EXAM

DECEMBER 19, 2018

INSTRUCTIONS. The presentation and the quality of the redaction, *the clarity and the precision of the arguments* will play an key role in the evaluation of the copy.

Any answer given without justification will receive no point.

Any handwritten notes are allowed. The rest, including electronic devises like smartphones, are prohibited.

Exercice 1 (Homotopy type of a product).

(1) Let (X, x_0) and (Y, y_0) be two pointed topological spaces. Show that, for every $n \geqslant 0$, there exists a canonical bijection
$$\pi_n\big(X \times Y, (x_0, y_0)\big) \cong \pi_n(X, x_0) \times \pi_n(Y, y_0) ,$$
which is a group isomorphism for $n \geqslant 1$.

Recall that the *real projective space* is defined as follows
$$\mathbb{P}^d\mathbb{R} := \left\{x \in \mathbb{R}^{d+1}\backslash\{0\}\right\} / \sim ,$$
where the equivalence relation is given by $x \sim \lambda.x$, for $\lambda \in \mathbb{R}^*$.

(2) Show that, for any $d \geqslant 1$, there exists a covering of the form
$$\mathbb{Z}/2\mathbb{Z} \to \mathrm{S}^d \to \mathbb{P}^d\mathbb{R} .$$

(3) Compare the homotopy groups $\pi_n\left(\mathrm{S}^2 \times \mathbb{P}^3\mathbb{R}\right)$ and $\pi_n\left(\mathrm{S}^3 \times \mathbb{P}^2\mathbb{R}\right)$, for every $n \in \mathbb{N}$.

(4) Are the two spaces $\mathrm{S}^2 \times \mathbb{P}^3\mathbb{R}$ and $\mathrm{S}^3 \times \mathbb{P}^2\mathbb{R}$ homotopy equivalent?

HINT. One can use the following computations of the homology groups with coefficients in $\mathbb{Z}/2\mathbb{Z}$:
$$\mathrm{H}_*\left(\mathrm{S}^n, \mathbb{Z}/2\mathbb{Z}\right) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} , & \text{for } * = 0, n , \\ 0 , & \text{otherwise} . \end{cases}$$
$$\mathrm{H}_*\left(\mathbb{P}^3\mathbb{R}, \mathbb{Z}/2\mathbb{Z}\right) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} , & \text{for } * = 0, 1, 2, 3 , \\ 0 , & \text{otherwise} . \end{cases}$$
$$\mathrm{H}_*\left(\mathbb{P}^2\mathbb{R}, \mathbb{Z}/2\mathbb{Z}\right) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} , & \text{for } * = 0, 1, 2 , \\ 0 , & \text{otherwise} . \end{cases}$$

(5) What does not this example say with respect to Whitehead theorem ?

Exercice 2 (Compatibility between the fiber sequence and the cofiber sequence).

(1) Describe the unit $\eta : X \to \Omega\Sigma X$ and the counit $\varepsilon : \Sigma\Omega X \to X$ of the Σ-Ω adjunction in the category of pointed topological spaces.

(2) Let $f : X \to Y$ be a pointed map. Show that the assignment
$$(x, \varphi) \mapsto \begin{cases} \varphi(2t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2} , \\ (x, 2(1-t) & \text{for } \frac{1}{2} \leqslant t \leqslant 1 , \end{cases}$$
defines a pointed map
$$\tilde{\eta} : \mathrm{Path}(f) \to \Omega\mathrm{Cone}(f) .$$

(3) Describe the adjoint map
$$\tilde{\varepsilon} : \Sigma\mathrm{Path}(f) \to \mathrm{Cone}(f)\ .$$

(4) Show that the following diagram is homotopy commutative.

$$\begin{array}{ccccccccccc}
 & & \Sigma\Omega\mathrm{Path}(f) & \xrightarrow{\Sigma\Omega f^1} & \Sigma\Omega X & \xrightarrow{\Sigma\Omega f} & \Sigma\Omega Y & \xrightarrow{\Sigma i(f)} & \Sigma\mathrm{Path}(f) & \xrightarrow{\Sigma f_1} & \Sigma X \\
 & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\tilde\varepsilon} & & \| \\
\Omega Y & \xrightarrow{i(f)} & \mathrm{Path}(f) & \xrightarrow{f^1} & X & \xrightarrow{f} & Y & \xrightarrow{f_1} & \mathrm{Cone}(f) & \xrightarrow{p(f)} & \Sigma X \\
\| & & \downarrow{\scriptstyle\tilde\eta} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} & & \\
\Omega Y & \xrightarrow{i(f)} & \Omega\mathrm{Cone}(f) & \xrightarrow{f^1} & \Omega\Sigma X & \xrightarrow{f} & \Omega\Sigma Y & \xrightarrow{f_1} & \Omega\Sigma\mathrm{Cone}(f) & &
\end{array}$$

Exercice 3 (Simplicial Eilenberg-MacLane spaces).
Let $n \geqslant 1$ be an integer and let G be a group which is supposed to be abelian when $n \geqslant 2$. An *Eilenberg–MacLane space* of type $K(G,n)$ is a connected topological space X such that

$$\pi_k(X) \cong \begin{cases} G\,, & \text{for } k = n\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

(1) Give an example of an Eilenberg–MacLane space of type $K(\mathbb{Z},1)$, of type $K(\mathbb{Z}/2\mathbb{Z},1)$, of type $K(\mathbb{Z}^2,1)$, and of type $K(\mathbb{Z},2)$.

Recall that the *nerve* $\mathrm{B}G$ of a group $(G,\cdot,\mathbb{1})$ is a simplicial set defined by
$$(\mathrm{B}G)_n \coloneqq G^{\times n}$$
with face and degeneracy maps given by
$$\begin{aligned}
d_i[g_1|\ldots|g_n] &\coloneqq \begin{cases} [g_2|\ldots|g_n]\,, & \text{for } i = 0\,, \\ [g_1|\ldots|g_i g_{i+1}|\ldots|g_n]\,, & \text{for } 1 \leqslant i \leqslant n-1\,, \\ [g_1|\ldots|g_{n-1}]\,, & \text{for } i = n\,, \end{cases} \\
s_i[g_1|\ldots|g_n] &\coloneqq [g_1|\ldots|g_i|\mathbb{1}|g_{i+1}|\ldots|g_n] \quad \text{and} \quad s_0(*) \coloneqq [\mathbb{1}]\ .
\end{aligned}$$

(2) Compute the simplicial homotopy groups of the nerve $\mathrm{B}G$.

(3) Show that $|\mathrm{B}G|$ is an Eilenberg–MacLane CW-complex of type $K(G,1)$.

To any group $(G,\cdot,\mathbb{1})$, we associated the collection of sets $(\mathrm{E}G)_n \coloneqq G^{n+1}$ endowed with the following face and degeneracy maps
$$\mathrm{d}_i(g_0,\ldots,g_n) \coloneqq (g_0,\ldots,\widehat{g_i},\ldots,g_n) \quad \text{and} \quad \mathrm{s}_i(g_0,\ldots,g_n) \coloneqq (g_0,\ldots,g_i,g_i,\ldots,g_n)\ .$$
We denote this data by $\mathrm{E}G$.

(4) Show that $\mathrm{E}G$ defines a functor from the category of groups to the category of simplicial sets.

(5) Show that $\mathrm{E}G$ admits a left adjoint and describe this latter one.

(6) Show that $\mathrm{E}G$ is contractible, that is there exist two simplicial maps $f : \mathrm{E}G \to *$ and $g : * \to \mathrm{E}G$ such that $fg \sim \mathrm{id}_*$ and $gf \sim \mathrm{id}_{\mathrm{E}G}$.

(7) Each n-simplex $(\mathrm{E}G)_n = G^{n+1}$ admits a (left) G-action by the formula
$$g.(g_0,\ldots,g_n) \coloneqq (gg_0,\ldots,gg_n)\ .$$
Show that this endows $\mathrm{E}G$ with simplicial G-module structure.

(8) We consider the orbits under this action and the induced face and degeneracy maps
$$\mathrm{E}G/G \coloneqq \left((\mathrm{E}G)_n/G = G^{n+1}/G, \bar{\mathrm{d}}_i, \bar{\mathrm{s}}_i\right)\ .$$

(9) Show that $\mathrm{E}G/G$ is a simplicial set isomorphic to the nerve $\mathrm{B}G$.

(10) Show that the canonical simplicial map $\mathrm{E}G \twoheadrightarrow \mathrm{E}G/G \cong \mathrm{B}G$ is a Kan fibration.

(11) Compute the fiber of this fibration $\mathrm{E}G \twoheadrightarrow \mathrm{B}G$.

(12) Give another proof of the fact that $|\mathrm{B}G|$ is an Eilenberg–MacLane space of type $K(G,1)$.