Persistence cohomology and Arrhenius law

D. Le Peutrec, Orsay
F. Nier, Paris 13
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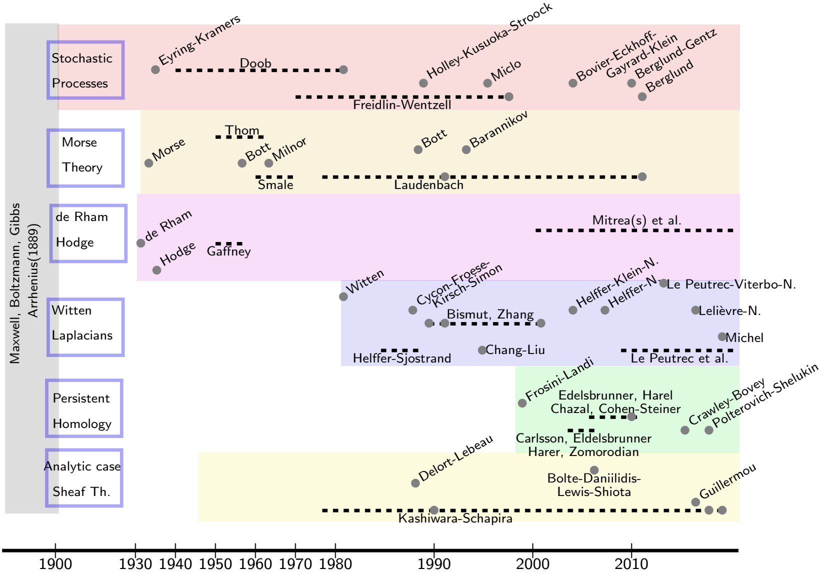
Nantes, 26/04/2019[1]

[1] I did my lecture in IHP for Martinez' conference, on the blackboard. Those slides essentially cover the same material.


Maxwell, Boltzmann, Gibbs
Arrhenius(1889)
Stochastic Processes
Eyring-Kramers
Doob
Freidlin-Wentzell
Holley-Kusuoka-Stroock
Miclo
Bovier-Eckhoff-Gayrard-Klein
Berglund-Gentz
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de Rham Hodge
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Gaffney
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Witten Laplacians
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Edelsbrunner, Harel
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Analytic case Sheaf Th.
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Bolte-Daniilidis-Lewis-Shiota
Guillermou
Kashiwara-Schapira
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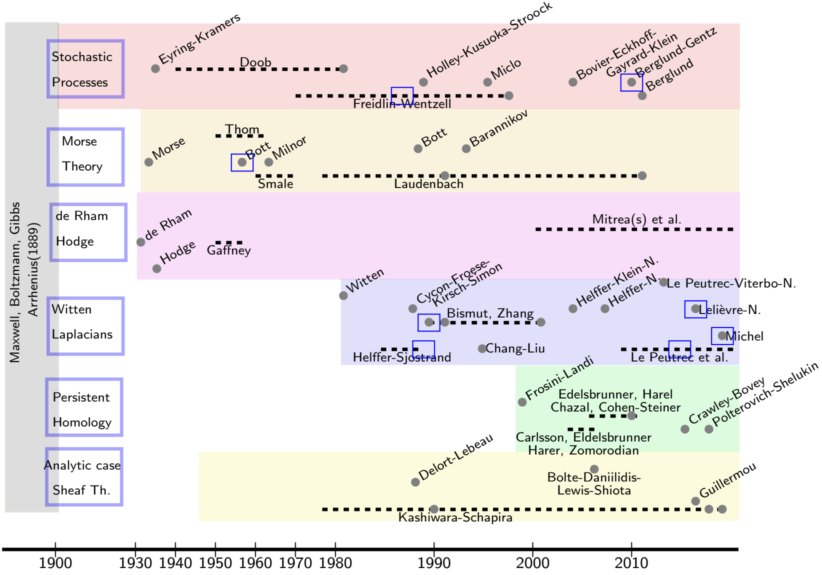
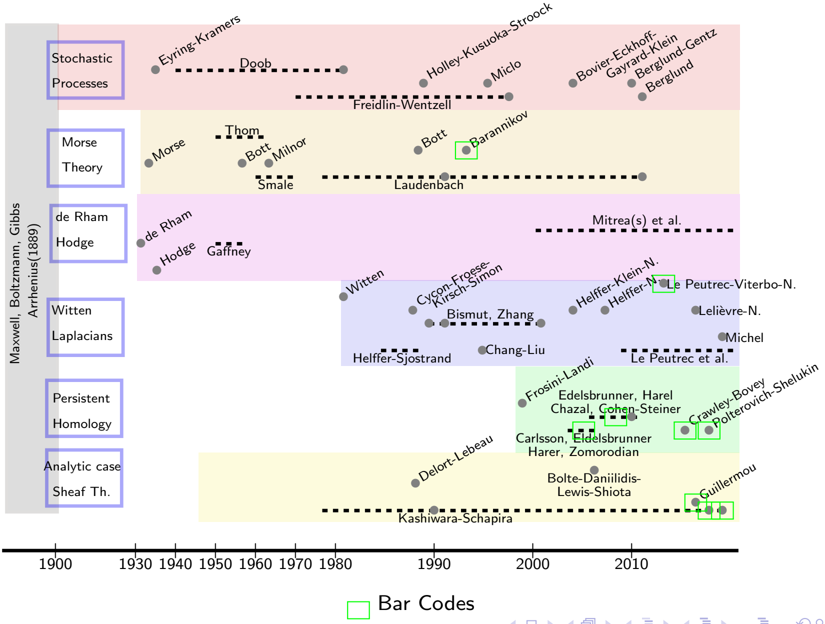


non generic Morse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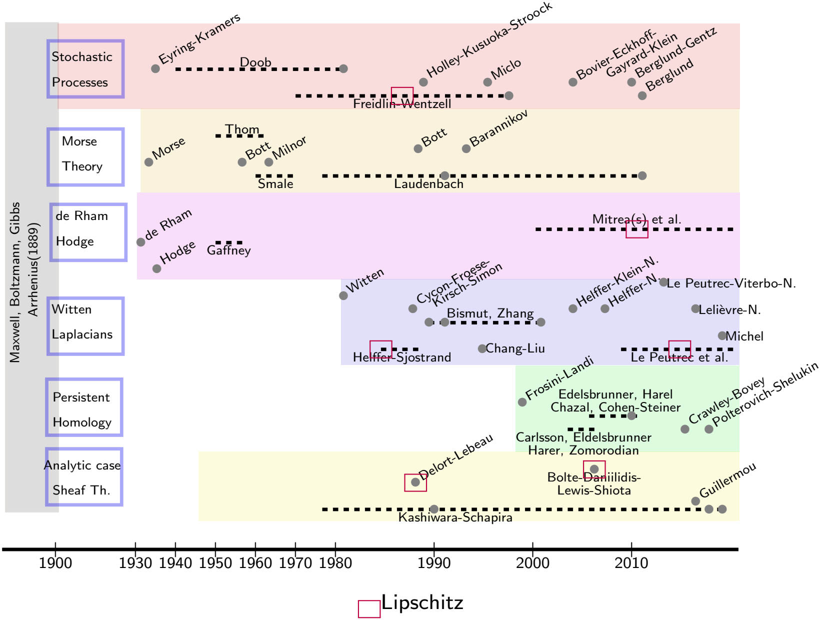
Bar Codes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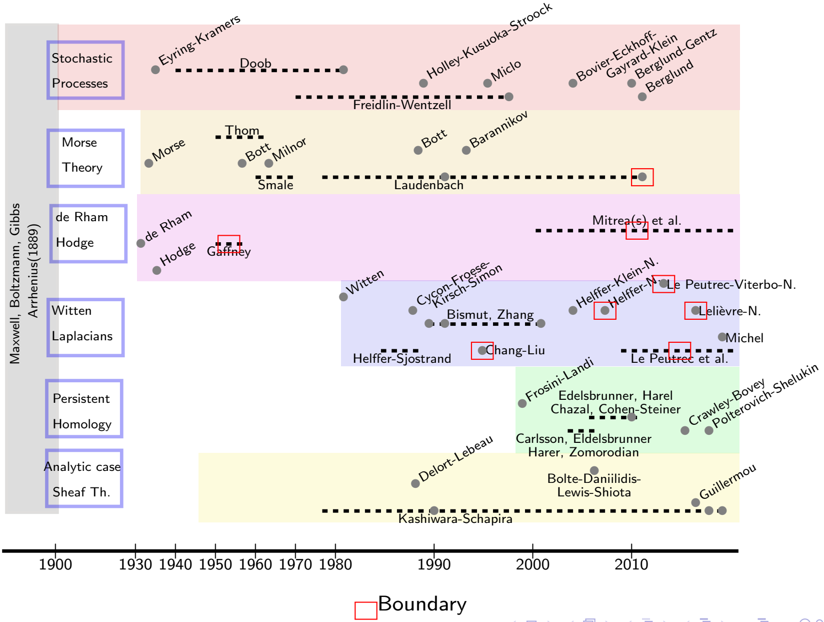
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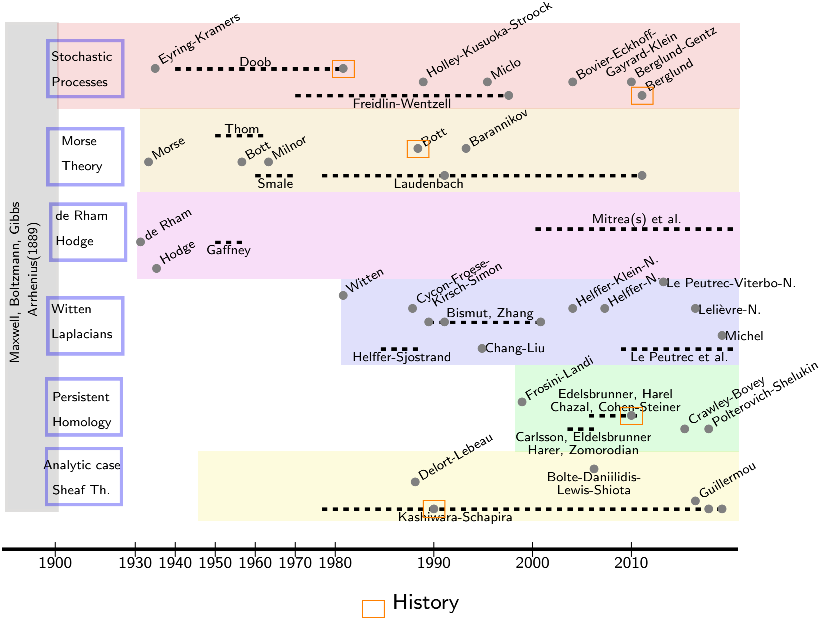
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History

Bar codes

Assumption

- (M, g) *is a closed (compact) riemannian manifold.*
- f *Lipschitz.*
- f $\mathcal{C}^\infty$ *and* $df \neq 0$ *in* $f^{-1}(\mathbb{R} \setminus \{c_1, \ldots, c_{N_f}\})$.
  $c_1, \ldots, c_{N_f}$: *"critical values".*

Alternatively, M *is real analytic (compact) and* f *is* $\underbrace{\text{Lipschitz and subanalytic}}_{\to \text{not yet}}$.

Definition

A bar code associated to f is a finite family $\mathcal{B}_f = (\underbrace{[a_\alpha^{(p)}, b_\alpha^{(p+1)}[}_{\text{degree } p})_{\alpha \in A}$ such that

$a_\alpha \in \{c_1, \ldots, c_{N_f}\}$, $b_\alpha \in \{c_2, \ldots, c_{N_f}, +\infty\}$ and

$$H^{(p)}(f^b, f^a; \mathbb{K}) \sim \bigoplus_{a_\alpha^{(p-1)} < a < b_\alpha^{(p)} < b} \mathbb{K} b_\alpha^{(p)} \bigoplus \bigoplus_{a < a_\alpha^{(p)} < b < b_\alpha^{(p+1)}} \mathbb{K} a_\alpha^{(p)} .$$

$\mathcal{B}_f$ unique modulo permutation and the addition of empty bars.

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Stability theorem

Definition

Bottleneck distance $\mathcal{B} = ([a_\alpha, b_\alpha[)_{\alpha\in A}$, $\mathcal{B}' = ([a'_\alpha, b'_\alpha[)_{\alpha\in A}$.

$$d_{bot}(\mathcal{B}, \mathcal{B}') = \min_{\sigma\in\mathfrak{S}(A)} \max_{\alpha\in A} \max(|a_\alpha - a'_{\sigma(\alpha)}|, 1_{\mathbb{R}}(\min(b_\alpha, b'_{\sigma(\alpha)}))|b_\alpha - b'_{\sigma(\alpha)}|) .$$

The same A is obtained after possibly adding empty bars.

Theorem (Cohen-Steiner–Edelsbrunner–Harrel (07), Kashiwara–Schapira (16))

For two functions f, g which satisfy our Assumption,

$$d_{bot}(\mathcal{B}_f, \mathcal{B}_g) \leq \|f - g\|_{\mathcal{C}^0}$$

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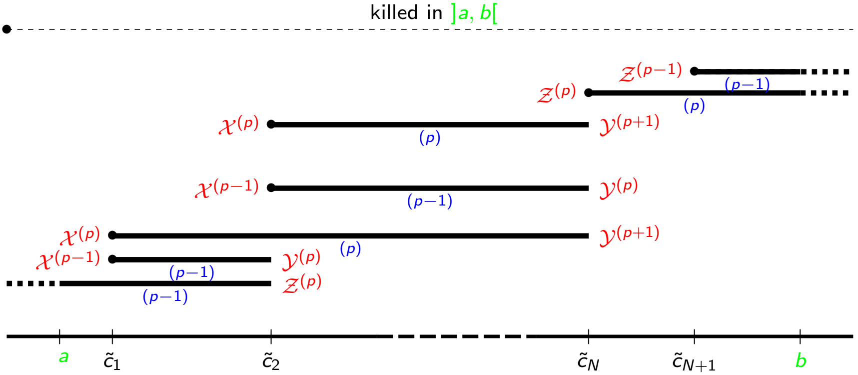
Some notations in $f^{-1}([a,b])$, $a<b$ non "critical"

Labelling of the endpoints: $\mathcal{J}(a,b) = \mathcal{X}(a,b) \sqcup \mathcal{Y}(a,b) \sqcup \mathcal{Z}(a,b)$



$\mathcal{X}^* = \mathcal{X}^*(a,b)$ (lower) , $\mathcal{Y}^* = \mathcal{Y}^*(a,b)$ (upper) , $\mathcal{Z}^* = \mathcal{Z}^*(a,b)$ (lonely)

$$\beta^{(p)}(f^b, f^a; \mathbb{K}) = \dim\ H^{(p)}(f^b, f^a; \mathbb{K}) = \sharp \mathcal{Z}^{(p)}(a,b)$$

Some notations in $f^{-1}([a,b])$, $a<b$ non "critical"

Definition

Differential operators

$$d_{f,h}=e^{-\frac{f}{h}}(hd)e^{\frac{f}{h}}=hd+df\wedge \quad , \quad d_{f,h}^{*}=e^{\frac{f}{h}}(hd^{*})e^{-\frac{f}{h}}=hd^{*}+\mathbf{i}_{\nabla f}$$

$$\Delta_{f,h}=(d_{f,h}+d_{f,h}^{*})^{2}=d_{f,h}^{*}d_{f,h}+d_{f,h}d_{f,h}^{*}\,.$$

Closed operators in $L^2(f_a^b,\Lambda T^*M)$

$$D(d_{f,f^{-1}([a,b],h)})=\left\{\omega\in L^2\,,\ d_{f,h}\omega\in L^2\,,\ \mathbf{t}\omega\big|_{f=a}=0\right\}\,.$$

$$D(d^{*}_{f,f^{-1}([a,b],h)})=\left\{\omega\in L^2\,,\ d^{*}_{f,h}\omega\in L^2\,,\ \mathbf{n}\omega\big|_{f=b}=0\right\}\,.$$

$$\left[\text{conseq. } D(d_{f,f^{-1}([a,b]),h})\cap D(d^{*}_{f,f^{-1}([a,b]),h})\overset{\text{Gaffney}}{\subset} W^{1,2}(f^{-1}([a,b]),\Lambda T^{*}M)\,.\right]$$

$$\Delta_{f,f^{-1}([a,b]),h}=d^{*}_{f,f^{-1}([a,b],h)}d_{f,f^{-1}([a,b],h)}+d_{f,f^{-1}([a,b]),h}d^{*}_{f,f^{-1}([a,b]),h}\,,$$

$$D(\Delta_{f,f^{-1}([a,b]),h})=\left\{\omega\in W^{1,2},\ \left[\begin{array}{ll}\mathbf{t}\omega\big|_{f=a}=0, & \mathbf{n}\omega\big|_{f=b}=0\,,\\ \underbrace{d^{*}_{f,h}\omega\in D(d_{f,f^{-1}([a,b]),h})}_{\mathbf{t}d^{*}_{f,h}\omega|_{f=a}=0}\,. & \underbrace{d_{f,h}\omega\in D(d^{*}_{f,f^{-1}([a,b]),h})}_{\mathbf{n}d_{f,h}\omega|_{f=b}=0},\end{array}\right.\right\}$$

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$$d_{f,h} = e^{-\frac{f}{h}}(hd)e^{\frac{f}{h}} = hd + df\wedge \quad , \quad d_{f,h}^{*} = e^{\frac{f}{h}}(hd^{*})e^{-\frac{f}{h}} = hd^{*} + \mathbf{i}_{\nabla f}$$

$$\Delta_{f,h} = (d_{f,h} + d_{f,h}^{*})^{2} = d_{f,h}^{*}d_{f,h} + d_{f,h}d_{f,h}^{*}\,.$$

Closed operators in $L^2(f_a^b, \Lambda T^*M)$

$$D(d_{f,f^{-1}([a,b],h)}) = \left\{\omega\in L^2\,,\ d_{f,h}\omega\in L^2\,,\ \mathbf{t}\omega\big|_{f=a}=0\right\}\,.$$

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$$\Delta_{f,f^{-1}([a,b]),h} = d^{*}_{f,f^{-1}([a,b],h)}d_{f,f^{-1}([a,b],h)} + d_{f,f^{-1}([a,b]),h}d^{*}_{f,f^{-1}([a,b]),h}\,,$$

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Basic properties, a, a', b non critical

$$d_{f,f^{-1}([a,b]),h}(z-\Delta_{f,f^{-1}([a,b],h)})^{-1}=(z-\Delta_{f,f^{-1}([a,b],h)})^{-1}d_{f,f^{-1}([a,b]),h}\,.$$

$$F^{(p)}_{[0,\alpha],[a,b],h}=\mathrm{Ran}\ 1_{[0,\alpha]}(\Delta^{(p)}_{f,f^{-1}([a,b],h)})\quad,\quad \delta_{[0,\alpha],[a,b],h}=d_{f,f^{-1}([a,b],h)}\big|_{F_{[0,\alpha],[a,b],h}}$$

$$\delta^{(p)}_{[0,\alpha],[a,b],h}:F^{(p)}_{[0,\alpha],[a,b],h}\to F^{(p+1)}_{[0,\alpha],[a,b],h}$$

$$\Delta_{f,f^{-1}([a,b]),h}=\underbrace{\delta^*_{[0,\alpha],[a,b],h}\delta_{[0,\alpha],[a,b],h}+\delta_{[0,\alpha],[a,b],h}\delta^*_{[0,\alpha],[a,b],h}}_{A+B\quad,\quad AB=BA=0}\,.$$

$$F_{[0,\alpha],[a,b],h}=\underbrace{\mathrm{Ran}\ \delta_{[0,\alpha],[a,b],h}\overset{\perp}{\oplus}\ker(\Delta_{[0,\alpha],[a,b],h})}_{\ker\,\delta_{[0,\alpha],[a,b],h}}\overset{\perp}{\oplus}\mathrm{Ran}\ \delta^*_{[0,\alpha],[a,b],h}$$

Consequences:

a) $\ker(\Delta^{(p)}_{f,f^{-1}([a,b],h)})=\ker(\delta^{(p)}_{[0,\alpha],[a,b],h})/\mathrm{Ran}\ \delta^{(p-1)}_{[0,\alpha],[a,b],h}\sim$ $\ker(d^{(p)}_{0,f^{-1}([a,b],h)})/\mathrm{Ran}\ d^{(p)}_{0,f^{-1}([a,b]),h}\sim H^{(p)}(f^b,f^a;\mathbb{R})$

b) Eigenvalues of $\Delta_{f,f^{-1}([a,b]),h}$ lying in $]0,\alpha]$ are the square of singular values of $\delta_{]0,\alpha],[a,b],h]}$ (counted twice).

$D(d_{f,f^{-1}([a,b]),h})\subset D(d_{g,f^{-1}([a',b]),h})$ for any Lipschitz g and $a'\leq a$.

Basic properties, a, a', b non critical

$$d_{f,f^{-1}([a,b]),h}(z-\Delta_{f,f^{-1}([a,b],h)})^{-1} = (z-\Delta_{f,f^{-1}([a,b],h)})^{-1} d_{f,f^{-1}([a,b]),h}\,.$$

$$F^{(p)}_{[0,\alpha],[a,b],h} = \mathrm{Ran}\ 1_{[0,\alpha]}(\Delta^{(p)}_{f,f^{-1}([a,b],h)}) \quad , \quad \delta_{[0,\alpha],[a,b],h} = d_{f,f^{-1}([a,b],h)}\big|_{F_{[0,\alpha],[a,b],h}}$$

$$\delta^{(p)}_{[0,\alpha],[a,b],h} : F^{(p)}_{[0,\alpha],[a,b],h} \to F^{(p+1)}_{[0,\alpha],[a,b],h}$$

$$\Delta_{f,f^{-1}([a,b]),h} = \underbrace{\delta^*_{[0,\alpha],[a,b],h}\delta_{[0,\alpha],[a,b],h} + \delta_{[0,\alpha],[a,b],h}\delta^*_{[0,\alpha],[a,b],h}}_{A+B \quad , \quad AB=BA=0}\,.$$

$$F_{[0,\alpha],[a,b],h} = \underbrace{\mathrm{Ran}\ \delta_{[0,\alpha],[a,b],h} \overset{\perp}{\oplus} \ker(\Delta_{[0,\alpha],[a,b],h})}_{\ker\ \delta_{[0,\alpha],[a,b],h}} \overset{\perp}{\oplus} \mathrm{Ran}\ \delta^*_{[0,\alpha],[a,b],h}$$

Consequences:

a) $\ker(\Delta^{(p)}_{f,f^{-1}([a,b],h)}) = \ker(\delta^{(p)}_{[0,\alpha],[a,b],h})/\mathrm{Ran}\ \delta^{(p-1)}_{[0,\alpha],[a,b],h} \sim \ker(d^{(p)}_{0,f^{-1}([a,b],h)})/\mathrm{Ran}\ d^{(p)}_{0,f^{-1}([a,b]),h} \sim H^{(p)}(f^b, f^a; \mathbb{R})$

b) Eigenvalues of $\Delta_{f,f^{-1}([a,b]),h}$ lying in $]0,\alpha]$ are the square of singular values of $\delta_{]0,\alpha],[a,b],h]}$ (counted twice).

$D(d_{f,f^{-1}([a,b]),h}) \subset D(d_{g,f^{-1}([a',b]),h})$ for any Lipschitz g and $a' \leq a$.

Basic properties, a, a', b non critical

$$d_{f,f^{-1}([a,b]),h}(z-\Delta_{f,f^{-1}([a,b],h)})^{-1} = (z-\Delta_{f,f^{-1}([a,b],h)})^{-1} d_{f,f^{-1}([a,b]),h}\,.$$

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Consequences:

a) $\ker(\Delta^{(p)}_{f,f^{-1}([a,b],h)}) = \ker(\delta^{(p)}_{[0,\alpha],[a,b],h})/\mathrm{Ran}\ \delta^{(p-1)}_{[0,\alpha],[a,b],h} \sim \ker(d^{(p)}_{0,f^{-1}([a,b],h)})/\mathrm{Ran}\ d^{(p)}_{0,f^{-1}([a,b]),h} \sim H^{(p)}(f^b, f^a; \mathbb{R})$

b) Eigenvalues of $\Delta_{f,f^{-1}([a,b]),h}$ lying in $]0,\alpha]$ are the square of singular values of $\delta_{]0,\alpha],[a,b],h]}$ (counted twice).

$D(d_{f,f^{-1}([a,b]),h}) \subset D(d_{g,f^{-1}([a',b]),h})$ for any Lipschitz g and $a' \leq a$.

Basic properties, a, a', b non critical

$$d_{f,f^{-1}([a,b]),h}(z-\Delta_{f,f^{-1}([a,b],h)})^{-1} = (z-\Delta_{f,f^{-1}([a,b],h)})^{-1} d_{f,f^{-1}([a,b]),h}\,.$$

$$F^{(p)}_{[0,\alpha],[a,b],h} = \mathrm{Ran}\ 1_{[0,\alpha]}(\Delta^{(p)}_{f,f^{-1}([a,b],h)}) \quad , \quad \delta_{[0,\alpha],[a,b],h} = d_{f,f^{-1}([a,b],h)}\big|_{F_{[0,\alpha],[a,b],h}}$$

$$\delta^{(p)}_{[0,\alpha],[a,b],h} : F^{(p)}_{[0,\alpha],[a,b],h} \to F^{(p+1)}_{[0,\alpha],[a,b],h}$$

$$\Delta_{f,f^{-1}([a,b]),h} = \underbrace{\delta^*_{[0,\alpha],[a,b],h}\delta_{[0,\alpha],[a,b],h} + \delta_{[0,\alpha],[a,b],h}\delta^*_{[0,\alpha],[a,b],h}}_{A+B \quad , \quad AB=BA=0}\,.$$

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$$F^{(p)}_{[0,\alpha],[a,b],h}=\mathrm{Ran}\ 1_{[0,\alpha]}(\Delta^{(p)}_{f,f^{-1}([a,b],h)})\quad,\quad \delta_{[0,\alpha],[a,b],h}=d_{f,f^{-1}([a,b],h)}\big|_{F_{[0,\alpha],[a,b],h}}$$

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Main result

Theorem

$\left\{c_1,\dots,c_{N_f}\right\}\cap[a,b]=\{\tilde{c}_1,\dots,\tilde{c}_N\}$. $\quad 0<\delta_1,\delta_2<\min\frac{c_n-c_{n-1}}{16}$.

$F^{(p)}_{[0,\tilde{o}(1)],[a,b],h}=Ran\ 1_{[0,e^{-\frac{\varepsilon}{h}}]}(\Delta^{(p)}_{f,f^{-1}([a,b]),h})$ *($\exists\varepsilon>0$ indep. of h).*

The singular values of $d_{f,f^{-1}([a,b]),h}\big|_{F^{(p)}_{[0,\tilde{o}(1)],[a,b],h}}$ *, are* $(\mu_j^{(p),h})_{j\in\mathcal{J}^{(p)}(a,b)}$,

$$\lim_{h\to 0}-h\log\mu_j^{(p),h}=b_\alpha^{(p+1)}-a_\alpha^{(p)}\quad \text{if } j=(\alpha,a_\alpha^{(p)})\in\mathcal{X}^{(p)}(a,b)\,,$$

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There exists a δ_1-family of quasimodes $(\varphi_j^h)_{j\in\mathcal{J}(a,b)}$ *which is* $\tilde{O}(e^{-\frac{\delta_1}{h}})$*-orthonormal and such that* $\mathcal{V}^{(p),h}=Vect(\varphi_j^h,j\in\mathcal{J}^{(p)}(a,b))$ *satisfies*

$$\vec{d}(\mathcal{V}^{(p),h};F^{(p)}_{[0,\tilde{o}(1)],[a,b],h})+\vec{d}(F^{(p)}_{[0,\tilde{o}(1)],[a,b],h};\mathcal{V}^{(p),h})=\tilde{O}(e^{-\frac{\delta_1}{h}})\,.$$

$$\mathcal{V}^{(p),h}\xrightarrow{\;d^{(p)}_{f,f^{-1}([a,b]),h}{}^{T}\delta_2\;}L^2(f^{-1}([a,b]))$$

$$\mathcal{V}^{(p),h}\xrightarrow{\;\Pi_{F^{(p)}_{[0,\tilde{o}(1)],[a,b],h}}\ d^{(p)}_{f,f^{-1}([a,b]),h}{}^{T}\delta_2\;}F^{(p)}_{[0,\tilde{o}(1)],[a,b],h}\xrightarrow{\;C^h\;}L^2(f^{-1}([a,b]))$$

$$\|C^h\|=\tilde{O}(e^{\frac{2\delta_2}{h}})\,.$$

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The singular values of $d_{f,f^{-1}([a,b]),h}\big|_{F^{(p)}_{[0,\tilde{o}(1)],[a,b],h}}$, *are* $(\mu_j^{(p),h})_{j\in\mathcal{J}^{(p)}(a,b)}$,

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$$\begin{array}{ccc}
\mathcal{V}^{(p),h} & \xrightarrow{\;d^{(p)}_{f,f^{-1}([a,b]),h}{}^{T_{\delta_2}}\;} & L^2(f^{-1}([a,b])) \\
 & \searrow{\scriptstyle \Pi_{F^{(p)}_{[0,\tilde{o}(1)],[a,b],h}}\, d^{(p)}_{f,f^{-1}([a,b]),h}{}^{T_{\delta_2}}} & \uparrow C^h \qquad \|C^h\|=\tilde{O}(e^{\frac{2\delta_2}{h}})\,. \\
 & & F^{(p)}_{[0,\tilde{o}(1)],[a,b],h}
\end{array}$$

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$\{c_1, \ldots, c_{N_f}\} \cap [a,b] = \{\tilde{c}_1, \ldots, \tilde{c}_N\}$. $\quad 0 < \delta_1, \delta_2 < \min \frac{c_n - c_{n-1}}{16}$.

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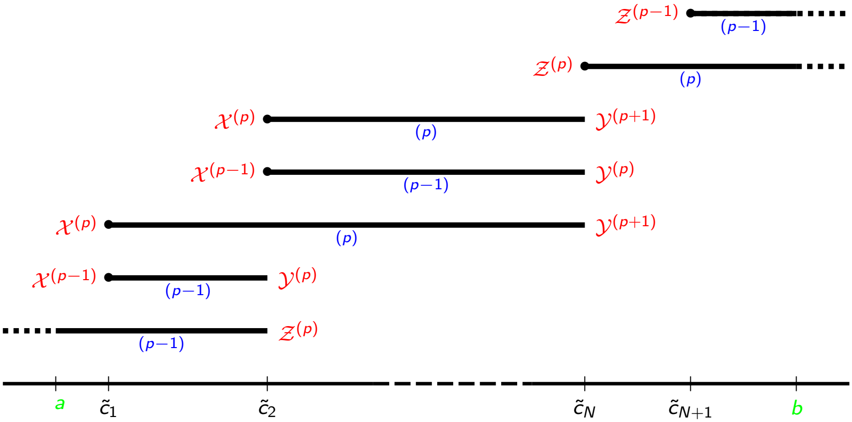
$$\vec{d}(\mathcal{V}^{(p),h}; F^{(p)}_{[0,\tilde{o}(1)],[a,b],h}) + \vec{d}(F^{(p)}_{[0,\tilde{o}(1)],[a,b],h}; \mathcal{V}^{(p),h}) = \tilde{O}(e^{-\frac{\delta_1}{h}})\,.$$

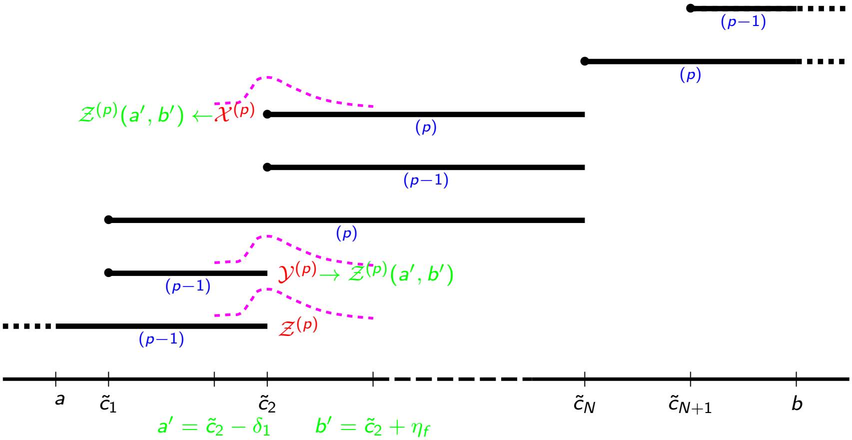
$$\begin{array}{ccc}
\mathcal{V}^{(p),h} & \xrightarrow{\ d^{(p)}_{f,f^{-1}([a,b]),h}\ T_{\delta_2}\ } & L^2(f^{-1}([a,b])) \\
 & \searrow^{\ \Pi_{F^{(p)}_{[0,\tilde{o}(1)],[a,b],h}}\ d^{(p)}_{f,f^{-1}([a,b]),h}\ T_{\delta_2}} & \uparrow C^h \\
 & & F^{(p)}_{[0,\tilde{o}(1)],[a,b],h}
\end{array}
\qquad \|C^h\| = \tilde{O}(e^{\frac{2\delta_2}{h}})\,.$$

Explanation



$$\mathcal{X}^* = \mathcal{X}^*(a,b) \text{ (lower)}, \ \mathcal{Y}^* = \mathcal{Y}^*(a,b) \text{ (upper)}, \ \mathcal{Z}^* = \mathcal{Z}^*(a,b) \text{ (lonely)}$$

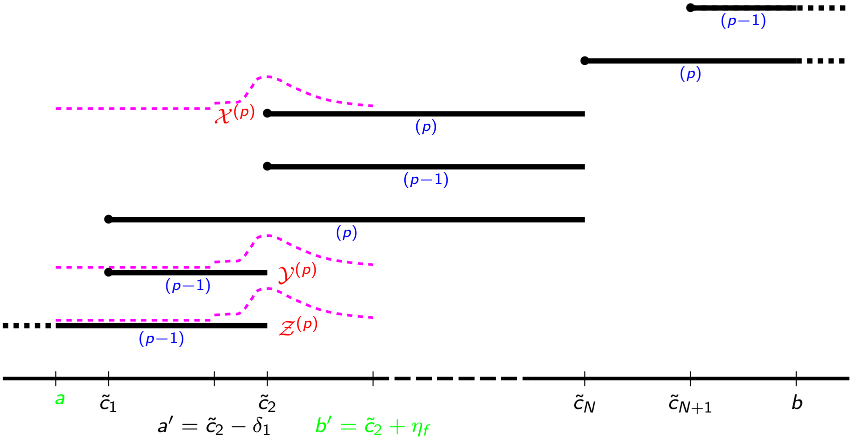
Explanation



$\ker(\Delta^{(p)}_{f,f^{-1}([a',b']),h})$, $a' = \tilde{c}_2 - \delta_1$, $b' = \tilde{c}_2 + \eta_f$, $c_1 = \tilde{c}_2$

D. Le Peutrec, Orsay
F. Nier, Paris 13
C. Viterbo, ENS

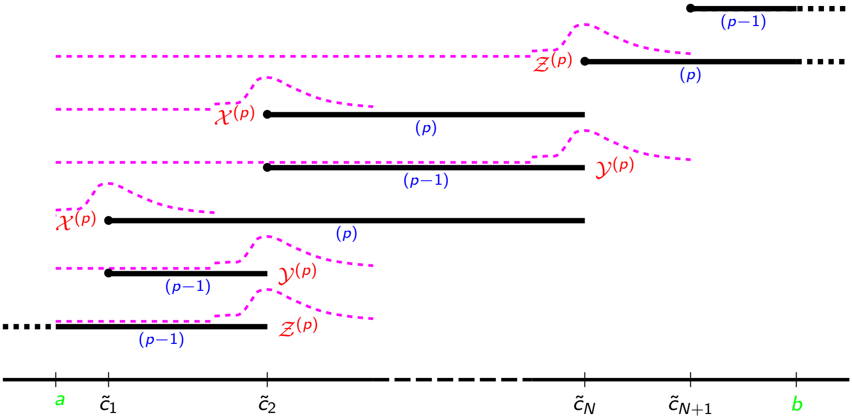
Explanation



Extension by 0 between a and $a' = \tilde{c}_2 - \delta_1$

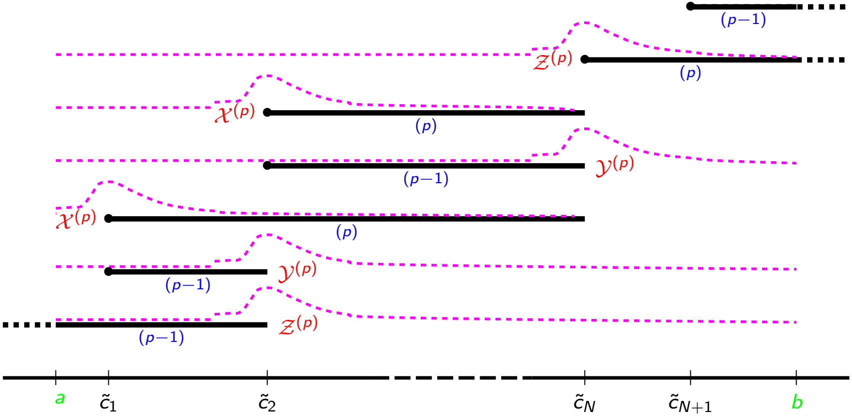
D. Le Peutrec, Orsay
F. Nier, Paris 13
C. Viterbo, ENS

Explanation



Same local construction for all $\tilde{c}_n$ in $]a, b[$

Explanation



Global construction.

The truncation operator T_{δ_2} truncates just before the upper end $\in \mathcal{Y}^{(p+1)}$ of the bar.