FACTORIZATION HOMOLOGY

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Contents

This is a survery talk of the paper [AF15] of Ayala and Francis, whose main result is an axiomatic characterization of factorization homology for topological manifolds (see Theorem 36 and Corollary 37 in §4). The first three sections mostly include definitions and constructions, while the last section contains a formulation and a proof of the main result.

1. Manifolds with tangent structures

Definition 1. Let $\mathbb{D}^n = \{x \in \mathbb{R}^n | |x| < 1\}$ be the open n-dimensional disc and $\mathbb{D}^n_+ := [0,1) \times \mathbb{D}^{n-1}$ the half-disc. By an n-manifold we will mean a (not necessarily compact) Hausdorff second countable space M which is locally homeomorphic at every point $x \in M$ to either $\mathbb{D}^n$ or $\mathbb{D}^n_+$. The boundary $\partial M \subseteq M$ is the subspace containing those points with local neighborhoods $\mathbb{D}^n_+$. We will say that M is **open** if $\partial M = \emptyset$.

A map $\iota : N \longrightarrow M$ between n-manifolds is said to be an **open embedding** if it is injective and sends open subsets to open subsets. Note if $\iota : N \longrightarrow M$ is an open embedding then $\iota^{-1}(\partial M) = \partial N$. The set $\mathrm{Emb}(M, N)$ of all open embeddings can topologized using the compact-open topology. We will denote by $\mathcal{M}\mathrm{fld}^{\partial}_n$ the ∞-category whose objects are the n-manifolds and whose mapping spaces are the spaces of open embeddings. We will denote by $\mathcal{M}\mathrm{fld}_n \subseteq \mathcal{M}\mathrm{fld}^{\partial}_n$ the full subcategory spanned by the open n-manifolds. We recall from the previous talk that both these ∞-categories carry natural symmetric monoidal structures, which are given each time by disjoint union.

Let $\mathrm{Top}(n)$ be the topological group of self homeomorphisms of $\mathbb{D}^n$. The natural map $\mathrm{Top}(n) \longrightarrow \mathrm{Map}_{\mathcal{M}\mathrm{fld}_n}(\mathbb{D}^n, \mathbb{D}^n)$ induces a map $\mathbb{B}\,\mathrm{Top}(n) \longrightarrow \mathcal{M}\mathrm{fld}_n$, where $\mathbb{B}\,\mathrm{Top}(n)$ is the classifying ∞-groupoid of $\mathrm{Top}(n)$. Consider the composed functor

$$(1) \quad \mathcal{M}\mathrm{fld}_n \longrightarrow \mathrm{Fun}(\mathcal{M}\mathrm{fld}_n^{\mathrm{op}}, \mathrm{Spaces}) \longrightarrow \mathrm{Fun}(\mathbb{B}\,\mathrm{Top}(n)^{\mathrm{op}}, \mathrm{Spaces}) \simeq \mathrm{Spaces}_{/\mathbb{B}\,\mathrm{Top}}$$

where the first functor is the Yoneda embedding. Given an open n-manifold $M \in \mathcal{M}\mathrm{fld}$ we will denote by $p : |M| \longrightarrow \mathbb{B}\,\mathrm{Top}(n)$ the image of M under (1). We will refer to $|M|$ as the **underlying space** of M, and the $\mathrm{Top}(n)$ bundle classified by p the **tangent bundle** of M. We note that $|M|$ can be identified with the homotopy quotient of $\mathrm{Emb}(\mathbb{D}^n, M)$ by the action of $\mathrm{Top}(n)$, and one can show that the evaluation at 0 map induces a homotopy equivalent $|M| \xrightarrow{\simeq} M$.

Remark 2. A classical theorem of Kister and Mazur asserts that the map $\mathrm{Top}(n) \longrightarrow \mathrm{Map}_{\mathcal{M}\mathrm{fld}_n}(\mathbb{D}^n, \mathbb{D}^n)$ is in fact an equivalence. We may hence consider the tangent bundle of M as capturing exactly the information on $M \in \mathcal{M}\mathrm{fld}_n$ "seen" by the object $\mathbb{D}^n$.

Definition 3. Let B be a space equipped with a map $\varphi : B \longrightarrow \mathbb{B}\,\mathrm{Top}(n)$ and let M be an open n-manifold. A B**-framing** of M is a lift of the form

$$\begin{tikzcd} & B \arrow[d] \\ |M| \arrow[ur, dashed] \arrow[r, "p"] & \mathbb{B}\,\mathrm{Top}(n) \end{tikzcd}$$

in the ∞-category of spaces. We will denote by $\mathcal{M}\mathrm{fld}_n^B$ the ∞-category of open B-framed n-manifolds, that is the ∞-category which sits in the Cartesian square

$$\begin{tikzcd} \mathcal{M}\mathrm{fld}_n^B \arrow[r] \arrow[d] & \mathcal{M}\mathrm{fld}_n \arrow[d] \\ \mathrm{Spaces}_{/B} \arrow[r] & \mathrm{Spaces}_{/\mathbb{B}\,\mathrm{Top}(n)} \end{tikzcd}$$

Remark 4. Using the equivalence $|M| \simeq M$ we may identify the data of a B-framing on M with the data of a continuous map $f : M \longrightarrow B$ together with an identification $TM \cong f^*E_\varphi$, where $E_\varphi \longrightarrow B$ is the $\mathrm{Top}(n)$-bundle classifying by $\varphi : B \longrightarrow \mathbb{B}\,\mathrm{Top}(n)$.

Example 5. When $\varphi : B \longrightarrow \mathbb{B}\,\mathrm{Top}(n)$ is an equivalence the notion of a B-framing is vacuous. On the other extreme, when $B \simeq *$ the notion of a B-framing coincides with a trivilization of the tangent bundle.

Example 6. When $\varphi : B \longrightarrow \mathbb{B}\,\mathrm{Top}(n)$ is the universal covering of $\mathbb{B}\,\mathrm{Top}(n)$ a B-framing is the same as an orientation. Similarly, when B is the 2-connected covering of $\mathrm{Top}(n)$ a B-framing is a topological spin structure.

Example 7. By smoothing theory, when $n \neq 5$ the data of a smooth structure on M is equivalent to the data of a framing with respect to the map $\mathbb{B}\,\mathrm{O}(n) \longrightarrow \mathbb{B}\,\mathrm{Top}(n)$, and the data of a piecewise linear structure is equivalent to the data of a framing with respect to the map $\mathbb{B}\,\mathrm{PL}(n) \longrightarrow \mathbb{B}\,\mathrm{Top}(n)$.

Example 8. If N is an open n-manifold then we can take $B = |N|$ with the map $\varphi : |N| \longrightarrow \mathbb{B}\,\mathrm{Top}(n)$ classifying the tangent bundle. In this case we will simplify notation and write N-framing instead of $|N|$-framing. If M is another open n-manifold then the data of an N-framing on M is equivalent to the data of a continuous map $f : M \longrightarrow N$ together with an identification $TM \xrightarrow{\cong} f^*TN$. For example, any **open immersion** $M \longrightarrow N$ (i.e., a continuous map which is locally a homeomorphism) gives an N-framing on M, although not every N-framing is obtained this way (e.g., the $\mathbb{R}^1$-framing of S^1 cannot be obtained by an immersion of S^1 in $\mathbb{R}^1$). Note however that by the Yoneda lemma every N-framing on $\mathbb{D}^n$ is equivalent to one which comes from an immersion, and even an embedding, of $\mathbb{D}^n$ in N.

It what follows it will be useful to have a notion of a framing for n-manifolds with possibly non-empty boundary. Consider the full subcategory $\mathcal{D} \subseteq \mathcal{M}\mathrm{fld}_n^{\partial}$ spanned by the objects $\mathbb{D}^n$ and $\mathbb{D}^n_+$. As mentioned in Remark 2 the $\mathbb{E}_1$-monoid $\mathrm{Emb}(\mathbb{D}^n, \mathbb{D}^n)$ is canonically equivalent to $\mathrm{Top}(n)$, and one can similarly show that the action of $\mathrm{Top}(n-1)$ on $\mathbb{D}^n_+ = [0,1) \times \mathbb{D}^{n-1}$ via its action on the right coordinate induces an equivalence $\mathrm{Top}(n-1) \hookrightarrow \mathrm{Emb}(\mathbb{D}^n_+, \mathbb{D}^n_+)$. Furthermore, the space $\mathrm{Emb}(\mathbb{D}^n_+, \mathbb{D}^n)$ is empty and the space $\mathrm{Emb}(\mathbb{D}^n, \mathbb{D}^n_+)$ is the same as $\mathrm{Emb}(\mathbb{D}^n, \mathbb{D}^n_+ \setminus \partial\mathbb{D}^n_+) \simeq \mathrm{Emb}(\mathbb{D}^n, \mathbb{D}^n) \simeq \mathrm{Top}(n)$, while the action of $\mathrm{Emb}(\mathbb{D}^n_+, \mathbb{D}^n_+)$ on $\mathrm{Emb}(\mathbb{D}^n, \mathbb{D}^n_+)$ is by the embedding $\mathrm{Top}(n-1) \hookrightarrow \mathrm{Top}(n)$ induced by the identification $\mathbb{D}^n \cong \mathbb{D}^1 \times \mathbb{D}^{n-1}$. We may hence identify functors $\mathcal{D}^{\mathrm{op}} \longrightarrow \mathrm{Spaces}$ with tuples (X, Y, f) where X is a $\mathrm{Top}(n-1)$-space, Y is a $\mathrm{Top}(n)$-space and $f : X \longrightarrow Y$ is a $\mathrm{Top}(n-1)$-equivariant map. Using the identification of $\mathrm{Top}(k)$-spaces with spaces over $\mathbb{B}\,\mathrm{Top}(k)$ we may finally identify the ∞-category $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathrm{Spaces})$ with the slice category $\mathrm{Spaces}^{[1]}_{/\mathbb{B}\,\mathrm{Top}(n-1) \to \mathbb{B}\,\mathrm{Top}(n)}$, where $\mathrm{Spaces}^{[1]} = \mathrm{Fun}([1], \mathrm{Spaces})$ is the arrow category of spaces. We hence obtain a natural composed functor

$$\mathcal{M}\mathrm{fld}_n^{\partial} \longrightarrow \mathrm{Fun}((\mathcal{M}\mathrm{fld}_n^{\partial})^{\mathrm{op}}, \mathrm{Spaces}) \longrightarrow \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathrm{Spaces}) \simeq \mathrm{Spaces}^{[1]}_{/\mathbb{B}\,\mathrm{Top}(n-1) \to \mathbb{B}\,\mathrm{Top}(n)} .$$

Given an n-manifold M we will write its image in $\mathrm{Spaces}^{[1]}_{/\mathbb{B}\,\mathrm{Top}(n-1) \to \mathbb{B}\,\mathrm{Top}(n)}$ as a diagram of the form

$$\begin{tikzcd} \partial|M| \arrow[r] \arrow[d] & |M| \arrow[d] \\ \mathbb{B}\,\mathrm{Top}(n-1) \arrow[r] & \mathbb{B}\,\mathrm{Top}(n) \end{tikzcd} \tag{2}$$

We will refer to $|M|$ as the underlying space of M and to $\partial|M|$ as the underlying boundary of M. We note that indeed $|M|$ is weakly equivalent to M and $\partial|M|$ is weakly equivalent to ∂M. We consider (2) as describing the tangent bundle of the open n-manifold $M \setminus \partial M$ together with its identification with $\mathbb{R} \oplus T(\partial M)$ near the boundary.

We will usually write a general object in $\mathrm{Spaces}^{[1]}_{/\mathbb{B}\,\mathrm{Top}(n-1) \to \mathbb{B}\,\mathrm{Top}(n)}$ as $[\partial B \to B]$, where ∂B is simply meant as suggestive terminology (it is just a space with a map to B), and the structure map to $[\mathbb{B}\,\mathrm{Top}(n-1) \to \mathbb{B}\,\mathrm{Top}(n)]$ will usually be omitted. We will refer to such an object as a **boundary tangent structure** of dimension n (when $\partial B = \emptyset$ we will simply say **tangent structures**). Given a boundary tangent structure $[\partial B \to B]$ and an n-manifold M, a $[\partial B \to B]$**-framing** on M is a diagram of the form

$$\begin{tikzcd} \partial|M| \arrow[r] \arrow[d] & |M| \arrow[d] \\ \partial B \arrow[r] \arrow[d] & B \arrow[d] \\ \mathbb{B}\,\mathrm{Top}(n-1) \arrow[r] & \mathbb{B}\,\mathrm{Top}(n) \end{tikzcd}$$

where the external rectangle is (2). More precisely, we define the ∞-category $\mathcal{M}\mathrm{fld}_n^{[\partial B\to B]}$ of $[\partial B \to B]$-framed n-manifolds as the fiber product

$$\begin{tikzcd} \mathcal{M}\mathrm{fld}_n^{[\partial B\to B]} \arrow[r] \arrow[d] & \mathcal{M}\mathrm{fld}_n^{\partial} \arrow[d] \\ \mathrm{Spaces}^{[1]}_{/\partial B\to B} \arrow[r] & \mathrm{Spaces}^{[1]}_{/\mathbb{B}\,\mathrm{Top}(n-1)\to\mathbb{B}\,\mathrm{Top}(n)} \end{tikzcd}$$

Remark 9. If $\partial B = \emptyset$ then any $[\partial B \to B]$-framed n-manifold is open, and so $\mathcal{M}\mathrm{fld}_n^{[\emptyset\to B]} \simeq \mathcal{M}\mathrm{fld}_n^B$ for every $B \longrightarrow \mathbb{B}\,\mathrm{Top}(n)$. In particular, we may consider the restriction to open n-manifold as a particular case of framing.

Example 10. If N is an n-manifold then we can take $[\partial B \to B] = [\partial|N| \to |N|]$ equipped with its natural map to $\mathbb{B}\,\mathrm{Top}(n-1) \to \mathbb{B}\,\mathrm{Top}(n)$. In this case we will simplify notation and write the corresponding boundary tangent structure as $[\partial N \to N]$. Similarly to Example 8, if U is either $\mathbb{D}^n$ or $\mathbb{D}^n_+$, then the data of an $[\partial N \to N]$-framing on U is essentially equivalent to the data of an embedding $U \hookrightarrow N$.

2. Disk algebras and manifold ∂-bundles

For a boundary tangent structure $[\partial B \to B]$, we let $\mathcal{D}\mathrm{isk}_n^{[\partial B\to B]} \subseteq \mathcal{M}\mathrm{fld}_n^{[\partial B\to B]}$ denote the full symmetric monoidal subcategory spanned by those $[\partial B \to B]$-framed n-manifolds which are homeomorphic to finite disjoint union of $\mathbb{D}^n$ and $\mathbb{D}^n_+$. When $\partial B = \emptyset$ we will also denote $\mathcal{D}\mathrm{isk}_n^{[\partial B\to B]}$ by $\mathcal{D}\mathrm{isk}_n^B$.

Definition 11. Given a symmetric monoidal ∞-category $\mathcal{C}$ and a boundary tangent structure $[\partial B \to B]$, a $[\partial B \to B]$**-disk algebra** in $\mathcal{C}$ is a symmetric monoidal functor

$$\mathcal{D}\mathrm{isk}_n^{[\partial B\to B]} \longrightarrow \mathcal{C}.$$

When $\partial B = \emptyset$ we will also refer to such a functor as a B**-disk algebra**.

Example 12. When $[\partial B \to B] = [\emptyset \to *]$ the symmetric monoidal ∞-category $\mathcal{D}\mathrm{isk}_n^B$ is the enveloping category of the $\mathbb{E}_n$-operad. Consequently, in this case a $[\partial B \to B]$-disk algebra is an $\mathbb{E}_n$-algebra in the usual sense. More generally, since $\mathrm{Top}(n)$ acts on the ∞-operad $\mathbb{E}_n$ the map $\varphi : B \longrightarrow \mathbb{B}\,\mathrm{Top}(n)$ determines a B-indexed family of $\mathbb{E}_n$-operads $b \mapsto \mathbb{E}_n(b)$. The notion of a B-disk algebra can then be identified with the notion of a φ-twisted B-indexed family of $\mathbb{E}_n$-algebras (i.e., for each b we have a $\mathbb{E}_n(b)$-algebra, depending coherently on $b \in B$).

Example 13. The symmetric monoidal ∞-category $\mathcal{D}\mathrm{isk}_n^{*\to *}$ is equivalent to the enveloping category of the n'th Swiss-cheese operad. Consequently, we may identify $[* \to *]$-disk algebras in $\mathcal{C}$ with a pair (R, S) where R is an $\mathbb{E}_n$-algebra and S is an $\mathbb{E}_{n-1}$-algebra over R. More generally, for a boundary tangent structure $f : \partial B \longrightarrow B$ of dimension n, we may consider $[\partial B \to B]$-algebras as pairs $(\{R_b\}_{b\in B}, \{S_b\}_{c\in\partial B})$ where R_b is an $\mathbb{E}_n(b)$-algebra and S_c is an $\mathbb{E}_{n-1}(c)$-algebras over $\mathbb{E}_n(f(c))$ (this makes sense since the structure maps of the boundary tangent structure specify a coherent collection of equivalences $\mathbb{E}_n(f(c)) \simeq \mathbb{E}_1 \otimes \mathbb{E}_{n-1}(c)$).

Example 14. Let $n = 1$ and $[\partial B \to B] = [\partial I \to I]$, where $I = [0, 1]$ is the unit interval (considered as a 1-manifold with boundary, see Example 10). In this case there is, up to equivalence, a single $[\partial I \to I]$-framed version of the open 1-disk, which we shall denote by $(0,1) \in \mathcal{D}\mathrm{isk}_1^{[\partial I \to I]}$, and **two** non-equivalent $[\partial I \to I]$-framed versions of the half-disk, which we shall denote by $[0,1),(1,0] \in \mathcal{D}\mathrm{isk}_1^{[\partial I \to I]}$ (here we use the suggestive notation to imply that the $[\partial I \to I]$-framing on $(0,1),[0,1)$ and $(0,1]$ is given by their inclusion in $I = [0,1]$). Unwinding the definitions, we see that in this case the corresponding notion of a disk algebra $\mathcal{F} : \mathcal{D}\mathrm{isk}_1^{[\partial I \to I]} \longrightarrow \mathcal{C}$ is equivalent to the data of a triple $(A, M_0, M_1) = (\mathcal{F}((0,1)), \mathcal{F}([0,1)), \mathcal{F}((0,1]))$, where A is an associative algebra object in $\mathcal{C}$, M_0 is a right A-module in $\mathcal{C}_{1_{\mathcal{C}}/}$ and M_1 is a left A-module in $\mathcal{C}_{1_{\mathcal{C}}/}$.

A particular case of interest is when the boundary tangent structure $[\partial B \to B]$ is of the form $[\partial N \to N]$ for some n-manifold N (e.g., the case of Example 14). Recall (see Example 10) that the notion of an $[\partial N \to N]$-framing on an n-manifold U which is either the disk or the half-disk is equivalent to that of an embedding $U \hookrightarrow N$. We may consequently try to construct a more rigid model for $\mathcal{D}\mathrm{isk}_n^{[\partial N \to N]}$ by considering the actual poset of these subsets. Better yet, we may consider them as a **colored operad**. More precisely, let $\mathrm{D}(N)$ denote the operad whose colors are the open subsets $U \subseteq N$ which are homeomorphic to either $\mathbb{D}^n$ or $\mathbb{D}^n_+$, and such that

$$\mathrm{Mul}(U_1, ..., U_n; V) = \begin{cases} * & U_i \cap U_j = \emptyset, U_i \subseteq V, \forall i, j \\ \emptyset & \text{otherwise} \end{cases}$$

Consider the enveloping symmetric monoidal category $\mathrm{Env}(\mathrm{D}(N))$. Explicitly, the objects of $\mathrm{Env}(\mathrm{D}(N))$ are tuples $\{U_i\}_{i\in I}$ of open subsets of N indexed by a finite set I, and the maps $\{U_i\}_{i\in I} \longrightarrow \{V_j\}_{j\in J}$ are given by maps $f : I \longrightarrow J$ such that for each $j \in J$ the collection $\{U_i\}_{i\in f^{-1}(j)}$ is pairwise disjoint and contained in V_j. The association $\{U_i\}_{i\in I} \mapsto \coprod_{i\in I} U_i$ determines a symmetric monoidal functor

$$\mathrm{Env}(\mathrm{D}(N)) \longrightarrow \mathcal{D}\mathrm{isk}_n^{[\partial N \to N]} \tag{3}$$

Let us denote by $\mathcal{I}_N$ the collection of those maps in $\mathrm{Env}(\mathrm{D}(N))$ whose image in $\mathcal{D}\mathrm{isk}_n^{[\partial N \to N]}$ is an equivalence. A direct inspection shows that $\mathcal{I}_N$ consists of exactly those maps $\{U_i\}_{i\in I} \longrightarrow \{V_j\}_{j\in J}$ whose associated $f : I \longrightarrow J$ is an isomorphism and such that for each $i \in J$ the inclusion $U_i \subseteq V_{f(i)}$ is an isotopy equivalence.

We then have the following lemma:

Lemma 15 ([AF15],[HA])**.** *The map* (3) *exhibits* $\mathcal{D}\mathrm{isk}_n^{[\partial N \to N]}$ *as the symmetric monoidal localization of* $\mathrm{Env}(\mathrm{D}(N))$ *with respect to* $\mathcal{I}_M$*.*

The main point of Lemma 15 is that we may identify the notion of a $|\partial N \to N]$-disk algebra with that of a $\mathrm{D}(N)$-algebra in which certain 1-ary operations act by equivalences. Our main interest in this idea is for the following geometric construction, which can be used to construct many examples of disk algebras.

Suppose first that N is an open k-manifold and E is an n-manifold, possibly with boundary. Then we have the notion of a **manifold bundle map** from E to N, which is, by definition a map $p : E \longrightarrow N$ such that for every open embedding $\iota : \mathbb{D}^k \hookrightarrow N$ the pullback $\iota^*E := E \times_N \mathbb{D}^k \longrightarrow \mathbb{D}^k$ admits a trivialization of the form $\tau : \iota^*E \cong P \times \mathbb{D}^k$ with P an $(n-k)$-manifold (here by trivialization we simply mean that τ commutes with the respective projections to $\mathbb{D}^k$). In this case the association $[\iota : \mathbb{D}^k \hookrightarrow N] \mapsto E \times_N \mathbb{D}^k$ determines a $\mathrm{D}(N)$-algebra object in $\mathcal{M}\mathrm{fld}_n^{\partial}$, which we shall call $[p^{-1}]$. The local triviality of E now implies that all the 1-ary operations act on $[p^{-1}]$ by equivalences, and so Lemma 15 tells us that $[p^{-1}]$ descends to an essentially unique N-disk algebra object in $\mathcal{M}\mathrm{fld}^{\partial}_n$.

We would like to have a similar story when N is a k-manifold which is not necessarily open (i.e., can have a boundary).

Definition 16. Let N be an k-manifold. By a **manifold ∂-bundle** over N we shall mean a map $E \longrightarrow N$ with E an n-manifold and such that the following conditions hold:

(1) For every open embedding $\iota : \mathbb{D}^k \hookrightarrow N$ the pullback $\iota^*E \longrightarrow \mathbb{D}^k$ admits a trivialization of the form $\iota^*E \cong P \times \mathbb{D}^k$ with P a $(n-k)$-manifold.
(2) For every open embedding $\iota : \mathbb{D}^k_+ = [0,1) \times \mathbb{D}^{k-1} \hookrightarrow N$ the pullback $\iota^*E \longrightarrow [0,1) \times \mathbb{D}^{k-1}$ admits an identification of the form

$$\begin{tikzcd} \iota^*E \arrow[rr, "\cong"] \arrow[dr] & & P \times \mathbb{D}^{k-1} \arrow[dl, "{(f,\mathrm{Id})}"] \\ & {[0,1) \times \mathbb{D}^{k-1}} & \end{tikzcd}$$

where P is an $(n-k+1)$-manifold equipped with a continuous map $f : P \longrightarrow [0,1)$.

In this case we will also say that $p : E \longrightarrow N$ is a **∂-bundle map**.

Remark 17. If $E \longrightarrow N$ is a manifold ∂-bundle then $E \times_N (N \setminus \partial N) \longrightarrow (N \setminus \partial N)$ is a manifold bundle in the usual sense. Furthermore, if $\iota : \partial N \times [0,1) \hookrightarrow N$ is a tubular neighborhood of the boundary of N then the composed map $\iota^*E \longrightarrow \partial N \times [0,1) \longrightarrow \partial N$ is a manifold bundle as well.

Example 18. If N is 1-dimensional then the boundary of N is 0-dimensional. In this case condition (2) of Definition 16 is vacuous, and so $p : E \longrightarrow N$ is a ∂-bundle map if and only if it restricts to a bundle map over the interior of N.

Remark 19. If we consider fiber bundles over N as analogous to locally constant sheaves, then the notion of a ∂-bundle can be considered as analogous to sheaves on N which are **constructible** with respect to the stratification $\partial N \subseteq N$.

Example 20. It is worthwhile to spell out what do ∂-manifold bundles over the unit interval I look like. Let M be a n-manifold and P an $(n-1)$-manifold. We will say that an open embedding $\iota : (0,1) \times P \hookrightarrow M$ is a **right P-collar** if $\iota([\varepsilon, 1) \times P)$ is closed in M for every $\varepsilon \in (0,1)$. Similarly, we will say that ι is a **left P-collar** if $\iota((0,\varepsilon] \times P)$ is closed in M for every $\varepsilon \in (0,1)$.

If M_0, M_1 are two n-manifolds, $\iota_0 : (0,1) \times P \hookrightarrow M_0$ a right P-collar and $\iota_1 : (0,1) \times P \hookrightarrow M_1$ a left P-collar then the topological space $M := M_0 \coprod_{(0,1)\times P} M_1$ is again an n-manifold which contains M_0 and M_1 as submanifolds. Following [AF15] we will refer to M as the **collar gluing** of M_0 and M_1 along $(0,1) \times P$. In this case, M admits a natural ∂-bundle map $M \longrightarrow [0,1]$ which extends the projection $(0,1) \times P \longrightarrow (0,1)$ and maps $M_0 \setminus \mathrm{Im}(\iota_0)$ and $M_1 \setminus \mathrm{Im}(\iota_1)$ to 0 and 1, respectively.

On the other hand, if $p : M \longrightarrow [0,1]$ is any ∂-bundle then by definition $p|_{(0,1)} : (0,1) \times_I M \longrightarrow (0,1)$ splits as a product $(0,1) \times_I M \cong (0,1) \times P$. If we now set $M_0 = p^{-1}[0,1)$ and $M_1 = p^{-1}(0,1]$ then the embedding $(0,1) \times P \hookrightarrow M_0$ is a right collar, the embedding $(0,1) \times P \hookrightarrow M_1$ is a left collar and $M \cong M_0 \coprod_{(0,1)\times P} M_1$ is a collar gluing of M_0 and M_1 along $(0,1) \times P$.

We shall now explain how the notion of a ∂-bundle can be used to construct $[\partial N \to N]$-disk algebras. Suppose that $p : E \longrightarrow N$ is a manifold ∂-bundle and that E is equipped with a $[\partial B \to B]$-framing for some boundary tangent structure $[\partial B \to B]$. Then the association $[\iota : U \hookrightarrow N] \mapsto U \times_N E$ for $U \cong \mathbb{D}^n, \mathbb{D}^n_+$ determines a $\mathrm{D}(N)$-algebra object $[p^{-1}]$ in $\mathcal{M}\mathrm{fld}_n^{[\partial B \to B]}$. The local models of Definition 16 imply that the 1-ary operations coming from inclusions of open sub-disks of N and from inclusion of sub half-disks of N act on $[p^{-1}]$ by equivalences, and so Lemma 15 tells us that $[p^{-1}]$ descends to an essentially unique $[\partial N \to N]$-disk algebra object in $\mathcal{M}\mathrm{fld}_n^{[\partial B \to B]}$.

If $\mathcal{C}$ is a presentably symmetric monoidal ∞-category and $\mathcal{F} : \mathcal{M}\mathrm{fld}_n^{[\partial B \to B]} \longrightarrow \mathcal{C}$ is a symmetric monoidal functor then the composed functor $p_*\mathcal{F} := \mathcal{F} \circ [p^{-1}] : \mathcal{D}\mathrm{isk}_n^{[\partial N \to N]} \longrightarrow \mathcal{C}$ gives an $[\partial N \to N]$-algebra in $\mathcal{C}$. This construction can be used to produce a variety of interesting examples of disk algebras. We will also make use of it in order to formulate the Fubini property of factorization homology in §3 and to define the property of being a homology theory for manifolds in §4.

3. Factorization homology

Let $[\partial B \to B] \in \mathrm{Spaces}_{/\mathbb{B}\,\mathrm{Top}(n-1) \to \mathbb{B}\,\mathrm{Top}(n)}$ be a boundary tangent structure. Given a $[\partial B \to B]$-framed n-manifold M, let us denote by

$$\mathcal{D}\mathrm{isk}_{n/M}^{[\partial B \to B]} = \mathcal{D}\mathrm{isk}_n^{[\partial B \to B]} \times_{\mathcal{M}\mathrm{fld}_n^{[\partial B \to B]}} (\mathcal{M}\mathrm{fld}_n^{[\partial B \to B]})_{/M}$$

the associated comma ∞-category. We now arrive to the main definition of this talk:

Definition 21. Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category. Let M be a $[\partial B \to B]$-framed manifold and $A : \mathcal{D}\mathrm{isk}_n^{[\partial B \to B]} \longrightarrow \mathcal{C}$ a $[\partial B \to B]$-disk algebra in $\mathcal{C}$. We define the **factorization homology** $\int_M A$ of M with coefficients in A as the colimit

$$\int_M A := \operatorname*{colim}_{\mathcal{D}\mathrm{isk}_{n/M}^{[\partial B \to B]}} \pi^* A \in \mathcal{C}.$$

where $\pi : \mathcal{D}\mathrm{isk}_{n/M}^{[\partial B \to B]} \longrightarrow \mathcal{D}\mathrm{isk}_n^{[\partial B \to B]}$ is the canonical projection.

Example 22 ([AF15, Corllary 3.12])**.** Let I be the unit interval. Recall (see Example 14) that a $[\partial I \to I]$-disk algebra in $\mathcal{C}$ is given by a triple (A, M_0, M_1) where A is an associative algebra object in $\mathcal{C}$, M_0 is a right A-module in $\mathcal{C}_{1_\mathcal{C}/}$ and M_1 is a left A-module in $\mathcal{C}_{1_\mathcal{C}/}$. In this case we have a natural equivalence

$$\int_M (A, M_0, M_1) \simeq M_0 \otimes_A M_1$$

Our next goal is to discuss the **Fubini property** of factorization homology. Let $[\partial B \to B]$ be a boundary tangent structure and let A be a $[\partial B \to B]$-disk algebra. Let M be a $[\partial B \to B]$-framed n-manifold and N a k-manifold. Given a ∂-bundle map $p : M \longrightarrow N$, let us denote by $p_*A : \mathcal{D}\mathrm{isk}_k^{[\partial N \to N]} \longrightarrow \mathcal{C}$ the composed functor

$$\mathcal{D}\mathrm{isk}_k^{[\partial N \to N]} \overset{[p^{-1}]}{\longrightarrow} \mathcal{M}\mathrm{fld}_n^{[\partial B \to B]} \overset{\int_{(-)} A}{\longrightarrow} \mathcal{C}$$

where $[p^{-1}]$ is the $[\partial N \to N]$-disk algebra object in $\mathcal{M}\mathrm{fld}_n^{[\partial B \to B]}$ associated to $p : M \longrightarrow N$ as in §2.

Proposition 23 (The Fubini property, [AF15, Proposition 3.23])**.** *In the above setting the natural map*
$$\int_N p_* A \longrightarrow \int_M A$$
is an equivalence.

Remark 24. Let $[\partial B \to B]$ be a tangent structure and $A : \mathcal{D}\mathrm{isk}_n^{[\partial B \to B]} \longrightarrow \mathcal{C}$ a $[\partial B \to B]$-disk algebra in $\mathcal{C}$. By the pointwise definition of left Kan extension we see that the functor $\int_{(-)} A : \mathcal{M}\mathrm{fld}_n^{[\partial B \to B]} \longrightarrow \mathcal{C}$ is the left Kan extension of $A : \mathcal{D}\mathrm{isk}_n^{[\partial B \to B]} \longrightarrow \mathcal{C}$ along the inclusion $\mathcal{D}\mathrm{isk}_n^{[\partial B \to B]} \hookrightarrow \mathcal{M}\mathrm{fld}_n^{[\partial B \to B]}$. Furthermore, Proposition 23 applied to $N = \{0,1\}$ shows that $\int_{(-)} A$ is symmetric monoidal. These two statements can in fact be combined: one can show that $\int_{(-)} A$ is the **symmetric monoidal** left Kan extension of A. In other words, it is initial in the ∞-category of symmetric monoidal functors $\mathcal{F} : \mathcal{M}\mathrm{fld}_n^{[\partial B \to B]} \longrightarrow \mathcal{C}$ equipped with a symmetric monoidal transformation $A \Rightarrow \mathcal{F}|_{\mathcal{D}\mathrm{isk}_n^{[\partial B \to B]}}$ (see [AF15, Proposition 3.7]).

Example 25. The Fubini property can help us to decipher what is the factorization homology along the circle. We first note that the notion of an $\mathbb{S}^1$-disk algebra object in $\mathcal{C}$ is equivalent to that of a pair (A, τ) where A is an associative algebra A and $\tau : A \longrightarrow A$ is an automorphism (associated to the monodromy along the circle). The projection $p : \mathbb{S}^1 \longrightarrow \mathbb{D}^1$ on x-axis is a ∂-bundle map and the $[\partial\mathbb{D}^1 \to \mathbb{D}^1]$-disk algebra $p_*(A, \tau)$ can be identified with the triple $(A^{\mathrm{op}} \otimes A, A_0, A_1)$ where A_0 is a copy of A considered as a right $A^{\mathrm{op}} \otimes A$-module in the usual way and A_1 is a copy of A considered as a left $A^{\mathrm{op}} \otimes A$-module via the equivalence $(\mathrm{Id} \otimes \tau) : A^{\mathrm{op}} \otimes A \longrightarrow A^{\mathrm{op}} \otimes A$. By the Fubini property and Example 22 we then have that
$$\int_{S^1} (A, \tau) \simeq \int_{\mathbb{D}^1} p_*(A, \tau) \simeq A \otimes_{A^{\mathrm{op}} \otimes A} A$$
is the τ-twisted Hochshild homology of A.

4. Homology theories for open manifolds

In this section we will focus attention on **open** n-manifolds, i.e., those which do not have boundary. Following [AF15], our goal to consider **homology theories** on suitably framed open n-manifolds. For this we will need to isolate a particular full subcategory of $\mathcal{M}\mathrm{fld}_n$ spanned by manifolds which can be built in finitely many steps by gluing discs of various dimensions. This gluing is defined via the notion of a **collar gluing** spelled out in Example 20.

Let $\mathbb{S}^{k-1}$ denote the standard $(k-1)$-sphere (where $\mathbb{S}^{-1} = \emptyset$ by convention), so that we have a canonical right $\mathbb{S}^{k-1}$-collar $(0,1) \times \mathbb{S}^{k-1} \hookrightarrow \mathbb{D}^k$ embedded as the complement of $\{0\} \subset \mathbb{D}^k$. Let M_0 be an open n-manifold. If $\iota : (0,1) \times \mathbb{S}^{k-1} \times \mathbb{D}^{n-k} \hookrightarrow M_0$ is a left $[\mathbb{S}^{k-1} \times \mathbb{D}^{n-k}]$-collar then we will say that
$$M = \mathbb{D}^k \times \mathbb{D}^{n-k} \coprod_{(0,1) \times \mathbb{S}^{k-1} \times \mathbb{D}^{n-k}} M_0$$
is obtained from M_0 by adding an **open handle** of index k.

Definition 26. Let M be an open n-manifold. We will say that M is of **finite type** if it can be obtained from $\emptyset$ by adding finitely many open handles. We will denote by $\mathcal{M}\mathrm{fld}_n^{\mathrm{fin}} \subseteq \mathcal{M}\mathrm{fld}_n$ the full subcategory spanned by open n-manifolds of finite type. Similarly, if $\varphi : B \longrightarrow \mathbb{B}\,\mathrm{Top}(n)$ is a tangent structure then we will denote by $\mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}} := \mathcal{M}\mathrm{fld}_n^B \times_{\mathcal{M}\mathrm{fld}_n} \mathcal{M}\mathrm{fld}_n^{\mathrm{fin}}$. We note that the inclusion $\mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}} \subseteq \mathcal{M}\mathrm{fld}_n^B$ is fully faithful and its essential image is spanned by those B-framed open n-manifolds which are of finite type.

Example 27. Adding to M_0 an open handle of index 0 is simply taking the coproduct $M = M_0 \coprod \mathbb{D}^n$. In particular, the n-disk $\mathbb{D}^n$ is an n-manifold of finite type.

Example 28. $\mathbb{S}^k \times \mathbb{D}^{n-k}$ is obtained from $\mathbb{D}^k \times \mathbb{D}^{n-k} \cong \mathbb{D}^n$ by adding a single open handle of index k. In particular, $\mathbb{S}^k \times \mathbb{D}^{n-k}$ is an n-manifold of finite type.

Warning 29. The notion of an open handle is closely related, but not identical to, the notion of a handle studied in classical geometric topology, which is usually applied only to compact manifolds. However, if M is a compact manifold with a finite handle decomposition in the classical sense, then the interior of M is of finite type in the sense of Definition 26. In particular, any closed manifold of dimension $\neq 4$ is of finite type and every closed piecewise linear 4-manifold is of finite type. We do not know if there exist closed non-piecewise linear 4-manifolds that are of finite type.

Definition 30. We will say that a manifold ∂-bundle $p : M \longrightarrow N$ is **open** if M is open, and we will say that p has **finite type** if for every $U \subseteq N$ which is homeomorphic to either $\mathbb{D}^k$ or $\mathbb{D}^k_+$ the fiber product $M \times_N U$ has finite type.

Now let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category and let $\varphi : B \longrightarrow \mathbb{B}\,\mathrm{Top}(n)$ be a tangent structure. In this section we will describe a certain class of symmetric monoidal functors $\mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}} \longrightarrow \mathcal{C}$ which are called **homology theories** in [AF15]. The defining property of these functors is that they satisfy $\otimes$**-excision**, a term we shall now define.

Definition 31. Let $\mathcal{F} : \mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}} \longrightarrow \mathcal{C}$ be a symmetric monoidal functor. We will say that $\mathcal{F}$ satisfies $\otimes$**-excision** if for every open finite type ∂-bundle $p : M \longrightarrow I$ the induced map

$$\int_I p_*\mathcal{F} \longrightarrow \mathcal{F}(M)$$

is an equivalence, where $p_*\mathcal{F}$ is the composed functor $\mathcal{D}\mathrm{isk}_1^{[\partial I \to I]} \xrightarrow{[p^{-1}]} \mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}} \xrightarrow{\mathcal{F}} \mathcal{C}$ as above.

Remark 32. In light of Example 22 and Example 20 we may also (somewhat informally) phrase the $\otimes$-excision property as saying that for every collar gluing $M = M_0 \coprod_{(0,1)\times P} M_1$ of finite type open n-manifolds the induced map

$$\mathcal{F}(M_0) \otimes_{\mathcal{F}((0,1)\times P)} \mathcal{F}(M_1) \longrightarrow \mathcal{F}(M)$$

is an equivalence.

Definition 33. Let $\varphi : B \longrightarrow \mathbb{B}\,\mathrm{Top}(n)$ be a tangent structure and $\mathcal{C}$ a presentably symmetric monoidal ∞-category. A B-framed **homology theory** is a symmetric monoidal functor $\mathcal{F} : \mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}} \longrightarrow \mathcal{C}$ which satisfies $\otimes$-excision. We will denote by $\mathbf{H}(\mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}}, \mathcal{C}) \subseteq \mathrm{Fun}^{\otimes}(\mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}}, \mathcal{C})$ the full subcategory spanned by the B-framed homology theories.

Example 34. Let $\varphi : B \longrightarrow \mathbb{B}\,\mathrm{Top}(n)$ be a boundary tangent structure and let A be a B-disk algebra. Then $\int_{(-)} : \mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}} \longrightarrow \mathcal{C}$ is a B-framed homology theory. This follows immediately from Proposition 23 and Remark 24.

Remark 35. If $\mathcal{F} : \mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}} \longrightarrow \mathcal{C}$ is a symmetric monoidal functor then $\mathcal{F}|_{\mathcal{D}\mathrm{isk}_n^B}$ is by definition a B-disk algebra object. Remark 24 then furnishes a symmetric monoidal natural transformation

$$\int_{(-)} \mathcal{F}|_{\mathcal{D}\mathrm{isk}_n^B} \Rightarrow \mathcal{F}(-)$$

of symmetric monoidal functors $\mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}} \longrightarrow \mathcal{C}$.

We now come to the main result of this talk.

Theorem 36 ([AF15])**.** *Let $\mathcal{F} : \mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}} \longrightarrow \mathcal{C}$ be a homology theory for manifolds and let $A = \mathcal{F}|_{\mathcal{D}\mathrm{isk}_n^B}$ be the associated B-disk algebra in $\mathcal{C}$. Then the natural map*

$$\int_M A \longrightarrow \mathcal{F}(M) \tag{4}$$

of Remark 35 is an equivalence for every B-framed n-manifold M of finite type.

Proof. We prove by double induction on the open handle decomposition of M. For integers $0 \leq k, m$ let us say that an open manifold M has **type** (k, m) if it can be obtained from $\emptyset$ by adding finitely many handles of index $\leq k$ out of which at most m handles are of index exactly k. We first note that an open n-manifold is of type $(0, 1)$ if and only if it is the n-disk, and the map (4) is an equivalence in this case by definition.

Now suppose we have proven that (4) is an equivalence for every B-framed manifold of type (k, m) where either $k > 0$ or $k = 0$ and $m \geq 1$, and let M be a B-framed manifold of type $(k, m+1)$. Then by definition there exists an n-manifold M_0 of type (k, m) and a left $[\mathbb{S}^{k-1} \times \mathbb{D}^{n-k}]$-collar $\iota : (0, 1) \times \mathbb{S}^{k-1} \times \mathbb{D}^{n-k} \hookrightarrow M_0$ such that

$$M := \mathbb{D}^k \times \mathbb{D}^{n-k} \coprod_{(0,1)\times\mathbb{S}^{k-1}\times\mathbb{D}^{n-k}} M_0.$$

In this case the B-framing on M restricts to B-framings on M_0, $(0, 1) \times \mathbb{S}^{k-1} \times \mathbb{D}^{n-k}$ and $\mathbb{D}^k \times \mathbb{D}^{n-k}$, so that we can consider all of them as B-framed sub-manifolds of M. Let $p : M \longrightarrow [0, 1]$ be the manifold ∂-bundle of Example 20, which is open and of finite type by construction (see Example 28 and Example 27), and consider the diagram

$$\begin{tikzcd} \int_I p_* A \arrow[r] \arrow[d, "\simeq"] & \int_I p_* \mathcal{F} \arrow[d, "\simeq"] \\ \int_M A \arrow[r] & \mathcal{F}(M) \end{tikzcd} \tag{5}$$

in which the vertical maps are equivalences since $\mathcal{F}$ and $\int_{(-)} A$ are homology theories. To show that the bottom horizontal map is an equivalence it will hence suffice to show that the top vertical map is an equivalence. We now observe that if $U \subseteq I$ is an open subset homeomorphic to either $\mathbb{D}^1$ or $\mathbb{D}^1_+$ then $p^{-1}(U)$ is an open manifold which is homeomorphic to either M_0, $\mathbb{D}^k \times \mathbb{D}^{n-k} \cong \mathbb{D}^n$ or $(0, 1) \times \mathbb{S}^{k-1} \times \mathbb{D}^{n-k}$, which are manifolds of types (k, m), $(k-1, 2)$ and $(0, 1)$, respectively By the induction hypothesis the map

$$\int_U p_* A \longrightarrow \mathcal{F}(U)$$

is an equivalence for every such $U \subseteq I$, and so the top vertical map of (5) is an equivalence, as desired. We may hence conclude that (4) is an equivalence for every manifold of type $(k, m+1)$. By induction on m we now get that (4) is an equivalence for every manifold of type (k, m') for $m' \geq 0$, and hence for every manifold of type $(k+1, 0)$. By induction on k we now get that (4) is an equivalence for any open n-manifold of finite type, as desired. $\square$

Corollary 37 ([AF15, Theorem 3.24])**.** *Restriction along* $\mathcal{D}\mathrm{isk}_n^B \hookrightarrow \mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}}$ *determines an equivalence*

$$\mathbf{H}(\mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Alg}_{\mathcal{D}\mathrm{isk}_n^B}(\mathcal{C})$$

between B*-framed homology theories with values in* $\mathcal{C}$ *and* B*-disk algebras in* $\mathcal{C}$*.*

References

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