Moduli spaces of curves and Grothendieck-Teichmüller group

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Let $n \geq 1$. Fix a compact Riemann surface $\Sigma_{0,n}$ of genus zero with n boundary components (this is unique up to diffeomorphism). Denote by $\Gamma_{0,n}$ the (topological) group of self-diffeomorphisms of $\Sigma_{0,n}$ that fix a neighborhood of the boundary pointwise. The map

$$\Gamma_{0,n} \to \pi_0(\Gamma_{0,n})$$

is a weak equivalence of topological groups.
We consider the classifying space $B\Gamma_{0,n}$. This is called the moduli space of genus zero Riemann surface with n boundary components. This space classifies fiber bundles with fiber $\Sigma_{0,n}$.

Moduli space of Riemann surfaces, the operad structure

Given two surfaces Σ and Σ' with respectively m and n boundary components and given a choice b of boundary component of Σ and b' of boundary component of Σ', we can glue Σ and Σ' along b and b' and form a new surface $\Sigma_b \circ_{b'} \Sigma'$ with $m + n - 2$ boundary components.
Now assume $\phi : \Sigma \to \Sigma$ and $\phi' : \Sigma' \to \Sigma'$ are self-diffeomorphisms that fix a neighborhood of the boundary. They can be glued together into a self-diffeomorphism

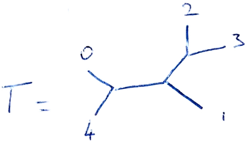
$$\phi_b \circ_{b'} \phi' : \Sigma_b \circ_{b'} \Sigma' \to \Sigma_b \circ_{b'} \Sigma'$$

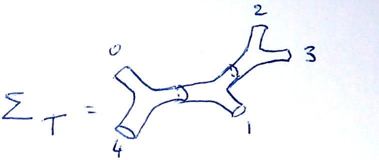
This induces maps

$$B\Gamma_{0,m} \times B\Gamma_{0,n} \to B\Gamma_{0,m+n-2}$$

Moduli space of Riemann surfaces, the operad structure

In order to get an operad, we introduce a variant of this construction. We fix once and for all a genus zero surface P with 3 boundary components (with a $\mathbb{Z}/3$-symmetry that permutes the boundary components). Let T be a planar trivalent graph without loops. We can assign to T a surface Σ_T by gluing together copies of P. The boundary components of Σ_T are in bijection with the legs of T.





Moduli space of Riemann surfaces, the operad structure

We define a groupoid $T\Gamma_{0,n+1}$. The objects of $T\Gamma_{0,n+1}$ are pairs (T,ϕ) where T is an isotopy class of planar trivalent graph without loops and ϕ is a bijection from $\{0,\dots,n\}$ to the set of legs of the tree. A morphism from (T,ϕ) to (T',ϕ') is a diffeomorphism between Σ_T and $\Sigma_{T'}$.
By convention $T\Gamma_{0,2}$ is the groupoid $\mathcal{B}\mathbb{Z}$ (recall that $\mathbb{Z}$ is the mapping class group of the cylinder $S^1\times[0,1]$). The collection of pairs (T,ϕ) as above form a cyclic operad through connected sum of graphs. In fact by gluing diffeomorphisms as above, the collection of groupoids $T\Gamma_{0,n+1}$ form a cyclic operad in groupoids.

Definition

The topological cyclic operad obtained by applying the classifying space construction to this cyclic operad is called the operad of moduli spaces of genus zero Riemann surfaces.

The framed little disks operad : topological point of view

For n a non-negative integer, the space $FD(n)$ is the space of orientation preserving embeddings of n disks of dimension 2 inside one disk. This is an operad in topological spaces in an obvious way.

Proposition

The space $FD(n)$ has the homotopy type of the classifying space of the pure ribbon braid group on n strands $PRB(n) = PB(n) \times \mathbb{Z}^n$.

Proposition

There is an isomorphism $\pi_0 \Gamma_{0,n+1} \to PRB(n)$.

Proposition

There is a weak equivalence of topological operads

$$FD(n) \simeq BT\Gamma_{0,n+1}$$

The main theorem

Let us denote by $X \mapsto \widehat{X}$ the completion of spaces (either profinite, pro-p or pro-unipotent). Our main result is the following.

Theorem (BHR)

The profinite (pro-p or pro-unipotent) Grothendieck-Teichmüller group is the gorup of automorphisms of $\widehat{FD}$ in the homotopy category of operads in profinite (pro-p, pro-unipotent) spaces.

Corollary

In the homotopy category of ∞-Hopf cooperads,

1. *the group of automorphisms of $C^*(FD, \mathbb{Q})$ is $\widehat{GT}_{\mathbb{Q}}$ (Fresse),*
2. *the group of automorphisms of $C^*(FD, \overline{\mathbb{F}}_p)$ is $\widehat{GT}_p$.*

Algebraic completion

Let C be the class of finite (finite p, unipotent) groups. We have an adjunction

$$\widehat{(-)} : \mathsf{Grp} \leftrightarrows \mathsf{Pro}(C) : \mathrm{Mat}$$

The left adjoint is called the pro-finite(p, unipotent) completion of a group. In homotopical terms, for H an element of C, we have a natural isomorphism

$$Map(BG, BH) \simeq Map(B\widehat{G}, BH) := \mathrm{colim}_i Map(BG_i, BH)$$

Homotopical completion

Let C be the class of π-finite (π-p-finite, π-unipotent) spaces.

Definition

A space is called

1. *π-finite if it has finitely many connected components and each component is truncated with finite homotopy groups.*
2. *π-p-finite if it is π-finite and all homotopy groups are p-groups.*
3. *π-unipotent if it has finitely many connected components and each component is truncated, nilpotent, with unipotent homotopy groups.*

We have an adjunction of ∞-categories

$$\widehat{(-)} : \mathsf{S} \leftrightarrows \mathsf{Pro}(C) : \mathrm{Mat}$$

the left adjoint is called the pro-finite(p, unipotent) completion.

Homotopical completion

Theorem

Let X be a space that has finitely many connected components that are nilpotent of finite type. Then the unit of the adjunction

$$X \to \mathrm{Mat}(\widehat{X})$$

1. *is a $\mathbb{F}_p$-homology localization in the π-p-finite case.*
2. *is a $\mathbb{Q}$-homology localization in the π-unipotent case.*

Theorem (Sullivan, Mandell)

Let K denote $\overline{\mathbb{F}}_p$ or $\mathbb{Q}$ depending on context. The functor

$$X \mapsto C^*(X, K)$$

is a fully faithful functor from $\mathrm{Pro}(C)^{op}$ *to* Alg_{E_∞} .

Homotopical completion of operads

The category Ω of trees (Moerdijk-Weiss).

Proposition

An operad in a category C with products is a functor

$$X : \Omega^{op} \to C$$

such that for any tree T, the map

$$X(T) \to \prod_{C_n \subset T} X(C_n)$$

is an isomorphism.

Concretely, the object $X(C_n)$ contains the arity n-operations of our operad. The rest of the functor encodes all the operadic structure.

Homotopical completion of operads

Proposition

An ∞-operad in an ∞-category C with products is a functor

$$X : \Omega^{op} \to \mathsf{C}$$

such that for any tree T, the map

$$X(T) \to \prod_{C_n \subset T} X(C_n)$$

is an equivalence. We denote by $\mathsf{Op}_\infty(\mathsf{C})$ the ∞-category of ∞-operads in C.

Completion of an operad in spaces is by definition the left adjoint of the functor

$$\mathrm{Mat} : \mathsf{Op}_\infty(\mathsf{Pro}(C)) \to \mathsf{Op}_\infty(\mathsf{S})$$

(this is often given by applying completion objectwise).

Operadic Sullivan-Mandell theorem

Let P be an operad in spaces that is arity-wise connected. Let $\widehat{P}$ be its pro-p (pro-unipotent) completion. Let K denote $\overline{\mathbb{F}}_p$ ($\mathbb{Q}$).

Theorem

The map $P \to \widehat{P}$ induces an equivalence

$$C^*(\widehat{P}, K) \to C^*(P, K)$$

in the ∞-category of ∞-operads in $\mathrm{Alg}_{E_\infty}^{op}$.

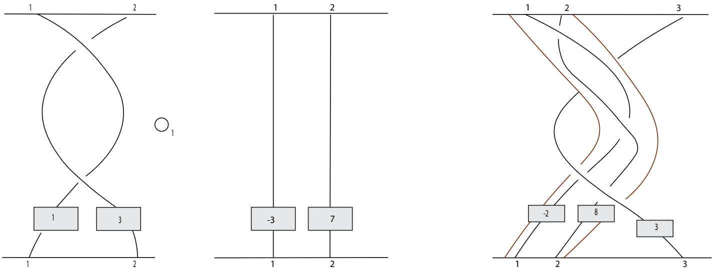
Definition

The category of ∞-operads in $\mathrm{Alg}_{E_\infty}^{op}$ is by definition the ∞-category of ∞-Hopf cooperads.

Conjecturally this is equivalent to Fresse's homotopy theory of strict Hopf cooperads in the rational case.

The framed little disks operad : categorical point of view

We can describe the operad in groupoids $T\Gamma_{0,\bullet+1}$ in a slighlty different way. The objects of $T\Gamma_{0,n+1}$ are well parenthesized words in the alphabet $\{1,\ldots,n\}$ with each letter appearing exactly once. A morphism between two such words is a ribbon braid connecting matching letters. The operad structure is via cabling.



The framed little disks operad : categorical point of view

Theorem (Joyal, Street)

An algebra over this operad in the category of categories is a balanced monoidal category (without unit).

Definition

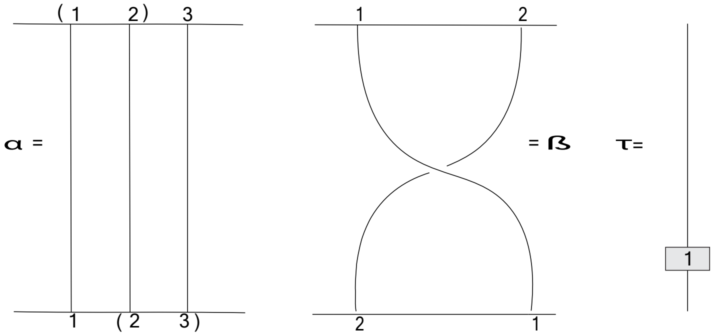
A balanced monoidal category is a braided monoidal category $(C, \otimes)$ equipped with the additional data of a natural transformation $\tau : id_C \to id_C$ such that for any two objects x and y, we have

$$\tau_{x\otimes y} = \beta_{y,x} \circ (\tau_y \otimes \tau_x) \circ \beta_{x,y}$$

A presentation of the operad of moduli spaces

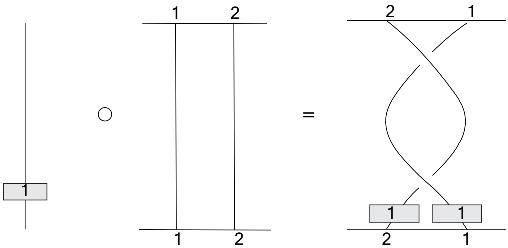
Proposition (BHR)

Let P be an operad in groupoids. Operad maps $T\Gamma_{0,\bullet+1} \to P$ are in one-to-one correspondance with quadruple (m, τ, β, α) satisfying the pentagon and two hexagon relation plus an additional relation.



A presentation of the operad of moduli spaces

The additional relation is :



As a consequence, we get

Proposition

The group GT is the group of automorphisms of the operad $T\Gamma_{0,\bullet+1}$ inducing the identity on object.

The group GT is defined by

Definition

The group GT is the group of automorphisms of the operad PaB (the analogue of $T\Gamma_{0,\bullet+1}$ with ribbon braids replaced by braids) inducing the identity on object.

With this definition, we have $GT = \mathbb{Z}/2$ (Drinfel'd) but we can replace $T\Gamma_{0,\bullet+1}$ by its completion (profinite, pro-p or pro-unipotent). In that case the proposition above remains true with GT replaced by its profinite, pro-p or pro-unipotent version.

Other ingredients in the proof

Definition

A group is called good if the map $BG \to B\widehat{G}$ exhibits $B\widehat{G}$ as the homotopical completion of BG.

Proposition

The pure ribbon braid groups are good.

Proposition

The map $\widehat{GT} \to \mathrm{Aut}(\widehat{T\Gamma}_{0,\bullet+1})$ induces an isomorphism

$$\widehat{GT} \cong \mathrm{Aut}(\widehat{T\Gamma}_{0,\bullet+1})/htpy$$

Proposition

The operad $T\Gamma_{0,\bullet+1}$ is cofibrant as an operad in groupoids.

Consequence of the theorem

Proposition

There is a non-trivial action of $\widehat{GT}$ on the operad of compactified moduli spaces $\overline{\mathcal{M}}_{0,\bullet+1}^{\wedge}$.

This is a consequence of our main theorem and the following theorem

Theorem (Drummond-Cole)

The operad $\overline{\mathcal{M}}_{0,\bullet+1}$ is obtained from FD by "killing" the space of arity one operations.

Theorem (BHR, Cirici-H.)

The dg-operad $C_*(FD, \mathbb{Q})$ is formal. The dg-operad $C_*(FD, \mathbb{F}_p)$ is $(p-2)$-formal.

Extensions of the main result and further questions

Theorem (BHR)

The theorem remains true if we replace FD by FD_+ (the unital version).

Conjecture

The theorem remains true for the ***cyclic*** *operad $BT\Gamma_{0,\bullet}$.*

We have injections $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{GT}_{prof}$ and $\mathrm{Gal}_{MTM} \to \widehat{GT}_{\mathbb{Q}}$.

Theorem (Vaintrob)

There is an algebro-geometric explanation for this action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ or Gal_{MTM} on the completions of FD.

Question

What are the automorphism of the completion of the ***modular*** *operad $BT\Gamma_{\bullet,\bullet}$?*