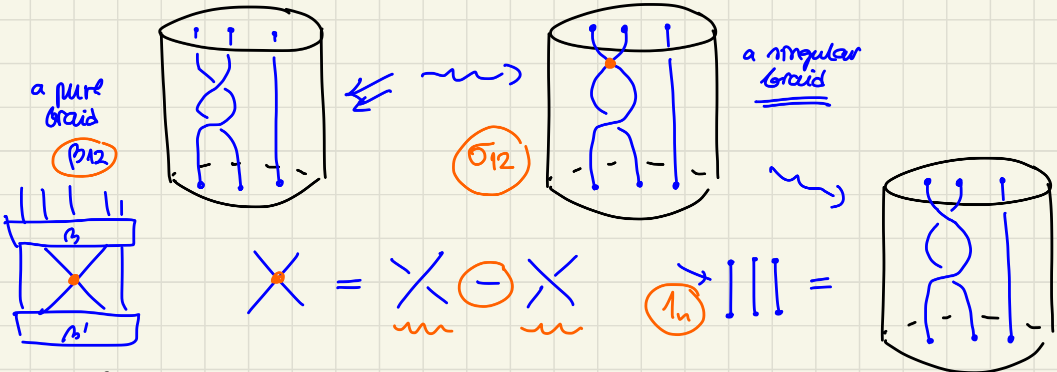
March 11: GT seminar at LAGA (Danica's Notes)

I. About braids & their Vassiliev invariants

$$PB_n = \pi_1(\mathrm{Conf}_n \mathbb{D}^2, 6) \longrightarrow \pi_1((\mathbb{D}^2)^{\times n}, 6) = \{1\}$$



* Can think of singular braids as linear combination of pure braids.

* This gives a natural filtration on $\mathbb{Z}[PB_n]$:

$$\mathcal{V}_k = \{\text{singular braid with} \geq k \text{ double points}\} \subseteq \mathbb{Z}[PB_n]$$

def. $v: PB_n \longrightarrow \underline{A}$ — map of sets "an invt of braids"; map of groups "an additive invt of braids"

$\bar{v}: \underline{\mathbb{Z}}[PB_n] \longrightarrow A$ linear extension.

We say that v is Vassiliev invt of type $\leq k-1$ if this extension vanishes on $\mathcal{V}_k$.

$\rightsquigarrow$ Vassiliev filtration on braid invts is dual to $\mathcal{V}_k \subseteq \mathbb{Z}[PB_n]$.

Lemma. $\mathcal{V}_k = I^k$ where $I := \ker(\mathbb{Z}[PB_n] \to \mathbb{Z})$

proof. β_{ij} generate PB_n, so $\underbrace{\beta_{ij} - 1_n}_{\sigma_{ij}}$ generate I. $\Rightarrow$ $\mathcal{V}_1 = I$ $\square$.

def. $gr_I \mathbb{Z}[PB_n] := \bigoplus_{k\geq 0} I^k/I^{k+1}$

Lemma. There is a (Hopf algebra) homomorphism

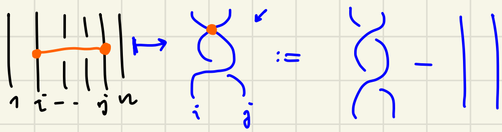
$$\mathcal{R}: \mathcal{A}^h \twoheadrightarrow \underline{gr_I \mathbb{Z}[PB_n]} = \bigoplus I^k/I^{k+1}$$

where

$$\mathcal{A}^h := \frac{\mathbb{Z}\text{-algebra gen. by } \underline{t_{ij}} \quad 1\le i<j\le n}{[t_{ij}, t_{rs}]=0, \ [t_{ij}, t_{ir}+t_{jr}]=0}$$

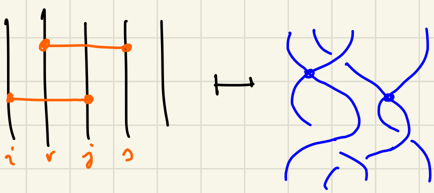
alg. commutator

proof. $\mathcal{R}(t_{ij}) = [\sigma_{ij}] = [\beta_{ij} - 1_n] \in I/I^2$



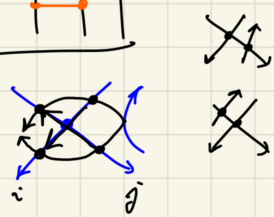
extend this mult. to $\mathbb{Z}$-alg gen. by t_{ij}.

Need to show that $\mathcal{R}$ vanishes on the relations.



✓

$\rightsquigarrow$ $4 \times \underbrace{\text{sing. braid w/ 2 d.p.}}_{= 2\times 2 \text{ braids}} = 16$



□.

Q: $\mathcal{R} : \mathcal{A}^h \twoheadrightarrow gr_I \mathbb{Z}[PB_n]$ Is this also injective? see Thm at the bottom.

Q: What is the geometric meaning of $\mathcal{V}_k$? → Gusarov-Habiro theory

Thm. Pure braid group has dimension property:

$$\beta - 1_n \in I^k = \mathcal{V}_k \overset{\Leftarrow}{\Leftrightarrow} \beta \in \Gamma_k PB_n \quad k\text{-th l.c.s. term.}$$

pf of this uses: free group have dim. property

$PB_n = \underbrace{PB_{n-1}}_{} \ltimes \mathbb{F}_n$ almost direct product

$PB_{n-1} \xrightarrow{0} \mathrm{Aut}\, \underbrace{\mathbb{F}_n^{ab}}_{\mathbb{Z}^n}$

Rem. $\underbrace{\beta \in (1+\mathcal{V}_k) \cap PB_n}_{} \overset{\text{always}}{\supseteq} \Gamma_k PB_n$

$\Rightarrow \beta$ has same Vass. invts up to deg k

"$\beta \sim_{\mathcal{V}_k} 1$"

$\beta \in \Gamma_k PB_n$

"$\beta \sim_k 1$"

k-equivalence of Gusarov and Habiro.

Cor. $gr PB_n = \bigoplus \Gamma_k / \Gamma_{k+1} \hookrightarrow gr_I \mathbb{Z}[PB_n]$

$[\beta] \longmapsto [\beta - 1]$

Thm.

$$\begin{tikzcd}
\underline{t_n := \mathrm{prim}\, \mathcal{A}_n^h} \arrow[r, leftrightarrow] \arrow[d, "\mathcal{R}^t"', "\cong"] & \mathcal{A}^h \arrow[d, "\mathcal{R}", "\cong"'] \\
gr PB_n \arrow[r, leftrightarrow] & gr_I PB_n
\end{tikzcd}$$

Lie alg. over $\mathbb{Z}$ — Hopf algebras over $\mathbb{Z}$

and vertical maps are isomorphisms.

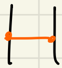
Rem. $\hat{R}^t_{\mathbb{Q}}$ Kohno. needs KZ

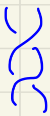
Ihara points out that holds over $\mathbb{Z}$

proof of Thm

Lemma. $$t_n = \frac{\text{Lie alg over } \mathbb{Z} \text{ gen by } t_{ij}}{[t_{ij}, t_{rs}]=0,\ \underbrace{[t_{ij}, t_{ir}+t_{jr}]=0}_{\text{infinitesimal braid rel. (4T)}}}$$ Drinfeld-Kohno Lie.

pf. $\mathcal{U} t_n = \mathcal{A}^n$ □





$$\begin{tikzcd}
t_{ij} \arrow[d, mapsto] & t_n \arrow[r, hook] \arrow[d, Rightarrow, "R^t"] & \mathcal{A}^n \arrow[d] \ni t_{ij} \arrow[dr, mapsto] & \\
\beta_{ij} \arrow[r, mapsto] & \mathrm{gr}PB_n \arrow[r, hook] & \mathrm{gr}_{\pm}\mathbb{Z}[PB_n] & \sigma_{ij} = \beta_{ij} - 1_n
\end{tikzcd}$$

$\beta \mapsto \beta - 1$

Claim R^t is an iso.

pf. Step 1 Show that apply gr to $\mathbb{F}_n \hookrightarrow PB_n \twoheadrightarrow PB_{n-1}$ (use almost direct pr) gives $\mathrm{gr}\mathbb{F}_n \hookrightarrow \mathrm{gr}PB_n \twoheadrightarrow \mathrm{gr}PB_{n-1}$, $\mathrm{gr}\mathbb{F}_n = \mathbb{L}_n$

Step 2 Do induction.

$$\begin{tikzcd}
\left\langle \begin{smallmatrix}\text{Lie ideal} \\ \text{gen. by} \\ t_{in}\end{smallmatrix} \right\rangle \arrow[r, hook] \arrow[d, "\cong"] & t_n \arrow[r, two heads, "\text{forget } t_{in}"] \arrow[d] & t_{n-1} \arrow[d, "\cong \text{ by ind. hyp.}"] \\
\mathbb{L}_n \arrow[r, hook] & \mathrm{gr}PB_n \arrow[r, two heads] & \mathrm{gr}PB_{n-1}
\end{tikzcd}$$

Use infinit. braid reln to show that this is the free Lie subalg. $\mathbb{L}\langle t_{in}\rangle$

□.

def. Universal Vass. invt for PB_n over R

is a map $\quad Z : PB_n \longrightarrow \hat{A}^h \otimes R \cong \prod_{k\geq 0} I^k/I^{k+1} \otimes R$

$\qquad\qquad\qquad \beta \longmapsto Z(\beta)_k$ diagrams of deg k

so that $\quad \bar{Z} : R[PB_n] \longrightarrow \hat{A}^h \otimes R$

filtration preserving R-linear

s.t.

induced map on assoc. graded

$$\begin{tikzcd} gr_{I,R} PB_n \arrow[r, "grZ"] & A^h \otimes R \arrow[l, bend left, "\cong", "\mathcal{R}"'] \end{tikzcd}$$

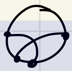
II. Universal invariants for tangles

If $\bar{Z}$ is also algebra map, then Z is a universal additive invt.

Rem Kohno constructed $Z_{\mathbb{Q}}$ additive. (KZ)

Rem It is easy to construct non-add. even over $\mathbb{Z}$.

Thm [Kontsevich, Le-Murakami] $\quad$ isotopy classes of knots



If $\mathbb{Q} \subseteq R$ then $Z_\Phi : \widehat{R[\mathbb{K}]} \xrightarrow{\cong} \hat{\mathcal{A}} \otimes R$

s.t. $\quad R[\mathbb{K}]/V_k \xrightarrow{\cong} \bigoplus_{i=0}^{k} \mathcal{A}(i) \otimes R$

Le-Murakami show Z_Φ does not depend on Φ.

↑ Drinfeld associator.

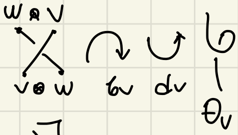
Thm [Kassel-Turaev '98]

· If $\mathbb{Q} \subseteq R$ then $\exists$ filtered isomorphism of categories R-linear

$$Z : \widehat{R[\mathcal{T}]} \xrightarrow{\cong} \widehat{\mathcal{A}^{tang} \otimes R} \text{ “}=\mathcal{A}(R)\text{”}$$

s.t. $\quad R[\mathcal{T}]/\mathcal{T}_{\geq m} \xrightarrow{\cong} \mathcal{A}^{tang} \otimes R/\mathcal{A}_{\geq m}$

def. $\mathcal{T}$ = ribbon cat of framed tangles.



$R[\mathcal{T}] = \begin{cases} \mathrm{ob}_{\ell} = \mathrm{ob}_j \mathcal{T} = + - + + \ldots = s \\ \mathrm{Mor}_{R[\mathcal{T}]}(s,s') = R[\mathrm{Mor}_{\mathcal{T}}(s,s')] \end{cases}$

def J is an ideal in R-linear cat $\mathcal{C}$

if $\forall V,W \in \mathrm{ob}_j \mathcal{C}$ $\quad \underbrace{J(V,W)}_{R\text{-submodule}} \subseteq \underbrace{\mathrm{Hom}_{\mathcal{C}}(V,W)}_{R\text{-module}}$

s.t. "closed" under $-\circ-$ and $-\otimes-$ with mor in $\mathcal{C}$.

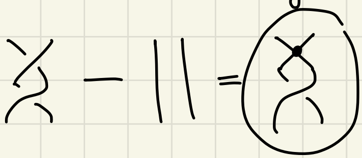
Let J^k be the ideal gen. by $\underbrace{f_1 * f_2 * \cdots * f_\ell}$ s.t. $* = -\circ-$ or $-\otimes-$ and at least k of f_i are in J

def. $J \subset \mathcal{C}$ ideal then $\mathcal{C}/J^k$ again R-linear cat.

def. $\mathcal{C}$ R-linear ribbon cat

$I :=$ ideal in $\mathcal{C}$ gen. by $c_{W,V} \circ c_{V,W} - \mathrm{Id}_{V\otimes W}$

augmentation ideal where $c_{V,W} : V\otimes W \to W \otimes V$ is the braiding.



Lemm. $\mathcal{T}_{\geq k} := \left\{ \begin{array}{l} \text{singular framed} \\ \text{tangles with} \\ \geq k \text{ d.p.} \end{array} \right\} = I^k$ pf as before. □

Recall: $\mathcal{B}$ = univ. braided monoidal cat

$\mathcal{T}$ = univ. ribbon cat.

$\widehat{R[\mathcal{T}]} = \lim R[\mathcal{T}]/I^k$

$= \left\{ \begin{array}{l} \mathcal{C} \text{ R-linear ribbon, } V \in \mathcal{C}, \ \mathcal{J} \subset \mathcal{C} \text{ contains } c_{W,V}\circ c_{V,W} - \mathrm{Id} \\ \text{then} \\ \exists !\ F : \hat{\mathcal{T}}(R) \to \lim_{k\to} \mathcal{C}/\mathcal{J}^k \end{array} \right.$

$+ \longmapsto V$

pf of Thm of Kassel-Turaev

first construct

$$\begin{tikzcd} & I^K \arrow[d, phantom, "\cap"] \\ & R[J] \arrow[dl, dashed, "\cancel{Z_\Phi}"'] \arrow[d, "Z_\Phi"] \\ A(R) \arrow[r, "K"] & A(R)[[h]]^{str}_{\Phi} \end{tikzcd}$$

R-linear ribbon cat.

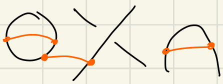
1°

2°

3° $\operatorname{im} Z_\Phi \subset m^K$

filtration by #chords $\cap$

R-linear cat. of chord diagram on tangles.



R-linear symmetric cat. with infinitesimal braiding $t_{V,W}: V\otimes W \circlearrowleft$

$C_{W,V}\circ C_{V,W} = \mathrm{Id}$

"derivative of a braiding"

new ribbon cat

Thm [Le-Murakami] Z does not depend on associator Φ. on links $\emptyset \longrightarrow \emptyset$

1° Universal property of J: $J \longrightarrow A(R)[[h]]^{str}_{\Phi}$ then take R-linear extension

5° Z is filtration preserving

6° iso on $R[J]/I^{K+1} \xrightarrow{Z_{K+1}} A(R)/A(R)_{\geq K+1}$

$$I^K/I^{K+1} \underset{R}{\overset{Z_{K,K+1}}{\rightleftarrows}} A(R)_{=K}$$

show $Z_{K,K+1}$ inverse to R.

$A(R)$ inf. braided.

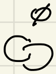
$A(R)[[h]]_\Phi$ non-strict ribbon cat

Rem 4.8 KT Φ & Φ' $\exists$ get equiv. cats.

$A(R)[[h]]^{str}_\Phi$ strict ribbon cat, associator pushed into morphisms.

$$\begin{tikzcd} J \arrow[r] \arrow[d, "Z_\Phi ="'] & A(R)[[h]]^{str}_\Phi \\ A(R) \arrow[ur] & \end{tikzcd}$$

$\emptyset$



$\emptyset$

$\in \mathrm{Mor}_J(\emptyset,\emptyset) \xmapsto{Z_\Phi}$ does not depend on Φ

$$\begin{tikzcd} & & A(R)[[h]]^{str}_\Phi \arrow[d] \\ J \arrow[urr, "Z_\Phi"] \arrow[r, shift left] \arrow[r, shift right] \arrow[drr, "Z_{\Phi'}"'] & A(R) \arrow[ur] \arrow[dr] & \\ & & A(R)[[h]]^{str}_{\Phi'} \end{tikzcd}$$