The twisting procedure for operads

I) What is twisting?

II) How to encode twisting with operads?

III) Why should I twist operads as well?

I) What is twisting?

First: explanation of def. theory. Theorem by Pridham and Lurie: dg-Lie alg encode def pb. in char = 0 via MC elements.

Given a dg-Lie alg $(\mathfrak{g}, [-,-], d_{\mathfrak{g}})$, a Maurer-Cartan element $\alpha \in \mathfrak{g}_{-1}$ is $\alpha \in \mathfrak{g}_{-1}$ such that:

$$\frac{1}{2}[\alpha, \alpha] + d_{\mathfrak{g}}(\alpha) = 0$$

They form the Maurer-Cartan space:

* Points: a possible structure.

What are the infinitesimal deformations of α.

* Example: V a vector space.
  - → The Hochschild complex encodes the via its MC the set of all associative alg. structures on V.
  - → Given a structure, how to get its infinitesimal deformations?

Q: How to get the "local" information at from $\mathfrak{g}$?

→ It is encoded by the twisted dg Lie algebra $\mathfrak{g}^{\alpha}$, given by:

$$\mathfrak{g}^{\alpha} := (\mathfrak{g}, [-,-], d^{\alpha}) \quad \text{where } d^{\alpha} := d_{\mathfrak{g}} + [\alpha, -]$$

→ Twisting answers the universal question.

• Properties of twisting: Let $\alpha \in \mathfrak{g}_{-1}$:

i) $(d^{\alpha})^2 = 0$ iff $\alpha \in MC(\mathfrak{g})$.

ii) $MC(\mathfrak{g}^{\alpha}) = \{\beta \in \mathfrak{g}_{-1} \mid \beta - \alpha \in MC(\mathfrak{g})\}$ ⊗

iii) Let $\beta \in \mathfrak{g}_{-1}$, then: $(\mathfrak{g}^{\alpha})^{\beta} = \mathfrak{g}^{(\alpha+\beta)}$ and $\mathfrak{g}^{0} = \mathfrak{g}$.

* There are other types of algebras for which it works. E.g: for associative algebras, where Maurer-Cartans are defined with respect to the antisymmetrization of the associative product.

* **Important example: sL_∞-algebras:**

A sL_∞-alg: $(g, \{\underbrace{l_n : g^{\otimes n} \to g}_{\deg -1}\}_{n\geq 2}, d_g)$ which satisfies:

$$\partial(l_n) + \sum_{\substack{p+q=n-1 \\ 2\leq p+q\leq n}} \sum_{\sigma\in Sh^{-1}(p,q)} (l_p \circ_1 l_q)^\sigma = 0$$

For them, the Maurer-Cartan eq. is given by:

$$\alpha \in g_0 \text{ st}: \ d_g(\alpha) + \sum_{n\geq 2} \frac{1}{n!} l_n(\alpha - \alpha) = 0$$

→ Generalizes the notion of a dg-Lie, has the "same" homotopical prop.

Given $\alpha \in g_0$, one can twist g with α:

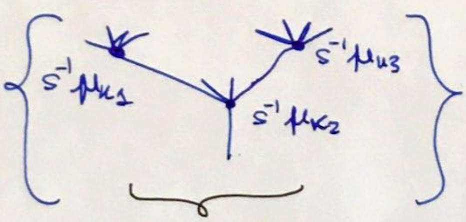
$g^\alpha := (g, \{l_n^\alpha : g^{\otimes n} \to g\}, d_g^\alpha)$ where:

$$\begin{cases} d_g^\alpha = d_g + \sum_{k\geq 0} \frac{1}{k!} l_{k+1}(\alpha^k, -) \\ l_n^\alpha = \sum_{k\geq 0} \frac{1}{k!} l_{n+k}(\alpha^k, \underbrace{\quad}_{n}) \end{cases}$$

→ This twisting is universal in a sense that will be made precise.

Operads: Encoded by:

$\Omega Com^* = (\mathcal{T}(s^{-1}\overline{Com^*}), d) =$



d: expanding a vertex in all possible ways: $d(Y_{\mu_n}) = \sum_{p+q=n-1} \ldots$

* **Examples of deformation cmpx:**

Let P be a Koszul operad, then $P_\infty = \Omega P^¡$, and we have that:

$$Hom_{op}(\Omega P^¡, End_A) \cong MC(\underbrace{Hom(P^¡, End_A)}_{})$$

[Particular example of the def. complex of morphisms of operads.] → def. $sLie$ alg encoding P_∞-alg structures.

In general, let Q be an operad:

$$Hom(\Omega P^¡, Q) \cong MC(\underbrace{Hom(P^¡, Q)}_{\text{convolution dg-Lie}})$$

And, given a morphism $\varphi: P_\infty \to Q$, we can twist this convolution algebra to get the def. complex of the morphism φ:

$$(Hom(P^¡, Q), [-,-], \partial^\varphi)$$

II) How to encode "twisting" with operads?

*Before this, we need to be able to twist operads themselves.

Def: [Maurer-Cartan of an operad]

Let P be an dg operad, a MC is an element $\Theta \in P(1)_{-1}$ such that:

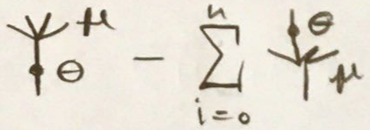
$$\boxed{d_P(\Theta) + \Theta \circ_1 \Theta = 0}$$ ← Left-nucleus condition in the pre-Lie alg $\prod_{n\geq 0} (P(n))^{\mathbb{S}_n}$.

Given $\Theta \in MC(P)$, we can define:

Avec des arbres!

$$\boxed{\begin{array}{l} P^{\Theta} := (P, \{\circ_i\}, d^{\Theta}) \text{ where} \\ \text{if } \mu \in P(n),\ d^{\Theta}_P(\mu) = d_P(\mu) + \Theta \circ_1 \mu - (-1)^{|\mu|} \sum_{i=1}^{n} \mu \circ_i \Theta \end{array}}$$

It forms again a dg-operad.



* **Lemma**: Let $f: P \to Q$, then:
  1) If $\alpha \in MC(P)$, then $f(\alpha) \in MC(Q)$.
  2) And $\underbrace{f: P^{\alpha} \longrightarrow}_{\boxed{\text{same } f \text{ as before}}} Q^{f(\alpha)}$ defines a morphism of dg operads.

* **Example**: Let (A, d_A) be a dg-mod Then a MC of $(\mathrm{End}_A, \partial)$ is a degree -1 map $h: A \longrightarrow A$ satisfying $\boxed{dm + md + m^2 = 0}$.

→ $\boxed{(\mathrm{End}_A^{m}, \partial^{m}) \text{ is simply } \mathrm{End}_{(A, d+m)}}$ Endomorphisms of the perturbed differential.

Now we are going to encode the twisting of sL_∞-alg at the operadic level. Recall the definition of $\Omega \mathrm{Com}^* = sL_\infty$.

Def: [$MCsL_\infty$]

The operad: $$MCsL_\infty := \Big(\hat{\mathcal{T}}\big(\overset{\alpha}{\bullet}, \overbrace{\underset{\mu_2}{Y}, \underset{\mu_3}{Y}, \dots}^{s^{-1}\overline{\mathrm{Com}}^*}\big), d\Big)$$

where $\overset{\alpha}{\bullet}$ has degree 0 and μ_n degree -1. It is endowed with the filtration $\overset{\alpha}{\bullet} \in F_1 MCsL_\infty(0)$ and $\mu_n \in F_0 MCsL_\infty(n)$. The differential is given by:

$$\begin{cases} d(\mu_n) := \displaystyle\sum_{p+q=n-1} \sum_{\sigma \in Sh^{-1}(p,q)} (\mu_p \circ \mu_q)^{\sigma} \\ d(\overset{\alpha}{\bullet}) := -\displaystyle\sum_{n\geq 2} \frac{1}{n!} \mu_n(\alpha, \dots, \alpha) \end{cases}$$

• **Lemma**: $\boxed{d^2 = 0}$ (By computation)

• **Rem**: $\hat{\mathcal{T}}(\bullet, Y_{\mu_2}, \dots) = \Omega \mathrm{Com}\, \hat{\sqcup}\, \bullet^{\alpha}$ with a derivation of $\bullet^{\alpha}$.

Encodes an sL_∞-alg together with a Maurer-Cartan element.

* MCsL∞-alg = sL∞-alg with a Maurer-Cartan $\alpha : \mathbb{K} \to \mathfrak{g}$.

⇝ Inside MCsL∞, we have the following elements:

$$\left\{ \mu_n^\alpha := \sum_{k \geq 0} \frac{1}{k!} \, \mu_{n+k}(\underbrace{\alpha, \dots, \alpha}_{k}, \underbrace{-, \dots, -}_{n}) \right\} \in \mathrm{MCsL}_\infty(n)$$

which correspond to a "formal" twisted sL∞ structure.

• **Lemma:**

$$d(\mu_n^\alpha) = \sum_{p+q=n-1} \sum_{Sh} \left(\mu_p^\alpha \circ \mu_q^\alpha \right)^\sigma$$

"sL∞-equation satisfied by the formal structure". (*)

⚠ Nevertheless, $C_\alpha : \mathrm{sL}_\infty \to \mathrm{MCsL}_\infty$ does not commute with the differentials! $\mu_n \mapsto \mu_n^\alpha$

Indeed:

$$C_\alpha(d(\mu_n)) = \sum \mu_q^\alpha \circ \mu_p + \sum \mu_q \circ \mu_p^\alpha .$$

→ We have to twist MCsL∞ to get the right operad!

(*) Implies that: $\boxed{\mu_1^\alpha \circ_1 \mu_1^\alpha + d(\mu_1^\alpha) = 0}$ hence $\mu_1^\alpha \in \mathrm{MC}(\mathrm{MCsL}_\infty)$.

Def: [TwsL∞]

The dg operad given by: $\boxed{\mathrm{TwsL}_\infty := (\mathrm{MCsL}_\infty)^{\mu_1^\alpha}}$

→ Its differential is given by:

* $d^{\mu_1^\alpha}(\mu_n) = d(\mu_n) + \mu_1^\alpha * \mu_n - \mu_n * \mu_1^\alpha$

$$\sum_{p+q=n-1}^{\wedge} \mu_p \circ \mu_q \qquad \sum_{k \geq 0}^{n} \frac{1}{k!} \, \mu_{k+1}(\mu_n, \alpha, \dots, \alpha) \qquad - \sum_{k \geq 0} \sum_{i=1}^{n} \frac{1}{k!} \, \mu_n \circ_i \mu_{k+1}(\alpha, \dots, \alpha, 1)$$

* $d^{\mu_1^\alpha}(\overset{\alpha}{\circ}) = \sum_{n \geq 0} \frac{(n-1)}{n!} \, \mu_n(\alpha, \dots, \alpha)$

• **Lemma:** $C_\alpha : \mathrm{sL}_\infty \to \mathrm{TwsL}_\infty$ commutes with the differentials.

* **Conclusion:** Let $\mathrm{MCsL}_\infty \xrightarrow{f} \mathrm{End}_{(\mathfrak{g},d)}$ be an sL∞-alg with a Maurer-Cartan $f(\alpha) : \mathbb{K} \to \mathfrak{g}$. Then:

$$f^{\mu_1^\alpha} : (\mathrm{MCsL}_\infty)^{\mu_1^\alpha} \longrightarrow \left(\mathrm{End}_{(\mathfrak{g},d)}\right)^{f(\mu_1^\alpha)} \cong \mathrm{End}_{(\mathfrak{g}, d + f(\mu_1^\alpha))}$$

is a morphism of operads and:

$$\boxed{\mathrm{sL}_\infty \xrightarrow{C_\alpha} (\mathrm{MCsL}_\infty)^{\mu_1^\alpha} \xrightarrow{f} \mathrm{End}_{(\mathfrak{g}, d + f(\mu_1^\alpha))}}$$

gives the twisted sL∞-alg structure on $\mathfrak{g}$ corresponding to $\mathfrak{g}^{f(\alpha)(1)}$.

→ We have encoded the twisting of sL∞-algebras with an operadic procedure.

Q: Can this be done for any operad?

We consider the category of multiplicative operads $s\mathcal{L}_\infty \xrightarrow{\lambda_P} P$, operads under $s\mathcal{L}_\infty$. We have operations

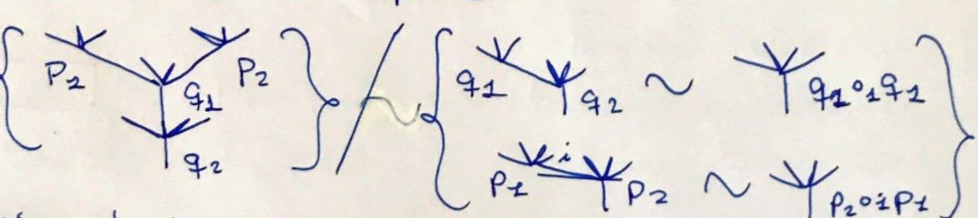
$$s\mathcal{L}_\infty \xrightarrow{\lambda_P} P$$
$$\mu_n \longrightarrow \lambda_P(\mu_n)$$

which for a P-algebra (A, γ_A, d_A) give a notion of MC equation:

$$\boxed{\alpha \in A_0 \text{ st } \quad d_A(\alpha) + \sum_{n\geq 2} \frac{1}{n!}\gamma_A(\lambda_P(\mu_n))(\alpha, \dots, \alpha) = 0}$$

* **Example:** $\Omega Com^* = s\mathcal{L}_\infty \twoheadrightarrow s\mathcal{L}ie \longrightarrow Gerst \longrightarrow Gra$, therefore $Gerst$ and Gra are multiplicative operads.

* **Coproduct of operads:** Let P and Q be operads:

$P \vee Q :=$ 

[P a complete operad]

Def: [MCP] The operad given by:

$$\boxed{MCP := (P \hat{\vee} \overset{\alpha}{\bullet}, d)} \text{ with } \overset{\alpha}{\bullet} \in F_1 MCP(0).$$

with the differential given by:

$$\begin{cases} d(\overset{\alpha}{\bullet}) = -\sum_{n\geq 2} \frac{1}{n!} \overset{\alpha \ \cdots \ \alpha}{\curlyvee}_{\lambda_P(\mu_n)} \\ d(\curlyvee_{p_1}) = d_P(p_1) \end{cases}$$

* **Lemma:** $\boxed{d^2 = 0.}$

* **Lemma:** $\boxed{\lambda_P(\mu_1^\alpha) = \sum_{n\geq 0} \frac{1}{n!} \overbrace{\overset{\alpha \cdots \alpha}{\curlyvee}}^{n}{}_{\lambda_P(\mu_{k+1})} \in MC(MCP)}$ — Image of the twisted formal diff. by the str. mor. λ_P.

Proof: $\lambda_P : s\mathcal{L}_\infty \to P$ can be extended into $\hat{\lambda}_P : MCs\mathcal{L}_\infty \longrightarrow MCsP_\infty$. The image of a MC is again a MC.

Def: [TwP] We define: $\boxed{TwP := (MCP)^{\lambda_P(\mu_1^\alpha)}}$

Again, the differential is given by:

$$d^{\lambda_P(\mu_1^\alpha)}(\overset{\alpha}{\bullet}) = \sum_{n\geq 2} \frac{(n-1)}{n!} \overset{\alpha \cdots \alpha}{\curlyvee}_{\lambda_P(\mu_n)} \; ; \; d^{\lambda_P(\mu_1^\alpha)}(p) = d_P(p) + \lambda_P(\mu_1^\alpha) * p - p * \lambda_P(\mu_1^\alpha)$$

$$= \sum_{n\geq 0} \frac{1}{n!} \overset{n \quad p \quad \alpha}{\curlyvee}_{\lambda_P(\mu_{n+1})} - \sum_{n\geq 0}\sum_{i=1}^{n} \frac{1}{n!} \overset{k \ \lambda_P(\mu_{n+1})}{\curlyvee_i}{}_p$$

We get a morphism of operads given by:

$$\boxed{\begin{aligned} \mathcal{C}^P_\alpha : s\mathcal{L}_\infty &\longrightarrow TwP \\ \mu_n &\longrightarrow \lambda_P(\mu_n^\alpha) \end{aligned}}$$

So far, on a twisted P-alg we "only" get an $s\mathcal{L}_\infty$ structure.

* **Prop:** $Tw : s\mathcal{L}_\infty/Op \longrightarrow s\mathcal{L}_\infty/Op$ defines an endofunctor.

If we apply it twice we get:

$$\boxed{Tw(Tw P) \cong \left(P \hat{\sqcup} \alpha \hat{\sqcup} \beta, d + ad_{\mu_1^{\alpha+\beta}}\right)}$$

The image by λ_P of –

* **Prop**: $Tw : s\mathcal{L}_\infty/Op \to s\mathcal{L}_\infty/Op$, together with:

$$\Delta(P) : Tw P \cong P \hat{\vee} \alpha \longrightarrow Tw(Tw P) \cong P \hat{\vee} \alpha \hat{\vee} \beta$$
$$\alpha \longmapsto \alpha + \beta$$
$$v \longmapsto v$$

and

$$\varepsilon(P) : Tw P \longrightarrow P$$
$$v \longmapsto v$$
$$\alpha \longmapsto 0$$

defines a comonad in the category of multiplicative operads.

Def: [Tw-stable operad]

A multiplicative operad $\lambda_P : s\mathcal{L}_\infty \to P$ is **tw-stable** if it can be endowed with a Tw-coalgebra structure.

→ This provides a "good" notion of twist for P-algebras:

↳ The map $\Delta_P : P \to Tw P$ allows to have a P-alg structure on the twisted P-alg over Tw P by pulling back.

↳ The condition: $P \xrightarrow{\Delta_P} Tw P \xrightarrow{\varepsilon(P)} P$ (composite $= id$) means that twisting by $0 \in MC$ changes nothing.

↳ The condition:

$$\begin{array}{ccc} P & \xrightarrow{\Delta_P} & Tw P \\ \Delta_P \downarrow & & \downarrow Tw(\Delta_P) \\ Tw P & \xrightarrow[\Delta(P)]{} & Tw(Tw P) \end{array}$$

means that twisting by α then by β is the same as twisting by $\alpha + \beta$.

→ We get back the properties of the twisting of Lie algebras given in the beginning.

* Examples:

1) Lie, $s\mathcal{L}_\infty$, Ass, A_∞ and Gerst are tw-stable.

2) BV is not tw-stable.

III - Why should I twist operads too?

Recall our beloved deformation complex:

$$\mathfrak{g}_P = \left(\mathrm{Hom}(\mathrm{Com}^*, P), \partial, [-,-]\right)$$

where:

$$[\alpha, \beta] = \alpha * \beta - (-1)^{|\alpha||\beta|} \beta * \alpha$$

$$\| \quad \sum_{p+q=n-1} \alpha(\mu_p) \text{ grafted with } \beta(\mu_q)$$

The MC of $\mathfrak{g}_P$ are morphisms $\mathcal{C}_\alpha : \Omega \mathrm{Com}^* \to P$ of operads. ($\Omega\mathrm{Com}^* = s\mathcal{L}_\infty$)

→ Given $\lambda_P : s\mathcal{L}_\infty \to P$, we get a canonical MC in $\mathfrak{g}_P$ by which we can twist the structure:

Def. complex of the struct. mor of a multiplicative operad.

$$\boxed{\mathrm{Def}(\Omega \mathrm{Com}^* \xrightarrow{\lambda_P} P) := \mathfrak{g}_P^{\lambda_P} = \left(\mathrm{Hom}(\mathrm{Com}^*, P), \partial^{\lambda_P}, [-,-]\right)}$$

* <u>Derivations of an operad</u>: $\left\{ \begin{array}{l} D : P \to P \text{ of degree } n \text{ such that } (\mathbb{S}\text{-mod}) \\ D(p_1 \circ_i p_2) = D(p_1) \circ_i p_2 + (-1)^{|p_1|} p_1 \circ_i D(p_2) \end{array} \right\}$

Derivations form a dg-Lie algebra $\mathrm{Der}\,P_*$ with the bracket given by:

$$\boxed{[D, D'] = D \circ D' - (-1)^{|D||D'|} D' \circ D}$$

* <u>Theorem</u>: There is a morphism of dg-Lie algebras:

$$\boxed{\Psi : \mathrm{Def}(\Omega \mathrm{Com}^* \xrightarrow{\lambda_P} P) \longrightarrow \left(\mathrm{Der}(\mathrm{Tw}P), d^{\lambda_P(\mu_1^*)}, [-,-]\right)}$$

given by, for $e : \mathrm{Com}^* \to P$, by:

$$\Psi(e) = D_e + \mathrm{ad}_{\lambda_P(\mu_1^*)}$$

where:

$$\left[\begin{array}{l} D_e : \mathrm{Tw}P \longrightarrow \mathrm{Tw}P \\ \qquad \nu \longmapsto 0 \\ \qquad \alpha \longmapsto -\sum_{n \geq 1} e(\mu_n) \end{array} \right.$$

We get an action of the def complex of λ_P on $\mathrm{Tw}P$ by derivations

Furthermore, thanks to $\mathrm{Tw}P$, one can <u>compute</u> the homology of $\mathrm{Def}(\Omega\mathrm{Com}^* \xrightarrow{\lambda_P} P)$:

* <u>Proposition</u>: If $P(0) = 0$, there is an iso <u>of dg modules</u>:

$$\boxed{\mathrm{Def}(\Omega \mathrm{Com}^* \xrightarrow{\lambda_P} P) \cong \left(\mathrm{Tw}P(0), d^{\lambda_P(\mu_1^*)}\right)}$$

⚠ Not an iso of dg Lie algebras: structure given by λ_P.

* Example: We had: $\chi : s\mathcal{L}_\infty \twoheadrightarrow s\mathcal{L}ie \rightarrowtail \mathcal{G}erst \rightarrowtail \mathcal{G}ra$

$$\begin{aligned} Y_{\mu_2} &\longmapsto ① \quad ② \\ [-,-] &\longmapsto ①—② \end{aligned}$$

Willwacher defines:

"full graph complex"

$$FGC_2 := \mathrm{Def}(s\mathcal{L}_\infty \xrightarrow{\chi} \mathcal{G}ra) \underset{dg}{\cong} (\mathrm{Tw}\mathcal{G}ra(0), d^{\chi(\mu_2^*)})$$

By the above, we have an action:

$$\mathrm{Def}(s\mathcal{L}_\infty \xrightarrow{\chi} \mathcal{G}ra) \to \mathrm{Der}(\mathrm{Tw}\mathcal{G}ra)$$

And a key result:

Thm [Kontsevich] $\mathcal{G}erst \xrightarrow{\sim} \mathrm{Tw}\mathcal{G}ra$ is a quasi-iso.

This implies that: $H^*(\mathrm{Der}(\mathcal{G}erst)) \cong H^*(\mathrm{Der}(\mathcal{G}erst_\infty)) \cong H^*(\mathrm{Der}(\mathrm{Tw}\mathcal{G}ra))$

So the results that are missing are (layout):

We have a well-defined map:

$$H^*(\mathrm{Def}(\Omega\mathcal{C}om^* \xrightarrow{\chi} \mathcal{G}ra)) \longrightarrow H^*(\mathrm{Der}(\mathcal{G}erst_\infty))$$

of dg-Lie algebras.

Willwacher shows that:

1) Thm: [Willwacher] $H^0(\mathrm{Def}(\Omega\mathcal{C}om^* \xrightarrow{\chi} \mathcal{G}ra) \cong H^0(FGC_2) \cong \mathfrak{grt}$ (Grothendieck-Teichmüller Lie algebra.)

2) Thm [Willwacher] $H^*(\mathrm{Der}(\mathcal{G}erst_\infty)) \cong S^+(H^*(\mathrm{Def}(\chi : \Omega\mathcal{C}om^* \to \mathcal{G}ra)) \oplus -)$ (small terms.)

This two facts together give that:

$$H^0(\mathrm{Der}(\mathcal{G}erst_\infty)) \cong \mathfrak{grt}$$

□