Deformation theory of Cohomological Field Theories

"Operad Pop-Up" Conf. 10th August 2020

Joint work with V. Dotsenko, S. Shadrin & A. Vaintrob [arXiv:2006.01649]

State of art

- Grothendieck programme to study $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ via the geometry of $\mathcal{M}_{g,n}$
- Drinfeld: (Algebraic) definitions of "Grothendieck-Teichmüller" groups
- Getzler-Kapranov: geometry of $\overline{\mathcal{M}}_{g,n} \dashrightarrow$ general notion of a modular operad
- Kontsevich-Manin: $H_\bullet(\overline{\mathcal{M}}_{g,n})$-algebras $=:$ CohFT
  ↳ applications: Gromov-Witten invariants, quantum cohomology
- Givental-Teleman: introduction of group, based on tautological classes of $H^\bullet(\overline{\mathcal{M}}_{g,n})$ acting on CohFTs.
  ↳ classification of semi-simple CohFTs.
- Willwacher: $GRT_1/grt_1 \cong H_0$ (graph complexes)

Combinatorics of graphs with genus | Tautological classes of $H^\bullet(\overline{\mathcal{M}}_{g,n})$

↓ | ↓

Grothendieck-Teichmüller grps | Givental-Teleman group.

way to unit them?

1. Operadic deformation theory
2. Cohomological field theories [CohFT]
3. Symmetry groups

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1. Deformation theory:

"Space" A

"structures of type $\mathcal{P}$"

equivalences between them

$\exists$ dg Lie \ $\mathcal{L}_\infty$ - algebra $\mathfrak{g}_{A,\mathcal{P}}$

$MC(\mathfrak{g}_{A,\mathcal{P}}) = \{ \underbrace{d\alpha + \frac{1}{2}[\alpha,\alpha] = 0}_{} \}$ Maurer-Cartan equation

gauge group $:= (\mathfrak{g}_0; BCH, 0)$

Ex: A : (dg) vector space
$\mathcal{P}$ = assoc algebra

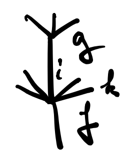
$\rightarrow$ $\mathfrak{g}_{A,\mathcal{P}}$ given by $\prod_{n\geq 1} \mathrm{Hom}(A^{\otimes n}; A)$

equipped with $f \star g := \sum_{i=1}^{k} \pm f \circ_i g$

pre Lie product

$\Downarrow$

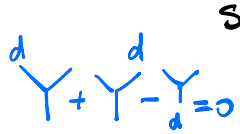
Lie bracket: $[\,;\,] := \star - \star^{(12)}$



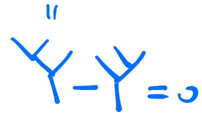
"Hochschild (co) chain complex"

with degree $1-n$: Maurer-Cartan element $\alpha =$  $\in \mathrm{Hom}(A^{\otimes 2}; A)$

st $d\alpha + \alpha \star \alpha = 0$



d derivation



actually, a Maurer-Cartan element here is

$$\underbrace{\alpha = (\alpha_n : A^{\otimes n} \to A)_{n \geq 2} \text{ st } d\alpha + \alpha \star \alpha = 0}$$

$$\Updownarrow$$

A_∞-algebra structure on A ....

↳ in fact not a surprise: comes from a conceptual deformation theory of morphisms of operads [Merkulov-V.]

→ which generalises the deformation theory of morphisms of commutative algebras of Quillen:

com. alg ⋯⋯> assoc alg ⋯⋯> operads.

↳ non-abelian derived functor of derivations

Usual input: cofibrant (usually quasi-free + triangulation) replacement.

here, we start from

$$\begin{tikzcd} & \mathrm{As} \arrow[r] & \mathrm{End}_A \\ & \underbrace{A_\infty} \arrow[u, "S"] \arrow[ur, dashed] & \end{tikzcd}$$

$$\mathfrak{g}_{A,\mathrm{As}} := \mathrm{Der}(\underset{\Omega \mathrm{As}^{¡}}{\underset{\shortparallel}{A_\infty}}, \mathrm{End}_A) \cong \mathrm{Hom}(\underbrace{\mathrm{As}^{¡}}_{\text{cooperad structure}}; \mathrm{End}_A) \cong \underbrace{\prod_{n \geq 1} \mathrm{Hom}(A^{\otimes n}; A)}_{\star_1[1] \text{ structure}}$$

cooperad structure ⟶ $\star_1[1]$ structure

Another important example [Willwacher] $H_0(\mathrm{Der}(s\mathcal{L}ie \to \mathrm{Gra})) \cong \mathfrak{grt}_1$

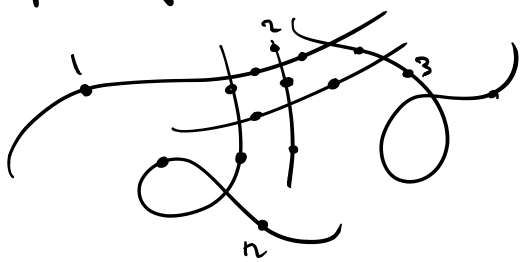
② CohFTs

$\mathcal{M}_{g,n}$: moduli space of curves of genus g with n marked points

Compactification [Deligne – Mumford – Knutsen]

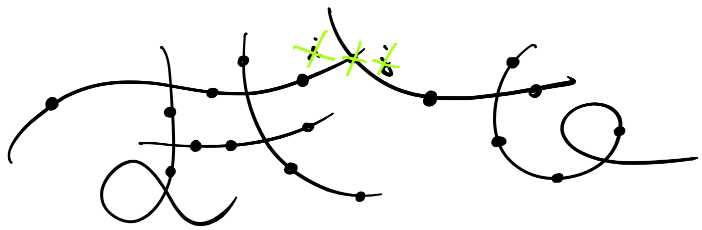
stratified by graphs ⟹ operadic in nature

$\overline{\mathcal{M}}_{g,n}$: moduli space of stable curves of genus g with n marked points

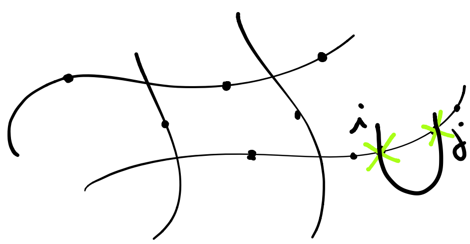


Operadic structure

$\circ_j^i$:



ξ_{ij}



satisfying ⊛

$$(\mu \circ_j^i \nu) \circ_\ell^k \omega = \begin{cases} \mu \circ_j^i (\nu \circ_\ell^k \omega) & \text{if } \ell \in \nu \\ (\mu \circ_\ell^k \omega) \circ_j^i \nu & \text{if } \ell \in \mu \end{cases}$$

$$\xi_{ij}\,\xi_{k\ell} = \xi_{k\ell}\,\xi_{ij}$$

[....]

General definition [Getzler – Kapranov '94]

A modular operad : collection $\{\mathcal{P}_g(n)\}$ " "

$\{ \ldots \}_{g,n \in \mathbb{N}^2}$

in any symmetric monoidal category

equipped with operations

$$\circ_j^i : \mathcal{P}_g(n) \otimes \mathcal{P}_{g'}(n') \longrightarrow \mathcal{P}_{g+g'}(n+n'-2) \quad : \text{"Composition"}$$

$$\xi_{ij} : \mathcal{P}_g(n) \longrightarrow \mathcal{P}_{g+1}(n-2) \quad \text{"contraction"}$$

st. $\circledast$

Ex: $\overline{\mathcal{M}} := \left(\{ \overline{\mathcal{M}}_{g,n} \}, \circ_j^i ; \xi_j^i \right) \xrightarrow[\text{monoidal functor}]{H_\bullet} H_\bullet(\overline{\mathcal{M}})$

in Top $\qquad$ in grVect

Apply the "usual" operadic method

(i) Encode the notion of a modular operad with a groupoid-colored operad $\mathcal{O}$, st. $\mathcal{O}\text{-alg} =$ modular op : $\mathcal{O} = \mathcal{T}\left(\circ_j^i \, ; \, \xi_j^i \right) / (\circledast \text{ quadratic})$

(ii) Apply the Koszul duality to it

$$\mathcal{O} \xleftarrow{k} \mathcal{O}^{i}$$

(iii) Descend one level to get good homotopical functors.

$$\mathrm{B} : \text{mod op} \underset{\perp}{\rightleftarrows} \text{(shifted) mod coop} : \Omega$$

Rk: $(\mathrm{B}\mathcal{P})^* \cong$ Feynman transform of [Getzler-Kapranov]

(iv) Apply: $\Omega \mathrm{B} \mathcal{P} \xrightarrow{\sim} \mathcal{P}$: functorial cofibrant replacement

any modular operad

Example: $(A, \langle , \rangle)$ pairing $\dashrightarrow$ $\mathrm{End}_A(g,n) := A^{\otimes n}$

o^i_j : $\pm\langle a_i, b_j\rangle \; a_1 \otimes \cdots \otimes \hat{a_i} \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes \hat{b_j} \otimes \cdots \otimes b_{n'}$

ξ^i_j : $\pm\langle a_i, a_j\rangle \; a_1 \otimes \cdots \otimes \hat{a_i} \otimes \cdots \otimes \hat{a_j} \otimes \cdots \otimes a_n$

$$\mathrm{Hom}_{\substack{\text{mod}\\ \text{op}}}\left(\Omega B H_\bullet(\overline{\mathcal{M}}), \mathrm{End}_A\right)$$

$$\Big\downarrow \wr$$

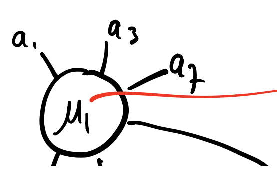
Def: CohFT_∞ structure on A : $\boxed{\mathrm{Hom}_{\substack{\text{modular}\\ \text{operads}}}\left(H_\bullet(\overline{\mathcal{M}}), \mathrm{End}_A\right)}$

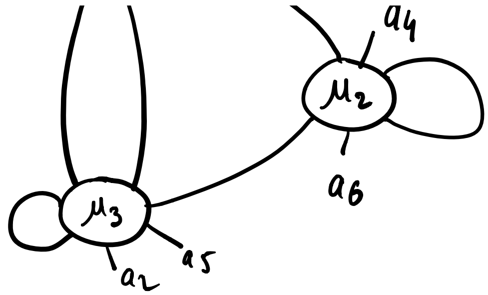
$\hookrightarrow$ **Deformation complex:**

$$\mathfrak{g}_A := \mathrm{Der}\left(\Omega B H_\bullet(\overline{\mathcal{M}}); \mathrm{End}_A\right) \cong \mathrm{Hom}\left(B H_\bullet(\overline{\mathcal{M}}); \mathrm{End}_A\right)$$

graphs, $|\text{edge}| = -1$

$$\cong \left(B H_\bullet(\overline{\mathcal{M}})\right)^*(A) \cong G\left(H^\bullet(\overline{\mathcal{M}})\right)(A)$$





Question: which algebraic structure governs the deformations of CohFTs (and more generally morphisms of modular operads)?

shifted modular operad $\xrightarrow{\text{totalisation}}$ shifted Δ-Lie algebras

$$\{\mathcal{P}_g(n)\}_{g,n} \longmapsto \prod_{g,n} \mathcal{P}_g(n)^{\mathbb{S}_n} =: \widehat{\mathcal{P}}$$

$$d \longmapsto d$$

symm Jacobi

$|\circ^i_j| = -1$ $\quad \circ^i_j \longmapsto \{\,;\,\} := \sum_{i,j} \circ^i_j$

$\Delta^2=0$

$|\xi_{ij}| = -1$ $\quad \xi_{ij} \longmapsto \Delta := \sum_{i,j} \xi_{ij}$

$=: \mathfrak{g}_{\mathcal{P}}$

shifted Δ-Lie alg

Def: [Master Equation] $\quad d\alpha + \underline{\Delta\alpha} + \frac{1}{2}\{\alpha,\alpha\} = 0$

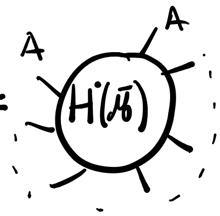
$(\partial, \{,\})$ dg Lie

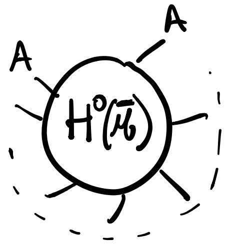
$\boxed{\partial := d + \Delta}$

$(\partial, \{,\})$ shifted dg Lie algebra

$$\boxed{\partial\alpha + \frac{1}{2}\{\alpha,\alpha\} = 0}$$

Maurer-Cartan eq

Ex: $\alpha =$  $\longleftrightarrow$ CohFT

$\alpha =$  $\longleftrightarrow$ Topological Field Theory

$\alpha =$ general case : $\boxed{\text{homotopy CohFT} = \text{CohFT}_\infty}$

↳ good homotopical properties
(no Koszul duality so far for modular operad $\Rightarrow$ no simplification yet ....)

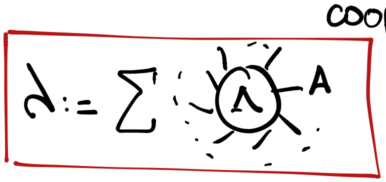
This setting $\Rightarrow$ general study of infinitesimal deformations, formal deformations, obstructions ....

Pandharipande - Zvonkine construction:

$A = H^\bullet(X, \mathbb{C})$ ← genus m curve $\Rightarrow$ carries a TFT structure α

ie $\boxed{d\alpha + \Delta\alpha + \frac{1}{2}\{\alpha,\alpha\} = 0}$

Λ : minimal class = primitive with respect to the modular cooperad structure on $H^\bullet(\overline{\mathcal{M}})$.

↳ 

Proposition: $\boxed{\underline{\alpha} + \underline{\lambda} \text{ is a CohFT ie } (d+\Delta)(\alpha+\lambda) + \tfrac{1}{2}\{\alpha+\lambda, \alpha+\lambda\} = 0}$

in this case:
Rk: infinitesimal deformation = global def.

↳ the present setting leads to generalisations of this P-Z construction.

Quantum versions: free ~~$k[[\hbar]]$~~-extensions (formal def)
↳ instead: some non-free $k[[\hbar]]$-extension.

shifted modular op /k
shifted modular op $/_{k[[\hbar]]}$

$$
\begin{tikzcd}
(\mathcal{P}, o^i_j, \xi_{ij}) \arrow[r, mapsto] \arrow[d, mapsto] & (\mathcal{P}, o^i_j, \hbar\, \xi_{ij}) \arrow[d] \\
g_{\mathcal{P}} := (\widehat{\mathcal{P}}, \{;\}, \Delta) \arrow[r, mapsto] & g^{\hbar}_{\mathcal{P}} := (g_{\mathcal{P}}; \{,\}; \hbar\Delta)
\end{tikzcd}
$$

shifted Δ-Lie-alg$/_k$
shifted Δ Lie-alg$/_{k[[\hbar]]}$

quantum Master Eq: $d\alpha + \hbar\Delta\alpha + \frac{1}{2}\{\alpha, \alpha\} = 0$

Ex: $g^{\hbar}_{A} \dashrightarrow \boxed{\text{quantum CohFT}_\infty}$ solution to.

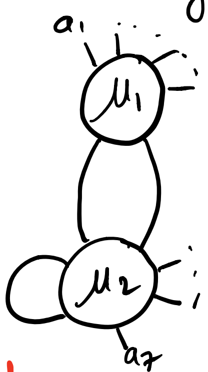
Ex: Buryak-Rossi functor (integrable hierarchies)

$\underbrace{\lambda_0 = 1, \lambda_1, \dots, \lambda_g}_{\lambda :=} \in H^\bullet(\overline{\mathcal{M}}_{g,n})$: Chern classes of the Hodge bundle

$\mathfrak{g}_A \longrightarrow \mathfrak{g}_A^{\hbar}$: morphism of shifted Δ-Lie algebras

$\Updownarrow$

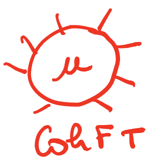
morphism of Deligne (∞)-groupoids



$\longmapsto$

$\cdot \hbar^{b_1}$ ← Betti number of the underlying graph.

in the special case

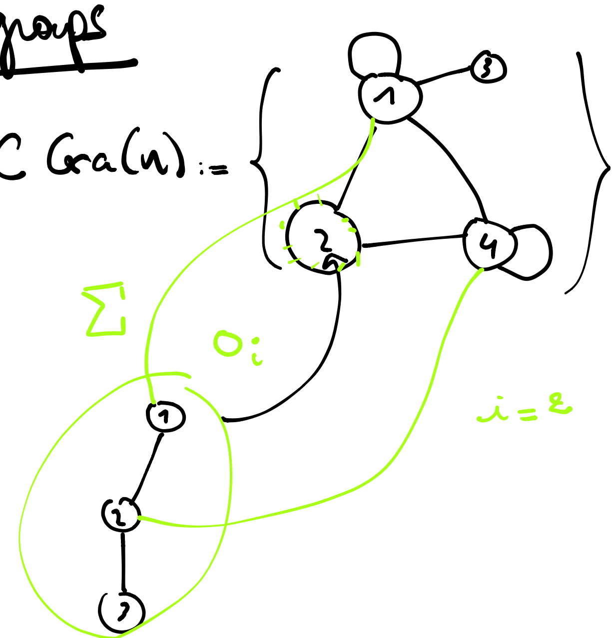


$\longmapsto$: Buryak-Rossi construction

Crucial point: $\mathbb{Z}$-grading $\longrightarrow$ $\mathbb{Z}/2\mathbb{Z}$-grading

③ Symmetry groups

Def: The operad [Kontsevich] $CGra(n) :=$

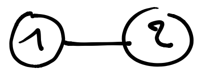


Ex: $\hat{P} = \prod P_g(n)^{S_n}$ is a natural CGra-algebra

$\hookrightarrow$ for any shifted modular operad.

$S\Delta Lie \hookrightarrow CGra$: inclusion of operads

$\Delta \longmapsto$ 

$\{\,;\,\} \longmapsto$ 

$$\begin{tikzcd} S\Delta Lie \arrow[rd] \arrow[d, hook] & \\ CGra \arrow[r] & End_{\hat{\mathfrak{g}}} \end{tikzcd}$$

Def: $\mathfrak{a}_{CGra} := \underbrace{\mathrm{Def}(s\Delta Lie \hookrightarrow CGra)}_{}$: $\underbrace{\text{dg pre Lie algebra}}_{}$

Deformation

$\boxed{\textbf{Thm}}$ $\underbrace{\boxed{(H_{even}(\mathfrak{a}_{CGra}), BCH, \circ)}}_{\text{group}} \xrightarrow{\text{morph of groups}} Aut(\underbrace{\mathcal{MC}}_{}(\mathfrak{g}^{h}))$

functorial in CGra-alg.

moduli space of MC element up to gauge equivalences

CGra-alg

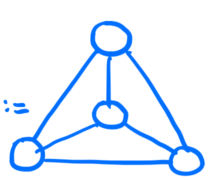
Ex: $\mathfrak{g}_P$

$(\mathfrak{g}_A)$

Def. Universal deformation group

$G :=$

extra cohomological dc

Ex: $\sigma_3 :=$  $\in G^0$

Proof: Methods of Merkulov - Willwacher + preLie calculus interpretation. $\square$

Rk: Action of σ_3 non-trivial (already on simple examples).

$\boxed{\text{Thm}}$ [Merkulov-Willwacher]

$G^0 \cong GRT_1$ $\Rightarrow$ GRT_1 acts universally on gauge equivalence classes of quantum CohFT_∞

$G^{<0} \cong 0$

$G^{>0} \cong \underbrace{???}_{\text{highly interesting and out of reach}}$

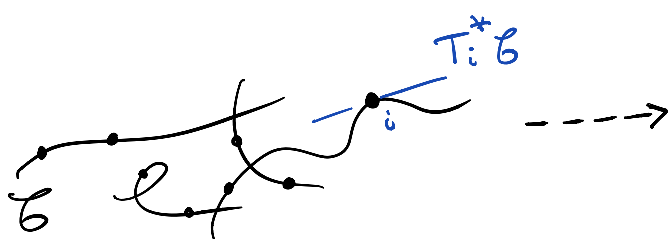
<u>Rk</u>[1]: specify at $\hbar = 1 \Leftrightarrow$ consider only genus preserving maps

$\hookrightarrow H_{even}(a^{\hbar=1}) = \mathbb{K}[8]$: one-dimensional

$\Rightarrow$ trivial theory in this case.

$\rightarrow$ So far, the theory does not see anything specific to the CGra-algebra $\underline{gA}$, i.e. uses nothing from $\underline{H^\bullet(\overline{\mathcal{M}})}$.

<u>For example</u>: Consider the line bundle $\mathbb{L}_i$:



$\psi_i := c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$

$\hookrightarrow$ characteristic class associated to the Chern class of $\mathbb{L}_i$.

<u>Def</u>: [Givental group] group of symplectomorphisms of Laurent series with coefficients in $(A, \langle ; \rangle)$:

$$\mathrm{Giv} := \{ R(z) = id + R_1 z + \cdots ;\ R^*(-z) R(z) = id \}$$

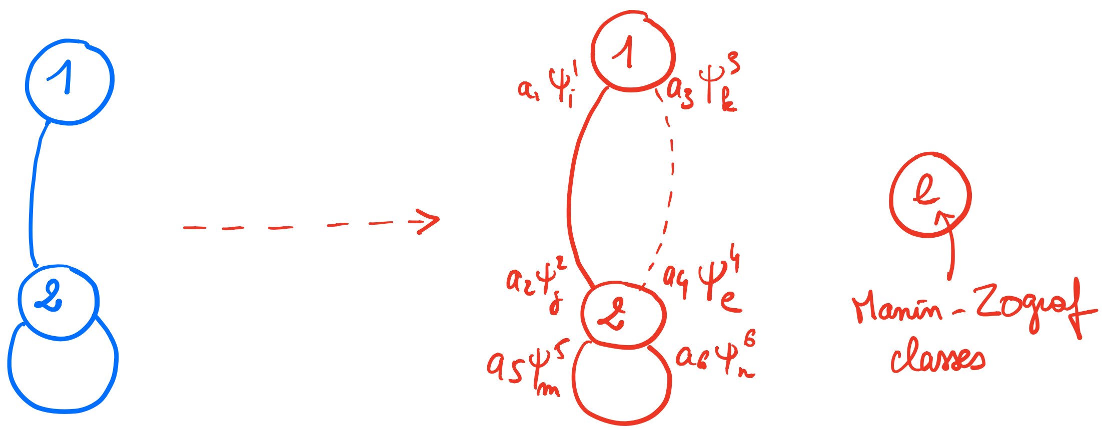
action via the ψ-classes.

CohFT_s

Idea: $\mathrm{CGra} \hookrightarrow \mathrm{TautGra}$

CGra: operad of all the natural operations acting on the totalisation of any modular operad

$\mathrm{TautGra}$: operad of all the natural operations acting on the deformation algebras $\mathfrak{g}_A = G(H^\bullet(\overline{\mathcal{M}}))(A)$



Now apply the same arguments as above (+ extra computations)

$$\begin{tikzcd}
S\Delta\mathrm{Lie} \arrow[d, hook] \arrow[dr] & \\
\mathrm{CGra} \arrow[r] \arrow[d, hook] & \mathrm{End}_{\mathfrak{g}_A} \\
\mathrm{TautGra} \arrow[ur] &
\end{tikzcd}$$

Def: $\boxed{a_{\mathrm{TautGra}} := \mathrm{Def}(S\Delta\mathrm{Lie} \hookrightarrow \mathrm{TautGra}) : \text{dg preLie algebra}}$

Thm $(H^{even}(a_{\mathrm{TautGra}}), BCH, 0) \longrightarrow \mathrm{Aut}(\mathcal{MC}(\mathfrak{g}_A^{\hbar}))$

$\overset{!!}{GGT}$

↳ functorial group action

Def: Givental-Grothendieck-Teichmüller group

$\boxed{\text{Thm}}$ $\boxed{\begin{array}{l} GRT_1 \subset \underline{GGT} \\ Giv \subset \end{array}}$

group that deserves further studies!

tentatively related to chain-level GW invariants

already $H^{even}(a^{\hbar=1}_{TautGra}) = Giv$

action = Givental action (surprizingly!)

CohFTs

not that we coinced the definitions for that! Rather upshot of the present theory.

$\Longrightarrow$ definition of Givental action on (quantum) $CohFT_\infty$.

Thank you for your attention!