Mardi 28 Février : (Claudio Sibilia)

I - A_∞ algebra

- A tensor alg on V is $(T^c(V[1]), \delta)$
- A tensor morphism $F : (T^c(V[1]), \delta^V) \to (T^c(W[1]), \delta^W)$
  Let $\mu^1 : T^c(V[1]) \otimes T^c(V[1]) \longrightarrow T^c(V[1])$
- A coalgebra on V is a tensor alg $(T^c(V[1]), \delta^V)$
  s.t $\delta^V(\mu^1(a \otimes b)) = 0 \quad \forall a,b \notin k$
- A C_∞ morphism is a morphism F of coalg s.t $F(\mu^1(a \otimes b)) = 0 \quad \forall a,b \notin k$
- $(T^c(V[1]), \delta)$

Let $(V, m_\cdot^V)$, $(A, m_\cdot^A)$ $\qquad \mathrm{Hom}_{gr}^\cdot(T^c(V[1]), A)$

define $M_n : \mathrm{Hom}_{gr}^\cdot(T^c(V[1]), A)^{\otimes n} \longrightarrow (\text{———})$
via $M_1(f) := -m_1^A(f) - (-1)^{|f|} f \circ \delta^V$

$$M_n(f_1, \dots, f_n) := (-1)^n m_n^A(f_1, \dots, f_n) \circ \Delta^{n-1}$$

Prop : $(\mathrm{Hom}_{dgVect}(T^c(V[1]), A), M_\cdot)$ is an A_∞-algebra
Corr : $($ " $, \ell_\cdot)$ is an L_∞-algebra

Let $L_{V[1]} * A \subset \mathrm{Hom}(T^c(V[1]), A)$ as the set of map which vanish on shuffles

$$f(\mu^1(a \otimes b)) = 0$$

Dictionary
Let V, A as before there is a bijection
① $F : T^c(V[1]) \to T^c(A[1])$ $\quad C_\infty$-morph
② $\tau \in MC(L_{V[1]} A, \ell_\cdot)$
We define $(L_{V[1]} A)^i \subseteq L_{V[1]} A$ as the space of $f(v) = 0$ for $|v| > i$

Prop: Assume that A is unital, V finite type
- $\mathrm{Hom}(T^c(V[1]), A) \simeq A \hat{\otimes} \hat{T}(V^*[-1])$
- $L_{V[1]} A \simeq A \hat{\otimes} \hat{\mathrm{Lie}}(V^*[-1])$
- Consider $(\delta_V)^* : \mathrm{Hom}(T^c(V[1]), A) \circlearrowleft$

Then $(\delta_V)^* : L_{V[1]} A \longrightarrow L_{V[1]} A$ is well defined and

$$\mathrm{Im}\big((\delta_V)^*\big)\big((L_{V[1]} A)^0 \oplus (L_{V[1]} A)^1\big)$$
$$\cong$$
$$A \hat{\otimes} R \quad \text{where } R \text{ is a Lie ideal of } \hat{\mathrm{Lie}}(V[-1])$$

Consider π defined by $\pi : A \otimes \hat{\mathrm{Lie}}(V[-1]) \xrightarrow{\text{restri}} A \hat{\otimes} \hat{\mathrm{Lie}}((V^0)^*[-1]) \xrightarrow{\text{proj}} A \hat{\otimes} \left(\hat{\mathrm{Lie}}((V^n)^*[-1] / R \right)$

Prop: $\left(A\hat{\otimes}\left(\widehat{\mathrm{Lie}}(V^\circ)^*[-1]\right)/_{\mathbb{R}}, \ell_\cdot\right)$ is again L_∞

π is a (strict) morphism of L_∞ algebra (preserve MC elements)

Ex:
- M manifold, connected
- $A = A^\cdot_{DR}(M)$
- $V = H^+(M) = \bigoplus_{p>0} H^p(M)$

1) $\tau \in A^1_{DR}(M) \otimes \widehat{\mathrm{Lie}}(((V^*)[-1])^\circ)/_{\mathbb{R}}$

consider the trivial bundle on M with fiber $\widehat{\mathrm{Lie}}(([V^*[-1]])^\circ/_{\mathbb{R}} = L$

Consider the action of $L \circlearrowright L$ given by

$$\mathrm{Ad} : L \to L \qquad\qquad d - \tau \text{ connection}$$
$$v \mapsto [v, -]$$

2) $\tau \in MC(\,'') \Rightarrow d - \tau$ is flat

$\tau \in A \hat{\otimes} L$

Choose basis $x_1, x_2, \dots$ for V

$$\tau = \sum w_i x_j + \sum w_{ij}[x_i, x_j] + \dots$$

<u>Existence of τ</u>

Thm: Let (A, m_A) be unital C_∞ algebra s.t $H^*(A, m_1^A)$ is of finite type + connected
Fix a diagram

$$H(A, m_1^A) \underset{i}{\overset{p}{\rightleftarrows}} (A, m_1^A) \circlearrowleft h$$

st: $pi = id \qquad ip - id = dh + hd$

By HTT we have $\underset{\overset{\uparrow}{C_\infty}}{i_\cdot} : (H(A, m_1^A), m_\cdot^!) \to (A, m_\cdot^A)$

Set $V \simeq H^+(A, m_1^A)$

- $(V, m_\cdot^V)$ C_∞ , $m_\cdot^V := m^!|_V$
- $i_\cdot : (V, m_\cdot^V) \longrightarrow (A, m_1^A)$

$$\Downarrow$$

$$F : T^c(V[1]) \to T^c(A[1])$$

By dictionary $\tau \in MC(A \hat{\otimes} \widehat{\mathrm{Lie}}(V^*[-1]), \ell_\cdot)$

By π (strict morphism)

$\pi(\tau) \in M(A \hat{\otimes} \widehat{\mathrm{Lie}}(V^*[-1])^\circ/_{\mathbb{R}}, \ell_\cdot)$

$\pi(\tau) = \sum w_i x_i + \sum w_{ij}[x_i, x_j] + \dots$

<u>Simplicial manifold</u> $\qquad x_i, \dots, x_j$ basis of V

$$M_\cdot = \begin{matrix} M_2 \\ \downarrow\uparrow\downarrow\uparrow\downarrow \\ M_1 \\ \downarrow\uparrow\downarrow \\ M_0 \end{matrix} \qquad \text{Daniel: } SL_{n-1} \otimes A_{DR}(M_{n-1})$$