A BORDISM APPROACH TO STRING TOPOLOGY

DAVID CHATAUR
LABORATOIRE PAINLEVÉ
UNIVERSITÉ DE LILLE1
FRANCE

Abstract. Using intersection theory in the context of Hilbert manifolds and geometric homology we show how to recover the main operations of string topology built by M. Chas and D. Sullivan. We also study and build an action of the homology of reduced Sullivan's chord diagrams on the singular homology of free loop spaces, extending previous results of R. Cohen and V. Godin and unifying part of the rich algebraic structure of string topology as an algebra over the partial Prop of these reduced chord diagrams.

1. Introduction

The study of spaces of maps is an important and difficult task of algebraic topology. Our aim is to study the algebraic structure of the homology of free loop spaces. The discovery by M. Chas and D. Sullivan of a Batalin Vilkovisky structure on the singular homology of free loop spaces [4] had a deep impact on the subject and has revealed a part of a very rich algebraic structure [5], [7]. The BV-structure consists of:
- A loop product $- \bullet -$, which is commutative and associative; it can be understood as an attempt to perform intersection theory of families of closed curves,
- A loop bracket $\{-,-\}$, which comes from a symmetrization of the homotopy controling the graded commutativity of the loop product.
- An operator Δ coming from the action of S^1 on the free loop space (S^1 acts by reparametrization of the loops).
M. Chas and D. Sullivan use (in [4]) "classical intersection theory of chains in a manifold". This structure has also been defined in a purely homotopical way by R. Cohen and J. Jones using a ring spectrum structure on a Thom spectrum of a virtual bundle over free loop spaces [8]. As discovered by S. Voronov [42], it comes in fact from a geometric operadic action of the cacti operad. Very recently J. Klein in [25] has extended the homotopy theoretic approach of R. Cohen and J. Jones to Poincaré duality spaces using A_∞-ring spectrum technology. For more algebraic approaches we refer the reader to [12] and [34].

Date: November 29, 2004.
The author was partially supported by DGESIC grant PB97-0202.

In this paper we adopt a different approach to string topology, namely we use a geometric version of singular homology introduced by M. Jakob [21]. We show how it is possible to define Gysin morphisms, exterior products and intersection type products (such as the loop product of M. Chas and D. Sullivan) in the setting of Hilbert manifolds. Let us point out that three different types of free loop spaces are used in the mathematical literature:
- Spaces of continuous loops ([4] for example),
- Spaces of smooth loops, which are Fréchet manifolds but not Hilbert manifolds ([3] for some details),
- Spaces of Sobolev class of loops [26],
These three spaces are very different from an analytical point of view, but they are homotopy equivalent. For our purpose, we deal with Hilbert manifolds in order to have a nice theory of transversality. The last space of maps is the one we use.

In order to perform such intersection theory we recall in section 2 what is known about transversality in the context of Hilbert manifolds. We also describe the manifold structure of free loop spaces used by W. Klingenberg [26] in order to study closed geodesics on Riemannian manifolds. The cornerstone of all the constructions of the next sections will be the "string pull-back", also used by R. Cohen and J. Jones [8, diagram 1.1].

Section 3 is devoted to the introduction and main properties of geometric homology. This theory is based upon bordism classes of singular manifolds. In this setting families of closed strings in M, which are families parametrized by smooth oriented compact manifolds, have a clear homological meaning. Of particular interest and crucial importance for applications to the topology of free loop spaces is the construction of an explicit Gysin morphism for Hilbert manifolds in the context of geometric homology (section 3.3). This construction does not use any Thom spaces and is based on the construction of pull-backs for Hilbert manifolds. Such approach seems completely new in this context. We give a comparison result of this geometric approach with more classical ones, this result has an interpretation in terms of bivariant theories over a topological category $\mathcal{M}_{Hilb}$ (I do not dare to write that this a kind of "motivic" interpretation).
We want to point out that all the constructions performed in this section work with a generalized homology theory h_* under some mild assumptions ([21]):
- the associated cohomology theory h^* is multiplicative,
- h_* satisfies the infinite wedge axioms.

In section 4 the operator Δ, the loop product, the loop bracket, the intersection morphism and the string bracket are defined and studied using the techniques introduced in section 2 and 3. This section is also concerned with string topology operations, these operations are parametrized by the topological space of reduced Sullivan's Chord diagrams $\overline{\mathcal{CF}^{\mu}_{p,q}(g)}$, which is closely related to the combinatorics of Riemann surfaces of genus g, with p-incoming boundary components and q-outgoing. A. Voronov (private communication) suggested introducing these spaces of diagrams because they form a partial Prop and the cacti appear as a sub-operad. Pushing the work of R. Cohen and V. Godin on the action of Sullivan's chord diagrams on free loop spaces further we prove our main result on free loop spaces:

Theorem: *Let $\mathcal{L}M$ be the free loop space over a compact oriented d dimensional manifold M. For $q > 0$ there exist morphisms:*

$$\mu_{n,p,q}(g) : H_n(\overline{\mathcal{CF}^{\mu}_{p,q}(g)}) \to Hom(H_*(\mathcal{L}M^{\times p}), H_{*+\chi(\Sigma).d+n}(\mathcal{L}M^{\times q})).$$

where $\chi(\Sigma) = 2 - 2g - p - q$.

Moreover as these operations are compatible with the partial gluing of reduced Sullivan's diagrams, $H_*(\mathcal{L}M)$ appear as an algebra over this partial Prop (when the homology is taken over a field). As a corollary one recovers the structure of Frobenius algebra on $H_{*+d}(\mathcal{L}M)$, build in [7], the BV-structure of M. Chas and D. Sullivan [4]. This theorem captures all the known algebraic structure of string topology under a same framework as an algebra over a partial Prop. Let us point out that this structure makes also appear new operations, for example we have a structure of coalgebra over the BV-operad.

Acknowledgments: I would like to thank Andy Baker and Martin Jakob for their help about infinite dimensional manifolds and geometric homology. Discussions about string topology with Ralph Cohen, Yves Félix, Véronique Godin and Jean-Claude Thomas and remarks of Muriel Livernet were also very useful. I am also thankful to Sasha Voronov for a careful reading of a preliminary draft of this paper and for many suggestions. I am very grateful to the algebraic topology group of Barcelona (CRM, UAB and UB) for organizing a seminar on this subject, this was my main motivation for writing this paper. Finally, I warmly thanks the "Centre de Recerca de Matematica de Barcelona" for its hospitality.

2. Infinite dimensional manifolds

2.1. **Recollections on Hilbert manifolds.** This section is mainly expository. We review basic facts about Hilbert manifolds. We refer to [31] (see also [30] for a general introduction to infinite dimensional manifolds). Moreover all the manifolds we consider in this paper are Hausdorff and second countable (we need these conditions in order to consider partitions of unity).

2.1.1. *Differential calculus.* Let E and F be two normed vector spaces and $L(E, F)$ the set of continuous linear maps of E into F. There is no difficulty extending the notion of differentiability to the infinite dimensional context. Let $f : E \to F$ be a continuous map we say that f is differentiable at $x \in E$ if there exists a continuous linear map λ of E into F such that:

$$(f(x+v) - f(x) - \lambda v)/|v|$$

tends to 0 as $v \in E$ tends to 0. One can define differentials, C^∞ morphisms, diffeomorphisms and so on.

2.1.2. *Hilbert manifolds.* A topological space X is a manifold modelled on a separable Hilbert space E if there exists an atlas $\{U_i, \phi_i\}_{i\in I}$ such that:
i) each U_i is an open set of X and $X = \bigcup_{i\in I} U_i$,
ii) $\phi_i : U_i \to E$ is an homeomorphism to an open subset of E,
iii) $\phi_i\phi_j^{-1}$ is a diffeomorphism whenever $U_i \bigcap U_j$ is not empty.

In this context one can define submanifolds, immersions, embeddings and submersions ([31], chapter 2). Moreover we only consider C^∞-Hilbert manifolds.

2.1.3. *Fredholm maps.* A continuous linear map $\lambda : E \to F$ between two normed vector spaces is a Fredholm operator if both $ker\lambda$ and $coker\lambda$ are finite dimensional. The index of the operator λ is defined by

$$index(\lambda) = dim(ker\lambda) - dim(coker\lambda).$$

The set of Fredholm operators is a topological subspace of the space of linear morphisms $L(E, F)$. The index is locally constant hence continuous.
A smooth map $f : X \to Y$ between two Hilbert manifolds is a Fredholm map if for each $x \in X$ the linear map

$$df_x : T_xX \longrightarrow T_{f(x)}Y$$

is a Fredholm operator. Recall that the index of a Fredholm map

$$index : X \to \mathbb{Z}$$

$$x \mapsto dim(kerdf_x) - dim(cokerdf_x)$$

is locally constant ([1, Prop 1.5]).

2.1.4. *Oriented morphisms.* Consider a Fredholm map $f : X \to Y$ between two Hilbert manifolds and assume for simplicity that X is connected. Let df be the map of vector bundles $df : TX \to f^*TY$. We suppose that $dim(Kerdf_x)$ and $dim(cokerdf_x)$ are constant, in this case one can define the vector bundles $Kerdf$ and $Cokerdf$. Two typical examples of this situation are given by immersions of finite codimension and submersions of finite rank. Now we want to define a virtual bundle associated to the map f. In order to proceed we suppose that the bundle $Cokerdf$ admits a finite trivialization i.e. we suppose that X has a finite open cover that trivializes the bundle; in this case we also say that $Cokerdf$ is of finite type ([31], chapter III,

section 5). By proposition 5.3 of [31] there exists a vector bundle α over X such that $df \oplus \alpha$ is trivializable.
The bundles $Kerdf$, $Cokerdf$ and α are finite dimensional of dimension k, c and l respectively. We define a virtual bundle of rank $index f = k - c$ over X:

$$V(f) = Kerdf - Cokerdf = Kerdf \oplus \alpha.$$

The bundle $V(f)$ defines a class $[V(f)]$ in the K-theory group $KO(X)$.

2.1.5. **Definition.** *A proper Fredholm map f is oriented if:*
i) $dim(kerdf)$ is constant over the connected components of X,
ii) the bundle $Cokerdf$ is of finite type,
under these hypotheses one can define a virtual bundle over X denoted by $V(f)$ and we suppose that it is oriented.

If we specialize to the case of a closed embedding it is Fredholm if it is proper by a result of S. Smale [38]. Hence the embedding is oriented if and only if a normal bundle of the embedding is finite dimensional, of finite type and oriented.

For convenience we have considered above the notion of oriented morphisms with respect to singular cohomology but we could have chosen to work by considering orientations with respect to a generalized cohomology theory. In this case let us consider E a ring spectrum and $E^*(-)$ the generalized cohomology theory associated to this spectrum. By adding trivial vector bundles over X to the bundle $Kerdf \oplus \alpha$ we define a Thom spectrum $TV(f)$. Let F_x be a fiber of $V(f)$ over x and $j_x : F_x \to V(f)$ the inclusion of this fiber. The map j_x induces a map of spectra $J_x : S^0 \to TV(f)$ from the sphere spectrum to the Thom spectrum of $V(f)$. We say that f is E-oriented is there is an element $u \in \tilde{E}^0(TV(f))$ such that the morphism

$$J_x^* : \tilde{E}^0(TV(f)) \to \tilde{E}^0(S^0) = \pi_0(E)$$

sends u to ± 1 for every $x \in X$.
For example, if we work with KO, a morphism f is KO-oriented if and only if the bundle $V(f)$ admits a Spin structure.

Example : Let us describe the following typical example of an oriented morphism. We consider a smooth map $f : M \to N$ between two compact differentiable manifolds and we suppose that this map is an embedding. Moreover we suppose that the normal bundle of this embedding is oriented. Now let X be a Hilbert manifold and $p : X \to N$ be a submersion or a smooth fiber bundle, then the pull-back of f along p :

$$f^* : X \times_N M \to X$$

is an embedding of Hilbert manifolds.The pull-back of the normal bundle of f gives a normal bundle for f^*, this vector bundle is clearly of finite type. Hence f^* is an oriented morphism.

2.1.6. *Partitions of unity.* A very nice feature of Hilbert manifolds is the existence of partitions of unity (see [31, chapter II,3] for a proof). Partitions of unity do not always exist in the case of Banach manifolds and other types of infinite dimensional manifolds. As a consequence mimicking techniques used in the finite dimensional case, one can prove that every continuous map

$$f : P \to X$$

from a finite dimensional manifold P to a Hilbert manifold X is homotopic to a smooth one. And we can also smooth homotopies.

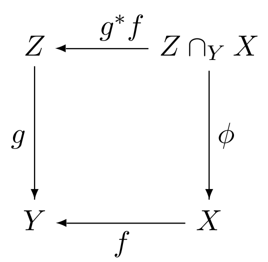
2.2. **Transversality.** We follow the techniques developed by A. Baker and C. Özel in [1] in order to deal with transversality in the infinite dimensional context.

2.2.1. *Transversal maps.* Let $f : X \to Y$ and $g : Z \to Y$ be smooth maps between two Hilbert manifolds. Then they are transverse at $y \in Y$ if

$$df(T_xX) + dg(T_zZ) = T_yY$$

for every pair (x, z) such that $f(x) = g(z) = y$. The maps are transverse if they are tranverse at any point $y \in Imf \cap Img$. It is also useful to notice that f and g are transverse if and only if $f \times g$ is transverse to $\Delta : Y \to Y \times Y$.

2.2.2. *Pull-backs.* Let us recall the main results about pull-backs of Hilbert manifolds. We consider the following diagram:



where Z is a finite dimensional manifold and $f : X \to Y$ is an oriented map.

Using an infinite dimensional version of the implicit function theorem [31, Chapter I,5], one can prove the following result:

2.2.3. **Proposition.** [1, prop. 1.17] *If the map*

$$f : X \to Y$$

is an oriented morphism and

$$g : Z \to Y$$

is a smooth map transverse to f, then $Z \cap_Y X$ is a smooth manifold and the pull-back map:

$$g^* f : Z \cap_Y X \longrightarrow Z$$

is an oriented morphism.

For the generecity of transversal maps in the context of Hilbert manifolds we refer the reader to [1, ch. 2]. The following result will be implicitly used throughout the proofs of this paper. It guarantees the existence of transversal maps up to homotopy.

2.2.4. **Theorem.** [1, Th. 2.1, 2.4] *Let*

$$f : X \to Y$$

be an oriented morphism and let

$$g : Z \to Y$$

be a smooth map from a finite dimensional manifold Z. Then there exists a smooth homotopy :

$$H : Z \times I \to Y$$

such that $H(.,0) = g$ and $H(.,1) = g'$ is transverse to f.

2.3. **Free loop spaces.** In what follows we identify S^1 with $\mathbb{R}/\mathbb{Z}$ an element of S^1 is denoted by t.
If we want to do intersection theory with spaces of closed curves, we need to consider them as smooth manifolds. Following [3, Chapter 3], one can consider the space $C^\infty(S^1, M)$ of all piecewise smooth curves as a Frechet manifold. But we prefer to enlarge this space and to consider for $k \in \mathbb{N}^{\geq 1}$ the spaces $\mathbf{H}^k(S^1, M)$ of $\mathbf{H}^k$ curves.
They are spaces of Sobolev maps whose $k-th$-derivative is square integrable with respect to the unique flat metric on S^1 inducing the measure dt on S^1. These spaces have the advantage of being Hilbert manifolds as we will see below for the case $k = 1$. With this choice we can apply all the techniques described in the sections 2.1 and 2.2. Let us also notice that:

$$C^\infty(S^1, M) = \bigcap_k \mathbf{H}^k(S^1, M).$$

2.3.1. **Proposition.** *The spaces* $\mathbf{H}^k(S^1, M)$ *are homotopy equivalent to the Frechet manifold* $C^\infty(S^1, M)$ *and to the space of continuous maps* $C^0(S^1, M)$ *equipped with the compact open topology.*

Proof This certainly depends on a deeper result about mapping spaces, but let us give a geometrical proof. P_kM denotes for $k = 0$ the space of continuous maps from $[0, 1]$ to M i.e. continuous paths with the compact open topology, for $k = \infty$ the space of piecewise C^∞-paths in M and of $\mathbf{H}^k$-paths in M otherwise. We have:

$$P_\infty M \subset \ldots \subset P_1 M \subset P_0 M$$

all these inclusions are continuous and as these spaces are all contractible the inclusions are homotopy equivalences.
Let $ev_{0,1} : P_kM \to M \times M$ be the evaluation map defined by $ev_{0,1}(c) = (c(0), c(1))$. Free loop spaces are obtained by taking the pull-back of this map along the diagonal map $\Delta : M \to M \times M$. The result follows from the fact that every map $ev_{0,1}$ is a fibration, pull-backs along fibrations being homotopy invariant. □

2.3.2. *Manifold structure.* Let us fix $k = 1$. In order to define a Hilbert manifold structure on free loop spaces we follow W. Klingenberg's approach [26].
Let M be a Riemannian manifold of dimension d. We set

$$\mathcal{L}M = \mathbf{H}^1(S^1, M).$$

The manifold $\mathcal{L}M$ is formed by the continuous curves $\gamma : S^1 \to M$ of class $\mathbf{H}^1$. The basic model is the Hilbert space $\mathcal{L}\mathbb{R}^d = \mathbf{H}^1(S^1, \mathbb{R}^d)$. The space $\mathcal{L}\mathbb{R}^d$ can be viewed as the completion of the space $C^\infty_p(S^1, \mathbb{R}^d)$ of piecewise differentiable curves with respect to the norm $\| - \|_1$. This norm is defined via the scalar product:

$$< \gamma, \gamma' >_1 = \int \gamma(t) \diamond \gamma'(t) dt + \int \delta\gamma(t) \diamond \delta\gamma'(t) dt,$$

where $\diamond$ is the canonical scalar product of $\mathbb{R}^d$ and $\delta\gamma$ is the derivative of γ with respect to the parameter t. As S^1 is 1-dimensional, we notice that by Sobolev's embedding theorem elements of $\mathcal{L}\mathbb{R}^d$ can be represented by continuous curves.
Let us describe an atlas $\{U_\gamma, u_\gamma\}$ of $\mathcal{L}M$. This atlas is modeled on spaces $\Gamma(\gamma^*TM)$ of $\mathbf{H}^1$-sections of pull-back bundles along $\gamma \in C^\infty_p(S^1, M)$ a piecewise differentiable curve in M (notice that $C^\infty_p(S^1, M) \subset \mathbf{H}^1(S^1, M)$) hence

consider the pullback:

$$\begin{array}{ccc} \gamma^*TM & \longrightarrow & TM \\ \downarrow & & \downarrow \pi_M \\ S^1 & \underset{\gamma}{\longrightarrow} & M. \end{array}$$

Choose a smooth Riemannian metric on M. Let $exp : U \to M$ be the smooth exponential mapping of this Riemannian metric, defined on a suitable open neighborhood U of the zero section. We may assume that U is chosen such that

$$(\pi_M, exp) : U \to M \times M$$

is a smooth diffeomorphism onto an open neighborhood V of the diagonal. Now define

$$U_\gamma = \{g \in \mathbf{H}^1(S^1, M) : \forall t \in S^1, (\gamma(t), g(t)) \in V\},$$

$$u_\gamma : U_\gamma \to \Gamma(\gamma^*(TM)),$$

$$u_\gamma(g)(t) = (t, exp^{-1}_{\gamma(t)}(g(t))) = (t, ((\pi_M, exp)^{-1} \circ (\gamma, g))(t)).$$

Then u_γ is a bijective mapping from U_γ onto the set

$$\{s \in \Gamma(\gamma^*TM) : s(S^1) \subset f^*U\}.$$

The set $u_\gamma(U_\gamma)$ is open in the Hilbert space $\Gamma(\gamma^*TM)$. Hence an atlas is given by $\{U_\gamma, u_\gamma\}_\gamma$.
In fact, the manifold structure on $\mathcal{L}M$ does not depend on a choice of a particular Riemannian metric on M.

2.3.3. *The tangent bundle.* Let $TM \to M$ be the tangent bundle of M. The tangent bundle of $\mathcal{L}M$ denoted by $T\mathcal{L}M$ can be identified with $\mathcal{L}TM$, this is an infinite dimensional vector bundle where each fiber is isomorphic to the Hilbert space $\mathcal{L}\mathbb{R}^d$. Let $\gamma \in \mathcal{L}M$ we have:

$$T\mathcal{L}M_\gamma = \Gamma(\gamma^*TM),$$

where $\Gamma(\gamma^*TM)$ is the space of sections of the pullback of the tangent bundle of M along γ (this is the space of $\mathbf{H}^1$ vector fields along the curve γ). A trivialization φ of γ^*TM induces an isomorphism:

$$T\mathcal{L}M_\gamma \cong \mathcal{L}\mathbb{R}^d.$$

The tangent bundle of $\mathcal{L}M$ has been studied in [9] and [35].
The manifold $\mathcal{L}M$ has a natural Riemannian metric. The scalar product on $T\mathcal{L}M_\gamma \cong \mathcal{L}\mathbb{R}^d$ comes from $< -, - >_1$.

2.3.4. *The S^1-action.* The circle acts on $\mathcal{L}M$:

$$\Theta : S^1 \times \mathcal{L}M \longrightarrow \mathcal{L}M$$

by reparametrization:

$$\Theta(\theta, \gamma) : t \mapsto \gamma(t + \theta).$$

Of course this action is not free. If we fix an element $\theta \in S^1$ the map $\Theta(\theta, .) : \mathcal{L}M \to \mathcal{L}M$ is smooth whereas the map Θ is continuous but not differentiable (see [26]). In fact the infinite dimensional Lie group $Diff^+(S^1)$ of orientation preserving diffeomorphims of S^1 acts continuously on $\mathcal{L}M$ by reparametrization.

2.3.5. *The string pullback.* Let us consider the evaluation map

$$ev_0 : \mathcal{L}M \to M$$

$$\gamma \mapsto \gamma(0).$$

This is a smooth fiber bundle. As the map $ev_0 \times ev_0$ is transverse to the diagonal map Δ (because $ev_0 \times ev_0$ is a smooth fiber bundle), we can form the *string pull-back* [8, (1.1)]:

$$\begin{array}{ccc}
\mathcal{L}M \times \mathcal{L}M & \xleftarrow{\widetilde{\Delta}} & \mathcal{L}M \cap_M \mathcal{L}M \\
\Big\downarrow{\scriptstyle ev_0 \times ev_0} & & \Big\downarrow{\scriptstyle ev} \\
M \times M & \xleftarrow{\Delta} & M,
\end{array}$$

by transversality this is a diagram of Hilbert manifolds. We have:

$$\mathcal{L}M \cap_M \mathcal{L}M = \{(\alpha, \beta) \in \mathcal{L}M \times \mathcal{L}M : \alpha(0) = \beta(0)\}.$$

This is the space of composable loops. The map

$$\widetilde{\Delta} : \mathcal{L}M \cap_M \mathcal{L}M \to \mathcal{L}M \times \mathcal{L}M$$

is a closed embedding of codimension d.
As the normal bundle $\nu_{\tilde{\Delta}}$ is the pull-back of ν_Δ and as this last one is isomorphic to TM , we deduce that $\tilde{\Delta}$ is an oriented morphism.

2.3.6. **Definition.** *A family of closed strings in M is a smooth map*

$$f : P \to \mathcal{L}M$$

from a compact oriented manifold P.

2.3.7. **Proposition.** *The family* $P \times Q \stackrel{f\times g}{\to} \mathcal{L}M \times \mathcal{L}M$ *is transverse to* $\widetilde{\Delta}$ *if and only if* $ev_0 f$ *and* $ev_0 g$ *are transverse in* M.

Now we suppose that (P, f) and (Q, g) are two families of dimensions p and q respectively. Moreover we suppose that they are such that $f \times g$ is transverse to $\widetilde{\Delta}$. We denote by $P * Q$ the pullback:

$$\begin{array}{ccc} P \times Q & \longleftarrow & P * Q \\ {\scriptstyle f\times g}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathcal{L}M \times \mathcal{L}M & \stackrel{\widetilde{\Delta}}{\longleftarrow} & \mathcal{L}M \cap_M \mathcal{L}M. \end{array}$$

Then $P*Q$ is a compact oriented submanifold of $P \times Q$ of dimension $p+q-d$.

2.3.8. *Intersection of families of closed strings.* Let us define the map:

$$\Upsilon : \mathcal{L}M \cap_M \mathcal{L}M \longrightarrow \mathcal{L}M.$$

Let (α, β) be an element of $\mathcal{L}M \cap_M \mathcal{L}M$ then $\Upsilon(\alpha, \beta)$ is the curve defined by:
$\Upsilon(\alpha, \beta)(t) = \alpha(2t)$ if $t \in [0, 1/2]$
$\Upsilon(\alpha, \beta)(t) = \beta(2t - 1)$ if $t \in [1/2, 1]$.
We notice that this map is well defined because we compose piecewise differential curves, hence no "dampening" constructions are needed as in [8, remark about construction (1.2)].
The construction of Υ comes from the co-H-space structure of S^1 i.e. the pinching map:

$$S^1 \longrightarrow S^1 \vee S^1.$$

Now consider two families of closed strings (P, f) and (Q, g), by deforming f and g one can produce a new family of closed strings $(P * Q, \Upsilon\psi)$ in M.

3. Geometric homology theories

As R. Thom proved it is not possible in general to represent singular homology classes of a topological space X by singular maps i.e continuous maps:

$$f : P \longrightarrow X$$

from a closed oriented manifold to X. But, M. Jakob in [21], [22] proves that if we add cohomological information to the map f (a singular cohomology class of P), then Steenrod's realizability problem with this additional cohomological data has an affirmative answer. In these two papers he develops a geometric version of homology. The geometric version seems to be very nice to dealing with Gysin morphisms, intersection products and so on.

All the constructions we give below and also their applications to string topology work out for more general homology theories: various theories of bordism, topological K-theory for example. We refer the reader to [21], [22] and [23] for the definitions of these geometric theories.

3.1. **An alternative description of singular homology.**

3.1.1. *Geometric cycles.* Let X be a topological space. A geometric cycle is a triple (P, a, f) where:

$$f : P \longrightarrow X$$

is a continuous map from a smooth compact connected oriented manifold P to X (i.e a singular manifold over X, P is without boundary, and $a \in H^*(P, \mathbb{Z})$. If P is of dimension p and $a \in H^m(P, \mathbb{Z})$ then (P, a, f) is a geometric cycle of degree $p - m$. Take the free abelian group generated by all the geometric cycles and impose the following relation:

$$(P, \lambda.a + \mu.b, f) = \lambda.(P, a, f) + \mu.(P, b, f).$$

Thus we get a graded abelian group.

3.1.2. *Relations.* In order to recover singular homology we must impose the two following relations on geometric cycles:

i) **Bordism relation** Given a map $h : W \to X$ where W is an oriented bordism between (P, f) and (Q, g) i.e.

$$\partial W = P \bigcup Q^-.$$

Let $i_1 : P \hookrightarrow W$ and $i_2 : Q \hookrightarrow W$ be the canonical inclusions, then for any $c \in H^*(W, \mathbb{Z})$ we impose:

$$(P, i_1^*(c), f) = (Q, i_2^*(c), g).$$

ii) **Vector bundle modification** Let (P, a, f) be a geometric cycle and consider a smooth oriented vector bundle $E \xrightarrow{\pi} P$ equipped with a Riemannian metric. Take the unit sphere bundle $S(E \oplus 1)$ of the Whitney sum of E with a copy of the trivial bundle over M. The bundle $S(E \oplus 1)$ admits a section σ. Let $\sigma^!$ be the Gysin morphism in cohomology associated to this section. Then we impose:

$$(P, a, f) = (S(E \oplus 1), \sigma^!(a), f\pi).$$

An equivalence class of geometric cycle is denoted by $[P, a, f]$. Let call it a geometric class. And $H'_q(X)$ is the abelian group of geometric classes of degree q.

3.1.3. **Theorem.** [21, Cor. 2.36] *The morphism:*

$$compar : H'_q(X) \longrightarrow H_q(X, \mathbb{Z})$$

$$[P, a, f] \mapsto f_*(a \cap [P])$$

where $[P]$ is the fundamental class of P is an isomorphism of abelian groups.

3.2. **Cap product and Poincaré duality** [22, 3.2]**.** The cap product between $H^*(X,\mathbb{Z})$ and $H'_*(X)$ is given by the following formula:

$$\cap : H^p(X,\mathbb{Z}) \otimes H'_q(X) \longrightarrow H'_{q-p}(X)$$

$$u \cap [P,a,f] = [P, f^*(u) \cup a, f].$$

Let M be a d-dimensional smooth compact orientable manifold without boundary then the morphism:

$$H^p(M,\mathbb{Z}) \longrightarrow H'_{d-p}(M)$$

$$x \mapsto [M, x, Id_M]$$

is an isomorphism.

3.3. **Gysin morphisms.** (see [23] for a finite dimensional version) We want to consider Gysin morphisms in the context of infinite dimensional manifolds.

Let us recall two possible definitions for Gysin morphisms in the finite dimensional context. The following one is only relevant to the finite dimensional case. Let us take a morphism:

$$f : M^m \longrightarrow N^n$$

of oriented Poincaré duality spaces. Then we define:

$$f_! : H_*(N^n) \xrightarrow{D} H^{n-*}(N^n) \xrightarrow{f} H^{n-*}(M^n) \xrightarrow{D^{-1}} H_{*+m-n}(M^m),$$

where D is the Poncaré duality isomorphism.

For the second construction, we suppose that f is an embedding of smooth oriented manifold then one can apply the Pontryagin-Thom collapse c to the Thom space of the normal bundle of f and then apply the Thom isomorphism th:

$$f_! : H_*(N^n) \xrightarrow{c} H_*(Th(\nu(f))) \xrightarrow{th} H_{*+m-n}(M^m).$$

Now we consider an oriented embedding of Hilbert manifolds. One can use the Pontryagin-Thom collapse and the Thom isomorphism in order to define a Gysin map. Let us give some details of this construction:

let $f : X \to Y$ be an oriented embedding of Hilbert manifolds, a tubular neighborhood of X in Y consists of a vector bundle $\pi : E \to X$, an open neighborhhod V of the zero section in the total space E, an open set U in Y containing X and a diffeomorphism $\phi : V \to U$ which commutes with the zero section. The tubular neighborhood is total if $V = E$.

Using the notion of sprays ([31, ch. IV.3]), its associated exponential map and restriction to the normal bundle of f one can prove the existence and uniqueness up to isotopy of tubular neighborhoods in the context of infinite dimensional manifolds ([31, ch. IV.5, IV.6]). Moreover, as Hilbert manifolds admit partitions of unity they admit a Riemannian metric, in the case of Riemannian manifolds one can always choose tubular neighborhoods to be total. Hence as in the finite dimensional context, we use the normal bundle of the ambedding, a Pontryagin-Thom collapse and apply the Thom isomorphism.

In the framework of geometric homology we prefer to use a very geometrical interpretation of the Gysin morphism which is to take pull backs of cycles along the map f.
So, we take $i : X \to Y$ an oriented morphism of Hilbert manifolds and we suppose that the virtual bundle $V(i)$ is of rank $-d$. Let us define:

$$i_! : H'_p(Y) \longrightarrow H'_{p-d}(X).$$

Let $[P, a, f]$ be a geometric class in $H'_p(Y)$, we can choose a representing cycle (P, a, f). If f is not smooth, we know that it is homotopic to a smooth map by the existence of partitions of unity on Y. Moreover we can choose it transverse to i, by the bordism relation all these cycles represent the same class. Now we can form the pull-back:

$$\begin{array}{ccc} P & \xleftarrow{f^*i} & P \cap_Y X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \phi} \\ Y & \xleftarrow[i]{} & X \end{array}$$

3.3.1. **Theorem.** *Let $i : X \to Y$ be an oriented morphism of Hilbert manifolds of codimension d, and $[P, a, f]$ a geometric cycle of $H'_{p-a}(Y)$ such that f and i are transverse then we set*

$$i_!([P, a, f]) = (-1)^{d.|a|}[P \cap_Y X, (f^*i)^*(a), \phi],$$

this give a well defined morphism:

$$i_! : H'_*(Y) \to H'_{*-d}(X)$$

which satisfies:
i) if we have a pull-back of Hilbert manifolds

$$\begin{array}{ccc} X' & \xrightarrow{j} & Y' \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \phi} \\ X & \xrightarrow[i]{} & Y \end{array}$$

then $i_!\phi_* = \psi_* j_!$,
ii) let i and j be two oriented morphisms of Hilbert manifolds then $i_! j_! = (ji)_!$,

iii) let $i_!^{PT}$ *be the composition of the Pontryagin-Thom collapse* c *and the Thom isomorphism* th *then the following diagram commutes:*

$$\begin{array}{ccc} H'_*(Y) & \xrightarrow{i_!} & H'_{*-d}(X) \\ {\scriptstyle compar}\downarrow & & \downarrow{\scriptstyle compar} \\ H_*(Y;\mathbb{Z}) & \xrightarrow[i_!^{PT}]{} & H_{*-d}(X;\mathbb{Z}). \end{array}$$

Proof First notice that the sign of $i_!$ is taken from [23, 3.2c)], the Gysin morphism can be viewed as a product for bivariant theories [13].
In order to prove that the morphism $i_!$ is well defined, one easily see that the only thing to verify is that $i_!$ respects bordisms (taking pull-backs clearly respects vector bundle modifications), this is exactly [1, theorem 2.4] (we also notice that this result is proved in the finite dimensional case in [23]).
i) and ii) follow easily from the properties of pull-backs.
iii) M. Jakob proved this result for compact manifolds in [23]. The result follows from the fact that each cycle is represented by a singular manifold $f : P \to X$ whith f transverse to i then we apply i) to the pull-back

$$\begin{array}{ccc} P \cap_Y X & \xrightarrow{j} & P \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \phi} \\ X & \xrightarrow[i]{} & Y \end{array}$$

then apply Jakob's comparison result for compact manifolds to $P \cap_Y X \to P$ and we conclude thanks to the commutativity of $i_!^{PT}$ with respect to pull-backs. □

Remarks: Let $\mathcal{M}_{Hilb}$ be the category whose objects are oriented morphisms of Hilbert manifolds and morphisms are pull-backs diagrams:

$$\begin{array}{ccc} X' & \xrightarrow{j} & Y' \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \phi} \\ X & \xrightarrow[i]{} & Y \end{array}$$

where ϕ and i are transverse. On $\mathcal{M}_{Hilb}$ there are two ways of producing bivariant theories (bifunctors from $\mathcal{M}_{Hilb}$ to $\mathbb{Z}$-graded abelian groups). The

first bivariant theory denoted by bH^{PT} uses singular homology and cohomology together with Gysin maps obtained by Pontryagin-Thom constructions. The second one bH^{Geom} uses geometric homology and singular homology and the geometric version of Gysin maps (we refer the reader to [23] for more precise definitions). The preceding theorem extends M. Jakob's comparison results of the restriction of bH^{PT} and bH^{geom} to the full subcategory $\mathcal{M}_{comp}$ of oriented morphisms of compact manifolds to the whole category $\mathcal{M}_{Hilb}$. As M. Jakob proved, bordisms theories are also universal among bivariant theories for $\mathcal{M}_{Hilb}$. Chas and Sullivan constructions are of bivariant nature as we will see from their geometric definitions, hence the bordism is in some sense universal for the Chas and Sullivan structure.
The preceding theorem also enables us to compare directly our geometric approach to Cohen-Jones'one [8] where the authors work with bH^{PT}. R. Cohen and J.D.S. Jones work in fact at the level of spectra, it is certainly worth to consider *stable bivariant theories* as functors from $\mathcal{M}_{Hilb}$ to the homotopy category of spectra, among stable bivariant theories there is a universal one obtained by considering the Thom spectra of the oriented morphisms, then bH^{PT} factors through this spectral universal bivariant theory. This is the bivariant interpretation of Cohen-Jones'approach to string topology.
The author wonders if there exists *chain bivariant theories* which factorize bH^{PT} or bH^{geom}. Even at level of $\mathcal{M}_{comp}$, that seems to be a rather difficult task which of course depends heavily on the modelization of Gysin maps. One can make such constructions at the level of the homotopy category of chain complexes, this is the road chosen by S.A. Merkulov in [34] who uses iterated integrals and by Y. Félix, J.-C. Thomas, M. Vigué in [12] who use rational models. But it seems worthwile to the author having such constructions at a strict level. Suppose that such a chain bivariant theory bC exists then $bC(M \stackrel{id}{\to} M)$ should compute the cohomology of M, $bC(M \to pt)$ should compute the homology of M.

3.4. **The cross product** [22, 3.1]**.** The cross product is given by the pairing:

$$\times : H'_q(X) \otimes H'_p(Y) \longrightarrow H'_{p+q}(X \times Y)$$

$$[P, a, f] \times [Q, b, g] = (-1)^{dim(P).|b|}[P \times Q, a \times b, f \times g].$$

The sign makes the cross product commutative. Let

$$\tau : X \times Y \to Y \times X$$

be the interchanging morphism. Then we have the formula :

$$\tau_*(\alpha \times \beta) = (-1)^{|\alpha||\beta|}\beta \times \alpha.$$

3.5. **The intersection product(**[23, sect.3]**).** Let us return to the finite dimensional case and consider M an oriented compact d-dimensional manifold. Let $[P, x, f] \in H'_{n_1}(M)$ and $[Q, y, g] \in H'_{n_2}(M)$, we suppose that f and

g are transverse in M, then we form the pull back:

$$\begin{array}{ccc} P\times Q & \xleftarrow{\;j\;} & P\cap_M Q \\ {\scriptstyle f\times g}\downarrow & & \downarrow{\scriptstyle \phi} \\ M\times M & \xleftarrow[\Delta]{} & M \end{array}$$

and define the pairing:

$$-\bullet - : H'_{n_1}(M)\otimes H'_{n_2}(M)\overset{\times}{\to} H'_{n_1+n_2}(M\times M)\overset{\Delta^!}{\to} H'_{n_1+n_2-d}(M).$$

Hence, we set:

$$[P,a,f]\bullet[Q,b,g]=(-1)^{d.(|a|+|b|)+dim(P).|b|}[P\cap_M Q, j^*(a\times b),\phi].$$

Let $l : P\cap_M Q\to P$ and $r : P\cap_M Q\to Q$ be the canonical maps, then we also have:

$$[P,a,f]\bullet[Q,b,g]=(-1)^{d.(|a|+|b|)+dim(P).|b|}[P\cap_M Q, l^*(a)\cup r^*(b),\phi].$$

With this sign convention the intersection product $\bullet$ makes $H'_{*+d}(M)$ into a graded commutative algebra:

$$[P\cap_M Q, l^*(a)\cup r^*(b),\phi]=(-1)^{(d-dim(P)-|a|)(d-dim(Q)-|b|)}[Q\cap_M P, l^*(b)\cup r^*(a),\phi].$$

4. String topology

In this section, using the theory of geometric cycles we show how to recover the BV-structure on

$$\mathbb{H}_*(\mathcal{L}M):=H'_{*+d}(\mathcal{L}M,\mathbb{Z})$$

introduced in [4] and studied from a homotopical point of view in [8].
We also define the intersection morphism, the string bracket of [4] and string topology operations (we extend the Frobenius structure given in [7] to a homological action of the space of Sullivan's chord diagrams).

Remark: In this section we use the language of operads and algebras over an operad in order to state some results in a nice and appropriate framework. For definitions and examples of operads and algebras over an operad we refer to [17], [18], [32] and [42].

4.1. **The operator Δ.** Let $[P,a,f]$ be geometric cycle representing a class in $\mathbb{H}_*(\mathcal{L}M)$. Let us consider the map :

$$\Theta_f : S^1\times P\overset{Id\times f}{\to} S^1\times\mathcal{L}M\overset{\Theta}{\to}\mathcal{L}M.$$

4.1.1. **Definition.** *Define the operator*

$$\Delta : H'_{n+d}(\mathcal{L}M) \to H'_{n+d+1}(\mathcal{L}M)$$

by the following formula:

$$\Delta([P, a, f]) = (-1)^{|a|}[S^1 \times P, 1 \times a, \Theta_f].$$

Notice, that the operator Δ is well defined, the construction is obviously invariant by bordism and vector bundles modifications. The operator Δ is the cross-product with the class $[S^1, 1, Id]$ followed by Θ_*. Later we will see that the operator δ has a very nice interpretation in term of S^1-equivariant geometric homology.

4.1.2. **Proposition** [4, prop. 5.1]**.** *The operator verifies:* $\Delta^2 = 0$.

Proof This follows from the associativity of the cross product and the nullity of $[S^1 \times S^1, 1 \times 1, \mu] \in H'_2(S^1)$ where μ is the product on S^1. □

4.2. **Loop product.** Let us take $[P, a, f] \in H'_{n_1+d}(\mathcal{L}M)$ and $[Q, b, g] \in H'_{n_2+d}(\mathcal{L}M)$. We can smooth f and g and make them transverse to $\widetilde{\Delta}$ then we form the pull-back $P * Q$.

4.2.1. **Definition.** *Let $j : P * Q \to P \times Q$ be the canonical maps. Then we have the pairing:*

$$- \bullet - : H'_{n_1+d}(\mathcal{L}M) \otimes H'_{n_2+d}(\mathcal{L}M) \longrightarrow H'_{n_1+n_2+d}(\mathcal{L}M)$$

$$[P, a, f] \bullet [Q, b, g] = (-1)^{d.(|a|+|b|)+dim(P).|b|}[P * Q, j^*(a \times b), \Upsilon\psi],$$

let call it the loop product.

4.2.2. **Proposition** [4, Thm. 3.3]**.** *The loop product is associative and commutative.*

Proof The associativity of the loop product follows from the associativity of the intersection product, the cup product and the fact that Υ is also associative up to homotopy.
In order to prove the commutativity of $\bullet$ we follow the strategy of [4, Lemma 3.2].
Let us consider the map:

$$Ev_1 : I \times \mathcal{L}M \times \mathcal{L}M \to M \times M$$

given by $Ev_1(t,\gamma_1,\gamma_2) = (\gamma(0),\gamma(t))$. let us take the pullback

$$\begin{array}{ccc} map(8_t, M) & \xrightarrow[\tilde{\Delta}_t]{} & I \times \mathcal{L}M \times \mathcal{L}M \\ \downarrow & & \downarrow Ev_1 \\ M & \xrightarrow[\Delta]{} & M \times M \end{array}$$

where $map(8_t, M) = \{(t,\gamma_1,\gamma_2) : \gamma_1(0) = \gamma_2(t)\}$. We have a map

$$comp_1 : map(8_t, M) \to \mathcal{L}M$$

which is given by:

$$comp_1(t,\gamma_1,\gamma_2) = \begin{cases} \gamma_2(2\theta), \theta \in [o, \frac{t}{2}) \\ \gamma_1(2\theta - t), \theta \in [\frac{t}{2}, \frac{t+1}{2}) \\ \gamma_2(2\theta), \theta \in [\frac{t+1}{2}, 1). \end{cases}$$

There is a smooth interchanging map:

$$\tau : \mathcal{L}M \cap_M \mathcal{L}M \to \mathcal{L}M \cap_M \mathcal{L}M.$$

Let $[P, a, f]$ and $[Q, b, g]$ be two geometric classes. By pulling back the family $I \times P \times Q \overset{Id\times f\times g}{\to} I \times \mathcal{L}M \times \mathcal{L}M$ over $\tilde{\Delta}$ we get a bordism between:

$$P * Q \overset{\psi}{\to} \mathcal{L}M \cap_M \mathcal{L}M \overset{\Upsilon}{\to} \mathcal{L}M$$

and

$$P * Q \overset{\psi}{\to} \mathcal{L}M \cap_M \mathcal{L}M \overset{\Upsilon\tau}{\to} \mathcal{L}M.$$

This bordism identifies the geometric class

$$[P * Q, j^*(a \times b), \Upsilon\psi]$$

and the geometric class

$$[P * Q, \tau^*(j^*(a \times b)), \Upsilon\tau\psi]$$

which is also equal to:

$$(-1)^{(dim(P)-d-a)(dim(P)-d-b)}[Q * P, j^*(b \times a), \Upsilon\psi].$$

□

4.3. **Loop bracket.** Let $[P, a, f]$ and $[Q, b, g]$ be two geometric classes. In the preceding section we have defined a bordism between

$$P * Q \overset{\psi}{\to} \mathcal{L}M \cap_M \mathcal{L}M \overset{\Upsilon}{\to} \mathcal{L}M$$

and

$$P * Q \overset{\psi}{\to} \mathcal{L}M \cap_M \mathcal{L}M \overset{\Upsilon\tau}{\to} \mathcal{L}M.$$

Using the same construction one can define another bordism between

$$P * Q \overset{\psi}{\to} \mathcal{L}M \cap_M \mathcal{L}M \overset{\Upsilon\tau}{\to} \mathcal{L}M$$

and

$$P * Q \xrightarrow{\psi} \mathcal{L}M \cap_M \mathcal{L}M \xrightarrow{\Upsilon_{\mathcal{T}^2}} \mathcal{L}M$$

by considering the map

$$Ev_2 : I \times \mathcal{L}M \times \mathcal{L}M \to M \times M$$

given by $Ev_2 = (\gamma_1(s), \gamma_2(0))$ and a map $comp_2 : map(8_s, M) \to \mathcal{L}M$ given by the formula:

$$comp_2(s, \gamma_1, \gamma_2) = \begin{cases} \gamma_1(2\theta), \theta \in [0, \frac{s}{2}) \\ \gamma_2(2\theta - s), \theta \in [\frac{s}{2}, \frac{s+1}{2}) \\ \gamma_1(2\theta), \theta \in [\frac{s+1}{2}, 1). \end{cases}$$

Then by gluing Ev_1 and Ev_2 one gets a map

$$Ev : S^1 \times \mathcal{L}M \times \mathcal{L}M \to M \times M$$

and a pull-back diagram:

$$\begin{array}{ccccc} \mathcal{L}M & \xleftarrow[comp]{} & map(8_{s,t}, M) & \longrightarrow & S^1 \times \mathcal{L}M \times \mathcal{L}M \\ & & \downarrow & & \downarrow Ev \\ & & M & \xrightarrow[\Delta]{} & M \times M \end{array}$$

where the map *comp* is also obtain by putting $comp_1$ and $comp_2$ together. Then one obtains a geometric class:

$$(-1)^{(d+1).(|a|+|b|)+dim(P).|b|}[S^1 * P * Q, j^*(1 \times a \times b), comp \circ (Id_{S^1} * f * g)].$$

4.3.1. **Definition.** *The loop bracket is the pairing:*

$$\{-,-\} : H'_{n_1+d}(\mathcal{L}M) \otimes H'_{n_2+d}(\mathcal{L}M) \longrightarrow H'_{n_1+n_2+d+1}(\mathcal{L}M)$$

$\{[P, a, f], [Q, b, g]\} =$
$(-1)^{(d+1).(|a|+|b|)+dim(P).|b|}[S^1 * P * Q, j^*(1 \times a \times b), comp \circ (Id_{S^1} * f * g)].$

Remark let us us notice that the loop bracket is anti-commutative, we have the formula:

$$\{a, b\} = (-1)^{(|a|+1).(|b|+1)}\{b, a\}.$$

4.3.2. **Proposition.** *For every elements a, b and c in $\mathbb{H}_*(\mathcal{L}M)$ we have the formula:*

$$\{a, b \bullet c\} = \{a, b\} \bullet c + (-1)^{|b|(|a|+1)} b \bullet \{a, c\}$$

Proof Let K be the simplex given by $K = \{(s,t) \in \mathbb{R}^2 / 0 \le t \le s \le 1\}$. Consider the map $f_1 : K \times \mathcal{L}M \times \mathcal{L}M \times \mathcal{L}M \to M^{\times 4}$ given by the formula:

$$f_1(s, t, \gamma_1, \gamma_2, \gamma_3) = (\gamma_1(0), \gamma_2(0), \gamma_3(s), \gamma_3(t)).$$

and the map $\Delta_{1,2,2,1} : M^{\times 2} \to M^{\times 4}$ defined by the formula:

$$\Delta_{1,2,2,1}(m,n) = (m,n,n,m)$$

Now we take the pull-back $\mathcal{K}_1$ of f_1 along $\Delta_{1,2,2,1}$. The space $\mathcal{K}_1$ is the space of loops $(\gamma_1, \gamma_2, \gamma_3)$ parametrized by (s,t) such that $\gamma_1(0) = \gamma_3(t)$ and $\gamma_2(0) = \gamma_3(s)$.
We define a composition map $comp_1 : \mathcal{K}_1 \to \mathcal{L}M$

$$comp_1(s,t,\gamma_1,\gamma_2,\gamma_3) = \begin{cases} \gamma_3(3\theta), \theta \in [0, \frac{t}{3}) \\ \gamma_1(3\theta - t), \theta \in [\frac{t}{3}, \frac{t+1}{3}) \\ \gamma_3(3\theta - 1), \theta \in [\frac{t+1}{3}, \frac{s+1}{3}) \\ \gamma_2(3\theta - (s+1)), \theta \in [\frac{s+1}{3}, \frac{s+2}{3}) \\ \gamma_3(3\theta - 2), \theta \in [\frac{s+2}{3}, 1) \end{cases}$$

Let us notice that such a construction is used in [4, lemma 4.6].
Let us consider the map $f_2 : K \times \mathcal{L}M \times \mathcal{L}M \times \mathcal{L}M \to M^{\times 4}$ given by the formula:

$$f_2(s,t,\gamma_1,\gamma_2,\gamma_3) = \begin{cases} (\gamma_1(0), \gamma_3(0), \gamma_1(2-t-s), \gamma_2(0)), t+s \geq 1 \\ (\gamma_1(0), \gamma_3(0), \gamma_2(1-t-s), \gamma_2(0)), t+s \leq 1 \end{cases}$$

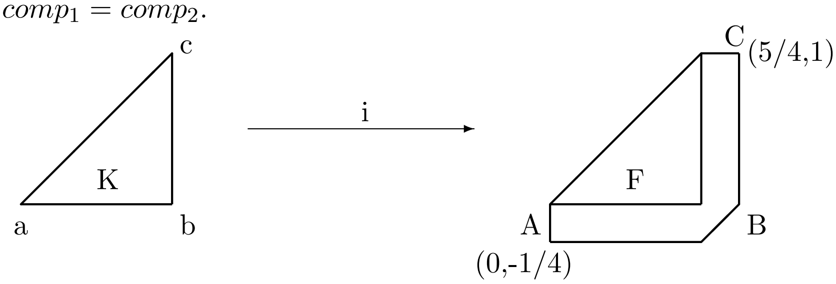
take the pull-back $\mathcal{K}_2$ of f_2 along $\Delta_{1,2,2,1}$. The space $\mathcal{K}_2$ is the space of loops $(\gamma_1, \gamma_2, \gamma_3)$ parametrized by (s,t) such that $\gamma_1(0) = \gamma_2(0)$ and $\gamma_3(0) = \gamma_1(2-t-s)$ if $t+s \geq 1$ and $\gamma_3(0) = \gamma_2(1-t-s)$ if $t+s \leq 1$.
We define a composition map $comp_2 : \mathcal{K}_2 \to \mathcal{L}M$ by

$$comp_2(s,t,\gamma_1,\gamma_2,\gamma_3) = \begin{cases} \gamma_1(3\theta), \theta \in [0, \frac{1-t-s}{3}) \\ \gamma_3(3\theta - (1-t-s)), \theta \in [\frac{1-t-s}{3}, \frac{2-t-s}{3}) \\ \gamma_1(3\theta - 1), \theta \in [\frac{2-t-s}{3}, \frac{2}{3}) \\ \gamma_2(3\theta - 2), \theta \in [\frac{2}{3}, 1) \end{cases}$$

if $t+s \leq 1$ and

$$comp_2(s,t,\gamma_1,\gamma_2,\gamma_3) = \begin{cases} \gamma_1(3\theta), \theta \in [0, \frac{1}{3}) \\ \gamma_2(3\theta - 1), \theta \in [\frac{1}{3}, \frac{3-t-s}{3}) \\ \gamma_3(3\theta - (3-t-s)), \theta \in [\frac{3-t-s}{3}, \frac{4-t-s}{3}) \\ \gamma_2(3\theta - 2), \theta \in [\frac{4-t-s}{3}, 1) \end{cases}$$

otherwise. At the points $a = (0,0)$, $b = (1,0)$ and $c = (0,1)$ we have $f_1 = f_2$ and $comp_1 = comp_2$.



we extend this morphisms on F in the following way:
$f_i(s,t,.,.) = f_i(1,t,.,.)$ if $t \geq 1$
$f_i(s,t,.,.) = f_i(s,0,.,.)$ if $t \leq 0$.

We get maps that are equal when restricted to A, B and C. If we identify two copies of K along A, B and C we obtain an oriented surface Σ of genus zero with three boundary components together with a map $f : \Sigma \times \mathcal{L}M^{\times 3} \to M^{\times 4}$, if $\mathcal{S}$ is the pull-back of this map along $\Delta_{1,2,2,1}$ we also have a map $comp : \mathcal{S} \to \mathcal{L}M$. Hence we have the maps of Hilbert manifolds:

$$\mathcal{L}M \overset{comp}{\leftarrow} \mathcal{S} \overset{i}{\to} \Sigma \times \mathcal{L}M^{\times 4}$$

where i is an oriented embedding.
We take $[P_1, c_1, f_1]$, $[P_2, c_2, f_2]$ $[P_3, c_3, f_3]$ three geometric classes in $H_*(\mathcal{L}M)$ denoted by α, β and γ. The pull-back of the map

$$\Sigma \times P_1 \times P_2 \times P_3 \overset{Id \times f_1 \times f_2 \times f_3}{\longrightarrow} \Sigma \times \mathcal{L}M^{\times 3}$$

along i gives a bordism. The boundary of this bordism corresponds to:
$\{\alpha \bullet \beta, \gamma\}$ at $t_i = -1/4$,
$\{\alpha, \gamma\} \bullet \beta$ at $s_i = 5/4$
and $\{\alpha \bullet \beta, \gamma\}$ at $t_i - s_i = 0$.
yielding the desired formula. The signs are given by the orientation of Σ. □

4.3.3. **Proposition.** *The bracket and the operator Δ are related by:*

$$\{\alpha, \beta\} = (-1)^{|\alpha|}\Delta(\alpha \bullet \beta) - (-1)^{|\alpha|}\Delta(\alpha) \bullet \beta - \alpha \bullet \Delta(\beta).$$

Proof Let K_1 and K_2 be two simplexes given by the equation $\{(s,t) \in \mathbb{R}^2 / 0 \leq t \leq s \leq 1\}$. Consider the map $f_1 : K_1 \times \mathcal{L}M \times \mathcal{L}M \to M^{\times 2}$ given by the formula:

$$f_1(s, t, \gamma_1, \gamma_2) = (\gamma_1(0), \gamma_2(s)).$$

Now we take the pull-back $\mathcal{K}_1$ of f_1 along the diagonal Δ. The space $\mathcal{K}_1$ is the space of loops (γ_1, γ_2) parametrized by (s,t) such that $\gamma_1(0) = \gamma_2(s)$. We define a composition map $comp_1 : \mathcal{K}_1 \to \mathcal{L}M$

$$comp_1(s, t, \gamma_1, \gamma_2) = \begin{cases} \gamma_2(2\theta + t), \theta \in [0, \frac{s-t}{2}) \\ \gamma_1(2\theta - s + t), \theta \in [\frac{s-t}{2}, \frac{s-t+1}{2}) \\ \gamma_2(2\theta + t), \theta \in [\frac{s-t+1}{2}, 1) \end{cases}$$

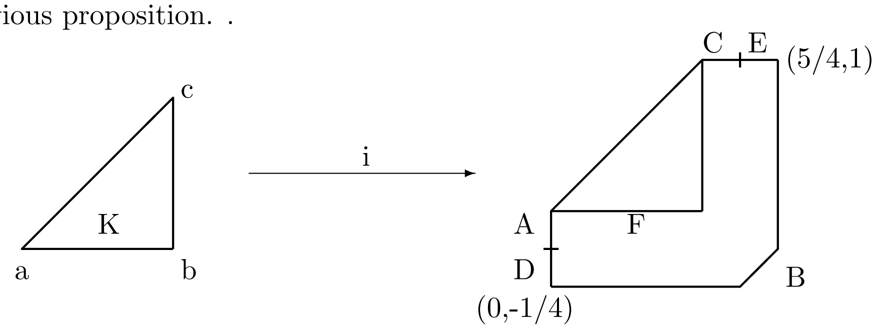
Let us notice that such construction is used in [4, lemma 5.2].
Consider the map $f_2 : K_2 \times \mathcal{L}M \times \mathcal{L}M \to M^{\times 2}$ given by the formula:

$$f_1(s, t, \gamma_1, \gamma_2) = (\gamma_1(1 - s), \gamma_2(0)).$$

Now we take the pull-back $\mathcal{K}_2$ of f_2 along the diagonal Δ. The space $\mathcal{K}_2$ is the space of loops (γ_1, γ_2) parametrized by (s,t) such that $\gamma_2(0) = \gamma_1(1-s)$. We define a composition map $comp_2 : \mathcal{K}_2 \to \mathcal{L}M$

$$comp_2(s, t, \gamma_1, \gamma_2) = \begin{cases} \gamma_1(2\theta - t), \theta \in [0, \frac{1-s+t}{2}) \\ \gamma_2(2\theta - (1 - s + t)), \theta \in [\frac{1-s+t}{2}, \frac{2-s+t}{2}) \\ \gamma_1(2\theta - t), \theta \in [\frac{2-s+t}{2}, 1) \end{cases}$$

We extend the morphisms f_1 on F_1 and f_2 on F_2 as in the proof of the previous proposition. .



We have $f_i(a) = f_i(c)$ and $f_1(x) + f_2(c)$ for $x = a, b, c$. Thus we can build a surface Σ of genus zero with four boundary components. This surface is obtained by gluing A_i and C_i, B_1 and B_2, D_1 and D_2, E_1 and E_2. Finally we get maps of Hilbert manifolds:

$$\mathcal{L}M \stackrel{comp}{\longleftarrow} \mathcal{S} \stackrel{i}{\rightarrow} \Sigma \times \mathcal{L}M \times \mathcal{L}M.$$

By studying the boundary of $\mathcal{S}$ we get.
$\{\alpha, \beta\}$ at $t_i = -1/4$
$\Delta(\alpha \bullet \beta)$ at $s_i = 5/4$,
$\alpha \bullet \Delta(\beta)$ on K_1 at $s_1 = t_1$,
$\Delta(\alpha) \bullet \beta$ on K_2 at $s_2 = t_2$.
□

It follows from the last two propositions the following results:

4.3.4. **Theorem** [4, Th. 5.4]**.** *The loop product* $\bullet$ *and the operator* Δ *makes* $\mathbb{H}_*(\mathcal{L}M)$ *into a Batalin Vilkovisky algebra, we have the following relations:*
i) $(\mathbb{H}_*(\mathcal{L}M), \bullet)$, *is a graded commutative associative algebra.*
ii) $\Delta^2 = 0$
iii) $(-1)^{|\alpha|}\Delta(\alpha \bullet \beta) - (-1)^{|\alpha|}\Delta(\alpha) \bullet \beta - \alpha \bullet \Delta(\beta)$ *is a derivation of each variable.*

Proof We use the following alternative definition of a BV-algebra given by Getzler in [16]:
A BV-algebra is a graded commutative algebra $(A, \bullet)$ with an operator Δ of degree $+1$ such that $\Delta^2 = 0$ and for every $\alpha, \beta, \gamma \in A$

$$\Delta(\alpha \bullet \beta \bullet \gamma) = \Delta(\alpha \bullet \beta) \bullet \gamma + (-1)^{|\alpha|}\alpha \bullet \Delta(\beta \bullet \gamma) + (-1)^{(|\alpha|-1).|\beta|}\beta \bullet \Delta(\alpha \bullet \gamma)$$

$$-\Delta(\alpha) \bullet \beta \bullet \gamma - (-1)^{|\alpha|}\alpha \bullet \Delta(\beta) \bullet \gamma - (-1)^{|\alpha|+|\beta|}\alpha \bullet \beta \bullet \Delta(\gamma).$$

This relation follows from propositions 4.3.2 and 4.3.3. □

4.3.5. **Corollary.** [4, Thm. 4.7] *The triple* $(\mathbb{H}_*(\mathcal{L}M), \bullet, \{-, -\})$ *is a Gerstenhaber algebra:*

i) $(\mathbb{H}_*(\mathcal{L}M), \bullet)$ *is a graded associative and commutative algebra.*
ii) The loop bracket $\{-,-\}$ *is a Lie bracket of degree* $+1$*:*

$$\{\alpha,\beta\} = (-1)^{(|\alpha|+1)(|\beta|+1)}\{\beta,\alpha\},$$

$$\{\alpha,\{\beta,\gamma\}\} = \{\{\alpha,\beta\}.\gamma\} + (-1)^{(|\alpha|+1)(|\beta|+1)}\{\beta,\{\alpha,\gamma\}\},$$

iii) $\{\alpha,\beta\bullet\gamma\} = \{\alpha,\beta\}\bullet\gamma + (-1)^{|\beta|(|\alpha|+1)}\beta\bullet\{\alpha,\gamma\}$.

4.3.6. **Remarks.** Let us recall that there are two important examples of Gerstenhaber algebras:
- The first one is the Hochschild cohomology of a differential graded associative algebra A:

$$HH^*(A,A),$$

this goes back to M. Gerstenhaber [15].
- The second example is the singular homology of a double loop space:

$$H_*(\Omega^2 X),$$

this is due to F. Cohen [6].
Both examples are related by the Deligne's conjecture proved in many different ways ([2], [29], [33], [40], [41], [28]). This conjecture states that there is a natural action of an operad C_2 quasi-isomorphic to the chain operad of little 2-discs on the Hochschild cochain complex of an associative algebra.
Hochschild homology enters the theory by the following results of R. Cohen and J.D.S. Jones [8, Thm. 13] (this result was also proved by completely different techniques in [12] and [34]):
$C^*(M)$ denotes the singular cochains of a manifold M, then there is an isomorphism of associative algebras:

$$HH^*(C^*(M), C^*(M)) \cong \mathbb{H}_*(\mathcal{L}M).$$

E. Getzler introduced BV-algebras in the context of 2-dimensional topological field theories [16]. And he proved that $H_*(\Omega^2 M)$ is a BV-algebra if M has a S^1 action. Other examples are provided by the de Rham cohomology of manifolds with S^1-action.
The BV-structure on $\mathbb{H}_*(\mathcal{L}M)$ comes in fact from a geometric action of the *cacti* operad [8], [42] (normalized cacti with spines in the terminology of R. Kaufmann [26]). Roughly speaking an element of $cacti(n)$ is a tree-like configuration of n-marked circles in the plane. The cacti operad is homotopy equivalent to the little framed discs operad [42]. And we know from the work of E. Getzler that the homology of the little framed discs operad gives the BV operad [16].
Let us explain this geometric action. First let us define the space $\mathcal{L}^{cacti(n)}M$ (denoted by L_nM in [8]) as:

$$\mathcal{L}^{cacti(n)}M = \{(c,f) : c \in cacti(n), f : c \to M\}.$$

We take the Gysin morphism associated to the map:

$$cacti(n) \times \mathcal{L}M^{\times n} \longleftarrow \mathcal{L}^{cacti(n)}M$$

since to any element $c \in cacti(n)$ one can associate a map:

$$S^1 \to c$$

we get:

$$\mathcal{L}^{cacti(n)}M \longrightarrow \mathcal{L}M.$$

For $n = 1, 2$ we know from R. Kaufmann's description of *cacti* [24] that $cacti(n)$ is a smooth manifold. In that case all the maps defined above are maps of Hilbert manifolds and they give also a very nice description of the action of $H'_*(cacti)$ on $H'_{*+d}(\mathcal{L}M)$.
So, it is certainly worth building a smooth structure on *cacti* or on an operad homotopy equivalent that acts in the same way. This would give a more conceptual proof of the preceding theorem.

4.4. **Constant strings.** We have a canonical embedding:

$$c : M \hookrightarrow \mathcal{L}M$$

c induces a map:

$$c_* : H'_{n+d}(M) \to H'_{n+d}(\mathcal{L}M).$$

The morphism c_* is a morphism of commutative algebras. This follows from the pullback diagram:

$$\begin{array}{ccc} M & \xrightarrow{\Delta} & M \times M \\ \downarrow & & \downarrow {\scriptstyle c \times c} \\ \mathcal{L}M \cap_M \mathcal{L}M & \xrightarrow[\tilde{\Delta}]{} & \mathcal{L}M \times \mathcal{L}M. \end{array}$$

4.5. **Intersection morphism.** Let recall that the map

$$ev_0 : \mathcal{L}M \longrightarrow M$$

is a smooth fiber bundle of Hilbert manifolds. Hence if we choose a base point $m \in M$ the fiber of ev_0 in m is the Hilbert manifold ΩM of based loops in M. Consider the morphism:

$$i : \Omega M \hookrightarrow \mathcal{L}M$$

from the based loops in M to the free loops in M, this is an orientable morphism of codimension d.
Let us describe the intersection morphism:

$$I = i_! : \mathbb{H}_*(\mathcal{L}M) \to H'_*(\Omega M).$$

Let $[P, a, f] \in H'_{n+d}(\mathcal{L}M)$ be a geometric class, one can define $I([P, a, f])$ in two ways:

i) using the Gysin morphism : $I([P,a,f]) = (-1)^{d.|a|}[P \cap_{\mathcal{L}M} \Omega M, (f^*i)^*(a), \phi]$.

Notice that this is the same as doing the loop product with $[c_m, 1, c]$ where c_m is a point and $c : c_m \to \mathcal{L}M$ is the constant loop at the point m, then we have:

ii) $I([P,a,f]) = (-1)^{d.|a|}[P * c_m, l^*(a), \psi]$.

4.5.1. **Proposition** [4, Prop 3.4]**.** *The intersection morphism I is a morphism of associative algebras.*

Proof. The algebra structure on $H'_*(\Omega M)$ comes from the Pontryagin product which is the restriction of Υ to $\Omega M \times \Omega M$, we have the following diagram:

$$\begin{array}{ccc} \Omega M \times \Omega M & \xrightarrow{\Upsilon_{\Omega M \times \Omega M}} & \Omega M \\ \downarrow{\scriptstyle i \times i} & & \downarrow{\scriptstyle i} \\ \mathcal{L}M \cap_M \mathcal{L}M & \xrightarrow{\Upsilon} & \mathcal{L}M. \end{array}$$

The Pontryagin product is given by the formula:

$$[P,a,f].[Q,b,g] = (-1)^{dim(P).|b|}[P \times Q, a \times b, \Upsilon_{\Omega M \times \Omega M}(f \times g)].$$

This product is associative but not commutative. The intersection morphism is a morphism of algebras by commutativity of the diagram above. □

This morphism has been studied in detail in [11], in particular it is proved that the kernel of I is nilpotent.

4.6. **String bracket.** Our goal in this section is to define, using geometric homology theory, the morphisms occurring in the Gysin sequence. These morphims were used By M. Chas and D. Sullivan to define a Lie algebra structure on the S^1-equivariant homology of free loop spaces.

4.6.1. *G-equivariant geometric homology.* Let G be a compact Lie group, and consider a paracompact topological left G-space X. We consider the locally trivial G-bundle:

$$G \to EG \times X \xrightarrow{p} EG \times_G X$$

where EG is a contractible free G-space. Let $[P,a,f] \in H'_*(EG \times_G X)$ then one can take the pullback of f along p we get a locally trivial G-bundle:

$$G \to f^*P \xrightarrow{f^*p} P$$

and a geometric class $(-1)^{dim(G).|a|}[f^*P, \phi^*(a), f \circ f^*p]$ this defines a morphism:

$$p_! : H'_*(EG \times_G X) \to H'_{*+dimG}(X).$$

Let $p_* : H'_*(X) \to H'_*(EG \times_G X)$ the morphism induced in geometric homology, as the action of G on $EG \times_G X$ is free one has:

$$p_! p_*([Q, a, f]) = [G \times Q, 1 \times a, \Theta_f]$$

where $\Theta_f : G \times Q \to X$ is given by $\Theta_f(g, q) = g.f(q)$. This transfer appears as a particular case of a gysin map in $\mathcal{M}_{Hilb}$, EG can be considered as a Hilbert manifold. There always exist faithful representations H of G such that H is an infinite dimensional Hilbert space we take $EG = H$. Now suppose that X is an Hilbert manifold, the space $EG \times_G X$ is not a priori a smooth Hilbert manifold because the action can be non-smooth, it is a topological manifold modelled on a Hilbert space E. But a topological manifold modelled on an infinite dimensional Hilbert space is always homeomorphic to a Hilbert manifold by the results of [10], thus we get a smooth locally trivial G-bundle:

$$G \to EG \times X \xrightarrow{p} EG \times_G X$$

and from the results of section 3, the geometric gysin map associated to the oriented morphism p coincides with the classical gysin map.

4.6.2. *String homology.* Let us specialize to the case:

$$S^1 \to ES^1 \times \mathcal{L}M \xrightarrow{\pi} ES^1 \times_{S^1} \mathcal{L}M$$

and Let $\mathcal{H}_i$ be the homology group $H'_{i+d}(ES^1 \times_{S^1} \mathcal{L}M)$, this is the string homology of M. In what follows we give explicit definitions of the morphism c, M, E of [4, 9].

The morphism c. Let $e \in H^2(ES^1 \times_{S^1} \mathcal{L}M)$ be the Euler class of the S^1-fibration defined above:

$$c : \mathcal{H}_i \to \mathcal{H}_{i-2}$$

$$c([P, a, f]) = [P, f^*(e) \cup a, f].$$

The morphism E. This morphism is $E = \pi_*$:

$$E : \mathbb{H}_i(\mathcal{L}M) \to \mathcal{H}_i$$

$$E([P, a, f]) = [P, a, \pi f].$$

The morphism M. The morphism M is $\pi_!$. We notice that $M \circ E = \Delta$.

We have the following exact sequence, which is the Gysin exact sequence associated to the S^1-fibration π:

$$\ldots \to \mathbb{H}_i(\mathcal{L}M) \xrightarrow{E} \mathcal{H}_i \xrightarrow{c} \mathcal{H}_{i-2} \xrightarrow{M} \mathbb{H}_{i-1}(\mathcal{L}M) \to \ldots$$

4.6.3. *The bracket.* The string bracket is given by the formula:

$$[\alpha,\beta]=(-1)^{|a|}E(M(\alpha)\bullet M(\beta)).$$

Together with this bracket $(\mathcal{H}_*,[-,-])$ is a graded Lie algebra of degree $(2-d)$ [4, Th. 6.1].

4.7. **Riemann surfaces operations.** These operations were defined by R. Cohen and V. Godin in [7] by means of Thom space technology.
Let Σ be an oriented surface of genus g with $p+q$ boundary components, p incoming and q outgoing. We fix a parametrization of these components.
Hence we have two maps:

$$\rho_{in}:\coprod_p S^1\to\Sigma,$$

and

$$\rho_{out}:\coprod_q S^1\to\Sigma.$$

If we consider a nice analytic model $map(\Sigma,M)$ for the space of maps of Σ into M, we can get a Hilbert manifold and the diagram of Hilbert manifolds:

$$\mathcal{L}M^{\times q}\stackrel{\rho_{out}}{\longleftarrow}map(\Sigma,M)\stackrel{\rho_{in}}{\longrightarrow}\mathcal{L}M^{\times p}.$$

Because the codimension of the map ρ_{in} is infinite we will work with graphs rather than surfaces.
Now, let $\chi(\Sigma)$ be the Euler characteristic of the surface. Using Sullivan's Chord diagrams it is proved in [7] that the morphism

$$map(\Sigma,M)\stackrel{\rho_{in}}{\longrightarrow}\mathcal{L}M^{\times p}$$

has a homotopy model:

$$\mathbf{H}^1(c,M)\stackrel{\rho_{in}}{\longrightarrow}\mathcal{L}M^{\times p}$$

that is an embedding of Hilbert manifolds of codimension $-\chi(\Sigma).d$. The space $\mathbf{H}^1(c,M)$ is obtained as a pull-back of Hilbert manifolds, we will give a definition of this space in the next sections.
Hence by using the Gysin morphism for Hilbert manifolds one can define the operation:

$$\mu_\Sigma:H'_*(\mathcal{L}M^{\times p})\stackrel{\rho_{in}^!}{\longrightarrow}H'_{*+\chi(\Sigma).d}(\mathbf{H}^2(\Sigma,M))\stackrel{\rho_{out*}}{\longrightarrow}H'_{*+\chi(\Sigma).d}(\mathcal{L}M^{\times q}).$$

All these operations are parametrized by the topological space of marked, metric chord diagrams $C\mathcal{F}^\mu_{p,q}(g)$ [7, sect1]. In the next section we introduce a reduced version of Sullivan's chord diagrams and give some algebraic properties of the associated operations.

4.7.1. *Sullivan chord diagrams.* In the preceding morphism $c \in \mathcal{CF}^{\mu}_{p,q}(g)$ is the *Sullivan chord diagram* associated to to the surface Σ. Let us recall the definitions of [7]. A *metric fat graph* is a graph whose vertices are at least trivalent such that the incoming edges at each vertex are equipped with a cyclic ordering, Moreover it has the structure of a compact metric space (details are given in [7, def. 1] and [20, chapter 8]),
This fat graph represents a surface of genus g with $p+q$ boundary components. The set of metric fat graphs is denoted by $\mathcal{F}at_{p,q}(g)$.
The cyclic ordering of the edges defines "boundary cycles". Pick an edge and an orientation on it, then traverse it in the direction of the orientation, this leads to a vertex, at this vertex take the next edge coming from the cycling ordering and so on. Then we get a cycle in the set of edges wich represent a boundary component of the Riemann surface associated to the fat graph. Hence on an element of $\mathcal{F}at_{p,q}(g)$ we have a partition of the cycles into p incoming cycles and q outgoing cycles.

Fat graphs (also called ribbon graphs) are a nice combinatoric tool in order to study Riemann surfaces [27], [36], [37] and [39].

4.7.2. **Definition.** *A metric Sullivan chord diagram c of type (g;p,q) consists in a metric fat graph $c \in \mathcal{F}at_{p,q}(g)$. c is a disjoint union of p parametrized circles of varying radii that are exactly the p incoming cycles of c, joined at a finite number of points by disjoint trees.*

On a metric Sullivan chord diagram the edges and the vertices lying on the incoming cycles are called the circular edges and the circular vertices. The other edges are the ghost edges.
If we contract all the ghost edges of a Sullivan chord diagram $c \in C\mathcal{F}_{p,q}(g)$ one gets a fat graph $S(c) \in \mathcal{F}at_{p,q}(g)$.

4.7.3. **Definition.** *A* **marking** *of a metric Sullivan chord diagram is a choice of a point on each boundary cycle of $S(c)$. Let us denote $C\mathcal{F}^{\mu}_{p,q}(g)$ the space of marked metric Sullivan chord diagrams.*

Following a suggestion of A. Voronov rather than using the space $C\mathcal{F}^{\mu}_{p,q}(g)$, we introduce the space of reduced metric marked Sullivan chord diagrams (in fact the utilisation of such daigrams is already implicit in the work of R. Cohen and V. Godin) .

4.7.4. **Definition.** *The space of reduced marked Sullivan chord diagram denoted by $\overline{C\mathcal{F}^{\mu}_{p,q}(g)}$ is the quotient of $C\mathcal{F}^{\mu}_{p,q}(g)$ by the following equivalence relation:*
let us consider the continuous map

$$S : c \mapsto S(c)$$

that collapses each ghost edge of a Sullivan chord diagram to a vertex, $c, c' \in C\mathcal{F}^{\mu}_{p,q}(g)$ are equivalent if and only if $S(c) = S(c')$. $\overline{C\mathcal{F}^{\mu}_{p,q}(g)}$ is equipped with the quotient topology.

As proved by V. Godin the spaces $\overline{C\mathcal{F}^{\mu}_{p,q}(g)}$ and $C\mathcal{F}^{\mu}_{p,q}(g)$ are homotopy equivalent [19].

4.7.5. **Proposition.** *The spaces of reduced and unreduced chord diagrams are homotopy equivalent.*

Proof. We sketch the proof of V. Godin. Let us consider the map

$$\pi : C\mathcal{F}^{\mu}_{p,q}(g) \to \overline{C\mathcal{F}^{\mu}_{p,q}(g)}.$$

This map can be viewed as the realization of the nerve of a functor:

$$\pi_{cat} : C\mathcal{F}at^{\mu}_{p,q}(g) \to \overline{C\mathcal{F}at^{\mu}_{p,q}(g)}$$

Let us describe these categorical constructions (see [7] and [20]). The objects of the category $C\mathcal{F}at^{\mu}_{p,q}(g)$ (respectively $\overline{C\mathcal{F}at^{\mu}_{p,q}(g)}$) are combinatorial Sullivan's chord diagrams with no length assigned to the edges (resp. combinatorial reduced Sullivan's chord diagrams with no length assigned to the edges). The morphisms of these categories are "collapse maps" i.e. a morphism $f : G \to H$ is such that:
- f is a map of graphs that preserve the cycling ordering, the p distinguished incoming edges and the marking,
- the inverse of a vertex is a tree,
- the inverse of an open edge is also an open edge.

In order to prove that π is a homotopy equivalence, it suffices to prove that for any reduced chord diagram c the over category π_{cat}/c is a retract of the fibre category $\pi_{cat}^{-1}(c)$, hence the realization of the fibre category is contractible. To conclude we apply Quillen's theorem A. □

4.7.6. **Definition.** *A collection $\{\mathcal{P}_{p,q}\}_{p,q\in N}$ of topological spaces is a partial PROP if we have :*
horizontal composition maps $\phi_{p,q,p',q'} : \mathcal{P}_{p,q} \times \mathcal{P}_{p',q'} \to \mathcal{P}_{p+p',q+q'}$,
partial vertical composition maps:

$$\mathcal{P}_{p,q} \times \mathcal{P}_{q,r} \stackrel{i}{\longleftarrow} U_{p,q,r} \stackrel{\psi_{p,q,r}}{\longrightarrow} \mathcal{P}_{p,r}$$

where $U_{p,q,r}$ is a sub-topological space of $\mathcal{P}_{p,q} \times \mathcal{P}_{q,r}$.
These composition maps satisfy:
Unit. *For every $l \in \mathbb{N}$ there is an element $1_l \in \mathcal{P}_{l,l}$ together with two maps: $j_1 : \mathcal{P}_{p,l} \times \{1_l\} \to U_{p,l,l}$ and $j_2 : \{1_l\} \times \mathcal{P}_{l,p} \to U_{l,l,p}$ such that $\psi_{p,l,l} j_1 = Id$ and $\psi_{l,l,p} j_2 = Id$. We also have $\phi_{l,l,m,m}(1_l, 1_m) = 1_{l+m}$.*
Associativity. *The horizontal composition maps are associative. Vertical*

composition maps verify a partial associativity condition:

$$\begin{array}{ccccc}
\mathcal{P}_{p,q}\times\mathcal{P}_{q,r}\times\mathcal{P}_{r,s} & \longleftarrow & \mathcal{P}_{p,q}\times U_{q,r,s} & \longrightarrow & \mathcal{P}_{p,q}\times\mathcal{P}_{q,s}\\
\uparrow & & & & \uparrow\\
U_{p.q.r}\times\mathcal{P}_{r,s} & & & & U_{p,q,s}\\
\downarrow & & & & \downarrow\\
\mathcal{P}_{p,r}\times\mathcal{P}_{r,s} & \longleftarrow & U_{p,r,s} & \longrightarrow & \mathcal{P}_{p,s}
\end{array}$$

Interchange. *For every p,q,r,p',q',r' we have*

$$\begin{array}{ccccc}
(\mathcal{P}_{p,q}\times\mathcal{P}_{q,r})\times(\mathcal{P}_{p',q'}\times\mathcal{P}_{q',r'}) & \longleftarrow & U_{p,q,r}\times U_{p',q',r'} & \longrightarrow & \mathcal{P}_{p,r}\times\mathcal{P}_{p',r'}\\
\downarrow & & & & \downarrow\\
\mathcal{P}_{p+p',q+q'}\times\mathcal{P}_{q+q',r+r'} & \longleftarrow & U_{p+p',q+q'.r+r'} & \longrightarrow & \mathcal{P}_{p+p',r+r'}.
\end{array}$$

Remarks - We have given a definition of partial PROPs in a non-symmetric framework, of course symmetric partial PROPs can be defined by requiring an action of the product of the symmmetric groups $\Sigma_p\times\Sigma_q$ on $\mathcal{P}_{p,q}$ and the equivariance of vertical and horizontal composition maps.
- The case of PROPs is recovered when $U_{p,q,r}=\mathcal{P}_{p,q}\times\mathcal{P}_{q,r}$.
- This definition can be adapted to every monoidal category. In particular for graded modules.
- In what follows we assume that $q>0$.

4.7.7. **Proposition.** *The space $\overline{C\mathcal{F}^{\mu}_{p,q}(g)}$ is a partial PROP.*

Proof. The structure of partial PROP is obtained via two kind of compatible composition products.
A horizontal composition

$$\coprod:\overline{C\mathcal{F}^{\mu}_{p,q}(g)}\times\overline{C\mathcal{F}^{\mu}_{p',q'}(g')}\to\overline{C\mathcal{F}^{\mu}_{p+p',q+q'}(g+g')}$$

which is obtained by taking the disjoint union of the chord diagrams.
A partial vertical composition

$$\sharp:\overline{C\mathcal{F}^{\mu}_{p,q}(g)}\times\overline{C\mathcal{F}^{\mu}_{q,r}(g')}\to\overline{C\mathcal{F}^{\mu}_{p,r}(g+g'+q-1)}$$

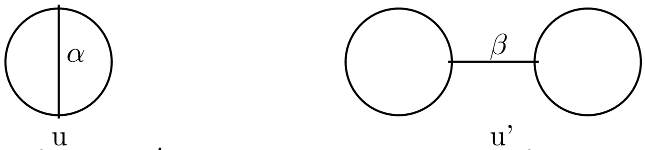
obtained by identifying the incoming boundary components of elements of $c_1\in\overline{C\mathcal{F}^{\mu}_{q,r}(g')}$ to the outgoing boundary components of $c_2\in\overline{C\mathcal{F}^{\mu}_{p,q}(g)}$,

At the level of unreduced diagrams this gluing is not continuous nor well-defined [7]. Let us recall the issues of this procedure for unreduced diagrams. Consider u, u' two Sullivan chord diagrams, and suppose that a circular vertex x of u' lying on an ingoing cycle coincides under the gluing procedure with a circular vertex v of u lying on a ghost edge of an outgoing cycle. We can put x at v or at the other vertex of the ghost edge. We get two different chord diagrams $(u\sharp u')_1$ and $(u\sharp u')_2$. Moreover, these chord diagrams may not be Sullivan chord diagrams because diagrams $(u\sharp u')_1$ and $(u\sharp u')_2$ can have a non-contractible connected component in their ghost edges, in that case the contraction $S(-)$ is not defined. When the gluing is a Sullivan chord diagram the quotient construction give a well-defined gluing operation at the level of reduced diagrans.
Let $U_{p,q,r} \subset \overline{C\mathcal{F}^{\mu}_{p,q}(g)} \times \overline{C\mathcal{F}^{\mu}_{q,r}(g')}$ be the open set of composable diagrams. We notice that the gluing is continuous. We also remark that if we take $(c_1, c_2) \in U_{p,q,r}$ then $S(c_1 \sharp c_2)$ is homeomorphic to the push-out of the maps

$$S(c_1) \leftarrow (S^1)^{\coprod q} \rightarrow S(c_2).$$

It follows easily that $\{\coprod_g \overline{C\mathcal{F}^{\mu}_{p,q}(g)}\}_{p,q}$ is a partial PROP. □

Remark. As noticed in the proof of the preceding proposition, the gluing is not possible when we have two ghost edges α and β with circular vertices α_1, α_2 lying on an outgoing cycle and β_1, β_2 lying on an ingoing cycle, if they are identified by the gluing procedure we can get a cycle in the ghost edges.
Take for example $u \in C\mathcal{F}^{\mu}_{1,2}(0)$ and $u' \in C\mathcal{F}^{\mu}_{2,1}(0)$



The gluing of u and u' depends on the markings of the incoming cycles of u' and of the outgoing cycles of u. These markings are parametrized by $(S^1)^{\times 4}$. One can easily show that the gluing is not defined on a closed subset of $(S^1)^{\times 4}$ homeomorphic to $(S^1)^{\times 2}$.

At the homological we get:

4.7.8. **Proposition.** *i) Over a field, the collection* $H'_*(\overline{C\mathcal{F}^{\mu}_{p,q}(g)})$ *is a partial PROP.*
ii) Moreover $\{H'_*(\overline{C\mathcal{F}^{\mu}_{1,1}(0)}), H'_*(\overline{C\mathcal{F}^{\mu}_{p,1}(g)})\}_{p>1}$ *form an operad.*

Proof. i) We have a partial vertical composition morphism

$$H'_*(\overline{C\mathcal{F}^{\mu}_{p,q}(g)}) \otimes H'_*(\overline{C\mathcal{F}^{\mu}_{q,r}(g')}) \leftarrow AU_{p,q,r} \rightarrow H'_*(\overline{C\mathcal{F}^{\mu}_{p,r}(g+g'+q-1)})$$

where $AU_{p,q,r}$ is the sub-group of $H'_*(\overline{C\mathcal{F}^\mu_{p,q}(g)}) \otimes H'_*(\overline{C\mathcal{F}^\mu_{q,r}(g')})$ generated by the elements $[M,c,f]\otimes[N,d,g]$ such that $f\times g$ factorizes through $U_{p,q,r}$.
ii) The composition product defining the structure comes from a topological operadic structure. We define $\overline{C}(1) = \overline{C\mathcal{F}^\mu_{1,1}(0)}$ and $\overline{C}(p) = \coprod_g \overline{C\mathcal{F}^\mu_{p,1}(g)}$ if $p>1$, we have a continuous map:

$$\gamma : \overline{C}(p)\times\overline{C}(i_1)\times\ldots\times\overline{C}(i_p) \to \overline{C}(i_1+\ldots+i_p)$$

$$(c,c_1,\ldots,c_p)\mapsto (c_1\coprod\ldots\coprod c_p)\sharp c.$$

The gluing is defined everywhere. passing to homology we get the operadic structure. □

4.7.9. **Proposition.** *The suboperads $\overline{C\mathcal{F}^\mu_{*,1}(0)}$ and $\overline{C\mathcal{F}^\mu_{1,*}(0)}$ are homeomorphic to the cacti operad.*

Proof. The homeorphisms are given by the contraction maps S, the images of this map in the space of metric fat graphs correspond exactly to cacti. Let us notice that any element in the preimage $\pi^{-1}(\overline{c})$ where

$$\pi : C\mathcal{F}^\mu_{1,q}(0) \to \overline{C\mathcal{F}^\mu_{1,q}(0)}$$

is a chord diagram associated to the cactus $\overline{c}$ as defined in [26]. □

4.7.10. **Theorem.** *For $q>0$ we have morphisms:*

$$\mu_{n,p,q}(g) : H'_n(\overline{C\mathcal{F}^\mu_{p,q}(g)}) \to Hom(H'_*(\mathcal{L}M^{\times p}), H'_{*+\chi(\Sigma).d+n}(\mathcal{L}M^{\times q}))$$

Proof First, let us recall the construction of Cohen-Godin (Cohen-Jones for cacti) of the space $\mathbf{H}^1(c,M)$ where c is a reduced Sullivan chord diagram. Let $v(c)$ be the set of circular vertices of c and $\sigma(c)$ the set of vertices of $S(c)$. We have a surjective map $\pi^* : v(c)\to\sigma(c)$ which induces a diagonal map:

$$\Delta_c : M^{\sigma(c)} \to M^{v(c)}.$$

Now if c has p ingoing cycles denoted by $c_1,\ldots,c_p$ we identify $\mathcal{L}M^{\times p}$ and $\mathbf{H}^1(c_1\coprod\ldots\coprod c_p, M)$. We get an evaluation map:

$$e_c : \mathcal{L}M^{\times p} \to M^{v(c)}$$

by evaluating at the circular vertices. This map is a smooth fiber bundle, the pul-back of Δ_c and e_c is a Hilbert manifold denoted by $\mathbf{H}^1(c,M)$. Moreover as the diagonal map Δ_c is an oriented embedding whose normal bundle is isomorphic to $TM^{v(c)-\sigma(c)}$, we get an oriented embedding of Hilbert manifolds:

$$\rho_{in}(c) : \mathbf{H}^1(c,M) \to \mathcal{L}M^{\times p}.$$

We introduce a parametrized version of the preceding constructions, let us define the space

$$map(\overline{C\mathcal{F}^{\mu}_{p,q}(g)},M)=\coprod_{c\in\overline{C\mathcal{F}^{\mu}_{p,q}(g)}}\mathbf{H}^1(c,M)$$

this space was first considered by R. Cohen and J. Jones for the case of cacti. The topology of this space is induced by the topology of $\overline{C\mathcal{F}^{\mu}_{p,q}(g)}$ and of $\mathcal{L}M^{\times p}$, an open neighborood of a point (c,γ_c) is the set of elements $(c',\gamma_{c'})\in map(\overline{C\mathcal{F}^{\mu}_{p,q}(g)},M)$ such that c' is in an open neighbourhood of c and $\rho_{in}(c')(\gamma_{c'})$ is in an open neighbourhood of $\rho_{in}(c)(\gamma_c)$. It follows from this construction:
- a projection map $p:map(\overline{C\mathcal{F}^{\mu}_{p,q}(g)},M)\to\overline{C\mathcal{F}^{\mu}_{p,q}(g)}$,
- a map $\rho_{out}:map(\overline{C\mathcal{F}^{\mu}_{p,q}(g)},M)\to\mathcal{L}M^{\times q}$.
- and an embedding $p\times\rho_{in}:map(\overline{C\mathcal{F}^{\mu}_{p,q}(g)},M)\to\overline{C\mathcal{F}^{\mu}_{p,q}(g)}\times\mathcal{L}M^{\times p}$ which corresponds to the map:

$$\coprod_{c\in\overline{C\mathcal{F}^{\mu}_{p,q}(g)}}\mathbf{H}^1(c,M)\to\coprod_{c\in\overline{C\mathcal{F}^{\mu}_{p,q}(g)}}\mathcal{L}M^{\times p}.$$

In order to define string topology operations we use the Thom collapse map

$$\tau:\overline{C\mathcal{F}^{\mu}_{p,q}(g)}\times\mathcal{L}M^{\times p}\to(map(\overline{C\mathcal{F}^{\mu}_{p,q}(g)},M))^{\nu}$$

associated to the preceding embedding where ν is an open neighborhood of the embedding $p\times\rho_{in}$ and let th be the Thom isomorphism:

$$th:H'_*((map(\overline{C\mathcal{F}^{\mu}_{p,q}(g)},M))^{\nu})\to H'_{*+\chi(\Sigma).d}(map(\overline{C\mathcal{F}^{\mu}_{p,q}(g)},M)).$$

Finally in homology we get the operation:

$$H'_*(\overline{C\mathcal{F}^{\mu}_{p,q}(g)})\otimes H'_*(\mathcal{L}M^{\times p})\to H'_{*+\chi(\Sigma).d}(map(\overline{C\mathcal{F}^{\mu}_{p,q}(g)},M))\to H'_*(\mathcal{L}M^{\times q})$$

We give an alternative description of this construction using geometric cycles. Consider an element of $H'_n(\overline{C\mathcal{F}^{\mu}_{p,q}(g)})$ and suppose that is represented by a geometric cycle (S,α,g) where $g:S\to\overline{C\mathcal{F}^{\mu}_{p,q}(g)}$. And let us define:

$$map(g,M)=\{(s,f):s\in S,f\in\mathbf{H}^1(g(s),M)\},$$

let us notice that we have a pull-back diagram:

$$\begin{array}{ccc}
map(g,M) & \longrightarrow & map(\overline{C\mathcal{F}^{\mu}_{p,q}(g)},M)\\
\downarrow & & \downarrow p\\
S & \xrightarrow[g]{} & \overline{C\mathcal{F}^{\mu}_{p,q}(g)}
\end{array}$$

we also have maps:

$$\rho_{in}:map(g,M)\to\mathcal{L}M^{\times p},$$
$$\rho_{out}:map(g,M)\to\mathcal{L}M^{\times q}.$$

we get an embedding of codimension $-\chi(\Sigma).d$:

$$p_g \times \rho_{in} : map(g, M) \to S \times \mathcal{L}M^{\times p}.$$

Let use the Thom collapse map of [7, lemma5]:

$$\tau_g : S \times \mathcal{L}M^{\times p} \to (map(g, M))^{\nu(g)}$$

where $\nu(g)$ is an open neighborhood of the embedding $p \times \rho_{in}$ and let th_g be the Thom isomorphism:

$$th_g : H'_*((map(g, M))^{\nu(g)}) \to H'_{*+\chi(\Sigma).d}(map(g, M)).$$

Let $[N_i, \alpha_i, f_i]$ be a geometric cycle of $H'_*(\mathcal{L}M)$ for $i = 1, \ldots, p$. Then the operation $\mu_{n,p,q}(g)$ is defined by:

$$\begin{array}{c} [N_1, \alpha_1, f_1] \otimes \ldots \otimes [N_p, \alpha_p, f_p] \\ \downarrow \\ \pm[S \times N_1 \times \ldots \times N_p, \alpha \times \alpha_1 \times \ldots \times \alpha_p, id_S \times f_1 \times \ldots \times f_p] \\ \downarrow \\ \pm[S \times N_1 \times \ldots \times N_p, \alpha \times \alpha_1 \times \ldots \times \alpha_p, \tau_g(id_S \times f_1 \times \ldots \times f_p)] \\ \downarrow \\ \pm\rho_{out*} th_g([S \times N_1 \times \ldots \times N_p, \alpha \times \alpha_1 \times \ldots \times \alpha_p, \tau_g(id_S \times f_1 \times \ldots \times f_p)]). \end{array}$$

The two constructions give the same operation in homology this follows from the pull-back diagram:

$$\begin{array}{ccc} map(g, M) & \xrightarrow{\phi} & map(\overline{C\mathcal{F}^{\mu}_{p,q}(g)}, M) \\ \Big\downarrow{\scriptstyle p_g \times \rho_{in}} & & \Big\downarrow{\scriptstyle p \times \rho_{in}} \\ S \times \mathcal{L}M^{\times p} & \xrightarrow[g \times Id]{} & \overline{C\mathcal{F}^{\mu}_{p,q}(g)} \times \mathcal{L}M^{\times p} \end{array}$$

and the commutativity of gysin morphism with pull-backs. We get:

$$\phi_*(p_g \times \rho_{in})_!([S \times N_1 \times \ldots \times N_p, \alpha \times \alpha_1 \times \ldots \times \alpha_p, id_S \times f_1 \times \ldots \times f_p]) =$$

$$(p \times \rho_{in})_!([S \times N_1 \times \ldots \times N_p, \alpha \times \alpha_1 \times \ldots \times \alpha_p, g \times f_1 \times \ldots \times f_p]).$$

□

4.7.11. **Definition.** *An algebra over a partial PROP $\mathcal{A}_{p,q}$ is a graded vector space V together with linear morphisms*

$$\psi_{p,q} : \mathcal{A}_{p,q} \to Hom(V^{\otimes p}, V^{\otimes q})$$

that respect the horizontal compositions. The compatibility with the partial vertical compositions is defined in the following way:
we suppose that the vertical partial composition product is given by:

$$\mathcal{A}_{p,q} \otimes \mathcal{A}_{q,r} \leftarrow AU_{p,q,r} \to A_{p,r}$$

the following diagram

$$\begin{array}{ccccc}
\mathcal{A}_{p,q} \otimes \mathcal{A}_{q,r} \otimes V^{\otimes p} & \longrightarrow & \mathcal{A}_{q,r} \otimes V^{\otimes q} & \longrightarrow & V^{\otimes r} \\
\uparrow & & & \nearrow & \\
AU_{p,q,r} \otimes V^{\otimes p} & \longrightarrow & \mathcal{A}_{p,r} \otimes V^{\otimes p} & &
\end{array}$$

commutes

4.7.12. **Proposition.** *Over a fiald $\mathbb{F}$ the graded vector space $H'_*(\mathcal{L}M, \mathbb{F})$ is an algebra over the partial PROP $H'_*(\overline{C\mathcal{F}^{\mu}_{p,q}(g)}, \mathbb{F})$.*

Proof We verify the compatibily of the operations $\mu_{n,p,q}(g)$ with the partial vertical composition product. Consider the composition (A):

$$\begin{array}{c}
AU_{p,q,r} \otimes H'_*(\mathcal{L}M)^{\otimes p} \\
\downarrow \phi \\
H'_*(\overline{C\mathcal{F}^{\mu}_{p,r}(g + g' + q - 1)}) \otimes H'_*(\mathcal{L}M)^{\otimes p} \\
\downarrow \mu_{*,p,r} \\
H'_*(\mathcal{L}M)^{\otimes r}.
\end{array}$$

Consider also the composition (B):

$$\begin{array}{c}
AU_{p,q,r} \otimes H'_*(\mathcal{L}M)^{\otimes p} \\
\Big\downarrow \\
H'_*(\overline{C\mathcal{F}^{\mu}_{p,q}(g)}) \otimes H'_*(\overline{C\mathcal{F}^{\mu}_{q,r}(g')}) \otimes H'_*(\mathcal{L}M)^{\otimes p} \\
{\scriptstyle Id \otimes \mu'_{*,p,q}} \Big\downarrow \\
H'_*(\overline{C\mathcal{F}^{\mu}_{q,r}(g)}) \otimes H'_*(\mathcal{L}M)^{\otimes q} \\
{\scriptstyle \mu_{*,q,r}} \Big\downarrow \\
H'_*(\mathcal{L}M)^{\otimes r}
\end{array}$$

with:

$$\mu'_{*,p,q}(c \otimes c' \otimes a_1 \otimes \ldots \otimes a_p) = c' \otimes \mu_{*,p,q}(c)(a_1 \otimes \ldots \otimes a_p).$$

Take $[S, a, f] \otimes [S', a', f'] \in AU_{p,q,r}$ we have $\phi([S, a, f] \otimes [S', a', f']) = [S \times S', a \times a', f \sharp f']$.
Consider also the diagram:

$$\begin{array}{ccccc}
\mathcal{L}M^{\times p} \times S \times S' & & & & \\
{\scriptstyle \rho_{in} \times p \times Id_{S'}} \Big\uparrow & \nwarrow {\scriptstyle \rho_{in} \times p} & & & \\
map(f, M) \times S' & \longleftarrow & map(f \sharp f', M) & & \\
{\scriptstyle \rho_{out}} \Big\downarrow & & \Big\downarrow & \searrow {\scriptstyle \rho_{out}} & \\
\mathcal{L}M^{\times q} \times S' & \xleftarrow{\rho_{in} \times p} & map(f', M) & \xrightarrow{\rho_{out}} & \mathcal{L}M^{\times r}
\end{array}$$

this diagram is commutative and $map(f \sharp f', M)$ can be identified with the pullback of the maps $\rho_{in} \times p$ and ρ_{out}. Moreover the maps:

$$\begin{array}{ccc}
\mathcal{L}M^{\times p} \times S \times S' & & \\
{\scriptstyle \rho_{in} \times p \times Id_{S'}} \Big\uparrow & \nwarrow {\scriptstyle \rho_{in} \times p} & \\
map(f, M) \times S' & \longleftarrow & map(f \sharp f', M)
\end{array}$$

are all embeddings. As Gysin maps in homology commute with pull-backs, it follows that composition (A) and composition (B) are the same, . □

This last result unifies some of the algebraic structures arising in string topology, as corollaries we get:

4.7.13. **Corollary** [7]. *If we fix $n = 0$ and if we work over a field, the action of homological degree 0 string topology operations induces on $H'_{*+d}(\mathcal{L}M)$ a structure of Frobenius algebra without co-unit.*

Proof This follows from the connectivity of the spaces of Sullivan's chord diagrams assoxiated to a fixed surface. This is a highly non-trivial fact proved by R. Cohen and V. Godin.. □

4.7.14. **Corollary.** *When restricted to $H'_*(\overline{C\mathcal{F}^{\mu}_{p,1}(0)})$ one recovers the BV-structure on $H'_{*+d}(\mathcal{L}M)$ induced by the string product and the operator Δ.*

Proof Let us notice that $\overline{C\mathcal{F}^{\mu}_{1,1}(0)}$ is homeomorphic to S^1, the operator Δ is given by the geometric cycle $[S^1, 1, Id_{S^1}] \in H'_1(\overline{C\mathcal{F}^{\mu}_{p,1}(0)})$. The loop product $- \bullet -$ is given by any cactus representing a pair of pants i.e a geometric class in $H'_0(\overline{C\mathcal{F}^{\mu}_{2,1}(0)}))$. □

let us conclude with new operations in fact cooperations in string topology.

4.7.15. **Corollary.** *Over a field, when restricted to $H'_*(\overline{C\mathcal{F}^{\mu}_{1,q}(0)})$ one has an operad isomorphic to BV and $H'_{*+d}(\mathcal{L}M)$ is a coalgebra over this operad.*

Proof The proof uses the same arguments as the preceding corollary. □

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E-mail address: `david.chataur@math.univ-lille1.fr`