# Exponentially small eigenvalues of Witten Laplacians 1: Results

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# Outline

- Thematic and chronological introduction
- Results for functions on manifolds without boundary
- Results for functions on manifolds with boundary
- Results for p-forms
- Open problems

# Thematic and chronological introduction

Statistical physics, stochastic processes, Brownian motion:

position $$dX = -2\nabla f(X)dt + \sqrt{2\beta^{-1}}dW_t \quad , \quad \beta = \frac{1}{k_B T} = \frac{1}{h}$$

phase-space Langevin $$dq = pdt \quad , \quad dp = -\nabla_q f(q)dt - \gamma_0 pdt + \sqrt{\frac{\gamma_0 m}{\beta}}dW_t$$

Invariant measure: $\frac{e^{-2f(x)/h}dx}{\int e^{-2f(x)/h}\,dx}$ concentrated at global minimum of $f$.

Metastability: Escape rate from a local minimum $\propto A(h)e^{-\frac{C}{h}}$.

- Arrhenius (1886, 1910)law : $C$ = energy gap to pass (activity)
- Eyring-Kramers(1935) law: leading term of $A(h)$ in some examples.
- Simulated annealing (1980's)
- Freidlin-Wentzell (1990's): $\lim_{h\to 0} h\log(E(\tau(X(t)|X(0)=x_0))) = C$, $x_0$ local minimum, $\tau$ = exit time from the corresponding valley.
- Bovier-Eckhoff-Gayrard-Klein (2004): Eyring-Kramers type law up to $\mathcal{O}(h^{1/2}\log(h))$-relative error.
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## PDE and spectral theory point of view: Witten Laplacian

$$-L_h = (-h\partial_x + 2\partial_x f(x)).(h\partial_x) \quad \text{on} \quad L^2(M, e^{-2\frac{f(x)}{h}}dx)$$
$$e^{-\frac{f(x)}{h}}(-L_h)e^{\frac{f(x)}{h}} = (-h\partial_x + \partial_x f(x)).(h\partial_x + \partial_x f(x)) \quad \text{on} \quad L^2(M, dx)\,.$$

Witten (1982): $(M, g)$ (compact) riemannian manifold
$d$ differential on $\mathcal{C}^\infty(M; \Lambda T^*M)$ codifferential $d^*$.

$$d_{f,h} = e^{-\frac{f}{h}}(hd)e^{\frac{f}{h}} = hd + df\wedge \qquad d^*_{f,h} = e^{\frac{f}{h}}(hd^*)e^{-\frac{f}{h}} = hd^* + \mathbf{i}_{\nabla f}\,.$$

Witten Laplacian

$$\Delta_{f,h} = (d_{f,h} + d^*_{f,h})^2 = d^*_{f,h}d_{f,h} + d_{f,h}d^*_{f,h} = \bigoplus_{p=0}^{\dim M} \Delta^{(p)}_{f,h}$$

$$\sharp\left\{\mathcal{O}(h^{3/2}) - \text{eigenvalues of} \Delta^{(p)}_{f,h}\right\} = \overbrace{\sharp\left\{\text{critical point with index} p\right\}}^{m_p}$$
$$m_p - m_{p-1} + \cdots + (-1)^p m_0 \underset{=\ \text{for } p=\dim M}{\geq} \beta_p - \beta_{p-1} + \cdots + (-1)^p\beta_0\,.$$

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# Thematic and chronological introduction



Quasi-stationary distributions and applications:

$\Omega$ open domain of $M$ .

$$dX = -2\nabla f(X)dt + \sqrt{2h}dW_t \,.$$

$\tau_x = \tau(X(t)|X(0) = x) = \min\{t\,, X(t,x) \in \partial\Omega\}$ exit time process; $X_\tau$ exit position process.

Definition: $\mu$ probability measure on $\Omega$ is a QSD if

$$E(u(X(t))|t < \tau) = \int_\Omega u(x)\ d\mu(x)$$

for all $t > 0$ when the law of $X(0)$ is $\mu$ .

Link with PDE: Here the QSD is unique and related with the Dirichlet Witten Laplacian. If $x = X(t=0)$ is distributed according to the QSD $\mu$ , the exit time follows a exponential law with parameter $\lambda_1$ and the density of $X_\tau$ on $\partial\Omega$ is given by the normal derivative $\partial_n u_1$ , where $(u_1, \lambda_1)$ firts eigenpair of the Dirichlet Witten Laplacian.

Barlett (1960): "Quasi-stationary distributions".

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# Thematic and chronological introduction



Algebraic topology and applications:

Morse theory: Thom-Smale , Milnor(1963), Laudenbach (1992)

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# Thematic and chronological introduction

Algebraic topology and applications:

- Morse theory: Thom-Smale , Milnor(1963), Laudenbach (1992)
- Barannikov (1994): Another presentation of Morse theory introducing some bar codes, $\beta_p = \dim H_p(M,\mathbb{K})$ , $\mathbb{K}$ a field.
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# Result for functions on manifolds without boundary



REF: Helffer-Klein-N.(04). Bovier-Eckhoff-Gayrard-Klein(04), Hérau-Hitrik-Sjöstrand(08), Michel(16)

$(M, g)$ (compact oriented) riemannian manifold.

$\Delta^{(0)}_{f,h} = d^*_{f,h} d_{f,h}$ restrictied to degree $p = 0$ .

Generic Assumption;

- $f$ is a Morse function
- All critical values of index 0 and 1 are distinct
- All difference $f(U^{(1)}_{j(k)}) - f(U^{(0)}_k)$ are distinct and ordered in the decreasing order (with $j(1) = +\infty$)

Pairing $k \to j(k)$: Consider $f^\lambda = \{x \in M\,, f(x) < \lambda\}$. Decrease $\lambda$ from $+\infty$ to $\min f$. When the number of connected components of $f^\lambda$ increases, $\lambda$ must be a critical value with $\lambda = f(U^{(1)}_j)$. The new global minimum of an appearing connected component is $U^{(0)}_k$ and $j = j(k)$.

The $k$-th, $m_0 \geq k \geq 2$, eigenvalue of $\Delta^{(0)}_{f,h}$ $(\lambda_1(h) = 0)$ equals

$$\lambda_{k\geq 2}(h) = \frac{h}{\pi} |\hat{\lambda}_1(U^{(1)}_{j(k)})| \sqrt{\frac{\left|\det(\mathrm{Hess}\, f(U^{(0)}_k))\right|}{\left|\det(\mathrm{Hess}\, f(U^{(1)}_{j(k)}))\right|}}\,(1 + c_k(h)) \exp\left(-\frac{2(f(U^{(1)}_{j(k)}) - f(U^{(0)}_k))}{h}\right)$$

with $c_k(h) \sim \sum_{\ell=1}^\infty c_\ell h^\ell$ , $\hat{\lambda}_1(U^{(1)}_j)$ negative eigenvalue of $\mathrm{Hess} f(U^{(1)}_j)$ .

## Results for functions on manifolds with boundary

Eyring-Kramers law for exp. small eigenvalues:

REF: Chang-Liu(95), Helffer-N.(06), Le Peutrec(10) $(M, g)$ (compact oriented) manifold with regular boundary $\partial M$.

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Eyring-Kramers law for exp. small eigenvalues:

REF: Chang-Liu(95), Helffer-N.(06), Le Peutrec(10) $(M,g)$ (compact oriented) manifold with regular boundary $\partial M$. Dirichlet and Neumann realizations of $\Delta_{f,h}^{(p)}$:

$$D(\Delta_{f,h}^{D,(p)}) = \left\{\omega \in W^{2,2}(M;\Lambda^p T^*M)\,, \quad \mathbf{t}\omega = 0 \quad , \quad \mathbf{t}d_{f,h}^{*}\omega = 0\right\}\,,$$

$$D(\Delta_{f,h}^{N,(p)}) = \{\omega \in W^{2,2}(M;\Lambda^p T^*M)\,, \quad \mathbf{n}\omega = 0 \quad , \quad \mathbf{n}d_{f,h}\omega = 0\}\,,$$

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$$(D) \quad \mathbf{t}\omega = 0, \mathbf{t}d_{f,h}^{*}\omega = 0 \quad , \quad (N) \quad \mathbf{n}\omega = 0\,, \mathbf{n}d_{f,h}\omega = 0\,.$$

Assumption: $f$ is a Morse function such that $\nabla f$ does not vanish on $\partial M$.
Generalized critical points $U^{(p)}$ of index $p$:

Dirichlet: $U^{(p)} \in M$ is a critical point of index $p$ or $U^{(p)} \in \partial M$ is a critical point of index $p-1$ of $f\big|_{\partial M}$ such that $\partial_n f(U^{(p)}) > 0$.

Neumann: $U^{(p)} \in M$ is a critical point of index $p$ or $U^{(p)} \in \partial M$ is a critical point of index $p$ of $f\big|_{\partial M}$ such that $\partial_n f(U^{(p)}) < 0$.

# Results for functions on manifolds with boundary

Eyring-Kramers law for exp. small eigenvalues:
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$$(D) \quad \mathbf{t}\omega = 0, \mathbf{t}d^{*}_{f,h}\omega = 0 \quad , \quad (N)\mathbf{n}\omega = 0\,, \mathbf{n}d_{f,h}\omega = 0\,.$$

Assumption: $f$ is a Morse function such that $\nabla f$ does not vanish on $\partial M$ .
Generalized critical points with index $p$ on $\partial M$ (D): $p-1, \partial_n f > 0$ , (N): $p, \partial_n f < 0$ .

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Assumption: $f$ is a Morse function such that $\nabla f$ does not vanish on $\partial M$.
Generalized critical points with index $p$ on $\partial M$ (D): $p-1,\partial_n f>0$, (N): $p,\partial_n f<0$.
Then same Generic Assumption and pairing process as for the boundaryless case while replacing critical points by generalized critical points. Result for Dirichlet:

$$\lambda_k(h)=\frac{h}{\pi}|\widehat{\lambda}_1(U_{j(k)}^{(1)})|\sqrt{\frac{\left|\det(\mathrm{Hess}\,f(U_k^{(0)}))\right|}{\left|\det(\mathrm{Hess}\,f(U_{j(k)}^{(1)}))\right|}}\,(1+hc_k^1(h))$$
$$\times\exp\left(-\frac{2\left(f(U_{j(k)}^{(1)})-f(U_k^{(0)})\right)}{h}\right)\quad,\quad \text{if } U_{j(k)}^{(1)}\notin\partial M\,,$$

$$\lambda_k(h)=\frac{2h^{1/2}|\nabla f(U_{j(k)}^{(1)})|}{\pi^{1/2}}\sqrt{\frac{\left|\det(\mathrm{Hess}\,f(U_k^{(0)}))\right|}{\left|\det(\mathrm{Hess}\,f\big|_{\partial M}(U_{j(k)}^{(1)}))\right|}}\,(1+hc_k^1(h))$$
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# Results for functions on manifolds with boundary

Quasi-stationary distribution:

REF: Lelièvre-N.(15), Di Gesu-Lelièvre-Le Peutrec-Nectoux

Generic Assumption on $f_1$ in the boundary case (Dirichlet BC on domain $\Omega$)
Assume that $\min_{x\in\partial\Omega} f_1$ is larger than all interior critical values of $f$ .
$f_2$ is a $\mathcal{C}^\infty$ perturbation of $f_1$ around the global minimum of $f_1$ ($f_2$ not necessarily Morse).

$$\frac{\lambda_1^{(0)}(f_2)}{\lambda_1^{(0)}(f_1)} = \frac{\int_\Omega e^{-2\frac{f_1(x)}{h}}\,dx}{\int_\Omega e^{-2\frac{f_2(x)}{h}}\,dx}(1+\mathcal{O}(e^{-\frac{c}{h}}))\,,$$

$$\frac{\partial_n\left[e^{-\frac{f_2}{h}}u_1^{(0)}(f_2)\right]\big|_{\partial\Omega}}{\|\partial_n\left[e^{-\frac{f_2}{h}}u_1^{(0)}(f_2)\right]\|_{L^1(\partial\Omega)}} = \frac{\partial_n\left[e^{-\frac{f_1}{h}}u_1^{(0)}(f_1)\right]\big|_{\partial\Omega}}{\|\partial_n\left[e^{-\frac{f_1}{h}}u_1^{(0)}(f_1)\right]\|_{L^1(\partial\Omega)}} + \mathcal{O}(e^{-\frac{c}{h}}) \quad \text{in } L^1(\partial\Omega)\,.$$

# Result for $p$-forms



REF: Le Peutrec-N.-Viterbo(13)

$(M,g)$ compact (oriented) manifold without boundary.

Consider $f^{\lambda}=\{x\in M, f(x)<\lambda\}$ and $f_{\lambda}=\{x\in M, f(x)>\lambda\}$.

For $-\infty\leq \mu<\lambda\leq +\infty$, $H_p(f^{\lambda}\big|f^{\mu})$ denotes the relative $p$-homology vector space (here $\mathbb{R}$-valued homology).

Assume that all the critical values are distinct $\to$ we identify the critical point $U$ with the critical value $f(U)=c$.

When $c$ is a critical value with index $p$ then $\dim H_p(f^{c+\varepsilon},f^{c-\varepsilon})=1$. Playing with long exact sequences one can partition critical points into upper,lower and homological critical points

$$\mathcal{U}^{(p)}=\mathcal{U}_U^{(p)}\sqcup\mathcal{U}_L^{(p)}\sqcup\mathcal{U}_H^{(p)}$$

The pairing is as follows: If $\mathcal{U}^{(p)}$ is an upper critical points we associate value $c'=\sup\left\{\lambda<c\,, H_p(f^{c+\varepsilon},f^{\lambda})\to H_p(f^{c+\varepsilon},f^{c-\varepsilon}) \text{ vanishes}\right\}$ then $c'$ is a lower critical value with index $p-1$. Then define $\partial_B c=c'$ (or $\partial_B U^{(p)}=U^{(p-1)}$ with $f(U^{(p-1)})=c'$) in this case and $\partial_B c=0$ (or $\partial_B U^{(p)}=0$) in all the other cases ($U^{(p)}$ a lower or homological critical points).

$(\mathrm{Vect}(U, U\in\mathcal{U}),\partial_B)$ is a chain complex and its degree $p$ homology vector space has the dimension $\beta_p=\dim H_p(M)$.

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## Result for $p$-forms

There is a one to one correspondance $j_p$ between $\mathcal{U}^{(p)}$ and the set of eigenvalues (counted with multiplicities) of $\Delta^{(p)}_{f,h}$ lying in $[0, h^{3/2})$ such that

$$j_p(U^{(p)}) = 0 \quad \text{if} \quad U^{(p)} \in \mathcal{U}^{(p)}_H$$

$$j_p(U^{(p)}) = \kappa^2(U^{(p+1)}) \frac{h}{\pi} \frac{|\lambda_1^{(p+1)} \ldots \lambda_{p+1}^{(p+1)}|}{|\lambda_1^{(p)} \ldots \lambda_p^{(p)}|} \frac{|\mathrm{Hess} f(U^{(p)})|^{1/2}}{|\mathrm{Hess} f(U^{(p+1)})|^{1/2}} (1 + \mathcal{O}(h)) e^{-2\frac{f(U^{(p+1)}) - f(U^{(p)})}{h}}$$

$$\text{if } \partial_B U^{(p+1)} = U^{(p)}$$

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Here the $\lambda$'s denote the negative eigenvalues of the $\mathrm{Hess} f$ at the corresponding points.

# Open problems



Accurate computations of exponentially small eigenvalues for $p$-forms in the case with boundary under Generic Assumption.

For the result on $p$-forms, are the topological constants $\kappa_p(k)^2$ equal to 1 (true for $p = 0$ or $p = \dim M \rightarrow$ true for all $p = 0, 1, 2$ when $\dim M = 2$) ?

Accurate computations of exponentially small eigenvalues for $p$-forms for the hypoelliptic Laplacian under the generic assumption (on manifolds 1-without boundary, 2-with regular boundary).

Extend the QSD results to the Langevin case (requires refinement on the analysis of boundary geometric Kramers-Fokker-Planck operators, parameter dependence).

Remove as much as possible the Generic Assumption and possibly the Morse assumption (connection with bar codes topology in persistent homology to be better understood).

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Francis Nier, LAGA, Univ. Paris 13

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For the result on $p$-forms, are the topological constants $\kappa_p(k)^2$ equal to 1 (true for $p = 0$ or $p = \dim M \to$ true for all $p = 0, 1, 2$ when $\dim M = 2$) ?

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