Why Higher Structures?

Bruno VALLETTE

Université Sorbonne Paris Nord

Math+ Berlin Colloquium

January 27, 2023

Foreword

Yuri Ivanovich MANIN (1937-2023)

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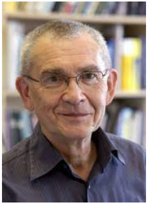

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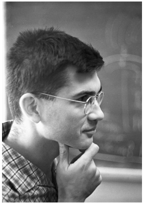

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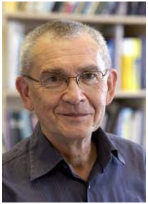

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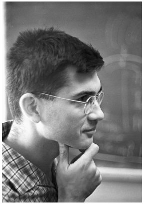

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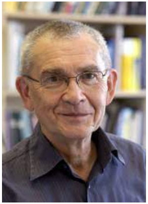

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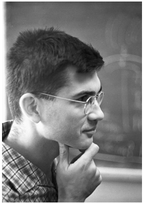

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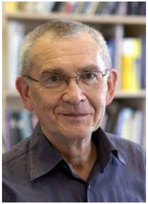

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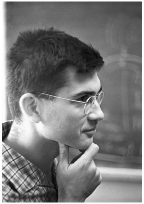

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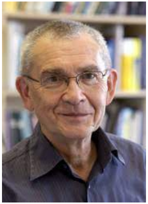

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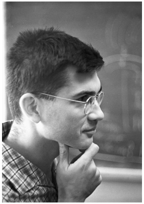

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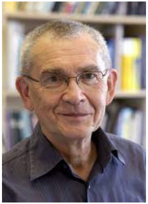

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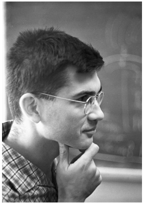

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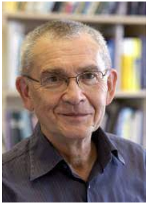

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- Quantise Poisson manifolds

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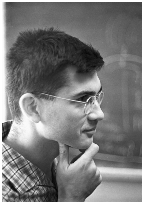

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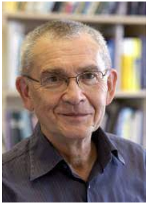

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- Fundamental theorem of deformation theory

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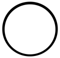

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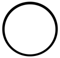

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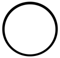

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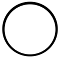

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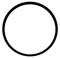

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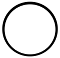 $\sim$ 

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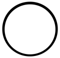 $\sim$ 

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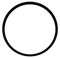 $\sim$ 

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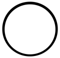

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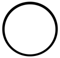 $\sim$ 

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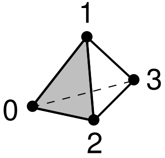
$\rightarrow$ Amount of algebra used: $\boxed{\mathbb{N}}$

Differential graded module

IDEA: Encode algebraically a cellular decomposition



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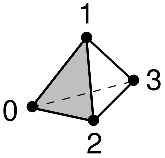

$$\mathbb{Z}\{0\text{-cells}\} \xleftarrow{d_0} \mathbb{Z}\{1\text{-cells}\} \xleftarrow{d_1} \mathbb{Z}\{2\text{-cells}\} \cdots$$

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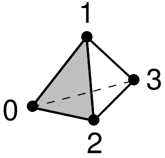

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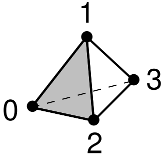

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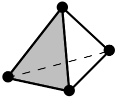

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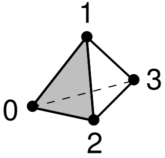

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Definition (differential graded module or chain complex)

$(C_\bullet = \{C_n\}_{n\in\mathbb{N}}, d = \{d_n\}_{n\in\mathbb{N}})$ s.t. $d^2 = 0$

Homology groups

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$$H_n(X, \mathbb{Z}) := \ker d_{n-1} / \operatorname{im} d_n$$

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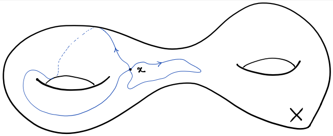
$\to$ Amount of algebra used: Linear algebra

Homotopy group



Homotopy group

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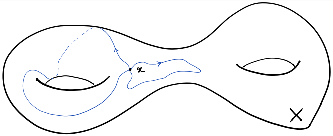

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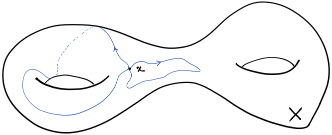

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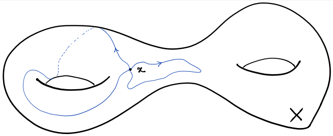

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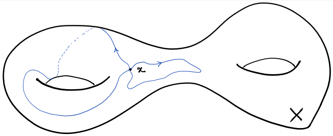

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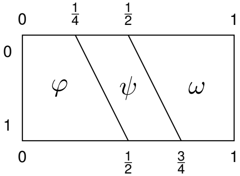

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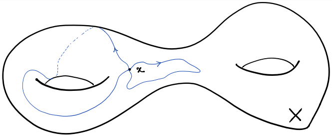

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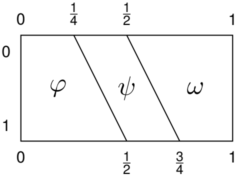

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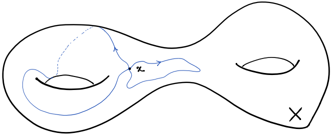

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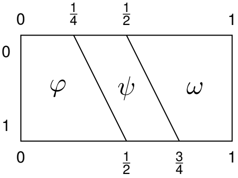

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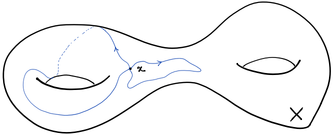

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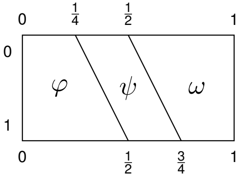

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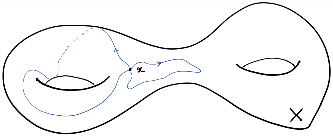

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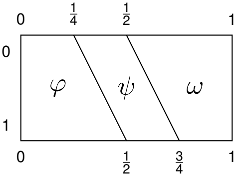

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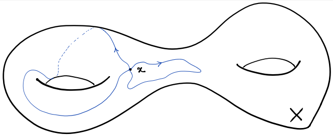

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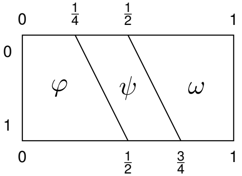

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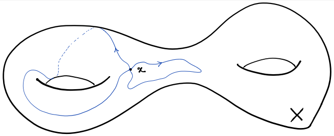

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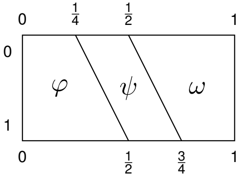

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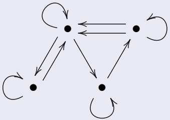

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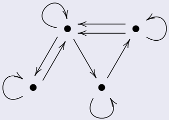

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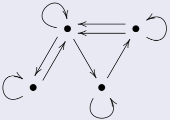

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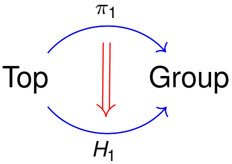

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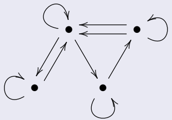

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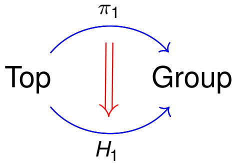

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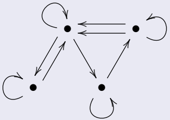

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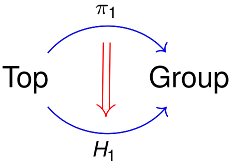

$\Longrightarrow$ 2-category (higher structure)

The three Graces

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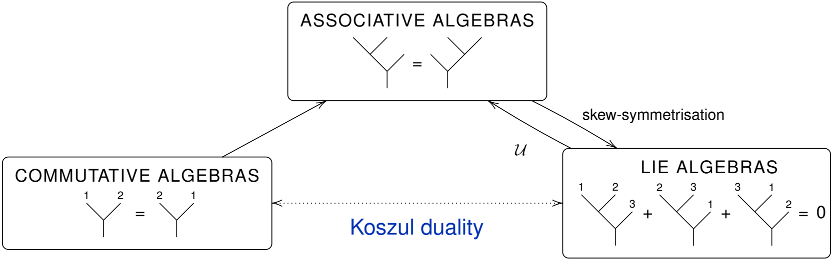

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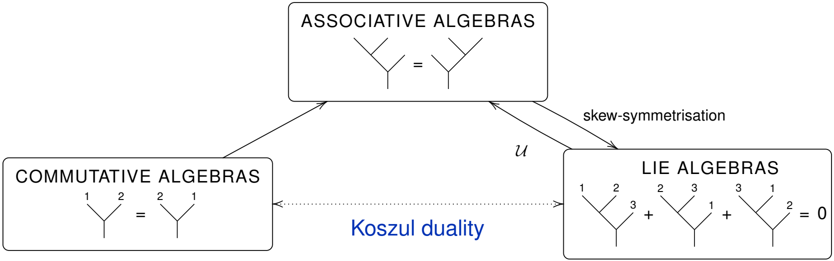

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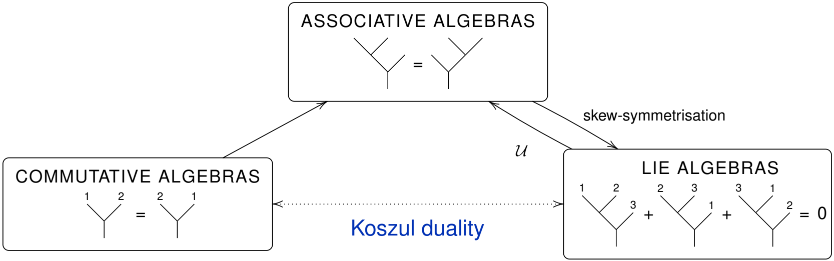

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Definition (Universal enveloping algebra)

$$\mathcal{U}\mathfrak{g} := \mathrm{T}(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y])$$

where $\mathrm{T}(\mathfrak{g}) = \bigoplus_{n \in \mathbb{N}} \mathfrak{g}^{\otimes n}$: free associative algebra (nc polynomials)

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Transfer of structure

- SIMPLEST ALGEBRAIC STRUCTURE: $\Delta : A \to A\,, \quad \Delta^2 = 0$

Proposition (Transfer of structure)

$$p : A \rightleftarrows H : i,\ pi = \mathrm{id}_H\ ,\ ip = \mathrm{id}_A$$

$$\Longrightarrow \boxed{\delta := p\Delta i\ \ ,\ \ \delta^2 = 0}$$

Proof.

$$\delta^2 = p\Delta \underbrace{ip}_{=\mathrm{id}_A} \Delta i = p \underbrace{\Delta^2}_{=0} i$$

$$= 0$$

□

- ALGEBRAIC HOMOTOPY EQUIVALENCE: Deformation retract

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

$$\mathrm{id}_A - ip = d_A h + h d_A \neq 0$$

- TRANSFERRED STRUCTURE: $\delta_1 := p\Delta i \qquad (\delta_1)^2 \neq 0$

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$(H, \delta_0 := -d_H, \delta_1, \delta_2, \ldots)$ graded vector space H endowed with a family of linear operators of degree $|\delta_n| = 2n - 1$ satisfying

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- MIXED COMPLEX OR BICOMPLEX: multicomplex s.t. $\delta_n = 0$, $n \geq 2$.

Multicomplexes are homotopy stable

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$\Longrightarrow$ Again a multicomplex, no need of further higher structure

Higher morphisms

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Proof.

$(1 - X)^{-1} = 1 + X + X^2 + X^3 + \cdots$ in $\mathbb{K}[[X]]$. □

Homotopy Transfer Theorem for multicomplexes

Theorem (Homotopy Transfer Theorem [Lapin 2001])

Given any deformation retract

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H) \qquad \mathrm{id}_A - ip = d_A h + h d_A$$

and any mixed complex (or multicomplex) structure on A, there exists a multicomplex structure on H such that i and p extend to ∞*-quasi-isomorphisms and such that h extends to an* ∞*-homotopy.*

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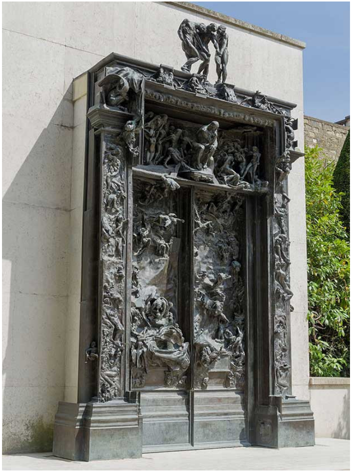
Doors of hell or pandora's box?



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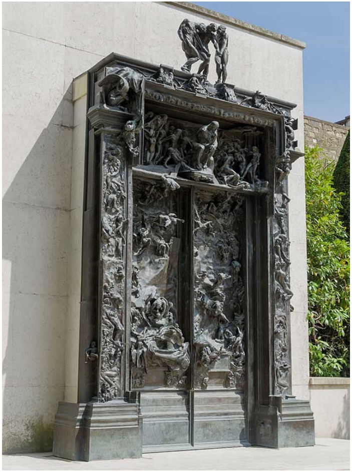

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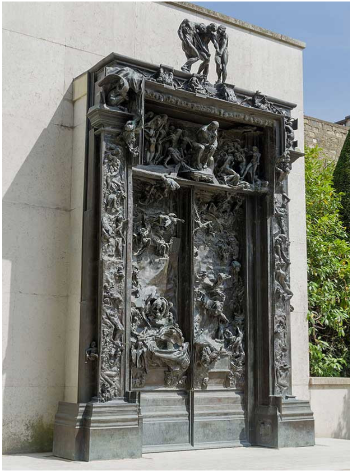

Verse-nous ton poison pour qu'il nous réconforte !
Nous voulons, tant ce feu nous brûle le cerveau,
Plonger au fond du gouffre, Enfer ou Ciel, qu'importe ?
Au fond de l'Inconnu pour trouver du nouveau !

Le voyage, Charles Baudelaire (Les fleurs du mal, 1861)

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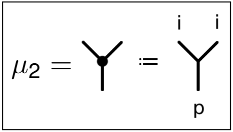

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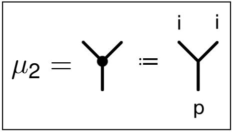
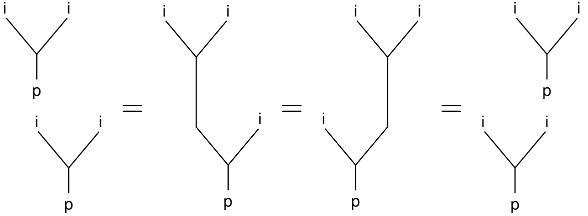


PROOF: 

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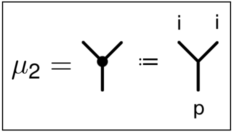
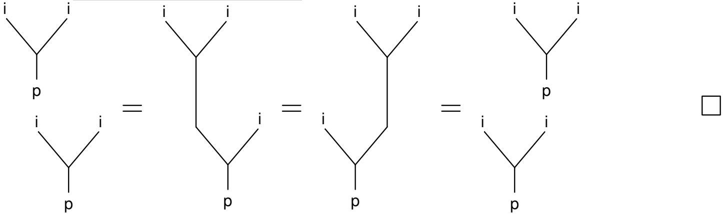


PROOF: 

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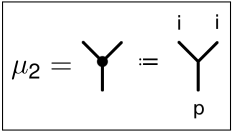

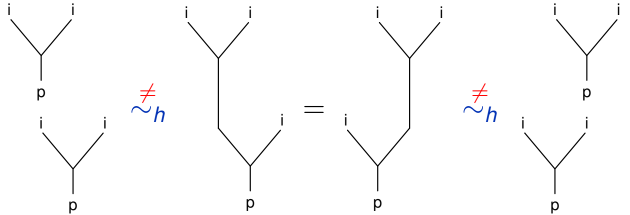

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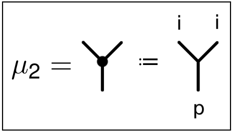

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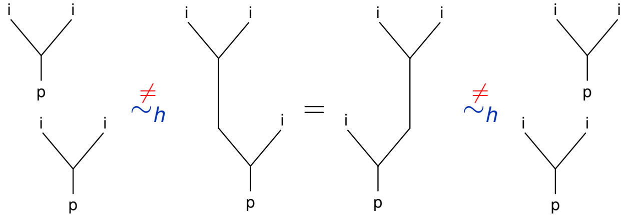

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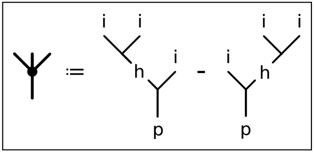

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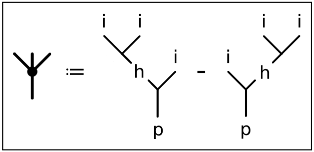

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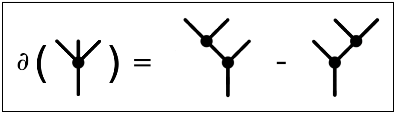

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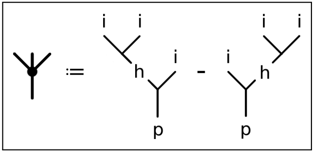

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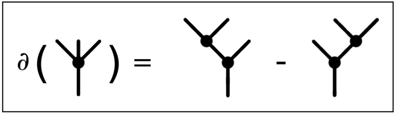

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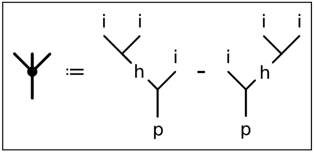

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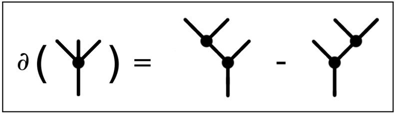

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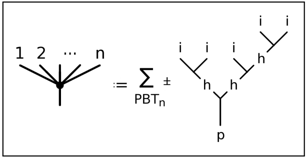
- Even higher operations:
  $\mu_n : H^{\otimes n} \to H, \ \forall n \geqslant 2$



Higher structure: homotopy associative algebras

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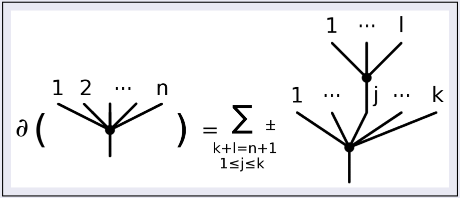
The operations $\{\mu_n\}_{n\geq 2}$ *satisfy*



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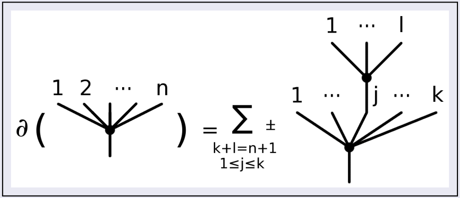

Définition (A_∞-algebras [Stasheff, 1963])

$$(H, \mu_1 = d_H, \mu_2, \mu_3, \ldots)$$

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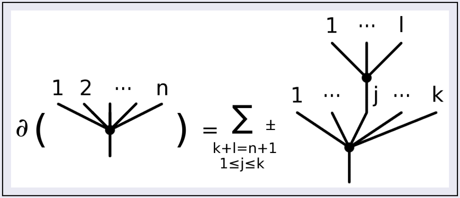

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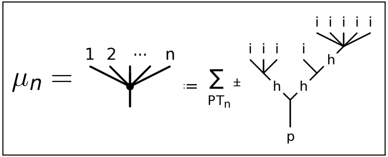
Consider 

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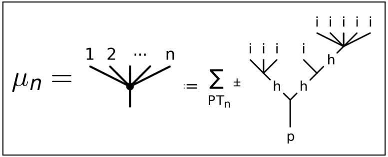

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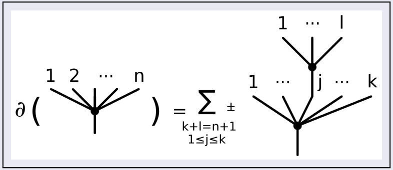

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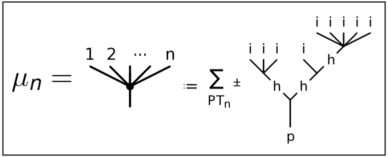

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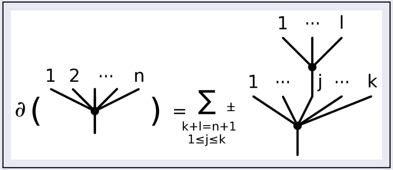

$\implies$ Again an A_∞-algebra, no need of further higher structure

Higher morphisms

$$\underbrace{(A, d_A, \nu_2, \nu_3, \ldots)}_{\text{Original structure}} \xleftarrow{\ i\ } \underbrace{(H, d_H, \mu_2, \mu_3, \ldots)}_{\text{Transferred structure}}$$

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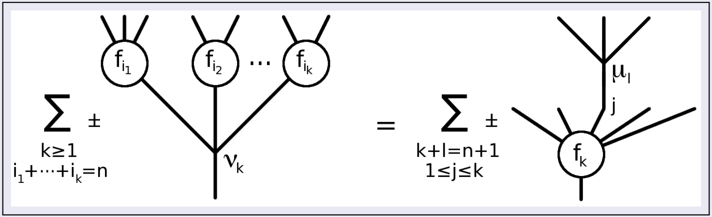

Homotopy Transfer Theorem for A_∞-algebras

A_∞-QUASI-ISOMORPHISM: $i : H \overset{\sim}{\rightsquigarrow} A$ s.t. $i_0 : H \xrightarrow{\sim} A$ homology iso.

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Given a A_∞-algebra A and a deformation retract

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- APPLICATION 2 : A_∞-categories
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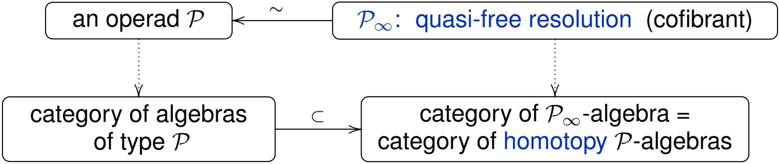
Operadic calculus [1994-now]



Operadic calculus [1994-now]

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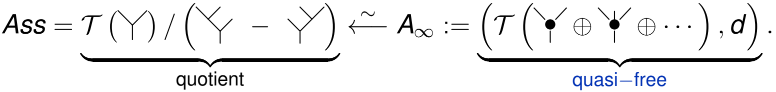

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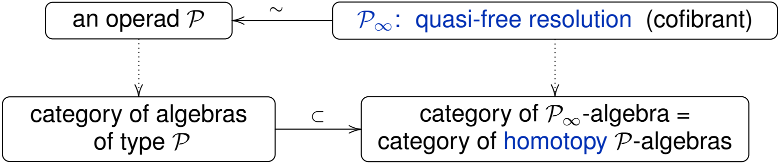

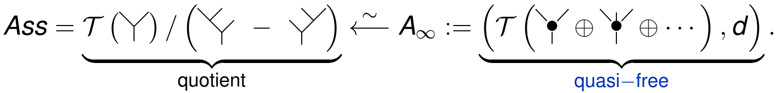

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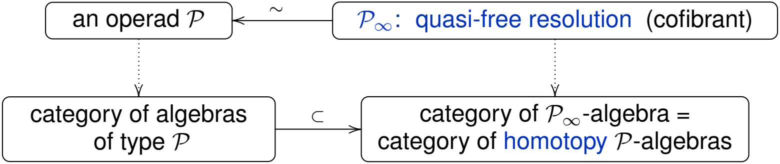

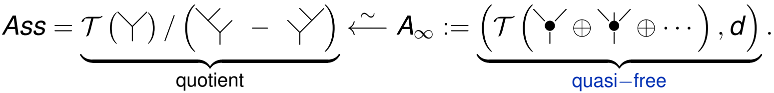

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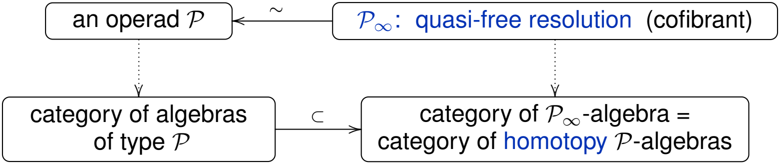

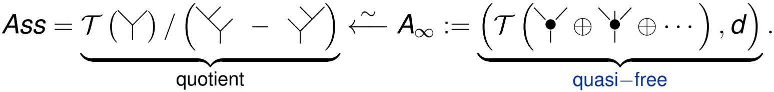

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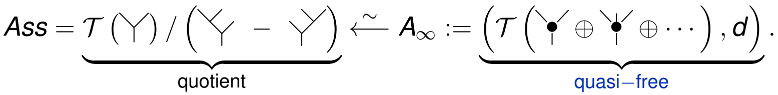

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Theorem (Mandell [2005])

The homotopy type of a topological space X is faithfully detected by the E_∞-algebra structure on its singular cochains $C^\bullet_{\text{sing}}(X, \mathbb{Z})$.

Table of contents

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$(\mathfrak{g},[\,,\,])$ complete Lie algebra $\Longrightarrow G := (\mathfrak{g}, \mathrm{BCH}, 0)$ Hausdorff group

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Any Poisson manifold (M, π) *can be quantised:*
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extends to a Lie_∞-quasi-isomorphism.

Deformation quantisation of Poisson manifolds

Theorem (Kontsevich [1997])

Any Poisson manifold (M, π) *can be quantised:*
$\exists$ *associative product* $*$ *on* $C^\infty(M)[[\hbar]]$ *s.t.* $*_0 = \cdot$ *and* $*_1 = \{\,,\}$

Proof.

- The functor: dg nilpotent Lie algebra $(\mathfrak{g}, [\,,], \mathrm{d}) \mapsto \mathrm{MC}(\mathfrak{g})/G$ sends quasi-isomorphisms to bijections.
- The Hochschild–Kostant–Rosenberg quasi-isomorphism

$$\Gamma(\Lambda^\bullet TM) \xrightarrow{\sim} Hoch^\bullet(C^\infty(M), C^\infty(M))$$

(fails to respect to Lie brackets)
extends to a Lie_∞-quasi-isomorphism.

- $\exists$ Lie_∞-quasi-isomorphism $\Leftrightarrow$ $\exists$ zig-zag of quasi-isomorphisms

□

Fundamental theorem of deformation theory

Definition (Deformation functor)

Given a dg Lie algebra $(\mathfrak{g}, [\,,\,], \mathrm{d})$:

$$\begin{array}{cccc} \mathrm{Def}_{\mathfrak{g}}: & \text{Artin rings} & \to & \text{groupoids} \\ & \mathfrak{R} \cong \mathbb{K} \oplus \mathfrak{m} & \mapsto & (\mathrm{MC}\ (\mathfrak{g} \otimes \mathfrak{m}), G) \end{array}$$

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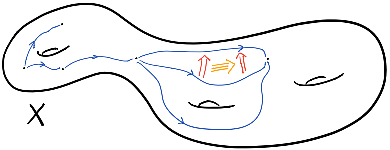

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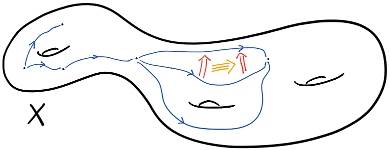

Theorem ([Pridham–Lurie 2010])

char $\mathbb{K} = 0 \implies$ *equivalence of ∞-categories:*

$$\boxed{\text{Formal moduli problems} \xleftrightarrow{\cong} \text{dg Lie algebras}}$$

Inventory “à la Prevert”

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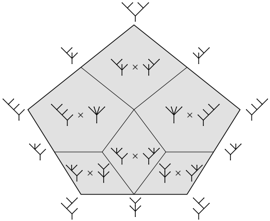

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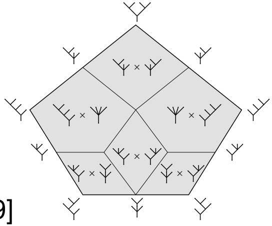

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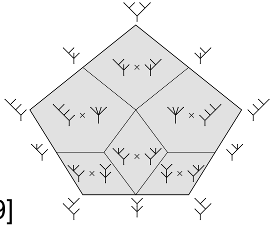

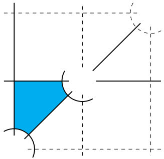

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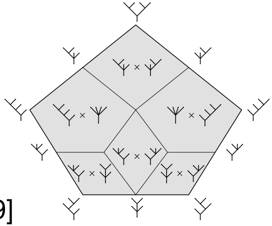

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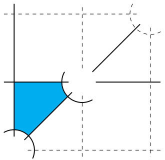

- Unification of the Grothendieck–Teichmüller group and the Givental group [Dotsenko–Shadrin–Vallette–Vaintrob 2020]

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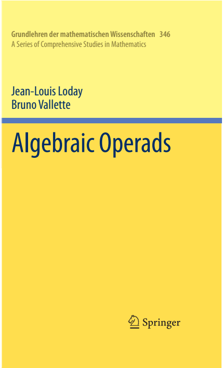
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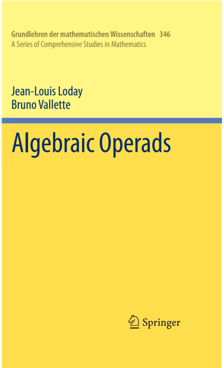

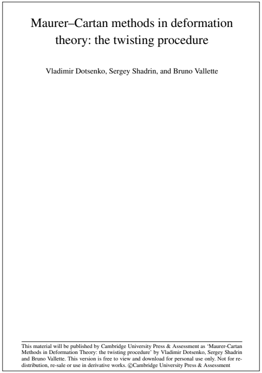
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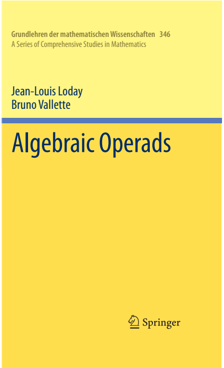

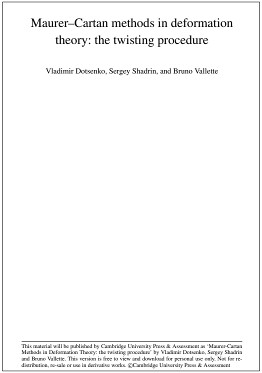
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