Phase-space approach to the bosonic mean field dynamics: review, new developments

Francis Nier,
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Joint works with Z. Ammari
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Outline

- Semiclassical and mean field asymptotics
- Wigner (semiclassical) measures
- Wick quantization, (PI)-condition, BBGKY hierarchy
- Propagation results
- Order of convergence
- Other developments

Aims and inspirations

Reconsider the old program: Bosonic QFT=infinite dimensional microlocal analysis (see e.g. Kree's seminar in the 70's).
Mean field=Semiclassical (easier).

Check the mean field convergence for dynamical problems with general initial data.

While doing so find assumptions and results which are invariant by the N-body and mean-field dynamics (when defined).
In the spirit of (semiclassical) propagation of singularities.

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Finite dimensional semiclassical asymptotics

For $w \in \mathcal{Z} = \mathbb{C}^d$ set $a(w) = \sum_j \overline{w}_j a_j \quad , \quad a^*(w) = \sum_j w_j a_j^*$,

$$[a(w), a^*(w')] = 2h\langle w\,,\, w'\rangle_{\mathcal{Z}} = \varepsilon\langle w\,,\, w'\rangle_{\mathcal{Z}} \quad , \quad \varepsilon = 2h$$

The Wick (resp. anti-Wick) quantization associates with the polynomial

$$b(z) \quad = \quad \sum_{\substack{|\beta| = p \\ |\alpha| = q}} b_{\alpha,\beta}\overline{z}^{\alpha} z^{\beta} = \langle z^{\otimes q}\,,\, \tilde{b} z^{\otimes p}\rangle \quad , \quad \tilde{b} = \frac{1}{q!p!}\partial_{\overline{z}}^{q}\partial_{z}^{p} b$$

the operator $b^{Wick} \quad = \quad \sum_{\alpha,\beta} b_{\alpha,\beta} a^{*\alpha} a^{\beta}$, (Wick)

Weyl operator $W(f)$:

$$\Phi(f) = \frac{a(f) + a^*(f)}{\sqrt{2}} = \sqrt{2}\,\mathrm{Re}\,\langle f\,,\, z\rangle^{Wick} \quad , \quad W(f) = e^{i\Phi(f)}\,.$$

Semiclassical annihilation-creation operators:

$$(PDE) \quad a_j = h\partial_{\nu_j} + \nu_j \quad , \quad a_j^* = -h\partial_{\nu_j} + \nu_j \quad , \quad \nu \in \mathbb{R}^d$$

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Example: $\mathbf{N} = (|z|^2)^{Wick} = \sum_j a_j^* a_j = \varepsilon \mathbf{N}_{\varepsilon=1} \quad , \quad \mathbf{N}\varphi_\alpha = \varepsilon|\alpha|\varphi_\alpha$ when φ_α is the α-th Hermite function $\alpha \in \mathbb{N}^d$, $|\alpha| = \sum_j \alpha_j$. $\mathbf{N} = \mathcal{O}(1) \leftrightarrow |\alpha| = \mathcal{O}(\frac{1}{\varepsilon})$.

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If $\hat{b}(\zeta) = \int_{\mathcal{Z}} b(z) e^{-2i\pi\,\mathrm{Re}\,\langle \zeta, z\rangle}\; dL_{\mathcal{Z}}(z)$ then $b(z) = \int_{\mathcal{Z}} \hat{b}(\zeta) e^{2i\pi\,\mathrm{Re}\,\langle \zeta, z\rangle}\; dL_{\mathcal{Z}}(\zeta)$

$$\text{and} \quad b^{Weyl} = b^{Weyl}(\sqrt{h}\nu, \sqrt{h}D_\nu) = \int_{\mathcal{Z}} \hat{b}(\zeta) W(\sqrt{2}\pi\zeta)\; dL_{\mathcal{Z}}(\zeta)\,.$$

Bosonic mean field asymptotics

Bosonic Fock space: Consider now the one particle (separable) complex Hilbert space $\mathcal{Z} = L^2(\mathbb{R}^D, dx; \mathbb{C})$.

$$\mathcal{H} = \Gamma_b(\mathcal{Z}) = \oplus_{n=0}^{\infty} \mathcal{S}_n \mathcal{Z}^{\otimes n} = \oplus_{n=0}^{\infty} \bigvee^n \mathcal{Z} \quad , \quad \bigvee^n \mathcal{Z} = L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C})$$

Energy: $\mathcal{E}(z,\overline{z}) = \langle z\,,\, -\Delta z\rangle + \frac{1}{2}\iint_{\mathbb{R}^{2D}} V(x-y)|z(x)|^2|z(y)|^2\, dxdy$

Nonlinear Hamiltonian dynamics: $i\partial_t z = \partial_{\overline{z}}\mathcal{E}$

Wick quantized Hamiltonian : Take $a = \sqrt{\varepsilon}a_{\varepsilon=1}$ with $\varepsilon > 0$ and set

$$H_\varepsilon = \mathcal{E}(z)^{Wick} = \langle z, -\Delta z\rangle^{Wick} + \frac{1}{2}\langle z^{\otimes 2}\,, V(x-y) z^{\otimes 2}\rangle^{Wick}$$

n-body evolution: For $\Psi_0 \in L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C}) = \bigvee^n \mathcal{Z}$

$\Psi(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\Psi_0 = \Psi(x_1,\ldots,x_n,t)$ solves

Formally "mean field limit"="semiclassical limit" with $\varepsilon = \frac{1}{n}$

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$$\mathcal{H} = \Gamma_b(\mathcal{Z}) = \oplus_{n=0}^{\infty} \mathcal{S}_n \mathcal{Z}^{\otimes n} = \oplus_{n=0}^{\infty} \bigvee^n \mathcal{Z} \quad , \quad \bigvee^n \mathcal{Z} = L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C})$$

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Nonlinear Hamiltonian dynamics: $i\partial_t z = \partial_{\overline{z}} \mathcal{E}$

Hartree (NLS) equation: $i\partial_t z = -\Delta z + (V * |z|^2) z$

Wick quantized Hamiltonian : Take $a = \sqrt{\varepsilon} a_{\varepsilon=1}$ with $\varepsilon > 0$ and set

$$H_\varepsilon = \mathcal{E}(z)^{Wick} = \langle z, -\Delta z \rangle^{Wick} + \frac{1}{2} \langle z^{\otimes 2}, V(x-y) z^{\otimes 2} \rangle^{Wick}$$

n-body evolution: For $\Psi_0 \in L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C}) = \bigvee^n \mathcal{Z}$

$\Psi(t) = e^{-i\frac{t}{\varepsilon} H_\varepsilon} \Psi_0 = \Psi(x_1, \ldots, x_n, t)$ solves

Formally "mean field limit"="semiclassical limit" with $\varepsilon = \frac{1}{n}$

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$$H_\varepsilon = \underbrace{\int_{\mathbb{R}^D} \nabla a^*(x)\nabla a(x) \; dx}_{d\Gamma(-\Delta)} + \frac{1}{2}\int_{\mathbb{R}^{2D}} V(x-y) a^*(x)a^*(y)a(x)a(y) \; dxdy\,.$$

$$d\Gamma(A) = \varepsilon d\Gamma_{\varepsilon=1}(A) \quad , \quad d\Gamma(\mathrm{Id}) = \mathbf{N} = \varepsilon \mathbf{N}_{\varepsilon=1}\,.$$

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Definition of infinite dimensional Wigner measures

Remember: $\mathcal{Z}$ is a separable complex Hilbert space (1 part. space)

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$$\Phi(f) = \frac{a(f) + a^*(f)}{\sqrt{2}} \quad , \quad W(f) = e^{i\Phi(f)} \, .$$

Consider a normal state in $\mathcal{H}$, $\varrho_\varepsilon \in \mathcal{L}^1(\mathcal{H})$, $\varrho_\varepsilon \geq 0$, $\mathrm{Tr}\,[\varrho_\varepsilon] = 1$.

Definition

For $\hat{E} \in (0, +\infty)$, $0 \in \overline{\hat{E}}$, and a family $(\varrho_\varepsilon)_{\varepsilon \in \hat{E}}$ of normal states in $\mathcal{H}$, $\mathcal{M}(\varrho_\varepsilon, \varepsilon \in \hat{E})$ is the set of Borel probability measures μ on $\mathcal{Z}$ for which there exists $\hat{E}' \subset \hat{E}$ such that

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Example: $\varrho_\varepsilon=|\Psi_\varepsilon\rangle\langle\Psi_\varepsilon|$, $\Psi_\varepsilon\in\mathcal{H}$,
Mean field coherent state $\Psi_\varepsilon=E(f)=W(\frac{\sqrt{2}}{i\varepsilon}f)|\Omega\rangle$
Mean field Hermite (atomic coherent) state: $\Psi_\varepsilon=\varphi^{\otimes n}$ with $\varepsilon=\frac{1}{n}$.

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If there exists $\delta > 0$ *and* $C_\delta > 0$ *s.t.*

$$\forall \varepsilon \in \hat{E}\,, \quad \mathrm{Tr}\left[\varrho_\varepsilon \langle \mathbf{N}\rangle^\delta\right] \leq C_\delta \tag{2.1}$$

then $\mathcal{M}(\varrho_\varepsilon\,, \varepsilon \in \hat{E}) \neq \emptyset$ *and every* $\mu \in \mathcal{M}(\varrho_\varepsilon\,, \varepsilon \in \hat{E})$ *satisfies*

$$\int_{\mathcal{Z}} (1+|z|^2)^\delta \; d\mu(z) \leq C_\delta\,.$$

Definition

$b \in \mathcal{S}_{cyl}(\mathcal{Z})$ if there exist a finite rank orth. proj. p and $a \in \mathcal{S}(p\mathcal{Z})$ s.t. $b = a \circ p$.

Corollary

Under the condition (2.1) *with* $\mathcal{M}(\varrho_\varepsilon\,, \varepsilon \in \hat{E}) = \{\mu\}$,

$$\forall b \in \mathcal{S}_{cyl}(\mathcal{Z})\,, \quad \lim_{\varepsilon\to 0\,, \varepsilon \in \hat{E}} \mathrm{Tr}\left[\varrho_\varepsilon\, b^{Weyl}\right] = \int_{\mathcal{Z}} b(z)\; d\mu(z)\,.$$

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Main ideas of the proof:

1. Separation of variables:

$$\begin{array}{rclcll} \mathcal{Z} & = & \mathcal{Z}_1 & \overset{\perp}{\oplus} & \mathcal{Z}_2 & \\ \mathcal{H} & = & \mathcal{H}_1 & \otimes & \mathcal{H}_2\,, & \mathcal{H}_* = \Gamma_b(\mathcal{Z}_*) \\ W(f_1 \oplus f_2) & = & W(f_1) & \otimes & W(f_2) & = W(f_1) \otimes \mathrm{Id}_{\mathcal{H}_2} \quad \text{if } f_2 = 0\,. \end{array}$$

2. $\mathcal{Z}$ is separable $->$ Borel σ-set and diagonal extraction.
3. Condition (2.1) is a tightness condition (see Prokhorov criterion)

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Remark: After a subsequence extraction we can assume $\mathcal{M}(\varrho_\varepsilon\,,\, \varepsilon \in \hat{E}) = \{\mu\}$.

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Wigner measures: Examples

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Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\hat{E})=\{\mu\}$ *and*

$$\forall k\in\mathbb{N}\,,\,\exists C_k>0\,,\,,\forall\varepsilon\in\hat{E}\,,\;\mathrm{Tr}\,\left[\varrho_\varepsilon\mathbf{N}^k\right]\leq C_k\,,$$

then for any cylindrical polynomial and with $Q=$*Weyl,Wick or anti-Wick*

$$\lim_{\varepsilon\to 0\,,\,\varepsilon\in\hat{E}}\mathrm{Tr}\,\left[\varrho_\varepsilon b^Q\right]=\int_{\mathcal{Z}}b(z)\;d\mu(z)\,.$$

Examples

Coherent states: $f\in\mathcal{Z}\,,|f|_{\mathcal{Z}}=1\,,\;E(f)=W(\frac{\sqrt{2}}{i\varepsilon}f)|\Omega\rangle=e^{\frac{a^*(f)-a(f)}{\varepsilon}}|\Omega\rangle\,,$
$\varrho_\varepsilon^C(f)=|E(f)\rangle\langle E(f)|\,,\;\mathrm{Tr}\,\left[\varrho_\varepsilon^C(f)b^{Wick}\right]=b(f)\,,\quad\mathcal{M}(\varrho_\varepsilon^C(f),\varepsilon\in\hat{E})=\{\delta_f\}\,.$

Hermite (atomic coherent) states: $f\in\mathcal{Z}\,,\;|f|_{\mathcal{Z}}=1\,,$
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$$\mathcal{M}(\varrho_\varepsilon^H(f)\,,\varepsilon\in\hat{E})=\left\{\delta_f^{\mathbb{S}^1}=\frac{1}{2\pi}\int_0^{2\pi}\delta_{e^{i\theta}f}\;d\theta\right\}.$$

mixed Hermite (twin Fock) states: $f_1,f_2\in\mathcal{Z}\,,\;\langle f_i\,,\,f_j\rangle=\delta_{ij}\,,$
$\varrho_\varepsilon^H(f_1,f_2)=|f_1^{\otimes n}\rangle\langle f_1^{\otimes n}|\otimes|f_2^{\otimes n}\rangle\langle f_2^{\otimes n}|\,,\varepsilon=\frac{1}{2n}\,,\;\hat{E}=\{\frac{1}{2n}\,,n\in\mathbb{N}^*\}\,,$

$$\mathcal{M}\left\{\varrho_\varepsilon^H(f_1,f_2)\,,\,\varepsilon\in\hat{E}\right\}=\left\{\delta_{2^{-1/2}f_1}^{\mathbb{S}^1}\otimes\delta_{2^{-1/2}f_2}^{\mathbb{S}^1}\right\}.$$

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Wigner measures: Examples

Corollary

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\hat{E})=\{\mu\}$ *and*

$$\forall k\in\mathbb{N}\,,\,\exists C_k>0\,,\,,\forall\varepsilon\in\hat{E}\,,\;\mathrm{Tr}\,\left[\varrho_\varepsilon\mathbf{N}^k\right]\leq C_k\,,$$

then for any cylindrical polynomial and with $Q=$*Weyl,Wick or anti-Wick*

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Definition

Fixed degrees: we say that $b(z) = \langle z^{\otimes q}\,,\,\tilde{b} z^{\otimes p}\rangle$ belongs to $\mathcal{P}_{p,q}(\mathcal{Z})$, if

$$\tilde{b} = \frac{1}{q!}\frac{1}{p!}\partial_{\bar{z}}^{q}\partial_{z}^{p} b \in \mathcal{L}(\bigvee^{p}\mathcal{Z};\bigvee^{q}\mathcal{Z}),$$

Polynomials: $\mathcal{P}(\mathcal{Z}) = \oplus^{alg}_{p,q\in\mathbb{N}}\mathcal{P}_{p,q}(\mathcal{Z})$

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Wick calculus, (PI)-condition, reduced density matrices

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Fixed degrees: we say that $b(z) = \langle z^{\otimes q}\,,\ \tilde{b} z^{\otimes p}\rangle$ belongs to $\mathcal{P}^r_{p,q}(\mathcal{Z})$, if

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Proposition

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\hat{E}) = \{\mu\}$ *and*

$$\forall k\in\mathbb{N}\,,\ \exists C_k>0\,,\ \forall\varepsilon\in\hat{E}\,,\ \mathrm{Tr}\,\left[\varrho_\varepsilon \mathbf{N}^k\right]\leq C_k\,,$$

then $\lim_{\varepsilon\to 0\,,\,\varepsilon\in E}\mathrm{Tr}\,\left[\varrho_\varepsilon b^{Wick}\right] = \int_{\mathcal{Z}} b(z)\,d\mu(z)$ *for all* $b\in\mathcal{P}^{\infty}(\mathcal{Z})$.

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A counter-example with $\tilde{b}$ not compact: Take $\varepsilon = \frac{1}{n}$, $\hat{E} = \left\{\frac{1}{n}\, , n \in \mathbb{N}^*\right\}$ and consider a normalized sequence $(f_n)_{n\in\mathbb{N}^*}$ converging weakly to 0 . Then

$$\mathcal{M}(\varrho^{C}_{\varepsilon}(f_n), \varepsilon \in \hat{E}) = \{\delta_0\} \; ,$$
$$\mathrm{Tr}\,\left[\varrho^{C}_{\varepsilon}(f_n)(|z|^{2p})^{Wick}\right] = |f_n|^{2p} = 1 \neq 0 = \int_{\mathcal{Z}} |z|^{2p}\;\delta_0(z) .$$

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Wick calculus, (PI) condition, reduced density matrices

Remember $(|z|^{2p})^{Wick}=\left(\langle z^{\otimes p}\,,\,\mathrm{Id}\, z^{\otimes p}\rangle\right)^{Wick}=\mathbf{N}(\mathbf{N}-\varepsilon)\cdots(\mathbf{N}-\varepsilon(p-1))\sim\mathbf{N}^p$

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Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\hat{E})=\{\mu\}$ *, with*

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Then

1. $\lim_{\varepsilon\to 0,\varepsilon\in\hat{E}}\mathrm{Tr}\,\left[\varrho_\varepsilon b^{Wick}\right]=\int_{\mathcal{Z}}b(z)\,d\mu(z)$ *for all* $b\in\mathcal{P}(\mathcal{Z})$ *;*
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with (assuming $\mu\neq\delta_0$*)*

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Remark: When $\varrho_\varepsilon \in \mathcal{L}^1(L^2_{sym}((\mathbb{R}^D)^n))\,,\ \varepsilon=\frac{1}{n}\,,$

$$\gamma_\varepsilon^p(x_1,\ldots,x_p;y_1,\ldots,y_p) = \int_{(\mathbb{R}^D)^{N-p}} \varrho_\varepsilon(x_1,\ldots,x_p,X;y_1,\ldots,y_p,X)\, dX$$

Mean field propagation of Wigner measures

Problem: After composition with a nonlinear flow, cylindrical (resp. polynomial symbols) do not remain cylindrical (resp. polynomials).

Take $\mathcal{E}(z) = \langle z\,,\, Az\rangle + Q(z)$ with A self-adjoint and $Q \in \mathcal{P}(\mathcal{Z})$ and set $H_\varepsilon = \mathcal{E}^{Wick}$ while Φ is the hamiltonian flow associated with $\mathcal{E}$.

Theorem Ammari-N. (JMPA 11)

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\hat{E}) = \{\mu\}$ *and the condition* (PI) , *then*

$$\mathcal{M}(e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}\,,\,\varepsilon\in\hat{E}) = \{\Phi(t)_*\mu\}$$

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Theorem Ammari-N. (Ann. della Sc. Norm. Pisa 15)

With $A=-\Delta$ *and* $V(x) = \frac{\alpha}{|x|}$, $x\in\mathbb{R}^3$, $\alpha\in\mathbb{R}$.

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Theorem Ammari-N. (JMPA 11)

Assume $\mathcal{M}(\varrho_\varepsilon\,,\, \varepsilon \in \hat{E}) = \{\mu\}$ and the condition (PI) , then

$$\mathcal{M}(e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}\,,\, \varepsilon \in \hat{E}) = \{\Phi(t)_*\mu\}$$

and the condition (PI) holds for all times.

Method: Truncated Dyson expansion after (Fröhlich-Graffi-Schwarz 07 and Fröhlich-Knowles-Schwarz 09) combined with a priori information on $\mu(t)$.

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With $A = -\Delta$ and $V(x) = \frac{\alpha}{|x|}$, $x \in \mathbb{R}^3$, $\alpha \in \mathbb{R}$.

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Mean field propagation of Wigner measures

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Method: Measure transportation adapted from Ambrosio-Gigli-Savaré (book 05).

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Some compactness is needed either on the interaction or on the initial data. In the 3D-Coulombic case, we used the compactness of $(1-\Delta)^{-1/2}\frac{1}{|x|}(1-\Delta)^{-1/2}$.

Propagation: Improvement by Q. Liard

In our last work with Z. Ammari for singular interactions, the assumptions on the interaction potential were

$$V(-x) = V(x) \quad V(1-\Delta)^{-1/2} \in \mathcal{L}(L^2) \quad (1-\Delta)^{-1/2} V (1-\Delta)^{-1/2} \text{ compact.}$$

While the usual assumptions for $-\Delta + V(x)$ are expressed in term of $V(1-\Delta)^{-1}$.

With a similar strategy but significant new ideas Q. Liard is able to treat one particle hamiltonians $H_0 = -\Delta + U(x)$ with assumptions on the interaction potential $V(x)$ similar to the one for the KLMN perturbative theorem for $H_0 + V$.

Significant difference: Infinite dimensional method of characteristics.
Z. Ammari, N.: Quadratic Wasserstein distance
$W^p(\mu_1, \mu_2) = \inf_{\pi_j \mu = \mu_j} \int\int |x-y|^p \, d\mu(x,y)$, quadratic means $p = 2$.
Q. Liard: Use of $W^1(\mu_1, \mu_2)$, inspired by finite dimensional results of Maniglia.
Tightness for families of probability measures on phase-space (tightness$->$ weak compactness) less obvious (coercivity replaced by Dunford-Pettis type arguments).

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Order of convergence and numerics

Ammari-Falconi-Pawilowski: Assume $\|\gamma_\varepsilon^{(p)} - \gamma_0^{(p)}\|_{\mathcal{L}^1} = C(\varepsilon)C^p$ for all $p \in \mathbb{N}$ with

$$\gamma_0^{(p)} = \frac{\int_{\mathcal{Z}} |z^{\otimes p}\rangle\langle z^{\otimes p}| \; d\mu_0(z)}{\int_{\mathcal{Z}} |z|^{2p} \; d\mu_0(z)} \quad \text{with} \, C(\varepsilon) \geq C^{-1}\varepsilon$$

then $\|\gamma_\varepsilon^{(p)}(t) - \gamma_0^{(p)}(t)\|_{\mathcal{L}^1} = C_T C^p C(\varepsilon)$ for all $p \in \mathbb{N}$ and all $t \in [-T, T]$.

Example of numerical results obtained by B. Pawilowski:

- $\mathcal{Z} = \ell^2(\mathbb{Z}/K\mathbb{Z}) \sim \mathbb{C}^K$, H_0 periodic discrete Laplacian.
- Use of exact formula for $\gamma_0^{(p)}(t)$ with numerical integration for the mean field flow.
- Heavy accurate computation of the quantum N-body problem $N \leq 20$

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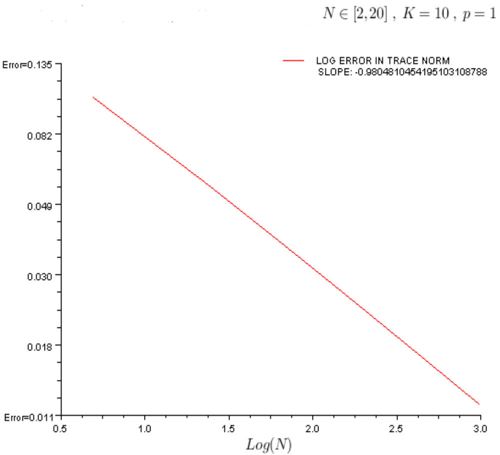
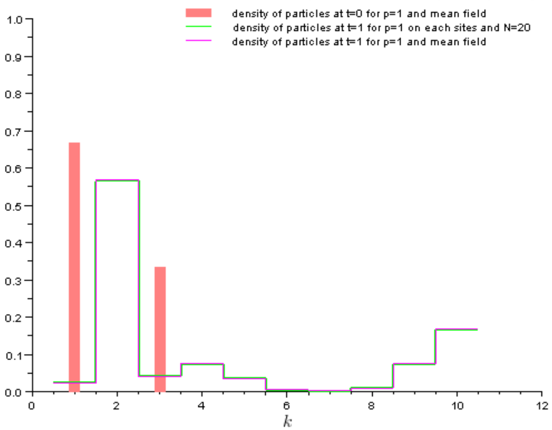


Order of convergence for $\sup_{t\in[0,T]} \|\gamma_\varepsilon^{(p)}(t) - \gamma_0^{(p)}(t)\|_{\mathcal{L}^1}$, here $p = 1$



Particle density: red=mean field $t = 0$,
green and purple = 20-body and mean field $t = 1$

Other developments

Semiclassical analysis is easier than microlocal analysis: It is possible to reconsider classical problems of bosonic quantum field theory by introducing scales and a semiclassical parameters.

- Z. Ammari, M. Zerzeri: $P(\Phi)_2$ and Hoegh-Krohn model.
- Z. Ammari, M. Falconi: Nelson model.

Work in progress with Z. Ammari and S. Breteaux: Use of multiscale (2nd microlocalized see e.g. C. Fermanian) semiclassical analysis for a more accurate description of all the $\gamma_\varepsilon^{(p)}$.
Observable looking like $\langle z^{\otimes p}, [K + a^{W,h}]z^{\otimes p}\rangle^{Wick}$ with K compact $\varepsilon = \varepsilon(h)$, $h \to 0$, $\varepsilon(h) \to 0$.

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Motivations:

- Mixture of BEC and non condensate phase
- Approach valid for the bosonic and fermionic case
- Another way of refining the mean field analysis, as compared with Bogoliubov 2nd order approximation.
- Possibly combine Ammari-N. propagation result (quantum part) with the recent result by Golse-Paul (macroscopic part).
- Double scales appear in random homogenization problems (see Breteaux' phD).

Thank you for your attention !