~ On a theorem of Kontsevich & Conant-Vogtmann ~

symplectic　cyclic operad　graph complex ass. to $\mathcal{O}$

THM. $H(\underbrace{\mathfrak{h}^{\omega}_{\infty}\mathcal{O}}_{\text{certain Lie algebra ass. to } \mathcal{O}}) \cong \Lambda H(\widetilde{\mathcal{G}}\mathcal{O})$

(Loday-Mahajan)

Cor. $PH(\mathfrak{c}_\infty) \cong H(\mathcal{G}Com) \cong$ (graph homology) $\oplus H(\mathfrak{sp}_\infty)$

$PH(\mathfrak{a}_\infty) \cong H(\mathcal{G}As) \cong \bigoplus_{\substack{m\geq 0\\ 2-2g-m<0}} H(\mathcal{M}_{g,m}; k) \oplus H(\mathfrak{sp}_\infty)$

← moduli space of genus g, m punc.

$PH(\ell_\infty) \cong H(\mathcal{G}Lie) \cong \bigoplus_{n\geq 2} H(\mathrm{Out}\,\mathbb{F}_n; k) \oplus H(\mathfrak{sp}_\infty)$

where: fix a $V_{2g} = k\{p_i, q_i : 1\leq i\leq g\}$ with a symplectic form $\omega = \begin{pmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{pmatrix}$ in this basis

non-degenerate $V \xrightarrow{\cong} V^*$

skew-symmetric.

$\omega = \sum p_i \wedge q_i \in \Lambda^2 V_{2g}$

$(\omega = \sum p_i\otimes q_i - q_i \otimes p_i \in V_{2g}^{\otimes 2})$

$(\omega = \sum [p_i, q_i])$

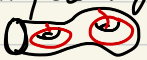
$\mathfrak{a}_g = \mathrm{Der}^{\omega}(T_{\geq 1}(V_{2g}))$ = derivations of $T_{\geq}(V_{2g})$ that preserve ω

$\ell_g = \mathrm{Der}^{\omega}(\mathbb{L}(V_g))$　　$\ell_\infty = \mathrm{colim}\, \ell_g$

$\mathfrak{c}_g = \mathrm{Der}^{\omega}(\mathrm{Sym}_{\geq 1}(V_{2g}))$ (& preserve codim 1 ideal $(p_1 .. p_g\, q_1 .. q_g)$)

$= k[p_i, q_i]_{\geq 2}$, $\{F, G\} = \sum \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}$

why? 1° non-commutative symplectic geometry

2° ℓ_g is related to mapping class groups $MCG_{g,1}$ of surface $S_{g,1}$ =  ←

$MCG_{g,1} = \pi_0 \mathrm{Diff}^+_\partial S_{g,1}$, $V_{2g} = H_1(S_{g,1}; \mathbb{Z})$

ω = intersection form

$J_{g,1} \rightarrowtail MCG_{g,1} \longrightarrow \mathrm{Aut}(V_{2g}, \omega)$

$= Sp(2g; \mathbb{Z})$　　$H_1(S_{g,1}) = \Gamma_0/\Gamma_1$　　Γ_k/Γ_{k+1}

m-th Johnson homomorphism $\mathrm{MCG}_{g,1}(m) \longrightarrow \mathcal{I}_g(m)$

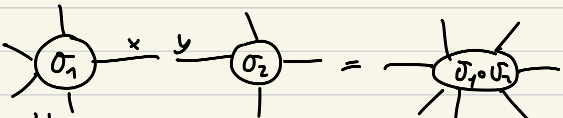
$=: \{\varphi \in \mathrm{MCG} : \forall \gamma \in \pi_1 \quad \varphi(\gamma)\gamma^{-1} \in \Gamma_m\}$

$\mathcal{I}_g(m) \leftarrow \Gamma_m / \Gamma_{m+1}$

$\boxed{1^\circ}$

"def" A collection $\mathcal{O}(m) \in \mathrm{Vect}_{\mathbb{k}}$ is a cyclic operad if $\exists\ \Sigma_m \curvearrowright \mathcal{O}(m)$ and

$\forall x \in \{1,\dots,m_1\}$
$y \in \{1 \dots m_2\}$
$\exists\ -{}_x\circ_y- : \mathcal{O}(m_1) \otimes \mathcal{O}(m_2) \to \mathcal{O}(m_1+m_2-2)$



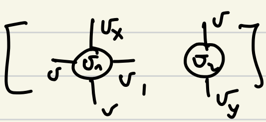
such that axioms hold.

think: $\mathcal{O} = \mathrm{Com}, \mathrm{As}, \mathrm{Lie}$

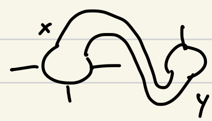
def. The symplectic Lie algebra associated to $\mathcal{O}$ (for fixed g) is

$$h_g^\omega \mathcal{O} := \mathcal{O}(V_{2g}) = \bigoplus_{m \geq 0} \left(\mathcal{O}(m) \otimes (V_{2g})^{\otimes m}\right)_{\Sigma_m}$$

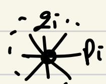
equipped with a "Poisson"-like bracket



$= \sum_{\substack{x \in \sigma_1 \\ y \in \sigma_2}} \omega(v_x, v_y)$



examples. $\mathcal{O} = \mathrm{Com}$  corolla

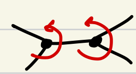
$h_g^\omega \mathrm{Com}(m)$ 

monomial of deg m

$[\,,\,] = \{\,,\,\}$

$\mathcal{O} = \mathrm{Ass}$ = planar corollas

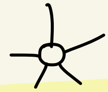
$\mathcal{O} = \mathrm{Lie}$ = planar binary trees / AS, IHX



$h_g^\omega \mathrm{Ass}$ = non-comm. analogue

$h_\infty^\omega \mathcal{O}$

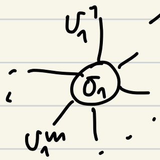
weight 0 ⇝  deg 0

 m legs = deg $m-2$

$\sum\sum$  $\wedge\,\sigma_3 \wedge$ $\in \mathcal{O}(2m-2)$ $\quad \mathcal{O}(m)$ $\boxed{m+1}$

$$\Lambda\, \mathfrak{h}_g^w \mathcal{O} = \Lambda \bigoplus_{m\geq 1} \mathfrak{h}_g^w \mathcal{O}(m) =: \bigoplus_{k,m} \Lambda_{k,m}$$

where $\Lambda_{k,m} \ni \sigma_1^{\bar{v}_1} \wedge \dots \wedge \sigma_k^{\bar{v}_k}$, $\quad \sigma_i \in \mathcal{O}(m_i)$, $\quad \sum m_i = m$



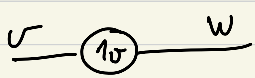
$\bar{v}_1 = (v_1^1, \dots, v_1^m)$

$$d_{CE} : \Lambda_{k,m} \to \Lambda_{k-1,m-2}$$

$$\Rightarrow \quad H_k(\mathfrak{h}_g^w \mathcal{O}) = \frac{\ker d_{CE}}{\operatorname{im} d_{CE}} = \bigoplus_{k,m} H_{k,m}$$

where
$$H_{k,m} = \frac{\Lambda_{k,m} \cap \ker d_{CE}}{\Lambda_{k,m} \cap \operatorname{im} d_{CE}} \begin{matrix} =: Z_{k,m} \\ =: B_{k,m} \end{matrix}$$

Prop 6. The subspace of $\mathfrak{h}_g^w \mathcal{O}$ gen. by spiders of the form



$\mapsto v \otimes w$

is isomorphic to $sp(2g, \mathbb{R})$.

Prop 8. $(\Lambda\, \mathfrak{h}_g^w \mathcal{O})^{sp(2g)} \hookrightarrow \Lambda\, \mathfrak{h}_g^w \mathcal{O}$ quasi-iso.

proof. $$H_k(\mathfrak{h}_g^w \mathcal{O}) = \bigoplus_{k,m} \frac{Z_{k,m}}{B_{k,m}} = \bigoplus \frac{Z_{k,m}^{sp(2g)}}{B_{k,m}^{sp(2g)}} \oplus \cancel{\frac{sp(2g) Z_{k,m}}{sp(2g) B_{k,m}}}$$
todo.

$sp(2g) \curvearrowright \Lambda_{k,m}$ and respect d_{CE}.

then $sp(2g) \curvearrowright \dfrac{Z_{k,m}}{B_{k,m}}$ ⇝ $Z_{k,m} = Z_{k,m}^{sp(2n)} \oplus sp(2g) Z_{k,m}$

$$B_{k,m} = B_{k,m}^{sp(2n)} \oplus sp(2g) B_{k,m}$$

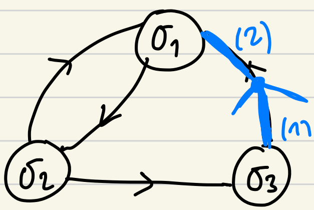
2°

def. The graph complex assoc. to $\mathcal{O}$ is a chain cx

$$\mathcal{G}\mathcal{O}_k := \frac{\mathbb{R}\{\text{oriented } \mathcal{O}\text{-graphs of deg } k\}}{\underset{\text{linearity}}{(X,-or) = -(X,or)}}$$

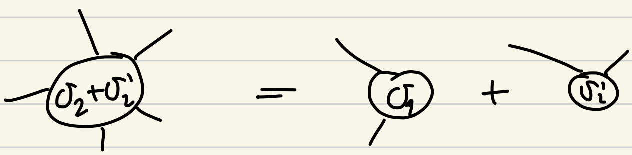
where
$X = \mathcal{O}$-graph $=$ a graph (finite 1-dim CW cx, no unival. vert) with each vertex labelled by an elt of $\mathcal{O}$ (valency of v)

$\deg X = \#$ vertices
orientation on X $=$ 1° choice of order on the set of vertices
2° orientation of all edges
(= order of two half-edges)



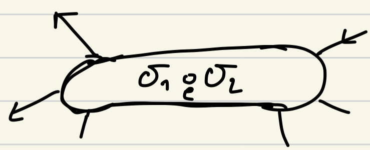
$\mathbb{R}^{|\text{vertices}|}$ $\qquad$ $\mathbb{R}^{|\text{half edges of } e|}$

linearity



differential $\quad \partial X = \sum\limits_{e \text{ edge of } X} X/e \quad$ collapse the edge e

$\mathcal{G}\mathcal{O}$ has product = disjoint union
has coproduct so that
conn. graphs are the primitives.



$V_{2g} = H_1(S_{g,1})$

$\mathcal{G}U = \bigoplus_{k} \mathcal{G}U_k = \bigoplus_{k,m} \mathcal{G}U_{k,m}$

k: number of ver

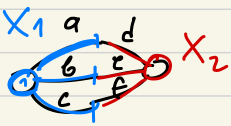
m: the number of half-edges

$\mathcal{G}U_k = \bigoplus_{m} \mathcal{G}U_{k,m}$

$$\partial : \mathcal{G}U_{k,m} \longrightarrow \mathcal{G}U_{k-1,m-2}$$

convention: $X \in \mathcal{G}U_{k,m}$ write it as $X = \bigcup_{\mathbb{1}_X} X_i$

where X_i = i-th vertex of X and its incident half-edges.



$\mathbb{1}_X$ = pairing of half-edges.

$= h(X) \circlearrowleft$

$$H(h^w_\infty U) \cong \Lambda\, H(\mathcal{G}U)$$

1° $\mathrm{cut}_{k,m} : \mathcal{G}U_{k,m} \longrightarrow \Lambda_{k,m} \subseteq \Lambda^k h^w_g U$ with total number of legs $= m+2$

2° $\mathrm{glue}_{k,m} : \Lambda_{k,m} \longrightarrow \mathcal{G}U_{k,m}$

3° $\mathrm{glue}_{k,m} \circ \mathrm{cut}_{k,m}$ = isomorphism for g large

4° $\mathrm{glue}_{k,m}$ is a chain map

proof. $H_k(h^w_\infty U) = \operatorname{colim}_{g\to\infty} H_k(h^w_g U) = \operatorname{colim} \bigoplus_{k,m} H_{k,m}$

$$H_{k,m} = \frac{(\Lambda_{k,m} \cap \ker d_{CE})^{Sp(2g)}}{(\Lambda_{k,m} \cap \operatorname{im} d_{CE})^{Sp(2g)}} \overset{\mathrm{glue}_{k,m}}{\cong} \frac{\mathcal{G}U_{k,m} \cap \ker \partial}{\mathcal{G}U_{k,m} \cap \operatorname{im} \partial}$$

$H_k(h^w_\infty U) \cong \bigoplus H_{k,m}(\mathcal{G}U) \Leftarrow H_{k,m}(\mathcal{G}U) = \dfrac{\mathcal{G}U_{k,m} \cap \ker \partial}{\mathcal{G}U_{k,m} \cap \operatorname{im} \partial}$

$= H_k(\mathcal{G}U)$

def. $\mathrm{cut}_{k,m} : \mathcal{G}\mathcal{V}_{k,m} \longrightarrow \Lambda_{k,m}$

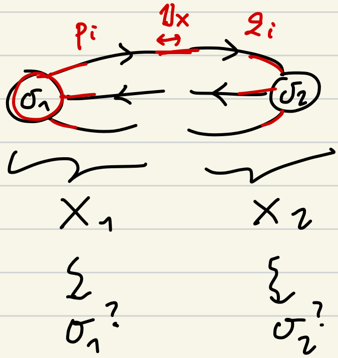
$\sigma_1^{\vec{v}_1} \wedge \dots \wedge \sigma_k^{\vec{v}_k}$

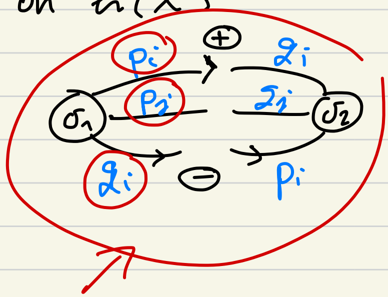
$\sigma_i \in \mathcal{O}(m_i)$

$\sum m_i = m$

$\vec{v}_i$ tuple of m_i elts in V_{2g}

$$X = \bigcup_{1_X} \sigma_i \longmapsto \sum_{\substack{S = \text{symplectic state} \\ \text{on } h(X)}} \mathrm{sgn}(S) \bigwedge_{i=1}^{k} \sigma_i^{\vec{v}_i}$$





total # legs

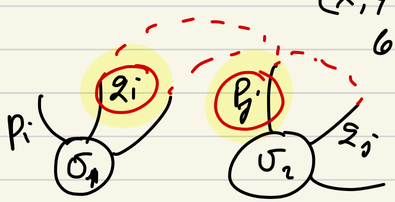
#half-edges

def $\mathrm{glue}_{k,m} : \Lambda_{k,m} \longrightarrow \mathcal{G}\mathcal{V}_{k,m}$

$$\bigwedge_{i=1}^{k} \sigma_i^{\vec{v}_i} \longmapsto \sum_{\substack{\pi = \text{pairing} \\ \text{on } h(\cup \sigma_i)}} w(\pi) \boxed{\bigcup_{\pi} \sigma_i}$$

orient in any way.

where $w(\pi) = \prod_{(x,y) \text{ paired by } \pi} \omega(v_x, v_y)$



Lemma. $\mathrm{glue}_{k,m} \circ \mathrm{cut}_{k,m} : \mathcal{G}\mathcal{O}_{k,m} \longrightarrow \mathcal{G}\mathcal{O}_{k,m}$

$$\underset{\mathbb{1}_X}{X = \bigcup \sigma_i} \longmapsto \sum_{\substack{\pi \text{ pairing} \\ \text{of } h(X)}} (2g)^{\#C_\pi} \bigcup_{\pi} \sigma_i$$

where C_π = the union of S^1's

$\# C_\pi$ = number of comp. of C_π

$$C_\pi : \; e_1^1 \xleftarrow{\mathbb{1}_X} e_1^2 \qquad \circ \!\rightarrow\! \circ \qquad e^1_{m/2} \longleftarrow e^2_{m/2}$$

Lemma. Moreover, this is an iso for large g.

Lemma. (Prop 11) $\boxed{\mathrm{im}\left(\bigoplus \mathrm{cut}_{k,m}\right) = \left(\Lambda h_g^{\omega} \mathcal{O}\right)^{sp(2g)}}$

g big enough

$$\begin{tikzcd} \mathcal{G}\mathcal{O}_{k,m} \arrow[r, bend left, two heads, "\mathrm{cut}_{k,m}"] & (sp(2g)\text{-invariants}) \arrow[l, bend left, two heads, "\mathrm{glue}_{k,m}"] \end{tikzcd}$$

Lemma. $\mathrm{glue}_{k,m}$ is a chain map.