Why Higher Structures?

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Homotopy theory

$\rightarrow$ Classification of topological spaces

- Strong equivalence : up to homeomorphisms no
- Weak equivalence : up to homotopy equivalence
  "continuous deformation without cutting" yes

METHODS: find a set of faithful invariants

- $H_\bullet(X), H^\bullet(X)$: homology and cohomology groups.
- $\pi_\bullet(X)$: homotopy groups.

$\rightarrow$ invariants but not faithful

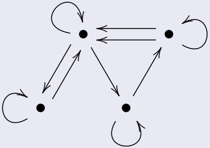
Category theory

→ **Notion of a category** [Eilenberg–MacLane, 1942]

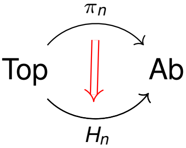
Définition (Category)

OBJECTS+ARROWS:

"monoid with many base points"



- PURPOSE 1: encode the functoriality of the invariants.
- PURPOSE 2: compare the invariants.
  $\Longrightarrow$ 2-categorical (higher category)



Classical algebraic structures

- $(C^\bullet_{\text{sing}}(X, \mathbb{Z}), \cup, \mathrm{d})$: singular cochains with the cup product.

differential graded **associative algebra**

- $(H^\bullet_{\text{sing}}(X, \mathbb{Z}), \overline{\cup})$: singular cohomology with the cup product.

graded **commutative algebra**

HEURISTIC REASON: $\exists \cup_1 : C^\bullet_{\text{sing}}(X, \mathbb{Z})^{\otimes 2} \to C^\bullet_{\text{sing}}(X, \mathbb{Z})$ s.t.

$$\mathrm{d} \circ \cup_1 + \cup_1 \circ (\mathrm{d} \otimes \mathrm{id}) + \cup_1 \circ (\mathrm{id} \otimes \mathrm{d}) = \cup - \cup^{(12)}.$$

- $(\pi_{\bullet+1}(X), [\,,])$: homotopy groups with the Whitehead bracket.

graded **Lie algebra**

None is a faithful invariant of the homotopy type.
$\to$ Need to consider higher structures.

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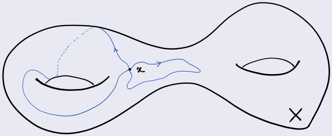
Loop spaces

Definition (Loop space)

$\Omega(X, x) := \{f\colon [0,1] \to X \mid f \text{ continuous}, f(0) = f(1) = x\}$

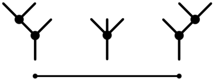


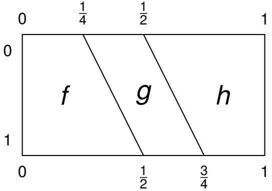
CONCATENATION PRODUCT: $\varphi \star \psi(t) := \begin{cases} \varphi(2t)\,, & \text{for } 0 \leqslant t \leqslant \frac{1}{2}\,, \\ \psi(2t-1)\,, & \text{for } \frac{1}{2} \leqslant t \leqslant 1\,. \end{cases}$

$\to$ is $\star$ associative?

no: $(\varphi \star \psi) \star \omega \neq \varphi \star (\psi \star \omega)$.

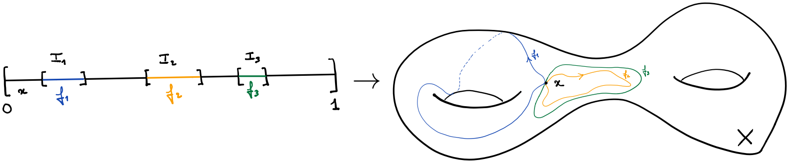
but: $(\varphi \star \psi) \star \omega \sim \varphi \star (\psi \star \omega)$.





Higher operations

$\rightarrow$ **More operations:** configurations of intervals in the unit interval



- LEFT-HAND SIDE: operations acting naturally on loop spaces.

$$\mathcal{D}_1(n) := \left\{ I_1, \ldots, I_n \text{ intervals of } [0,1] \mid \mathring{I}_k \cap \mathring{I}_l = \emptyset,\ 1 \leqslant k < l \leqslant n \right\}$$

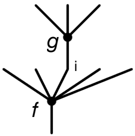
- RIGHT-HAND SIDE: all the operations acting on $Y = \Omega X$.

$$\mathrm{End}_Y(n) := \mathrm{Top}(Y^{\times n}, Y)$$

- ACTION: $\mathcal{D}_1 \to \mathrm{End}_Y$

Operad

$\rightarrow$ **Algebraic structure** on End_Y :



SEQUENTIAL AXIOM: $$(\lambda \circ_i \mu) \circ_{i-1+j} \nu = \lambda \circ_i (\mu \circ_j \nu)$$

for $\lambda \in \mathrm{End}_Y(l), \mu \in \mathrm{End}_Y(m), 1 \leqslant i \leqslant l, 1 \leqslant j \leqslant m.$

PARALLEL AXIOM: $$(\lambda \circ_i \mu) \circ_{k-1+m} \nu = (\lambda \circ_k \nu) \circ_i \mu$$

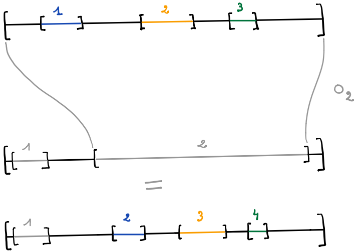
for $\lambda \in \mathrm{End}_Y(l), \mu \in \mathrm{End}_Y(m), 1 \leqslant i < k \leqslant l.$

Definition (Operad)

- **Collection :** $\{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ of $\mathbb{S}_n$-modules
- **Compositions :** $\circ_i : \mathcal{P}(n) \times \mathcal{P}(m) \to \mathcal{P}(n+m-1)$ satisfying the sequential and the parallel axioms.

Algebra over an operad

$\rightarrow$ **Same operadic structure** on $\mathcal{D}_1$:



Definition (Algebra over an operad)

Structure of a $\mathcal{P}$-algebra : morphism of operads $\mathcal{P} \rightarrow \mathrm{End}_Y$.

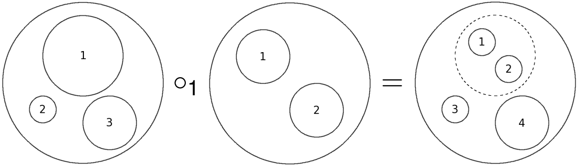
EXAMPLE: ΩX is a $\mathcal{D}_1$-algebra.

$\rightarrow$ Definitions hold in any symmetric monoidal category.
$\rightarrow$ "Multilinear" representation theory: $\mathcal{P}(n) \rightarrow \mathrm{Hom}(Y^{\times n}, Y)$.

Recognition principle

Definition (Little d-discs operad)

$$\mathcal{D}_d(n) := \left\{ D_1, \ldots, D_n \ d\text{-discs of } D^d \mid \mathring{D}_k \cap \mathring{D}_l = \emptyset,\ 1 \leqslant k < l \leqslant n \right\}$$



Theorem (Recognition principle [Stasheff, Boardman–Vogt, May])

$$Y\ \mathcal{D}_d\text{-algebra} \iff Y \sim \Omega^d(X)$$

$\longrightarrow$ Algebraic structure faithfully detects the homotopical form.

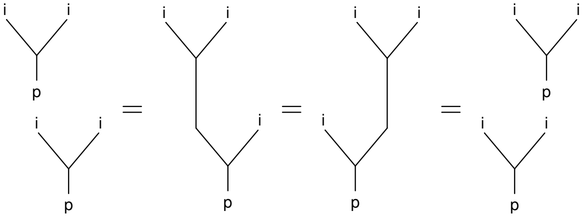
Differential graded world

$\to$ **Transfer of structure:** under isomorphisms

$$A \underset{i}{\overset{p}{\rightleftarrows}} H \ , \quad ip = \mathrm{id}_A \quad \text{et} \quad pi = \mathrm{id}_H \ .$$

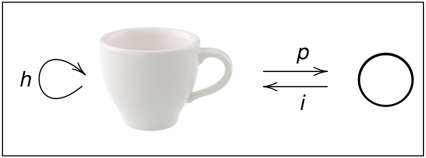
$$\mu_2 := p\,\nu\, i^{\otimes 2} \ : \ H^{\otimes 2} \to H : \ \text{associative}$$

$\nu =$  $\mu_2 =$  $:=$ 

PROOF: 

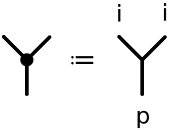
Naive homotopy transfer

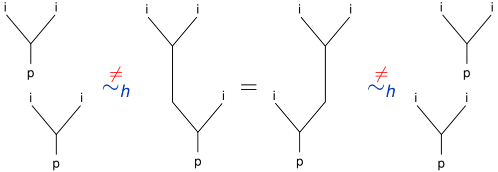
- **Algebraic homotopy equivalence:** Deformation retract



$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

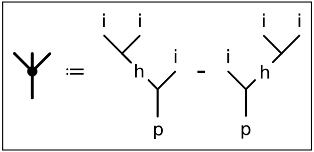
$$\mathrm{id}_A - ip = d_A h + h d_A \neq 0$$

- **Transferred product:**  : not associative



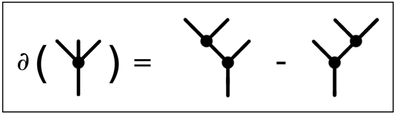
Higher homotopy transfer

- **Idea:** introduce $\mu_3 : H^{\otimes 3} \to H$



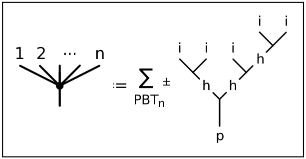
mesures the failure of associativity for μ_2 .

In Hom$(H^{\otimes 3}, H)$, it satisfies:



$\iff$ μ_3 is a homotopy for the associativity relation of μ_2.

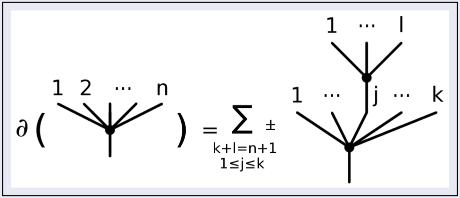
And so on:
$\mu_n : H^{\otimes n} \to H$,
for any $n \geqslant 2$.



Homotopy associative algebras

Définition (A_∞-algebras [Stasheff, 1963])

$(H, \mu_1 = d_H, \mu_2, \mu_3, \ldots)$
satisfying



Theorem (Homotopy transfer [Kadeishvili, 1982])

H deformation retract on a dg associative algebra (A, ν):

$(H, \mu_1, \mu_2, \mu_3, \ldots)$ *A_∞-algebra.*

Applications

- **Higher Massey products:** $A = C^\bullet_{\text{Sing}}(X, \mathbb{K})$, $H = H^\bullet(X, \mathbb{K})$

  $\cup \mapsto (\overline{\cup} = \mu_2, \mu_3, \mu_4, \ldots)$: (lifting of) Higher Massey products

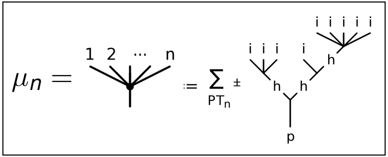
$\to$ Non-triviality of the Borromean rings, Galois cohomology, elliptic curves, etc.

- **Floer cohomology for Lagrangian submanifolds** [Fukaya–Oh–Ohta–Ono, 2009]
- A_∞**-categories:** higher version of dg category
  $\to$ Homological mirror symmetry conjecture [Kontsevich]

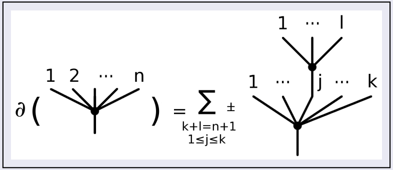
A_∞-algebras are homotopy stable

→ **Starting from an A_∞-algebra** $(A, d_A, \nu_2, \nu_3, \dots)$:

Consider



Proposition



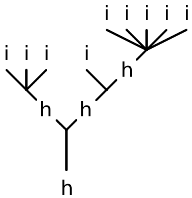
$\Longrightarrow$ Again an A_∞-algebra, no need of further higher structure.

Compatibility with the transferred structure

$$\underbrace{(A, d_A, \nu_2, \nu_3, \ldots)}_{\text{Original structure}} \xleftarrow{\ i\ } \underbrace{(H, d_H, \mu_2, \mu_3, \ldots)}_{\text{Transferred structure}}$$

- i chain map $\Longrightarrow$ $\boxed{d_A i = id_H}$
- **Question:** Does i commutes with the higher ν's and μ's?
  Anwser: not in general!
- Define $\boxed{i_1 := i}$ and consider in $\mathrm{Hom}(H^{\otimes n}, A)$, for $n \geq 2$:

$$i_n := \sum_{\mathrm{PT_n}} \pm$$



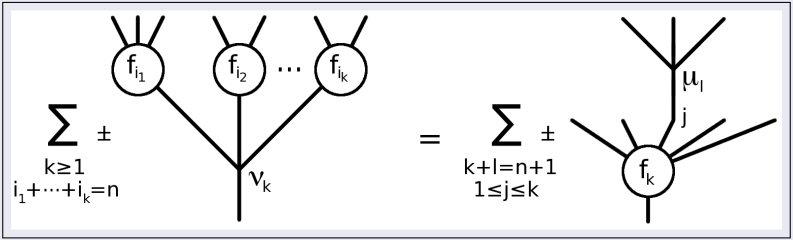
A_∞-morphism

Définition (A_∞-morphism)

$(H, d_H, \{\mu_n\}_{n\geq 2}) \rightsquigarrow (A,\, d_A,\, \{\nu_n\}_{n\geq 2})$ is a **collection** of linear maps

$$\{f_n \,:\, H^{\otimes n} \to A\}_{n\geq 1}$$

of degree $|f_n| = n - 1$ satisfying



EXAMPLE: The aforementioned $\{i_n \,:\, H^{\otimes n} \to A\}_{n\geq 1}$.

Homotopy Transfer Theorem for A_∞-algebras

A_∞-QUASI-ISOMORPHISM: $i : H \stackrel{\sim}{\rightsquigarrow} A$ s.t. $i_0 : H \xrightarrow{\sim} A$ quasi-iso.

Theorem (HTT for A_∞-algebras, Kadeshvili '82 $\to$ Markl '04)

Given a A_∞-algebra A and a deformation retract

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H) \qquad \mathrm{id}_A - ip = d_A h + h d_A\ ,$$

there exists an A_∞-algebra structure on H such that i, p, and h extend to A_∞-quasi-isomorphisms and A_∞-homotopy respectively.

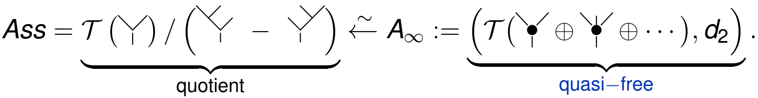
$\to$ no loss of algebro-homotopical data & explicit formulas.

Théorème (Munkholm '78, Lefèvre-Hasegawa '03)

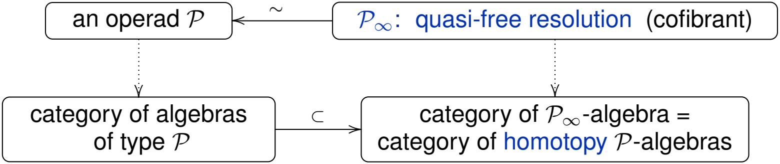
- *Every ∞-qi of A_∞-algebras admits a homotopy inverse.*
- $\boxed{\mathrm{Ho}(\text{dga alg}) := \text{dga alg}\,[qi^{-1}] \cong \infty\text{-dga alg}/\sim_h}$.

Operadic calculus

$\rightarrow$ Previous structures encoded by dg operads:



- **General method:** Koszul duality theory for dg operads



- **Examples:** Lie_∞, Com_∞, $Gerstenhaber_\infty$, $Batalin\text{-}Vilkovisky_\infty$, $LieBi_\infty$, $Frobenius_\infty$, $DoublePoisson_\infty$, etc.

- **All the results for A_∞ hold for any Koszul (pr)operads**

Further applications

- **Applications of the homotopy transfer theorem:** spectral sequences, cyclic homology (definition and Chern characters), formality statements, Feynman diagrams, etc.
- **Applications of ∞-morphisms:** cumulants in non-commutative probability.

$\to$ NOTION OF E_∞-ALGEBRA: $E_\infty \xrightarrow{\sim} \mathit{Com}$
associativity *and* commutativity relaxed up to homotopy
EXAMPLE: $(C^\bullet_{\text{sing}}(X,\mathbb{Z}), \cup, \mathrm{d})$ extends to a natural E_∞-algebra

Theorem (Mandell, 2005)

The homotopy category of (some) topological spaces embeds inside the homotopy category of E_∞-algebras under $C^\bullet_{\text{sing}}(X,\mathbb{Z})$.

$\Longrightarrow$ First family of faithful invariants!

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Classical Lie theory

- **Integration:** Lie algebra $\mathfrak{g} \xrightarrow{\exp}$ Lie Group G.
  UNIVERSAL FORMULA:

$$\begin{aligned}\mathrm{BCH}(x,y) &:= \ln\left(\exp(x).\exp(y)\right)\\ &= x + y + \tfrac{1}{2}[x,y]\tfrac{1}{12}[x,[x,y]] + \tfrac{1}{12}[y,[x,y]] + \cdots\\ &\in \widehat{Lie}(x,y) \subset \widehat{Ass}(x,y)\ .\end{aligned}$$

$\to \mathrm{BCH}(\mathrm{BCH}(x,y),z) = \mathrm{BCH}(x,\mathrm{BCH}(y,z))$ and
$\mathrm{BCH}(x,0) = x = \mathrm{BCH}(0,x)$

Definition (Hausdorff group)

$(\mathfrak{g},[\,,])$ complete Lie algebra: $G := (\mathfrak{g},\mathrm{BCH},0)$ Hausdorff group.

Deformation theory

→ **Differential graded Lie algebra:** $(\mathfrak{g}, [\,,\,], \mathrm{d})$

Definition (Maurer–Cartan elements)

$$\mathrm{MC}(\mathfrak{g}) := \left\{\alpha \in \mathfrak{g}_{-1} \mid \mathrm{d}\alpha + \tfrac{1}{2}[\alpha, \alpha] = 0\right\} \ .$$

Proposition

The Hausdorff group G of $\mathfrak{g}_0$ acts on $\mathrm{MC}(\mathfrak{g})$.

→ PHILOSOPHY: "Any deformation problem over a field of characteristic 0 can be encoded by a dg Lie algebra."

$$\begin{array}{ccc} \text{structures} & \longleftrightarrow & \mathrm{MC}(\mathfrak{g}) \\ \text{equivalence} & \longleftrightarrow & G \end{array}$$

- $(Hoch^\bullet(A, A), [\,,\,]_{Gerst})$: associative algebras up to iso.
- $(\Gamma(\Lambda^\bullet TM), [\,,\,]_{SN})$: Poisson structure up to diffeomorphisms.

Deformation quantisation of Poisson manifolds

Theorem (Kontsevich, 1997)

Any Poisson manifold (M, π) *can be quantised:* $\exists$ *associative product* $*$ *on* $C^\infty(M)[[\hbar]]$ *such that* $*_0 = \cdot$ *and* $*_1 = \{\ ,\ \}$.

PROOF:

- The functor: dg Lie algebra $(\mathfrak{g}, [\ ,\ ], \mathrm{d}) \mapsto \mathrm{MC}(\mathfrak{g})/G$ sends quasi-isomorphisms to bijections.
- The Hochschild–Kostant–Rosenberg quasi-isomorphism
$$\Gamma(\Lambda^\bullet TM) \xrightarrow{\sim} Hoch^\bullet(C^\infty(M), C^\infty(M))$$
extends to a Lie_∞-quasi-isomorphism.
- $\exists$ Lie_∞-qi $\iff$ $\exists$ zig-zag of qis.

Deformation functor

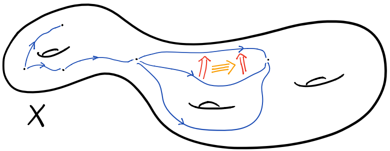
$\to$ Make this "philosophy" into a theorem.

Definition (Deformation functor)

Given a dg Lie algebra $(\mathfrak{g}, [\,,\,], d)$:

$$\begin{array}{rccc} \mathrm{Def}_{\mathfrak{g}}: & \text{Artin rings} & \to & \text{groupoids} \\ & \mathfrak{R} \cong \mathbb{K} \oplus \mathfrak{m} & \mapsto & (\mathrm{MC}(\mathfrak{g} \otimes \mathfrak{m}), G) \end{array}$$

$\to$ no enough: need a notion of an ∞-groupoid.

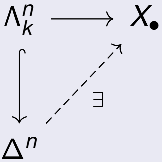


HEURISTIC: ∞-groupoid $\leftrightarrow$ topological space $\leftrightarrow$ Kan complex.

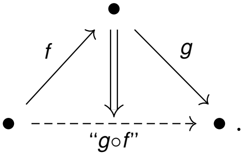
∞-groupoid

Definition (∞-groupoid)

A simplicial set $X_\bullet$ having horn fillers



objects X_0,
1-morphisms X_1,
2-morphisms X_2, etc.



∞-CATEGORY: no horn filler for $k = 0$ or $k = n$.

Fundamental theorem of deformation theory

Definition (Sullivan algebra)

Ω_n: polynomial differential forms on $|\Delta^n|$.

$\rightarrow \Omega_\bullet$: simplicial dg commutative algebra.

Theorem (Hinich, 1997)

$$\mathrm{MC}(\mathfrak{g} \otimes \mathfrak{m} \otimes \Omega_\bullet) \textit{ : } \infty\textit{-groupoid.}$$

$$\Rightarrow \text{functor: dg Lie algebras} \rightarrow \underbrace{(\text{dg Artin rings} \rightarrow \infty\text{-groupoids})}_{\text{formal moduli problems}}$$

Theorem (Pridham–Lurie, 2010)

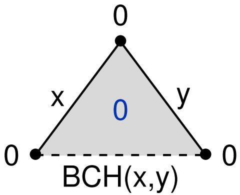
$\exists$ *equivalence of* ∞*-categories*

$$\mathit{FMP} \xleftrightarrow{\cong} \textit{dg Lie algebras}$$

Higher Lie theory

→ **Refinement of Hinich's functor:** Getzler [2009]

ALGEBRAIC ∞-GROUPOID: horn fillers are given.



→ **Higher BCH formulas [Robert-Nicoud–V. 2020].**

THANK YOU FOR YOUR ATTENTION!