, f_1, f_2)$ a basis on M as in Lemma 2.7, and $\mu_1^{-1}, x, y, z \in \mathbb{F}$ be the elements defined by $\mathrm{Mat}_{\underline{f}}(\phi_\bullet)$ as in (2.1.3).

The Fontaine-Laffaille parameter associated to $\overline{\rho}_0$ is defined as

$$\mathrm{FL}(\overline{\rho}_0) = \left[-x : \det \begin{pmatrix} \mu_1^{-1} & x \\ z & y \end{pmatrix} \right] \in \mathbb{P}^1(\mathbb{F}).$$

Remark 2.9. Let $\overline{\rho}_0$ be as in (2.8). The isomorphism class of $\overline{\rho}_0$ is completely determined by the pair $(\mu_0\mu_1, \mu_2)$ and the Fontaine-Laffaille parameter $\mathrm{FL}(\overline{\rho}_0)$ as well as their Hodge–Tate weights.

2.2. p-adic Hodge theory: Preliminaries. We place ourselves in the framework of strongly divisible lattices, Breuil module, étale φ-modules with coefficients and descent data, having [EGH13] Section 3.1 and [HLM] Section 2 as a main reference.

2.2.1. *Preliminaries in characteristic zero.* The ring $S_{W(k)}$ (cf. [Bre97], Section 4.1, [Car08], Section 2.1) is defined as the p-adic completion of the divided power envelope of the polynomial ring $W(k)[u]$ with respect to the ideal generated by $E(u)$ (compatibly with the standard divided powers on $pW(k)[u]$).

It is canonically isomorphic to the following sub-algebra of $K_0[[u]]$:

$$S_{W(k)} = \left\{ \sum_{i=0}^{\infty} w_i \frac{E(u)^i}{i!}, \qquad w_i \in W(k)[u], \lim_{i \to \infty} w_i = 0 \right\}$$

where $W(k)[u]$ is endowed with the topology of the pointwise convergence.

The ring $S_{W(k)}$ is endowed with a continuous, semilinear Frobenius endomorphism $\varphi :$ $S_{W(k)} \to S_{W(k)}$ (semilinear with respect to the absolute Frobenius on $W(k)$), uniquely characterized by $u \mapsto u^p$ and a $W(k)$-linear derivation N, uniquely determined by $N(u) = -u$ (hence $N\varphi = p\varphi N$). This ring is naturally endowed with a filtration $\{\mathrm{Fil}^i S_{W(k)}\}_{i \in \mathbb{N}}$, where $\mathrm{Fil}^i S_{W(k)}$ is the closure of the ideal generated by $\frac{E(u)^j}{j!}$, $j \geq i$, and with a residual Galois action by $W(k)$-algebra endomorphisms, defined by $\widehat{g}(u) = \widetilde{\omega}_\varpi(g)u$ for any $g \in \mathrm{Gal}(K/\mathbb{Q}_p)$. In particular, the action of any $g \in \mathrm{Gal}(K/\mathbb{Q}_p)$ is compatible with the Frobenius, the filtration and the monodromy on S. Note that, by extension of scalars, the ring $S_{\mathbb{Q}_p} \overset{\mathrm{def}}{=} S_{W(k)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is endowed with the evident additional structures inherited from $S_{W(k)}$.

We will be mainly concerned with objects having E-coefficients. Concretely, we write $S \overset{\mathrm{def}}{=} S_{W(k)} \otimes_{\mathbb{Z}_p} \mathcal{O}_E$, $S_E \overset{\mathrm{def}}{=} S \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, so that the additional structures on $S_{W(k)}$ induce, by $\mathcal{O}_E$ and E-linearity respectively, a Frobenius, a derivation, a filtration and a compatible residual Galois action on S, S_E.

Recall that a strongly divisible lattice in weights $(0, r)$ is the datum of a free S-module of finite type $\widehat{\mathcal{M}}$, an S-submodule $\mathrm{Fil}^r \widehat{\mathcal{M}} \subseteq \widehat{\mathcal{M}}$, together with additive morphisms φ_r, N such that:

(i) $\mathrm{Fil}^r S \cdot \widehat{\mathcal{M}} \subseteq \mathrm{Fil}^r \widehat{\mathcal{M}}$ and $\widehat{\mathcal{M}}/\mathrm{Fil}^r \widehat{\mathcal{M}}$ is ϖ_E-torsion free;

(ii) the morphism $\varphi_r : \mathrm{Fil}^r \widehat{\mathcal{M}} \to \widehat{\mathcal{M}}$ is semilinear with respect to the Frobenius on S and its image contains a family of S-generators for $\widehat{\mathcal{M}}$;

(iii) the morphism $N : \widehat{\mathcal{M}} \to \widehat{\mathcal{M}}$ is $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$-linear and verifies

    (a) $N(sx) = N(s)x + sN(x)$ for all $x \in \widehat{\mathcal{M}}$, $s \in S$;

    (b) $E(u)N(\mathrm{Fil}^r \widehat{\mathcal{M}}) \subseteq \mathrm{Fil}^r \widehat{\mathcal{M}}$;

    (c) $\varphi_r(E(u) \cdot N) = cN \circ \varphi_r$ where $c \overset{\mathrm{def}}{=} \frac{\varphi(E(u))}{p} \in S^\times$.

Let $K' \in \{K_0, \mathbb{Q}_p\}$. A descent data from K to K' on $\widehat{\mathcal{M}}$ is the datum of an action of $\mathrm{Gal}(K/K')$ by additive automorphisms on $\widehat{\mathcal{M}}$, which are semilinear (with respect to the descent data on S) and compatible with the additional structures on $\widehat{\mathcal{M}}$ (i.e. with the Frobenius, monodromy, and the filtration). We write $\mathcal{O}_E\text{-}\mathrm{Mod}^r_{\mathrm{dd}}$ to denote the category of strongly divisible lattices in weights $(0, r)$, with descent data from K to K'.

We have a contravariant functor

$$\mathrm{T}^{*,K'}_{\mathrm{st}} : \mathcal{O}_E\text{-}\mathrm{Mod}^r_{\mathrm{dd}} \to \mathrm{Rep}^{K\text{-st},[-r,0]}_{\mathcal{O}_E}(G_{K'})$$

where $\mathrm{Rep}^{K\text{-st},[-r,0]}_{\mathcal{O}_E}(G_{K'})$ is the category of $G_{K'}$-stable $\mathcal{O}_E$-lattices inside E-valued, finite dimensional p-adic Galois representation of $G_{K'}$ becoming semi-stable over K and with Hodge–Tate weights in $\{-r, 0\}$ (cf. [EGH13], Section 3.1), and which establish an antiequivalence of categories if $r < p - 1$ (cf. [EGH13], Proposition 3.1.4, building on work of Liu [Liu08]).

2.2.2. **p-adic Hodge theory: preliminaries in characteristic p.** The residual Breuil ring $\overline{S} \overset{\mathrm{def}}{=} (k \otimes_{\mathbb{F}_p} \mathbb{F})[u]/(u^{ep})$ is equipped with an action of $\mathrm{Gal}(K/\mathbb{Q}_p)$ by $k \otimes_{\mathbb{F}_p} \mathbb{F}$-semilinear automorphisms. Explicitly if $g \in \mathrm{Gal}(K/\mathbb{Q}_p)$ and $a \in k \otimes_{\mathbb{F}_p} \mathbb{F}$, we have

$$\widehat{g}(au) \overset{\mathrm{def}}{=} (g \cdot a)(\omega_\varpi(g) \otimes 1)u$$

where $g \cdot a$ denotes the natural $\mathrm{Gal}(K/\mathbb{Q}_p)$ action on $k \otimes_{\mathbb{F}_p} \mathbb{F}$.

We recall that $\overline{S}$ is equipped with an $k \otimes_{\mathbb{F}_p} \mathbb{F}$-linear derivation N defined by $N(u) = -u$ and with a semilinear Frobenius φ defined by $u \mapsto u^p$ (semilinear with respect to the absolute Frobenius on $k \otimes_{\mathbb{F}_p} \mathbb{F}$).

Fix $r \in \{0, \ldots, p-2\}$ and let $\overline{S}_k \overset{\text{def}}{=} k[u]/u^{ep}$. A *Breuil module over* $\mathbb{F}$ is the datum of a quadruple $(\mathcal{M}, \text{Fil}^r \mathcal{M}, \varphi_r, N)$ where

   (i) $\mathcal{M}$ is a finitely generated $\overline{S}$-module which is free over $\overline{S}_k$;

   (ii) $\text{Fil}^r \mathcal{M}$ is a $\overline{S}$-submodule of $\mathcal{M}$, verifying $u^{er}\mathcal{M} \subseteq \text{Fil}^r \mathcal{M}$;

   (iii) the morphism $\varphi_r : \text{Fil}^r \mathcal{M} \to \mathcal{M}$ is φ-semilinear and the associated fibered product $\overline{S} \otimes_{k \otimes_{\mathbb{F}_p} \mathbb{F}} \text{Fil}^r \mathcal{M} \to \mathcal{M}$ is surjective;

   (iv) the operator $N : \mathcal{M} \to \mathcal{M}$ is $k \otimes_{\mathbb{F}_p} \mathbb{F}$-linear and satisfies the following properties:

      (a) $N(P(u)x) = P(u)N(x) + N(P(u))x$ for all $x \in \mathcal{M}$, $P(u) \in \overline{S}$;

      (b) $u^e N(\text{Fil}^r \mathcal{M}) \subseteq \text{Fil}^r \mathcal{M}$;

      (c) $\varphi_r(u^e N(x)) = N(\varphi_r(x))$ for all $x \in \text{Fil}^r \mathcal{M}$.

A morphism of Breuil modules is defined as an $\overline{S}$-linear morphism which is compatible, in the evident sense, with the additional structures (monodromy, Frobenius, filtration).

As above, we let $K' \in \{\mathbb{Q}_p, K_0\}$. A descent data relative to K' on a Breuil module $\mathcal{M}$ is the datum of an action of $\text{Gal}(K/K')$ on $\mathcal{M}$ by $\mathbb{F}$-linear automorphisms which are semilinear with respect to the residual Galois action on $\overline{S}$ and which are compatible, in the evident sense, with the additional structures on $\mathcal{M}$. We write $\mathbb{F}\text{-BrMod}^r_{\text{dd}}$ to denote the category of Breuil modules over $\mathbb{F}$ with descent data to K'.

We recall that we have an exact, faithful, contravariant functor

$$\begin{aligned} \text{T}^*_{\text{st}} : \mathbb{F}\text{-BrMod}^r_{\text{dd}} &\to \text{Rep}_{\mathbb{F}}(G_{K'}) \\ \mathcal{M} &\mapsto \text{T}^*_{\text{st}}(\mathcal{M}) \overset{\text{def}}{=} \text{Hom}(\mathcal{M}, \widehat{A}) \end{aligned}$$

where $\widehat{A}$ is a certain period ring (cf. [EGH13], Section 3.2 building on [Bre99a], Section 2.2; see also [HLM], appendix A).

The functor T^*_{st} respects the rank on both sides, i.e. $\dim_{\mathbb{F}} \text{T}^*_{\text{st}}(\mathcal{M}) = \text{rank}_{\overline{S}} \mathcal{M}$ (cf. [Car11], Théorème 4.2.4 and the Remarque following it, see also [EGH13] Lemma 3.2.2)

We have a natural compatibility between strongly divisible lattices and Breuil modules:

Proposition 2.10. *Let $\widehat{\mathcal{M}}$ be an object in $\mathcal{O}_E\text{-Mod}^r_{\text{dd}}$. Then $\widehat{\mathcal{M}} \otimes_S S/(\varpi_E, \text{Fil}^p S)$ is an object in $\mathbb{F}\text{-BrMod}^r_{\text{dd}}$ in a natural way and one has a natural isomorphism:*

$$\text{T}^{*,K'}_{\text{st}}(\widehat{\mathcal{M}}) \otimes_{\mathcal{O}_E} \mathbb{F} \cong \text{T}^*_{\text{st}}(\widehat{\mathcal{M}} \otimes_S S/(\varpi_E, \text{Fil}^p S)).$$

Proof. This is contained in [EGH13], Section 3.2 (Lemma 3.2.2 and Definition 3.2.8). $\square$

In the rest of this paper we will be mainly interested in the covariant version of the previous functors toward Galois representations. For this reason we define $\text{T}^{K',r}_{\text{st}} : \mathcal{O}_E\text{-Mod}^r_{\text{dd}} \to \text{Rep}^{K\text{-st},[-r,0]}_{\mathcal{O}_E}(G_{K'})$ and $\text{T}^r_{\text{st}} : \mathbb{F}\text{-BrMod}^r_{\text{dd}} \to \text{Rep}_{\mathbb{F}}(G_{K'})$ via

$$\text{T}^{K',r}_{\text{st}}(\widehat{\mathcal{M}}) \overset{\text{def}}{=} \left(\text{T}^{*,K'}_{\text{st}}(\widehat{\mathcal{M}}) \right)^{\vee} \otimes \varepsilon^r_p, \qquad \text{T}^r_{\text{st}}(\mathcal{M}) \overset{\text{def}}{=} \left(\text{T}^*_{\text{st}}(\mathcal{M}) \right)^{\vee} \otimes \omega^r$$

(where we write $\bullet^{\vee}$ to denote the usual linear dual for an $\mathbb{F}$-linear space $\bullet$).

We remark that this definition is compatible with the notion of duality on Breuil and strongly divisible modules as defined in [Car05] and [Car11], namely $\text{T}^{*,\mathbb{Q}_p}_{\text{st}}(\widehat{\mathcal{M}}^*) \cong \text{T}^{\mathbb{Q}_p,r}_{\text{st}}(\widehat{\mathcal{M}})$ and $\text{T}^r_{\text{st}}(\mathcal{M}) = \text{T}^*_{\text{st}}(\mathcal{M}^*)$.

We recall the crucial notion of *type* associated to a Breuil module.

Definition 2.11. Let $n \in \mathbb{N}$ and let $(a_0, \ldots, a_{n-1}) \in \mathbb{Z}^n$ be an n-tuple. A rank n Breuil module $\mathcal{M} \in \mathbb{F}\text{-BrMod}_{\mathrm{dd}}^r$ *is of (framed) type* $\omega_\varpi^{a_0} \oplus \cdots \oplus \omega_\varpi^{a_{n-1}}$ if $\mathcal{M}$ has an $\overline{S}$-basis $(e_0, \ldots, e_{n-1})$ such that $\widehat{g}e_i = (\omega_\varpi^{a_i}(g) \otimes 1)e_i$ for all i and all $g \in \mathrm{Gal}(K/K_0)$. We call such a basis a *framed basis of* $\mathcal{M}$.

We also say that $(f_0, \ldots, f_{n-1})$ is a *framed system of generators of* $\mathrm{Fil}^r \mathcal{M}$ if $(f_0, \ldots, f_{n-1})$ is a system of $\overline{S}$-generators for $\mathrm{Fil}^r \mathcal{M}$ and $\widehat{g}f_i = (\omega_\varpi^{p^{-1}a_i}(g) \otimes 1)f_i$ for all i and all $g \in \mathrm{Gal}(K/K_0)$.

A key tool in local to global compatibility is that the inertial type on a Breuil module $\mathcal{M}$ is closely related to the Weil-Deligne representation associated to a potentially crystalline lift of $\mathrm{T}_{\mathrm{st}}^r(\mathcal{M})$.

Proposition 2.12. *Let* $\widehat{\mathcal{M}}$ *be an object in* $\mathcal{O}_E\text{-Mod}_{\mathrm{dd}}^r$ *and let* $\mathcal{M} \overset{\mathrm{def}}{=} \widehat{\mathcal{M}} \otimes_S S/(\varpi_E, \mathrm{Fil}^p S)$ *be the Breuil module associated to* $\widehat{\mathcal{M}}$ *via the base change* $S \twoheadrightarrow \overline{S}$.

Assume that $\mathrm{T}_{\mathrm{st}}^{\mathbb{Q}_p, r}(\widehat{\mathcal{M}})$ *has inertial type* $\oplus_{i=0}^{n-1} \omega_f^{a_i}$. *Then the Breuil module* $\mathcal{M}$ *is of type* $\oplus_{i=0}^{n-1} \omega_\varpi^{a_i}$ *and* $\mathrm{Fil}^r \mathcal{M}$ *admits a framed system of generators.*

Proof. This can be spelled out from, e.g. [EGH13], Section 3.3 (proof of Theorem 3.3.13). See also [HLM], Lemma 2.3.8. $\square$

2.2.3. *Comparison between Breuil and Fontaine-Laffaille modules.* We now recall the following categories of étale φ-modules, first introduced by Fontaine ([Fon90]).

Let $k((\underline{p}))$ be the field of norms associated to (K_0, p). In particular, $\underline{p}$ is identified with a sequence $(p_n)_n \in (\overline{\mathbb{Q}}_p)^{\mathbb{N}}$ verifying $p_n^p = p_{n-1}$ for all n. We define the category $(\varphi, \mathbb{F} \otimes_{\mathbb{F}_p} k((\underline{p})))\text{-}\mathfrak{Mod}$ of étale $(\varphi, \mathbb{F} \otimes_{\mathbb{F}_p} k((\underline{p})))$-modules as the category of free $\mathbb{F} \otimes_{\mathbb{F}_p} k((\underline{p}))$-modules of finite rank $\mathfrak{D}$ endowed with a semilinear map $\varphi : \mathfrak{D} \to \mathfrak{D}$ (semilinear with respect to the Frobenius on $k((\underline{p}))$) and inducing an isomorphism $\varphi^* \mathfrak{D} \to \mathfrak{D}$ (with obvious morphisms between objects).

By work of Fontaine [Fon90], we have an anti-equivalence

$$
\begin{aligned}
\left(\varphi, \mathbb{F} \otimes_{\mathbb{F}_p} k((\underline{p}))\right)\text{-}\mathfrak{Mod} &\overset{\sim}{\longrightarrow} \mathrm{Rep}_{\mathbb{F}}(G_{(\mathbb{Q}_p)_\infty}) \\
\mathfrak{D} &\longmapsto \mathrm{Hom}\left(\mathfrak{D}, k((\underline{p}))^{\mathrm{sep}}\right).
\end{aligned}
$$

Let us consider $\varpi \overset{\mathrm{def}}{=} \sqrt[e]{-p} \in K$. We can fix a sequence $(\varpi_n)_n \in (\overline{\mathbb{Q}}_p)^{\mathbb{N}}$ such that $\varpi_n^e = p_n$ for all $n \in \mathbb{N}$ and which is compatible with the norm maps $K(\varpi_{n+1}) \to K(\varpi_n)$ (cf. [Bre14], Appendix A).

By letting $K_\infty \overset{\mathrm{def}}{=} \cup_{n \in \mathbb{N}} K(\varpi_n)$ and $(K_0)_\infty \overset{\mathrm{def}}{=} \cup_{n \in \mathbb{N}} K_0(p_n)$, we have a canonical isomorphism $\mathrm{Gal}(K_\infty/(K_0)_\infty) \to \mathrm{Gal}(K/K_0)$ and we will identify ω_ϖ as a character on $\mathrm{Gal}(K_\infty/(K_0)_\infty)$.

The field of norms $k((\underline{\varpi}))$ associated to (K, ϖ) is then endowed with a residual action of $\mathrm{Gal}(K_\infty/(K_0)_\infty)$, which is completely determined by $\widehat{g}(\underline{\varpi}) = \omega_\varpi(g)\underline{\varpi}$.

We can therefore define the category $(\varphi, \mathbb{F} \otimes_{\mathbb{F}_p} k((\underline{\varpi})))\text{-}\mathfrak{Mod}_{\mathrm{dd}}$ of étale $(\varphi, \mathbb{F} \otimes_{\mathbb{F}_p} k((\underline{\varpi})))$-modules with descent data: an object $\mathfrak{D}$ is defined in the analogous, evident way as for the category $(\varphi, \mathbb{F} \otimes_{\mathbb{F}_p} k((\underline{p})))\text{-}\mathfrak{Mod}$, but we moreover require that $\mathfrak{D}$ is endowed with a semilinear action of $\mathrm{Gal}(K_\infty/(K_0)_\infty)$ (semilinear with respect to the residual action on $\mathbb{F} \otimes_{\mathbb{F}_p} k((\underline{\varpi}))$, where $\mathbb{F}$ is endowed with the trivial $\mathrm{Gal}(K_\infty/(K_0)_\infty)$-action) and the Frobenius φ is $\mathrm{Gal}(K_\infty/(K_0)_\infty)$-equivariant.

From [HLM], Appendix A.2 (which builds on the classical result of Fontaine) we have an anti-equivalence

$$\left(\varphi, \mathbb{F} \otimes_{\mathbb{F}_p} k((\underline{\varpi}))\right)\text{-}\mathfrak{Mod}_{\mathrm{dd}} \;\xrightarrow{\sim}\; \mathrm{Rep}_{\mathbb{F}}(G_{(K_0)_\infty})$$
$$\mathfrak{D} \;\mapsto\; \mathrm{Hom}\left(\mathfrak{D}, k((\underline{\varpi}))^{\mathrm{sep}}\right).$$

The main result concerning the relations between the various categories and functors introduced so far is summarized by the following proposition ([HLM], Proposition 2.2.9).

Proposition 2.13. *There exist faithful functors*

$$M_{k((\underline{\varpi}))} : \mathbb{F}\text{-}\mathrm{BrMod}^r_{\mathrm{dd}} \to \left(\varphi, \mathbb{F} \otimes_{\mathbb{F}_p} k((\underline{\varpi}))\right)\text{-}\mathfrak{Mod}_{\mathrm{dd}}$$

and

$$\mathcal{F} : \mathbb{F}\text{-}\mathcal{FL}^{[0,p-2]} \to \left(\varphi, \mathbb{F} \otimes_{\mathbb{F}_p} k((\underline{p}))\right)\text{-}\mathfrak{Mod}$$

fitting in the following commutative diagram:

(2.2.1)

where the descent data is relative to K_0 and the functor $\mathrm{Res} \circ \mathrm{T}_{\mathrm{cris}}$ is fully faithful.

The functors $M_{k((\underline{\varpi}))}$, $\mathcal{F}$ are defined in [HLM], Appendix A, building on the classical work of Breuil [Bre99b] and Caruso-Liu [CL09].

Corollary 2.14. *Let $r \leq p - 2$ and let $\mathcal{M}$, M be objects in $\mathbb{F}\text{-}\mathrm{BrMod}^r_{\mathrm{dd}}$ and $\mathbb{F}\text{-}\mathcal{FL}^{[0,p-2]}$ respectively. Assume that $\mathrm{T}^*_{\mathrm{st}}(\mathcal{M})$ is Fontaine-Laffaille. If*

$$M_{k((\underline{\varpi}))}(\mathcal{M}) \cong \mathcal{F}(M) \otimes_{k((\underline{p}))} k((\underline{\varpi}))$$

then one has an isomorphism of G_{K_0}-representations

$$\mathrm{T}^*_{\mathrm{st}}(\mathcal{M}) \cong \mathrm{T}^*_{\mathrm{cris}}(M).$$

2.2.4. ***Linear algebra with descent data.*** We recall here some formalism on linear algebra with descent data which was introduced in [HLM]. In what follows we fix a residual Galois type $\overline{\tau} : I_{\mathbb{Q}_p} \to \mathrm{GL}_n(\mathbb{F})$, with a framing $\overline{\tau} = \oplus_{i=0}^{n-1} \omega_f^{a_i}$.

Definition 2.15. Let $\mathcal{M} \in \mathbb{F}\text{-}\mathrm{BrMod}^r_{\mathrm{dd}}$ be of type $\oplus_{i=0}^{n-1} \omega_{\varpi}^{a_i}$. Let $\underline{e} \stackrel{\mathrm{def}}{=} (e_0, \ldots, e_{n-1})$, $\underline{f} = (f_0, \ldots, f_{n-1})$ be a framed basis and a framed system of generators of $\mathcal{M}$, $\mathrm{Fil}^r \mathcal{M}$ respectively.

The *matrix of the filtration*, with respect to $\underline{e}, \underline{f}$, is the element $\mathrm{Mat}_{\underline{e}, \underline{f}}(\mathrm{Fil}^r \mathcal{M}) \in M_n(\overline{S})$ verifying

$$\underline{f} = \underline{e} \cdot \mathrm{Mat}_{\underline{e}, \underline{f}}(\mathrm{Fil}^r \mathcal{M}).$$

Similarly, we define the *matrix of the Frobenius* with respect to $\underline{e}$, $\underline{f}$ as the element $\mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r) \in \mathrm{GL}_n(\overline{S})$ characterized by

$$\varphi_r(\underline{f}) = \underline{e} \cdot \mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r).$$

As we require $\underline{e}$, $\underline{f}$ to be compatible with the framing, the coefficients in the matrix of the filtration verify important additional properties:

$$\left(\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r \mathcal{M})\right)_{i,j} \in \left(\overline{S}\right)_{\omega_f^{p^{-1}a_j - a_i}}.$$

Concretely, one has $\left(\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r \mathcal{M})\right)_{i,j} = u^{[p^{-1}a_j - a_i]} s_{i,j}$ where for any $x \in \mathbb{Z}$ we define $[x] \in \{0, \ldots, e-1\}$ by $[x] \equiv a_j - a_i \mod e$ and $s_{i,j} \in \left(\overline{S}\right)_{\omega_\varpi^0} = k \otimes_{\mathbb{F}_p} \mathbb{F}[u^e]/(u^{ep})$.

We can therefore introduce the subspace $M_n^\square(\overline{S})$ of "matrices with framed type $\overline{\tau}$":

Definition 2.16. Let $\overline{\tau}$ be a framed tame Galois type.
The space $M_n^\square(\overline{S})$ is defined as

$$M_n^\square(\overline{S}) \stackrel{\text{def}}{=} \left\{ V \in M_n(\overline{S}), \ V_{i,j} \in \left(\overline{S}\right)_{\omega_\varpi^{a_j - a_i}} \ \text{for all } 0 \le i, j \le n-1 \right\}.$$

Similarly, we set

$$\mathrm{GL}_n^\square(\overline{S}) \stackrel{\text{def}}{=} \mathrm{GL}_n(\overline{S}) \cap M_n^\square(\overline{S})$$

which is a subgroup in $\mathrm{GL}_n(\overline{S})$.

As $\overline{\tau}$ is a residual Galois type, there exists an element $w_{\overline{\tau}} \in \mathfrak{S}_n$ such that $\widehat{g} f_{w_{\overline{\tau}}(j)} = (\omega_\varpi^{a_j} \otimes 1) f_{w_{\overline{\tau}}(j)}$ for all $g \in \mathrm{Gal}(K/K_0)$ and $0 \le j \le n-1$. Moreover as $\varphi_r(f_i)$ is a $\omega_f^{a_i}$ eigenvector for the residual Galois action we deduce that

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r \mathcal{M}) \cdot w_{\overline{\tau}} \in \mathrm{Mat}_n^\square(\overline{S}), \qquad \mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r) \in \mathrm{GL}_n^\square(\overline{S})$$

where we used the same notation $w_{\overline{\tau}}$ for the permutation matrix associated to $w_{\overline{\tau}}$.
Given $A, B \in M_{n+1}^\square(\overline{S})$ and $x \in \left(\overline{S}\right)_{\omega_\varpi^0}$ we write, with a slight abuse of notation,

$$A \equiv B \mod x$$

meaning that there exists an element $C \in M_{n+1}^\square(\overline{S})$ such that $A = B + xC$.

Lemma 2.17. *Let $\mathcal{M}$ be a Breuil module of framed type $\oplus_{i=0}^{n-1} \omega_\varpi^{a_i}$, and let $\underline{e}$, $\underline{f}$ be a framed basis for $\mathcal{M}$ and a framed system of generators for $\mathrm{Fil}^r \mathcal{M}$ respectively.*
Let $V = \mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r \mathcal{M}) \in M_n(\overline{S})$ and $A \stackrel{\text{def}}{=} \mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r) \in \mathrm{GL}_n^\square(\overline{S})$ be the matrices for the filtration and the Frobenius action respectively.
Then there exists a basis $\mathfrak{e}$ for $M_{k((\varpi))}(\mathcal{M}^*)$, framed with respect to the type $\oplus_{i=0}^{n-1} \omega_\varpi^{-a_i}$, such that the Frobenius action is described by

$$\mathrm{Mat}_{\mathfrak{e}}(\phi) = \widehat{V}^t \left(\widehat{A}^{-1}\right)^t \in M_n(\mathbb{F} \otimes_{\mathbb{F}_p} k[[\varpi]])$$

where $\widehat{V}, \widehat{A}$ are lifts of V, A in $M_n(\mathbb{F} \otimes_{\mathbb{F}_p} k[[\varpi]])$ via the reduction morphism $\mathbb{F} \otimes_{\mathbb{F}_p} k[[\varpi]] \twoheadrightarrow \overline{S}_{\mathbb{F}}$ and $\left(\mathrm{Mat}_{\mathfrak{e}}(\phi)\right)_{ij} \in \left(\mathbb{F} \otimes_{\mathbb{F}_p} k[[\varpi]]\right)_{\omega_\varpi^{a_i - pa_j}}$.

Proof. This is Lemma 2.2.12 in [HLM] $\qquad\square$

Lemma 2.18. *Let $M \in \mathbb{F}\text{-}\mathcal{FL}^{[0,p-2]}$ be a rank n Fontaine-Laffaille module in parallel Hodge-Tate weights $0 \le m_0 \le \cdots \le m_{n-1} \le p-2$ (counted with multiplicity).*

Let $\underline{e} = (e_0, \ldots, e_{n-1})$ be a $k \otimes_{\mathbb{F}_p} \mathbb{F}$ basis for M_i, compatible with the Hodge filtration $\mathrm{Fil}^\bullet M$ and let $F \in M_n(k \otimes_{\mathbb{F}_p} F)$ be the associated matrix of the Frobenius $\phi_\bullet : \mathrm{gr}^\bullet M \to M$

There exists a basis $\underline{\mathfrak{e}}$ for $\mathfrak{M} \overset{def}{=} \mathcal{F}(M)$ such that the Frobenius ϕ on $\mathfrak{M}$ is described by

$$\mathrm{Mat}_{\underline{\mathfrak{e}}}(\phi) = \mathrm{Diag}(\underline{p}^{m_0} \ldots \underline{p}^{m_{n-1}})F.$$

Proof. This is Lemma 2.2.13 in [HLM]. $\qquad\square$

Finally, we need a technical result which lets us keep track of base changes on Breuil modules with descent data.

Lemma 2.19. *Let $\mathcal{M} \in \mathbb{F}\text{-}\mathrm{BrMod}_{\mathrm{dd}}^r$ be of type $\oplus_{i=0}^{n-1} \omega_{\overline{\omega}}^{a_i}$ and let $\underline{e}$, $\underline{f}$ be respectively a framed basis for $\mathcal{M}$ and a framed system of generators for $\mathrm{Fil}^r \mathcal{M}$.*

Write $V = \mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r \mathcal{M})$, $A \overset{def}{=} \mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r)$ to denote the matrix of the filtration and of the Frobenius respectively.

Assume that there exists an element $V' \in M_n^\square(\overline{S})$ such that

(2.2.2)
$$A \cdot V' \equiv V \cdot w_{\overline{\tau}} \cdot B \bmod u^{e(r+1)}.$$

for some $B \in \mathrm{GL}_n^\square(\overline{S})$.

Then the element

$$\underline{e}' \overset{def}{=} \underline{e} \cdot A.$$

defines a framed basis on $\mathcal{M}$. Moreover:

(i) *$V' \cdot w_{\overline{\tau}}^{-1} = \mathrm{Mat}_{\underline{e}',\underline{f}'}(\mathrm{Fil}^r \mathcal{M})$ is a matrix of the filtration with respect to $\underline{e}'$ and a system $\underline{f}'$ of generators for $\mathrm{Fil}^r \mathcal{M}$;*

(ii) *$\varphi(B)$ is the matrix of the Frobenius with respect to $\underline{e}'$, $\underline{f}'$.*

Proof. It easily follows from Lemma 2.2.14 in [HLM]. $\qquad\square$

2.3. **Classification of simple Breuil modules of rank** 2. By [Car11], Théorème 4.2.4 and the Remarque following it, the category $\mathbb{F}\text{-}\mathrm{BrMod}_{\mathrm{dd}}^r$ is additive and admits kernels and cokernels. In particular a complex

$$0 \to \mathcal{M}_0 \overset{f_0}{\to} \mathcal{M}_1 \overset{f_1}{\to} \mathcal{M}_2 \to 0$$

in $\mathbb{F}\text{-}\mathrm{BrMod}_{\mathrm{dd}}^r$ is *exact* if the morphisms f_i induce exact sequences on the underlying $\overline{S}$-modules $\mathcal{M}_j$ and $\mathrm{Fil}^r \mathcal{M}_j$ ($j \in \{0, 1, 2\}$). This endows $\mathbb{F}\text{-}\mathrm{BrMod}_{\mathrm{dd}}^r$ with the structure of an exact category.

In what follows, we give a slight improvement of a technical result in [HLM] (*loc. cit.*, Lemma 2.2.2) concerning the submodule structure of a given Breuil module $\mathcal{M} \in \mathbb{F}\text{-}\mathrm{BrMod}_{\mathrm{dd}}^r$ and which will be crucial to provide the classification of rank two irreducible objects in $\mathbb{F}\text{-}\mathrm{BrMod}_{\mathrm{dd}}^r$.

We recall the definition of *Breuil submodule*:

Definition 2.20. Let $\mathcal{M}$ be an object in $\mathbb{F}\text{-}\mathrm{BrMod}_{\mathrm{dd}}^r$. An $\overline{S}$-submodule $\mathcal{N} \subseteq \mathcal{M}$ is said to be a *Breuil submodule* if $\mathcal{N}$ fulfills the following conditions:

(i) $\mathcal{N}$ is an $\overline{S}_k$-direct summand in $\mathcal{M}$;

(ii) $\mathcal{N}$ is stable under the descent data action and the monodromy operator on $\mathcal{M}$;

(iii) the Frobenius φ_r on $\mathrm{Fil}^r \mathcal{M}$ restricts to a φ-semilinear morphism $\mathcal{N} \cap \mathrm{Fil}^r \mathcal{M} \to \mathcal{N}$.

The importance of Definition 2.20 is explained in the following two propositions.

Lemma 2.21 ([HLM], Lemma 2.2.8)**.** *Let*

$$0 \to \mathcal{M}_1 \xrightarrow{f} \mathcal{M} \to \mathcal{M}_2 \to 0$$

be an exact sequence in $\mathbb{F}\text{-BrMod}_{\mathrm{dd}}^r$. *Then the* $\overline{S}$-*module* $f(\mathcal{M}_1)$ *is a Breuil submodule of* $\mathcal{M}$.

Conversely if $\mathcal{M}$ *is an object in* $\mathbb{F}\text{-BrMod}_{\mathrm{dd}}^r$ *and* $\mathcal{N} \subseteq \mathcal{M}$ *is a Breuil submodule of* $\mathcal{M}$, *the pair* $(\mathcal{N}, \mathrm{Fil}^r \mathcal{N} \stackrel{def}{=} \mathrm{Fil}^r \mathcal{M} \cap \mathcal{N})$ *with the induced structures is an object of* $\mathbb{F}\text{-BrMod}_{\mathrm{dd}}^r$ *in a natural way and the complex*

$$0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{M}/\mathcal{N} \to 0$$

is an exact sequence in $\mathbb{F}\text{-BrMod}_{\mathrm{dd}}^r$.

In particular, if $\mathcal{N}$ *is a Breuil submodule in* $\mathcal{M}$, *then* $\mathcal{N}$ *is an* $\overline{S}$-*direct summand of* $\mathcal{M}$.

Recall that we have a faithful, covariant functor $\mathrm{T}_{\mathrm{st}}^r : \mathbb{F}\text{-BrMod}_{\mathrm{dd}}^r \to \mathrm{Rep}_{\mathbb{F}}(G_{\mathbb{Q}_p})$ (cf. Section 2.2.2)

Proposition 2.22 ([HLM], Proposition 2.2.5)**.** *Let* $K' \in \{K_0, \mathbb{Q}_p\}$. *With the above notion of exact sequence, the category* $\mathbb{F}\text{-BrMod}_{\mathrm{dd}}^r$ *is an exact category in the sense of* [Kel90] *and* $\mathrm{T}_{\mathrm{st}}^r$ *is an exact functor. Moreover, if* $\mathcal{M}$ *an object in* $\mathbb{F}\text{-BrMod}_{\mathrm{dd}}^r$ *the functor* $\mathrm{T}_{\mathrm{st}}^r$ *induces an order preserving bijection*

$$\Theta : \{\textit{Breuil submodules in } \mathcal{M}\} \xrightarrow{\sim} \{G_{K'} \textit{ subrepresentations of } \mathrm{T}_{\mathrm{st}}^r(\mathcal{M})\}$$

sending $\mathcal{N} \subseteq \mathcal{M}$ *to the image of* $\mathrm{T}_{\mathrm{st}}^r(\mathcal{N}) \hookrightarrow \mathrm{T}_{\mathrm{st}}^r(\mathcal{M})$ *and canonically identifying* $\Theta(\mathcal{M})/\Theta(\mathcal{N})$ *with* $\mathrm{T}_{\mathrm{st}}^r(\mathcal{M})/\mathrm{T}_{\mathrm{st}}^r(\mathcal{N})$.

We now establish the main result of this section, namely the complete classification of rank 2 Breuil modules with descent data of niveau 2 relative to $\mathbb{Q}_p$. We start with a preliminary lemma:

Lemma 2.23. *Let* $e = p^2 - 1$, $K_0 = \mathbb{Q}_{p^2}$, $K = K_0(\sqrt[e]{-p})$, *and* $\overline{S} = (\mathbb{F}_{p^2} \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^{ep}$. *Let* $\mathcal{M} \in \mathbb{F}\text{-BrMod}_{\mathrm{dd}}^s$ *be a rank two Breuil module, with descent data relative to* K_0. *Assume that* $\mathrm{T}_{\mathrm{st}}^s(\mathcal{M})|_{I_{K_0}} \cong \omega_2^{r+1} \oplus \omega_2^{p(r+1)}$ *and the integers* $r, s \in \mathbb{N}$ *satisfy* $n(p+1) + (s+1) < r + 1 < (n+1)(p+1) - (s+1)$ *for some* $n \in \mathbb{Z}$.

Then we have a decomposition of Breuil modules $\mathcal{M} \cong \mathcal{M}_k \oplus \mathcal{M}_l$ *where* $\mathrm{T}_{\mathrm{st}}^s(\mathcal{M}_k)|_{I_{K_0}} = \omega_2^{r+1}$ *and* $\mathrm{T}_{\mathrm{st}}^s(\mathcal{M}_l)|_{I_{K_0}} = \omega_2^{p(r+1)}$.

Note that the numerical assumption on r, s implies $s < \frac{p-1}{2}$.

Proof. By Proposition 2.22, there exist Breuil submodules $\mathcal{M}_k$ and $\mathcal{M}_l$ in $\mathcal{M}$ such that $\mathrm{T}_{\mathrm{st}}^s(\mathcal{M}_k)|_{I_{\mathbb{Q}_p}} \simeq \omega_2^{r+1}$ and $\mathrm{T}_{\mathrm{st}}^s(\mathcal{M}_l)|_{I_{\mathbb{Q}_p}} \simeq \omega_2^{p(r+1)}$. Let us write $\mathcal{M}_k = \overline{S}m_k$ (resp. $\mathcal{M}_l = \overline{S}m_l$) with descent data $\hat{g}(m_k) = \sum_{i=0}^1 (\omega_2(g)^{k_i} \otimes 1)m_k$ (resp. $\hat{g}(m_l) = \sum_{i=0}^1 (\omega_2(g)^{l_i} \otimes 1)m_l$), filtration $\mathrm{Fil}^s \mathcal{M}_k = \langle (u^{r_0}e_0 + u^{r_1}e_1)m_k \rangle$ (resp. $\mathrm{Fil}^s \mathcal{M}_l = \langle (u^{s_0}e_0 + u^{s_1}e_1)m_l \rangle$), Frobenius map $\varphi_s : (u^{r_0}e_0 + u^{r_1}e_1)m_k \mapsto \lambda m_k$ (resp. $\varphi_s : (u^{s_0}e_0 + u^{s_1}e_1)m_l \mapsto \eta m_l$), and monodromy operator $N : m_k \mapsto 0$ (resp. $N : m_l \mapsto 0$). Note that the integers k_i, l_i, r_i, s_i satisfy $r_i \equiv pk_{i+1} - k_i \bmod (e)$ and $s_i \equiv pl_{i+1} - l_i \bmod e$ (cf. [EGH13], Lemma 3.3.2).

Assume first that $\{m_k, m_l\}$ is linearly independent in $\mathcal{M}$ over $\overline{S}$. By comparing the cardinalities, it is clear that $\overline{S}(m_k, m_l) = \mathcal{M}$, and so it is obvious that the Frobenius map φ_s and the monodromy operator N on $\mathcal{M}$ are immediately determined by the ones on $\mathcal{M}_k$ and $\mathcal{M}_l$. We have $\mathrm{Fil}^s \mathcal{M} \supset \langle (u^{r_0}e_0 + u^{r_1}e_1)m_k, (u^{s_0}e_0 + u^{s_1}e_1)m_l \rangle$. As the Frobenius on $\mathrm{Fil}^s \mathcal{M}_k$, $\mathrm{Fil}^s \mathcal{M}_l$ is induced from the Frobenius on $\mathrm{Fil}^s \mathcal{M}$, and since the Frobenius acts via

$\lambda, \eta \in \mathbb{F}_{p^2} \otimes_{\mathbb{F}_p} \mathbb{F}$ on $\mathrm{Fil}^s \mathcal{M}_k$, $\mathrm{Fil}^s \mathcal{M}_l$, the previous inclusion is an equality. Hence, the Breuil module $\mathcal{M}$ is a direct sum of these two Breuil submodules in the obvious way.

We now check that $\{m_k, m_l\}$ is linearly independent over $\overline{S}$. Assume on the contrary that $\alpha' m_l = \beta' m_l$ for $\alpha', \beta' \in \overline{S} \setminus \{0\}$. Then the minimal degree of α' and β' should be the same (if not, $\mathcal{M}_k$ and $\mathcal{M}_l$ would not have the same cardinality): more precisely, $u^i \alpha e_0 m_k = u^i \beta e_0 m_l$, $u^j \alpha e_1 m_k = u^j \beta e_1 m_l$, or both, for $\alpha, \beta \in \overline{S}^\times$ and for $i, j \in [0, ep)$. Say, $u^i \alpha e_0 m_k = u^i \beta e_0 m_l$. Then this immediately implies that $k_0 \equiv l_0 \bmod (e)$. We check that this violates our numerical assumption on r and s. Since $p r_0 + r_1 \equiv 0 \bmod (e)$ and $p s_0 + s_1 \equiv 0 \bmod (e)$, we let $p r_0 + r_1 = ae$ and $p s_0 + s_1 = be$ for $0 \le a, b \le s(p+1)$. Since $\mathrm{T}^s_{\mathrm{st}}(\mathcal{M}_k)|_{I_{\mathbb{Q}_p}} \simeq \omega_2^{r+1}$ and $\mathrm{T}^s_{\mathrm{st}}(\mathcal{M}_l)|_{I_{\mathbb{Q}_p}} \simeq \omega_2^{p(r+1)}$, we also have

$$\begin{cases} k_0 + pa \equiv r + 1 \bmod (e); \\ l_0 + pb \equiv p(r+1) \bmod (e). \end{cases}$$

Subtracting the first one from the second one, $(p-1)(r+1) \equiv p(b-a) \bmod (e)$ and so we may let $b - a = \epsilon(p-1)$, and $-(s+1) \le \epsilon \le s+1$ since $s < p - 1$. Hence, $r + 1 \equiv -\epsilon \bmod (p+1)$ and so we may let $r + 1 = -\epsilon + \delta(p+1)$ for $\delta \in \mathbb{Z}$. Our assumption on r and s implies that $n(p+1) < \delta(p+1) = r + 1 + \epsilon < (n+1)(p+1)$, which is obviously impossible. $\square$

Proposition 2.24. *Let $e = p^2 - 1$, $K_0 = \mathbb{Q}_{p^2}$, $K = K_0(\sqrt[e]{-p})$, and $\overline{S} = (\mathbb{F}_{p^2} \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^{ep}$. We let x and y be integers with $x \not\equiv y \bmod (e)$ and $\mathcal{M} \in \mathbb{F}\text{-BrMod}^s_{\mathrm{dd}}$ be a Breuil module of type $\tau \simeq \omega_\varpi^x \oplus \omega_\varpi^y$ such that $\mathrm{T}^s_{\mathrm{st}}(\mathcal{M})$ is an absolutely irreducible 2-dimensional representation of $G_{\mathbb{Q}_p}$, i.e, $\mathrm{T}^s_{\mathrm{st}}(\mathcal{M})|_{I_{\mathbb{Q}_p}} \simeq \omega_2^{r+1} \oplus \omega_2^{p(r+1)}$. Assume further that $n(p+1) + (s+1) < r + 1 < (n+1)(p+1) - (s+1)$ for some $n \in \mathbb{Z}$.*

Then there exists a framed basis $\underline{e} = (e_x, e_y)$ for $\mathcal{M}$ and a framed system of generators $\underline{f} = (f_{px}, f_{py})$ for $\mathrm{Fil}^2 \mathcal{M}$ such that

- $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^s \mathcal{M}) = \begin{pmatrix} 0 & u^{r_x} \\ u^{r_y} & 0 \end{pmatrix}$ *where $0 \le r_x, r_y \le es$ with $r_x \equiv py - x \bmod (e)$ and $r_y \equiv px - y \bmod (e)$;*
- $\mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_s) = \begin{pmatrix} \lambda_x & 0 \\ 0 & \lambda_y \end{pmatrix}$ *where $\lambda_x, \lambda_y \in (\mathbb{F}_{p^2} \otimes_{\mathbb{F}_p} \mathbb{F})^\times$;*
- $\mathrm{Mat}_{\underline{e}}(\hat{g}) = \begin{pmatrix} \omega_\varpi^x(g) \otimes 1 & 0 \\ 0 & \omega_\varpi^y(g) \otimes 1 \end{pmatrix}$ *for all $g \in G(K/K_0)$;*
- $N(e_x) = 0 = N(e_y)$;
- $\mathrm{T}^s_{\mathrm{st}}(\mathcal{M})|_{I_{\mathbb{Q}_p}} \simeq \omega_2^{x + p\frac{pr_x + r_y}{e}} \oplus \omega_2^{y + p\frac{pr_y + r_x}{e}}$.

Proof. By Lemma 2.23, we deduce that $\mathcal{M}$ has a basis $\underline{e} = (m_k, m_l)$ over $\overline{S}$, and a system of generators $\underline{f} = (f_k, f_l)$ for $\mathrm{Fil}^2 \mathcal{M}$ such that:

- $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^s \mathcal{M}) = \begin{pmatrix} u^{r_0}e_0 + u^{r_1}e_1 & 0 \\ 0 & u^{s_0}e_0 + u^{s_1}e_1 \end{pmatrix}$ where $0 \le r_i, s_i \le es$ with $r_i \equiv pk_{i-1} - k_i \bmod (e)$ and $s_i \equiv pl_{i-1} - l_i \bmod (e)$;
- $\mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_s) = \begin{pmatrix} \lambda & 0 \\ 0 & \eta \end{pmatrix}$ where $\lambda, \eta \in (\mathbb{F}_{p^2} \otimes_{\mathbb{F}_p} \mathbb{F})^\times$;
- $\mathrm{Mat}_{\underline{e}}(\hat{g}) = \begin{pmatrix} \sum_{i=0}^1 (\omega_\varpi^{k_i}(g) \otimes 1)e_i & 0 \\ 0 & \sum_{i=0}^1 (\omega_\varpi^{l_i}(g) \otimes 1)e_i \end{pmatrix}$ for all $g \in G(K/K_0)$;
- $N(m_k) = 0 = N(m_l)$.

Let σ be the unique lift in $G(K/\mathbb{Q}_p)$ of the arithmetic Frobenius in $G(K_0/\mathbb{Q}_p)$ such that $\sigma(\sqrt[e]{-p}) = \sqrt[e]{-p}$, and let us try to recover the action of σ on $\mathcal{M}$. Let $\hat{\sigma}(m_k) = \alpha_k m_k + \alpha_l m_l$

and $\hat{\sigma}(m_l) = \beta_k m_k + \beta_l m_l$ where $\alpha_*, \beta_* \in \overline{S}$. The identity $\sigma g \sigma^{-1} = g^p$ for $g \in G(K/K_0)$ gives rise to the following two identities:

○: $[(\omega_{\varpi}^{pk_0}(g) \otimes 1)e_1 + (\omega_{\varpi}^{pk_1}(g) \otimes 1)e_0](\alpha_k m_k + \alpha_l m_l) = \hat{\sigma}\hat{g}(m_k) = \hat{g}^p\hat{\sigma}(m_k) = \hat{g}^p(\alpha_k)[(\omega_{\varpi}^{pk_0}(g) \otimes 1)e_0 + (\omega_{\varpi}^{pk_1}(g) \otimes 1)e_1]m_k + \hat{g}^p(\alpha_l)[(\omega_{\varpi}^{pl_0}(g) \otimes 1)e_0 + (\omega_{\varpi}^{pl_1}(g) \otimes 1)e_1]m_l$;

○: $[(\omega_{\varpi}^{pl_0}(g) \otimes 1)e_1 + (\omega_{\varpi}^{pl_1}(g) \otimes 1)e_0](\beta_k m_k + \beta_l m_l) = \hat{\sigma}\hat{g}(m_l) = \hat{g}^p\hat{\sigma}(m_l) = \hat{g}^p(\beta_k)[(\omega_{\varpi}^{pk_0}(g) \otimes 1)e_0 + (\omega_{\varpi}^{pk_1}(g) \otimes 1)e_1]m_k + \hat{g}^p(\beta_l)[(\omega_{\varpi}^{pl_0}(g) \otimes 1)e_0 + (\omega_{\varpi}^{pl_1}(g) \otimes 1)e_1]m_l$.

Comparing the coefficients in these two identities, we have the following relations of descent data:

(i) $\begin{cases} k_1 \equiv a_0 + k_0 \bmod(e) \text{ and } e_0\alpha_k \in e_0 u^{a_0}(\overline{S}_0)^\times & \text{if } e_0\alpha_k \neq 0; \\ k_0 \equiv a_1 + k_1 \bmod(e) \text{ and } e_1\alpha_k \in e_1 u^{a_1}(\overline{S}_0)^\times & \text{if } e_1\alpha_k \neq 0, \end{cases}$

(ii) $\begin{cases} k_1 \equiv b_0 + l_0 \bmod(e) \text{ and } e_0\alpha_l \in e_0 u^{b_0}(\overline{S}_0)^\times & \text{if } e_0\alpha_l \neq 0; \\ k_0 \equiv b_1 + l_1 \bmod(e) \text{ and } e_1\alpha_l \in e_1 u^{b_1}(\overline{S}_0)^\times & \text{if } e_1\alpha_l \neq 0, \end{cases}$

(iii) $\begin{cases} l_1 \equiv c_0 + k_0 \bmod(e) \text{ and } e_0\beta_k \in e_0 u^{c_0}(\overline{S}_0)^\times & \text{if } e_0\beta_k \neq 0; \\ l_0 \equiv c_1 + k_1 \bmod(e) \text{ and } e_1\beta_k \in e_1 u^{c_1}(\overline{S}_0)^\times & \text{if } e_1\beta_k \neq 0, \end{cases}$

(iv) $\begin{cases} l_1 \equiv d_0 + l_0 \bmod(e) \text{ and } e_0\beta_l \in e_0 u^{d_0}(\overline{S}_0)^\times & \text{if } e_0\beta_l \neq 0; \\ l_0 \equiv d_1 + l_1 \bmod(e) \text{ and } e_1\beta_l \in e_1 u^{d_1}(\overline{S}_0)^\times & \text{if } e_1\beta_l \neq 0. \end{cases}$

It is immediate that $a_0 + a_1 \equiv 0 \bmod(e)$, $b_0 + c_1 \equiv 0 \bmod(e)$, $b_1 + c_0 \equiv 0 \bmod(e)$, and $d_0 + d_1 \equiv 0 \bmod(e)$.

Since $\mathrm{Fil}^r \mathcal{M}$ is stable under the action of σ, we have

$$\sigma(\mathrm{Fil}^r \mathcal{M}) = \big\langle (u^{r_0}e_1 + u^{r_1}e_0)(\alpha_k m_k + \alpha_l m_l), (u^{s_0}e_1 + u^{s_1}e_0)(\beta_k m_k + \beta_l m_l) \big\rangle$$
$$\subset \mathrm{Fil}^r \mathcal{M} = \big\langle (u^{r_0}e_0 + u^{r_1}e_1)m_k, (u^{s_0}e_0 + u^{s_1}e_1)m_l \big\rangle,$$

which immediately implies the following inequalities:

(a) $r_1 + a_0 \geq r_0$ and $r_0 + a_1 \geq r_1$;
(b) $r_1 + b_0 \geq s_0$ and $r_0 + b_1 \geq s_1$;
(c) $s_1 + c_0 \geq r_0$ and $s_0 + c_1 \geq r_1$;
(d) $s_1 + d_0 \geq s_0$ and $s_0 + d_1 \geq s_1$.

Since $\sigma^2 = 1$, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha_k\sigma(\alpha_k) + \beta_k\sigma(\alpha_l) & \alpha_k\sigma(\beta_r) + \beta_k\sigma(\beta_l) \\ \alpha_l\sigma(\alpha_k) + \beta_l\sigma(\alpha_l) & \alpha_l\sigma(\beta_k) + \beta_l\sigma(\beta_l) \end{pmatrix}.$$

From the $(1,1)$- and $(2,2)$-entries, we have the equations:

(2.3.1) $\qquad\qquad \alpha_k\sigma(\alpha_k) = \beta_l\sigma(\beta_l) \text{ and } \beta_k\sigma(\alpha_l) = \alpha_l\sigma(\beta_k),$

and so at least one of $\alpha_k\sigma(\alpha_k)$ and $\beta_k\sigma(\alpha_l)$ are in $\overline{S}_0^\times$. (Note that σ fixes the quantities in (2.3.1).)

Assume that $\alpha_k\sigma(\alpha_k) \in \overline{S}_0^\times$, i.e., $a_0 + a_1 = 0$. By the identity (2.3.1), $d_0 + d_1 = 0$. Hence, we have $a_0 = a_1 = d_0 = d_1 = 0$. Then, by (i) and (iv), $k_0 \equiv k_1 \bmod(e)$ and $l_0 \equiv l_1 \bmod(e)$, and we also have $r_0 = r_1$ and $s_0 = s_1$ by (a) and (d). But this is impossible since we assume that the Breuil submodules $\overline{S}m_k$ and $\overline{S}m_l$ correspond to characters of niveau 2. Hence, $\alpha_k\sigma(\alpha_k) \notin \overline{S}_0^\times$, i.e., either $\alpha_k\sigma(\alpha_k) = 0$ or $a_0 + a_1 > 0$.

Assume now that $\beta_k\sigma(\alpha_{pr}) \in \overline{S}_0^\times$, i.e., $b_0 + c_1 = 0 = b_1 + c_0$. Thus, $b_0 = b_1 = c_0 = c_1 = 0$. Then, by (ii) and (iii), $k_0 \equiv l_1 \bmod(e)$ and $k_1 \equiv l_0 \bmod(e)$, and we also have $r_0 = s_1$ and $r_1 = s_0$ by (b) and (c). We let $x = k_0$, $y = l_0$, $r_x = r_0$, and $r_y = s_0$. Then, by change of basis: $e_x = e_0 m_k + e_1 m_l$ and $e_y = e_1 m_k + e_0 m_l$, we get the description in the statement. $\quad\square$

The following lemma lets us specialize the result of Proposition 2.24 to a niveau 1 descent data:

Lemma 2.25. *For $i \in \{1, 2\}$, let $e_i \overset{def}{=} p^i - 1$, $K_i \overset{def}{=} \mathbb{Q}_{p^i}(\sqrt[e_i]{-p})$ and $\overline{S}_i \overset{def}{=} \mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^i}[u]/(u^{pe_i})$. Let $\iota : \overline{S}_1 \to \overline{S}_2$ be the morphism defined by the embedding $\mathbb{F}_p \hookrightarrow \mathbb{F}_{p^2}$ and $u \mapsto u^{p+1}$.*

If $\mathcal{M} \in \mathbb{F}\text{-BrMod}_{\mathrm{dd}}^s$ is a Breuil module of niveau one of niveau one type, then $\mathcal{M} \otimes_{\overline{S}_1, \iota} \overline{S}_2$ has a natural structure of a Breuil module of niveau 2 of niveau two type and the functor $\mathcal{M} \mapsto \mathcal{M} \otimes_{\overline{S}_1, \iota} \overline{S}_2$ is fully faithful. Moreover, one has $\mathrm{T}_{\mathrm{st}}^*(\mathcal{M}) \cong \mathrm{T}_{\mathrm{st}}^*(\mathcal{M} \otimes_{\overline{S}_1, \iota} \overline{S}_2)$

Proof. Just for the duration of this proof, let us write $\mathbb{F}\text{-BrMod}_{\mathrm{dd}}^{s,i}$ to denote the category of Breuil modules with $\mathbb{F}$-coefficients and descent data from K_i to $\mathbb{Q}_p$.

The exact sequence

$$1 \to \mathrm{Gal}(K_2/K_1) \to \mathrm{Gal}(K_2/\mathbb{Q}_p) \to \mathrm{Gal}(K_1/\mathbb{Q}_p) \to 0$$

shows that any object in $\mathbb{F}\text{-BrMod}_{\mathrm{dd}}^{s,i}$ is naturally endowed, by inflation, with a niveau two descent datum. In particular, the natural morphism $\overline{S}_1 \hookrightarrow \overline{S}_2$ factors through $(\overline{S}_2)^{\mathrm{Gal}(K_2/K_1)}$; by the explicit definition of the descent data action on $\overline{S}_2$, one checks that the previous factorization is indeed a isomorphism: $\overline{S}_1 \overset{\sim}{\longrightarrow} (\overline{S}_2)^{\mathrm{Gal}(K_2/K_1)}$.

Hence, by endowing $\mathcal{M} \otimes_{\overline{S}_1} \overline{S}_2$ with the diagonal residual action of $\mathrm{Gal}(K_2/\mathbb{Q}_p)$, we deduce that the natural morphism $\mathcal{M} \hookrightarrow \mathcal{M} \otimes_{\overline{S}_1} \overline{S}_2$ factors through a (functorial) isomorphism $\mathcal{M} \overset{\sim}{\longrightarrow} (\mathcal{M} \otimes_{\overline{S}_1} \overline{S}_2)^{\mathrm{Gal}(K_2/K_1)}$. It follows that the functor $\mathcal{M} \mapsto \mathcal{M} \otimes_{\overline{S}_1} \overline{S}_2$, defined on $\mathbb{F}\text{-BrMod}_{\mathrm{dd}}^{s,1}$ is fully faithful.

As for the last statement, we recall the functor $\mathrm{T}_{\mathrm{st}}^{*,i} : \mathbb{F}\text{-BrMod}_{\mathrm{dd}}^{s,i} \to \mathrm{Gal}_{\mathbb{F}}(G_{\mathbb{Q}_p})$ is defined by $\mathcal{M} \mapsto \mathrm{Hom}(\mathcal{M}, \widehat{A}_{K_i} \otimes_{\mathbb{F}_{p^i}} \mathbb{F})$, where $\widehat{A}_{K_i} = \left(\mathbb{F}_{p^i} \otimes \mathcal{O}_{\overline{\mathbb{Q}}_p}/p \right) \langle X \rangle$ is a certain a period ring described in [Car11], Section 2.1 (where is simply noted as $\widehat{A}$, as in *loc. cit.* the extension $\mathbb{F}_{p^i}/\mathbb{F}_p$ has been fixed).

More importantly, one has $\widehat{A}_{K_i} \cong \widehat{A}_{\mathrm{st}}/p \otimes_{\mathbb{F}_{p^i}\langle u \rangle} \mathbb{F}_{p^i}[u]/u^{e_i p}$ (cf. [HLM], Section A.2). By virtue of the fully faithfulness of $\mathcal{M} \mapsto \mathcal{M} \otimes_{\overline{S}_1} \overline{S}_2$, the last statement follows once we show that

$$\widehat{A}_{K_1} \otimes_{\overline{S}_1} \overline{S}_2 \to \widehat{A}_{K_2}$$

is an isomorphism, which can be verified by a direct computation on the definition of $\widehat{A}_{K_i}$. □

We deduce:

Corollary 2.26. *Let $e = p - 1$, $K = \mathbb{Q}_p(\sqrt[e]{-p})$, and $\overline{S} = \mathbb{F}[u]/u^{ep}$. We also let x and y be integers with $x \not\equiv y \mod (e)$, and let $\mathcal{M} \in \mathbb{F}\text{-BrMod}_{\mathrm{dd}}^s$ be a Breuil module of type $\overline{\tau} \simeq \omega^x \oplus \omega^y$ such that $\mathrm{T}_{\mathrm{st}}^s(\mathcal{M})$ is an absolutely irreducible 2-dimensional representation of $G_{\mathbb{Q}_p}$, i.e, $\mathrm{T}_{\mathrm{st}}^s(\mathcal{M})|_{I_{\mathbb{Q}_p}} \simeq \omega_2^{r+1} \oplus \omega_2^{p(r+1)}$. Assume further that $n(p+1) + (s+1) < r + 1 < (n+1)(p+1) - (s+1)$ for some $n \in \mathbb{Z}$.*

Then there exists a framed basis $\underline{e} = (e_x, e_y)$ for $\mathcal{M}$ and a framed system of generators $\underline{f} = (f_x, f_y)$ for $\mathrm{Fil}^2 \mathcal{M}$ such that

- $\mathrm{Mat}_{\underline{e}, \underline{f}}(\mathrm{Fil}^s \mathcal{M}) = \begin{pmatrix} 0 & u^{r_x} \\ u^{r_y} & 0 \end{pmatrix}$ *where $0 \leq r_x, r_y \leq es$ with $r_x \equiv y - x \mod (e)$ and $r_y \equiv x - y \mod (e)$;*

- $\mathrm{Mat}_{\underline{e}, \underline{f}}(\varphi_s) = \begin{pmatrix} \lambda_x & 0 \\ 0 & \lambda_y \end{pmatrix}$ *where $\lambda_x, \lambda_y \in \mathbb{F}^\times$;*

- $\mathrm{Mat}_{\underline{e}}(\hat{g}) = \begin{pmatrix} \omega^x(g) \otimes 1 & 0 \\ 0 & \omega^y(g) \otimes 1 \end{pmatrix}$ *for all* $g \in G(K/\mathbb{Q}_p)$;
- $N(e_x) = 0 = N(e_y)$.
- $\mathrm{T}^s_{\mathrm{st}}(\mathcal{M})|_{I_{\mathbb{Q}_p}} \simeq \omega_2^{(p+1)x + p\frac{pr_x + r_y}{e}} \oplus \omega_2^{(p+1)y + p\frac{pr_y + r_x}{e}}$.

Proof. Using the notation of Lemma 2.25, it suffices to apply Proposition 2.24 to $\mathcal{M} \otimes_{\overline{S}_1} \overline{S}_2$ and then take the $\mathrm{Gal}(K_2/K_1)$-fixed part. $\square$

2.4. Crystalline lifts.
We end this section with certain results for crystalline lifts of $\overline{\rho}_0$. The results in this subsection will be used in Section 6.5.

Proposition 2.27. *Let $\overline{\rho}_0$ be as in Definition 2.4. Then $\overline{\rho}_0$ admits a crystalline lift ρ : $G_{\mathbb{Q}_p} \to \mathrm{GL}_3(\mathbb{Q}_p)$ such that $\rho|_{G_{\mathbb{Q}_{p^2}}}$ is ordinary crystalline, with parallel Hodge-Tate weights $\{a_2 + 1, a_1 + 1, a_0 + 1\}$. In particular ρ is potentially diagonalizable.*

Moreover, if $\mathrm{FL}(\overline{\rho}_0) = [0 : 1]$ then $\overline{\rho}_0$ admits a crystalline lift with Hodge-Tate weights $\{p + a_0 + 1, a_2 + 1, a_1\}$.

Finally if $\overline{\rho}_0$ is split then then $\overline{\rho}_0$ admits further crystalline lift with Hodge-Tate weights $\{p + a_1, p + a_0, a_2 + 1\}$.

The proof of Proposition 2.27 will occupy the reminder of this section.

Let $\alpha, \beta \in \mathbb{Z}$. By [GS] Lemma 6.2, there is a crystalline character $\varepsilon_{(\alpha,\beta)} : G_{\mathbb{Q}_{p^2}} \to \mathcal{O}_E^\times$, unique up to unramified twist such that $\mathrm{HT}_{\sigma_0}(\varepsilon_{(\alpha,\beta)}) = \alpha$, $\mathrm{HT}_{\sigma_1}(\varepsilon_{(\alpha,\beta)}) = \beta$; such a character verifies moreover $\overline{\varepsilon}_{(\alpha,\beta)}|_{I_{\mathbb{Q}_p}} = \omega_2^{\alpha + p\beta}$. If $V_{(\alpha,\beta)} \overset{\mathrm{def}}{=} \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \varepsilon_{(\alpha,\beta)}$ then $V_{(\alpha,\beta)} \otimes_{\mathcal{O}_E} \mathbb{F} = \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{\alpha + p\beta}$ up to an unramified twist and we have the following particular case of [GHS], Corollary 7.1.3:

Lemma 2.28. *The representation $V_{(\alpha,\beta)}|_{G_{\mathbb{Q}_{p^2}}}$ is crystalline with parallel Hodge-Tate weights $\{\alpha, \beta\}$.*

Proof. Indeed, we have $V_{(\alpha,\beta)}|_{G_{\mathbb{Q}_{p^2}}} = \varepsilon_{(\alpha,\beta)} \oplus \varepsilon_{(\alpha,\beta)}^{(1)}$, where we have defined the $G_{\mathbb{Q}_{p^2}}$-character $\varepsilon_{(\alpha,\beta)}^{(1)}$ by $g \mapsto \varepsilon_{(\alpha,\beta)}(\mathrm{Frob}_p^{-1} \cdot g \cdot \mathrm{Frob}_p)$ where Frob_p denotes a geometric Frobenius. By [GHS], Lemma 7.1.2 we have that $\mathrm{HT}_{\sigma_0}(\varepsilon_{(\alpha,\beta)}^{(1)}) = \beta$, $\mathrm{HT}_{\sigma_1}(\varepsilon_{(\alpha,\beta)}^{(1)}) = \alpha$. The representation $V_{(\alpha,\beta)}|_{G_{\mathbb{Q}_{p^2}}}$ is crystalline, as crystalline property is insensitive to unramified base change. $\square$

If $\gamma \in \mathbb{Z}$ we define the space of $\mathcal{O}_E$-valued crystalline extensions $\mathrm{Ext}^1_{\mathcal{O}_E[G_{\mathbb{Q}_p}],\mathrm{cris}}(V_{(\alpha,\beta)}, \varepsilon_p^\gamma)$ as the inverse image (under base change $\mathcal{O}_E \to E$) of $\mathrm{Ext}^1_{E[G_{\mathbb{Q}_p}],\mathrm{cris}}(V_{(\alpha,\beta)} \otimes_{\mathcal{O}_E} E, \varepsilon_p^\gamma \otimes_{\mathcal{O}_E} E)$.

By an immediate application of Hochschild-Serre spectral sequence and since the crystalline condition is insensitive with respect to restriction to unramified base change, we have

the following commutative diagram:

$$(2.4.1) \qquad \mathrm{Ext}^1_{\mathcal{O}_E[G_{\mathbb{Q}_p}],\mathrm{cris}}(V_{(\alpha,\beta)},\varepsilon_p^\gamma) \xrightarrow{\ \sim\ } \left(\mathrm{Ext}^1_{\mathcal{O}_E[G_{\mathbb{Q}_{p^2}}],\mathrm{cris}}(\varepsilon_{(\alpha,\beta)}\oplus\varepsilon_{(\alpha,\beta)}^{(1)},\varepsilon_{(\gamma,\gamma)})\right)^{G_2}$$

$$\mathrm{Ext}^1_{\mathcal{O}_E[G_{\mathbb{Q}_p}]}(V_{(\alpha,\beta)},\varepsilon_p^\gamma) \xrightarrow{\ \sim\ } \left(\mathrm{Ext}^1_{\mathcal{O}_E[G_{\mathbb{Q}_{p^2}}]}(\varepsilon_{(\alpha,\beta)}\oplus\varepsilon_{(\alpha,\beta)}^{(1)},\varepsilon_{(\gamma,\gamma)})\right)^{G_2}$$

$$\mathrm{Ext}^1_{\mathbb{F}[G_{\mathbb{Q}_p}]}(\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}\omega_2^{\alpha+p\beta},\omega^\gamma) \xrightarrow{\ \sim\ } \left(\mathrm{Ext}^1_{\mathbb{F}[G_{\mathbb{Q}_{p^2}}]}(\omega_2^{\alpha+p\beta}\oplus\omega_2^{\beta+p\alpha},\omega_2^{(p+1)\gamma})\right)^{G_2}$$

where the bottom vertical arrows are the mod ϖ_E-reduction maps and $G_2 \overset{\mathrm{def}}{=} \mathrm{Gal}(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$.

The following technical lemma is a simple manipulation with Fontaine-Laffaille modules. In its statement, we set $e_0 \overset{\mathrm{def}}{=} e_{\sigma_0}$, $e_1 \overset{\mathrm{def}}{=} e_{\sigma_0\circ\mathrm{Frob}_p}$ for the standard idempotent elements of $\mathbb{F}_{p^2}\otimes_{\mathbb{F}_p}\mathbb{F}$, following the notation of Section 1.1.

Lemma 2.29. *Let $M \in \mathbb{F}\text{-}\mathcal{FL}^{[0,p-2]}$ be a Fontaine-Laffaille module over $\mathbb{F}_p\otimes_{\mathbb{F}_p}\mathbb{F}$, with Hodge-Tate weghts (β,α,γ). Assume that*

$$(2.4.2) \qquad\qquad \mathrm{Mat}_{\underline{f}}(\phi_\bullet) = \begin{pmatrix} 0 & \lambda_1 & x \\ \lambda_0 & 0 & y \\ 0 & 0 & \lambda_2 \end{pmatrix}$$

in a basis $\underline{f} = (f_0,\,f_1,\,f_2)$ which is compatible with the Hodge filtration on M. Then if we write M' for the induced Breuil module $\mathbb{F}_{p^2}\otimes_{\mathbb{F}_p}M$, we have two Fontaine-Laffaille quotients $M' \twoheadrightarrow N$, $M' \twoheadrightarrow N^{(1)}$ of rank two over $\mathbb{F}_{p^2}\otimes_{\mathbb{F}_p}\mathbb{F}$. Explicitly, we have $N = Ne_0 \oplus Ne_1$ where Ne_i are $\mathbb{F}$-linear spaces, with Hodge-Tate weights (α,γ) and (β,γ) for $i=0$ and $i=1$ respectively, and

$$\mathrm{Mat}(Ne_1 \overset{\phi_0}{\to} Ne_0) = \begin{pmatrix} \lambda_0 & y \\ 0 & \lambda_2 \end{pmatrix} \quad \& \quad \mathrm{Mat}(Ne_0 \overset{\phi_1}{\to} Ne_1) = \begin{pmatrix} \lambda_1 & x \\ 0 & \lambda_2 \end{pmatrix}$$

We have a similar description for $N^{(1)} = N^{(1)}e_0 \oplus N^{(1)}e_1$:

$$\mathrm{Mat}(N^{(1)}e_1 \overset{\phi_0}{\to} N^{(1)}e_0) = \begin{pmatrix} \lambda_1 & x \\ 0 & \lambda_2 \end{pmatrix} \quad \& \quad \mathrm{Mat}(N^{(1)}e_0 \overset{\phi_1}{\to} N^{(1)}e_1) = \begin{pmatrix} \lambda_0 & y \\ 0 & \lambda_2 \end{pmatrix}$$

and $N^{(1)}e_0$, $N^{(1)}e_1$ have Hodge-Tate weights (β,γ), (α,γ) respectively.

Proof. This is elementary. Let $\underline{f} = (f_0,f_1,f_2)$ be a basis on M, compatible with the Hodge filtration, such that the matrix of the Frobenius on M is given by (2.4.2). In particular, we have

$$\mathrm{Fil}^{i+1}M = \begin{cases} M & \text{if} & i < \beta \\ \langle f_1,\ f_2\rangle_{\mathbb{F}} & \text{if} & \beta \le i < \alpha \\ \langle f_2\rangle_{\mathbb{F}} & \text{if} & \alpha \le i < \gamma \\ 0 & \text{if} & i \ge \gamma \end{cases}$$

Then, considering the change of basis we get

$$1 \otimes \underline{f} \overset{\mathrm{def}}{=} (1\otimes f_0, 1\otimes f_1, 1\otimes f_2)\cdot \begin{pmatrix} e_0 & e_1 & 0 \\ e_1 & e_0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we obtain

$$\mathrm{Mat}_{1\otimes \underline{f}}(\phi_\bullet) = \begin{pmatrix} \lambda_1 e_0 + \lambda_0 e_1 & 0 & xe_0 + ye_1 \\ 0 & \lambda_0 e_0 + \lambda_1 e_1 & ye_0 + xe_1 \\ 0 & 0 & \lambda_2 \end{pmatrix}.$$

We define N to be the Fontaine-Laffaille quotient characterized by

$$\ker(M' \twoheadrightarrow N) = \langle (1\otimes f_0)\cdot e_0 + (1\otimes f_1)\cdot e_1\rangle.$$

This is well-defined since the kernel is a rank one submodule. Note that, by construction, we have

$$\mathrm{Fil}^{i+1} Ne_0 \stackrel{\mathrm{def}}{=} \frac{\mathrm{Fil}^{i+1} M'e_0 + \langle (1\otimes f_0)\cdot e_0\rangle}{\langle (1\otimes f_0)\cdot e_0\rangle} = \begin{cases} \frac{M'e_0}{\langle (1\otimes f_0)\cdot e_0\rangle} = Ne_0 & \text{if} \quad i < \alpha \\[2mm] \frac{\langle (1\otimes f_2)\cdot e_0, (1\otimes f_0)\cdot e_0\rangle}{\langle (1\otimes f_0)\cdot e_0\rangle} & \text{if} \quad \alpha \le i < \gamma \\[2mm] 0 & \text{if} \quad i \ge \gamma \end{cases}$$

$$\mathrm{Fil}^{i+1} Ne_1 \stackrel{\mathrm{def}}{=} \frac{\mathrm{Fil}^{i+1} M'e_1 + \langle (1\otimes f_1)\cdot e_1\rangle}{\langle (1\otimes f_1)\cdot e_1\rangle} = \begin{cases} \frac{M'e_1}{\langle (1\otimes f_1)\cdot e_1\rangle} = Ne_1 & \text{if} \quad i < \beta \\[2mm] \frac{\langle (1\otimes f_1)\cdot e_1, (1\otimes f_2)\cdot e_1\rangle}{\langle (1\otimes f_1)\cdot e_1\rangle} & \text{if} \quad \beta \le i < \gamma \\[2mm] 0 & \text{if} \quad i \ge \gamma. \end{cases}$$

Hence, N has Hodge-Tate weights $\mathrm{HT}_{\sigma_0} = \{\alpha, \gamma\}$ and $\mathrm{HT}_{\sigma_0 \circ \mathrm{Frob}_p} = \{\beta, \gamma\}$.

Similarly, one takes $N^{(1)}$ to be the Fontaine-Laffaille quotient of M characterized by

$$\ker(M' \twoheadrightarrow N^{(1)}) = \langle (1\otimes f_0)\cdot e_1 + (1\otimes f_1)\cdot e_0\rangle.$$

This is well-defined by the same reason as N. $\qquad\square$

We deduce from Lemma 2.29:

Lemma 2.30. *Assume that $\overline{\rho}_0$ is as in Definition 2.4. Let $M \in \mathbb{F}\text{-}\mathcal{FL}^{[0,p-2]}$ be the associated Fontaine-Laffaille module and fix a basis on it in such a way that $\mathrm{Mat}_{\underline{f}}(\phi_\bullet)$ has the form (2.1.3), with moreover $z = 0$. Let τ be the image of $\overline{\rho}_0|_{G_{\mathbb{Q}_{p^2}}}$ via the projection map*

$$(2.4.3) \qquad \mathrm{Ext}^1_{\mathbb{F}[G_{\mathbb{Q}_{p^2}}]}\left(\omega_2^{(a_1+1)+p(a_0+1)} \oplus \omega_2^{(a_0+1)+p(a_1+1)}, \; \omega_2^{(p+1)(a_2+1)}\right)$$

$$\downarrow$$

$$\mathrm{Ext}^1_{\mathbb{F}[G_{\mathbb{Q}_{p^2}}]}\left(\omega_2^{(a_1+1)+p(a_0+1)}, \; \omega_2^{(p+1)(a_2+1)}\right).$$

Then τ has a crystalline lift with Hodge-Tate weights $\mathrm{HT}_{\sigma_0} = \{a_2 + 1, a_1 + 1\}$, $\mathrm{HT}_{\sigma_1} = \{a_2 + 1, a_0 + 1\}$.

If moreover $\mathrm{FL}(\overline{\rho}_0) = [0:1]$ then τ has also a crystalline lift with Hodge-Tate weights $\mathrm{HT}_{\sigma_0} = \{a_2 + 1, a_1\}$, $\mathrm{HT}_{\sigma_1} = \{p + a_0 + 1, a_2 + 1\}$.

If finally $\overline{\rho}_0$ is split then τ admits further a crystalline lift with the following Hodge-Tate weights $\mathrm{HT}_{\sigma_0} = \{p + a_1, a_2 + 1\}$, $\mathrm{HT}_{\sigma_1} = \{p + a_0, a_2 + 1\}$.

Proof. We can assume that $a_0 = -1$ and set $c \stackrel{\mathrm{def}}{=} a_2 - a_0 - 1$, $r \stackrel{\mathrm{def}}{=} a_1 - a_0 - 1$.

By Lemma 2.29 we see that the Fontaine-Laffaille module $N = Ne_0 + Ne_1$ associated to τ has Hodge-Tate weights $\mathrm{HT}_{\sigma_0}(Ne_0) = \{r+1, c+1\}$, $\mathrm{HT}_{\sigma_1}(Ne_1) = \{0, c+1\}$ and Frobenius described by

$$(2.4.4) \qquad \mathrm{Mat}(Ne_1 \overset{\phi_0}{\to} Ne_0) = \begin{pmatrix} \mu_0^{-1} & y \\ 0 & \mu_2^{-1} \end{pmatrix}$$

$$(2.4.5) \qquad \mathrm{Mat}(Ne_0 \overset{\phi_1}{\to} Ne_1) = \begin{pmatrix} \mu_1^{-1} & x \\ 0 & \mu_2^{-1} \end{pmatrix}$$

We now use the explicit description of the set of modular weights for τ, given in [Bre14] pag. 26. Following the notation in *loc. cit.* we deduce from (2.4.4) that the weight $(c - r - 1, c) \otimes \det^{r+1}$ (which would be written as $\sigma_{(c,c),(r+1,0)}$ in the notation of [GLS], Definition 4.1.1) is always modular, while the weight $(c - r, p - 2 - c) \otimes \det^{r+p(c+1)}$ (i.e. $\sigma_{(c,p-1),(r,c+1)}$ in the notation of [GLS]) is modular when $x = 0$. For sake of completeness, the weight $(p - 2 - c + r, p - 3 - c) \otimes \det^{c+1+p(c+1)}$ i.e. $\sigma_{(p-1+r,p-2),(c+1,c+1)}$ in the notation of [GLS] is modular when $x = y = 0$. We now can globalize τ: by [GK], Corollary A.3 there is a totally real field F^+ such that $F_v^+ \cong \mathbb{Q}_p$ for all places $v|p$, and a RAESDC automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_{F^+})$ such that the mod p reduction of the associated p-adic Galois representation $\bar{r}_{p,\imath}(\pi) : G_{F^+} \to \mathrm{GL}_2(\mathbb{F})$ (cf. [BLGGT] §2.1) is absolutely irreducible (modular) and verifies $\bar{r}_{p,\imath}(\pi)|_{G_{F_v^+}} \cong \tau$ for all places $v|p$. The conclusion follows from [GLS], Theorem A. $\qquad\square$

Proof of Proposition 2.27. The existence of the crystalline lifts as in the statement of Proposition 2.27 follows now from Lemma 2.30 and the diagram (2.4.1). More precisely, let $\tau \oplus \tau^{(1)}$ be the image of $\bar{\rho}_0$ in $\mathrm{Ext}^1_{\mathbb{F}[G_{\mathbb{Q}_{p^2}}]}\left(\omega_2^{(a_1+1)+p(a_0+1)}, \omega_2^{(p+1)(a_2+1)}\right)$ via the isomorphism in the bottom line of the diagram (2.4.1). By Lemma 2.30, τ admits a crystalline lift $\hat{\tau} : G_{\mathbb{Q}_{p^2}} \to \mathrm{GL}_2(\mathcal{O}_E)$ with Hodge-Tate weights $\mathrm{HT}_{\sigma_0} = \{\alpha, \gamma\}$, $\mathrm{HT}_{\sigma_1} = \{\beta, \gamma\}$ where the integers α, β, γ are suitably specialized according to $\bar{\rho}_0$ (e.g. specialized at $\alpha = a_1 + 1$, $\beta = a_0 + 1$, $\gamma = a_2 + 1$ for the first case of Proposition 2.27). By letting $\hat{\tau}^{(1)} : G_{\mathbb{Q}_{p^2}} \to \mathrm{GL}_2(\mathcal{O}_E)$ be defined by $\hat{\tau}^{(1)}(g) \overset{\mathrm{def}}{=} \hat{\tau}(\mathrm{Frob}_p^{-1} g \,\mathrm{Frob}_p)$ we see that $\hat{\tau}^{(1)}$ is a crystalline lift of $\tau^{(1)}$ with Hodge-Tate weights $\mathrm{HT}_{\sigma_1} = \{\alpha, \gamma\}$, $\mathrm{HT}_{\sigma_0} = \{\beta, \gamma\}$. By construction $\hat{\tau} \oplus \hat{\tau}^{(1)} \in \mathrm{Ext}^1_{\mathcal{O}_E[G_{\mathbb{Q}_{p^2}}],\mathrm{cris}}(\varepsilon_{(\alpha,\beta)} \oplus \varepsilon_{(\alpha,\beta)}^{(1)}, \varepsilon_{(\gamma,\gamma)})$ is fixed under the G_2-action on the Ext^1-space. Its inverse image via the isomorphism in the first line of the diagram (2.4.1) provides the required crystalline lift.

Moreover, any element of $\mathrm{Ext}^1_{E[G_{\mathbb{Q}_p}],\mathrm{cris}}(V_{(\alpha,\beta)} \otimes_{\mathcal{O}_E} E, \varepsilon_p^\gamma)$ becomes ordinary when restricted to $G_{\mathbb{Q}_{p^2}}$, as it can be directly checked on the associated filtered ϕ-module. $\qquad\square$

Remark 2.31. The existence of the crystalline lift for $\bar{\rho}_0$ with Hodge-Tate weights $\{a_2 + a_1 + 1, a_0 + 1\}$ can be obtained without invoking [GLS]. Indeed if $(\alpha_0, \ldots, \alpha_{f-1}), (\alpha_0', \ldots, \alpha_{f-1}') \in \mathbb{Z}^f$ are such that $\alpha_i - \alpha_i' > 1$ for all i, then we are in the setting of [Nak], Lemma 4.2(1) and Lemma 4.3(3), so that

$$\dim_E \left(\mathrm{Ext}^1_{E[G_{\mathbb{Q}_{p^f}}],\mathrm{cris}}(\varepsilon_{(\alpha_0,\ldots,\alpha_{f-1})}, \varepsilon_{(\alpha_0',\ldots,\alpha_{f-1}')}) \right) = f$$

(cf. also *loc. cit.*, Definition 2.4 and Remark 2.5). On the other hand, under the previous hypotheses on $\alpha_i - \alpha_i'$, we have also $\dim_E \left(\mathrm{Ext}^1_{E[G_{\mathbb{Q}_{p^f}}]}(\varepsilon_{(\alpha_0,\ldots,\alpha_{f-1})}, \varepsilon_{(\alpha_0',\ldots,\alpha_{f-1}')}) \right) = f$ ([Nak], Proposition 2.15).

3. Elimination of Galois types

The aim of this section is to perform elimination of Galois types for a niveau 2, generic representation $\overline{\rho}_0 : G_{\mathbb{Q}_p} \to \mathrm{GL}_3(\mathbb{F})$ (cf. Definition 2.4), by means of integral p-adic Hodge theory.

For $K' \in \{\mathbb{Q}_p, K_0\}$ we recall the category $\mathrm{Mod}_E^{\mathrm{w.a.}}(\varphi, N, K/K')$ of weakly admissible filtered $(\varphi, N, K/K', E)$-modules (see e.g. [EGH13], Section 3.1). We have a contravariant equivalence of categories $\mathrm{D}_{\mathrm{st}}^{*,K'} : \mathrm{Rep}_E^{K\text{-st}}(G_{K'}) \to \mathrm{Mod}_E^{\mathrm{w.a.}}(\varphi, N, K/K')$, where $\mathrm{Rep}_E^{K\text{-st}}(G_{K'})$ denotes the category of finite dimensional E-representations of $G_{K'}$ that become semistable over K. If $\rho \in \mathrm{Rep}_E^{K\text{-cris}}(G_{K'})$ has Hodge-Tate weights in $\{-r, 0\}$, we define $\mathrm{D}_{\mathrm{st}}^{K',r}(\rho) \overset{\mathrm{def}}{=} \mathrm{D}_{\mathrm{st}}^{*,K'}(\rho^\vee \otimes \varepsilon_p^r)$.

The following result will be particularly useful to us:

Proposition 3.1. *Let $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_3(\mathcal{O}_E)$ be a potentially semistable Galois representation, becoming crystalline over K in Hodge-Tate weights in $\{-r, 0\}$. Let $\widehat{\mathcal{M}}$ be a strongly divisible $\mathcal{O}_E$-module in $\mathcal{O}_E\text{-}\mathrm{Mod}_{\mathrm{dd}}^r$ such that $\mathrm{T}_{\mathrm{st}}^{\mathbb{Q}_p,r}(\widehat{\mathcal{M}}) \otimes_{\mathcal{O}_E} E \cong \rho$.*

Then $\mathrm{D}_{\mathrm{st}}^{\mathbb{Q}_p,r}(\rho) \cong \widehat{\mathcal{M}}[\frac{1}{p}] \otimes_{S_{\mathbb{Q}_p}, s_0} \mathbb{Q}_p$ and and $\widehat{\mathcal{M}}$ has inertial type $\mathrm{WD}(\rho \otimes \varepsilon_p^{-r})|_{I_{\mathbb{Q}_p}} = \mathrm{WD}(\rho)|_{I_{\mathbb{Q}_p}}$ (where $s_0 : S_{\mathbb{Q}_p} \to \mathbb{Q}_p$ is the morphism defined by "$u \mapsto 0$").

Proof. The isomorphism $\mathrm{D}_{\mathrm{st}}^{\mathbb{Q}_p,r}(\rho) \cong \widehat{\mathcal{M}}[\frac{1}{p}] \otimes_{S_{\mathbb{Q}_p}, s_0} \mathbb{Q}_p$ is proved in [EGH13], proof of Proposition 3.1.4.

As for the second part of the proposition, let us write $\mathrm{WD}(\rho)|_{I_{\mathbb{Q}_p}} \cong \chi_1 \oplus \cdots \oplus \chi_n$ for the inertial type associated to ρ.

By definition of type on a strongly divisible lattice $\widehat{\mathcal{M}}$, we have to prove that there exists a basis $(\widehat{e}_1, \ldots, \widehat{e}_n)$ of $\mathcal{M}$ such that $\widehat{g} \cdot \widehat{e}_i = 1 \otimes \chi_i(g) \widehat{e}_i$ for all $g \in \mathrm{Gal}(K/K_0)$ and $i = 1, \ldots, n$.

For $r = 1$ this is proved in [GS11], Proposition 5.1 (note that the functors $\mathcal{M} \mapsto \mathrm{T}_{\mathrm{st}}^{\mathbb{Q}_p,r}(\widehat{\mathcal{M}})$, $\rho \mapsto \mathrm{D}_{\mathrm{st}}^{\mathbb{Q}_p,r}(\rho)$ would be written as $\mathrm{T}_{\mathrm{st},r+1}^{\mathbb{Q}_p}$, $\mathrm{D}_{\mathrm{st},r+1}^{\mathbb{Q}_p}(\rho)$ in *loc. cit.*). But the proof in *loc. cit.* generalizes *verbatim* for higher Hodge-Tate weights. See also [EGH13], Proof of Proposition 3.3.1. $\square$

Recall that the restriction functor $\overline{\rho}_0 \mapsto \overline{\rho}_0|_{G_{K_0}}$ is not full. The following elementary lemma shows that in our situation, the Fontaine-Laffaille invariant $\mathrm{FL}(\overline{\rho}_0)$ can be deduced from $\overline{\rho}_0|_{G_{K_0}}$.

Lemma 3.2. *Let $\overline{\rho}_0$ be as in Definition 2.4 and let $F \in \mathrm{GL}_3(\mathbb{F})$ be the matrix describing the Frobenius action on the associated Fontaine-Laffaille module as in (2.1.3).*

Assume that the Fontaine-Laffaille module M' associated to $\overline{\rho}_0|_{G_{K_0}}$ has parallel Hodge-Tate weights $\{0, r+1, c+1\}$ and Frobenius action described by

$$F' \overset{\mathrm{def}}{=} \begin{pmatrix} 0 & \lambda_1 & X \\ \lambda_0 & 0 & Y \\ 0 & 0 & \lambda_2 \end{pmatrix} \in \mathrm{GL}_3(k \otimes_{\mathbb{F}_p} \mathbb{F}).$$

Then $X = 0$ if and only if $x = 0$, and $Y = 0$ if and only if $y = 0$.

Proof. In the given hypotheses, we have an isomorphism of Fontaine-Laffaille modules (in parallel Hodge-Tate weights $\{0, r+1, c+1\}$) over $k \otimes_{\mathbb{F}_p} \mathbb{F}$. This means that there exists a lower triangular matrix $B \in \mathbf{B}^{\mathrm{opp}}(k \otimes_{\mathbb{F}_p} \mathbb{F})$ such that

$$(3.0.6) \qquad\qquad B \cdot F' \cdot (\varphi \otimes 1)(\mathrm{gr}(B)) = F \otimes_{\mathbb{F}_p} k,$$

where $\mathrm{gr}(B) \in \mathbf{T}(k \otimes_{\mathbb{F}_p} \mathbb{F})$ is defined by $(\mathrm{gr}(B))_{ii} = (B)_{ii}$ for $i = 0, 1, 2$ and $\varphi \otimes 1$ denotes the induced Frobenius automorphism on $k \otimes_{\mathbb{F}_p} \mathbb{F}$.

By an immediate computation we deduce that condition (3.0.6) forces B to be diagonal. In particular, there exists units α, β, $\gamma \in k \otimes_{\mathbb{F}_p} \mathbb{F}$ such that $1 \otimes x = \alpha\sigma(\gamma)X$ and $\beta\sigma(\gamma)Y$.

As the natural morphism $\mathbb{F} \to k \otimes_{\mathbb{F}_p} \mathbb{F}$ is injective, the result follows. $\square$

For the reminder of this section, we assume that $a_0 = -1$ and define $c \overset{\text{def}}{=} a_2 - a_0 + 1$, $r \overset{\text{def}}{=} a_1 - a_0 - 1$ (it is always possible to reduce to this case by twisting by $\omega^{-(a_0+1)}$).

3.1. **Elimination of Galois types of niveau** 1.
We start this subsection by recalling the following (cf. [MP], Lemma 3.3): let i, j, k be integers , and let ρ be a potentially crystalline representation with Hodge–Tate weights $\{-2, -1, 0\}$ and of inertial type $\widetilde{\omega}^i \oplus \widetilde{\omega}^j \oplus \widetilde{\omega}^k$ such that $\overline{\rho}_0^{ss} \simeq \overline{\rho}^{ss}$. Then we have the identity

$$(3.1.1) \qquad \omega^{3+i+j+k} = \det \overline{\rho}|_{I_{\mathbb{Q}_p}} = \omega^{(r+1)+(c+1)}.$$

In this subsection, we fix $e = p - 1$ and $K = \mathbb{Q}_p(\sqrt[e]{-p})$. We also let $\overline{S} = \mathbb{F}[u]/u^{ep}$ and $\overline{S}_0 = \mathbb{F}[u^e]/u^{ep}$. Recall that by $[m]_1$ for an integer m we mean the unique integer in $[0, e)$ congruent to $m \mod (e)$.

Proposition 3.3. *Let* $\mathcal{M} \in \mathbb{F}\text{-BrMod}_{\mathrm{dd}}^2$ *be a Breuil module of type* $\tau \cong \omega^z \oplus \omega^x \oplus \omega^y$ *such that* $\mathrm{T}_{\mathrm{st}}^2(\mathcal{M})^{ss} \cong \overline{\rho}_0^{ss}$ *and* $\overline{\rho}_2 \subset \mathrm{T}_{\mathrm{st}}^2(\mathcal{M})$, *where* $\overline{\rho}_2$ *is the one-dimensional subrepresentation of* $\overline{\rho}_0$. *Assume moreover that the submodule corresponding to* $\overline{\rho}_2$ *is of type* ω^z.

Then there exists a framed basis $\underline{e} = (e_z, e_x, e_y)$ *and a framed system of generators* $\underline{f}$ *for* $\mathrm{Fil}^2 \mathcal{M}$ *such that*

$$(3.1.2) \qquad \mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^2 \mathcal{M}) = \begin{pmatrix} u^{se} & u^{[x-z]_1} \cdot v_y & u^{[y-z]_1} \cdot v_x \\ 0 & 0 & u^{r_x} \\ 0 & u^{r_y} & 0 \end{pmatrix};$$

$$(3.1.3) \qquad \mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_2) = \begin{pmatrix} \alpha_z & u^{[x-z]_1} \cdot \eta_x & u^{[y-z]_1} \cdot \eta_y \\ 0 & \alpha_x & 0 \\ 0 & 0 & \alpha_y \end{pmatrix},$$

where $\alpha_x, \alpha_y, \alpha_z \in \mathbb{F}^\times$, $v_x, v_y, \eta_x, \eta_y \in \overline{S}_0$, *and the tuple* (x, y, z, r_x, r_y, s) *satisfies one of the following properties:*

(a): $x \equiv r + 1 - m_0 \mod (p - 1)$, $y \equiv 0 \mod (p - 1)$, $z \equiv c + 1 - m_2 \mod (p - 1)$, *and*

$$\begin{cases} r_x = (p-1)m_0 - (r + 1 - m_0); \\ r_y = r + 1 - m_0; \\ s = m_2, \end{cases}$$

where $m_0, m_2 \in \{1, 2\}$ *satisfy* $m_0 + m_2 = 3$;

(b): $x \equiv r - m_0 \mod (p - 1)$, $y \equiv p - 2 \mod (p - 1)$, $z \equiv c + 1 - m_2 \mod (p - 1)$, *and*

$$\begin{cases} r_x = (p-1)(m_0 + 1) - (r + 1 - m_0); \\ r_y = (p-1) + (r + 1 - m_0); \\ s = m_2, \end{cases}$$

where $m_0, m_2 \in \{0, 1\}$ *satisfy* $m_0 + m_2 = 1$.

Proof. Since $\overline{\rho}_0$ is an extension of a two-dimensional irreducible representation by a character of niveau 1, $\mathcal{M}$ is also an extension of a simple Breuil module of rank 2 by a Breuil module of rank 1 by Proposition 2.22. Hence, it is immediate that the filtration and the Frobenius map φ_2 of $\mathcal{M}$ are described as in (3.1.2) and (3.1.3) respectively, by using the classification of simple Breuil modules of rank 2 in Corollary 2.26 and the classification of simple Breuil modules of rank 1 in [MP], Lemma 3.1.

By Corollary 2.26 we have $r_x \equiv y - x \bmod e$ and $r_y \equiv x - y \bmod e$, $r_x + r_y \equiv 0 \bmod e$. We let $r_x + r_y = ae$ for $a \in \{0, 1, 2, 3, 4\}$. Again by Corollary 2.26, we have

$$(3.1.4) \qquad \begin{cases} (p+1)x + pr_x + pa \equiv r + 1 \bmod (p^2 - 1); \\ (p+1)y + pr_y + pa \equiv p(r+1) \bmod (p^2 - 1); \\ z + s \equiv c + 1 \bmod (p - 1). \end{cases}$$

By the determinant condition (3.1.1), $3(p+1) + (r+1) - p(r_x + a) + p(r+1) - p(r_y + a) + (p+1)(c+1-s) \equiv (p+1)(c+1+r+1) \bmod (p^2 - 1)$. Hence, we get $a + s = 3$, and so $a \in \{1, 2, 3\}$ since $s \in \{0, 1, 2\}$.

Via the equations (3.1.4) we now write r_y in terms of a and the inertial weights z, x, y. We have $(p+1)r_y \equiv (p+1)(x-y) \equiv (1-p)(r+1) - p(r_x - r_y) \bmod (p^2 - 1)$. So $r_y \equiv -(p-1)(r+1) - p(ae - r_y) \bmod (p^2 - 1)$. Solving this for r_y, we get $r_y \equiv r + 1 - a \bmod (p+1)$. We let $r_y = r + 1 - a + \epsilon(p+1)$ for $\epsilon \in \{0, 1\}$ (since $0 \le r_y \le 2e$). Then $r_x = ae - (r+1-a) - \epsilon(p+1)$. Moreover, by the equations (3.1.4), we also have $x \equiv r+1-a+\epsilon$ mod e and $y \equiv e - \epsilon$ mod e. We let $s = m_2$. Then we have $a + m_2 = 3$.

Assume that $\epsilon = 0$. If $m_2 = 0$, then $a = 3$, and so $r_x = 3e - (r+1-3) > 2e$, which contradicts to $r_x \in [0, 2e]$. Hence, $a, m_2 \in \{1, 2\}$ and this gives rise to the case (**a**), letting $m_0 = a$.

Assume that $\epsilon = 1$. If $m_2 = 2$, then $a = 1$, and so $r_x = (p-1) - r - (p+1) < 0$, which contradicts to $r_x \in [0, 2e]$. Hence, $m_2 \in \{0, 1\}$ and $a \in \{2, 3\}$. Letting $m_0 = a - 2$, this gives rise to the case (**b**). $\qquad \square$

Lemma 3.4. *Keep the notation as in Proposition 3.3 (in particular, recall the elements v_x and v_y in the matrix (3.1.2)) and let $s = 1$.*

  (i) *If $r_x - [y-z]_1 > e$ then there is a framed basis for which $v_x = 0$.*
  (ii) *If $r_y - [x-z]_1 > e$ then there is a framed basis for which $v_y = 0$.*

Proof. Since $s = 1$, we may assume that $v_x, v_y \in \mathbb{F}$. We only give a proof for (i), but one can prove (ii) by the same argument.

Assume that $v_x \ne 0$. Then the matrix (3.1.2) is column-equivalent to

$$\begin{pmatrix} 0 & u^{[x-z]_1} \cdot v_y & u^{[y-z]_1} \cdot v_x \\ u^{r_x + e - [y-z]_1} & 0 & u^{r_x} \\ 0 & u^{r_y} & 0 \end{pmatrix},$$

which implies that

$$\mathrm{Fil}^2 \mathcal{M} \otimes_{\overline{S}} \overline{S}/u \cong \omega^x \oplus \omega^x \oplus \omega^y,$$

since $r_x + e - [y-z]_1 > 2e$. But this is impossible unless $x \equiv z \bmod (p-1)$. $\qquad \square$

Lemma 3.5. *Keep the notation as in Proposition 3.3. If*

$$p([y-z]_1 + r_y - se) > [x-z]_1 \quad \text{and} \quad p([x-z]_1 + r_x - se) > [y-z]_1$$

then there is a framed basis such that $\eta_x = 0 = \eta_y$ in the matrix (3.1.3). Moreover, this change of basis does not affect the vanishing of v_x and v_y.

Proof. We let V_0 be the matrix in (3.1.2) and A_0 the matrix in (3.1.3). We also let

$$V_1 = \begin{pmatrix} u^{se} & u^{[x-z]_1} \cdot v'_y & u^{[y-z]_1} \cdot v'_x \\ 0 & 0 & u^{r_x} \\ 0 & u^{r_y} & 0 \end{pmatrix}$$

and

$$B_1 = \begin{pmatrix} \alpha_z & u^{[x-z]_1} \cdot \eta'_x & u^{[y-z]_1} \cdot \eta'_y \\ 0 & \alpha_y & 0 \\ 0 & 0 & \alpha_x \end{pmatrix}.$$

One can easily check that the equation

(3.1.5) $A_0 V_1 = V_0 B_1$

holds if and only if the following two equalities hold:

$$\alpha_z u^{[x-z]_1} v'_y + u^{[y-z]_1 + r_y} \eta_y = u^{se+[x-z]_1} \eta'_x + \alpha_y u^{[x-z]_1} v_y;$$

$$\alpha_z u^{[y-z]_1} v'_x + u^{[x-z]_1 + r_x} \eta_x = u^{se+[y-z]_1} \eta'_y + \alpha_x u^{[y-z]_1} v_x.$$

Hence, the equation (3.1.5) holds true if we let $v'_x = \alpha_x \alpha_z^{-1} v_x$, $v'_y = \alpha_y \alpha_z^{-1} v_y$,

$$u^{[x-z]_1} \eta'_x = u^{[y-z]_1 + r_y - se} \eta_y \in \overline{S}, \text{ and } u^{[y-z]_1} \eta'_y = u^{[x-z]_1 + r_x - se} \eta_x \in \overline{S}.$$

Note that our assumption implies that $[x-z]_1 + r_x - se \geq 0$ and $[y-z]_1 + r_y - se \geq 0$. Now let us consider the new basis $\underline{e}' \stackrel{\text{def}}{=} \underline{e} A_0$. Then $V_1 = \text{Mat}_{\underline{e}', \underline{f}'}(\text{Fil}^2 \mathcal{M})$ and $A_1 \stackrel{\text{def}}{=} \varphi(B_1) = \text{Mat}_{\underline{e}', \underline{f}'}(\varphi_2)$, where $\underline{f}'$ is the system of generators given by the column vectors of V_1. By our hypothesis the $(1,2)$-entry and $(1,3)$-entry of A_1 can be written as follows:

$$\varphi(u^{[x-z]_1} \eta'_x) = u^{[x-z]_1} u^{p([y-z]_1 + r_y - se) - [x-z]_1} \varphi(\eta_y)$$

and

$$\varphi(u^{[y-z]_1} \eta'_y) = u^{[y-z]_1} u^{p([x-z]_1 + r_x - se) - [y-z]_1} \varphi(\eta_x).$$

As $p([y-z]_1 + r_y - se) - [x-z]_1, p([x-z]_1 + r_x - se) - [y-z]_1 > 0$, by iterating the previous procedure, we end up with a basis with the required properties. For the last statement, it is obvious that $v_x = 0$ (resp. $v_y = 0$) if and only if $v'_x = 0$ (resp. $v'_y = 0$).   □

Proposition 3.6. *Keep the notation as in Proposition 3.3 and assume $\overline{\rho}_0 \cong \text{T}^2_{\text{st}}(\mathcal{M})$.*

   (i) *If $s = 1$ in the case (**a**) and $\overline{\rho}_0$ is non-split, then $\text{FL}(\overline{\rho}_0) = [0 : 1]$.*
   (ii) *If $s = 0$ in the case (**b**), then $\overline{\rho}_0$ splits as a sum of a two-dimensional irreducible representation and a character.*

Proof. Assume that $s = 1$ in the case (**a**), Proposition 3.3, i.e., $(m_2, m_0) = (1, 2)$. Then $x \equiv r - 1 \mod e$, $y \equiv 0 \mod e$, $z \equiv c \mod e$, $r_x = 2e - (r-1)$, $r_y = r - 1$, and $s = 1$. Clearly, $[x-z]_1 = e - c + (r-1)$ and $[y-z]_1 = e - c$. Then by Lemma 3.5, we can assume $v_x = 0$ in the matrix (3.1.2), and by the Lemma 3.5, we can assume $\eta_x = 0 = \eta_y$ in the matrix (3.1.3). We can also assume that $v_y \in \mathbb{F}$ as $s = 1$.

Let V be the matrix (3.1.2) and A the matrix (3.1.3). By Proposition 2.13, the ϕ-module over $\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_p((\varpi))$ defined by $\mathfrak{M} \stackrel{\text{def}}{=} M_{\mathbb{F}_p((\varpi))}(\mathcal{M}^*)$ is described by

(3.1.6) $\text{Mat}_{\underline{e}}(\phi) = \hat{V}^t (\hat{A}^{-1})^t = \begin{pmatrix} \alpha_z^{-1} \underline{\varpi}^e & 0 & 0 \\ \alpha_z^{-1} \underline{\varpi}^{[x-z]_1} \cdot v_y & 0 & \alpha_y^{-1} \underline{\varpi}^{r_y} \\ 0 & \alpha_x^{-1} \underline{\varpi}^{r_x} & 0 \end{pmatrix}$

in an appropriate basis $\mathfrak{e} = (\mathfrak{e}_z, \mathfrak{e}_x, \mathfrak{e}_y)$. By considering the change of basis $\mathfrak{e}' = (\underline{\varpi}^c \mathfrak{e}_z, \underline{\varpi}^{r-1} \mathfrak{e}_x, \mathfrak{e}_y)$ we have:

$$\mathrm{Mat}_{\mathfrak{e}'}(\phi) = \begin{pmatrix} \alpha_z^{-1} \underline{\varpi}^{e(c+1)} & 0 & 0 \\ \alpha_z^{-1} v_y \underline{\varpi}^{e(c+1)} & 0 & \alpha_y^{-1} \\ 0 & \alpha_x^{-1} \underline{\varpi}^{e(r+1)} & 0 \end{pmatrix}.$$

We easily see that the ϕ-module $\mathfrak{M}$ is the base change via $\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_p((\underline{p})) \to \mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_p((\underline{\varpi}))$ of the ϕ-module $\mathfrak{M}_0$ over $\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_p((\underline{p}))$ described by

$$\mathrm{Mat}(\phi_0) = \begin{pmatrix} \alpha_z^{-1} \underline{p}^{(c+1)} & 0 & 0 \\ \alpha_z^{-1} v_y \underline{p}^{(c+1)} & 0 & \alpha_y^{-1} \\ 0 & \alpha_x^{-1} \underline{p}^{(r+1)} & 0 \end{pmatrix}.$$

Now we can find a basis for $\mathfrak{M}_0$ such that

$$\mathrm{Mat}(\phi_0) = \mathrm{Diag}(1, \underline{p}^{r+1}, \underline{p}^{c+1}) \begin{pmatrix} 0 & \alpha_x^{-1} & 0 \\ \alpha_y^{-1} & 0 & \alpha_z^{-1} v_y \\ 0 & 0 & \alpha_z^{-1} \end{pmatrix},$$

and so $\mathrm{FL}(\overline{\rho}_0) = [0:1]$ as $\overline{\rho}_0$ is non-split.

Assume that $s = 0$ in the case (**b**), Proposition 3.3, i.e., $(m_2, m_0) = (0, 1)$. Since $s = 0$, we can assume $v_x = 0 = v_y$. One can readily check that we can assume $\eta_x = 0 = \eta_y$ as well, using Lemma 3.5. By the same argument as above, it is easy to check that

$$\mathrm{Mat}(\phi_0) = \mathrm{Diag}(1, \underline{p}^{r+1}, \underline{p}^{c+1}) \begin{pmatrix} 0 & \alpha_x^{-1} & 0 \\ \alpha_y^{-1} & 0 & 0 \\ 0 & 0 & \alpha_z^{-1} \end{pmatrix}$$

(the only difference is the base change: $\mathfrak{e}' = (\underline{\varpi}^{c+1} \mathfrak{e}_z, \underline{\varpi}^{r-1} \mathfrak{e}_x, \underline{\varpi}^{-1} \mathfrak{e}_y)$). Hence, the corresponding representation $\overline{\rho}_0$ splits as a sum of a two-dimensional irreducible representation and a character. $\qquad\square$

3.2. Elimination of Galois types of niveau 2.
We start this subsection by recalling the following (cf. [MP], Lemma 3.3): let j, k be integers with $k \not\equiv 0 \bmod (p+1)$, and let ρ be a potentially crystalline representation with Hodge–Tate weights $\{-2, -1, 0\}$ and inertial type $\widetilde{\omega}^j \oplus \widetilde{\omega}_2^k \oplus \widetilde{\omega}_2^{pk}$ such that $\overline{\rho}_0^{ss} \simeq \overline{\rho}^{ss}$. Then we have the identity

$$(3.2.1) \qquad\qquad \omega^{3+j+k} = \det \overline{\rho}|_{I_{\mathbb{Q}_p}} = \omega^{(r+1)+(c+1)}.$$

In this section, we fix $e = p^2 - 1$, $K_0 = \mathbb{Q}_{p^2}$, and $K = K_0(\sqrt[e]{-p})$. We also let $\overline{S} = (\mathbb{F}_{p^2} \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^{ep}$ and $\overline{S}_0 = (\mathbb{F}_{p^2} \otimes_{\mathbb{F}_p} \mathbb{F})[u^e]/u^{ep}$. Recall that by $[m]_2$ for an integer m we mean the unique integer in $[0, e)$ congruent to $m \bmod (e)$.

Proposition 3.7. *Let* $\mathcal{M} \in \mathbb{F}\text{-BrMod}_{\mathrm{dd}}^2$ *be a Breuil module over* $\overline{S}$ *of type* $\tau \simeq \omega_{\varpi}^z \oplus \omega_{\varpi}^x \oplus \omega_{\varpi}^y$ *such that* $\mathrm{T}_{\mathrm{st}}^2(\mathcal{M})^{ss} \cong \overline{\rho}_0^{ss}$ *and* $\overline{\rho}_2 \subset \mathrm{T}_{\mathrm{st}}^2(\mathcal{M})$, *where* $\overline{\rho}_2$ *is the one-dimensional subrepresentation of* $\overline{\rho}_0$. *Assume that the submodule corresponding to* $\overline{\rho}_2$ *has descent data* ω_2^z.
Then there exists a framed basis $\underline{e} = (e_z, e_x, e_y)$ *and a framed system of generators* $\underline{f}$ *such that*

$$(3.2.2) \qquad \mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^2 \mathcal{M}) = \begin{pmatrix} u^{s(p-1)} & u^{[px-z]_2} \cdot v_y & u^{[py-z]_2} \cdot v_x \\ 0 & 0 & u^{r_x} \\ 0 & u^{r_y} & 0 \end{pmatrix};$$

$$(3.2.3) \qquad \mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_2) = \begin{pmatrix} \alpha_z & u^{[x-z]_2} \cdot \eta_x & u^{[y-z]_2} \cdot \eta_y \\ 0 & \alpha_x & 0 \\ 0 & 0 & \alpha_y \end{pmatrix},$$

where $\alpha_x, \alpha_y, \alpha_z \in (\mathbb{F}_{p^2} \otimes \mathbb{F})^\times$, $v_x, v_y \in \overline{S}_0$, and the tuple (x, y, z, r_x, r_y, s) satisfies the following properties:

(a): if $x \equiv k \bmod(e)$, $y \equiv pk \bmod(e)$, and $z \equiv (p+1)j \bmod(e)$, then

$$j \equiv c + 1 - m_2 \bmod(p-1), \quad k \equiv r + 1 - m_0 - pm_1 \bmod(e),$$

and

$$\begin{cases} r_x = m_0 e; \\ r_y = m_1 e; \\ s = m_2(p+1), \end{cases}$$

where $m_i \in \{0, 1, 2\}$ satisfy $m_0 + m_1 + m_2 = 3$.

(b): if $x \equiv (p+1)j \bmod(e)$, $y \equiv k \bmod(e)$, and $z \equiv pk \bmod(e)$, then

$$j \equiv r + 1 - \epsilon \bmod(p-1), \quad k \equiv (\delta + \epsilon - 3) + p(c + 1 - \delta) \bmod(e),$$

and

$$\begin{cases} r_x = (c - r - \delta + \epsilon) + p(\delta + 2\epsilon - r - 4) + \epsilon e; \\ r_y = (r + 4 - \delta - 2\epsilon) + p(r - c + \delta - \epsilon) + (3 - \delta - \epsilon)e; \\ s = (c + 4 - \epsilon - \delta) + p\delta, \end{cases}$$

where $\epsilon \in \{1, 2\}$ and $\delta \in \{0, 1\}$ with $\epsilon + \delta \neq 3$.

(c): if $x \equiv k \bmod(e)$, $y \equiv (p+1)j \bmod(e)$, and $z \equiv pk \bmod(e)$, then

$$j \equiv \epsilon + \delta - 3 \bmod(p-1), \quad k \equiv (r + 1 - \epsilon) + p(c + 1 - \delta) \bmod(e),$$

and

$$\begin{cases} r_x = (2\epsilon + \delta - r - 4) + p(\epsilon + 2\delta - c - 4) + \epsilon e; \\ r_y = (c + 4 - \epsilon - 2\delta) + p(r + 4 - \delta - 2\epsilon) + (3 - \delta - \epsilon)e; \\ s = (c - r + \epsilon) + p\delta, \end{cases}$$

where $\epsilon \in \{1, 2\}$ and $\delta \in \{0, 1\}$ with $\epsilon + \delta \neq 1$.

Proof. Since $\overline{\rho}_0$ is an extension of a two-dimensional irreducible representation by a character of niveau 1, $\mathcal{M}$ is also an extension of a simple Breuil module of rank 2 by a Breuil module of rank 1 by Proposition 2.22. Hence, it is immediate that the filtration and the Frobenius map φ_2 of $\mathcal{M}$ are described as in (3.2.2) and (3.2.3) respectively, by using the classification of simple Breuil modules of rank 2 in Proposition 2.24 and the classification of simple Breuil modules of rank 1 in [MP], Lemma 3.1.

Recall from Proposition 2.24 that

$$(3.2.4) \qquad r_x \equiv py - x \bmod(e), \ r_y \equiv px - y \bmod(e), \ \text{and} \ z + ps \equiv 0 \bmod(p+1).$$

We also recall that $0 \le r_x, r_y \le 2e$, $0 \le s \le 2(p+1)$, and by Lemma 3.3.2 in [EGH13] and by Proposition 2.24 we have:

$$(3.2.5) \qquad \begin{cases} x + p\frac{pr_x + r_y}{e} \equiv r + 1 \bmod(e); \\ z + ps \equiv (p+1)(c+1) \bmod(e). \end{cases}$$

For case **(a)**, assume that $x \equiv k \bmod(e)$, $y \equiv pk \bmod(e)$, and $z \equiv (p+1)j \bmod(e)$. We let $r_x = m_0 e$, $r_y = m_1 e$, and $s = (p+1)m_2$ for $m_i \in \{0, 1, 2\}$, due to the equation (3.2.4).

Then it is immediate from the equation (3.2.5) that

$$\begin{cases} k + p(pm_0 + m_1) \equiv r + 1 \bmod(e); \\ j + pm_2 \equiv (c+1) \bmod(p-1). \end{cases}$$

Hence, $j \equiv c + 1 - m_2 \bmod(p-1)$ and $k \equiv r + 1 - m_0 - pm_1 \bmod(e)$. The determinant condition (3.2.1) gives rise to the condition $m_0 + m_1 + m_2 \equiv 3 \bmod(p-1)$ and so $m_0 + m_1 + m_2 = 3$ since $p > 5$.

For case (**b**), assume that $x \equiv (p+1)j \bmod(e)$, $y \equiv k \bmod(e)$, and $z \equiv pk \bmod(e)$. From equation (3.2.4) we can write $pr_x + r_y = ae$ for $0 \le a \le 2(p+1)$. From the equation (3.2.5) we get

$$\begin{cases} (p+1)j + pa \equiv r + 1 \bmod(e); \\ pk + ps \equiv (p+1)(c+1) \bmod(e). \end{cases}$$

From the determinant condition (3.2.1), we have

(3.2.6) $\qquad (p+1)(c-r+3) - (p+1)s \equiv pa - (r+1) \bmod(e),$

and so $a \equiv -(r+1) \bmod(p+1)$. We let $a = \epsilon(p+1) - (r+1)$ where $\epsilon \in \{1, 2\}$ (recall that $0 \le a \le 2(p+1)$).

We now determine j, k, and s in terms of $a = \epsilon(p+1) - (r+1)$ and the inertial weights. We have $(p+1)j \equiv (r+1) - pa \equiv (r+1) - p[\epsilon(p+1) - (r+1)] \equiv (p+1)(r+1-\epsilon) \bmod(e)$ and hence $j \equiv r + 1 - \epsilon \bmod(p-1)$. From equation (3.2.6) we have $(p+1)s \equiv (p+1)(c-r+3) - p[\epsilon(p+1) - (r+1)] + (r+1) \equiv (p+1)(c+4-\epsilon) \bmod(e)$ and so we have $s \equiv c + 4 - \epsilon \bmod(p-1)$. We write $s = c + 4 - \epsilon + \delta(p-1)$ for $\delta \in \{0, 1\}$ (again, since $0 \le s \le 2(p+1)$). Finally k is immediately deduced from s: $k \equiv (p+1)(c+1) - s \equiv (p+1)(c+1) - [c+4-\epsilon+\delta(p-1)] = (\epsilon + \delta - 3) + p(c+1-\delta) \bmod(e)$.

We now describe r_x, r_y in the filtration. From the equation (3.2.4), $r_x \equiv pk - (p+1)j \equiv (c-r+\epsilon-\delta) + p(\delta+2\epsilon-r-4) \bmod(e)$ and $r_y \equiv (p+1)j - k \equiv (r+4-\delta-2\epsilon) + p(\delta-\epsilon+r-c) \bmod(e)$. Hence we have $r_x = (c-r+\epsilon-\delta) + p(\delta+2\epsilon-r-4) + m_0 e$ and $r_y = (r+4-\delta-2\epsilon) + p(\delta-\epsilon+r-c) + m_1 e$ for some $m_0, m_1 \in \{1, 2\}$ (since $0 \le r_x, r_y \le 2e$).

We finally determine m_0, m_1. We have $ae = pr_x + r_y = (\delta+2\epsilon-r-4+pm_0+m_1)e$ and so $\epsilon(p+1) - (r+1) = a = \delta+2\epsilon-r-4+pm_0+m_1$. Hence, we have $\delta+\epsilon-3+m_1 = p(\epsilon-m_0)$ which immediately implies that $m_0 = \epsilon$ and $m_1 = 3 - \delta - \epsilon$. The requirement $m_1 \in \{1, 2\}$ implies that $(\delta, \epsilon) \ne (1, 2)$.

For case (**c**), assume that $x \equiv k \bmod(e)$, $y \equiv (p+1)j \bmod(e)$, and $z \equiv pk \bmod(e)$. We write $pr_x + r_y = ae$ for $0 \le a \le 2(p+1)$ from the equation (3.2.4). From the equation (3.2.5) we get

$$\begin{cases} k + pa \equiv r + 1 \bmod(e); \\ pk + ps \equiv (p+1)(c+1) \bmod(e). \end{cases}$$

We now determine j, k, s in terms of a and the inertial weights. From the determinant condition (3.2.1), we have $j \equiv r + c - 1 - k \equiv r + c - 1 - [r+1-pa] \equiv c - 2 + a \bmod(p-1)$. We also have $p(r+1-pa) \equiv pk \equiv (p+1)(c+1) - ps$ which gives $s \equiv (p+1)(c+1) - (r+1) + pa \equiv (c-r) + p(c+1+a) \bmod(e)$. Hence we can write $s = (c-r) + p(c+1+a) - \epsilon e = (c-r+\epsilon) + p(c+1+a-p\epsilon)$ where $\epsilon \in \{1, 2\}$ since $1 \le s, a \le 2(p+1)$. Define $\delta := c+1+a-p\epsilon$. Then $\delta \in \{0, 1\}$ (since $0 \le s \le 2(p+1)$) and we have $a = \delta + p\epsilon - (c+1)$. We finally obtain $j \equiv \epsilon + \delta - 3 \bmod(p-1)$ and $k \equiv r + 1 - pa \equiv r + 1 - \epsilon + p(c+1-\delta) \bmod(e)$.

We now describe r_x, r_y in the filtration. From the equation (3.2.4), $r_x \equiv (p+1)j - k \equiv (2\epsilon+\delta-r-4) + p(2\delta+\epsilon-c-4) \bmod(e)$ and $r_y \equiv pk - (p+1)j \equiv (c+4-2\delta-\epsilon) + p(r+4-\delta-2\epsilon) \bmod(e)$. So we can write $r_x = (2\epsilon+\delta-r-4) + p(2\delta+\epsilon-c-4) + m_0 e$ and $r_y = (c+4-2\delta-\epsilon) + p(r+4-\delta-2\epsilon) + m_1 e$ for some $m_0 \in \{1, 2\}$ and $m_1 \in \{0, 1\}$

(since $0 \leq r_x, r_y \leq 2e$). We have $ae = pr_x + r_y = (2\delta + \epsilon - c - 4 + pm_0 + m_1)e$ so that $\delta - (c+1) + p\epsilon = a = 2\delta + \epsilon - c - 4 + pm_0 + m_1$. Hence, we have $\delta + \epsilon - 3 + m_1 = p(\epsilon - m_0)$ which easily implies $m_0 = \epsilon$ and $m_1 = 3 - \delta - \epsilon$. The requirement $m_1 \in \{0, 1\}$ implies that $(\delta, \epsilon) \neq (0, 1)$.   □

Lemma 3.8. *Keep the notation as in Proposition 3.7 (in particular, recall the elements v_x and v_y in the matrix (3.2.2)) and assume $s \leq p + 1$.*

   (i) *If $r_x + s(p-1) - [py - z]_2 > 2e$ then there is a basis such that $v_x = 0$.*
   (ii) *If $r_y + s(p-1) - [px - z]_2 > 2e$ then there is a basis such that $v_y = 0$.*

Proof. The same argument as in Lemma 3.4 works.   □

Lemma 3.9. *Keep the notation as in Proposition 3.7 (in particular, recall the elements η_x and η_y in the matrix (3.2.3)).*

   (i) *If $[x - z]_2 + r_x - s(p-1) + e \geq 0$ and $[y - z]_2 + r_y - s(p-1) - e \geq 0$ then there is a basis such that $\eta_x \in \mathbb{F}_{p^2} \otimes_{\mathbb{F}_p} \mathbb{F}$ and $\eta_y = 0$.*
   (ii) *If $[x - z]_2 + r_x - s(p-1) - e \geq 0$ and $[y - z]_2 + r_y - s(p-1) + e \geq 0$ then there is a basis such that $\eta_x = 0$ and $\eta_y \in \mathbb{F}_{p^2} \otimes_{\mathbb{F}_p} \mathbb{F}$.*
   (iii) *If $p([x - z]_2 + r_x - s(p-1)) > [y - z]_2$ and $p([y - z]_2 + r_y - s(p-1)) > [x - z]_2$ then there is a basis such that $\eta_x = 0$ and $\eta_y = 0$.*

Moreover, the change of basis does not affect the vanishing of v_x and v_y.

Proof. One can prove case (iii) by the same argument as in Lemma 3.5, and case (i) is similar to case (ii). We only provide with a proof for case (ii).

Let V_0 be the matrix (3.2.2) and A_0 the matrix (3.2.3). We define $\tilde{\eta}_y \in u^e \cdot \overline{S}_0$ by $\eta_y = \eta_y^0 + \tilde{\eta}_y$ with $\eta_y^0 \in \mathbb{F}_{p^2} \otimes_{\mathbb{F}_p} \mathbb{F}$ and let $\tilde{A}_0$ be the matrix obtained from A_0 by replacing η_y in A_0 by $\tilde{\eta}_y$. We also let

$$
B_1 = \begin{pmatrix} \alpha_z & u^{[p^{-1}(x-z)]_2} \cdot \eta_x' & u^{[p^{-1}(y-z)]_2} \cdot \eta_y' \\ 0 & \alpha_y & 0 \\ 0 & 0 & \alpha_x \end{pmatrix}
$$

for some $\eta_x', \eta_y' \in \overline{S}_0$.

One can easily check that the equation

(3.2.7) $\tilde{A}_0 V_1 = V_0 B_1$

holds true if and only if the following two equalities hold:

$$
\alpha_z u^{[px-z]_2} v_y' + u^{[y-z]_2 + r_y} \tilde{\eta}_y = u^{s(p-1) + [p^{-1}(x-z)]_2} \eta_x' + \alpha_y u^{[px-z]_2} v_y;
$$

$$
\alpha_z u^{[py-z]_2} v_x' + u^{[x-z]_2 + r_x} \eta_x = u^{s(p-1) + [p^{-1}(y-z)]_2} \eta_y' + \alpha_x u^{[py-z]_2} v_x.
$$

Hence, the equation (3.2.7) holds if we choose $v_x' = \alpha_x \alpha_z^{-1} v_x$, $v_y' = \alpha_y \alpha_z^{-1} v_y$,

$$
u^{[p^{-1}(x-z)]_2} \eta_x' = u^{[y-z]_2 + r_y - s(p-1)} \tilde{\eta}_y, \quad \text{and} \quad u^{[p^{-1}(y-z)]_2} \eta_y' = u^{[x-z]_1 + r_x - s(p-1)} \eta_x.
$$

Here, both $u^{[p^{-1}(x-z)]_2} \eta_x'$ and $u^{[p^{-1}(y-z)]_2} \eta_y'$ are well-defined elements in $\overline{S}$ by our assumption on (x, y, z) and (s, r_x, r_y).

Now let us consider the new basis $\underline{e}' \stackrel{\text{def}}{=} \underline{e} \tilde{A}_0$. Then $V_1 = \mathrm{Mat}_{\underline{e}', \underline{f}'}(\mathrm{Fil}^2 \mathcal{M})$, where $\underline{f}'$ be the system of generators given by the column vectors of V_1. Note that $\varphi(u^{[p^{-1}(y-z)]_2} \eta_y') = 0$,

again by our assumption. We compute $\mathrm{Mat}_{\underline{e}',\underline{f}'}(\varphi_2)$ as follows:

$$\varphi_2(\underline{e}'V_1) = \underline{e}A_0\varphi(B_1)$$

$$= \underline{e}\left[\tilde{A}_0 + \begin{pmatrix} 0 & 0 & \eta_y^0 u^{[y-z]} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] \begin{pmatrix} \varphi(\alpha_z) & u^{p[p^{-1}(x-z)]_2}\varphi(\eta_x') & 0 \\ 0 & \varphi(\alpha_y) & 0 \\ 0 & 0 & \varphi(\alpha_x) \end{pmatrix}$$

$$= \underline{e}\left[\tilde{A}_0 + \tilde{A}_0\begin{pmatrix} 0 & 0 & \frac{\eta_y^0}{\alpha_z} u^{[y-z]} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] \begin{pmatrix} \varphi(\alpha_z) & u^{p[p^{-1}(x-z)]_2}\varphi(\eta_x') & 0 \\ 0 & \varphi(\alpha_y) & 0 \\ 0 & 0 & \varphi(\alpha_x) \end{pmatrix}$$

$$= \underline{e}\tilde{A}_0\begin{pmatrix} \varphi(\alpha_z) & u^{p[p^{-1}(x-z)]_2}\varphi(\eta_x') & \frac{\eta_y^0\varphi(\alpha_x)}{\alpha_z} u^{[y-z]} \\ 0 & \varphi(\alpha_y) & 0 \\ 0 & 0 & \varphi(\alpha_x) \end{pmatrix}$$

$$= \underline{e}'\underbrace{\begin{pmatrix} \varphi(\alpha_z) & \varphi(u^{[y-z]_2+r_y-s(p-1)}\tilde{\eta}_y) & \frac{\eta_y^0\varphi(\alpha_x)}{\alpha_z} u^{[y-z]} \\ 0 & \varphi(\alpha_y) & 0 \\ 0 & 0 & \varphi(\alpha_x) \end{pmatrix}}_{=\mathrm{Mat}_{\underline{e}',\underline{f}'}(\varphi_2)}.$$

Hence, for $\mathrm{Mat}_{\underline{e}',\underline{f}'}(\varphi_2)$, we see that $\eta_y = \eta_y^0$, i.e., $\tilde{\eta}_y = 0$. Performing the above procedure one more time, we see that $\eta_x = 0$ and $\eta_y \in \mathbb{F}_{p^2} \otimes_{\mathbb{F}_p} \mathbb{F}$. It is obvious that the above procedure does not affect the vanishing of v_x and v_y. $\square$

Proposition 3.10. *Keep the notation as in Proposition 3.7 and assume that $\overline{\rho}_0 \cong \mathrm{T}_{\mathrm{st}}^2(\mathcal{M})$.*

(i) *If $m_2 = 0$ in the case (**a**), then $\overline{\rho}_0$ splits as a sum of a two-dimensional irreducible representation and a character.*

(ii) *If $(m_2, m_1, m_0) = (1, 0, 2)$ in the case (**a**) and $\overline{\rho}_0$ is non-split, then $\mathrm{FL}(\overline{\rho}_0) = [0 : 1]$.*

(iii) *If $(m_2, m_1, m_0) = (1, 2, 0)$ in the case (**a**) and $\overline{\rho}_0$ is non-split, then $\mathrm{FL}(\overline{\rho}_0) = [1 : 0]$.*

(iv) *If $(\epsilon, \delta) = (2, 0)$ in the case (**b**) and $\overline{\rho}_0$ is non-split, then $\mathrm{FL}(\overline{\rho}_0) = [0 : 1]$.*

(v) *If $(\epsilon, \delta) = (2, 0)$ in the case (**c**) and $\overline{\rho}_0$ is non-split, then $\mathrm{FL}(\overline{\rho}_0) = [0 : 1]$.*

Proof. Let V be the matrix (3.2.2) and A the matrix (3.2.3), and assume that $s(p-1) \leq e$. Since $s \leq (p+1)$, we may assume that $v_x, v_y \in \mathbb{F}_{p^2} \otimes_{\mathbb{F}_p} \mathbb{F}$. By Proposition 2.13, the ϕ-module over $\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}((\varpi))$ defined by $\mathfrak{M} \overset{\text{def}}{=} M_{\mathbb{F}_{p^2}((\varpi))}(\mathcal{M}^*)$ is described by

$$(3.2.8) \qquad \mathrm{Mat}_{\mathfrak{e}}(\phi) = \hat{V}^t(\hat{A}^{-1})^t = \begin{pmatrix} \frac{1}{\alpha_z}\varpi^{s(p-1)} & 0 & 0 \\ \frac{v_y}{\alpha_z}\varpi^{[px-z]_2} + \frac{\eta_y}{\alpha_z\alpha_y}\varpi^{r_y+[y-z]_2} & 0 & \frac{1}{\alpha_y}\varpi^{r_y} \\ \frac{v_x}{\alpha_z}\varpi^{[py-z]_2} + \frac{\eta_x}{\alpha_z\alpha_x}\varpi^{r_x+[x-z]_2} & \frac{1}{\alpha_x}\varpi^{r_x} & 0 \end{pmatrix}$$

in an appropriate basis $\mathfrak{e} = (\mathfrak{e}_z, \mathfrak{e}_x, \mathfrak{e}_y)$.

We now prove case (iii). Assume that $(m_2, m_1, m_0) = (1, 2, 0)$. Then we have $x \equiv r+1-2p \pmod{e}$, $y \equiv p(r+1) - 2 \pmod{e}$, $z \equiv (p+1)c$, $s \equiv (p+1)$, $r_x = 0$, and $r_y = 2e$. So we have $[x-z]_2 = e + r + 1 - 2p - (p+1)c$ and $[y-z]_2 = e + p(r+1) - 2 - (p+1)c$. By lemma 3.9, case (i), we may assume that $\eta_y = 0$ and $\eta_x \in \mathbb{F}_{p^2} \otimes_{\mathbb{F}_p} \mathbb{F}$, and, by Lemma 3.8, case (ii), we may assume that $v_y = 0$ as well. Hence, in this specific case, we have

$$\mathrm{Mat}_{\mathfrak{e}}(\phi) = \begin{pmatrix} \frac{1}{\alpha_z}\varpi^e & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha_y}\varpi^{2e} \\ \frac{v_x}{\alpha_z}\varpi^{e+r+1-2p-(p+1)c} + \frac{\eta_x}{\alpha_z\alpha_x}\varpi^{e+r+1-2p-(p+1)c} & \frac{1}{\alpha_x} & 0 \end{pmatrix}.$$

By considering the change of basis $\mathfrak{e}' = (\underline{\varpi}^{(p+1)c}\mathfrak{e}_z, \underline{\varpi}^{p(r+1)-2}\mathfrak{e}_x, \underline{\varpi}^{r+1-2p}\mathfrak{e}_y)$ we have:

$$\mathrm{Mat}_{\mathfrak{e}'}(\phi) = \begin{pmatrix} \frac{1}{\alpha_z}\underline{\varpi}^{e(c+1)} & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha_y} \\ \frac{v_x}{\alpha_z}\underline{\varpi}^{e(c+1)} + \frac{\eta_x}{\alpha_z\alpha_x}\underline{\varpi}^{e(c+1)} & \frac{1}{\alpha_x}\underline{\varpi}^{e(r+1)} & 0 \end{pmatrix}.$$

We easily see that the ϕ-module $\mathfrak{M}$ is the base change via $\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}((\underline{p})) \to \mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}((\underline{\varpi}))$ of the ϕ-module $\mathfrak{M}_0$ over $\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}((\underline{p}))$ described by

$$\mathrm{Mat}(\phi_0) = \begin{pmatrix} \frac{1}{\alpha_z}\underline{p}^{(c+1)} & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha_y} \\ \frac{v_x}{\alpha_z}\underline{p}^{(c+1)} + \frac{\eta_x}{\alpha_z\alpha_x}\underline{p}^{(c+1)} & \frac{1}{\alpha_x}\underline{p}^{(r+1)} & 0 \end{pmatrix}.$$

Now we can find a basis for $\mathfrak{M}_0$ such that

$$\mathrm{Mat}(\phi_0) = \mathrm{Diag}(1, \underline{p}^{r+1}, \underline{p}^{c+1}) \begin{pmatrix} 0 & \frac{1}{\alpha_x} & \frac{v_x}{\alpha_z} + \frac{\eta_x}{\alpha_z\alpha_x} \\ \frac{1}{\alpha_y} & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha_z} \end{pmatrix},$$

and so $\mathrm{FL}(\overline{\rho}_0) = [1:0]$, by Lemma 3.2, as $\overline{\rho}_0$ is non-split.

Case (ii) is very similar to the previous one. We now have $v_x = 0 = \eta_x$ and $\eta_y \in \mathbb{F}_{p^2} \otimes_{\mathbb{F}_p} \mathbb{F}$. By the same argument as above, one can check that

$$\mathrm{Mat}(\phi_0) = \mathrm{Diag}(1, \underline{p}^{r+1}, \underline{p}^{c+1}) \begin{pmatrix} 0 & \frac{1}{\alpha_x} & 0 \\ \frac{1}{\alpha_y} & 0 & \frac{v_y}{\alpha_z} + \frac{\eta_y}{\alpha_z\alpha_y} \\ 0 & 0 & \frac{1}{\alpha_z} \end{pmatrix},$$

and so $\mathrm{FL}(\overline{\rho}_0) = [0:1]$, by Lemma 3.2, as $\overline{\rho}_0$ is non-split.

Assume that $s = 0$, i.e., $m_2 = 0$. Since $s = 0$, we may let $v_x = 0 = v_y$. One can readily check $\eta_x = 0 = \eta_y$ as well, using Lemma 3.9, case (iii). By the same argument as above, it is easy to check that

$$\mathrm{Mat}(\phi_0) = \mathrm{Diag}(1, \underline{p}^{r+1}, \underline{p}^{c+1}) \begin{pmatrix} 0 & \frac{1}{\alpha_x} & 0 \\ \frac{1}{\alpha_y} & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha_z} \end{pmatrix}.$$

Hence, the corresponding representation $\overline{\rho}_0$ splits as a sum of a two-dimensional irreducible representation and a character.

Assume that $(\epsilon, \delta) = (2, 0)$ in the case (**b**). By Lemma (3.8), case (i), we have $v_x = 0$, and, by Lemma (3.9), case (iii), $\eta_x = \eta_y = 0$. By the same argument as above, one can check that

$$\mathrm{Mat}(\phi_0) = \mathrm{Diag}(1, \underline{p}^{r+1}, \underline{p}^{c+1}) \begin{pmatrix} 0 & \frac{1}{\alpha_x} & 0 \\ \frac{1}{\alpha_y} & 0 & \frac{v_y}{\alpha_z} \\ 0 & 0 & \frac{1}{\alpha_z} \end{pmatrix},$$

and so $\mathrm{FL}(\overline{\rho}_0) = [0:1]$, by Lemma 3.2, as $\overline{\rho}_0$ is non-split.

Assume that $(\epsilon, \delta) = (2, 0)$ in the case (**c**). In this case, we may let $v_x = 0$ since $s(p-1) \leq [py - z]_2$. By Lemma (3.9), case (iii), $\eta_x = \eta_y = 0$. By the same argument as above, one can check that

$$\mathrm{Mat}(\phi_0) = \mathrm{Diag}(1, \underline{p}^{r+1}, \underline{p}^{c+1}) \begin{pmatrix} 0 & \frac{1}{\alpha_x} & 0 \\ \frac{1}{\alpha_y} & 0 & \frac{v_y}{\alpha_z} \\ 0 & 0 & \frac{1}{\alpha_z} \end{pmatrix},$$

and so $\mathrm{FL}(\overline{\rho}_0) = [0:1]$, by Lemma 3.2, as $\overline{\rho}_0$ is non-split. $\square$

4. Fontaine-Laffaille parameter and crystalline Frobenius

The aim of this section is to explicitly determine the Fontaine-Laffaille module associated to the mod-p reduction of a potentially crystalline lift of $\overline{\rho}_0$, with a carefully chosen inertial type. The main result is Theorem 4.5, whose proof relies on some direct manipulation in semilinear algebra (cf. Section 2.2.4, Lemmas 2.17, 2.18).

As we did in Section 3, in the reminder of this section we may and do assume $a_0 = -1$ and define $c \overset{\mathrm{def}}{=} a_2 - a_0 - 1$, $r \overset{\mathrm{def}}{=} a_1 - a_0 - 1$.

4.1. Filtration on strongly divisible modules.
We go back to the setting of section 2.1 and we let $\overline{\rho}_0 : G_{\mathbb{Q}_p} \to \mathrm{GL}_3(\mathbb{F})$ be as in (2.1.1) with the genericity condition as in Definition 2.4.

Proposition 4.1. *Let $\mathcal{M} \in \mathbb{F}\text{-BrMod}_{\mathrm{dd}}^2$ be a Breuil module of type $\overline{\tau} = \omega^c \oplus \omega^r \oplus \omega^{-1}$ such that $\mathrm{T}_{\mathrm{st}}^2(\mathcal{M}) \cong \overline{\rho}_0$.*

Then there exists a framed basis $\underline{e} = (e_c, e_r, e_{-1})$ on $\mathcal{M}$ and a framed system of generators $\underline{f} = (f_c, f_r, f_{-1})$ for $\mathrm{Fil}^2 \mathcal{M}$ such that

$$
\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}\,\mathcal{M}) = \begin{pmatrix} u^e & u^{e-(c-r)}\lambda & u^{e-(c+1)}\mu \\ 0 & 0 & u^{e-(r+1)} \\ 0 & u^{e+(r+1)} & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_2) = \begin{pmatrix} \alpha_c & 0 & 0 \\ 0 & \alpha_r & 0 \\ 0 & 0 & \alpha_{-1} \end{pmatrix}
$$

where $\lambda_i \in \mathbb{F}^\times$ and $\lambda, \mu \in \mathbb{F}$.

Moreover, we have the following properties:

    (i) *$\lambda = 0 = \mu$ if and only if $\overline{\rho}_0$ splits;*

    (ii) *if $\overline{\rho}_0$ is non-split, then $\mathrm{FL}(\overline{\rho}_0) = [\mu\alpha_r : -\lambda] \in \mathbf{P}^1(\mathbb{F})$.*

Proof. From Proposition 3.3, (**b**) for $m_0 = 0$ and $m_2 = 1$, it is immediate to get $\mathrm{Mat}_{\underline{e}}(\mathrm{Fil}^2 \mathcal{M})$ as above. By Lemma 3.5, it is also easy to check that $\eta_x = 0 = \eta_y$ in the matrix (3.1.3), and so we get $\mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_2)$ as above.

By the same argument as in Proposition 3.6, one can readily compute the following ϕ-module over $\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_p((\underline{p}))$ from the Breuil module structure as above:

$$
\mathrm{Mat}(\phi_0) = \mathrm{Diag}(1, \underline{p}^{r+1}, \underline{p}^{c+1}) \begin{pmatrix} 0 & \frac{1}{\alpha_r} & \frac{\mu}{\alpha_c} \\ \frac{1}{\alpha_{-1}} & 0 & \frac{\lambda}{\alpha_c} \\ 0 & 0 & \frac{1}{\alpha_c} \end{pmatrix}.
$$

The second part is immediate from this matrix. $\square$

From now on in this section, we restrict our attention to $\overline{\rho}_0$ that is non-split. We easily deduce the following:

Lemma 4.2. *Let $\mathcal{M} \in \mathbb{F}\text{-BrMod}_{\mathrm{dd}}^2$ and $\lambda, \mu \in \mathbb{F}$ as in the statement of Proposition 4.1. Assume that $\overline{\rho}_0$ is non-split, i.e., not both λ and μ are zero.*

Then the elementary divisors for $\mathcal{M}/\mathrm{Fil}^2 \mathcal{M}$ are described by one of the following possibilities:

    (i) *if $\lambda\mu \neq 0$, by $(u^{e-(c+1)}, u^e, u^{e+(c+1)})$;*

    (ii) *if $\lambda = 0$, by $(u^{e-(c+1)}, u^{e+(c-r)}, u^{e+(r+1)})$;*

    (iii) *if $\mu = 0$, by $(u^{e-(c-r)}, u^{e-(r+1)}, u^{e+(c+1)})$.*

In particular, one has:

(iv) $\left(\mathrm{Fil}^2\,\mathcal{M}\right)_{\omega^{-1}} \subseteq u^{e-(c+1)}\mathcal{M}$; moreover, $\left(\mathrm{Fil}^2\,\mathcal{M}\right)_{\omega^{-1}} \subseteq u^{e-(r+1)}\mathcal{M}$ holds true if and only if $\mu = 0$;

(v) $\left(\mathrm{Fil}^2\,\mathcal{M} \cap u^e\mathcal{M}\right)_{\omega^{-1}} \subseteq u^{2e-(c+1)}\mathcal{M}$;

(vi) $\left(\mathrm{Fil}^2\,\mathcal{M}\right)_{\omega^c} \subseteq u^e\mathcal{M}$.

Proof. The elementary divisors are immediately deduced from the Smith normal forms of $\mathrm{Mat}_{\underline{e}}(\mathrm{Fil}^2\,\mathcal{M})$ in Proposition 4.1.

It is easy to check the following computation:

$$(\mathrm{Fil}^2\,\mathcal{M})_{\omega^c} = \left\langle u^e e_c, u^e \lambda e_c + u^{e+(c-1)}e_{-1}, u^e \mu e_c + u^{e+(c-r)}e_{-1} \right\rangle;$$

$$(\mathrm{Fil}^2\,\mathcal{M})_{\omega^r} = \left\langle u^{e+r} e_c, u^{e-(c-r)} \lambda e_c + u^{e+(r+1)}e_{-1}, u^{e-(c-r)} \mu e_c + u^e e_{-1} \right\rangle;$$

$$(\mathrm{Fil}^2\,\mathcal{M})_{\omega^{-1}} = \left\langle u^{2e-(c+1)} e_c, u^{2e-(c+1)} \lambda e_c + u^{2e}e_{-1}, u^{e-(c+1)} \mu e_c + u^{e-(r+1)}e_{-1} \right\rangle.$$

The second part is also immediate from the computation above. $\qquad\square$

Proposition 4.3. *Let $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_3(\mathcal{O}_E)$ be a p-adic Galois representation becoming crystalline over K, with inertial type $\tau = \widetilde{\omega}^c \oplus \widetilde{\omega}^r \oplus \widetilde{\omega}^{-1}$ and Hodge-Tate weights $\{-2, -1, 0\}$ such that $\overline{\rho} \cong \overline{\rho}_0$. Let $\widehat{\mathcal{M}} \in \mathcal{O}_E\text{-}\mathrm{Mod}^2_{\mathrm{dd}}$ be a strongly divisible lattice such that $\mathrm{T}^{\mathbb{Q}_p,2}_{\mathrm{st}}(\widehat{\mathcal{M}}) = \rho$.*

Then there exists a framed basis $(\widehat{e}_c, \widehat{e}_r, \widehat{e}_{-1})$ for $\widehat{\mathcal{M}}$ and a framed system of generators $(\widehat{f}_c, \widehat{f}_r, \widehat{f}_{-1})$ for $\mathrm{Fil}^2\,\widehat{\mathcal{M}} / \mathrm{Fil}^2\,S \cdot \widehat{\mathcal{M}}$ whose coordinates are described as follows:

A : *if $\mathrm{FL}(\overline{\rho}_0) \in \mathbf{P}^1(\mathbb{F}) \setminus \{[0:1], [1:0]\}$ then*

$$\widehat{f}_c = \begin{pmatrix} -\frac{p^2}{\alpha} \\ 0 \\ pu^{c+1} \end{pmatrix} + E(u)\begin{pmatrix} 0 \\ 0 \\ u^{c+1} \end{pmatrix}$$

$$\widehat{f}_r = E(u)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\widehat{f}_{-1} = \begin{pmatrix} u^{e-(c+1)} \\ 0 \\ \alpha \end{pmatrix}$$

where $0 < \mathbf{v}_\mathrm{p}(\alpha) < 2$.

B : *if $\mathrm{FL}(\overline{\rho}_0) = [1:0]$ then*

$$\widehat{f}_c = \begin{pmatrix} -\frac{p\beta}{\alpha} \\ 0 \\ \beta u^{c+1} \end{pmatrix} + E(u)\begin{pmatrix} 0 \\ u^{c-r} \\ 0 \end{pmatrix}$$

$$\widehat{f}_r = E(u)\begin{pmatrix} 0 \\ -\frac{p}{\beta} \\ u^{r+1} \end{pmatrix}$$

$$\widehat{f}_{-1} = \begin{pmatrix} u^{e-(c+1)} \\ 0 \\ \alpha \end{pmatrix}$$

where $0 < \mathbf{v}_\mathrm{p}(\beta)$ and $0 < \mathbf{v}_\mathrm{p}(\alpha) < \mathbf{v}_\mathrm{p}(\beta) + 1 < 2$.

C : *if* $\mathrm{FL}(\overline{\rho}_0) = [0:1]$ *then*

$$\widehat{f}_c = \begin{pmatrix} -\frac{p^2\alpha}{\beta} \\ -\frac{p^2}{\beta}u^{c-r} \\ pu^{c+1} \end{pmatrix} + E(u)\begin{pmatrix} 0 \\ 0 \\ u^{c+1} \end{pmatrix}$$

$$\widehat{f}_r = \begin{pmatrix} u^{e-(c-r)} \\ -\frac{p}{\alpha} \\ \frac{\beta}{\alpha}u^{r+1} \end{pmatrix} + E(u)\begin{pmatrix} 0 \\ 0 \\ \gamma u^{r+1} \end{pmatrix}$$

$$\widehat{f}_{-1} = \begin{pmatrix} \alpha u^{e-(c+1)} \\ u^{e-(r+1)} \\ \beta \end{pmatrix}$$

where $0 < \mathbf{v}_{\mathrm{p}}(\alpha) < 1$, $0 < \mathbf{v}_{\mathrm{p}}(\gamma)$, *and* $0 < \mathbf{v}_{\mathrm{p}}(\alpha) < \mathbf{v}_{\mathrm{p}}(\beta) < 2$.

Proof. Let $\underline{e} \overset{\text{def}}{=} (e_c, e_r, e_{-1})$ be a framed basis for $\widehat{\mathcal{M}}$. We write the elements of $\widehat{\mathcal{M}}$ in terms of coordinates with respect to $\underline{e}$. Moreover, we let $\mathcal{M} \overset{\text{def}}{=} \widehat{\mathcal{M}} \otimes_S S/(\varpi_E, \mathrm{Fil}^p S)$ denote the Breuil module associated to $\widehat{\mathcal{M}}$, define $\mathcal{D} \overset{\text{def}}{=} \widehat{\mathcal{M}} \otimes_{\mathcal{O}_E} S_E$ and, if $\chi : \mathbb{F}_p^\times \to \mathcal{O}_E$ is a tame character, we write $X_\chi \overset{\text{def}}{=} \left(\mathrm{Fil}^2 \mathcal{D}/ \mathrm{Fil}^2 S \cdot \mathcal{D}\right)_\chi$, which is a $E[E(u)]/(E(u)^2)$-module explicitly described in [HLM], Proposition 2.2.24.

By [HLM], Proposition 2.2.24 we have an element $f_{-1} \in X_{\widetilde{\omega}^{-1}} \cap \widehat{\mathcal{M}}$ of the form

$$f_{-1} : \begin{pmatrix} xu^{e-(c+1)} \\ yu^{e-(r+1)} \\ z \end{pmatrix} + E(u)\begin{pmatrix} x'u^{e-(c+1)} \\ y'u^{e-(r+1)} \\ z' \end{pmatrix}$$

where $x, y, z, x', y', z' \in \mathcal{O}_E$ and $(x, y, z) \neq (0, 0, 0)$. By Lemma 4.2-(iv) we necessarily have $z \equiv 0$ modulo ϖ_E.

Case **A** : Assume that $\mathrm{FL}(\overline{\rho}_0) \neq [1:0], [0:1]$, or equivalently, by Proposition 4.1, that $\lambda\mu \neq 0$. Then $\mathbf{v}_{\mathrm{p}}(x) = 0$ as $u^{e-(c+1)}$ is an elementary divisor for $\mathcal{M}/\mathrm{Fil}^2 \mathcal{M}$ and $\mathbf{v}_{\mathrm{p}}(y) > 0$ by Lemma 4.2-(iv). We define $e_c' \in \widehat{\mathcal{M}}$ as follows:

$$e_c' : \begin{pmatrix} x + x'E(u) \\ u^{c-r}(y + y'E(u)) \\ u^{c+1}z' \end{pmatrix}.$$

As $\mathbf{v}_{\mathrm{p}}(x) = 0$, $\underline{e}' \overset{\text{def}}{=} (e_c', e_r, e_{-1})$ is again a framed basis for $\widehat{\mathcal{M}}$. By letting $\alpha \overset{\text{def}}{=} z + pz'$ we therefore have the following coordinates for f_{-1} in the basis $\underline{e}'$:

$$f_{-1} : \begin{pmatrix} u^{e-(c+1)} \\ 0 \\ \alpha \end{pmatrix}$$

where $\mathbf{v}_{\mathrm{p}}(\alpha) > 0$. From now onwards we use the basis $\underline{e}'$ to write the coordinates of the elements in $\widehat{\mathcal{M}}$.

By [HLM], Proposition 2.2.24 we easily deduce:

$$\left(\frac{\mathrm{Fil}^2 \widehat{\mathcal{M}}}{\mathrm{Fil}^2 S\widehat{\mathcal{M}}}\right)_{\widetilde{\omega}^{-1}} = \left\langle \begin{pmatrix} u^{e-(c+1)} \\ 0 \\ \alpha \end{pmatrix}, E(u)\begin{pmatrix} u^{e-(c+1)} \\ 0 \\ \alpha \end{pmatrix}, E(u)\begin{pmatrix} 0 \\ \gamma u^{e-(r+1)} \\ \beta \end{pmatrix} \right\rangle_{\mathcal{O}_E}$$

where $\beta, \gamma \in \mathcal{O}_E$. Moreover, by Lemma 4.2-(v) we necessarily have $\mathbf{v}_{\mathrm{p}}(\beta) > 0$ so that, without loss of generality, we can assume $\gamma = 1$.

By [HLM], Proposition 2.2.24 we have

$$X_{\widetilde{\omega}^r} = \left\langle \begin{pmatrix} u^{e-(c-r)} \\ 0 \\ \alpha u^{r+1} \end{pmatrix}, E(u) \begin{pmatrix} u^{e-(c-r)} \\ 0 \\ \alpha u^{r+1} \end{pmatrix}, E(u) \begin{pmatrix} 0 \\ p \\ \beta u^{r+1} \end{pmatrix} \right\rangle_E.$$

If $0 < \mathbf{v}_\mathrm{p}(\beta) < 1$, then one can easily check that it violates Lemma 4.2-(i). Assume that $\mathbf{v}_\mathrm{p}(\beta) \geq 1$. Then the element e_r' defined by

$$e_r' : \begin{pmatrix} 0 \\ 1 \\ -\frac{\beta}{p} u^{r+1} \end{pmatrix}$$

is in $\left(\frac{\mathrm{Fil}^2 \widehat{\mathcal{M}}}{\mathrm{Fil}^2 S\widehat{\mathcal{M}}} \right)_{\widetilde{\omega}^r}$ and the family $\underline{e}'' \overset{\text{def}}{=} (e_c', e_r', e_{-1})$ is again a framed basis for $\widehat{\mathcal{M}}$. Until the end of case $\mathbf{A}$ we use the basis $\underline{e}''$ to write the coordinates of the elements in $\widehat{\mathcal{M}}$.

Hence, $\left(\frac{\mathrm{Fil}^2 \widehat{\mathcal{M}}}{\mathrm{Fil}^2 S\widehat{\mathcal{M}}} \right)_{\widetilde{\omega}^{-1}}$ is generated by

$$\begin{pmatrix} u^{e-(c+1)} \\ 0 \\ \alpha \end{pmatrix}, E(u) \begin{pmatrix} u^{e-(c+1)} \\ 0 \\ \alpha \end{pmatrix}, E(u) \begin{pmatrix} 0 \\ u^{e-(r+1)} \\ 0 \end{pmatrix}$$

over $\mathcal{O}_E$, and $\left(\frac{\mathrm{Fil}^2 \widehat{\mathcal{M}}}{\mathrm{Fil}^2 S\widehat{\mathcal{M}}} \right)_{\widetilde{\omega}^r}$ by

$$\begin{pmatrix} u^{e-(c-r)} \\ 0 \\ \alpha u^{r+1} \end{pmatrix}, E(u) \begin{pmatrix} u^{e-(c-r)} \\ 0 \\ \alpha u^{r+1} \end{pmatrix}, E(u) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

over $\mathcal{O}_E$. Again by [HLM], Proposition 2.2.24 we further deduce

$$X_{\widetilde{\omega}^c} = \left\langle \begin{pmatrix} -p \\ 0 \\ \alpha u^{c+1} \end{pmatrix} + E(u) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, E(u) \begin{pmatrix} -p \\ 0 \\ \alpha u^{c+1} \end{pmatrix}, E(u) \begin{pmatrix} 0 \\ u^{c-r} \\ 0 \end{pmatrix} \right\rangle_E,$$

and an immediate manipulation provides us with:

$$\begin{pmatrix} -\frac{p^2}{\alpha} \\ 0 \\ pu^{c+1} \end{pmatrix} + E(u) \begin{pmatrix} 0 \\ 0 \\ u^{c+1} \end{pmatrix} \in X_{\widetilde{\omega}^c}.$$

By Lemma 4.2-(vi) we necessarily have $\mathbf{v}_\mathrm{p}(\frac{p^2}{\alpha}) > 0$, in particular

$$\begin{pmatrix} -\frac{p^2}{\alpha} \\ 0 \\ pu^{c+1} \end{pmatrix} + E(u) \begin{pmatrix} 0 \\ 0 \\ u^{c+1} \end{pmatrix} \in \left(\frac{\mathrm{Fil}^2 \widehat{\mathcal{M}}}{\mathrm{Fil}^2 S\widehat{\mathcal{M}}} \right)_{\widetilde{\omega}^c}.$$

Hence, we obtain the following inclusion:

$$\left\langle \begin{pmatrix} u^{e-(c+1)} \\ 0 \\ \alpha \end{pmatrix}, E(u) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{p^2}{\alpha} \\ 0 \\ pu^{c+1} \end{pmatrix} + E(u) \begin{pmatrix} 0 \\ 0 \\ u^{c+1} \end{pmatrix} \right\rangle_{\mathcal{O}_E} \subseteq \frac{\mathrm{Fil}^2 \widehat{\mathcal{M}}}{\mathrm{Fil}^2 S\widehat{\mathcal{M}}}.$$

By Nakayama's lemma and noticing that the elementary divisors of $\mathcal{M}/\mathrm{Fil}^2 \mathcal{M}$ are described by Lemma 4.2-(i) we conclude that the inclusion is indeed an equality.

Case $\mathbf{B}$: Assume that $\mathrm{FL}(\overline{\rho}_0) = [1 : 0]$, or equivalently, by Proposition 4.1, that $\lambda = 0$ and $\mu \neq 0$. By exactly the same argument as in the proof of case $\mathbf{A}$, we get the same

$\left(\frac{\mathrm{Fil}^2\,\widehat{\mathcal{M}}}{\mathrm{Fil}^2\,S\widehat{\mathcal{M}}} \right)_{\widetilde{\omega}^{-1}}$ as well as $X_{\widetilde{\omega}^r}$ as in case **A**. If $\mathbf{v}_{\mathrm{p}}(\beta) \geq 1$, then one can easily check that it violates Lemma 4.2-(ii). Assume $0 < \mathbf{v}_{\mathrm{p}}(\beta) < 1$.

As in case **A** we easily deduce

$$E(u)\begin{pmatrix} 0 \\ -\frac{p}{\beta} \\ u^{r+1} \end{pmatrix} \in \left(\frac{\mathrm{Fil}^2\,\widehat{\mathcal{M}}}{\mathrm{Fil}^2\,S\widehat{\mathcal{M}}} \right)_{\widetilde{\omega}^r}$$

and

$$X_{\widetilde{\omega}^c} = \left\langle \begin{pmatrix} -p \\ 0 \\ \alpha u^{c+1} \end{pmatrix} + E(u)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\, E(u)\begin{pmatrix} -p \\ 0 \\ \alpha u^{c+1} \end{pmatrix},\, E(u)\begin{pmatrix} 0 \\ -\frac{p}{\beta}u^{c-r} \\ u^{c+1} \end{pmatrix} \right\rangle_E.$$

In particular,

$$\begin{pmatrix} -\frac{p\beta}{\alpha} \\ 0 \\ \beta u^{c+1} \end{pmatrix} + E(u)\begin{pmatrix} 0 \\ u^{c-r} \\ 0 \end{pmatrix} \in \left(\frac{\mathrm{Fil}^2\,\widehat{\mathcal{M}}}{\mathrm{Fil}^2\,S\widehat{\mathcal{M}}} \right)_{\widetilde{\omega}^c}$$

and, by Lemma 4.2-(vi) we necessarily have $\mathbf{v}_{\mathrm{p}}(\beta) > 0$ and $\mathbf{v}_{\mathrm{p}}(\beta) + 1 > \mathbf{v}_{\mathrm{p}}(\alpha)$.

Hence, we obtain the following inclusion:

$$\left\langle \begin{pmatrix} u^{e-(c+1)} \\ 0 \\ \alpha \end{pmatrix},\, E(u)\begin{pmatrix} 0 \\ -\frac{p}{\beta} \\ u^{r+1} \end{pmatrix},\, \begin{pmatrix} -\frac{p\beta}{\alpha} \\ 0 \\ \beta u^{c+1} \end{pmatrix} + E(u)\begin{pmatrix} 0 \\ u^{c-r} \\ 0 \end{pmatrix} \right\rangle_{\mathcal{O}_E} \subseteq \frac{\mathrm{Fil}^2\,\widehat{\mathcal{M}}}{\mathrm{Fil}^2\,S\widehat{\mathcal{M}}}.$$

which implies that the elementary divisors for $\mathcal{M}/\mathrm{Fil}^2\,\mathcal{M}$ are necessarily of the form described by Lemma 4.2-(ii). It follows, as for case **A**, that the inclusion is actually an equality and the case **B** claimed in the statement of the proposition follows.

Case **C** :Assume that $\mathrm{FL}(\overline{\rho}_0) = [0 : 1]$, or equivalently, by Proposition 4.1, that $\lambda \neq 0$ and $\mu = 0$. We may assume that $y = 1$ as $u^{e-(r+1)}$ is an elementary divisor for $\mathcal{M}/\mathrm{Fil}^2\,\mathcal{M}$ and $\mathbf{v}_{\mathrm{p}}(x), \mathbf{v}_{\mathrm{p}}(y) > 0$ by Lemma 4.2-(iv). We define $e'_r \in \widehat{\mathcal{M}}$ as follows:

$$e'_r : \begin{pmatrix} x'u^{e-(c-r)} \\ 1 + y'E(u) \\ z'u^{r+1} \end{pmatrix}.$$

Then $\underline{e}' \overset{\mathrm{def}}{=} (e_c, e'_r, e_{-1})$ is again a framed basis for $\widehat{\mathcal{M}}$. By letting $\alpha \overset{\mathrm{def}}{=} x + px'$ and $\beta \overset{\mathrm{def}}{=} z + pz'$ we therefore have the following coordinates for f_{-1} in the basis $\underline{e}'$:

$$f_{-1} : \begin{pmatrix} \alpha u^{e-(c+1)} \\ u^{e-(r+1)} \\ \beta \end{pmatrix}$$

where $\mathbf{v}_{\mathrm{p}}(\alpha) > 0$ and $\mathbf{v}_{\mathrm{p}}(\beta) > 0$. From now onwards we use the basis $\underline{e}'$ to write the coordinates of the elements in $\widehat{\mathcal{M}}$.

By [HLM], Proposition 2.2.24 we easily deduce:

$$\left(\frac{\mathrm{Fil}^2\,\widehat{\mathcal{M}}}{\mathrm{Fil}^2\,S\widehat{\mathcal{M}}} \right)_{\widetilde{\omega}^{-1}} = \left\langle \begin{pmatrix} \alpha u^{e-(c+1)} \\ u^{e-(r+1)} \\ \beta \end{pmatrix},\, E(u)\begin{pmatrix} \alpha u^{e-(c+1)} \\ u^{e-(r+1)} \\ \beta \end{pmatrix},\, E(u)\begin{pmatrix} \delta u^{e-(c+1)} \\ 0 \\ \gamma \end{pmatrix} \right\rangle_{\mathcal{O}_E}$$

where $\gamma, \delta \in \mathcal{O}_E$. Moreover, by Lemma 4.2-(v) we necessarily have $\mathbf{v}_{\mathrm{p}}(\gamma) > 0$ so that, without loss of generality, we can assume $\delta = 1$.

By [HLM], Proposition 2.2.24 we have

$$X_{\widetilde{\omega}^r} = \left\langle \left(\begin{array}{c} \alpha u^{e-(c-r)} \\ -p \\ \beta u^{r+1} \end{array} \right) + E(u) \left(\begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right), \ E(u) \left(\begin{array}{c} \alpha u^{e-(c-r)} \\ -p \\ \beta u^{r+1} \end{array} \right), \ E(u) \left(\begin{array}{c} u^{e-(c-r)} \\ 0 \\ \gamma u^{r+1} \end{array} \right) \right\rangle_E.$$

If $\min\{1, \mathbf{v}_{\mathrm{p}}(\beta)\} \le \mathbf{v}_{\mathrm{p}}(\alpha)$, then one can easily check that it violates Lemma 4.2-(iii). Assume that $0 < \mathbf{v}_{\mathrm{p}}(\alpha) < \min\{1, \mathbf{v}_{\mathrm{p}}(\beta)\}$. Then easy manipulations provide us with

$$E(u) \left(\begin{array}{c} 0 \\ 1 \\ -\frac{\beta-\alpha\gamma}{p} u^{r+1} \end{array} \right), \ \left(\begin{array}{c} u^{e-(c-r)} \\ -\frac{p}{\alpha} \\ \frac{\beta}{\alpha} u^{r+1} \end{array} \right) + E(u) \left(\begin{array}{c} 0 \\ 0 \\ \frac{\beta-\alpha\gamma}{p\alpha} u^{r+1} \end{array} \right) \in X_{\widetilde{\omega}^r}.$$

Again by [HLM], Proposition 2.2.24 we further deduce

$$X_{\widetilde{\omega}^c} = \left\langle \left(\begin{array}{c} -p \\ -\frac{p}{\alpha} u^{c-r} \\ \frac{\beta}{\alpha} u^{c+1} \end{array} \right) + E(u) \left(\begin{array}{c} 1 \\ 0 \\ \frac{\beta-\alpha\gamma}{p\alpha} u^{c+1} \end{array} \right), \ E(u) \left(\begin{array}{c} 0 \\ u^{c-r} \\ -\frac{\beta-\alpha\gamma}{p} u^{c+1} \end{array} \right), \ E(u) \left(\begin{array}{c} -p \\ 0 \\ \gamma u^{c+1} \end{array} \right) \right\rangle_E,$$

and an immediate manipulation provides us with:

$$\left(\begin{array}{c} -\frac{p^2\alpha}{\beta} \\ -\frac{p^2}{\beta} u^{c-r} \\ p u^{c+1} \end{array} \right) + E(u) \left(\begin{array}{c} 0 \\ 0 \\ u^{c+1} \end{array} \right) \in X_{\widetilde{\omega}^c}.$$

By Lemma 4.2-(vi) we necessarily have $\mathbf{v}_{\mathrm{p}}(\frac{p^2}{\beta}) > 0$, in particular

$$\left(\begin{array}{c} -\frac{p^2\alpha}{\beta} \\ -\frac{p^2}{\beta} u^{c-r} \\ p u^{c+1} \end{array} \right) + E(u) \left(\begin{array}{c} 0 \\ 0 \\ u^{c+1} \end{array} \right) \in \left(\frac{\mathrm{Fil}^2 \, \widehat{\mathcal{M}}}{\mathrm{Fil}^2 \, S\widehat{\mathcal{M}}} \right)_{\widetilde{\omega}^c}.$$

Hence, we obtain that $\frac{\mathrm{Fil}^2 \, \widehat{\mathcal{M}}}{\mathrm{Fil}^2 \, S\widehat{\mathcal{M}}}$ contains

$$\left(\begin{array}{c} -\frac{p^2\alpha}{\beta} \\ -\frac{p^2}{\beta} u^{c-r} \\ p u^{c+1} \end{array} \right) + E(u) \left(\begin{array}{c} 0 \\ 0 \\ u^{c+1} \end{array} \right), \ \left(\begin{array}{c} u^{e-(c-r)} \\ -\frac{p}{\alpha} \\ \frac{\beta}{\alpha} u^{r+1} \end{array} \right) + E(u) \left(\begin{array}{c} 0 \\ 0 \\ \frac{\beta-\alpha\gamma}{p\alpha} u^{r+1} \end{array} \right), \ E(u) \left(\begin{array}{c} \alpha u^{e-(c+1)} \\ u^{e-(r+1)} \\ \beta \end{array} \right).$$

By Nakayama's lemma and noticing that the elementary divisors of $\mathcal{M}/\mathrm{Fil}^2 \, \mathcal{M}$ are described by Lemma 4.2-(iii) we conclude that the inclusion is indeed an equality. Note that $\mathbf{v}_{\mathrm{p}}(\beta - \alpha\gamma) > 1 + \mathbf{v}_{\mathrm{p}}(\alpha)$ by Lemma 4.2-(iii). $\qquad\square$

Corollary 4.4. *Let ρ and $\widehat{\mathcal{M}}$ be respectively a Galois representation and a strongly divisible lattice as in Proposition 4.3. Write $(\lambda_c, \lambda_r, \lambda_{-1})$ for the Frobenius eigenvalue on the $(\widetilde{\omega}^c, \widetilde{\omega}^r, \widetilde{\omega}^{-1})$-isotypic component of the filtered (φ, N)-module $\mathrm{D}_{\mathrm{st}}^{\mathbb{Q}_p, 2}(\rho)$.*

Then the valuation of the Frobenius eigenvalues on $\mathrm{D}_{\mathrm{st}}^{\mathbb{Q}_p, 2}(\rho)$ is described as follows:

A : *if $\mathrm{FL}(\overline{\rho}_0) \in \mathbf{P}^1(\mathbb{F}) \setminus \{[0:1], [1:0]\}$ then*

$$(\mathbf{v}_{\mathrm{p}}(\lambda_c), \mathbf{v}_{\mathrm{p}}(\lambda_r), \mathbf{v}_{\mathrm{p}}(\lambda_{-1})) = (\mathbf{v}_{\mathrm{p}}(\alpha), 1, 2 - \mathbf{v}_{\mathrm{p}}(\alpha))$$

where $0 < \mathbf{v}_{\mathrm{p}}(\alpha) < 2$.

B : *if $\mathrm{FL}(\overline{\rho}_0) = [1:0]$ then*

$$(\mathbf{v}_{\mathrm{p}}(\lambda_c), \mathbf{v}_{\mathrm{p}}(\lambda_r), \mathbf{v}_{\mathrm{p}}(\lambda_{-1})) = (1 + \mathbf{v}_{\mathrm{p}}(\alpha) - \mathbf{v}_{\mathrm{p}}(\beta), \mathbf{v}_{\mathrm{p}}(\beta), 2 - \mathbf{v}_{\mathrm{p}}(\alpha))$$

where $0 < \mathbf{v}_{\mathrm{p}}(\beta)$ and $0 < \mathbf{v}_{\mathrm{p}}(\alpha) < \mathbf{v}_{\mathrm{p}}(\beta) + 1 < 2$.

C : *if* $\mathrm{FL}(\overline{\rho}_0) = [0:1]$ *then*

$$(\mathbf{v}_\mathrm{p}(\lambda_c), \mathbf{v}_\mathrm{p}(\lambda_r), \mathbf{v}_\mathrm{p}(\lambda_{-1})) = (\mathbf{v}_\mathrm{p}(\beta) - \mathbf{v}_\mathrm{p}(\alpha), 1 + \mathbf{v}_\mathrm{p}(\alpha), 2 - \mathbf{v}_\mathrm{p}(\beta))$$

where $0 < \mathbf{v}_\mathrm{p}(\alpha) < 1$ *and* $0 < \mathbf{v}_\mathrm{p}(\alpha) < \mathbf{v}_\mathrm{p}(\beta) < 2$.

Proof. Let us write $s_0 : S_{\mathbb{Q}_p} \to E$ to denote the morphism defined by $u \mapsto 0$. Then one has $\mathrm{D}_\mathrm{st}^{\mathbb{Q}_p,2}(\rho) \cong \widehat{\mathcal{M}}[\frac{1}{p}] \otimes_{S_{\mathbb{Q}_p}, s_0} E$. Moreover, the Frobenius φ on $\widehat{\mathcal{M}}[\frac{1}{p}] \otimes_{S_{\mathbb{Q}_p}, s_0} E$ is uniquely determined by the condition

$$\varphi(\widehat{e}_i \otimes_{s_0} 1) = p^2 (\varphi_2 \otimes_{s_0} 1)(\widehat{f}_i \otimes_{s_0} \kappa_i)$$

for $i \in \{c, r, -1\}$, where the elements $\widehat{e}_i$, $\widehat{f}_i$ can be chosen to be as in Proposition 4.3 and the $\kappa_i \in E$ are such that $\widehat{f}_i \otimes_{s_0} \kappa_i = \widehat{e}_i \otimes_{s_0} 1$.

The result is therefore immediate from the explicit description of the elements $\widehat{f}_i$ given in Proposition 4.3. $\qquad\square$

4.2. From Frobenius eigenvalues to Fontaine–Laffaille parameters.

We are now ready to state the main local result on the Galois. Let $\mathrm{red} : \mathbf{P}^1(\mathcal{O}_E) \to \mathbf{P}^1(\mathbb{F})$ be the natural reduction map on the rational points of the projective lines over $\mathcal{O}_E$. Namely, $\mathrm{red}([x:y])$ is defined as $[\overline{(x/y)} : 1]$ if $\mathbf{v}_\mathrm{p}(x) \geq \mathbf{v}_\mathrm{p}(y)$ and $[1 : \overline{(y/x)}]$ if $\mathbf{v}_\mathrm{p}(x) \leq \mathbf{v}_\mathrm{p}(y)$. We fix a coordinate on $\mathbf{P}^1(\mathcal{O}_E)$ (hence on $\mathbf{P}^1(\mathbb{F})$).

Theorem 4.5. *Let* $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_3(\mathcal{O}_E)$ *be a potentially crystalline Galois representation with parallel Hodge-Tate weights* $\{-2, -1, 0\}$ *and inertial type* $\mathrm{WD}(\rho)|_{I_{\mathbb{Q}_p}} \cong \tau \overset{def}{=} \widetilde{\omega}^c \oplus \widetilde{\omega}^r \oplus \widetilde{\omega}^{-1}$ *such that* $\overline{\rho} \cong \overline{\rho}_0$. *We also let* $(\lambda_c, \lambda_r, \lambda_{-1}) \in (\mathcal{O}_E)^3$ *be the Frobenius eigenvalues on the* $(\widetilde{\omega}^c, \widetilde{\omega}^r, \widetilde{\omega}^{-1})$-*isotypic component of* $\mathrm{D}_\mathrm{st}^{\mathbb{Q}_p,2}(\rho)$.

Then the Fontaine-Laffaille parameter associated to $\overline{\rho}_0$ *is computed by:*

$$\mathrm{FL}(\overline{\rho}_0) = \mathrm{red}\left([\lambda_r : p]\right).$$

The rest of this subsection is devoted to the proof of Theorem 4.5. In the case where $\mathrm{FL}(\overline{\rho}_0) = [0:1]$ or $\mathrm{FL}(\overline{\rho}_0) = [1:0]$, it is straightforward to prove it from the results in the previous subsection (see the end of this subsection) and in what follows we will be firstly interested in the case where $\mathrm{FL}(\overline{\rho}_0) \notin \{[1:0], [0:1]\}$.

Lemma 4.6. *Keep the notation of Proposition 4.3. Define* $\alpha_\bullet \in \mathbb{F}^\times$ *by the condition* $\alpha_\bullet \widehat{e}_\bullet = \frac{\lambda_\bullet}{p^2} \widehat{f}_\bullet$ *modulo* (ϖ_E, u) *for all* $\bullet \in \{c, r, -1\}$ *(note that the* α_i *here is not necessarily the same as the ones in Proposition 4.1), and assume that* $\mathrm{FL}(\overline{\rho}_0) \notin \{[1:0], [0:1]\}$.

If $\mathcal{M} \in \mathbb{F}\text{-BrMod}_\mathrm{dd}^2$ *denotes the associated Breuil module to* $\widehat{\mathcal{M}}$, *then there exists a framed basis* $\underline{e} = (e_c, e_r, e_{-1})$ *on* $\mathcal{M}$ *and a framed system of generators* $\underline{f} = (f_c, f_r, f_{-1})$ *for* $\mathrm{Fil}^2 \mathcal{M}$ *such that* $\mathrm{Mat}_{\underline{e}, \underline{f}}(\varphi_2) = \mathrm{Diag}(\alpha_c, \alpha_r, \alpha_{-1})$ *and*

$$\mathrm{Mat}_{\underline{e}, \underline{f}}(\mathrm{Fil}\,\mathcal{M}) = \begin{pmatrix} 0 & 0 & u^{e-(c+1)} \\ 0 & u^e & u^{e-(r+1)}y \\ u^{e+(c+1)} & u^{e+(r+1)}x & u^e z \end{pmatrix}$$

for some $x, y, z \in \mathbb{F}$.

Proof. The proof follows closely the argument of [HLM], Lemma 2.3.2, which we outline here for the comfort of the reader.

Let $\widehat{\mathcal{M}} \in \mathcal{O}_E\text{-Mod}_\mathrm{dd}^2$ be a strongly divisible lattice as in the statement of Proposition 4.3. In particular we have a framed basis $\widehat{\underline{e}}$ on $\widehat{\mathcal{M}}$ and a framed family $\widehat{\underline{f}}$ of generators for

$\mathrm{Fil}^2\,\widehat{\mathcal{M}}/\mathrm{Fil}^2\,S\cdot\widehat{\mathcal{M}}$ which is explicitly described in terms of $\widehat{\underline{e}}$-coordinates according to the value of $\mathrm{FL}(\overline{\rho}_0)$.

Write $\underline{e}_0$, $\underline{f}_0$ for the base change of $\widehat{\underline{e}}$, $\widehat{\underline{f}}$ via $S \twoheadrightarrow \overline{S}$ and set

$$V_0 \overset{\mathrm{def}}{=} \mathrm{Mat}_{\underline{e}_0,\underline{f}_0}(\mathrm{Fil}^2\,\mathcal{M}), \quad A_0 \overset{\mathrm{def}}{=} \mathrm{Mat}_{\underline{e}_0,\underline{f}_0}(\varphi_2).$$

Note that, by construction, we have $(A_0)_{00}\,\widehat{e}_c \equiv \alpha_c\widehat{e}_c \equiv \frac{\lambda_c}{p^2}\widehat{f}_c$ modulo (u, ϖ_E), and, similarly, $(A_0)_{11}\,\widehat{e}_r \equiv \alpha_r\widehat{e}_r \equiv \frac{\lambda_r}{p^2}\widehat{f}_r$, $(A_0)_{22}\,\widehat{e}_{-1} \equiv \alpha_{-1}\widehat{e}_{-1} \equiv \frac{\lambda_{-1}}{p^2}\widehat{f}_{-1}$. Moreover, by the height condition, we can write $V_0^{adj} = u^e W_0$ where $W_0 \in \mathrm{Mat}_3^{\square}(\overline{S})$ is well defined modulo $u^{e(p-1)}$.

We deduce from Proposition 4.3-Case **A** that the matrix of the filtration for $\mathrm{Fil}^2\,\mathcal{M}$ has the form

$$V_0 = \begin{pmatrix} 0 & 0 & u^{e-(c+1)} \\ 0 & u^e & 0 \\ u^{e+(c+1)} & 0 & 0 \end{pmatrix}.$$

Then there exists $b_{12}, b_{21}, b_{22} \in \mathbb{F}$ such that

(4.2.1)

(4.2.2) $\qquad -W_0 \cdot A_0 \cdot \underbrace{\begin{pmatrix} 0 & 0 & u^{e-(c+1)} \\ 0 & u^e & u^{e-(r+1)}b_{12} \\ u^{e+(c+1)} & u^{e+(r+1)}b_{21} & u^e b_{22} \end{pmatrix}}_{\overset{\mathrm{def}}{=}V_1} = u^{2e}B_0$

where $B_0 \in \mathrm{GL}_3^{\square}(\overline{S})$ verifies moreover

$$B_0 \equiv \begin{pmatrix} \alpha_{-1} & u^{e-(c-r)}\beta_{01} & u^{e-(c+1)}\beta_{02} \\ 0 & \alpha_r & u^{e-(r+1)}\beta_{12} \\ 0 & 0 & \alpha_c \end{pmatrix} \bmod u^e$$

for some $\beta_{ij} \in \mathbb{F}$. Indeed, an elementary computation shows that it suffices to take $b_{12} \equiv -\alpha_r^{-1}a_{10}$, $b_{21} \equiv -\alpha_{-1}a_{21}$ and $b_{22} \equiv -\alpha_{-1}^{-1}(a_{21}b_{12} + a_{20})$ modulo u^e, where the a_{ij}'s denote the corresponding entries of A_0.

By Lemma 2.19 we deduce that V_1 describes the coordinates of a framed system of generators $\underline{f}_1$ for $\mathrm{Fil}^2\,\mathcal{M}$ with respect to the basis $\underline{e}_1 \overset{\mathrm{def}}{=} \underline{e}_0 \cdot A_0$ and moreover $A_1 \overset{\mathrm{def}}{=} \mathrm{Mat}_{\underline{e}_1,\underline{f}_1}(\varphi_2) = \varphi(B_0)$ is the matrix for the associated Frobenius action.

We now iterate the previous procedure: as $A_1 \in \mathrm{Diag}(\alpha_{-1},\alpha_r,\alpha_c) + u^3\,\mathrm{Mat}_3^{\square}(\overline{S})$ (by the genericity assumption (2.1.2)), we easily find $V_2 \in \mathrm{Mat}_3^{\square}(\overline{S})$ as in the statement, and $B_1 \in \mathrm{Diag}(\alpha_c,\alpha_r,\alpha_{-1}) + u\,\mathrm{Mat}_3^{\square}(\overline{S})$ verifying:

$$A_1 V_2 \equiv B_1 V_1 \pmod{u^{3e}}.$$

By virtue of Lemma 2.19, this completes the proof. $\qquad\qquad\square$

Lemma 4.7. *Keep the notation of Lemma 4.6 and assume that* $\mathrm{FL}(\overline{\rho}_0) \notin \{[1:0],\,[0:1]\}$. *Let* $M \in \mathbb{F}\text{-}\mathcal{FL}^{[0,p-2]}$ *be the contravariant Fontaine-Laffaille module associated to* $\overline{\rho}_0$.

Then there exists a basis $\underline{f}$ on M, compatible with its Hodge filtration, such that the Frobenius action on M is described by

$$
\mathrm{Mat}_{\underline{f}}(\phi_\bullet) = \begin{pmatrix} 0 & y\alpha_r^{-1} & \alpha_c^{-1} \\ \alpha_{-1}^{-1}x & 0 & -\dfrac{\alpha_c^{-1}}{y} \\ 0 & 0 & \dfrac{\alpha_c^{-1}}{xy} \end{pmatrix}
$$

for some $x, y \in \mathbb{F}^\times$.

Proof. By Lemma 4.6 and Lemma 2.17, the Frobenius action on the $(\phi, \mathbb{F}((\varpi)))$-module $\mathfrak{M} \overset{\mathrm{def}}{=} M_{\mathbb{F}_p((\varpi))}(\mathcal{M}^*)$ is described by

$$
\mathrm{Mat}_{\underline{\mathfrak{e}}}(\phi) \;=\; \begin{pmatrix} 0 & 0 & \varpi^{e+(c+1)}\alpha_{-1}^{-1} \\ 0 & \varpi^e\alpha_r^{-1} & \varpi^{e+(r+1)}x\alpha_{-1}^{-1} \\ \varpi^{e-(c+1)}\alpha_c^{-1} & \varpi^{e-(r+1)}y\alpha_r^{-1} & \varpi^e z\alpha_{-1}^{-1} \end{pmatrix}
$$

where $\underline{\mathfrak{e}} = (\mathfrak{e}_{-c}, \mathfrak{e}_{-r}, \mathfrak{e}_1)$ is a framed basis for the dual type $\tau^\vee$ and $x, y, z \in \mathbb{F}$.

By performing the change of basis $\underline{\mathfrak{e}}' \overset{\mathrm{def}}{=} (\varpi^c\mathfrak{e}_c, \varpi^r\mathfrak{e}_r, \varpi^{-1}\mathfrak{e}_1)$, it can be easily checked that $\mathfrak{M} = \mathfrak{M}_0 \otimes_{\mathbb{F}((\underline{p}))} \mathbb{F}((\varpi))$ where the $(\phi, \mathbb{F}((\underline{p})))$-module $\mathfrak{M}_0$ is described by

$$
\mathrm{Mat}(\phi_0) = \begin{pmatrix} 0 & 0 & \alpha_{-1}^{-1} \\ 0 & \alpha_r^{-1} & x\alpha_{-1}^{-1} \\ \alpha_c^{-1} & y\alpha_r^{-1} & z\alpha_{-1}^{-1} \end{pmatrix} \mathrm{Diag}(\underline{p}^{c+1}, \underline{p}^{r+1}, 1)
$$

i.e., by an evident change of basis over $\mathbb{F}$,

$$
\mathrm{Mat}(\phi_0) = \mathrm{Diag}(1, \underline{p}^{r+1}, \underline{p}^{c+1}) \underbrace{\begin{pmatrix} z\alpha^{-1} & y\alpha_r^{-1} & \alpha_c^{-1} \\ x\alpha_{-1}^{-1} & \alpha_r^{-1} & 0 \\ \alpha_{-1}^{-1} & 0 & 0 \end{pmatrix}}_{\overset{\mathrm{def}}{=} F}.
$$

By Lemma 2.14 we deduce that $\mathfrak{M}_0 \cong \mathcal{F}(M)$ for a rank 3 Fontaine-Laffaille module $M \in \mathbb{F}\text{-}\mathcal{FL}^{[0,p-2]}$ with Hodge-Tate weights $\{0, r+1, c+1\}$ and $\mathrm{Mat}_{\underline{f}}(\phi_\bullet) = F$ for a basis $\underline{f}$ on M compatible with the Hodge filtration.

On the other hand the condition $\mathrm{T}^*_{\mathrm{cris}}(M) = \overline{\rho}_0$ implies, by Lemma 2.5, the existence of another basis $\underline{f}'$ on M such that $\mathrm{Mat}_{\underline{f}'}(\phi_\bullet)$ is the one described in (2.1.3). Equivalently, there exists of a change of basis $A \in \mathrm{GL}_3(\mathbb{F})$ from $\underline{f}$ to $\underline{f}'$, compatible with the Hodge filtration (i.e. $A = (a_{ij})_{i,j}$ is lower unipotent) and such that

$$
(4.2.3) \qquad\qquad A \cdot F = \begin{pmatrix} 0 & \eta_1^{-1} & \gamma \\ \eta_0^{-1} & 0 & \delta \\ 0 & 0 & \eta_2^{-1} \end{pmatrix}
$$

for some $\gamma, \delta \in \mathbb{F}$, $\eta_i \in \mathbb{F}^\times$.

This can be easily verified: condition (4.2.3) holds true if and only if one has

$$
z = 0, \quad x, \, y, \, a_{20} \in \mathbb{F}^\times, \quad 1 + ya_{10} = 0
$$
$$
1 + xa_{21} = 0 \quad a_{21} + ya_{20} = 0,
$$

which completes the proof. $\qquad\square$

Proof of Theorem 4.5. First of all, note that Proposition 4.3 and its corollary apply in our context. If $\mathrm{FL}(\overline{\rho}_0) = [1:0]$, then it is immediate that

$$\mathrm{FL}(\overline{\rho}_0) = [1:0] = \mathrm{red}([\lambda_r : p]),$$

since $\mathbf{v}_\mathrm{p}(\lambda_r) < 1$ by Corollary 4.4, Case **B**. Similarly, one can prove the case $\mathrm{FL}(\overline{\rho}_0) = [0:1]$ by Corollary 4.4, Case **C**.

For the case that $\mathrm{FL}(\overline{\rho}_0) \notin \{[1:0], [0:1]\}$ it is also easy to check that

$$\mathrm{FL}(\overline{\rho}_0) = [\alpha_r : 1] = \mathrm{red}([\lambda_r : p]),$$

by Lemma 4.7 and by Definition 2.8. $\square$

5. The local automorphic side

We now need to recall certain group algebra operators for $\mathcal{O}_E[\mathrm{GL}_3(\mathbb{F}_p)]$, $\mathbb{F}[\mathrm{GL}_3(\mathbb{F}_p)]$ which are needed to obtain local-global compatibility in terms of *Hecke* action. In order to introduce such operators, we need some notation. In what follows, we have [Jan03] as a main reference for the notation and terminology.

5.1. **Basic set up.** We let $G \overset{\mathrm{def}}{=} \mathrm{GL}_{3/\mathbb{Z}_p}$, T be the maximal split torus consisting of diagonal matrices and $B \supset T$ the Borel subgroup of upper triangular matrices. The character and cocharacter groups $X^*(T)$, $X_*(T)$ are identified with $\mathbb{Z}^3$ in the usual way. In particular the positive simple roots $\{\alpha_1, \alpha_2\}$ for the pair (B, T) become $\alpha_1 = (1, -1, 0)$, $\alpha_2 = (0, 1, -1)$. Finally, we let $\overline{G}$, $\overline{B}$, ... denote the base change of G, B, ... via $\mathbb{Z}_p \twoheadrightarrow \mathbb{F}_p$.

The Weyl group W_G of G is canonically isomorphic to the Weyl group of $\overline{G}$. We write $w_0 \in W_G$ for the longest element and define

$$\dot{s}_1 \overset{\mathrm{def}}{=} \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}, \qquad\qquad \dot{s}_2 \overset{\mathrm{def}}{=} \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}$$

which are lifts in $G(\mathbb{Z}_p)$ of the simple reflections $s_1, s_2 \in W_G$ corresponding to α_1, α_2. In particular $\dot{w}_0 \overset{\mathrm{def}}{=} \dot{s}_1 \dot{s}_2 \dot{s}_1$ is a lift of $w_0 \in W_G$.

For any dominant character $\lambda \in X^*(\overline{T})$ we let

$$H^0(\lambda) \overset{\mathrm{def}}{=} \left(\mathrm{Ind}_{\overline{B}}^{\overline{G}} w_0 \lambda \right)^{\mathrm{alg}} \otimes_{\mathbb{F}_p} \mathbb{F}$$

be the associated dual Weyl module. It is an algebraic representation of $\overline{G}$ (or more precisely of $\overline{G}_{/\mathbb{F}}$) and we write $F(\lambda) \overset{\mathrm{def}}{=} \mathrm{soc}_{\overline{G}} \left(H^0(\lambda) \right)$ for its irreducible socle. If the weight λ is p-restricted, i.e. if $0 \leq \langle \lambda, \alpha_i^\vee \rangle \leq p - 1$ for $i = 1, 2$, then $F(\lambda)$ is irreducible as a $\overline{G}(\mathbb{F}_p)$-representation (see for example [Her09], Corollary 3.17).

As in [HLM] we let I be the Iwahori subgroup of $G(\mathbb{Z}_p)$ (preimage of $\overline{B}(\mathbb{F}_p)$ under the reduction map $G(\mathbb{Z}_p) \twoheadrightarrow \overline{G}(\mathbb{F}_p)$) and $I_1 \leq I$ for its maximal pro-p subgroup. If V is a smooth representation of $G(\mathbb{Z}_p)$ over $\mathcal{O}_E$ and $a_i \in \mathbb{Z}$ we write $V^{I,(a_2, a_1, a_0)}$ to denote the $\widetilde{\omega}^{a_2} \otimes \widetilde{\omega}^{a_1} \otimes \widetilde{\omega}^{a_0}$-isotypic component for the I-action on V^{I_1}.

5.2. **Group algebra operators and the automorphic parameter.** Let $(a, b, c) \in \mathbb{Z}^3$ be a triple satisfying condition (2.1.2) (when specialized at $(a_2, a_1, a_0) = (a, b, c)$). In this case the weight (a, b, c) is in particular restricted. In [HLM] the following elements of $\mathbb{F}[\overline{G}(\mathbb{F}_p)]$

are defined:

$$S \stackrel{\mathrm{def}}{=} \sum_{x,y,z \in \mathbb{F}_p} x^{p-(a-c)} z^{p-(b-c)} \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \dot{w}_0$$

$$S' \stackrel{\mathrm{def}}{=} \sum_{x,y,z \in \mathbb{F}_p} x^{p-(a-b)} z^{p-(a-c)} \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \dot{w}_0$$

as well as their characteristic zero counterparts

$$\widehat{S} \stackrel{\mathrm{def}}{=} \sum_{x,y,z \in \mathbb{F}_p} \widetilde{x}^{p-(a-c)} \widetilde{z}^{p-(b-c)} \begin{pmatrix} 1 & \widetilde{x} & \widetilde{y} \\ 0 & 1 & \widetilde{z} \\ 0 & 0 & 1 \end{pmatrix} \dot{w}_0$$

$$\widehat{S}' \stackrel{\mathrm{def}}{=} \sum_{x,y,z \in \mathbb{F}_p} \widetilde{x}^{p-(a-b)} \widetilde{z}^{p-(a-c)} \begin{pmatrix} 1 & \widetilde{x} & \widetilde{y} \\ 0 & 1 & \widetilde{z} \\ 0 & 0 & 1 \end{pmatrix} \dot{w}_0.$$

The behavior of such operators is described in [HLM], §3 and we include here the statements for the comfort of the reader.

Proposition 5.1. *Let $(a,b,c) \in \mathbb{Z}^3$ be a triple satisfying (2.1.2)) (when specialized at $(a_2, a_1, a_0) = (a,b,c)$) and consider the associated operators S, $S' \in \mathbb{F}[\bar{G}(\mathbb{F}_p)]$.*

(i) *There is a unique non-split extension of irreducible $\bar{G}(\mathbb{F}_p)$-representations*

$$0 \to F(a-1, b, c+1) \to V \to F(b+(p-1), a, c) \to 0$$

and S induces an isomorphism $S : V^{I,(b,a,c)} \xrightarrow{\sim} V^{I,(a-1,b,c+1)}$ of one-dimensional vector spaces.

(ii) *There is a unique non-split extension of irreducible $\bar{G}(\mathbb{F}_p)$-representations*

$$0 \to F(a-1, b, c+1) \to V' \to F(a, c, b-(p-1)) \to 0$$

and S' induces an isomorphism $S' : (V')^{I,(a,c,b)} \xrightarrow{\sim} (V')^{I,(a-1,b,c+1)}$ of one-dimensional vector spaces.

In characteristic zero, we have:

Proposition 5.2. *Let $(a,b,c) \in \mathbb{Z}^3$ be a triple satisfying (2.1.2)) (when specialized at $(a_2, a_1, a_0) = (a,b,c)$). Let $\pi_p \stackrel{\mathrm{def}}{=} \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \left(\chi_b \otimes \chi_a \otimes \chi_c \right)$ be a principal series representation, where the smooth characters $\chi_\bullet : \mathbb{Q}_p^\times \to E^\times$ verify $\chi_\bullet|_{\mathbb{Z}_p^\times} = \widetilde{\omega}^\bullet$ for $\bullet \in \{a,b,c\}$.*

On the one-dimensional isotypic component $\pi_p^{I,(b,a,c)}$ we have

$$(5.2.1) \qquad \widehat{S}' \circ \begin{pmatrix} & & 1 \\ & 1 & \\ p & & \end{pmatrix} = p\chi_b(p)\eta\,\widehat{S},$$

where the element $\eta \in \mathbb{Z}_p^\times$ verifies $\eta \equiv (-1)^{b-c} \cdot \frac{a-b}{b-c} \pmod p$.

Recall that if σ is a smooth representation of $G(\mathbb{Q}_p)$ we can define certain U_p-operators on isotypic components of σ^{I_1}. Concretely, by letting $t_1 \stackrel{\mathrm{def}}{=} \begin{pmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $t_2 \stackrel{\mathrm{def}}{=} \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the U_i operator is defined as the double coset operator $[I_1 t_i I_1]$, i.e.

$$U_i(v) = \sum_{x \in I_1/(t_i I_1 t_i^{-1} \cap I_1)} x t_i v.$$

Lemma 5.3 ([HLM] Lemma 3.1.8). *Let $(a, b, c) \in \mathbb{Z}^3$ be a triple with $a - b > 0$, $b - c > 0$, $a - c < p - 1$ and define $\overline{\tau} \stackrel{def}{=} \mathrm{Ind}_I^K \left(\omega^b \otimes \omega^a \otimes \omega^c \right)$. Let σ be a representation of $G(\mathbb{Q}_p)$ over $\mathbb{F}$. Then*

$$\mathrm{Hom}_K(\overline{\tau}, \sigma)[U_i] = \mathrm{Hom}_K(\overline{\tau}/M_i, \sigma)$$

for $i \in \{1, 2\}$, where M_1 (resp. M_2) is the minimal subrepresentation of $\overline{\tau}$ containing $F(a, c, b - p + 1)$ (resp. $F(c + p - 1, b, a - p + 1)$) as subquotient.

In characteristic zero, we have:

Lemma 5.4 ([HLM] Lemma 3.2.8). *Let $\pi_p \stackrel{def}{=} \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \left(\chi_b \otimes \chi_a \otimes \chi_c \right)$ be a principal series representation, where the smooth characters $\chi_\bullet : \mathbb{Q}_p^\times \to E^\times$ verify $\chi_\bullet|_{\mathbb{Z}_p^\times} = \widetilde{\omega}^\bullet$ for $\bullet \in \{a, b, c\}$ and where a, b, c are distinct modulo $p - 1$.*

  (i) *On the one-dimensional isotypic component $\pi_p^{I,(b,a,c)}$ we have $U_1 = \chi_b(p)^{-1}$ and $U_2 = \chi_b(p)^{-1}\chi_a(p)^{-1}$.*
  (ii) *On the one-dimensional isotypic component $\pi_p^{I,(a,c,b)}$ we have $U_1 = p\chi_a(p)^{-1}$ and $U_2 = p^2\chi_a(p)^{-1}\chi_c(p)^{-1}$.*

6. Local-global compatibility

This section contains the main global application of the local results obtained in Section 4. We follow closely the setup of [HLM], which we reproduce in Section 6.1, Section 6.2 for the convenience of the reader.

6.1. Automorphic forms on unitary groups.

Let $F/\mathbb{Q}$ be a CM field, $F^+ \neq \mathbb{Q}$ its maximal totally real subfield. We write c for the generator of $\mathrm{Gal}(F/F^+)$ and assume that all places v of F^+ above p further decompose as $v = ww^c$ in F. We let S_p^+ (resp. S_p) the set of places of F^+ (resp. F) above p. For v (resp. w) a finite place of F^+ (resp. F) we write k_v (resp. k_w) for the residue field of F_v^+ (resp. F_w).

We let $G_{/F^+}$ be a reductive group, which is an outer form for GL_3, and which splits over F. We assume that $G(F_v^+) \simeq U_3(\mathbb{R})$ for all $v | \infty$. By [CHT08], Section 3.3, G admits an integral model $\mathcal{G}$ such that $\mathcal{G} \times \mathcal{O}_{F_v^+}$ is reductive if v is a finite place of F^+ which splits in F. If v is such a place and $w|v$ is a place of F, we obtain and fix an isomorphism

$$(6.1.1) \qquad\qquad \iota_w : \mathcal{G}(\mathcal{O}_{F_v^+}) \stackrel{\sim}{\to} \mathcal{G}(\mathcal{O}_{F_w}) \stackrel{\sim}{\to} \mathrm{GL}_3(\mathcal{O}_{F_w}).$$

Define $F_p^+ \stackrel{def}{=} F^+ \otimes_\mathbb{Q} \mathbb{Q}_p$ and $\mathcal{O}_{F^+,p} \stackrel{def}{=} \mathcal{O}_{F^+} \otimes_\mathbb{Z} \mathbb{Z}_p$.

If W is a $\mathcal{O}_E$-module endowed with an action of $\mathcal{G}(\mathcal{O}_{F^+,p})$ and $U \leq G(\mathbf{A}_{F^+}^{\infty,p}) \times \mathcal{G}(\mathcal{O}_{F^+,p})$ is a compact open subgroup, the space of algebraic automorphic forms on G of level U and coefficients in W is the following $\mathcal{O}_E$-module:

$$(6.1.2) \quad S(U, W) \stackrel{def}{=} \left\{ f : G(F^+) \backslash G(\mathbf{A}_{F^+}^\infty) \to W \mid f(gu) = u_p^{-1} f(g) \ \forall \ g \in G(\mathbf{A}_{F^+}^\infty), u \in U \right\}$$

(with the usual notation $u = u^p u_p$ for the elements in U).

Recall that the level U is *sufficiently small* if $t^{-1}G(F^+)t \cap U$ has order prime to p for all $t \in G(\mathbf{A}_{F^+}^\infty)$. For a finite place v of F^+ we say that U is *unramified* at v if one has a decomposition $U = \mathcal{G}(\mathcal{O}_{F_v^+})U^v$ for some compact open $U^v \leq G(\mathbf{A}_{F^+}^{\infty,v})$. If w is a finite place of F we say, with an abuse, that w is an unramified place for U if its restriction $w|_{F^+}$ is unramified for U.

Let $\mathcal{P}_U$ denote the set consisting of finite places w of F such that $v \stackrel{def}{=} w|_{F^+}$ is split in F, $v \notin S_p^+$ and U is unramified at v. For a subset $\mathcal{P} \subseteq \mathcal{P}_U$ of finite complement and closed

with respect to complex conjugation we write $\mathbf{T}^{\mathcal{P}} = \mathcal{O}_E[T_w^{(i)}, \; w \in \mathcal{P}, \; i \in \{0,1,2,3\}]$ for the universal Hecke algebra on $\mathcal{P}$, where the Hecke operator $T_w^{(i)}$ acts on $S(U,W)$ as the usual double coset operator

$$\iota_w^{-1} \left[\mathrm{GL}_3(\mathcal{O}_{F_w}) \begin{pmatrix} \varpi_w \mathrm{Id}_i & 0 \\ 0 & \mathrm{Id}_{3-i} \end{pmatrix} \mathrm{GL}_3(\mathcal{O}_{F_w}) \right].$$

Remark 6.1. It important to note that for places v which split as $v = ww^c$ in F the composite $c \circ \iota_w$ is conjugate by an element of $\mathrm{GL}_3(\mathcal{O}_{F_{w^c}})$ to the transpose inverse of ι_{w^c} (cf. [EGH13], Section 7.1.1).

We briefly recall the relation between the space $\mathcal{A}$ of classical automorphic forms and the previous spaces of algebraic automorphic forms, in the particular case which is relevant to us.

Let $S \stackrel{\text{def}}{=} \mathrm{Hom}(F, \overline{\mathbb{Q}}_p)$ and, for any place $w|p$, let $S_w \stackrel{\text{def}}{=} \mathrm{Hom}(F_w, \overline{\mathbb{Q}}_p)$, $\overline{S}_w \stackrel{\text{def}}{=} \mathrm{Hom}(k_w, \overline{\mathbb{F}}_p)$. Following [EGH13], Section 7.3 we consider the subset $(\mathbb{Z}_+^3)_0^S$ of dominant weights $\underline{\lambda} = (\underline{\lambda}_\sigma)_\sigma$ verifying the condition

$$(6.1.3) \qquad \lambda_{1,\sigma c} + \lambda_{3,\sigma} = 0, \quad \lambda_{2,\sigma} + \lambda_{2,\sigma c} = 0, \quad \lambda_{3,\sigma c} + \lambda_{1,\sigma} = 0$$

for all triples $\underline{\lambda}_\sigma = (\lambda_{1,\sigma}, \lambda_{2,\sigma}, \lambda_{3,\sigma})$ and all $\sigma \in S$. If $w|p$ and $\lambda \in (\mathbb{Z}_+^3)_0^S$ we write λ_w for the projection of λ on $(\mathbb{Z}_+^3)_0^{S_w}$ and $W_{\underline{\lambda}_w}$ for the $\mathcal{O}_{F_w}$-specialization of the dual Weyl module associated to $\underline{\lambda}_w$ (cf. [EGH13], Section 4.1.1); by condition (6.1.3) and Remark 6.1 one deduces an isomorphism of $\mathcal{G}(\mathcal{O}_{F_v^+})$-representations $W_{\underline{\lambda}_w} \circ \iota_w \cong W_{\underline{\lambda}_{w^c}} \circ \iota_{w^c}$. Therefore, by letting $W_{\underline{\lambda}_v} \stackrel{\text{def}}{=} W_{\underline{\lambda}_w} \circ \iota_w$ for any place $w|v$, the $\mathcal{G}(\mathcal{O}_{F^+,p})$-representation

$$W_{\underline{\lambda}} \stackrel{\text{def}}{=} \bigotimes_{v|p} W_{\underline{\lambda}_v}$$

is well defined.

For a weight $\underline{\lambda} \in (\mathbb{Z}_+^3)_0^S$ and an irreducible smooth $\mathcal{G}(\mathcal{O}_{F^+,p})$-representation τ over $\overline{\mathbb{Q}}_p$, let us write $S_{\underline{\lambda},\tau}(\overline{\mathbb{Q}}_p)$ to denote the inductive limit of the spaces $S(U, W_{\underline{\lambda}} \otimes_{\mathcal{O}_E} \tau)$ over the compact open subgroups $U \leq G(\mathbf{A}_{F^+}^{\infty,p}) \times \mathcal{G}(\mathcal{O}_{F^+,p})$ (note that the latter is an inductive system in a natural way, with injective transition maps induced from the inclusions between levels). Then $S_{\underline{\lambda},\tau}(\overline{\mathbb{Q}}_p)$ has a natural left action of $G(\mathbf{A}_{F^+}^\infty)$ induced by right translation of functions.

Fix an isomorphism $\imath : \overline{\mathbb{Q}}_p \stackrel{\sim}{\to} \mathbf{C}$. As we did for the $\mathcal{O}_{F_w}$-specialization of the dual Weyl modules, we define a smooth $G(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$-representation $\sigma_{\underline{\lambda}} \cong \bigoplus_{v|\infty} \sigma_{\underline{\lambda}_v}$ with $\mathbf{C}$-coefficients, where $\sigma_{\underline{\lambda}_v}$ depends only on $\underline{\lambda}_w$ for a place $w|v$ (we invite the reader to refer to [EGH13], Section 7.1.4 for the precise definition of $\sigma_{\underline{\lambda}}$).

Lemma 6.2. *The isomorphism* $\imath : \overline{\mathbb{Q}}_p \stackrel{\sim}{\to} \mathbf{C}$ *induces an injective morphism of smooth* $G(\mathbf{A}_{F^+}^\infty)$-*representations*

$$S_{\underline{\lambda},\tau}(\overline{\mathbb{Q}}_p) \otimes_{\overline{\mathbb{Q}}_p, \imath} \mathbf{C} \stackrel{\imath}{\longrightarrow} \mathrm{Hom}_{G(F^+ \otimes_{\mathbb{Q}} \mathbb{R})}(\sigma_{\underline{\lambda}}^\vee, \mathcal{A}).$$

If Π *is an irreducible automorphic representation of* $G(\mathbf{A}_{F^+})$, Π_p *contains* $\tau \otimes_{\overline{\mathbb{Q}}_p, \imath} \mathbf{C}$ *if and only if the isotypic space* $\mathrm{Hom}_{G(F^+ \otimes_{\mathbb{Q}} \mathbb{R})}(\sigma_{\underline{\lambda}}^\vee, \Pi)$ *is in the image of* $\imath$.

6.2. **Serre weights.** We recall the notion of Serre weights of $\bar{r} : G_F \to \mathrm{GL}_3(\mathbb{F})$ and relate constituents of $\mathrm{GL}_3(\mathcal{O}_{F_w})$-types and potentially crystalline lifts of $\bar{r}|_{G_{F_w}}$.

Definition 6.3. A *Serre weight for* $\mathcal{G}$ (or just Serre weight if $\mathcal{G}$ is clear from the context) is an isomorphism class of a smooth, absolutely irreducible representation V of $\mathcal{G}(\mathcal{O}_{F^+,p})$. If $v|p$ is a place of F^+, a *Serre weight at* v is an isomorphism class of a smooth, absolutely irreducible representation V_v of $\mathcal{G}(\mathcal{O}_{F_v^+})$. Finally, if $w|p$ is a place of F, a *Serre weight at* w is an isomorphism class of a smooth, absolutely irreducible representation V_w of $\mathrm{GL}_3(\mathcal{O}_{F_w})$.

In particular, if V_v is a Serre weight at v, the Serre weights at w^c defined by $V_v \circ \iota_w^{-1} \circ c$, $V_v \circ \iota_{w^c}^{-1}$ are dual to each other by Remark 6.1.

As explained in [EGH13], Section 7.3, a Serre weight V admits an explicit description in terms of $\mathrm{GL}_3(k_w)$-representations. More precisely, let w be a place of F above p and write $v \overset{\text{def}}{=} w|_{F^+}$. The element $c \in \mathrm{Gal}(F/F^+)$ induces an involution $\overline{S}_w \overset{\sim}{\to} \overline{S}_{w^c}$ and we define the set $\oplus_{w|p}(\mathbb{Z}_+^3)_0^{\overline{S}_w}$ as the set of tuples $(a_w, b_w, c_w)_w$ (where each triple (a_w, b_w, c_w) is dominant) verifying:

$$(6.2.1) \qquad a_{w,\sigma} + c_{w^c,\sigma c} = 0, \; b_{w,\sigma} + b_{w^c,\sigma c} = 0, \; c_{w,\sigma} + a_{w^c,\sigma c} = 0$$

for all $\sigma \in \overline{S}_w$. If the triple $\underline{a}_w \overset{\text{def}}{=} (a_w, b_w, c_w) \in \mathbb{Z}_+^3$ is *restricted* (i.e. $0 \le a_{w,\sigma} - b_{w,\sigma}, \; b_{w,\sigma} - c_{w,\sigma} \le p - 1$ for all $w|p$, $\sigma \in \overline{S}_w$) we consider the Serre weight $F_{\underline{a}_w} = F(a_w, b_w, c_w)$ as defined in [EGH13], Section 4.1.2. It is an irreducible representation of $\mathrm{GL}_3(k_w)$, hence of $\mathcal{G}(k_v)$ and (by inflation) of $\mathcal{G}(\mathcal{O}_{F_v^+})$ via the morphism ι_w.

As above, condition (6.2.1) implies that $F(a_w, b_w, c_w)^\vee \circ \iota_{w^c} \cong F(a_w, b_w, c_w) \circ \iota_w$ as $\mathcal{G}(k_v)$-representations (i.e. $F(a_{w^c}, b_{w^c}, c_{w^c}) \circ \iota_{w^c} \cong F(a_w, b_w, c_w) \circ \iota_w$) and the smooth $\mathcal{G}(\mathcal{O}_{F_v^+})$-representation $F_{\underline{a}_v} \overset{\text{def}}{=} F_{\underline{a}_w} \circ \iota_w$ is well defined.

We set

$$F_{\underline{a}} \overset{\text{def}}{=} \bigotimes_{v|p} F_{\underline{a}_v}$$

which is a Serre weight for $\mathcal{G}(\mathcal{O}_{F^+,p})$. From [EGH13], Lemma 7.3.4 if V is a Serre weight for $\mathcal{G}$, there exists a tuple $\underline{a} = (a_w, b_w, c_w)_w \in \bigoplus_{w|p}(\mathbb{Z}_+^3)_0^{\overline{S}_w}$ and a decomposition $V \cong \bigotimes_{v|p} V_v$ where the V_v are Serre weights at v verifying $V_v \circ \iota_w^{-1} \cong F(a_w, b_w, c_w)$. Again, thanks to condition (6.2.1) and Remark 6.1 we deduce that V_v is well defined.

Definition 6.4. Let $\bar{r} : G_F \to \mathrm{GL}_3(\mathbb{F})$ be a continuous, absolutely irreducible Galois representation and let V be a Serre weight for $\mathcal{G}$. We say that $\bar{r}$ is *automorphic of weight* V (or that V *is a Serre weight of* $\bar{r}$) if there exists a compact open subset U in $G(\mathbf{A}_F^{\infty,p}) \times \mathcal{G}(\mathcal{O}_{F^+,p})$ unramified above p and a cofinite subset $\mathcal{P} \subseteq \mathcal{P}_U$ such that $\bar{r}$ is unramified at each place of $\mathcal{P}$ and

$$S(U, V)_{\overline{\mathfrak{m}_{\bar{r}}}} \ne 0$$

where $\overline{\mathfrak{m}_{\bar{r}}}$ is the kernel of the system of Hecke eigenvalues $\overline{\alpha} : \mathbf{T}^{\mathcal{P}} \to \mathbb{F}$ associated to $\bar{r}$, i.e.

$$\det\left(1 - \bar{r}^\vee(\mathrm{Frob}_w)X\right) = \sum_{j=0}^{3} (-1)^j (\mathbf{N}_{F_w/\mathbb{Q}_p}(w))^{\binom{j}{2}} \overline{\alpha}(T_w^{(j)}) X^j$$

for all $w \in \mathcal{P}$.

In what follows (sections 6.3, 6.4) we will be needing the notion of Serre weight above a specific place $w|p$. That is the reason for the following:

Definition 6.5. Let $\bar{r}: G_F \to \mathrm{GL}_3(\mathbb{F})$ be a continuous Galois representation and let $w_0|v_0$ be places of F, F^+ respectively, above p.

If V_{w_0} is a Serre weight at w_0, we say that $\bar{r}$ is *automorphic of weight V_{w_0} at w_0* (or that V_{w_0} is a Serre weight of $\bar{r}$ at w_0) if for all $v|p$, $v \neq v_0$ there exist Serre weights V_v such that by letting $V^{v_0} \overset{\text{def}}{=} \bigotimes_{v|p,\ v \neq v_0} V_v$, the smooth $\mathcal{G}(\mathcal{O}_{F_p^+})$ representation $V^{v_0} \otimes V_{v_0}$ is a Serre weight of $\bar{r}$ as in Definition 6.4, where $V_{v_0} = V_{w_0} \circ \iota_{w_0}$.

As above, we write $W_w(\bar{r})$ for the set of all Serre weights of $\bar{r}$ at a place $w|p$. Note that condition 6.2.1 implies that $W_w(\bar{r})$ and $W_{w^c}(\bar{r})$ are in natural bijection via the involution $c \in \mathrm{Gal}(F/F^+)$: $V_w \in W_w(\bar{r})$ if and only if $(V_w)^\vee \circ c \in W_{w^c}(\bar{r})$.

We recall some formalism related to Deligne-Lusztig representations and potentially crystalline lifts for $\bar{r}|_{G_{F_{w_0}}}$. We refer the reader to [Her09], Section 4 for a precise reference.

Let $w|p$ be a place of F, $n \in \{1,2,3\}$ and let $k_{w,n}/k_w$ be an extension verifying $[k_{w,n} : k_w] = n$. Let T be a maximal torus in GL_{3/k_w}. Following [Her09], Lemma 4.7 we have an identification

$$(6.2.2) \qquad T(k_w) \overset{\sim}{\longrightarrow} \prod_j k_{w,n_j}^\times$$

where $3 \geq n_j > 0$ and $\sum_j n_j = 3$; the isomorphism is unique up to $\prod_j \mathrm{Gal}(k_{w,n_j}/k_w)$-conjugacy. In particular, any character $\theta: T(k_w) \to \overline{\mathbb{Q}}_p^\times$ can be written as $\theta = \otimes_j \theta_j$ where $\theta_j: k_{w,n_j}^\times \to \overline{\mathbb{Q}}_p^\times$. We say that θ is *primitive* if θ_j is primitive as in [Her09], Section 4.2 for all j.

Given a maximal torus T and a primitive character θ we consider the Deligne-Lusztig representation R_T^θ of $\mathrm{GL}_3(k_w)$. By letting $\Theta(\theta_j)$ be the cuspidal representation of $\mathrm{GL}_{n_j}(k_w)$ associated to the primitive character θ_j via [Her09], Lemma 4.7, we have

$$R_T^\theta \cong (-1)^{n-r} \mathrm{Ind}_{P_{\underline{n}}(k_w)}^{\mathrm{GL}_3(k_w)} \left(\otimes_j \Theta(\theta_j)\right)$$

where $P_{\underline{n}}$ is the standard parabolic subgroup containing the Levi $\prod_j \mathrm{GL}_{n_j}$ and r denotes the number of its Levi factors.

Let $F_{w,n} \overset{\text{def}}{=} W(k_{w,n})[\frac{1}{p}]$; we consider θ_j as a character on $\mathcal{O}_{F_{w,n_j}}^\times$ by inflation and we define the following character $\mathrm{rec}(\theta)$:

(i) $\mathrm{rec}(\theta) \overset{\text{def}}{=} \bigoplus_{j=1}^3 \theta_j \circ \mathrm{Art}_{F_w}^{-1}$ if $\theta_j: k_w \to \overline{\mathbb{Q}}_p^\times$ are niveau one characters;

(ii) $\mathrm{rec}(\theta) \overset{\text{def}}{=} \theta_1 \circ \mathrm{Art}_{F_w}^{-1} \oplus \bigoplus_{\sigma \in \mathrm{Gal}(k_{w,2}/k_w)} \sigma\left(\theta_2 \circ \mathrm{Art}_{F_{w,2}}^{-1}\right)$ if θ_1 is a niveau one character and θ_2 is a niveau 2, primitive character on $k_{w,2}^\times$;

(iii) $\mathrm{rec}(\theta) \overset{\text{def}}{=} \bigoplus_{\sigma \in \mathrm{Gal}(k_{w,3}/k_w)} \sigma\left(\theta_1 \circ \mathrm{Art}_{F_{w,3}}^{-1}\right)$ if θ_1 is a niveau three, primitive character.

From now on we assume that p is unramified in F^+. In particular, the set of embeddings S_w, $\overline{S}_w$ are in natural bijection.

Theorem 6.6. *Assume that p is unramified in F^+ and let w be a place of F above p. Let V_w be a Serre weight at w for the Galois representation $\bar{r}: G_F \to \mathrm{GL}_3(\mathbb{F})$ and assume that V_w is a Jordan-Hölder constituent in the mod-p reduction of a Deligne-Lusztig representation R_T^θ of $\mathrm{GL}_3(k_w)$, where T is a maximal torus in GL_{3/k_w} and $\theta: T(k_w) \to \overline{\mathbb{Q}}_p^\times$ is a primitive character. If $\mathrm{rec}(\theta)$ is as in item (i) above, we assume the characters θ_j are pairwise distinct.*

Then $\bar{r}|_{G_{F_w}}$ has a potentially crystalline lift with parallel Hodge–Tate weights $\{-2,-1,0\}$ and Galois type $\mathrm{rec}(\theta)$.

Proof. This is the statement of [MP], Theorem 5.5. Note that in *loc. cit.* one assumes further that p splits completely in F, but this condition is unnecessary as long as p is unramified in F^+ (the statement of *loc. cit.*, Proposition 5.2 holds true for p unramified in F^+). □

6.3. **Weight elimination.** Let $w_0|v_0$ be places above p of F and F^+ respectively with $F_{w_0} \cong \mathbb{Q}_p$. We define a predicted set of Serre weights $W^?_{w_0}(\bar{r})$ for $\bar{r}$ at w_0. Assume that $\bar{r}|_{G_{F_{w_0}}}$ is of the form (2.1.1). We write $\bar{\rho}_0$ for $\bar{r}|_{G_{F_{w_0}}}$ in this subsection. Recall that we defined in Section 2.1 the Fontaine-Laffaille parameter $\mathrm{FL}(\bar{\rho}_0) \in \mathbf{P}^1(\mathbb{F})$. From now onwards, we fix an affine coordinate in $\mathbf{P}^1(\mathbb{F}) \cong \mathbf{A}^1(\mathbb{F}) \cup \{\infty\}$ via $[x_0 : x_1] \mapsto \frac{x_1}{x_0}$ if $x_0 \neq 0$ and $[0 : 1] \mapsto \infty$.

If $\bar{\rho}_0$ is split, then we let
$$W^?_{w_0}(\bar{r}) = W_L \cup W_U \cup W_S$$
where
$$W_L \stackrel{\mathrm{def}}{=} \{F(a_1 - 1, a_0, a_2 + 2 - p), \; F((p-1) + a_0, a_2, a_1), \; F(a_2 - 1, a_1, a_0 + 1)\};$$
$$W_U \stackrel{\mathrm{def}}{=} \{F((p-1) + a_0, a_1 - 1, a_2 + 2 - p), \; F((p-1) + a_1, a_2, a_0), \; F(a_2 - 1, a_0 + 1, a_1 - (p-1))\};$$
$$W_S \stackrel{\mathrm{def}}{=} \{F(a_2, a_0, a_1 - (p-1)), \; F(p - 2 + a_1, a_2, a_0 + 1), \; F(p - 1 + a_0, a_1, a_2 - (p-1))\}.$$
If $\bar{\rho}_0$ is non-split, then
$$W^?_{w_0}(\bar{r}) = \left\{ \begin{array}{c} F(a_2 - 1, a_1, a_0 + 1), \;\; F((p-1) + a_0, a_1, a_2 - (p-1)), \\ F(a_2 - 1, a_0 + 1, a_1 - (p-1)) \end{array} \right\} \bigcup W$$
where
$$W \stackrel{\mathrm{def}}{=} \begin{cases} \left\{ \begin{array}{c} F(p - 1 + a_0, a_2, a_1), \;\; F(p - 2 + a_1, a_2, a_0 + 1), \\ F(a_2, a_0, a_1 - (p-1)) \end{array} \right\} & \text{if } \mathrm{FL}(\bar{\rho}_0) = \infty; \\[2em] \{F((p-1) + a_1, a_2, a_0)\} & \text{if } \mathrm{FL}(\bar{\rho}_0) = 0; \\[1em] \emptyset & \text{otherwise.} \end{cases}$$

Moreover, we define the set of *obvious* weights at w_0 as
$$W^{?,obv}_{w_0}(\bar{r}) \stackrel{\mathrm{def}}{=} W^?_{w_0}(\bar{r}) \cap (W_L \cup W_U).$$

Theorem 6.7. *Let $w_0|v_0$ be a place above p on F and F^+ respectively with $F_{w_0} \cong \mathbb{Q}_p$, and assume that $\bar{r}|_{G_{F_{w_0}}}$ is of the form (2.1.1) with the generic condition (2.1.2). If V_{w_0} is a modular weight for $\bar{r}$ at w_0, then $V_{w_0} \in W^?_{w_0}(\bar{r})$.*

In what follows, we prove the inclusion $W_{w_0}(\bar{r}) \subseteq W^?_{w_0}(\bar{r})$ under the assumption $a_0 = -1$, $c = a_2 - a_0 - 1$, and $r = a_1 - a_0 - 1$. This assumption is harmless since $W_{w_0}(\bar{r} \otimes \omega^a) = W_{w_0}(\bar{r}) \otimes \omega^a$. The proof is performed case by case, by series of lemmas.

Lemma 6.8. *Keep the assumption as in Theorem 6.7 and assume that $\bar{\rho}_0$ is semisimple. If V_{w_0} is a Serre weight of $\bar{r}$ at w_0, then $V_{w_0} \in W^?_{w_0}(\bar{r})$.*

Proof. Proposition 3.3 tells us the possible Galois types of niveau 1 for the potentially crystalline lifts with Hodge–Tate weights $\{-2, -1, 0\}$ of $\bar{\rho}_0$. Hence, the Serre weights of $\bar{\rho}_0$ are constituents of $\overline{R}^\theta_T$ for θ determined in Proposition 3.3. Moreover, we can restrict our attention to the obvious weights in $\mathrm{JH}\left(\overline{R}^\theta_T\right)$ since a shadow weight is either non-modular or an obvious weight of $\overline{R}^\theta_T$ for another θ. Thus, there are 24 weights we need to consider.

It is easy to check the following:

$$F(p-1, c, r-1) \in \mathrm{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^c \otimes \widetilde{\omega}^{r-2} \otimes \widetilde{\omega}^1;$$

$$F((p-1)+r, p-1, c-1) \in \mathrm{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^{c-2} \otimes \widetilde{\omega}^{r+1} \otimes \widetilde{\omega}^0;$$

$$F(p-1, c-1, r) \in \mathrm{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^{c-1} \otimes \widetilde{\omega}^{r-1} \otimes \widetilde{\omega}^1;$$

$$F(p-2, c+1, r-1) \in \mathrm{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^{c+1} \otimes \widetilde{\omega}^{r-2} \otimes \widetilde{\omega}^0;$$

$$F(c+1, r-1, -1) \in \mathrm{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^{c+2} \otimes \widetilde{\omega}^{r-1} \otimes \widetilde{\omega}^{-2};$$

$$F(c, r, -1) \in \mathrm{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^{c+1} \otimes \widetilde{\omega}^{r} \otimes \widetilde{\omega}^{-2};$$

$$F((p-1)+r, p-2, c) \in \mathrm{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^{c-1} \otimes \widetilde{\omega}^{r+1} \otimes \widetilde{\omega}^{-1}.$$

None of the Galois types θ of niveau 1 above appears in Proposition 3.3. Hence, we can eliminate all of the weights listed above so that we now have 17 weights survived.

Proposition 3.7 tells us the possible Galois types of niveau 2 for the potentially crystalline lifts with Hodge–Tate weights $\{-2, -1, 0\}$ of $\overline{\rho}_0$. It is also easy to check the following:

$$F(c, r-1, 0) \in \mathrm{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^{r-1} \otimes \widetilde{\omega}_2^{-2+p(c+2)};$$

$$F((p-1)+r-1, p-1, c) \in \mathrm{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^{r-1} \otimes \widetilde{\omega}_2^{pc};$$

$$F((p-1), r-1, c-(p-1)) \in \mathrm{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^{r-1} \otimes \widetilde{\omega}_2^{pc};$$

$$F((p-1)+c, p-1, r-1) \in \mathrm{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^{p-1} \otimes \widetilde{\omega}_2^{c+1+p(r-2)};$$

$$F((p-1), r, c-1-(p-1)) \in \mathrm{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^{r} \otimes \widetilde{\omega}_2^{p(c-1)};$$

$$F((p-1)+r, c-1, 0) \in \mathrm{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^{p-1} \otimes \widetilde{\omega}_2^{c-1+pr};$$

$$F((p-1)+r-1, c+1, -1) \in \mathrm{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^{r-1} \otimes \widetilde{\omega}_2^{-2+p(c+2)};$$

$$F(c+1, -1, r-1-(p-1)) \in \mathrm{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^{p-2} \otimes \widetilde{\omega}_2^{c+2+p(r-2)}.$$

None of the Galois types θ of niveau 2 above appears in Proposition 3.7. Hence, we can further eliminate the weights listed above so that there are 9 weights survived, which are exactly the same as the set $W_{w_0}^?(\overline{r})$ for $\overline{\rho}_0$ split. This completes the proof. $\square$

Lemma 6.9. *Keep the assumption as in Theorem 6.7 and assume that $\overline{\rho}_0$ is non-split with* $\mathrm{FL}(\overline{\rho}_0) \neq 0$. *If V_{w_0} is a Serre weight of $\overline{r}$ at w_0, then V_{w_0} is isomorphic to one of the weights in the following list:*

$$F(c-1, r, 0), \ F(p-2, r, c-(p-1)), \ F(c-1, 0, r-(p-1)),$$

$$F(p-2, c, r), \ F(p-2+r, c, 0), \ F((p-1)+c, p-2, r).$$

Proof. It is enough to consider in the set of Serre weights listed in Lemma 6.8. Proposition 3.6, (ii) tells us that we can further eliminate the Galois type $\omega^{c+1} \oplus \omega^{r-1} \oplus \omega^{-1}$. It is

easy to check the following:

$$F((p-1)+(r-1), p-2, c+1) \in \text{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^{c+1} \otimes \widetilde{\omega}^{r-1} \otimes \widetilde{\omega}^{-1};$$

$$F((p-2), r-1, c+1-(p-1)) \in \text{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^{c+1} \otimes \widetilde{\omega}^{r-1} \otimes \widetilde{\omega}^{-1}.$$

Hence, we can eliminate the two weights above.

Proposition 3.10 tells us that we can further eliminate the Galois type $\omega^c \oplus \omega_2^{r+1-2p} \oplus \omega_2^{p(r+1)-2}$. It is easy to check the following:

$$F((p-1)+r, c, -1) \in \text{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^c \otimes \widetilde{\omega}_2^{r+1-2p}.$$

Hence, we can further eliminate this weight as well, so that there are only the six weights in the statement of this lemma remaining. □

Lemma 6.10. *Keep the assumption as in Theorem 6.7 and assume that $\overline{\rho}_0$ is non-split with $\text{FL}(\overline{\rho}_0) \neq \infty$. If V_{w_0} is a Serre weight of $\overline{r}$ at w_0, then V_{w_0} is isomorphic to one of the weights in the following list:*

$$F(c-1, r, 0), \ F(p-2, r, c-(p-1)), \ F(c-1, 0, r-(p-1)),$$

$$F((p-1)+r, c, -1).$$

Proof. It is, again, enough to consider in the set of Serre weights listed in Lemma 6.8. Proposition 3.6 tells us that we can further eliminate the Galois types $\omega^{c+1} \oplus \omega^{r-1} \oplus \omega^{-1}$ and $\omega^c \oplus \omega^{r-1} \oplus \omega^0$. It is easy to check the following:

$$F((p-1)+(r-1), p-2, c+1) \in \text{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^{c+1} \otimes \widetilde{\omega}^{r-1} \otimes \widetilde{\omega}^{-1};$$

$$F((p-2), r-1, c+1-(p-1)) \in \text{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^{c+1} \otimes \widetilde{\omega}^{r-1} \otimes \widetilde{\omega}^{-1};$$

$$F((p-1)+(r-1), c, 0) \in \text{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^c \otimes \widetilde{\omega}^{r-1} \otimes \widetilde{\omega}^0;$$

$$F((p-2), c, r) \in \text{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^c \otimes \widetilde{\omega}^{r-1} \otimes \widetilde{\omega}^0.$$

Hence, we can eliminate the four weights above.

Proposition 3.10, (v) tells us that we can further eliminate the Galois type $\omega^{p-2} \oplus \omega_2^{c+1+p(r-1)} \oplus \omega_2^{r-1+p(c+1)}$. It is easy to check the following:

$$F((p-1)+c, p-2, r) \in \text{JH}\left(\overline{R}_T^\theta\right) \text{ for } \theta = \widetilde{\omega}^{p-2} \otimes \widetilde{\omega}_2^{c+1+p(r-1)}.$$

Hence, we can further eliminate this weight as well in this case, so that there are only the four weights in the statement of this lemma remaining. □

Proof of Theorem 6.7. The lemma 6.8 provides with a complete proof for the case $\overline{\rho}_0$ split. If $\text{FL}(\overline{\rho}_0) = \infty$ then it holds by Lemma 6.9, and if $\text{FL}(\overline{\rho}_0) = 0$ then it holds by Lemma 6.10. Finally, if $\text{FL}(\overline{\rho}_0) \neq 0, \infty$ then, by Lemmas 6.9 and 6.10, the Serre weights are isomorphic to the weights listed in both lemmas. □

6.4. **Local-global compatibility.** *From now on we assume that p is totally split in the CM field F. We fix a place w_0 of F above p and let $v_0 \overset{\text{def}}{=} w_0|_{F^+}$. The aim of this section is to prove that under suitable local hypotheses, the Fontaine-Laffaille invariant $\mathrm{FL}(\bar{\rho}_0)$ defined in Section 2.1 can be recovered from a refined Hecke action when $\bar{\rho}_0 : G_{\mathbb{Q}_p} \to \mathrm{GL}_3(\mathbb{F})$ is realized as a local parameter in an automorphic Galois representation $\bar{r} : G_F \to \mathrm{GL}_3(\mathbb{F})$.*

From now on we assume that the Galois representation $\bar{r} : G_F \to \mathrm{GL}_3(\mathbb{F})$ is automorphic of weight $V_{w_0} = F(a_{2,w_0}, a_{1,w_0}, a_{0,w_0})$ at w_0 (cf. Definition 6.5) Let $\widetilde{V}^{v_0} \overset{\text{def}}{=} \bigotimes_{v|p,\, v \neq v_0} W_{\lambda_v}$ where $W_{\lambda_v} \overset{\text{def}}{=} W_{(a_{2,w}, a_{1,w}, a_{0,w})} \circ \iota_w$ for any $w|v$ (cf. Section 6.1).

We fix a sufficiently small subgroup U of $G(\mathbf{A}_F^{\infty,p}) \times \mathcal{G}(\mathcal{O}_{F^+,p})$, unramified at all places dividing p, and such that

$$W_{w_0}(\bar{r}) = \{\text{Serre weights } V \text{ at } w_0 \text{ such that } S(U, (V \circ \iota_{w_0}) \otimes_{\mathbb{F}} V^{v_0})_{\mathfrak{m}_{\bar{r}}} \neq 0\}$$

where $\mathfrak{m}_{\bar{r}}$ is the system of Hecke eigenvalues associated to $\bar{r}$ in the Hecke algebra $\mathbf{T}^{\mathcal{P}}$ as in Section 6.1 (such a subgroup exists, cf. [EGH13], Remark 7.3.6.). Note that we can write $U = U^{v_0} \times \mathcal{G}(\mathcal{O}_{F_{v_0}^+})$ where $U^{v_0} \leq G(\mathbf{A}^{\infty,v_0})$ is compact open.

We first prove the modularity of certain Serre weights, which will be needed to prove Theorem 6.13. We introduce the following useful notation. If W (resp. V) is a $\mathrm{GL}_3(\mathcal{O}_{F_{w_0}})$-representation over $\mathcal{O}_E$ (resp. over $\mathbb{F}$), we write

$$S(W) \overset{\text{def}}{=} S(U, W \circ \iota_{w_0} \otimes_{\mathcal{O}_E} \widetilde{V}^{v_0}) \quad \left(\text{resp. } S(V) \overset{\text{def}}{=} S(U, V \circ \iota_{w_0} \otimes_{\mathcal{O}_E} V^{v_0})\right).$$

Lemma 6.11. *Assume that $\bar{r} : G_F \to \mathrm{GL}_3(\mathbb{F})$ is absolutely irreducible and automorphic, and that $\bar{\rho}_0 \overset{\text{def}}{=} \bar{r}|_{G_{F_{w_0}}}$ is of the form (2.1.1) with the generic condition (2.1.2). Assume further that $\bar{\rho}_0$ is non-semisimple. Then*

$$\{F(a_2 - 1, a_1, a_0 + 1),\ F(a_2 - 1, a_0 + 1, a_1 - (p-1))\} \subseteq W_{w_0}(\bar{r}).$$

Proof. The argument is not new: it is a "weight cycling" technique for GL_3, first used in [EGH13], Theorem 6.2.3 for a niveau three Galois representation, and recently adapted in the niveau two semisimple case in upcoming work by Hui Gao [Gao].

As in [Gao] one proves that the commuting operators $\overline{T}_1, \overline{T}_2$ (acting on $S(V)_{\mathfrak{m}_{\bar{r}}}$ for any $V \in W_w(\bar{r})$ and defined as in [EGH13], Section 4.2) act nilpotently on $S(V)_{\mathfrak{m}_{\bar{r}}}$ whenever $V \notin \{F(a_2 - 1, a_1, a_0 + 1),\ F(a_2 - 1, a_0 + 1, a_1 - p + 1)\}$ (as $\bar{\rho}$ is not semisimple, it can not have ω^{a_2+1} as a quotient); if else $V \in \{F(a_2 - 1, a_1, a_0 + 1),\ F(a_2 - 1, a_0 + 1, a_1 - p + 1)\}$ then $\overline{T}_2 \neq 0$ (while $\overline{T}_1$ still acts nilpotently).

As $\overline{T}_1$ acts nilpotently on both $S(F(a_2 - 1, a_0 + 1, a_1 - p + 1))_{\mathfrak{m}_{\bar{r}}}$ and $S(F(a_2 - 1, a_1, a_0 + 1))_{\mathfrak{m}_{\bar{r}}}$ we deduce from [EGH13], Proposition 6.1.3 and the upper bound on $W_{w_0}(\bar{r})$ (Theorem 6.7) that $F(a_2 - 1, a_0 + 1, a_1 - p + 1) \in W_{w_0}(\bar{r})$ if and only if $F(a_2 - 1, a_1, a_0 + 1) \in W_{w_0}(\bar{r})$ i.e. that these two weights cycle to each other (this is independent on the value of $\mathrm{FL}(\bar{\rho}_0)$).

Assume that $\mathrm{FL}(\bar{\rho}_0) \notin \{0, \infty\}$ and that $F(p - 1 + a_0, a_1, a_2 - p + 1) \in W_{w_0}(\bar{r})$. As $\overline{T}_i$ act nilpotently on $S(F(p - 1 + a_0, a_1, a_2 - p + 1))_{\mathfrak{m}_{\bar{r}}}$ for $i = 1, 2$ we conclude by [EGH13], Proposition 6.1.3 and the weight elimination above that $F(a_2 - 1, a_1, a_0 + 1) \in W_{w_0}(\bar{r})$.

Assume that $\mathrm{FL}(\bar{\rho}_0) = 0$ and that one of $F(p - 1 + a_0, a_1, a_2 - p + 1)$, $F(a_1 + p - 1, a_2, a_0)$ is modular. By Theorem 6.7 (and again [EGH13] Proposition 6.1.3(ii)) we deduce that $F(a_1 + p - 1, a_2, a_0)$ can be cycled to $F(a_2 - 1, a_1, a_0 + 1)$ via $\overline{T}_2$ (cf. Remark 6.12 (iv) and (v)). Similarly, $F(p - 1 + a_0, a_1, a_2 - p + 1)$ can be cycled to $F(a_2 - 1, a_1, a_0 + 1)$ via $\overline{T}_1$.

Finally, consider the case $\mathrm{FL}(\bar{\rho}_0) = \infty$. As above, the weight $F(a_2, a_0, a_1 - p + 1)$ (resp. $F(p - 1 + a_0, a_1, a_2 - p + 1)$) cycles to $F(a_2 - 1, a_1, a_0 + 1)$ via $\overline{T}_2$ (resp. $\overline{T}_1$). Similarly, $F(a_0 + p - 1, a_2, a_1)$ cycles to $F(a_2, a_0, a_1 - p + 1)$ via $\overline{T}_2$ (resp. to $F(p - 1 + a_0, a_1, a_2 - p + 1)$

via $\overline{T}_1$). Finally, $F(a_1 + p - 2, a_1, a_0 + 1)$ cycles to $F(a_0 + p - 1, a_2, a_1)$ via both $\overline{T}_1$ and $\overline{T}_2$. □

Remark 6.12. In the semisimple case it is easy to prove, along the argument of Lemma 6.11, that either $\{F(a_2 - 1, a_1, a_0 + 1),\ F(a_2 - 1, a_0 + 1, a_1 - p + 1)\} \subseteq W_{w_0}(\bar{r})$ or $\{F(a_0 + p - 1, a_1 - 1, a_2 + 2 - p),\ F(a_1 - 1, a_0, a_2 + 2 - p)\} \subseteq W_{w_0}(\bar{r})$.

Indeed, the *only* weights where $\overline{T}_1$, $\overline{T}_2$ need not both act by zero are $F(a_2 - 1, a_1, a_0 + 1)$, $F(a_2 - 1, a_0 + 1, a_1 - p + 1)$ (where $\overline{T}_1$ may be non-zero, according to the normalizations) and $F(a_0 + p - 1, a_1 - 1, a_2 + 2 - p)$, $F(a_1 - 1, a_0, a_2 + 2 - p)$ (where $\overline{T}_2$ may be non-zero).

By weight cycling an easy but tedious check, using [EGH13] Proposition 6.1.3 and Theorem 6.7 shows that:

(i) $F(a_2 - 1, a_1, a_0 + 1)$, $F(a_2 - 1, a_0 + 1, a_1 - p + 1)$ (resp. $F(a_0 + p - 1, a_1 - 1, a_2 + 2 - p)$, $F(a_1 - 1, a_0, a_2 + 2 - p)$) cycle to each other via $\overline{T}_1$ (resp. via $\overline{T}_2$);

(ii) $F(a_1 - 2 + p, a_2, a_0 + 1)$ cycles to $F(a_0 + p - 1, a_2, a_1)$ (via both $\overline{T}_1$ and $\overline{T}_2$);

(iii) $F(a_0 + p - 1, a_2, a_1)$ can be cycled to either $F(a_0 + p - 1, a_1, a_2 - p + 1)$ (via $\overline{T}_1$) and $F(a_2, a_0, a_1 - p + 1)$ (via $\overline{T}_2$);

(iv) both $F(a_2, a_0, a_1 - p + 1)$ and $F(a_0 + p - 1, a_1, a_2 - p + 1)$) can be cycled to one of the weights in $\{F(a_1 - 1, a_0, a_2 - p + 2),\ F(a_2 - 1, a_1, a_0 + 1)\}$, via $\overline{T}_2$ and $\overline{T}_1$ respectively.

(v) $F(a_1 + p - 1, a_2, a_0)$ can be cycled to one of the weights in $\{F(a_1 - 1, a_0, a_2 - p + 2),\ F(a_2 - 1, a_1, a_0 + 1),\ F(a_0 + p - 1, a_1, a_2 - p + 1)\}$ via $\overline{T}_1$ (resp. to one of the weights in $\{F(a_1 - 1, a_0, a_2 - p + 2),\ F(a_2 - 1, a_1, a_0 + 1),\ F(a_2, a_0, a_1 - p + 1)\}$ via $\overline{T}_2$).

In the following picture, we draw the Hasse diagram of the cosocle filtration in the principal series $\mathrm{Ind}_{\mathbf{B}(\mathbb{F}_p)}^{\mathbf{GL}_3(\mathbb{F}_p)} \omega^{a_2} \otimes \omega^{a_1} \otimes \omega^{a_0}$: letting $e \overset{\text{def}}{=} p - 1$ for brevity,
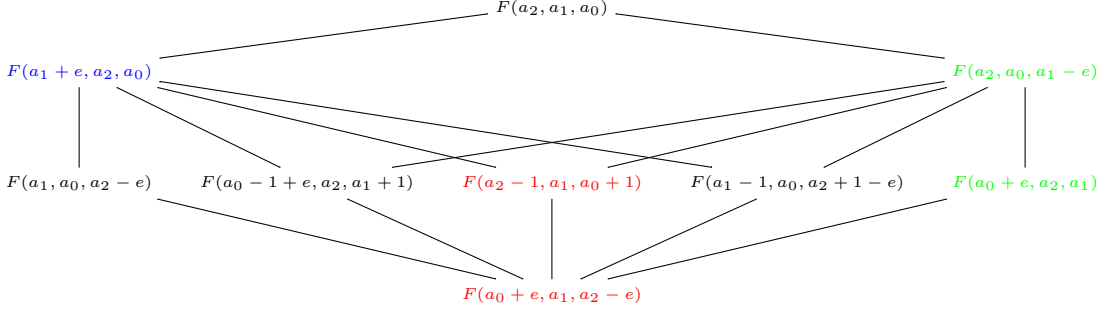
Provided that $\bar{\rho}_0$ is *non*-semisimple as in the statement of Theorem 6.7, the weights in red are always in $W_{w_0}^?(\bar{r})$; the weight in blue is in $W_{w_0}^?(\bar{r})$ if and only if $\mathrm{FL}(\bar{\rho}_0) = 0$ and the weights in green are in $W_{w_0}^?(\bar{r})$ if and only if $\mathrm{FL}(\bar{\rho}_0) = \infty$.

Using the notation and convention of Section 6.2 we define

$$S(U^{v_0}, V^{v_0}) \overset{\text{def}}{=} \{f : G(F^+)\backslash G(\mathbf{A}_{F^+}^\infty) \to V^{v_0} \mid f(gu) = u_p^{-1} f(g) \ \forall\ g \in G(\mathbf{A}_{F^+}^\infty), u \in U^{v_0}\}$$

which is $G(F_v^+)$ representation by right translation of functions. We write $S^{\mathrm{sm}}(U^{v_0}, V^{v_0})$ to denote the submodule of $S(U^{v_0}, V^{v_0})$ consisting of locally constant functions (i.e. the submodule of smooth vectors for the $G(F_{v_0}^+)$ action on $S(U^{v_0}, V^{v_0})$).

Theorem 6.13. *Let F be a CM field in which p splits completely and let $\bar{r} : G_F \to \mathrm{GL}_3(\mathbb{F})$ be an absolutely irreducible and automorphic Galois representation. Let $w_0 | p$ be a place of*

F with $v_0 \overset{\text{def}}{=} w_0|_{F^+}$ and fix a sufficiently small compact open $U = U_{v_0} \times U^{v_0} \le G(\mathbf{A}_F^{\infty,p}) \times \mathcal{G}(\mathcal{O}_{F^+,p})$ where $U^{v_0} \subset G(\mathbf{A}_{F^+}^{\infty,v_0})$. We make the following two assumptions:

(i) $\bar{r}|_{G_{F_{w_0}}}$ is indecomposable of the form (2.1.1) with the strongly generic condition (2.1.2);

(ii) $\mathrm{FL}(\bar{r}|_{G_{F_{w_0}}}) \notin \{0, \infty\}$;

(iii) The $\mathcal{O}_E$-dual of $S^{\mathrm{sm}}(U^{v_0}, \widetilde{V}^{v_0})_{\mathfrak{m}_{\bar{r}}}^{I,(-a_1,-a_0,-a_2)}$ is free over $\mathbf{T}$, where $\mathbf{T}$ denotes the $\mathcal{O}_E$-subalgebra of $\mathrm{End}\left(S^{\mathrm{sm}}(U^{v_0}, \widetilde{V}^{v_0})_{\mathfrak{m}_{\bar{r}}}^{I,(-a_1,-a_0,-a_2)}\right)$ generated by $\mathbf{T}^{\mathcal{P}}$, U_1 and U_2.

Let S, S' be the operators defined in Section 5 specialised to $(a, b, c) = (-a_0, -a_1, -a_2)$. Then

$$(6.4.1) \qquad S' \circ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & 0 & 0 \end{pmatrix} = (-1)^{a_2 - a_1} \cdot \frac{a_1 - a_0}{a_2 - a_1} \cdot \mathrm{FL}(\bar{r}|_{G_{F_{w_0}}}) \cdot S$$

on $S^{\mathrm{sm}}(U^{v_0}, V^{v_0})[\mathfrak{m}_{\bar{r}}]^{I,(-a_1,-a_0,-a_2)}[U_1, U_2]$. Moreover, $S^{\mathrm{sm}}(U^{v_0}, V^{v_0})[\mathfrak{m}_{\bar{r}}]^{I,(-a_1,-a_0,-a_2)}[U_1, U_2]$ is embedded into $S^{\mathrm{sm}}(U^{v_0}, V^{v_0})[\mathfrak{m}_{\bar{r}}]^{I,(-a_0-1,-a_1,-a_2+1)}$ under the map S.

Proof. The proof follows closely the proof of the local-global compatibility statement of [HLM] (Theorem 4.4.1 in *loc. cit.*). We sketch here the argument.

We identify $G(F_{v_0}^+)$ with $\mathrm{GL}_3(\mathbb{Q}_p)$ via ι_{w_0} without further comment. Let $\theta : T(\mathbb{F}_p) \to \mathcal{O}_E^\times$ be the character $\widetilde{\omega}^{a_1} \otimes \widetilde{\omega}^{a_0} \otimes \widetilde{\omega}^{a_2}$, (where T is the maximal split torus in GL_3) and consider the Deligne-Lusztig representation R_T^θ (which will be considered as a smooth $\mathrm{GL}_3(\mathbb{Z}_p)$-representation by inflation).

Recall that we have fixed at the beginning of Section 6.4 the weights $\lambda_w = (a_{w,2}, a_{w,1}, a_{w,0})$ for places $w|v$ above p with $v \ne v_0$. By letting $\lambda_{w_0} \overset{\text{def}}{=} (0,0,0)$ we define the tuple $\underline{\lambda} \overset{\text{def}}{=} \left((\lambda_w)_{v|p,v\ne v_0}, \lambda_{w_0}\right) \in \oplus_{v|p}(\mathbb{Z}^3)$ and set

$$M \overset{\text{def}}{=} S^{\mathrm{sm}}(U^v, \widetilde{V}^{v_0})_{\mathfrak{m}_{\bar{r}}}^{I,(-a_1,-a_0,-a_2)}.$$

We write M_E, $M_{\mathbb{F}}$, $\mathbf{T}_E$ etc. to denote the extension of scalars of M, $\mathbf{T}$ to E, $\mathbb{F}$ etc.

By Lemma 6.11 we have that $S(U, F(a_2-1, a_1, a_0+1))_{\mathfrak{m}_{\bar{r}}} \ne 0$. As $F(a_2-1, a_1, a_0+1) \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ is a constituent of $\overline{R}_T^\theta$ we can lift the system of Hecke eigenvalues associated to $\mathfrak{m}_{\bar{r}}$ to deduce the following:

(i) $M_E = \bigoplus_{\mathfrak{p}} M_E[\mathfrak{p}_E]$ where the direct sum runs over the minimal primes of $\mathbf{T}$;

(ii) For each minimal prime $\mathfrak{p}$ of $\mathbf{T}$ we have $M_E[\mathfrak{p}_E] = \bigoplus_{\pi} m(\pi) \pi_{v_0}^{I,(-a_1,-a_0,-a_2)} \otimes \left(\pi^{\infty,v_0}\right)^{U^{v_0}}$, where $\pi \otimes_E \mathbf{C}$ runs among the cuspidal automorphic representations such that the representation $\pi_\infty \otimes_E \mathbf{C}$ is algebraic, of weight determined by $(\widetilde{V}^{v_0})^\vee$, $r_\pi^\vee$ lifts $\bar{r}$, and the Satake parameters of the base change of π_v to $G(F_w)$ (for $v = w|_{F^+}$ with $w \in \mathcal{P}$) are determined by $\mathfrak{p}_E$;

(iii) there are smooth, E-valued characters $\psi_{a_i} : \mathbb{Q}_p^\times \to \overline{\mathbb{Q}}_p^\times$ such that $\psi_{a_i}|_{\mathbb{Z}_p^\times} = \widetilde{\omega}^{-i}$ for $i \in \{0, 1, 2\}$ and such that for any π as in item (ii) we have

$$\pi_{v_0} \cong \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \psi_{a_1} |\cdot|^2 \otimes \psi_{a_2} |\cdot| \otimes \psi_{a_0};$$

(iv) for π as in item (ii), $r_\pi^\vee|_{G_{F_{w_0}}}$ is potentially crystalline, with Hodge-Tate weights $\{-2, -1, 0\}$ lifting $\bar{r}$, and moreover $\mathrm{WD}(r_\pi|_{G_{F_{w_0}}})^{\text{F-ss}} \cong \psi_{a_1} \oplus \psi_{a_2} \oplus \psi_{a_0}$.

From (iii)-(iv) above and Corollary 4.4 we deduce (cf. Lemma 5.4) that the eigenvalues of the U_p-operators have positive valuation. In particular $\mathbf{T}$ is a finite reduced, local $\mathcal{O}_E$-algebra, with maximal ideal $\mathfrak{m}$ generated by the image of $\mathfrak{m}_{\bar{r}}$, U_1, and U_2.

Moreover, from (iii)-(iv) above and Corollary 4.4 the φ-eigenvalue on $\mathrm{D}_{\mathrm{st}}^{\mathbb{Q}_p,2}(r_\pi^\vee)^{I_{F_{w_0}}=\widetilde{\omega}^{a_1}}$ is given by $p^2\psi_{a_1}(p)^{-1}$ and hence

$$\mathrm{FL}(\bar{r}|_{G_{F_{w_0}}}) = \mathrm{red}(\frac{\psi_{a_1}(p)}{p}).$$

By Proposition 5.2 specialized at $(a,b,c) = (-a_0,-a_1,-a_2)$ we have

(6.4.2) $$\widehat{S'} \circ \Pi\widetilde{v} = \frac{\psi_{a_1}(p)}{p}\eta\widehat{S}\widetilde{v}$$

on $M_E[\mathfrak{p}_E]$.

Assume now that $\mathrm{Hom}_{\mathcal{O}_E}(M,\mathcal{O}_E)$ is free of rank $d \geq 1$ over $\mathbf{T}$. The argument of [HLM], Theorem 4.4.1 shows that $M[\mathfrak{p}]$ is free of rank d and we have an isomorphism

$$M_{\mathbb{F}}[\mathfrak{p}] \xrightarrow{\sim} M_{\mathbb{F}}[\mathfrak{m}]$$

which implies the desired relation (6.4.1) on $M_{\mathbb{F}}[\mathfrak{m}] = (S^{\mathrm{sm}}(U^{v_0},V^{v_0})[\mathfrak{m}_{\bar{r}}])^{I,(-a_1,-a_0,-a_2)}[U_1,U_2]$. Let $N \overset{\mathrm{def}}{=} S^{\mathrm{sm}}(U^v,\widetilde{V}^{v_0})_{\mathfrak{m}_{\bar{r}}}^{I,(-a_0,-a_2,-a_1)}$, $\mathbf{T}'$ the $\mathcal{O}_E$-subalgebra of $\mathrm{End}(N)$ generated by $\mathbf{T}^{\mathcal{P}}$, U_1, U_2, $\mathfrak{m}'$ the maximal ideal of $\mathbf{T}'$ generated by $\mathfrak{m}_{\bar{r}}$, U_1, U_2. Then one sees that Π induces an injective morphism $M_{\mathbb{F}}[\mathfrak{m}] \hookrightarrow N_{\mathbb{F}}[\mathfrak{m}']$.

Let $v \in M_{\mathbb{F}}[\mathfrak{m}]$ be non-zero. Then by the upper bound of Theorem 6.7 we see by Lemma 5.3 and [Le], Proposition 2.2.2 that $\langle K \cdot v \rangle$ is uniserial, of shape $F(-a_0-1,-a_1,-a_2+1)$—$F(-a_1+p-1,-a_0,-a_2)$ and $\langle K \cdot \Pi v \rangle$ is uniserial, of shape $F(-a_0-1,-a_1,-a_2+1)$—$F(-a_0,-a_2,-a_1-p+1)$. Hence Sv, $S' \circ \Pi v$ are non-zero by Proposition 5.1 and the result follows. $\square$

Remark 6.14. There is a symmetry under the involution $w_0 \mapsto w_0^c$. Indeed, if w_0 is a place where $\bar{\rho}_{w_0} \overset{\mathrm{def}}{=} \bar{r}|_{G_{F_{w_0}}}$ admits a Fontaine-Laffaille parameter (in particular, it is non-semisimple, and maximally non-split if its niveau is moreover one) then $\mathrm{FL}(\bar{\rho}_{w_0}) = \iota\left(\mathrm{FL}(\bar{\rho}_{w_0^c})\right)$ where $\iota : \mathbb{P}^1(\mathbb{F}) \to \mathbb{P}^1(\mathbb{F})$ denotes the standard involution on the projective line. Similarly, the role of the group algebra operators is exchanged: one has $S_{w_0} = S'_{w_0^c}$ and $S_{w_0^c} = S'_{w_0}$ (in the obvious notation).

From the proof of Theorem 6.13, we deduce the following modularity result:

Corollary 6.15. *Assume that $\bar{r}$ satisfies the assumption (i) in Theorem 6.13. Then*

$$\{F(a_2-1,a_1,a_0+1),\ F(a_2-1,a_0+1,a_1-(p-1))\} \subseteq W_{w_0}(\bar{r}).$$

Furthermore,

$$\begin{cases} F((p-1)+a_1,a_2,a_0) \in W_{w_0}(\bar{r}), & \text{if } \mathrm{FL}(\bar{r}|_{G_{F_{w_0}}}) = 0; \\ F(a_2,a_0,a_1-(p-1)) \in W_{w_0}(\bar{r}), & \text{if } \mathrm{FL}(\bar{r}|_{G_{F_{w_0}}}) = \infty. \end{cases}$$

Assume moreover that F is unramified at all finite places of F^+ and that there is a RACSDC automorphic representation Π of $\mathrm{GL}_3(\mathbf{A}_F)$ of level prime to p such that

- *$\bar{r} \simeq \bar{r}_{p,i}(\Pi)$;*
- *For each place $w|p$ of F, $r_{p,i}(\Pi)|_{G_{F_w}}$ is potentially diagonalizable;*
- *$\bar{r}(G_{F(\zeta_p)})$ is adequate.*

Then $W_{w_0}^{?,obv}(\bar{r}) \subseteq W_{w_0}(\bar{r})$.

Proof. The first part is immediate from Lemma 6.11. Assume now that $\mathrm{FL}(\bar{r}|_{G_{F_{w_0}}}) = \infty$. The argument is now similar to [HLM], Proposition 4.5.8.

We claim that $F(a_2, a_0, a_1 - p + 1) \in W_{w_0}(\bar{r})$. Suppose that $\langle K \cdot v \rangle$ contains the weight $F(-a_1, -a_2, -a_0 - p + 1)$. Then an easy check (as in the proof of Lemma 6.11) shows that both Hecke operators $\overline{T}_1$ and $\overline{T}_2$ act by zero on $F(a_0 + p - 1, a_2, a_1)$, which implies, by weight cycling and Theorem 6.7 above, that $F(a_2, a_0, a_1 - p + 1)$ is in $W_{w_0}(\bar{r})$.

We now suppose that $\langle K \cdot v \rangle$ does not contain the weight $F(-a_1, -a_2, -a_0 - p + 1)$. Then both $\langle K \cdot v \rangle$ and $\langle K \cdot \Pi v \rangle$ are quotients of the uniserial representations $F(-a_0 - 1, -a_1, -a_2 + 1)$—$F(-a_1 + p - 1, -a_0, -a_2)$ and $F(-a_0 - 1, -a_1, -a_2 + 1)$—$F(-a_0, -a_2, -a_1 - p + 1)$, respectively (by Lemma [HLM], 3.1.8 and Theorem 6.7 above). As $\frac{\psi_{a_1}(p)}{p} = -\frac{1}{\alpha}$ where $0 < \mathbf{v}_p(\alpha) < 1$, the equality (6.4.2) on $M_E[\mathfrak{p}_E]$ implies that $Sv = 0$ for some non-zero $v \in M_{\mathbb{F}}[\mathfrak{m}']$. By Proposition 5.1 (cf. [HLM] Proposition 3.1.2) and the previous observation on $\langle K \cdot v \rangle$ this forces $\langle K \cdot v \rangle$ to have length one, i.e. $F(a_2, a_0, a_1 - p + 1)$ is modular. The case $\mathrm{FL}(\bar{r}|_{G_{F_{w_0}}}) = 0$ is easier and treated similarly.

As for the last statement (which needs to be proved only if $\mathrm{FL}(\bar{\rho}_0) = \infty$), it is enough to remark that for $\mathrm{FL}(\bar{\rho}_0) = \infty$, the representation $\bar{\rho}_0$ admits a potentially diagonalizable lift with Hodge-Tate weights $\{p + 1 + a_0, a_2 + 1, a_1\}$ by Proposition 2.27, and the conclusion follows from [BLGG], Theorem 4.1.9 and Lemma 5.1.1. $\qquad\square$

6.5. Freeness over the Hecke algebra.
In this section, we prove Theorem 6.16, which states that the dual
$$\mathrm{Hom}_{\mathcal{O}_E}(S(U^v, \widetilde{V}^{v_0})_{\mathfrak{m}}^{I, (-a_1, -a_0, -a_2)}, \mathcal{O}_E)$$
of the space of automorphic forms is free over a Hecke algebra for certain choices of compact open subgroup U^{v_0} ($\widetilde{V}^{v_0}$ and $\mathfrak{m}_{\bar{r}}$ are as defined in Section 6.2).

We keep the notation of Section 6.4. Hence $F/\mathbb{Q}$ is a CM field in which p splits, F^+ its maximal totally real field, with F/F^+ unramified at all finite places and $[F : F^+] \equiv 0$ mod 4. Fix a place $w|p$ of F, and let $v \overset{\mathrm{def}}{=} w|_{F^+}$. Let $\bar{r} : G_F \to \mathrm{GL}_3(\mathbb{F})$ be a Galois representation with $\bar{r}|_{G_{F_w}}$ niveau two non-split as in Theorem 6.13 (i) satisfying the following additional properties.

(i) $\bar{r}$ is unramified at places away from p.
(ii) $\bar{r}$ is Fontaine-Laffaille and regular at all places dividing p.
(iii) $\bar{r}$ has image containing $\mathrm{GL}_3(k)$ for some $k \subset \mathbb{F}$ with $\#k > 9$.
(iv) $\overline{F}^{\ker \mathrm{ad}\bar{r}}$ does not contain $F(\zeta_p)$.

By condition (iii) (stronger than the usual condition of adequacy (see Definition 2.3 of [Tho12])) we can choose a place v_1 of F^+ which is prime to p satisfying the following properties (see Section 2.3 of [CEG$^+$]).

$\circ$ v_1 splits in F as $v_1 = w_1 w_1^c$.
$\circ$ v_1 does not split completely in $F(\zeta_p)$.
$\circ$ $\bar{\rho}(\mathrm{Frob}_{w_1})$ has distinct $\mathbb{F}$-rational eigenvalues, no two of which have ratio $(\mathbb{N}v_1)^{\pm 1}$.

We now fix an unitary group $G_{\backslash F^+}$ and a model $\mathcal{G}$ over $\mathcal{O}_{F^+}$ as in Section 6.1. We require moreover that G is quasi-split at all finite places (which is possible by the choice of F). Let $U^{v_0} \leq G(\mathbf{A}_{F^+}^{\infty, v_0})$ be a compact open subgroup satisfying the following properties.

(v) $U_{v_0} = \mathcal{G}(\mathcal{O}_{v_0})$ for all places v which split in F other than v_1 and those dividing p;
(vi) U_{v_1} is the preimage of the upper triangular matrices under the map
$$\mathcal{G}(\mathcal{O}_{v_1}) \to \mathcal{G}(k_{v_1}) \xrightarrow[\iota_{w_1}]{\sim} \mathrm{GL}_3(k_{w_1});$$

(vii) U_v is a hyperspecial maximal compact open subgroup of $G(F_v)$ if v is inert in F.

The choice of the compact open set U_{v_1} implies that $U^{v_0}U_{v_0}$ is sufficiently small in the sense of Section 6.1 for any compact open subgroup U_{v_0} of $G(F_{v_0})$.

Let $\mathcal{P}$ denote the set consisting of finite places w' of F such that $v' \overset{\text{def}}{=} w'|_{F^+}$ is split in F and w' does not divide p or v_1. We define the maximal ideal $\mathfrak{m}_{\bar{r}}$ of $\mathbf{T}^{\mathcal{P}}$ as in 6.4. Recall the space of automorphic forms $S^{\text{sm}}(U^{v_0}, \widetilde{V}^{v_0})_{\mathfrak{m}_{\bar{r}}}^{I,(-a_1,-a_0,-a_2)}$ defined in Section 6.1, which carries a natural action of the algebra $\mathbf{T}^{\mathcal{P}}$ and the operators U_1, U_2. *From now on, we assume that the highest weights $\lambda_w \in (Z_+^3)_0^{S_p}$ appearing in the constituents of $\widetilde{V}^{v_0} \cong \bigotimes_{v|p,\, v \neq v_0} W_{\lambda_v}$ all lie in the lowest alcove (i.e. for all $w|v$, $v \in S_p^+ \setminus \{v_0\}$ we have $a_{2,w} - a_{0,w} < p - 2$).*

We make finally the following assumption:

(viii) $S^{\text{sm}}(U^v, V')_{\mathfrak{m}_{\bar{r}}}$ is nonzero.

Let $\mathbf{T}_a$ (resp. $\mathbf{T}$, resp. $\mathbf{T}_i$) denote the $\mathcal{O}_E$-subalgebra of

$$\text{End}(S^{\text{sm}}(U^{v_0}, \widetilde{V}^{v_0})_{\mathfrak{m}_{\bar{r}}}^{I,(-a_1,-a_0,-a_2)})$$

generated by $\mathbf{T}^{\mathcal{P}}$ (resp. $\mathbf{T}^{\mathcal{P}}$, U_1, and U_2, resp. U_i). Here the subscript a stands for the "anemic" Hecke algebra. See Section 5.2 of [HLM] for the definitions of M_∞ and R_∞. As in [HLM], we let R_i be the R_∞-subalgebra of $\text{End}_{R_\infty}(M_\infty(\tau))$ generated by U_i.

Theorem 6.16. *Let $\bar{r}$ be as in Theorem 6.13 (i).Assume (i)-(viii) in the setup above. If $\text{FL}(\bar{r}|_{G_{F_w}}) \neq \infty$ (resp. $\text{FL}(\bar{r}|_{G_{F_w}}) \neq 0$) then the space*

$$(S^{\text{sm}}(U^{v_0}, \widetilde{V}^{v_0})_{\mathfrak{m}_{\bar{r}}}^{I,(-a_1,-a_0,-a_2)})^{\text{d}} \qquad (resp. \; (S^{\text{sm}}(U^{v_0}, \widetilde{V}^{v_0})_{\mathfrak{m}_{\bar{r}}}^{I,(-a_0,-a_2,-a_1)})^{\text{d}})$$

is free over $\mathbf{T}$, where the superscript "d" stands for Schikhof duality (see Section 1.8 of [CEG+]). Moreover, if $\text{FL}(\bar{r}|_{G_{F_w}}) \notin \{0, \infty\}$ then $R_i = R_\infty$, $\mathbf{T}_i = \mathbf{T}$ and

$$S^{\text{sm}}(U^{v_0}, V^{v_0})[\mathfrak{m}_{\bar{r}}]^{I,(-a_1,-a_0,-a_2)}[U_2] = S^{\text{sm}}(U^{v_0}, V^{v_0})[\mathfrak{m}_{\bar{r}}]^{I,(-a_1,-a_0,-a_2)}[U_1,U_2]$$
$$= S^{\text{sm}}(U^{v_0}, V^{v_0})[\mathfrak{m}_{\bar{r}}]^{I,(-a_0,-a_2,-a_1)}[U_1].$$

Proof. The proof is exactly as in Section 5 of [HLM]. The key point is that Lemma 5.3.3 of [HLM] still holds using Theorem 6.7 in place of Theorem 4.3.1 of [HLM]. $\qquad\square$

Note that by combining Proposition 2.27, Theorems 6.13, 6.16 and [EG] Corollary A.7 we can infer the following:

Theorem 6.17. *Let $\bar{\rho}_0$ be as in Definition 2.4.Then there is a CM field F, an automorphic Galois representation $\bar{r} : G_F \to \text{GL}_3(\mathbb{F})$, verifying $\bar{r}|_{F_w} \cong \bar{\rho}_0$ for all $w|p$, such that all the hypotheses in the setup of Section 6.5 are satisfied.*

In particular Theorem 6.13 applies to $\bar{r}$: if $\text{FL}(\bar{r}|_{G_{F_w}}) \neq \infty$ (resp. $\text{FL}(\bar{r}|_{G_{F_w}}) \neq 0$) then $S^{\text{sm}}(U^{v_0}, V^{v_0})[\mathfrak{m}_{\bar{r}}]^{I,(-a_1,-a_0,-a_2)}[U_2]$ (resp. $S^{\text{sm}}(U^{v_0}, V^{v_0})[\mathfrak{m}_{\bar{r}}]^{I,(-a_0,-a_2,-a_1)}[U_1]$) is free over $\mathbf{T}$ and if moreover $\text{FL}(\bar{r}|_{G_{F_w}}) \notin \{0, \infty\}$ then the equality (6.4.1) of refined Hecke operators on $S^{\text{sm}}(U^{v_0}, V^{v_0})[\mathfrak{m}_{\bar{r}}]^{I,(-a_1,-a_0,-a_2)}[U_1, U_2]$ holds true.

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