Le complexe de graphes & grt, d'après Willwacher

Goal: **Part I**: describe $\mathrm{Der}^h(e_2)$ via graph complexes

$\uparrow$ $H_*(\mathbb{E}_2)$ = Gerstenhaber operad

Part II: relate $H_0\mathrm{Der}^h(e_2)$ to $\mathfrak{grt}_1$.

Recollections and conventions

1) P dg-operad $\rightsquigarrow$ dgla of derivations

$$\begin{tikzcd} \mathrm{Der}^+(P) \arrow[d, "\in \mathrm{ho}(\mathrm{dgla's})"] & = & \mathrm{Der}(P) & \ltimes & P(1)[1] \arrow[ll, bend right, "ad"'] \\ \mathrm{Der}^h(P) & = & \mathrm{Der}^+(P^{cof}) & & \end{tikzcd}$$

dgla of homotopy der^s

If $P = \Omega(\mathcal{C})$, then:

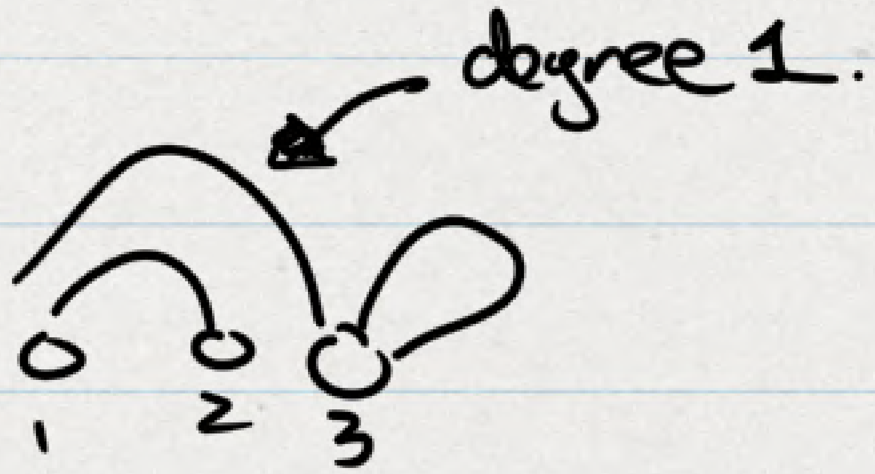
$$\mathrm{Der}^{n}(P) = \mathrm{Der}^{+}(P) = \left(\prod_{n \geq 1} \mathrm{Hom}(\mathcal{C}(n)[-1], P(n))^{\Sigma_n}, d \right)$$

$$= \mathrm{Def}(\Omega\mathcal{C} \to P)[1]$$

⚠ Two different Lie brackets, $[-,-]$ for bracket from deformation complex.

2) Graph complexes

Gra operad



full graph complex $\mathrm{Def}(L_\infty\{-1\}, \mathrm{Gra}) \cong \mathrm{TwGra}(0)$

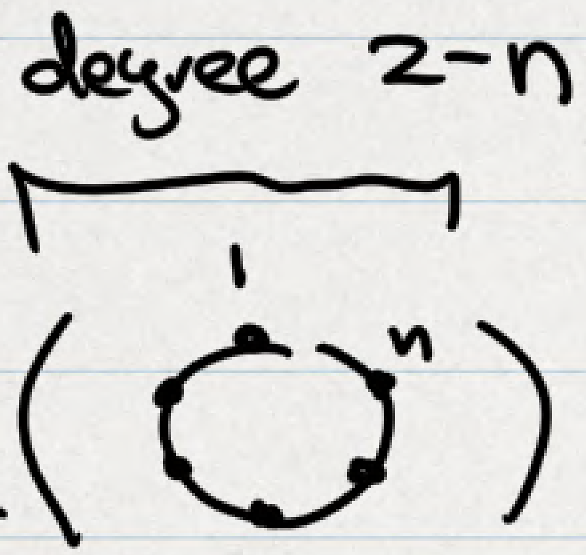
$\rotatebox{90}{$\subseteq$}$

connected graphs fGC_{cn}

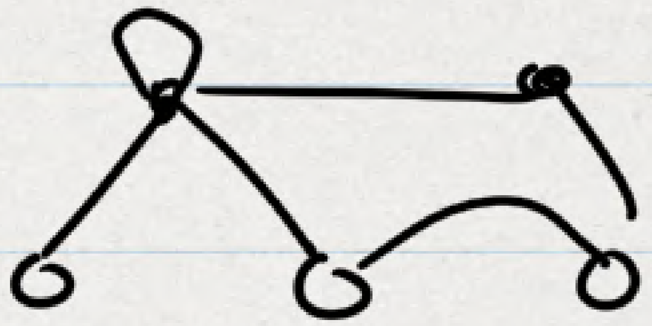
$\rotatebox{90}{$\subseteq$}$

valence ≥ 3. GC

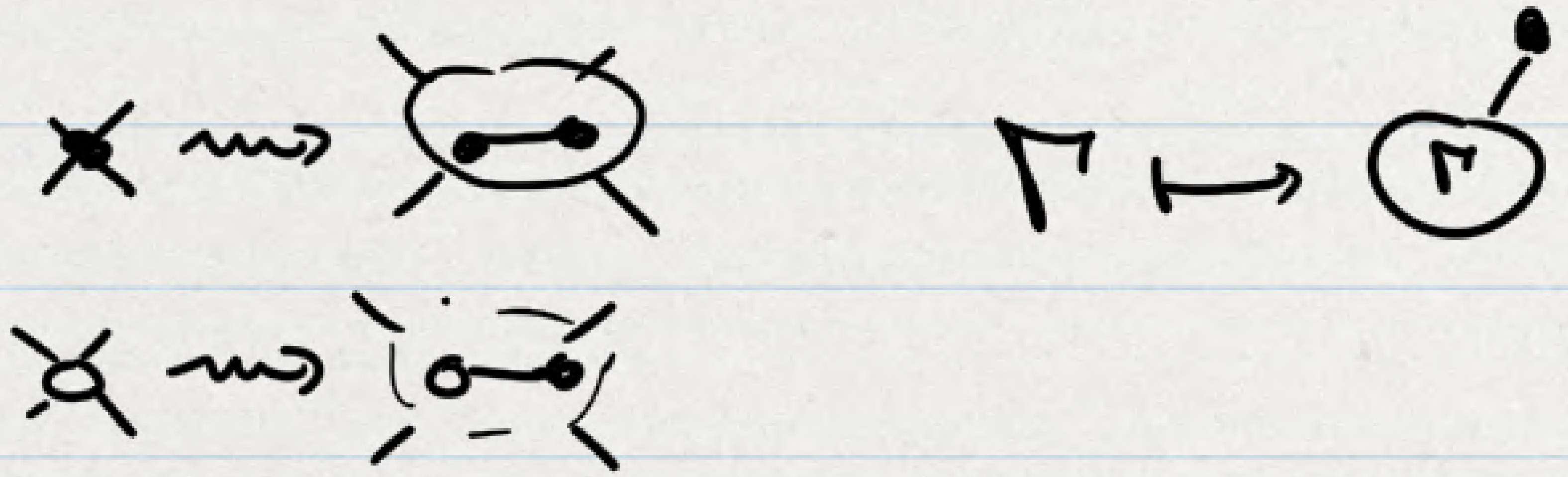
no tadpoles !

Prop (Kontsevich). $H_*(\mathrm{fGC}_{cn}) \cong H_*(\mathrm{GC}) \oplus \bigoplus_{n=1 \bmod 4} \mathbb{Q}\Big($  $\Big)$

3) TwGra



differential = local − global

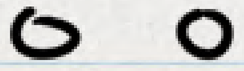


$$\begin{tikzcd} \mathrm{TwGra} \\ \mathrm{TwGra}_c \arrow[u, hook] \end{tikzcd}$$

TwGra_c graphs s.t everything connected to o.

<u>Prop</u> : $e_2 \xrightarrow{\sim} \mathrm{TwGra}_c$

(Kontsevich). $\lambda_2 \longmapsto$ 

$\mu_2 \longmapsto$ 

Have: $\mathrm{ho}e_2 = \Omega(e_2^\vee\{-2\}) \xrightarrow{\sim} e_2 \xrightarrow{\sim} \mathrm{TwGra}_c$

By twisting procedure

$$\begin{tikzcd}
 & & \mathrm{Der}^+(\mathrm{ho}e_2) = \mathrm{Der}^h(e_2) \arrow[d, "\sim"] \\
\mathrm{fGC}_{cn} \arrow[r] \arrow[urr, bend left, "\bar{\Phi}"] \arrow[rr, bend right, "\Psi"'] & \mathrm{Der}^+(\mathrm{TwGra}_c) \arrow[r] \arrow[ur, dotted, "\exists \in \mathrm{ho}(\mathrm{dgLa})"] & \mathrm{Def}(\mathrm{ho}e_2 \to \mathrm{TwGra}_c)[1]
\end{tikzcd}$$

Thm (Willwacher). The map in ho(dgLa) $\bar{\Phi}$ is given at the level of homology by the linear map

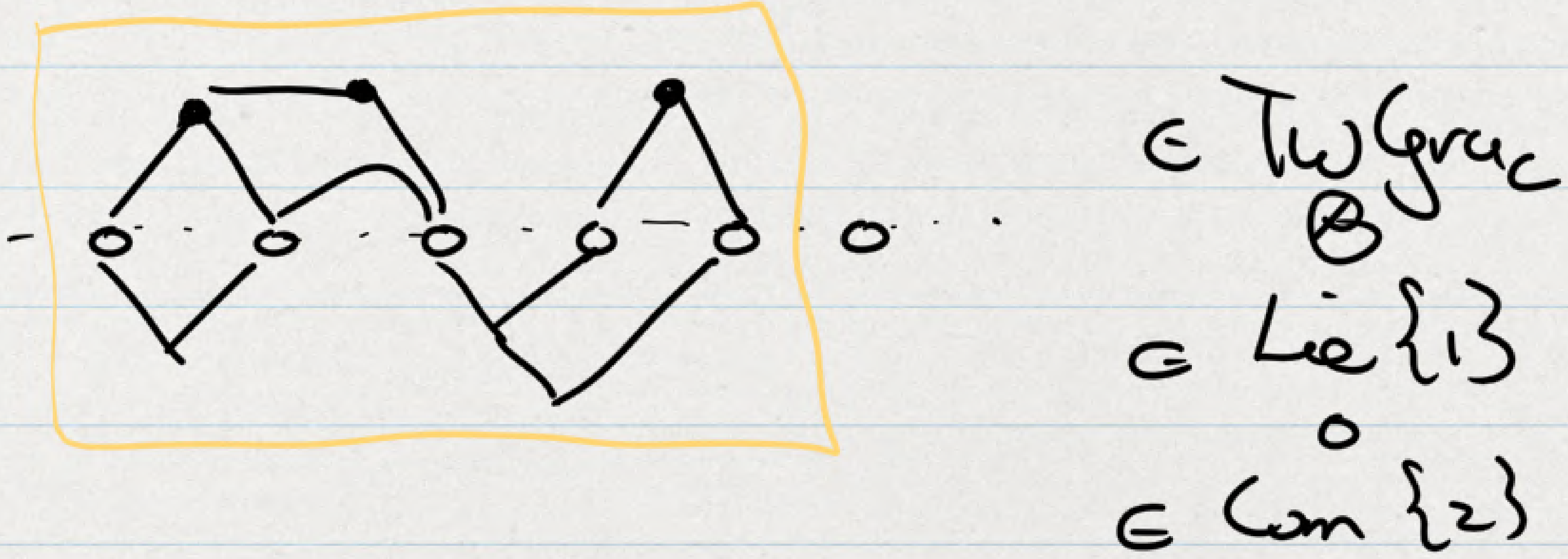
$$H_*(\mathrm{fGC}_{cn}) \hookrightarrow \bar{S}\left((H_*(\mathrm{fGC}_{cn}) \oplus \mathbb{K})[-3] \right)[3]$$

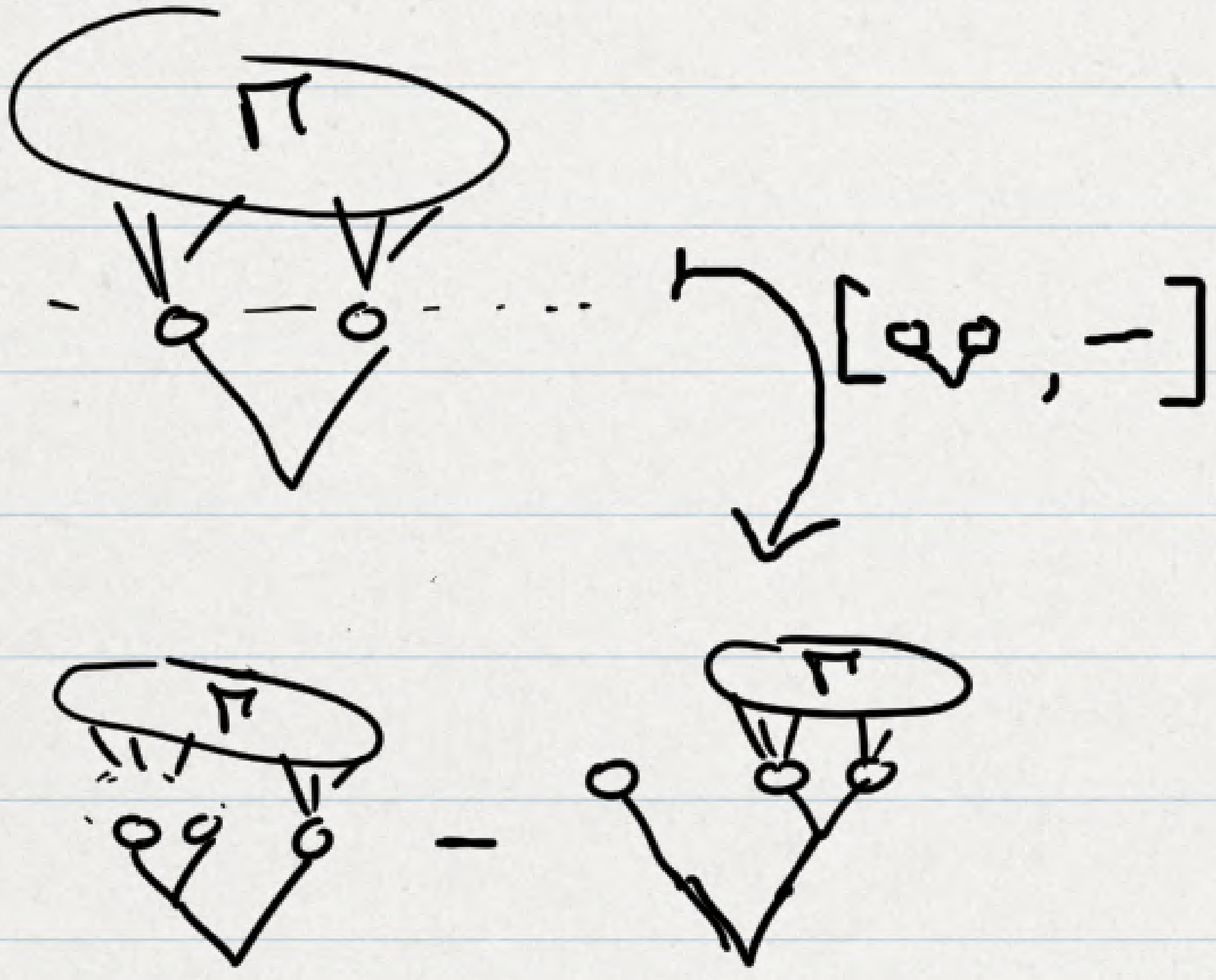
In part: to prove this, just have to understand $\Psi : \mathrm{fGC}_{cn} \to \mathrm{Def}(\mathrm{ho}e_2 \to \mathrm{TwGra})$

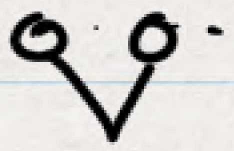
Description of $\mathrm{Def}(\mathrm{hoe}_2 \to \mathrm{TwGra}_c)$:

given by $\prod_{n\geq 1} \left(e_2\{2\} \otimes \mathrm{TwGra}_c\right)(n)^{\Sigma_n}$

elements



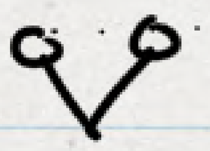


differential: $d = d_{\mathrm{TwGra}_c} + [$  $, -] + [$  $, -]$

Def: $\mathrm{Def}(\mathrm{hoe}_2 \to \mathrm{TwGra}_c)_{cn}$ subcomplex of pictures that are totally connected.

Lem: $\mathrm{Def}(\mathrm{hoe}_2 \to \mathrm{TwGra}_c) \cong \bar{S}\left(\mathrm{Def}(\mathrm{hoe}_2 \to \mathrm{TwGra}_c)_{cn}[-2]\right)[2]$.

In these terms, the map $\Psi : \mathrm{fGC}_{cn}[-1] \longrightarrow \mathrm{Def}(\mathrm{hoe}_2 \to \mathrm{TwGra}_c)_{cn}$



$$\Gamma \longmapsto \Big[\ \cdot\ ,\ \underset{\downarrow}{\Gamma_{\bullet\to\circ}}\Big] + \Big[\ \cdot\ ,\ \Gamma_{\bullet\to\circ}\Big].$$

(sum of)

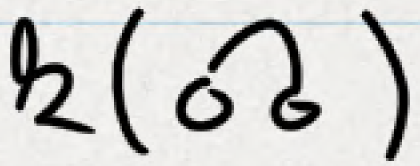
Γ, where replace one $\bullet\!\longrightarrow\!\circ$.

<u>Outline of proof:</u>

$$\begin{tikzcd}
\mathrm{fGC}_{cn}[-1] \arrow[r, "\Psi"] & \mathrm{Def}(\mathrm{hoe}_2 \to \mathrm{TwGra}_c)_{cn} \arrow[r] \arrow[d] \arrow[dr, phantom, "\textcircled{C}"] & \mathrm{Def}(\mathrm{hoe}_2 \to \mathrm{Gra})_{cn} \arrow[d] & = \textcircled{B} \\
\textcircled{D} \arrow[u, bend right] & \mathrm{Def}(L_\infty\{-1\} \to \mathrm{TwGra}_c)_{cn} \arrow[r] & \mathrm{Def}(L_\infty\{-1\} \to \mathrm{Gra})_{cn} = \mathrm{fGC}_{cn}.
\end{tikzcd}$$

$\textcircled{A} =$

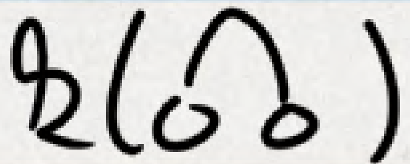
Claims: Ⓐ is acyclic

Ⓑ $\simeq \Bbbk($  $)$ 1-dim

Ⓒ homotopy cartesian

Ⓓ Ψ "connecting map".

$\Longrightarrow$ LES on homology groups splits into SES

$0 \longrightarrow H_*(fGC) \xrightarrow{\Psi} H_{*-1}\mathrm{Def}(hoe_2 \to \mathrm{TwGra}_c)_{cn} \longrightarrow \Bbbk($  $) \longrightarrow$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \subset * = 0.$

Lemma $\Longrightarrow$ $H_*\mathrm{Der}^h(e_2) \underset{\uparrow}{\cong} H_*\mathrm{Def}_-(hoe_2 \to \mathrm{TwGra}_c)[1] \cong \bar{S}((H_*(fGC_{cn}) \oplus \Bbbk)[-3])[3]$

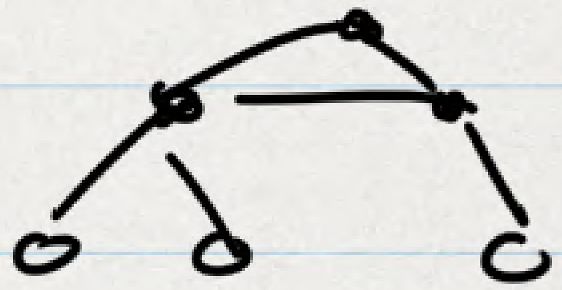
$e_2 \simeq \mathrm{TwGra}_c$

About the claims

General method: put some decreasing on graphs s.t on associated graded, d simplifies.

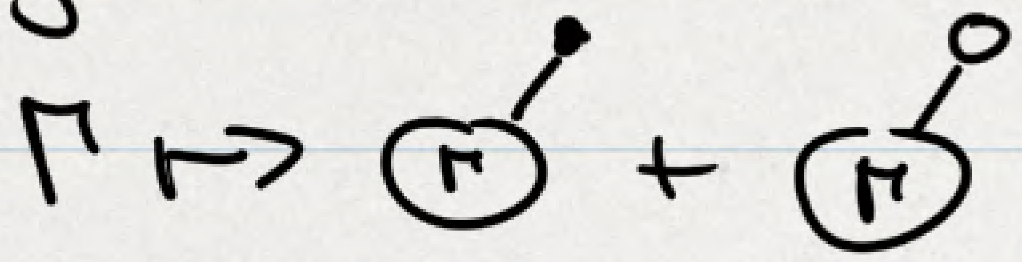
Claim Ⓐ: $\mathrm{Def}(L_\infty\{-1\} \longrightarrow \mathrm{Tw}(\mathrm{gra}_c)_{cn} \simeq 0$.

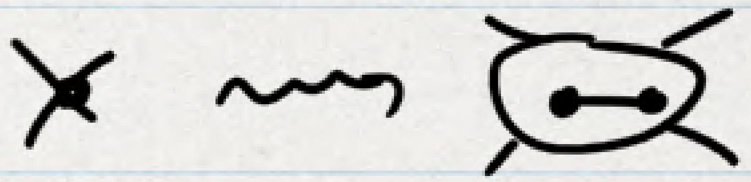
Pf: **Elements:**

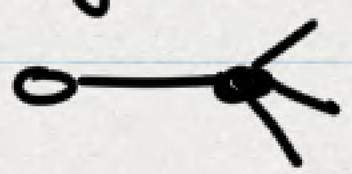


with at least one $\circ$.

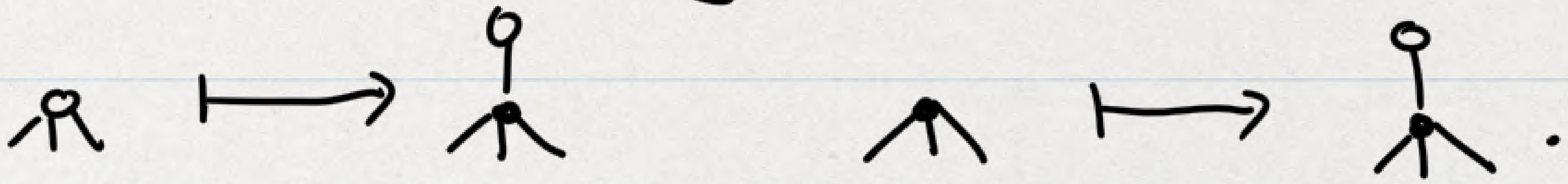
Differential: Local — global

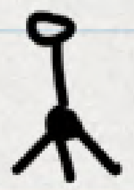




Can filter this by # { edges not of the form  }

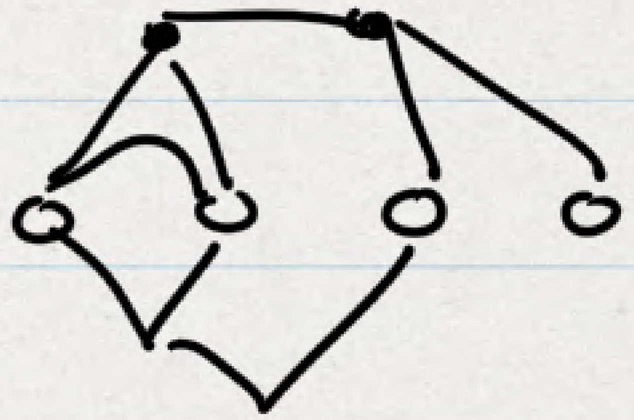
⇝ on associated graded, the differential is given by:



On assoc. graded, $h(\Gamma) =$ { contract  to  }.

$$\rightsquigarrow dh + hd(\Gamma) = \Gamma_{\bullet \to \circ} + (\# \circ) \cdot \Gamma$$

$$\rightsquigarrow \underset{\text{nilpotent}}{(-)_{\bullet \to \circ}} \underset{\text{chain homotopy.}}{\simeq} \underset{\text{not zero}}{\#} \cdot \mathrm{id}. \qquad \Longrightarrow \text{associated graded acyclic.}$$

Claim Ⓑ: Say that  $\in \mathrm{Def}(\mathrm{hoe}_2 \to \mathrm{TwGra}_c)_{cn}$

is simple if a) no Lie brackets in bottom half.

b)  valence 1.

Then: • $\mathrm{Def}(\mathrm{hoe}_2 \to \mathrm{TwGra}_c)_{cn,\,simple} \overset{\sim}{\hookrightarrow} \mathrm{Def}(\mathrm{hoe}_2 \to \mathrm{TwGra}_c)_{cn}$

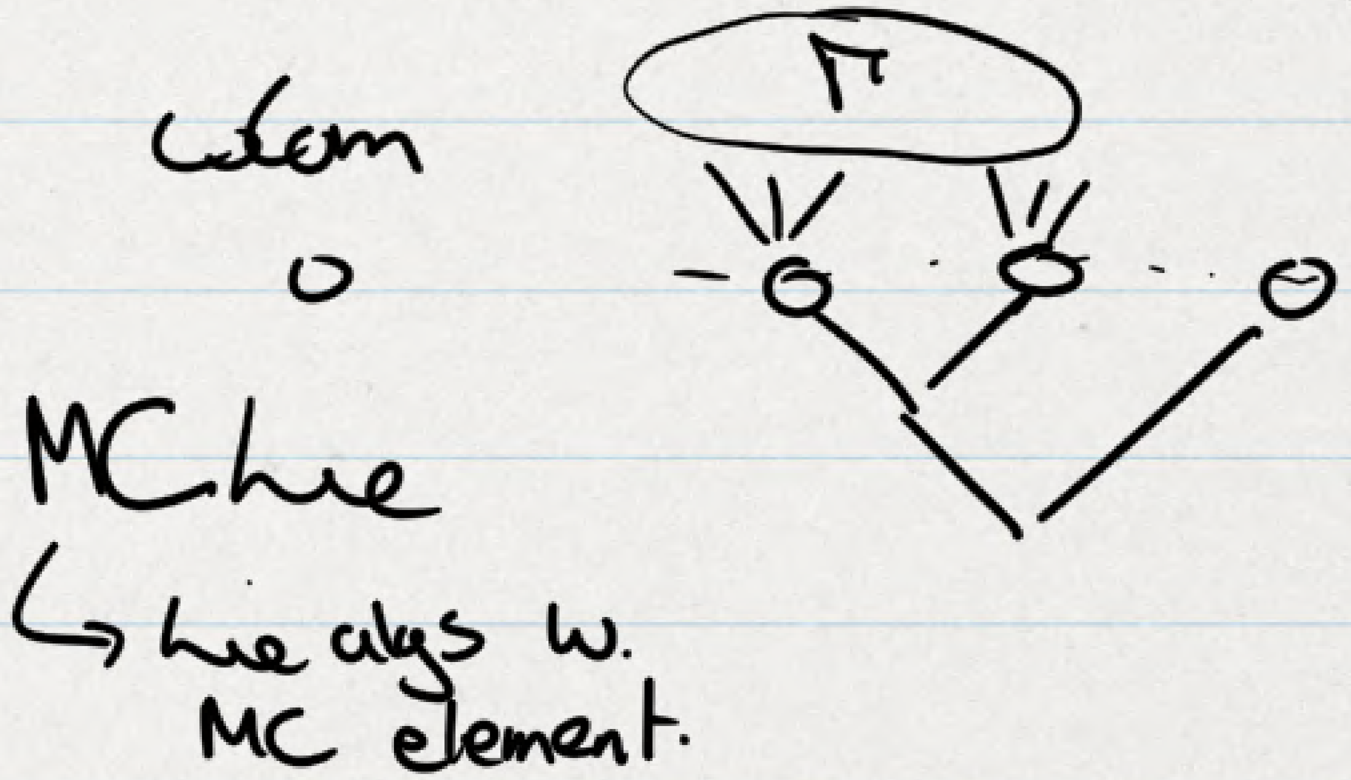
• $\mathrm{Def}(\mathrm{hoe}_2 \to \mathrm{Gra})_{cn,\,simple} \overset{\sim}{\hookrightarrow} \mathrm{Def}(\mathrm{hoe}_2 \to \mathrm{Gra})_{cn}$

$\shortparallel$

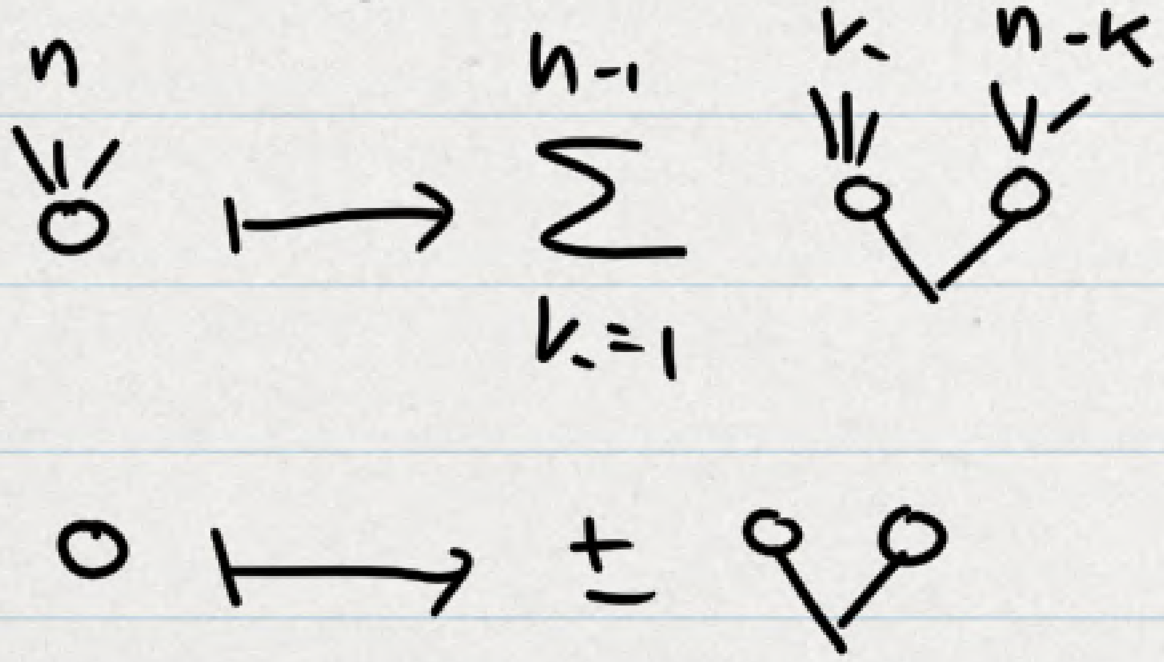
$\frac{1}{2}($  $)$.

<u>Proof of this</u>: Filter s.t. on associated graded

$d = [$  $, -].$ This only concerns bottom half:



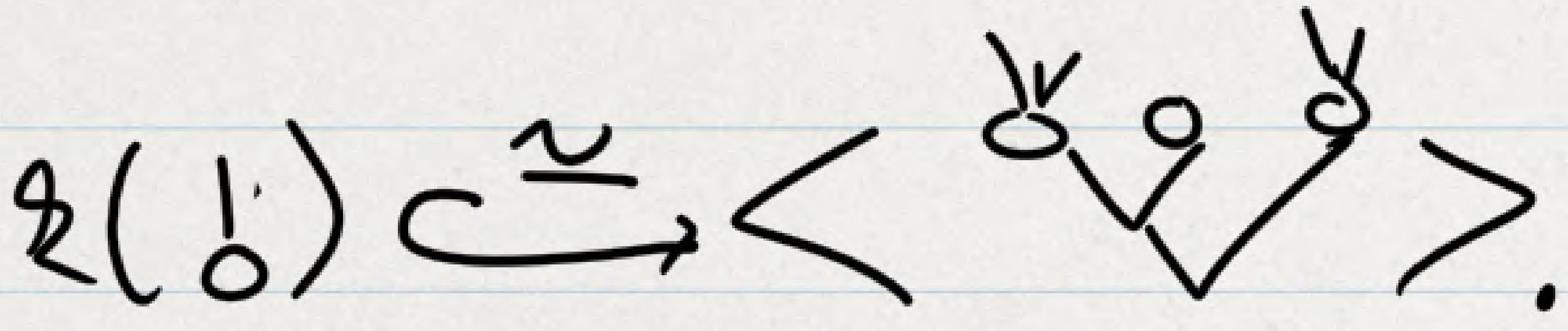
<u>differential</u>:



In fact, this is exactly the Koszul complex for

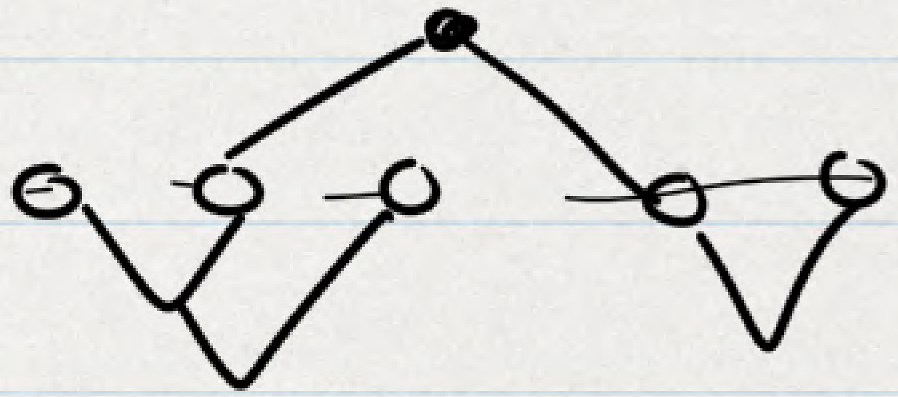
$$\mathrm{coCom}\{1\} \longrightarrow \mathrm{Lie}\{1\} \xrightarrow{\sim} \mathrm{MCLie}\{1\}$$

⇒ Koszul complex is acyclic:



Part II: $H_0 \mathrm{Der}^h(e_2)$ and grt_1

Goal: Identify $H_0 \mathrm{Der}^h(e_2)_{cn} = H_{-1}\mathrm{Def}(hoe_2 \to \mathrm{TwGra}_c)_{cn}$.

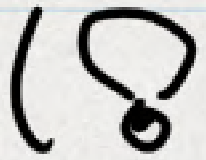
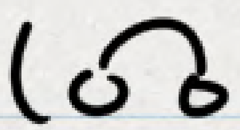
Method: filter  by (# conn comp below) − 1.

$\rightsquigarrow$ spectral sequence $H_j(\mathrm{gr}_p \mathrm{Def}(hoe_2 \to \mathrm{TwGra}_c)_{cn}) \Rightarrow H_{j+1}\mathrm{Der}^h(e_2)$

Note:
- $\mathrm{gr}\,\mathrm{Def}(\dots)_{cn} \cong \mathrm{Def}(hoe_2 \xrightarrow{\pi} C_\infty \longrightarrow \mathrm{TwGra}_c)_{cn}$
- $d = d_{\mathrm{TwGra}_c} + [$  $, -]$
- $\mathrm{gr}_0 \mathrm{Def}(\dots) \cong \mathrm{Def}(C_\infty \to \mathrm{TwGra}_c)$

Main calculation:

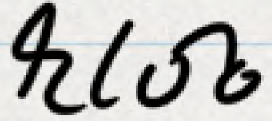
$H_j \, gr_p \, Def(\ldots) =$

- 0 : $j \geq 1$ or $j > 1-2p$
- $H_j Def(C_\omega \to TwGra_2)$: $p = 0$
- $k($$)$: $p = 0, j = 0$
- $k($$)$: $j = -1, p = 1$
- 0 : $j = -2, p = 1$.

Cor: $H_j(Der^h(e_2)_{cn}) = 0 \quad j \geq 2$

$H_1 Der^h(e_2) = k($$)$

Rmk: SES splits naturally:



$$\begin{tikzcd} & & k \arrow[d, phantom, "\wr"] \\ k(\,) \arrow[r] \arrow[rr, bend right, "iso"'] & H_0 Der^h(e_2) \arrow[r] & End(H_1(e_2)(2)) \end{tikzcd}$$

There is a SES $k($$) \longrightarrow H_0 Der^h(e_2) \longrightarrow H_{-1} Def(C_\omega \to TwGra_2)$.

Relation to grt_1

Recall: $\mathrm{GRT}_1 = e^{\mathrm{grt}_1} \curvearrowright \mathrm{PaCD}$ fixing $\overset{1\ \ 2}{\underset{1\ \ 2}{H}}{}^{t_{12}}$.

$\rightsquigarrow \mathrm{GRT}_1 \curvearrowright C_*(\mathrm{PaCD}) \xleftarrow{\sim} \xrightarrow{\sim} e_2$

differentiate

$$\begin{tikzcd} \rightsquigarrow \mathrm{grt}_1 \arrow[r] \arrow[rr, bend right, "0"'] & \mathrm{Der}^h(e_2) \arrow[r] & \mathrm{End}(\underbrace{H_1(e_2(2))}_{k\cdot\lambda_2}) \end{tikzcd}$$

$\boxed{\underline{\text{Thm}}}$ (Willwacher + Tamarkin, Kontsevich) $\mathrm{grt}_1 \longrightarrow \mathrm{Der}^h(e_2) \xrightarrow{\sim} H_{-1}\mathrm{Def}(C_\infty \to \mathrm{TwGra}_c)$

is an isomorphism

<u>Cor</u>: There are SES of Lie algebras

$$\mathfrak{grt}_1 \longrightarrow H_0\mathrm{Der}^h(e_2) \xrightarrow[\lambda_2]{\text{act on}} \mathrm{End}(H_1(e_2(2)))$$

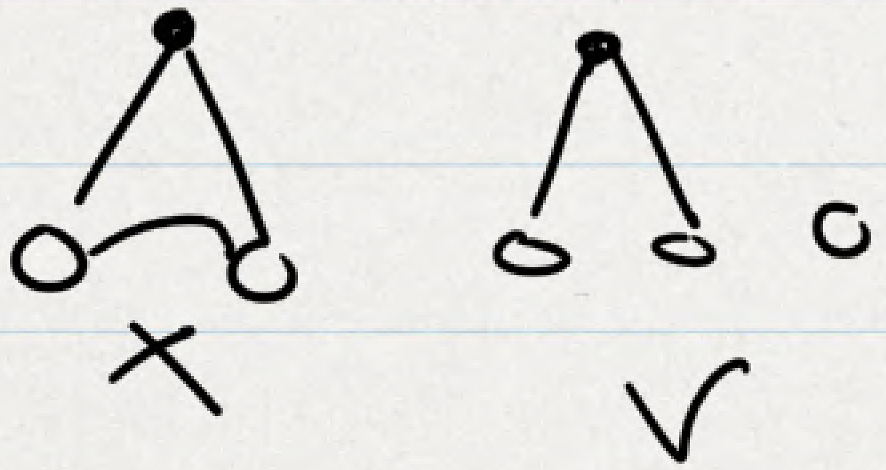
$$H_0(GC) \cong H_0(fGC_{cn}) \longrightarrow H_0\mathrm{Der}^h(e_2) \longrightarrow \mathrm{End}(H_1(e_2(2))). \qquad \text{(Part I)}$$

$\Longrightarrow$ $\mathfrak{grt}_1 \cong H_0(GC)$ as Lie algebras.

Ingredients of Main calculation and Thm.

① → TwGra_c operad in $\overset{\text{dg-}}{\text{coalgebras}}$, primitives ICG

"internally connected graphs"



→ In fact,

$$\mathrm{TwGra}_c = S^c(\mathrm{ICG})$$

$\Longrightarrow$ ICG operad in $\left(L_\infty\{1\}\text{-alg}^s, \oplus\right)$.
(with ∞-maps)

② $\mathfrak{g}$ operad in $L_\infty\{1\}$-algs, then ∃ maps of dg operads

$$hoe_2 \xrightarrow{\pi} C_\infty \longrightarrow Com = C_*(0) \longrightarrow C_*(\mathfrak{g})$$

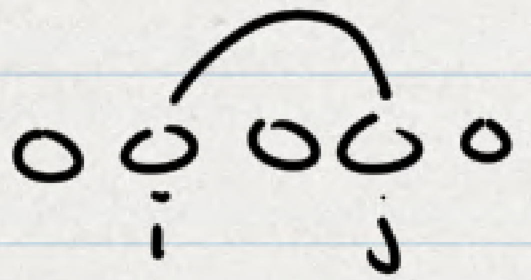
Subcomplex

$$\begin{tikzcd} C_*(ICG) \arrow[d, equal] & \text{“}Def(hoe_2 \to \mathfrak{g})\text{”} \arrow[r, hook] \arrow[d, phantom, sloped, "\cong"] & Def(hoe_2 \to C_*(\mathfrak{g})) \arrow[d, phantom, sloped, "\cong"] \\ TwGra_c & \prod (e_2\{2\} \otimes \mathfrak{g})(n)^{\Sigma_n} \arrow[r, hook] & \prod (e_2\{2\} \otimes C_*(\mathfrak{g}))(n)^{\Sigma_n} \end{tikzcd}$$

Prop: $Def(hoe_2 \longrightarrow ICG)_{cn} \xhookrightarrow{\sim} \underbrace{Def(hoe_2 \to C_\infty \longrightarrow TwGra_c)_{cn}}_{\mathbb{Z}\text{ graded we want to understand.}}$

③ **Thm** (Ševera-Willwacher). There is a zig-zag of ∞-quasi-iso

$$\mathrm{ICG} \leftarrow \tau_{\geq 1}\mathrm{ICG} \longrightarrow H_1(\mathrm{ICG})[1] \cong \mathfrak{t}[1]$$



$\mapsfrom t_{ij}$

where $\mathfrak{t}(n) = \dfrac{\widehat{\mathrm{Lie}}(t_{ij})}{(\ \cdot - \ )}.$

Drinfeld-Kohno operad.

Cor: $H_* \, \mathrm{gr}\, \mathrm{Def}(\mathrm{hoe}_2 \to \mathrm{TwGra}_c) \cong H_*(\mathrm{hoe}_2 \to \mathfrak{t}[1])$

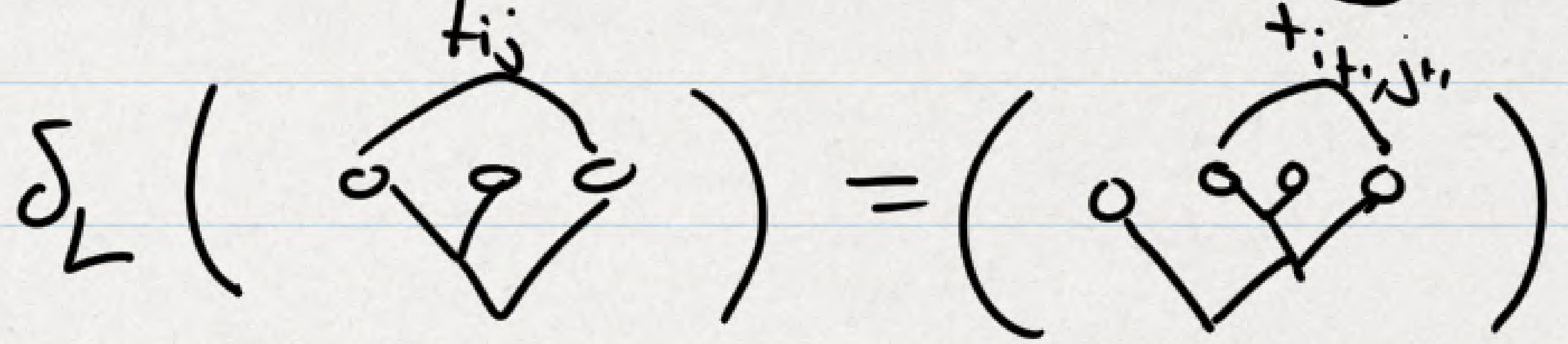
$\uparrow$ 0 in lots of degrees

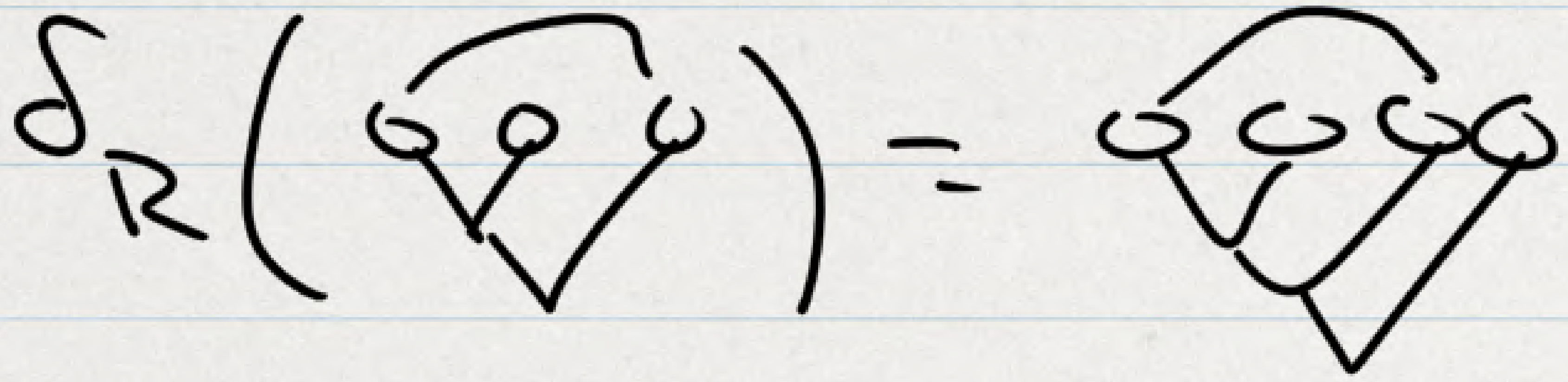
Final computation $H_{-1}\mathrm{Def}(C_\infty \to \mathrm{ImGra}_c) \cong H_{-1}\mathrm{Def}(C_\infty \to \mathfrak{t}[1])$.

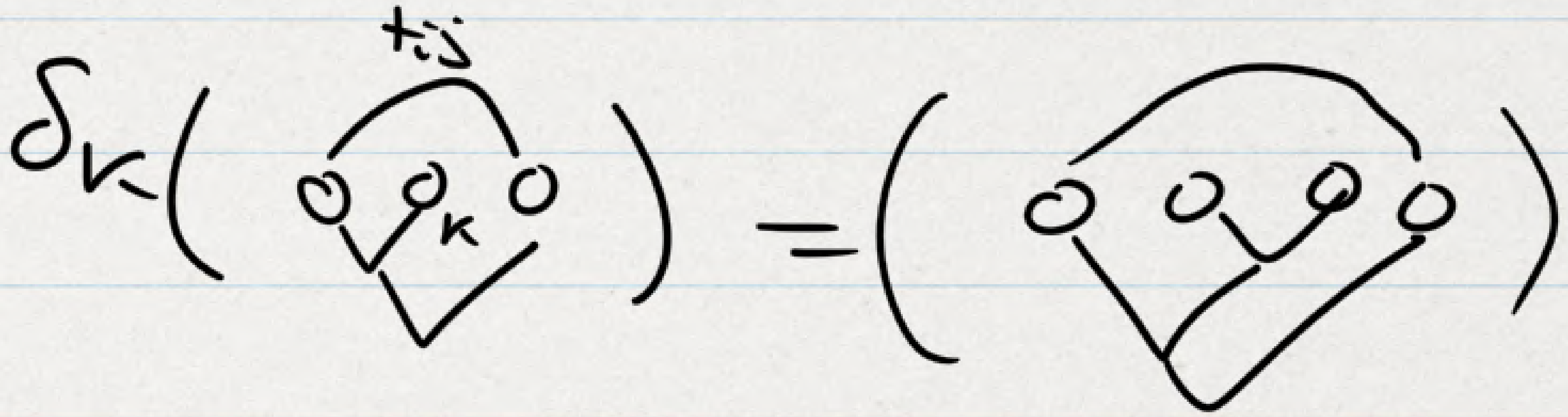
Complex:

$$\begin{array}{ccccc}
\overset{0}{(\mathrm{Lie}\{1\}(2)\otimes \mathfrak{t}_2)^{\Sigma_2}} & \xrightarrow{\delta=0} & \overset{-1}{(\mathrm{Lie}\{1\}(3)\otimes \mathfrak{t}_3)^{\Sigma_3}} & \xrightarrow{\delta} & (\mathrm{Lie}\{1\}(4)\otimes \mathfrak{t}_4)^{\Sigma_4} \\
\| & & \cap & & \cap \\
\mathbb{k}(\overset{t_{12}}{\circ\,\circ}) = \mathfrak{t}_2 & & \mathfrak{t}_3 & & \mathfrak{t}_4
\end{array}$$

Each δ sum of Lie algebra maps: $\delta = \delta_L + \delta_R + \sum_k (-1)^k \delta_k$







$\delta_k(t_{kj}) = t_{k,j+1} + t_{k+1,j+1}$

Compute $H_{-1} = Z_{-1} \subseteq \mathfrak{t}_3 = \widehat{\mathfrak{Lie}}(t_{12}, t_{23}) \oplus \Bbbk(t_{12}+t_{23}+t_{31})$.

Consists of elements $F(t_{12}, t_{23})$ satisfying:

symmetry: · $F(t_{12}, t_{23}) + F(t_{23}, t_{12}) = 0$.

· $F(t_{12}, t_{23}) + F(t_{23}, -t_{12}-t_{23}) + F(-t_{12}-t_{23}, t_{12}) = 0$

cocycle: $F(t_{12}, t_{23}) + F(t_{23}, t_{34}) + F(t_{12}+t_{13}, t_{24}+t_{34})$

$$= F(t_{13}+t_{23}, t_{34}) + F(t_{12}, t_{23}+t_{24})$$

Conclusion: this is exactly $\mathfrak{grt}_1$!