Deformation theory of algebraic structures

Second Congrès Canada-France

June 2008

Data:

- An algebraic structure $\mathcal{P}$
- A vector space A

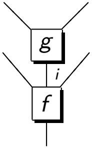
Goal: **Study the deformation theory of $\mathcal{P}$-structures on A**

- Define a dg Lie algebra $\mathfrak{g}(\mathcal{P}, A)$ [Deligne]
- Maurer-Cartan elements $\longleftrightarrow$ $\mathcal{P}$-structures on A
- $H^\bullet(\mathfrak{g}(\mathcal{P}, A))$: obstruction space for the possible deformations

Paradigm: Associative algebras

• Let A be a $\mathbb{K}$-module, consider $\mathrm{End}_A := \{\mathrm{Hom}(A^{\otimes n}, A)\}_{n\geq 0}$.
For $f \in \mathrm{Hom}(A^{\otimes n}, A)$, $g \in \mathrm{Hom}(A^{\otimes m}, A)$

$$f \star g := \sum_{i=1}^{n} \pm$$



$$= \sum_{i=1}^{n} \pm f \circ_i g.$$

Theorem (Gerstenhaber)

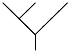
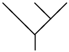
$$(f \star g) \star h - f \star (g \star h) = (f \star h) \star g - f \star (h \star g)$$

$(\mathrm{End}\, V, \star)$ *is a preLie algebra* $\Longrightarrow$ *with* $[f, g] := f \star g - (-1)^{|f|.|g|} g \star f$,

$(\mathrm{End}\, V, [\ ,\ ])$ **is a Lie algebra.**

- Associative algebra structure on A :

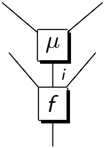
$$\mu \ : \ A^{\otimes 2} \to A \quad ,$$ 

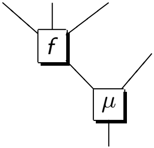
 $-$  $= 0 \quad \text{in} \quad \mathrm{Hom}(A^{\otimes 3}, A) \Longleftrightarrow \boxed{\mu \star \mu = 0} \Longleftrightarrow \boxed{[\mu, \mu] = 0}$

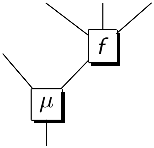
In this case,

$$d_\mu(f) := [\mu, f] \quad \text{satisfies} \quad d_\mu(f)^2 = 0.$$

Explicitly, for $f \in \mathrm{Hom}(A^{\otimes n}, A)$

$d_\mu(f) = \sum_{i=1}^{n} \pm$ 

$\pm$ 

$\pm$ 

$\in \mathrm{Hom}(A^{\otimes n+1}, A)$

$$\mathrm{Hom}(\mathbb{K}, A) \xrightarrow{d_\mu} \mathrm{Hom}(A, A) \xrightarrow{d_\mu} \mathrm{Hom}(A^{\otimes 2}, A) \xrightarrow{d_\mu} \cdots$$

$\rightarrow$ **Hochschild cohomology of "A with coefficients into itself"**

- Deformation complex of the associative algebra structure μ on A :

$$C^\bullet(\mathcal{A}ss, A) := (End_A, d_\mu, [\,,\,]) \text{ dg Lie algebra (twisted by } \mu).$$

▷ Interpretation of $\mathrm{H}^0, \mathrm{H}^1, \mathrm{H}^2$ in terms of deformations of μ
$\longrightarrow$ **Deformation Quantization** [Kontsevich]

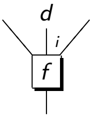
- More operations on $C^\bullet(\mathcal{A}ss, A)$:

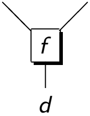
  ▷ Cup product $\cup$: associative operation

  ▷ Braces operations

$\longrightarrow$ **Deligne Conjecture**

• (A, d) dg module $\Longrightarrow$ End_A dg module

$D(f) := \sum_{i=1}^{n}$ 

$- (-1)^{|f|}$ 

$(\mathrm{End}_A, D, \star)$ is a dg preLie algebra and $(\mathrm{End}_A, D, [\ ,\ ])$ is a dg Lie algebra.

(A, d, μ) is a dg associative algebra $\Longleftrightarrow$

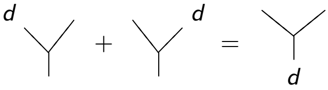
$$\boxed{D\mu + \mu \star \mu = 0} \Longleftrightarrow \boxed{D\mu + \frac{1}{2}[\mu, \mu] = 0}$$: **Maurer-Cartan equation**

General solutions:

$$\mu \in \{\mathrm{Hom}(V^{\otimes n}, V)\}_{n\geq 2}, \quad \mu_n : V^{\otimes n} \to V$$

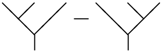
$D\mu + \mu \star \mu = 0 \Longleftrightarrow$

<u>$n=2$</u> :



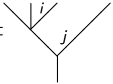
: d is a derivation for μ_2

<u>$n=3$</u> :



$= D(\mu_3)$: μ_2 is associative up to the homotopy μ_3

<u>n</u> : $\sum_{\substack{i+j=n+1\\ i,j\geq 2}} \pm$



$= D(\mu_n)$

Definition (Stasheff)

Maurer-Cartan element μ in $(\mathrm{End}_A, D, \star) \iff$
associative algebra up to homotopy or A_∞*-algebra* structure on $(A, d, \mu = \{\mu_n\}_n)$.

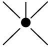
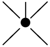
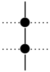
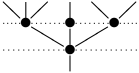
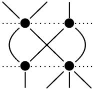
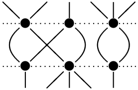
Viewpoint : An associative algebra = very particular A_∞-algebra.

Once again, $d_\mu(f) := D(f) + [\mu, f]$ satisfies $d_\mu^2 = 0$.
$\Longrightarrow$ $(\mathrm{End}_A, d_\mu, [\,,\,])$ dg Lie algebra (twisted by μ)
$\rightarrow$ **deformation complex of A_∞-algebra structures**

- Other kind of algebraic structures :
  - ▷ Lie algebras, commutative algebras, Poisson algebras, Gerstenhaber algebras, PreLie algebras, Batalin-Vilkovisky algebras, ...
  - ▷ Lie bialgebras, Frobenius bialgebras, associative bialgebras, ...

For any algebraic structure $\mathcal{P}$:

$$\begin{array}{c}
\text{dg module } A \\
\downarrow \\
\text{dg Lie algebra} \ : \ \mathfrak{g}(\mathcal{P}, A) \\
\downarrow \\
\text{Maurer-Cartan elements} = \ \text{homotopy } \mathcal{P} - \text{algebra structure on } A \\
\downarrow \\
\text{twisted dg Lie} : \ \text{deformation complex (cohomology)}
\end{array}$$

Operations				
Composition				
Monoidal category	$(\text{Vect}, \otimes)$	$(\mathbb{S}\text{-Mod}, \circ)$	$(\mathbb{S}\text{-biMod}, \boxtimes_c)$	$(\mathbb{S}\text{-biMod}, \boxtimes)$
Monoid	$R \otimes R \to R$ Associative Algebra	$\mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ **Operad**	$\mathcal{P} \boxtimes_c \mathcal{P} \to \mathcal{P}$ **Properad**	$\mathcal{P} \boxtimes \mathcal{P} \to \mathcal{P}$ **Prop**
Modules	Modules	Algebras	Bialgebras	Bialgebras
Examples		Associative, Lie, Gerstenhaber algebras	Lie, associative bialgebras	
Free monoid	Ladders (Tensor module)	Trees	Connected graphs	Graphs

Properadic homological algebra

$\mathcal{C}$ dg coproperad $\mathcal{C} \to \mathcal{C} \boxtimes_c \mathcal{C}$, $\mathcal{P}$ a dg properad $\mathcal{P} \boxtimes_c \mathcal{P} \to \mathcal{P}$.

Theorem (Merkulov-V.)

- $\mathrm{Hom}(\mathcal{C}, \mathcal{P}) := \{\mathrm{Hom}(\mathcal{C}(m,n), \mathcal{P}(m,n)\}_{m,n}$ *is a dg properad :* **convolution properad**
- $(\mathrm{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{P}), D, [\,,\,])$ *is a dg Lie algebra*

$\mathrm{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{P})$ is a "symmetric non-differential B_∞-algebra" :
$\to$ non-symmetric operations $\{x_1, \ldots, x_k\}\{y_1, \ldots, y_l\}$.

Solution of the Maurer-Cartan equation : **twisting morphism** $\mathrm{Tw}(\mathcal{C}, \mathcal{P})$.

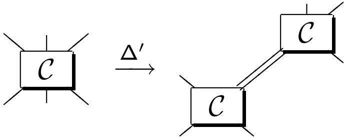
Theorem (Merkulov-V., V.)

$$\mathrm{Hom}_{dg\ properads}(\Omega(\mathcal{C}), \mathcal{P}) \cong Tw(\mathcal{C}, \mathcal{P}) \cong \mathrm{Hom}_{dg\ coproperads}(\mathcal{C}, B(\mathcal{P}))$$

PROOF. *Cobar construction of a coproperad* : $\Omega(\mathcal{C}) := (\mathcal{F}(s^{-1}\bar{\mathcal{C}}), d)$, where

- $\mathcal{F}$: free properad (graphs)
- s^{-1} : homological desuspension
- $\bar{\mathcal{C}}$: augmentation coideal

d : unique coderivation which extends



□

Theorem (V.)

Canonical bar-cobar resolution (counit of adjunction)

$$\Omega(B(\mathcal{P})) \xrightarrow{\sim} \mathcal{P}$$

Definition

- *Quadratic resolution* : $\mathcal{P}_\infty = \Omega(\mathcal{C}) \xrightarrow{\sim} \mathcal{P}$
- *Homotopy $\mathcal{P}$-algebra* : $\mathcal{P}_\infty$-algebra

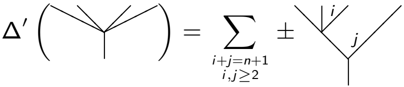
$$\begin{array}{ccc} \mathcal{P}_\infty = \Omega(\mathcal{C}) & \xrightarrow{\sim} & \mathcal{P} \\ & \searrow & \vdots \\ & & \mathrm{End}_A \end{array}$$

$$\boxed{\mathfrak{g}(\mathcal{P}, A) := (\mathrm{Hom}_{\mathbb{S}}(\mathcal{C}, \mathrm{End}_A), D, [\,,\,])}$$

$\mathrm{MC}(\mathfrak{g}(\mathcal{P}, A)) = \mathrm{Tw}(\mathcal{C}, \mathrm{End}_A) \longleftrightarrow$ homotopy $\mathcal{P}$-algebra structures on A

PARTICULAR CASE : $\alpha : \mathcal{C}_0 \to \mathrm{End}_A$, $\alpha(\mathcal{C}_n) = 0 \longleftrightarrow$ strict $\mathcal{P}$-algebra structures on A

Example : $\mathcal{P} = \mathcal{A}ss$, $\mathcal{C} = \mathcal{A}ss^*$



$$\Longrightarrow \mathfrak{g}(\mathcal{A}ss, A) \cong \mathrm{End}_A \ \ : \ \ \text{Gerstenhaber dg Lie algebra}$$

Problems:

- Well defined ?
- How to make $\mathcal{C}$ or $\mathcal{P}_\infty$ explicit?

Homotopy theory for properads

Problem : No model category structure on $\mathcal{P}$-bialgebras over a properad (no coproduct)

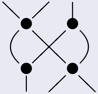
Theorem (Merkulov-V.)

The category of dg prop(erad)s has a cofibrantly generated model category structure
$\Longrightarrow \mathfrak{g}(\mathcal{P}, A)$ *well defined in the homotopy category of dg Lie algebras*

Definition

Quasi-free properad $(\mathcal{F}(X), d)$

- $\mathcal{F}(X)$ free properad
- d derivation $\longleftrightarrow d_{|X} : X \to \mathcal{F}(X)$

$d_{|X}$:  $\mapsto \sum$  : **not only two vertices**

Theorem (Merkulov-V.)

- *Cofibrant properads = rectract of quasi-free properads*
  $\Longrightarrow$ *Quasi-free properads are cofibrant*
- *Any properad admits a quasi-free resolution*

Explicit quasi-free resolution

- **Koszul duality theory**

Associative algebras [Priddy], Operads [Ginzburg-Kapranov], Properads [V.])
When $\mathcal{P} = \mathcal{F}(V)/(R)$ with $R \subset \mathcal{F}(V)^{(2)}$: quadratic relations

- $\mathcal{C} = \mathcal{P}^{!^*} =$, with $\mathcal{P}^! := \mathcal{F}(V^*)/(R^\perp)$
- $d \longleftrightarrow \Delta'_{\mathcal{C}} = \gamma^t_{\mathcal{P}^{¡}}$
- small criterion

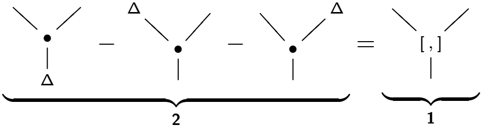
Examples

- $\mathcal{P} = \mathcal{A}ss$, $\mathcal{C} = \mathcal{A}ss^*$: Hochschild cohomology of associative algebras
- $\mathcal{P} = \mathcal{L}ie$, $\mathcal{C} = \mathcal{C}om^*$: Chevalley-Eilenberg cohomology of Lie algebras
- $\mathcal{P} = \mathcal{C}om$, $\mathcal{C} = \mathcal{L}ie^*$: Harrison cohomology of commutative algebras
- $\mathcal{P} = \mathcal{B}i\mathcal{L}ie$, $\mathcal{C} = \mathcal{F}rob^*_\diamond$: Lecomte-Roger cohomology of Lie bialgebras

• **Linear Koszul duality theory** [Galvez-Tonks-V.]

When $\mathcal{P} = \mathcal{F}(V)/(R)$ with $R \subset \mathcal{F}(V)^{(2)} \oplus \mathcal{F}(V)^{(1)}$: quadratic and linear relations

Example : Batalin-Vilkovisky algebras



• Homotopy Koszul

$(\mathcal{F}(\mathcal{C}_\infty), d)$ quasi-free properad $\Longrightarrow \mathcal{C}_\infty$ is a **homotopy coproperad** $\mathcal{C}_\infty \to \mathcal{F}(\mathcal{C}_\infty)$

When $\mathcal{P} = \mathcal{F}(V)/(R)$ with $R \subset \mathcal{F}(V)^{(\geq 2)}$

Example : $\mathcal{P} = \mathcal{B}i\mathcal{A}ss$

Theorem (Merkulov-V.)

$\mathfrak{g}(\mathcal{B}i\mathcal{A}ss, A) \cong$ *Gerstenhaber-Shack bicomplex*

Deformation L_∞-algebra

$$\begin{array}{ccc} \mathcal{P}_\infty = \mathcal{F}(\mathcal{C}_\infty) & \xrightarrow{\sim} & \mathcal{P} \\ & \searrow & \vdots \\ & & \mathrm{End}_A \end{array}$$

$$\mathfrak{g}(\mathcal{P}, A) := (\mathrm{Hom}_{\mathbb{S}}(\mathcal{C}_\infty, \mathrm{End}_A), D, \{l_n\}_n) \ : \ L_\infty - \textbf{algebra}$$

Well defined in the homotopy category of L_∞-algebras

Example : $\mathcal{P} = \mathcal{B}i\mathcal{A}ss$

Theorem (Merkulov-V.)

L_∞-algebra structure on Gerstenhaber-Shack bicomplex

Deformation theory of morphisms of properads

$$
\begin{array}{ccc}
\mathcal{P}_\infty = \mathcal{F}(\mathcal{C}_\infty) & \xrightarrow{\sim} & \mathcal{P} \\
& \searrow & \vdots \\
& & \mathcal{Q}
\end{array}
$$

$$\mathfrak{g}(\mathcal{P}, \mathcal{Q}) := (\mathrm{Hom}_{\mathbb{S}}(\mathcal{C}_\infty, \mathcal{Q}), D, \{l_n\}_n) \ : \ L_\infty - \textbf{algebra}$$

Extension of Quillen-Illusie deformation theory of morphisms

$$\mathrm{Der}(\mathcal{P}_\infty, \mathcal{Q}) \cong \mathrm{Hom}_{\mathbb{S}}(\mathcal{C}_\infty, \mathcal{Q}) \quad \cong \quad \mathrm{Hom}_{\mathcal{P}-\mathrm{bimodules}}(\underbrace{\mathbb{L}_{\mathcal{P}_\infty}}_{\textbf{Cotangent complex}}, \mathcal{Q})$$

based on the module of Kähler differentials of the properad $\mathcal{P}$.
Example : $\mathcal{P} = \mathcal{A}ss$, $\mathcal{Q} = \mathcal{P}oisson$: Invariant of knots [Vassiliev, Turchine]