Université Sorbonne Paris Nord

MASTER 2 : HOMOTOPY THEORIES   JANUARY 4TH, 2022

EXAM

INSTRUCTIONS. The presentation and the quality of the redaction, *the clarity and the precision of the exposition* will play an important part in the evaluation of the copy. Any answer given without justification will receive no point. Only handwritten notes and paper documents are allowed. All electronic devices are forbidden.

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Exercise 1 (Fundamental group of SO(3)). Let us recall that the canonical euclidean space $(\mathbb{R}^3, \langle -,- \rangle)$ is isomorphic to the euclidean space

$$\mathfrak{h} := \left\{ M \in \mathrm{M}_2(\mathbb{C}) \mid M = -{}^t\overline{M} \,, \mathrm{tr}\, M = 0 \right\}$$

of skew-Hermitian matrices with vanishing trace endowed with the scalar product $< M, N > := \frac{1}{2}\mathrm{tr}\left(M\, {}^t\overline{N}\right)$, via the application $(x, y, z) \mapsto \begin{pmatrix} ix & y+iz \\ -y+iz & -ix \end{pmatrix}$. The *special unitary group* is defined by

$$\mathrm{SU}(2) := \left\{ M \in \mathrm{M}_2(\mathbb{C}) \mid M\, {}^t\overline{M} = I_2 \,, \det M = 1 \right\} ,$$

where I_2 is the identity matrix; it admits a topology induced by that of matrices.

(1) Show that the conjugaison action $(M, H) \in \mathrm{SU}(2) \times \mathfrak{h} \mapsto MHM^{-1} \in \mathfrak{h}$ defines a continuous group morphism $\Phi\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$.

(2) Compute the fundamental group of SO(3) .
HINT. One can admit that the morphism $\Phi\colon \mathrm{SU}(2) \twoheadrightarrow \mathrm{SO}(3)$ is surjectif.

(3) Is the topological space SO(3) homotopy equivalent to $S^1 \times S^2$?

(4) Show that the manifold S^2 does not admit a continuous non-vanishing tangent vector field.

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Exercise 2 (Relative CW-complex). Let (X, A) be a relative CW-complex.

(1) Let Y be a pathwise connected topological space satisfying $\pi_{n-1}(Y) \cong 0$, for any $n \geqslant 1$ such that $X \backslash A$ has an n-dimensional cell. Show that any continuous application $f : A \to Y$ extends to a continuous application $F : X \to Y$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & Y \\ X \arrow[ur, dashed, "\exists F"'] & \end{tikzcd} .$$

(2) Show that X retracts onto A when A is contractible.

(3) Show that the canonical projection $X \twoheadrightarrow X/A$ is a homotopy equivalence when A is contractible.
INDICATION. One can use the fact that $A \rightarrowtail X$ is a cofibration.

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Exercise 3 (Join). The *join* of two categories C, D is the category $\mathsf{C} \star \mathsf{D}$ whose objects

$$\mathrm{Obj}(\mathsf{C} \star \mathsf{D}) := \mathrm{Obj}(\mathsf{C}) \sqcup \mathrm{Obj}(\mathsf{D})$$

are that of C and D and whose morphisms are the following ones:

$$\mathrm{Hom}_{\mathsf{C}\star\mathsf{D}}(x, y) := \begin{cases} \mathrm{Hom}_{\mathsf{C}}(x, y)\,, & \text{for } x, y \in \mathrm{Obj}(\mathsf{C})\,, \\ \mathrm{Hom}_{\mathsf{D}}(x, y)\,, & \text{for } x, y \in \mathrm{Obj}(\mathsf{D})\,, \\ \{*\}\,, & \text{for } x \in \mathrm{Obj}(\mathsf{C})\,, y \in \mathrm{Obj}(\mathsf{D})\,, \\ \emptyset\,, & \text{for } x \in \mathrm{Obj}(\mathsf{D})\,, y \in \mathrm{Obj}(\mathsf{C})\,. \end{cases}$$

The composite of morphisms is induced by that of C and D .

(1) For any $n \in \mathbb{N}$, we consider the category $\mathsf{Cat}[n] := \{0 \to 1 \to \cdots \to n\}$ associated to the poset $[n] := \{0 < 1 < \cdots < n\}$. Make explicit $\mathsf{Cat}[m] \star \mathsf{Cat}[n]$.

We denote by Cat the category of small categories and we fix D a small category. We consider the category D/Cat of *categories under* D whose objects are the functors $\mathrm{F} : \mathsf{D} \to \mathsf{C}$ from D to a category C and whose morphisms between $\mathrm{F} : \mathsf{D} \to \mathsf{C}$ and $\mathrm{F}' : \mathsf{D} \to \mathsf{C}'$ are the functors $\mathrm{G} : \mathsf{C} \to \mathsf{C}'$ such that $\mathrm{G} \circ \mathrm{F} = \mathrm{F}'$. Since the category D is a full sub-category of the join $\mathsf{C} \star \mathsf{D}$, this latter one is canonically a category under D under the inclusion $\mathsf{D} \to \mathsf{C} \star \mathsf{D}$.

(2) Show that the functor $- \star \mathsf{D} : \mathsf{C} \mapsto \mathsf{C} \star \mathsf{D}$ is left adjoint to the the functor $\mathsf{Cone} : (\mathrm{F} : \mathsf{D} \to \mathsf{C}) \mapsto \mathsf{Cone}(\mathrm{F})$:

$$- \star \mathsf{D} \ : \ \mathsf{Cat} \ \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \ \mathsf{D}/\mathsf{Cat} \ : \ \mathsf{Cone} \ .$$

The *join* $\mathfrak{X} \star \mathfrak{Y}$ of two simplicial sets $\mathfrak{X}, \mathfrak{Y}$ is defined by

$$(\mathfrak{X} \star \mathfrak{Y})_n := \bigsqcup_{p+1+q=n} X_p \times Y_q \ ,$$

under the convention $X_{-1} = Y_{-1} := \{*\}$, and it is endowed with face and degeneracy applications which send $(x, y) \in X_p \times Y_q$ respectively as follows:

$$d_i(x,y) := \begin{cases} \left(d_i^{\mathfrak{X}}(x), y\right) \ , & \text{for } i \leqslant p \ , \\ \left(x, d_{i-p-1}^{\mathfrak{Y}}(y)\right) \ , & \text{for } i \geqslant p+1 \ , \end{cases} \quad \text{and} \quad s_i(x,y) := \begin{cases} \left(s_i^{\mathfrak{X}}(x), y\right) \ , & \text{for } i \leqslant p \ , \\ \left(x, s_{i-p-1}^{\mathfrak{Y}}(y)\right) \ , & \text{for } i \geqslant p+1 \ , \end{cases}$$

under the convention $d_0^{\mathfrak{X}} : X_0 \twoheadrightarrow X_{-1} = \{*\}$ and $d_0^{\mathfrak{Y}} : Y_0 \twoheadrightarrow Y_{-1} = \{*\}$.

(3) Show that the join $\mathfrak{X} \star \mathfrak{Y}$ forms a simplicial set.

(4) Represent $|\Delta^1 \star \Delta^1|$ et $|\Delta^0 \star \partial\Delta^2|$. These two geometric realisations correspond to the geometric realisations of which classical simplicial sets?

(5) Let $m, n \in \mathbb{N}$ be two non-negative integers. Make explicit $\Delta^m \star \Delta^n$.

Let $\mathfrak{X}$ be a simplicial set. For any simplicial set $\mathfrak{Y}$, we consider the canonical inclusion $\mathfrak{X} \hookrightarrow \mathfrak{X} \star \mathfrak{Y}$ explicitly given by $X_n \to X_n \times \{*\}$. The category $\mathfrak{X}/\mathsf{sSet}$ of *simplicial sets under* $\mathfrak{X}$ admits for objects the morphisms of simplicial sets $\mathrm{f} : \mathfrak{X} \to \mathfrak{Y}$ and for morphisms between $\mathrm{f} : \mathfrak{X} \to \mathfrak{Y}$ and $\mathrm{f}' : \mathfrak{X} \to \mathfrak{Y}'$ the morphisms $\mathrm{g} : \mathfrak{Y} \to \mathfrak{Y}'$ such that $\mathrm{g} \circ \mathrm{f} = \mathrm{f}'$.

(6) Show that the functors

$$\begin{array}{rccc} - \star \mathfrak{X} \ : & \mathsf{sSet} & \to & \mathfrak{X}/\mathsf{sSet} \\ & \mathfrak{Y} & \mapsto & \mathfrak{Y} \star \mathfrak{X} \end{array} \quad \text{and} \quad \begin{array}{rccc} \mathfrak{X} \star - \ : & \mathsf{sSet} & \to & \mathfrak{X}/\mathsf{sSet} \\ & \mathfrak{Y} & \mapsto & \mathfrak{X} \star \mathfrak{Y} \end{array}$$

are cocontinuous, that is they preserve colimits.

(7) Show that the two properties mentioned in the questions (5) and (6) characterise the notion of the join of simplicial sets.

(8) Show that the join $\mathfrak{X} \star \mathfrak{Y}$ of two ∞-categories $\mathfrak{X}$ and $\mathfrak{Y}$ is again an ∞-category.

(9) Let C and D be two small categories. Show that there exists an isomorphism of simplicial sets

$$\mathfrak{N}(\mathsf{C} \star \mathsf{D}) \cong (\mathfrak{N}\mathsf{C}) \star (\mathfrak{N}\mathsf{D}) \ ,$$

where $\mathfrak{N} : \mathsf{Cat} \to \mathsf{sSet}$ is the nerve functor.

(10) Let $\mathfrak{Y}$ be a simplicial set. Show that the functor $- \star \mathfrak{Y} : \mathfrak{X} \mapsto \mathfrak{X} \star \mathfrak{Y}$ admits a right adjoint functor, that will be denoted by $\mathfrak{Cone} : (\mathrm{f} : \mathfrak{Y} \to \mathfrak{X}) \mapsto \mathfrak{Cone}(\mathrm{f})$:

$$- \star \mathfrak{Y} \ : \ \mathsf{sSet} \ \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \ \mathfrak{Y}/\mathsf{sSet} \ : \ \mathfrak{Cone} \ ,$$

and make explicit its n-simplicies $\mathfrak{Cone}(\mathrm{f})_n$.

(11) Let $\mathrm{F} : \mathsf{D} \to \mathsf{E}$ be a functor. Show that there exists an isomorphism of simplicial sets

$$\mathfrak{N}\mathsf{Cone}(\mathrm{F}) \cong \mathfrak{Cone}(\mathfrak{N}\mathrm{F}) \ .$$