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« Twisting the operad of graphs and the formality of the little discs »

<u>Plan</u>

A) Twisting an operad

B) The operad Gra

C) Its twisted version Tw Gra

D) Formality of the little discs

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$\boxed{A}$ Recall from Victor's talk

1) The twisting procedure

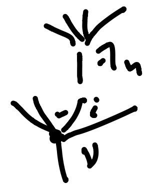
$$s\mathcal{L}_\infty \longrightarrow \mathcal{P} \quad \Rightarrow \quad \mathrm{Tw}\,\mathcal{P} = (\mathcal{P} \,\hat{\sqcup}\, \overset{\alpha}{\mathfrak{i}}, d^{\lambda_1^\alpha})$$

complete coproduct

$$d^{\lambda_1^\alpha}(\overset{\alpha}{\mathfrak{i}}) = \sum_{n\geq 2} \frac{n-1}{n!} \lambda_n(\alpha, \ldots, \alpha)$$

$$d^{\lambda_1^\alpha}(\nu) = d_{\mathcal{P}}(\nu) + \sum_{n\geq 2} \frac{1}{(n-1)!} \lambda_n(\underbrace{\alpha, \ldots, \alpha}_{n-1}, \nu) \ldots$$

$\in \mathcal{P}(k)$

$$+ (-1)^{|v|} \sum_{i=1}^{k} \sum_{n \geq 2} \frac{1}{(n-1)!}$$ 

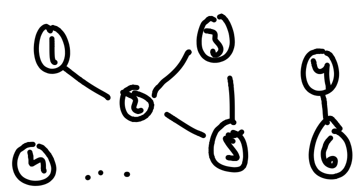
2) It gives a dg Lie action

$$\mathrm{Def}(s\mathcal{L}_\infty \to \mathcal{P}) \underset{\mathrm{Der}}{\circlearrowright} \mathrm{Tw}\,\mathcal{P}$$

3) If $\mathcal{P}(0) = 0$, then

$$\mathrm{Def}(s\mathcal{L}_\infty \to \mathcal{P}) \cong \left(\mathrm{Tw}\,\mathcal{P}(0),\ d_1^{\lambda_1^*}\right)$$

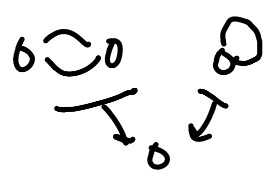
$\boxed{B}$ The operad Gra

$$\mathrm{Gra}(n) := \Bbbk \left\{ \right.$$  $$\left. \right\}$$

$|\text{edges}| = 1$ $\quad$ $\mathbb{S}_n \circlearrowright$ vertices

orientation

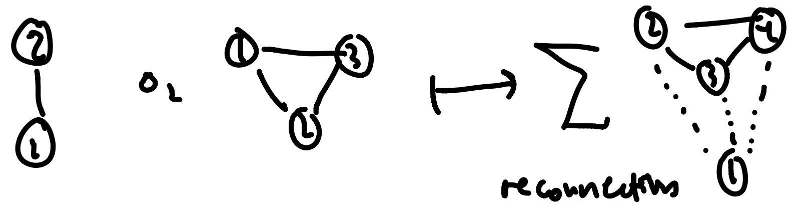
$\Downarrow$



differential $= 0$

operadic composition = insertion

$$\circ_i \; : \; Gra(k) \otimes Gra(l) \longrightarrow Gra(k+l-1)$$



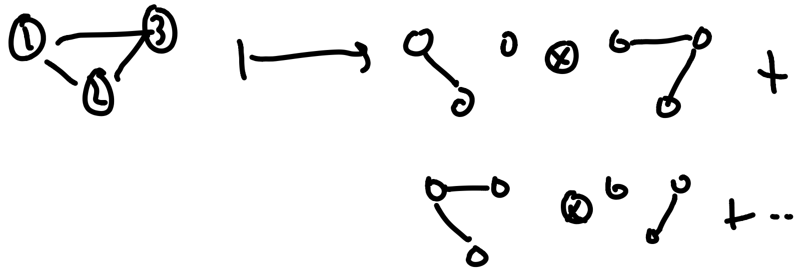
reconnections of edges adj. to i

+ renumbering

+ orientation : "first the base graph, then the inserted one"

In fact it is a Hopf operad, i.e. $Gra(n)$ is cocom. coalg.

$$T \longmapsto \underbrace{\sum}_{\text{edges distributions}} T' \otimes T''$$



$\boxed{B'}$ Why Gra?

Operad of natural operations acting on polyvector fields

$$A = \mathbb{k}[p_1, \dots, p_n, q_1, \dots, q_n] \quad \text{graded com.-alg}$$

$|p_i| = 0$

$|q_i| = 1$

$$[q_i, p_j] = \delta_{ij}$$

$\uparrow$

"partial derivatives wrt p_i"

Let $\Gamma \in \mathrm{Gra}(n)$

$$\Omega = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial q_i} + \frac{\partial}{\partial q_i} \otimes \frac{\partial}{\partial p_i}$$

To each factor of $A^{\otimes n}$ we associate a vertex of Γ

To each edge of Γ, we associate Ω acting on the 2 adj. vertices

We multiply the resulting expression

$$① \quad ② \longmapsto A \otimes A \xrightarrow{\mu} A$$

$$①\!-\!② \longmapsto \text{Schouten bracket}$$

What is the Schouten bracket?

"the <u>unique</u> extension of the Lie bracket on the space of polyvector fields that make it into a <u>Gerstenhaber alg.</u>"

At the operad level,

$$\mathrm{Gerst} \hookrightarrow \mathrm{Gra}$$

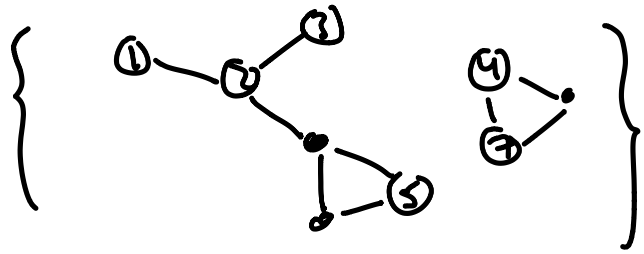
com. prod. $|\mu| = 0$ $\qquad \mu \longmapsto ① \quad ②$

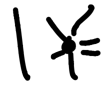
Lie bracket $|\lambda| = 1$ $\qquad \lambda \longmapsto ① — ②$

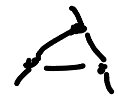
↪ induces $s\mathcal{L}_\infty \to s\mathcal{L}ie \to \mathrm{Gerst}$

$\Rightarrow$ Gra is a multiplicative operad and we can twist it!

$\boxed{C}$ Who is TwGra?

$\mathrm{TwGra}(n) :=$ 

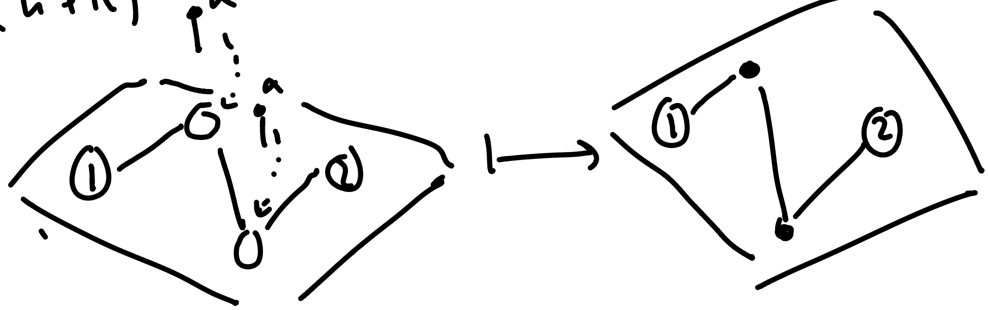
$|\text{edges}| = 1$  $= -2$ $\quad S_n \circlearrowleft$ white vertices

orientation :  can have automorphisms

a graph with aut. inducing odd perm. on edges $\mapsto 0$

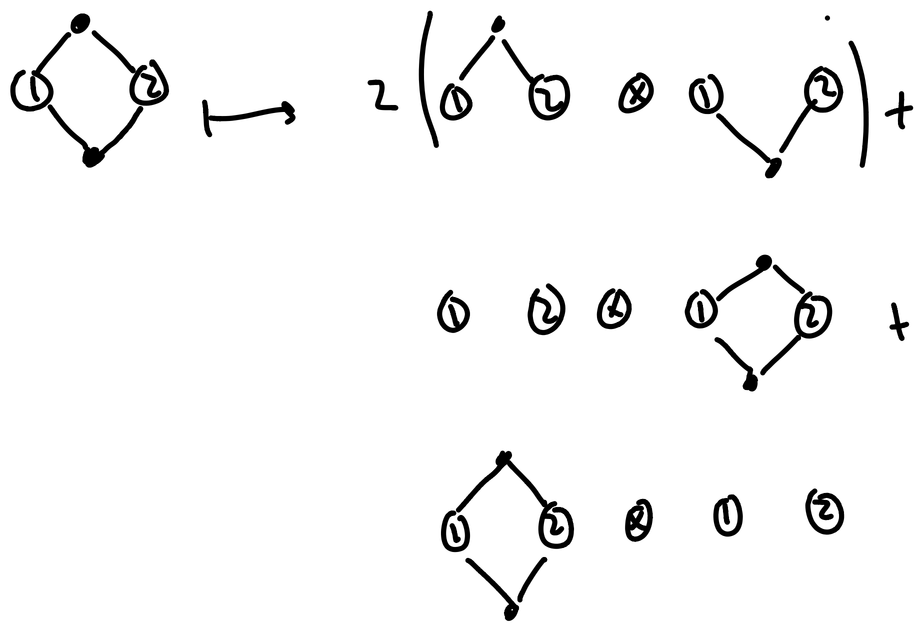
$$\mathrm{TwGra} = \left(\mathrm{Gra} \,\hat{\sqcup}\, \bullet^{\alpha},\ d^{\lambda_i^{\alpha}}\right)$$

We think of its elements as living in $\mathrm{Gra}(n+k)$

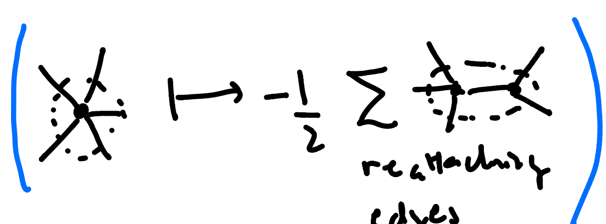


Operadic composition = insertion (white vertices only!)

Hopf operad again

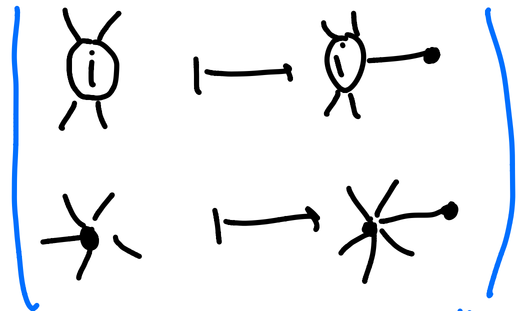


Differential = sum of 3 types of terms

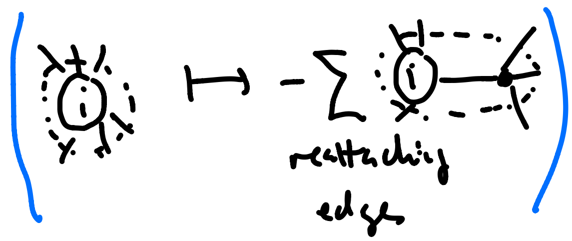
$d^{\lambda_i^*}(\hat{i}) = \sum_{n \geq 2} \frac{n-1}{n!}$  $\longleftrightarrow$ 

$d^{\lambda_i^*}(\Psi_v) = d_P(\Psi_v) \longleftrightarrow 0$ (a)

$\in P(k)$

$+\sum_{n\geq 2}\frac{1}{(n-1)!}$  $\longleftrightarrow$ 

(b)

$+(-1)^{|v|}\sum_{i=1}^{K}\sum_{n\geq 2}\frac{1}{(n-1)!}$  $\longleftrightarrow$ 

(c)

$$d^{\lambda_i^\alpha}(\Gamma) = \sum (a) + \sum (b) + \sum (c)$$

* orientation induced by Γ : new edge = first one

2) We obtain an action

$$\mathrm{Def}(s\mathcal{L}_\infty \to \mathrm{Gra}) \underset{\mathrm{Der}}{\curvearrowright} \mathrm{TwGra}$$

3) Since $\mathrm{Gra}(0) = 0$, we have

$$\mathrm{Def}(s\mathcal{L}_\infty \to \mathrm{Gra}) \cong (\mathrm{TwGra}(0), d^{\partial, \alpha})$$

graphs with only black vertices.

<u>THM</u> [Kontsevich, Lambrechts-Volić]

$$\mathrm{Gerst} \longrightarrow \mathrm{TwGra}_c$$ is a quasi-isomorphism

$\boxed{D}$ Formality of little discs

We look at the linear dual picture

$$\mathrm{Gra}^* = \text{Hopf cooperad}$$

(Gra is finite dim. in each arity) graph insertion $\longrightarrow$ subgraph contraction

edge distribution $\longrightarrow$ edge union

vertex splitting $\longrightarrow$ edge contraction

$$H^\bullet(D_2) \longleftarrow Gra^* \longrightarrow C^\bullet(D_2)$$

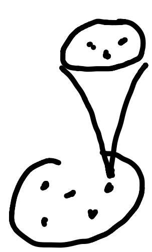
$$\cong$$

$$\Omega_{PA}(FM)$$

piecewise semi-alg. diff forms (analogue of de Rham)

Fulton-MacPherson operad; quasi-iso to D_2

$\|$



compactification of $Conf_k(\mathbb{R}^n)$

« infinitesimal insertion »

$$\underbrace{\hspace{20em}}_{\text{Hopf "almost" cooperad}}$$

or Hopf homotopy cooperad

$H^\bullet(D_2)$ has classical presentation due to Arnold

$$\frac{S(\omega_{ij})}{(\omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij})}$$

The red map sends edges of a graph to generators

$$e_{ij} \longmapsto \omega_{ij}$$

$$\left(\text{dual map of} \quad \begin{array}{l} \mu \longmapsto (1)\ \ (2) \\ \lambda \longmapsto (1)-(2) \end{array} \right)$$

The blue map is given by

$$\bigwedge_{\text{edges of } \Gamma} p^*(\mathrm{vol}_{n-1})$$

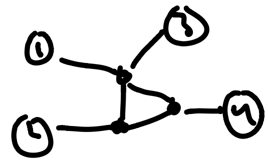
↑ volume form of $FM_n(2) \cong S^{n-1}$

Then it remains to extend these maps to TwGra^*

$$
\begin{tikzcd}
H^\bullet(D_2) & \mathrm{Gra}^* \arrow[l] \arrow[r] \arrow[d] & \Omega^\bullet_{PA}(FM) \\
& \mathrm{TwGra}^* \Big/ \mathrm{TwGra}^*(0) \arrow[ul, dashed, "\text{internal vertices} \mapsto 0"] \arrow[ur, dashed, "\text{integration of forms}"'] &
\end{tikzcd}
$$

The two dotted arrows are shown to be quasi-isomorphisms by KLV.

In conclusion, the twisting procedure has produced "the right replacement" of Gra inducing an iso in (co)homology.

ICG $\subset$ TwGra(n) 

Is the algebra TwGra(n) free?

ICG(n) require L_∞-alg struct.

<u>THM</u> [Severa–Willwacher]

$$\boxed{H_\bullet(\mathrm{ICG}) \cong \mathrm{Lie}(\mathrm{DK})}$$

$\uparrow$ Lie alg

$$\mathrm{DK}^! \cong \mathrm{AOS} \xrightarrow{s} H^\bullet(B_2)$$

BKW $\leftrightarrow$ Gra

DK $\leftrightarrow$ Gerst

$\boxed{E}$ Epilogue / Preview

$GC_2 \subset \mathrm{TwGra}(0)$ connected graphs with vertices at least trivalent

<u>THM</u> [Willwacher]

$$\boxed{H^0(GC_2) \cong \mathfrak{grt}_1}$$

$$GC_2 \xrightarrow{\text{Der}} \mathrm{TwGra} \xrightarrow{\sim} \mathrm{Gerst}$$

$\Rightarrow \quad \mathfrak{grt}_1 \cong H^0(GC_2) \longrightarrow H^0(\mathrm{Der}(\mathrm{Gerst}))$

In fact

<u>THM</u> [Willwacher]

$$\boxed{\mathfrak{grt} \cong H^0(\mathrm{Der}(\mathrm{Gerst}))}$$

$\Rightarrow \mathrm{Aut}(\widehat{D_2}) \cong \widehat{GRT}$ .... TBC (Joost).