Loday-Quillen-Tsygan theorem

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$\Bbbk$ field of char 0

<u>Thm</u>. [Loday-Quillen, Tsygan, '83]

A unital assoc. alg over $\Bbbk$

Then there is a canonical iso of graded Hopf alg

$$H_*(\mathfrak{gl}(A), \Bbbk) \cong \Lambda^\cdot HC_{*-1}(A)$$

Generalizations :
- to dga [Burghelea '86]
- to H-unital alg [Hanlon, Feigin-Tsygan]

§ Strategy of the proof

1. Koszul's trick : action of reductive subalg $\mathfrak{gl}(\Bbbk) < \mathfrak{gl}(A)$

$$\Lambda^\cdot \mathfrak{gl}(A) \xrightarrow{\sim} \left(\Lambda^\cdot \mathfrak{gl}(A)\right)_{\mathfrak{gl}(\Bbbk)}$$

2. Invariant theory

$$\left(\Lambda^n \mathfrak{gl}(A)\right)_{\mathfrak{gl}(\Bbbk)} \cong \left(\Bbbk[S_n] \otimes A^{\otimes n}\right)_{S_n} = L_n$$

3. Hopf alg structure and Cartier-Milnor-Moore

$$L_* \cong \Lambda^\cdot \operatorname{Prim} L_*$$

4. Computation of the primitives

$$\operatorname{Prim} L_* \cong C^\lambda_{*-1}(A).$$

§ Lie algebra homology

$\mathfrak{g}$ Lie alg, V $\mathfrak{g}$-module

Def. Coinvariants $V_{\mathfrak{g}} = V/V\mathfrak{g}$

↓ left derived functor

$$CE_*(\mathfrak{g},V) = k \overset{L}{\underset{U\mathfrak{g}}{\otimes}} V$$

Its homology is $H_*(\mathfrak{g},V)$.

Chevalley–Eilenberg complex:

$$CE_*(\mathfrak{g},V) = (V\otimes\Lambda^*\mathfrak{g},\ d_{CE})$$

$$d_{CE}(\underset{\substack{\cap \\ V}}{g_0} \otimes \underbrace{g_1\wedge\dots\wedge g_n}_{\in\Lambda^n\mathfrak{g}}) = \sum_{0\le i<j\le n} (-1)^j g_0 \otimes g_1\wedge\dots\wedge \underset{\substack{\uparrow \\ i^{\text{th}}\text{ position}}}{[g_i,g_j]}\wedge\dots\wedge\hat{g}_j\wedge\dots\wedge g_n$$

Prop. $\mathfrak{g} \curvearrowright V\otimes\Lambda^*\mathfrak{g}$ adjoint action is trivial on homology.

Lemma. (Koszul's trick) $\mathfrak{h} < \mathfrak{g}$ reductive subalg. Then

$V\otimes\Lambda^*\mathfrak{g} \xrightarrow{\sim} (V\otimes\Lambda^*\mathfrak{g})_{\mathfrak{h}}$ is a q-iso.

Sketch. Since $\mathfrak{g}$ and V are completely reducible $\mathfrak{h}$-modules:

$$V\otimes\Lambda^*\mathfrak{g} \cong (V\otimes\Lambda^*\mathfrak{g})_{\mathfrak{h}} \oplus \underbrace{R_*}_{\substack{\text{non-trivial} \\ \text{repr of } \mathfrak{h}}}$$

Claim. R_* acyclic

$R_* = \bigoplus M_i$ simple modules, $\mathfrak{h} \circlearrowleft$ non-trivial

In homology, M_i gives $\underline{0}$ or M_i. □

Csq. $\mathfrak{gl}(A) = \operatorname{colim}_r \mathfrak{gl}_r(A)$, $\mathfrak{gl}_r = \mathfrak{gl}_r(k)$ reductive

$[A,B] = AB - BA$

$\mathfrak{gl}_r(A) := (M_r(A), [-,-])$

$$\boxed{\rightsquigarrow \quad \Lambda^\cdot \mathfrak{gl}(A) \xrightarrow{\sim} (\Lambda^\cdot \mathfrak{gl} A)_{\mathfrak{gl}} .}$$

§ Cyclic homology

Hochschild homology: M A^e-module, $A^e = A \otimes A^{op}$.

$$C_\cdot(A,M) = A \overset{\mathbb{L}}{\underset{A^e}{\otimes}} M, \quad C_\cdot(A) = C_\cdot(A,A)$$

Cyclic homology:

$$CC_\cdot(A) = C_\cdot(A)_{hS^1}$$

Models:

$$C_\cdot(A,M) = (M \otimes A^{\otimes \cdot}, b)$$

$$b(a_0\otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$

Define $b' = \sum_{i=0}^{n-1} (-1)^i$ ———

Def. $t, N : A^{\otimes n+1} \longrightarrow A^{\otimes n+1}$

- $t(a_0 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$
- $N = \sum_{i=0}^{n} t^i$: "circle action"

$$CC_\cdot(A) = \mathrm{Tot}\left(\begin{tikzcd}
\vdots & \vdots & \vdots \\
A^{\otimes 4} \arrow[d, "b"] & & \\
A^{\otimes 3} \arrow[d, "b"] & A^{\otimes 3} \arrow[l] \arrow[d, "b'"] & A^{\otimes 3} \arrow[l] \arrow[d, "b"] \\
A^{\otimes 2} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "1-t"] \arrow[d, "b'"] & A^{\otimes 2} \arrow[l, "N"] \arrow[d, "b"] & \cdots \arrow[l] \\
A & A \arrow[l, "1-t"] & A \arrow[l, "N"] & \cdots \arrow[l, "1-t"]
\end{tikzcd}\right)$$

$$\begin{tikzcd}
\vdots \arrow[d, "b"] \\
A^{\otimes 3}/1-t \arrow[d, "b"] \\
A^{\otimes 2}/1-t \arrow[d, "b"] \\
A/1-t
\end{tikzcd} = C^\lambda_\cdot(A)$$

Thm. If $\mathbb{Q} \subseteq k$, then $CC_\cdot(A) \simeq C^\lambda_\cdot(A)$. Connes' complex

<u>Trace map</u>: LQT ~~to~~ is induced by

$$H_{\cdot+1}(\mathfrak{gl}_r(A)) \xrightarrow{\rho} HC_\cdot(M_r(A)) \xrightarrow[\cong]{tr} HC_\cdot(A)$$

$$\rho(\alpha_0 \wedge \dots \wedge \alpha_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\,(\alpha_0, \alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)})$$

$$tr : M_r(A)^{\otimes n+1} \longrightarrow A^{\otimes n+1}$$

$$\beta^0 \otimes \dots \otimes \beta^n \longmapsto \sum_{\substack{(i_0, \dots, i_n) \\ \text{all indices}}} \beta^0_{i_0 i_1} \otimes \beta^1_{i_1 i_2} \otimes \dots \otimes \beta^n_{i_n i_0}.$$

§ Invariant theory

$$\left(\Lambda^n \mathfrak{gl}_r(A)\right)_{\mathfrak{gl}_r} \cong \left(\left(\mathfrak{gl}_r(A)^{\otimes n}\right)_{S_n}\right)_{\mathfrak{gl}_r} \cong \left(\left(\mathfrak{gl}_r(A)^{\otimes n}\right)_{\mathfrak{gl}_r}\right)_{S_n} \cong \left(\left(\mathfrak{gl}_r^{\otimes n}\right)_{\mathfrak{gl}_r} \otimes A^{\otimes n}\right)_{S_n}$$

$\Lambda^n = (-)^{\otimes n}_{S_n}$

$\mathfrak{gl}_r$-action commutes with S_n-action

$$\cong \left(\left(\mathfrak{gl}_r^{\otimes n}\right)_{GL_r} \otimes A^{\otimes n}\right)_{S_n}$$

<u>Question</u>. How to compute $\left(\mathfrak{gl}_r^{\otimes n}\right)_{GL_r}$?

V k-v. sp of dim r

$S_n \curvearrowright V^{\otimes n}$, $\sigma(v_1, \dots, v_n) = (v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(n)})$

$$\rightsquigarrow \begin{tikzcd} k[S_n] \arrow[r] \arrow[dr, dotted, "\mu"'] & \mathrm{End}(V^{\otimes n}) \circlearrowleft GL(V) \\ & \mathrm{End}(V^{\otimes n})^{GL(V)} \arrow[u, hook] \end{tikzcd}$$

by conjugation (diagonally on $V^{\otimes n}$)

<u>Thm.</u> If k is infinite and $r \geq n$, then μ is an iso.

<u>Sketch of proof</u>: $(e_1, \dots, e_n, \dots, e_r)$ basis of V

Let $f \in \mathrm{End}(V^{\otimes n})^{GL(V)}$. $f(e_1, \dots, e_n) = \sum_{I=(i_1,\dots,i_n)} a_I e_I$

Since $r \geq n$, every basis vector of $V^{\otimes n}$ is $\alpha(e_1, \dots, e_n)$ for $\alpha \in GL(V)$.

k infinite $\Rightarrow$ I is a permutation of $\{1, \dots, n\}$.

$$f(e_1, \dots, e_n) = \sum_{\sigma \in S_n} a_\sigma \, \sigma(e_1, \dots, e_n)$$

By GL-invariance of f:

$$f(\alpha(e_1, \dots, e_n)) = \sum_{\sigma \in S_n} a_\sigma \, \sigma(\alpha(e_1, \dots, e_n))$$

so $f = \sum_{\sigma \in S_n} a_\sigma \sigma \in \operatorname{im}(\mu)$.

Hence μ is surjective.

Since $\{\sigma(e_1, \dots, e_n)\}_{\sigma \in S_n}$ linearly independent (as $r \geq n$), so μ is injective. $\square$

It gives, for $r \geq n$

$$T : \mathbb{k}[S_n] \xrightarrow[\cong]{\mu} \operatorname{End}(V^{\otimes n})^{GL} \cong \left(\operatorname{End}(V^{\otimes n})^*\right)^{GL} \cong \left(\left(\operatorname{End}(V)^{\otimes n}\right)_{GL}\right)^*$$

If $\sigma = (i_1 \dots i_s)(j_1 \dots j_s)$

then

$$T(\sigma) : (\alpha_1, \dots, \alpha_n) \longmapsto \operatorname{tr}(\alpha_{i_1} \cdots \alpha_{i_s}) \operatorname{tr}(\alpha_{j_1} \cdots \alpha_{j_s})$$

Assume $r \geq n$ for now on.

<u>Dualize</u> : $T^* : \left(\operatorname{End}(V)^{\otimes n}\right)_{GL} \xrightarrow{\cong} \mathbb{k}[S_n]^* \cong \mathbb{k}[S_n]$

<u>Fact</u>. $(E_{1\sigma(1)}, \dots, E_{n\sigma(n)}) \longmapsto \sigma$

We obtain $\Theta : \left(\Lambda^n \mathfrak{gl}_r A\right)_{\mathfrak{gl}_r} \cong \left(\mathbb{k}[S_n] \otimes A^{\otimes n}\right)_{S_n} =: L_n$

§ Hopf algebra structure on the coinvariant CE complex

- <u>Comultiplication</u>: $\Delta : \mathfrak{gl}A \longrightarrow \mathfrak{gl}A \times \mathfrak{gl}A$

  yields $(\Lambda^\cdot \mathfrak{gl}A)_{\mathfrak{gl}} \longrightarrow (\Lambda^\cdot \mathfrak{gl}A)_{\mathfrak{gl}}^{\otimes 2}$

  cocommutative, coassociative.

- <u>Multiplication</u>: $\oplus : \mathfrak{gl}A^{\times 2} \longrightarrow \mathfrak{gl}A$

$$\alpha = \begin{pmatrix} \ast & \ast & \cdots \\ \ast & \ast & \cdots \\ \vdots & \vdots & \end{pmatrix}, \quad \beta = \begin{pmatrix} \times & \times & \cdots \\ \times & \times & \\ \vdots & \vdots & \end{pmatrix}$$

$$\alpha \oplus \beta = \begin{pmatrix} \ast & 0 & \ast & 0 & \ast & \\ 0 & \times & 0 & \times & 0 & \cdots \\ \ast & 0 & \ast & 0 & & \ddots \\ 0 & \times & 0 & \times & & \\ \vdots & & \vdots & & & \end{pmatrix}$$

$(\Lambda^\cdot \mathfrak{gl}A)^{\otimes 2} \longrightarrow \Lambda^\cdot \mathfrak{gl}(A)$ not assoc., nor comm.

$\oplus$ is assoc and comm up to conjugation by permutation matrices.

$((\Lambda^\cdot \mathfrak{gl}A)_{\mathfrak{gl}}, d)$ is a cocomm. comm. dg Hopf algebra.

Via $\Theta : (\Lambda^\cdot \mathfrak{gl}A)_{\mathfrak{gl}} \simeq L_*$, comultiplication becomes

$$\Delta(\sigma \otimes (a_1, \dots, a_n)) = \sum_{(I,J)} (\sigma_I \otimes (\dots a_i \dots)) \otimes (\sigma_J \otimes (\dots a_j \dots))$$

(I,J) ordered partitions of $\{1, \dots, n\}$ s.t. $\sigma(I) = I$, $\sigma(J) = J$

for $\sigma \otimes (a_1, \dots, a_n) \in L_n = (k[S_n] \otimes A^{\otimes n})_{S_n}$

Cartier – Milnor – Moore thm. / Borel

$\Bbbk \xrightarrow{\cong} \mathcal{H}_0$

$\mathcal{H}_{<0} = 0$

$\Bbbk$ char 0.

$\mathcal{H}$ connected cocomm. dg Hopf alg

Then $U\,\mathrm{Prim}\,\mathcal{H} \cong \mathcal{H}$.

If $\mathcal{H}$ is moreover commutative, then

$$\Lambda\,\mathrm{Prim}\,\mathcal{H} \cong \mathcal{H}.$$

§ Computation of primitives of L_*

U_n = conjugacy class of $\tau = (1 \ldots n)$ in S_n
= {n-cycles}

$$P_* = \bigoplus_{n \geq 1} \left(\Bbbk[U_n] \otimes A^{\otimes n} \right)_{S_n}$$

Prop. $P_* = \mathrm{Prim}\, L_*$

Sketch: for an n-cycle σ

$$\Delta(\sigma \otimes a_1 \ldots a_n)$$
$$= (\sigma \otimes (a_1 \ldots a_n)) \otimes 1 + 1 \otimes (\sigma \otimes (a_1 \ldots a_n)).$$

so $P_* \subseteq \mathrm{Prim}\, L_*$.

Hence $\Lambda P_* \hookrightarrow \Lambda\,\mathrm{Prim}\, L_* = L_*$.

Since every permutation is a product of cycles, this is an iso. □

Prop. $P_* \cong C^{\lambda}_{*-1}(A)$

$$P_* \cong \bigoplus_{n \geq 1} \left(A^{\otimes n}\right)_{\mathbb{Z}/n\mathbb{Z}} \cong \bigoplus_{n \geq 1} \left(A^{\otimes n}\right)_{/1-t} = C^{\lambda}_{*-1}(A).$$ □

$U_n \cong S_n / \mathbb{Z}/n\mathbb{Z}$ as S_n-sets

$\mathbb{Z}/n\mathbb{Z} = \langle \tau \rangle$

so $\Bbbk[U_n] \cong \Bbbk \otimes_{\Bbbk[\mathbb{Z}/n\mathbb{Z}]} \Bbbk[S_n]$

$\tau \longleftrightarrow t$.

action by permutation and multiplication by $\mathrm{sgn}(\tau) = (-1)^{n-1}$

Prop. $\Theta : \mathrm{Prim}\left(\Lambda^{\cdot}\mathfrak{gl}(A)\right)_{\mathfrak{gl}} \cong C^{\lambda}_{*-1}(A)$

sends $d_{CE} \longmapsto b$

Conclusion

$$\Lambda^{\cdot}\mathfrak{gl}(A) \xrightarrow{\sim} \left(\Lambda^{\cdot}\mathfrak{gl}(A)\right)_{\mathfrak{gl}} \quad \text{as dg Hopf alg.}$$

$$\cong \Lambda^{\cdot}\,\mathrm{Prim}\left(\Lambda^{\cdot}\mathfrak{gl}A\right)_{\mathfrak{gl}} \cong \Lambda^{\cdot} C^{\lambda}_{*-1}(A)$$

$\downarrow$ taking Homology and using that $H_*\mathrm{Prim} = \mathrm{Prim}\,H_*$ for cocomm. Hopf.

$$H_*\mathfrak{gl}(A) \cong \Lambda^* HC_{*-1}(A).$$

One can verify that this iso is indeed induced by the trace map.

Remark.

Consider $BGL_\infty \longrightarrow K$ as a natural transformation

$$BGL_\infty(F_A) \xrightarrow{\vartheta_A} K(F_A).$$

$$\left(\begin{array}{l} \text{dg Art} \longrightarrow \text{Spaces} \\ \text{where} \\ F_A : \text{dg Art} \longrightarrow \text{dg Alg}^{\leq 0} \\ \qquad - \otimes_k A \qquad \downarrow K \\ \qquad\qquad\qquad\qquad S \end{array}\right)$$

Then ϑ induces the trace map via

$$\mathbb{T}_{K(F_A)} \simeq HC_{\cdot}(A)$$

[Hennion] Tangent of K-theory

$$\mathrm{tr} : \mathfrak{gl}A \longrightarrow \underbrace{HC_{\cdot-1}(A)}_{\text{abelian Lie alg}}$$