HOMOTOPY THEORIES

$[id_X] = id$ et assoc

(I) Homotopy Theory of Topological Spaces

(1) Homotopy categories

Top : cat of Top. with C^0 maps
iso = homeo [...]

Ex: $\mathbb{R}^n \supset D^n \supset S^{n-1}$ $\mathbb{P}\mathbb{R}^n$
$I^n, \partial I^n, |\Delta^n|, |\partial\Delta^n|$ $\mathbb{P}\mathbb{C}^n$

Def: [Homotopy]

$f, g : X \to Y$: $H : X \times I \to Y$ C^0
$f \overset{H}{\curvearrowright} g$
tq $H(-,0) = f$
$H(-,1) = g$

↳ f, g are homotopic

Prop: $f \sim g$ equivalence relation on $Top(X,Y)$

↳ $[f]$ h eq class.

Def prop (naive) Homotopy Category:

$$hoTop := \begin{cases} obj : \text{top spaces} \\ morph : Top(X,Y)/_\sim =: [X,Y] \end{cases}$$

Issue: composition of morphs.

$$X \xrightarrow{[f]} Y \xrightarrow{[g]} Z \quad : \quad [g \circ f] =: [g] \circ [f]$$

well defined since $\overset{f}{\underset{f'}{\wr H}}$ $\overset{g}{\underset{g'}{\wr K}}$ $\Rightarrow$ $\overset{gf}{\underset{g'f'}{\wr}}$

compatibility with composite

Lemma: $gf \sim gf'$ puis $gf' \sim g'f'$

$\overset{f}{\underset{f'}{H \wr}} \Rightarrow gH : \overset{gf}{\underset{gf'}{\wr}}$ et idem $\overset{g}{\underset{g'}{K \wr}}$ $\overset{gf}{\underset{g'f}{\wr Kf}}$

Proposition: only cat structure with makes the assignment $Top \to hoTop$ a functor

$$\begin{tikzcd} X \arrow[d, "f"] \arrow[r, mapsto] & X \arrow[d, "{[f]}"] \\ Y \arrow[r, mapsto] & Y \end{tikzcd}$$

Rq: // $G \to G/_N$ ~ group structure

Def: [Homotopy Equivalence]

$f : X \xrightarrow{\sim} Y$ si $[f]$ iso

$\Leftrightarrow$ $\exists g : Y \to X$ (not nec. unique)
st $gf : X \to X \sim id_X$
$fg \sim id_Y$

Def: Homotopy equivalent spaces if iso in hoTop (i.e. $\exists f : X \xrightarrow{\simeq} Y$)
(say: same homotopy type)

Def: X contractible if $X \simeq *$.
f null homotopic if $[f] = [cst]$

Rq: "bons foncteurs" = $F : hoTop \to \mathcal{C}$
iso $\longrightarrow$ iso
heq $\longmapsto$ iso

Ex: H_n, π_1

mais hoTop n'est pas caractérisée par cette propriété (modèles)

Def: The catégorie homotopique : HoTop together with $Top \xrightarrow{P} HoTop$ P.heq $\mapsto$ iso st.

$$\begin{tikzcd} Top \arrow[r, "P"] \arrow[dr, "F"'] & HoTop \arrow[d, dashed, "\exists ! \tilde{F}"] \\ & \mathcal{C} \end{tikzcd}$$

F : heq $\mapsto$ iso

1) if $\exists$ ⇒ unique (univ property)

2) Construction : Objet les mêmes. F
beau
Morph : il faut que les h. equiv. soient inversibles.
$f\ X \xrightarrow{\sim} Y$. $\uparrow$ $X \overset{"f^{-1}"}{\longleftarrow} Y$ tq "$f f^{-1}$" $= id_Y$; "f^{-1}" $f = id_X$

↳ $\xrightarrow{f_1} \xrightarrow{f_2} \xleftarrow{f_3^{-1}} \xleftarrow{f_4^{-1}}$ . . . . . .

$\xrightarrow{g} \xrightarrow{f} = \xrightarrow{fg}$
$\xleftarrow{f^{-1}} \xleftarrow{g^{-1}} = (g \cdot f)^{-1}$

Def. $Top[heq^{-1}] =: HoTop$

Rk: 1) Arnaque : pq en catégorie HoTop(X, Y) : ensemble ?
2) no loss of data. ("pas un quotient")

Goal : $\boxed{CW\ cx/_\sim \cong HoTop}$ (Compactly gen weak Haus) ↑ weq

3) $\exists !$ $HoTop \to hoTop$

4) $hoTop \cong HoTop$ Strøm. TBC ...
quotient pute vs localisation

5) Bonne question : fonctionne pour d'autres notions d'eq. faible (plus qu'iso)

W : classes de morph stable par composition et id ⊂ W

à suivre

"Pb" HoTop does not ~~[illegible]~~ keep track of the homotopies

$Top(X,Y)$;

$$\begin{tikzcd} X \arrow[r, bend left=50, "f"{name=U}] \arrow[r, "g"{name=M}] \arrow[r, bend right=50, "h"'{name=D}] & Y \arrow[from=U, to=M, Rightarrow, shorten=2pt] \arrow[from=M, to=D, Rightarrow, shorten=2pt] \end{tikzcd}$$

Def

$\Pi(X,Y)$: obj : $f : X \to Y$
morph : $H : f \times I \to Y$

Prop: groupoïde
cat ou $\forall$ iso

Pb^f : Compo paramétrized by $I \Rightarrow$ pas strictly assoc (à tue $\pi_1 \neq \Omega$)

Def: $Top_{(2)}$

Obj: pairs (A, X) $A \subset X$

Morph : $f : X \to Y$ C^0 tq $f(A) \subset B$

Ex : # subcat with #A = 1

Top_* : | obj : (x, X)
morph
$\mapsto y$

Def. Homotopy relative à $A \subset X$

$f, g : X \to Y$ tq $f_{|A} = g_{|A}$

$H : f \sim g$ rel$_A$ si $H : f \sim g$ constante sur A ie — more restrictive

$H(a,t) = f(a) = g(a)$ $\forall t \in [0,1]$

better notion

A homotopy $(A,X) \underset{g}{\overset{f}{\rightrightarrows}} (B,Y)$: $H : X \times I \to Y$ st $\forall t$, H_t morph of Top(2) ie $H_t(A) \subset B$

⇝ hoTop(2)

Ex. a homotopy in Top$_*$: $H_t(x) = y$

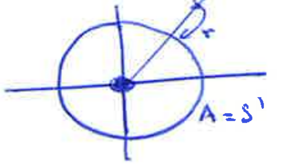
hoTop$_*$

Ex: Def Deformation retract

$A \overset{i}{\hookrightarrow} X$ s.t $\exists r : X \to A$

tq $\begin{cases} r \circ i = id_A & (r : \text{"retract" of } X) \\ i \circ r \sim id_X \text{ rel } A \end{cases}$

Ex.



$H(x,t) := x + t(\bar{x} - r(x))$: $X \times I \to X$

$H(x,0) = x$ ⇝ id

$H(x,1) = r(x)$

Def: Compact-Open Topology

sur $Top(X,Y)$:

subbasis of opens : $W(K,U) := \{f : X \to Y \mid f(K) \subset U\}$

compact, open

↳ basis = finite ∩ of

Exercice: $f : X \to Y \in$ Top

$\Rightarrow f^* : Y^Z \to X^Z$ et $f_* : X^Z \to Y^Z \in$ Top

Prop: $f : X \times Y \to Z \Rightarrow \tilde{f} : X \to Z^Y$ C^0

↳ adjunction in Top

$\underbrace{Top(X \times Y; Z)} \overset{\uparrow *}{\cong} \underbrace{Top(X, Z^Y)}$

But more ↑ top ↑ top

(Top: enriched over itself)

(exponential law)

Prp: X, Y locally compact ⇒ $*$ homeomorphism

Dans Top : coproduit : $X \sqcup Y$

coproduit : $X \times Y$

pushout :

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d] \\ Y \arrow[r] & X \sqcup Y / {f(a) \sim g(a)} \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} =: X \sqcup_A Y$$

pullback :

$$\{(x,y) \mid f(x) = g(y)\} =: \begin{tikzcd} X \times_A Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & A \end{tikzcd}$$

Proof. set puis bonne topo □

Dans $\boxed{Top_*}$ produit $X \times Y$ avec (x,y) comme pt base   pullback idem

coproduit $\dfrac{X \sqcup Y}{x \sim y} =: X \vee Y$ wedge ou bouquet   pushout idem

Rappel. locally compact if ∀ neighborhood of a pt ⊃ a compact neighborhood.

Ex. notion of a homotopy $X \times I \to Y$ in sets: internal hom :

$Set(X \times I, Y) \cong Set(X; Y^I)$ où $Y^I := \{f : I \to Y\}$

↑ natural iso

$\times I : Set \rightleftarrows Set : \_^I$

Rk: ∃ Top(3) ... objects $(A \subset B \subset X)$.

Retour au (2-)groupoid: $\Pi(X,Y) = $ Morph $= \begin{tikzcd} X \arrow[r, bend left, "f"{name=F}] \arrow[r, bend right, "g"'{name=G}] & Y \arrow[from=F, to=G, Rightarrow, "\alpha"] \end{tikzcd}$

homotopies $X \times I \to Y$ / homotopies relative to $X \times \partial I$

↑ we keep unchanged f & g.

~~Rq~~

⇒ // π_1 become assoc, un

Rk. ∀ inversible : cat que dis : "groupoïde"

Prop " La donnée objects : X topo

1-Morph : $X \xrightarrow{f} Y$

2-Morph : $\begin{tikzcd} X \arrow[r, bend left] \arrow[r, bend right] & Y \end{tikzcd}$

forme une 2-catégorie où tout est inversible"

↳ structure

compo

longs des Z ...

et cohérence

Mais higher op ???

cohérence ???

2 here all the spaces will be locally compact

Ex: $\mathbb{R}^n, D^n, S^n$ ...

Rk: ∃ weaker notion of compactly generated (weak) Hausdorff (see Toby)

$\hookrightarrow g(k)$ in X $\forall g: k \to X$
closed compact

tout sous espace $A \subset X$
$g^{-1}(A)$ closed in K
$\forall g: k \to X$
compact

T1 ≤
pt are closed

≤ T2
Hausdorff
separation of points

& ∃ new topology: $k(X)$: compactly generated ⤳ $k(X \times Y)$

Ex: X locally compact ($\overset{+\text{weak}}{\text{Hausdorff}}$) et $k(X \times Y) = X \times Y$
with countable neighbourhood basis

continue → in Top*

② (Co)fiber Sequences

Construction to express pointed homotopy in the category Top* (some constrain)

Rappel $H: X \times I \to Y$ tq $x_0 \times I \to y_0$

Ex: Cyl(S¹)

$$\boxed{\text{Def: } Cyl(X) := \frac{X \times I}{x_0 \times I}}$$

$Top_*(Cyl(I), Y) = \{\text{pointed } H\}$

H: pointed ho $\overset{1-1}{\Longleftrightarrow}$ $Cyl(I) \to Y$
dans Top*

X locally compact

$$\text{Prop. } \boxed{e: Y^X \times X \to Y \ \mathcal{C}^0}$$

Def "co" homotopy $f, g: X \to Y$

$H: X \to Y^I$ st. $e_0: X \to Y = f$
$e_1: g$

Prop: $\boxed{\text{equivalent to } H \text{ since } I \text{ locally cpt}}$

Cours 1 | cours 2 ↓

Def: smash product:

$$X \wedge Y := \frac{X \times Y}{x_0 \times Y \cup X \times y_0}$$

unit S^0
$S^0 \wedge X \cong X$.

Rks: assoc $(X \wedge Y) \wedge Z \underset{\text{homeo}}{\simeq} X \wedge (Y \wedge Z)$ on locally cpt

et aussi $\boxed{\text{Thm}}$ [Exponential law]
X, Y locally cpt

$Top_*(X \wedge Y; Z) \cong Top_*(X; Z^Y)$
homeo

same but
homotopy with prescribed "~~target~~" (target): the cst map i.e. $Top_*(Cone(X), Y) = \{\text{pted homotopy to cst map}\}$

$H: X \times I \to Y$ tq $x_0 \times I \to y$
$x \times 1 \mapsto y_0$

Ex: Cone(S¹)

$$\text{Def: } Cone(X) := \frac{X \times I}{x_0 \times I \cup X \times \partial}$$

1
0

~~$Cyl(X) \to Cone(X) \to \Sigma X$~~

What about in Top*?

~~then Top~~ $Set_*(?; Z) \cong Set_*(X; Z^Y)$

$Y \to Z$
$y_0 \mapsto z_0$
pt $y \mapsto z_0$

$X \times Y \to Z$
$(x_0, y) \longmapsto z_0$

⟺

$(x, y_0) \longmapsto z_0$
$(x_0, y) \longmapsto z_0$

Prop: f ho eq $X \xrightarrow{\sim} Y$
$\Rightarrow f^*: Y^Z \xrightarrow{\sim} X^Z$, $f_*: X^Z \xrightarrow{\sim} Y^Z$

same in Top*

Symmetric version: want pointed homotopy $X \overset{cst}{\underset{cst}{\rightrightarrows}} Y$

Def: Suspension of $X := X \wedge {}^{I}/_{\partial I}$ =

$$\Sigma X := \frac{X \times I}{x_0 \times I \cup X \times \partial I}$$

$\Sigma * \simeq *$

$\Sigma\, x_0 \cup x_0 \simeq S^1$

Ex: $\Sigma S^1 \cong S^2$

Ex. $\Sigma^n S^1 \simeq S^{n+1}$

~~they recall functors. $Top_* \to Top_*$~~

~~$Top_*(\Sigma X, Y)$~~
~~{homotopy H_0 id → id}~~

③

①

Ex: $S^0 = \{*;u\}$ $\quad \Sigma S^0 \cong S^1 \overset{\simeq}{} {}^{I/\partial I}$

$$\Sigma^2 S^0 \cong \Sigma S^1 \cong S^2 \cong I^2/\partial I^2$$

$$? \ \Sigma^n S^0 \cong S^n$$

Rk: ~~X~~ $\Sigma X \cong X \wedge I/\partial I$

Proposition: $\boxed{\Sigma^n X \cong X \wedge I^n/\partial I^n}$

Proof: induction $n=1$ ok

$n \Rightarrow n+1$

$$\Sigma^{n+1} X \cong \Sigma(\Sigma^n X) \cong \Sigma(X \wedge I^n/\partial I^n) \cong (X \wedge I/\partial I) \wedge I^n/\partial I^n$$

$$\cong X \wedge (I/\partial I \wedge I^n/\partial I^n) \cong X \wedge I^{n+1}/\partial I^{n+1}$$

(I is locally compact)

$I^k/\partial I^k \wedge I^\ell/\partial I^\ell \cong I^{k+\ell}/\partial I^{k+\ell}$

identify for one coord $=0,1$ ☐

$S^0 \xrightarrow{\Sigma} S^1 \xrightarrow{\Sigma} S^2 \to \cdots$

Cor: $\Sigma^n S^0 \cong S^n \cong I^n/\partial I^n$

"Sphere spectrum" (initial object in the cat of ~~Sp~~ spectra)

Proof: induction

$$\Sigma^n S^0 = \Sigma^{n-1} S^0 \wedge I^1/\partial I^1 \cong I^n/\partial I^n$$

unit pa

For $\cong S^n$: $\quad S^n \cong \partial D^{n+1} \cong \partial I^{n+1}$

$2(n+1)$ faces $\cong I^n$

$I^n \cong \bigsqcup_{2(n+1)-1} I^n$ views as faces of $\quad I^n/\partial I^n \cong \partial I^{n+1}$ identify last face ☐

②

$\Sigma : \mathrm{Top}_* \longrightarrow \mathrm{Top}_*$ et

$$\begin{tikzcd} \mathrm{Cyl} \arrow[d, Rightarrow] & \mathrm{Cyl}(X) \arrow[d] \\ \mathrm{Cone} \arrow[d, Rightarrow] & \mathrm{Cone}(X) \arrow[d] \\ \Sigma & \Sigma X \end{tikzcd}$$

nat. transfo

even "better"

any pointed homotopy

$H : X \times I \longrightarrow Y$ $\quad$ $\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow H"] & Y \end{tikzcd}$

induces a pointed homotopy

$\Sigma X \times I \longrightarrow \Sigma Y$ $\quad$ $\begin{tikzcd} \Sigma X \arrow[r, bend left, "\Sigma f"] \arrow[r, bend right, "\Sigma g"'] & \Sigma Y \end{tikzcd}$

$(x, t, s) \longmapsto (H(x,s), t)$

$\hookrightarrow \Sigma : \mathrm{hoTop}_* \longrightarrow \mathrm{hoTop}_*$

③

$\left\{ \begin{array}{l} \text{pointed homotopy } X \times I \to Y \\ \text{from } X \to y_0 \text{ to itself} \end{array} \right\} \cong \mathrm{Top}_*(\Sigma X; Y)$

pointed homotopies $X \times I \xrightarrow{H} Y$ relative to $X \times \partial I$ $\quad \overset{1\text{-}1}{\longleftrightarrow}$ $\quad \begin{tikzcd} \Sigma X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "H"] & Y \end{tikzcd}$

pointed homotopy

$\wr$

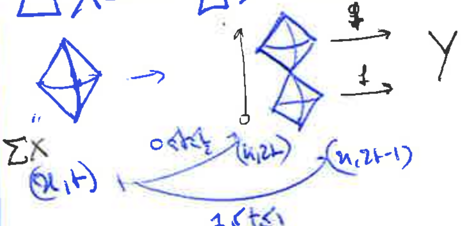
$[\Sigma X; Y]_* := \mathrm{hoTop}_*(\Sigma X; Y)$

$f : \Sigma X \to Y$
$g : \Sigma X \to Y$ $\quad \leftrightarrow \quad f+g : \Sigma X \to Y$

④

$\boxed{\text{Thm}}$ The pinch map

$$\Sigma X \xrightarrow{p_1} \Sigma X \vee \Sigma X$$



induces a group structure on $[\Sigma X; Y]_*$ under

$$\begin{tikzcd} \mathrm{Top}_*(\Sigma X; Y) \times \mathrm{Top}_*(\Sigma X; Y) \arrow[d, "\cong"] & (f,g) \arrow[dd, mapsto] \\ \mathrm{Top}_*(\Sigma X \vee \Sigma X; Y) \arrow[d, "p_1^*"] & \\ \mathrm{Top}_*(\Sigma X; Y) & f+g \end{tikzcd}$$

But open the doors to more general, for instance

Explicitely

$$f+g : (x,t) \longmapsto \begin{cases} f(x,2t) & 0 \leq t \leq \frac{1}{2} \\ g(x,2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

☐

⑤

$\boxed{\text{Proposition}}$ under these bijections (theoretically from the 2-groupoid:

$$\begin{tikzcd}[column sep=huge] X \arrow[r, bend left=60, "\mathrm{cst}_{X \to y_0}"] \arrow[r, "\mathrm{cst}" description] \arrow[r, bend right=60, "\mathrm{cst}"'] & y_0 \end{tikzcd}$$

$\Downarrow F \quad \Downarrow G \quad \Downarrow G \circ F$)

$c\mathrm{Top}_*(X \times I, Y)$: espace

$\cong \mathrm{Top}_*(I, Y^X)$

$\Omega(Y^X, \mathrm{cst})$

$\Rightarrow$ same proof as for π_1 ☐

well… $[\Sigma^1 S^0; Y]_* = [\underset{I/\partial I}{S^1}, Y]_*$

$= \pi_1(Y)$

④

$$\mathrm{Top}_*(\Sigma^k X; Y) \twoheadrightarrow [\Sigma^k X; Y]_*$$

$$\begin{tikzcd}
X \wedge I^k/\partial I^k \arrow[r] & Y \\
X \times I^k \arrow[u, two heads] &
\end{tikzcd}$$

en terme de pinch map ??

$$f +_i g : (x, t_1, \dots, t_k) \longmapsto \begin{cases} f(x, t_1, \dots, t_{i-1}, 2t_i, \dots, t_k) & 0 \leq t_i \leq \frac{1}{2} \\ g(x, t_1, \dots, t_{i-1}, 2t_i - 1, \dots t_k) & \frac{1}{2} \leq t_i \leq 1 \end{cases}$$

$1 \leq i \leq k$

(bien défini : $(x_0, \overset{0}{1} \dots \overset{0}{1}) \mapsto y_0$)

Lemme: vérifient l'interchange law : ~~$(\alpha +_i \beta) +_j (\alpha +$~~

$$(\alpha +_i \beta) +_j (\gamma +_i \delta) = (\alpha +_j \gamma) +_i (\beta +_j \delta) \quad \forall i \neq j$$

Proof: $e_2 = (e_2 +_1 e_1) +_2 (e_1 +_1 e_2) \overset{IL}{=} (e_2 +_2 e_1) +_1 (e_1 +_2 e_2) = e_1 = e$

$a +_2 b = (a +_1 e) +_2 (e +_1 b) \overset{IL}{=} (a +_2 e) +_1 (e +_2 b) = a +_1 b$

$a + b = (e + a) + (b + e) \overset{IL}{=} (e + b) + (a + e) = b + a$

$(a+b)+c = (a+b)+(e+c) \overset{IL}{=} (a+e)+(b+c) = a+(b+c)$ $\square$

Rk: $\Sigma_* : [\Sigma A; Y]_* \to [\Sigma^2 A; \Sigma Y] \Rightarrow$ group (homo)morphism

$+ \longmapsto +_1$

Proposition (Thm)

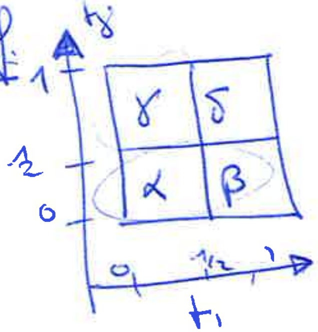
$\pi_1(X, x_0) := [(I/\partial I), (X, x_0)]_*$ group "+" = concat Ω the one you know

$[\Sigma S^0; X]_*$

$\pi_n(X, x_0) := [(I^n, \partial I); (X, x_0)]_*$ abelian group

$[\Sigma^n S^0; X]_*$

Proof:



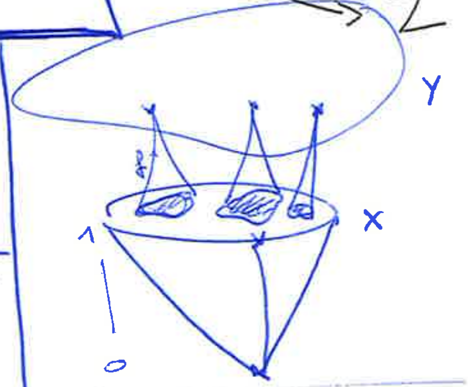
$$\mathrm{LHS}(x, \dots) \longmapsto \begin{cases} \alpha +_i \beta(\quad 2t_j \quad) & 0 \leq t_j \leq \frac{1}{2} \\ \gamma +_i \delta(\quad 2t_j - 1 \quad) & \frac{1}{2} \leq t_j \leq 1 \end{cases} = \begin{cases} \alpha(\dots 2t_i \quad 2t_j) & 0 \leq t_i \leq \frac{1}{2} \\ \beta(\quad 2t_i - 1 \quad 2t_j) & \frac{1}{2} \leq t_i \leq 1 \end{cases}$$

$$\begin{cases} \gamma(\dots 2t_i \quad 2t_j - 1) & 0 \leq t_i \leq \frac{1}{2} \\ \delta(\dots 2t_i - 1 \quad 2t_j - 1) & \frac{1}{2} \leq t_i \, 1 \end{cases}$$

Def: mapping Cone(f) $f : X \to Y$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook, "i_1"'] & Y \arrow[d, "h"] \arrow[ddr, bend left, "g"] & \\
\mathrm{Cone}(X) \arrow[r, "j"'] \arrow[drr, bend right, "h"'] & \mathrm{Cone}(f) \arrow[dr, dashed, "\exists ! \varphi"] & \\
& & Z
\end{tikzcd}$$

$$\mathrm{Cone}(f) \cong \frac{\mathrm{Cone}(X) \vee Y}{(x,1) \sim f(x)}$$



$Y \overset{f_1}{\hookrightarrow} \mathrm{Cone}(f)$ en bedding (inclusion + homeo $Y \to i(Y)$)

Proposition: [Eckmann-Hilton argument]

For any ~~composite~~ (composition rule) $+_1$ $+_2$ satisfying the product unit e_1 e_2

interchange law $\Longrightarrow$ $+_1 = +_2$, $e_1 = e_2$ & commutative associative

~~Def: [mapping cylinder] of pointed map $X \xrightarrow{f} Y$~~

not here $\to$ cofibrations

verifie $X \xrightarrow{f} Y \overset{f_1}{\hookrightarrow} \mathrm{Cone}(f)$ h-coexact

$\mathrm{Con}(X) \to j$ null homotopy for $f_1 f$

pullback Data $- \begin{cases} g : Y \to Z \\ h \Leftrightarrow \text{null homotopy for } g f \end{cases}$

$\Leftrightarrow \mathrm{Cone}(f) \xrightarrow{\varphi} Z$ tq $\begin{cases} \varphi f_1 = g \\ \varphi j = h \end{cases}$

$\Rightarrow gf$ null homotopic iff $\exists \varphi$ tq $g = \varphi f_1$

⑤

Def: [h-coexact sequence]

- **exact**: $(A,a) \xrightarrow{\alpha} (B,b) \xrightarrow{\beta} (C,c)$ in Set$_*$
  ↳ if $\beta^{-1}(c) = \alpha(A)$
  (// exact sequence of groups, modules, etc..)
- **h-coexact**: $U \xrightarrow{f} V \xrightarrow{g} W$ in Top$_*$
  if $[U,Z]_* \xleftarrow{f^*} [V,Z]_* \xleftarrow{g^*} [W,Z]_*$
  exact for any $Z \in$ Top$_*$

ie: $f^{*-1}([U \to x_0]) = \{[\varphi: V \to Z] \text{ tq } \varphi f: U \to Z$ null homotope$\}$

$$\overset{\shortparallel}{g^*([W,X]_*)} = [V \xrightarrow{g} W \xrightarrow{\psi} Z]_*$$

Rk: $X = W$ et $\psi = id_W \Rightarrow$ gf null homotopic

$\boxed{\text{Thm}}$ [(Co)fibre sequence] $\forall$ pt $f: X \to Y$ The sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{Cone}(f) \xrightarrow{p(f)} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma f_1} \Sigma \mathrm{Cone}(f) \xrightarrow{\Sigma p(f)} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \to \cdots$$

is h-coexact, where $\mathrm{Cone}(f) \overset{p(f)}{\twoheadrightarrow} \mathrm{Cone}(f)/_{f_1(Y)} \cong \Sigma X$ quotient map

moreover $\forall Z$

$$\underbrace{[X,Z]_* \xleftarrow{f^*} [Y;Z]_* \xleftarrow{f_1^*} [\mathrm{Cone}(f),Z]_*}_{\text{exact in Sets}_*} \xleftarrow{p(f)^*} \underbrace{[\Sigma X;Z]_* \xleftarrow{(\Sigma f)^*} [\Sigma Y;Z]_* \xleftarrow{(\Sigma f_1)^*} [\Sigma \mathrm{Cone}(f);Z]}_{\text{morph of groups}} \xleftarrow{()} \underbrace{[\Sigma^2 X;Z] \xleftarrow{()} [\Sigma^2 Y;Z] \leftarrow \cdots}_{\text{morph of abelian groups}}$$

Dual constructions (notions)

under the adjunction $\{X \times I \to Y\} \overset{1-1}{\longleftrightarrow} \{X \to Y^I\}$

$$\mathrm{Top}(X \times I, Y) \cong \mathrm{Top}(X, Y^I)$$

$\cup$

homotopies from cst $=$ homotopies from cst

$\updownarrow$ $\updownarrow$

$$\mathrm{Top}_*(\mathrm{Cone}(X);Y) \cong \mathrm{Top}_*(X;\mathrm{Path}(Y))$$

②

where $\mathrm{Path}(Y) := \{\varphi: I \to Y \mid \varphi(0) = y_0\}$

Path: Top$_*$ $\overset{\perp}{\rightleftarrows}$ Top$_*$: Cone

homotopies cst → cst = homotopes cst → cst

$\updownarrow$

$$\mathrm{Top}_*(\Sigma X;Y) \cong \mathrm{Top}_*(X;\Omega Y)$$

$$\Omega Y := \{\varphi: I \to Y \mid \varphi(0) = \varphi(1) = y_0\}$$

fact: Ω: Top$_*$ $\overset{\perp}{\rightleftarrows}$ Top$_*$: Σ

like Σ induces

Ω: hoTop$_*$ $\overset{\perp}{\rightleftarrows}$ hoTop$_*$: Σ

Def: [Path space]
Mapping
of a map

$\forall f: X \to Y$

$$\begin{tikzcd} \mathrm{Path} f \arrow[r, "q"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Path} Y \arrow[d, "e'"] \\ X \arrow[r] & Y \end{tikzcd}$$

$\boxed{\text{Thm}}$ [Fibre Sequence] $\forall f: X \to Y$ in Top$_*$ the following sequence is h-exact

$$\cdots \to \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{\Omega i(f)} \Omega \mathrm{Path}(f) \xrightarrow{\Omega f'} \Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{i(f)} \mathrm{Path}(f) \xrightarrow{f^1} X \xrightarrow{f} Y$$

ie, $\forall Z$

$$\underbrace{\cdots \to [Z,-] \to}_{\text{abelian group morphisms}} \underbrace{[Z,\ ] \to \qquad\qquad}_{\text{group morph.}} \to [Z,X] \xrightarrow{f_*} [Z,Y]$$

where $i(f): \Omega Y \cong (f')^{-1}(*) \hookrightarrow \mathrm{Path}(f)$

$\mathrm{Path}(f) \xrightarrow{f'} X \xrightarrow{f} Y$ is h exact ie

$[\mathrm{Path}(f);Z]_* \xrightarrow{f'_*} [X;Z] \xrightarrow{f_*} [Y;Z]$ exact in Set$_*$

$$\mathrm{Path}(f) = \left\{ \left(u, \underset{\varphi(0)=y_0}{\varphi : I \to Y}\right) \,\middle|\, \varphi(1) = f(u) \right\}$$

③ (Co)Fibrations

Cone(f) ⇝ approximation of $X \xrightarrow{f} Y \xrightarrow{p_1} \mathrm{Cone}(f) \dashrightarrow \cdots$ cofibre sequence

⦚

similar construction : Cone ⟷ cylindre : non-pointé

Def. [Mapping cylinder] $f : X \to Y$ continuous (not nec. pointed)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i_1"'] & Y \arrow[d, "j"] \\ X \times I \arrow[r, "h"'] & \mathrm{Cyl}(f) \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

$\mathrm{Cyl}(f) \overset{\text{def}}{=} \dfrac{X \times I \sqcup Y}{(x,1) \sim f(x)}$

Comes equipped with seminal maps:

$$Y \overset{j}{\hookrightarrow} \mathrm{Cyl}(f) \xrightarrow{p} Y$$

embedding $\begin{cases} (x,t) \longmapsto f(x) \\ y \longmapsto y \end{cases}$

well-defined and continuous.

$y \longmapsto y$

so $pj = id_Y$ ie p retraction of j

in the other way round

$$\mathrm{Cyl}(f) \xrightarrow{p} Y \xrightarrow{j} \mathrm{Cyl}(f) \sim id_{\mathrm{Cyl}(f)}$$

$(x,t) \longmapsto f(x) \longmapsto f(x)$

$y \longmapsto y \longmapsto y$

with the homotopy being for instance → relative to Y

⇓

deformation retract.

$H : \mathrm{Cyl}(f) \times I \to \mathrm{Cyl}(f)$

$((x,t),s) \longmapsto (x, s+(1-s)t)$

$(y,s) \longmapsto y$

well defined. $(x,1,s) \longmapsto (x,1)$

 is is

 $f(x) \longmapsto f(x)$

$H(-,0) = id_{\mathrm{Cyl}(f)}$

$H(-,1) = jp$

Def: Extension of f along i

$$\begin{tikzcd} & X \arrow[dl, dashed, "F"'] \\ Y & A \arrow[l, "f"] \arrow[u, hook, "i"'] \end{tikzcd}$$

← condition pour que ext. de f dépende que de $[f]$ ⇝ Cof

Lifting of f along p:

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "F"] \arrow[r, "f"'] & B \end{tikzcd}$$

← idem ⇝ Fib

Def. [Homotopy Extension Property]

$i : A \to X$ HEP % Z if $\forall g : X \to Z$ tq $h : A \times I \to Z$

$$\begin{tikzcd} A \arrow[r, "i_0^A"] \arrow[d, "i"'] & A \times I \arrow[d, "i \times id_I"] \arrow[ddr, bend left, "h"] & \\ X \arrow[r, "i_0^X"] \arrow[drr, bend right, "g"'] & X \times I \arrow[dr, dashed, "\exists H"] & \\ & & Z \end{tikzcd}$$

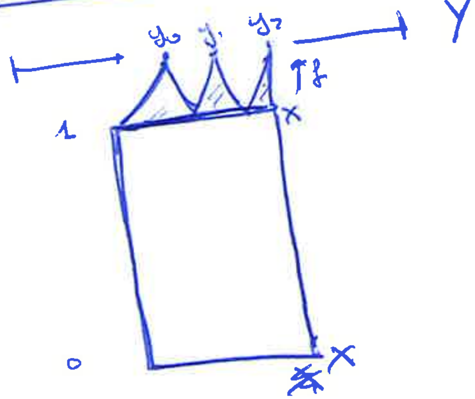
extension of h with initial condition g

⇕

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i"'] & Z^I \arrow[d, "e_0"] \\ X \arrow[r, "g"'] \arrow[ur, dashed, "H"] & Z \end{tikzcd}$$

extension of h along i

Ex:

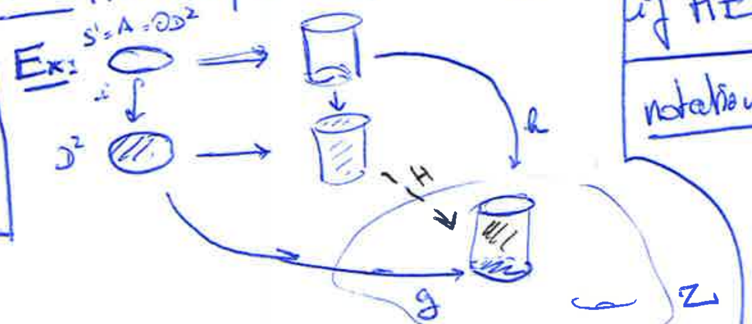


So up to homotopy: can replace f

by: $\underbrace{X \overset{i_0}{\hookrightarrow} \mathrm{Cyl}(f)}_{\text{universal property ?}} \xrightarrow[\sim]{p} Y$

Rk: H not unique (up to homotopy ...)

Ex:



Def: i [cofibration]

if HEP $\forall Z$

notation: $A \overset{i}{\rightarrowtail} X$

⑦

Example

Test space = Cyl(i)

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, tail, "i"'] & A\times I \arrow[d, tail, "i\times id_I"] \arrow[ddr, bend left, "k"] \arrow[dddrr, bend left=40, "h"] & & \\
X \arrow[r, "i_0"] \arrow[drr, bend right, "j"'] \arrow[ddrrr, bend right=40, "g"'] & X\times I \arrow[dr, dashed, "\exists r"'] & & \\
& & Cyl(i) \arrow[ul, "\exists ! s"] \arrow[dr, dashed, "\exists ! k"] & \\
& & & Z
\end{tikzcd}$$

by pullback property

by HEP

by pullback

Proposition TFAE

① i Cofibration

② i satisfies HEP % Cyl(i)

③ s: Cyl(i) ⟶ X×I admits a retraction

Proof: ① ⟹ ② ⟹ ③ ⟶ ①

$rs = id_{Cyl(i)}$ ⟹ r retract of s.

s embedding

□

14/11/18

et ③ → ① ok

Rk: i cofibration ⟹ i embedding
⟸ et i(A) closed (i/X Hausdorff)

$i_0: A \hookrightarrow Cyl(i)$

$i_0 = rsi_0 = ri_0^* i \Rightarrow i$ inj.

①

Proposition: i: A ↪ X

① i cof ⟹ ∃ retraction X×I ⟶ X×0 ∪ A×I

② (A closed) ∃ — ⟹ i cof

②

Proof ①

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & A\times I \arrow[d, "i\times I"] \arrow[ddr, bend left, "h"] & \\
X \arrow[r] \arrow[drr, bend right, "g"'] & X\times I \arrow[dr, dashed, "H"] & \\
& & X\times 0 \cup A\times I \arrow[ul, dotted]
\end{tikzcd}$$

②

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & A\times I \arrow[d, "i\times I"] \arrow[ddr, bend left, "h"] & \\
X \arrow[r] \arrow[drr, bend right, "g"'] & X\times I \arrow[dr, dashed, "\text{[illegible]}"] & \\
& & Z
\end{tikzcd}$$

$\varphi: X\times 0 \cup A\times I \longrightarrow Z$ continuous (A closed)

$(x,0) \longmapsto g(x)$

$(a,t) \longmapsto h(a,t)$

(bien déf/cont)

$H := \varphi . r$ □

Ex: cof $S^{n-1} \hookrightarrow D^n$ ③

$\partial I^n \hookrightarrow I^n$

id_X, any homeo

Proposition: [Cobase change] ⑤

Cofibrations stable under pushout:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, tail, "i"'] & B \arrow[d, tail] \\
X \arrow[r] & X\sqcup_A B \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

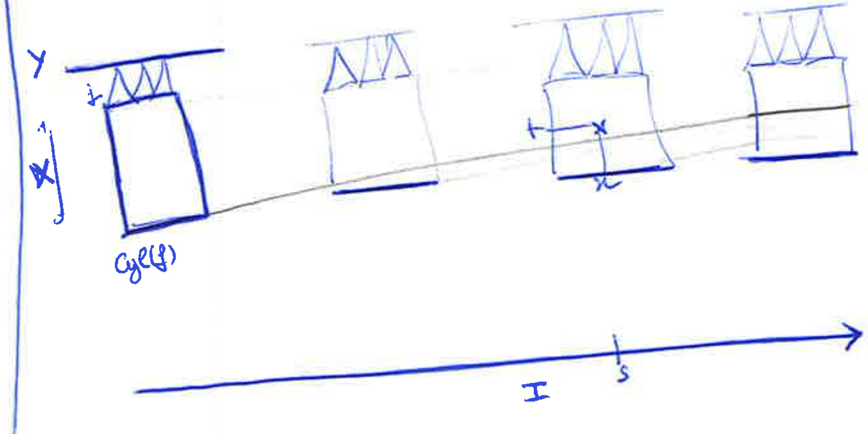
② **Proof:** ~~Exercise~~ . □

Proposition ④

$X \overset{i_0}{\hookrightarrow} Cyl(f) \xrightarrow{p} Y$

cofibration

Proof: find a retraction to [illegible] $Cyl(f)\times 0 \cup X\times I \hookrightarrow Cyl(f)\times I$



□

Proof ①

Rk: up to homotopy, any map changed to a cofibration.

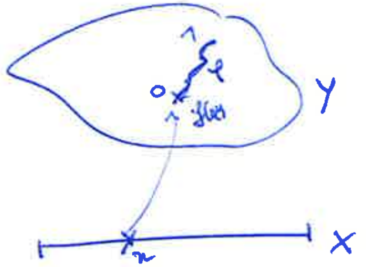
Ex: stable under ∘.

Duality: Def [(unpointed) Mapping Path space] f: X ⟶ Y ⑥

$$\begin{tikzcd}
\{(x,\varphi),\ \varphi(0)=f(x)\} \cong Path(f) \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
Y^I \arrow[r, two heads, "e_0"'] & Y
\end{tikzcd}$$

$\varphi: I \to Y$

Path(f) =



f factors as

$$\begin{tikzcd}
X \arrow[r, hook, "I"'] \arrow[rr, bend left, "f"] & Path(f) \arrow[r, "e_1"] & Y
\end{tikzcd}$$

$x \longmapsto (x, cst = x)$

$(x,\varphi) \longmapsto \varphi(1)$

① si on pose $Path(f) \xrightarrow{p} X$, $(x,\varphi) \longmapsto x$ alors $X \xrightarrow{I} Path(f) \xrightarrow{p} X = id_X$

$$\mathrm{Path}(f) \xrightarrow{p} X \xrightarrow{\Sigma} \mathrm{Path}(f)$$

$$(x,\varphi) \longmapsto x \longmapsto (x, cst = x)$$

$$H : \mathrm{Path}(f) \times I \longrightarrow \mathrm{Path}(f)$$

$$(x,\varphi,s) \longmapsto (x, \varphi(ts))$$

homotopy $s=0$: $H_0 = IP$

$s=1$ $H_1 = id_{\mathrm{Path}(f)}$

↳ Def retract.

$\Rightarrow$ I hoeq quid de e^0 ?

Def: [Homotopy lifting property] $E \xrightarrow{p} B$

HLP % Z:

H : lifting of h with initial condition g

$$\begin{tikzcd}
Z \arrow[drr, bend left, "g"] \arrow[dr, dashed, "\exists H"] \arrow[ddr, bend right, "h"'] & & \\
& E^I \arrow[r, "e^0"] \arrow[d, "p_*"'] & E \arrow[d, "p"] \\
& B^I \arrow[r] & B
\end{tikzcd}$$

$\Updownarrow$

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "j_0"'] & E \arrow[d, "p"] \\
Z \times I \arrow[r, "h"'] \arrow[ur, dashed, "H"] & B
\end{tikzcd}$$

H : lifting of h along p

Def: [(Hurewicz) fibration] HLP % $\forall Z$

$\cap$

[Serre fibration] HLP % I^n, $n \geq 0$ (homotopy groups)

notation: $p : E \twoheadrightarrow B$

E: total space; B: base space

Example: Test space $Z = \mathrm{Path}(p)$

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\exists ! \text{ pullback}"] \arrow[ddrr, bend left, "g"] \arrow[dddrr, bend right, "h"'] & & & \\
& \mathrm{Path}(p) \arrow[drr, bend left] \arrow[ddr, bend right] \arrow[dr, dashed, "\exists \Delta"] & & \\
& & E^I \arrow[r, "e^0"] \arrow[d, "p_*"] \arrow[ul, dashed, bend left, "\exists ! r \text{ by pullback}"] & E \arrow[d, "p"] \\
& & B^I \arrow[r, "e^0"'] & B
\end{tikzcd}$$

Proposition TFAE

① p fibration

② p has the HLP wrt Path(p)

③ $E^I \xrightarrow{r} \mathrm{Path}(p)$ admits a section $(r\Delta = id_{\mathrm{Path}(p)})$

Proof: ① ⇒ ② ⇒ ③ ⇒ ① □

Def: $p : E \to B$ [covering] (cover, covering space) when

. p surj

. $\forall b \in B$, $\exists V$ open neighborhood of b s.t.

$p^{-1}(V)$: each connected component $\xrightarrow[p]{\simeq} V$

Paradigm: $\mathbb{R} \longrightarrow S^1$

$\theta \longmapsto e^{i\theta}$

$\mathbb{Z}/2\mathbb{Z}$ or $\to S^n \longrightarrow \mathbb{R}P^n$

~~**Recall:** universal cover if E simply connected~~

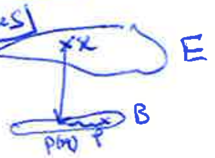
~~$\pi_1(E) = *$~~

~~↳ group of covering transfo $\cong F_b = p^{-1}(b)$~~

~~$\pi_1(S^1) \cong \mathbb{Z}$, $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}$, $n \geq 2$~~

Proposition: {coverings} ⊂ {fibrations}

p satisfies (iii) with a uniques



Proof: section s :

$$E^I \xrightarrow{r} \mathrm{Path}(p)$$

$$(\varphi : I \to E) \longmapsto (\varphi(0), p\varphi)$$

$$E^I \xleftarrow{\Delta} \mathrm{Path}(p)$$

$$\psi \longleftarrow (x, \varphi)$$

Lemma: $\boxed{\exists \psi : I \to E \quad \psi(0) = x \text{ et } p\psi = \varphi}$

(universal property of coverings)

Proof (lemma).

Consider an open cover of $\varphi(I)$ in B

$\varphi^{-1}(-)$ = open cover of I

I compact ⇒ extract a finite one.

step by step use the lemma with the copies of the fiber □

Rk: difficult to characterise fibrations (≠ cofibrations) → so good example

↓ never the less

Prop: $\mathrm{Path}(f) \xrightarrow{e^f} Y$ fibration ⊛

③

Proof: [May]

Rk: same : factorisation

↓ and

Proposition [Base Change]

$$\begin{tikzcd} E\times_f B' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "p"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

fibration fibration

Proposition: $i: A\subset X$ cofibration A, X locally compact

$\Rightarrow i^*: Z^X \twoheadrightarrow Z^A$ fibration

- $p: E\twoheadrightarrow B$ fibration $\Rightarrow p_*: E^Z \twoheadrightarrow B^Z$ for Z locally compact

Application: $\partial I \rightarrowtail I$ $\Rightarrow X^I \twoheadrightarrow X^{\partial I} = X\sqcup X$ fibration

local^t compact

(ok useful)

- e': ~~$\mathrm{Path}(f) \to$~~ $Y^I \twoheadrightarrow Y$ case $e' = i^*$ with $i:\{*\}\to I$ locally cpt.

~~$= \{I \xrightarrow{\varphi} Y, \dots\}$~~

$\Longrightarrow \mathrm{Path}(f) \twoheadrightarrow X$ fibration

Rk: $E \to B$ $\overset{\text{fibration}}{\text{quotient map}}$ (surj) dual argument

Def: [Fiber Bundle] of fibre F

$p: E\twoheadrightarrow B$ surjective st

$\forall b\in B$, $\exists U$ neighborhood such $h: p^{-1}(U) \xrightarrow{\simeq} F\times U$ "local trivialisation"

s.t.

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "h"] \arrow[dr, "p"'] & & F\times U \arrow[dl, "pr_U"] \\ & U & \end{tikzcd}$$

notation: $F \to E \to B$ "short exact seq of spaces"

fiber, total space, base

Ex: $B\times F$ ⇝ "twisted product"

Rk: $B\times F \twoheadrightarrow B$ fibration

Ex: Moebius Band : $M := \dfrac{I\times[-1,1]}{(0,v)\sim(1,-v)}$

$[-1,1] \to M \to I$

~~**Ex:** $d\geq 2$ $\mathbb{Z}/2\mathbb{Z} \to S^d \to \mathbb{P}^d\mathbb{R}$~~

Prop:
- fiber bundle with discrete fiber $\Longrightarrow$ covering
- covering with fibers of same cardinality $\Rightarrow$ fiber bundle
  $\Uparrow$ B connected

Thm [Adams] $F\to E\to B$ fiber bundle with B paracompact ⊛ $\Longrightarrow$ fibration

⊛ ∀ recouvrement ouvert ∃ un local^t fini

∀pt ∃ ouvert tq ∩ ≠ 0 fini.

Ex: compact, CW cx.

Coverings ⊂ fibre bundles ⊂ fibrations ⊂ Serre fibrations

discrete fiber, cst fiber, HLP % I^n

④ Higher homotopy groups

Recall:

$\pi_n(X) := [S^n, X]_* = [\Sigma^n S^0, X]_* = [S^0, \Omega^n X]$

$\pi_n(X,x)$

$\cong \pi_{n-1}(\Omega X) \cong \dots \cong \pi_0(\Omega^n X)$

Apply fiber sequence to the inclusion $A \overset{i}{\hookrightarrow} X$

Prop: $f: X \xrightarrow{\sim} Y$ $\Rightarrow$ $\pi_n X \xrightarrow{\simeq} \pi_n Y$

$\mathcal{P} := \mathrm{Path}(i) = \left\{(a, \varphi: I\to X) \,\middle|\, \begin{matrix}\varphi(0)=*\\ \varphi(1)=a\end{matrix}\right\} = \mathrm{Path}(X; *; A)$

$$\cdots \to \Omega^2 A \xrightarrow{\Omega^2 i} \Omega^2 X \xrightarrow{\Omega i(\iota)} \Omega \mathcal{P} \xrightarrow{\Omega \iota'} \Omega A \xrightarrow{\Omega \iota} \Omega X \xrightarrow{i(\iota)} \mathcal{P} \xrightarrow{\iota'} A \xrightarrow{\iota} X$$

↓ $[S^0,-]$

$$\to [S^0,\Omega^2 A] \to [S^0,\Omega^2 X] \to [S^0,\Omega \mathcal{P}] \to [S^0,\Omega A] \to [S^0,\Omega X] \to [S^0,\mathcal{P}] \to [S^0,A] \to [S^0,X]$$

$$\left.\to \pi_2 A \xrightarrow{\iota_*} \pi_2 X \to \underset{\pi_2(X,A)}{\underset{\parallel}{\pi_1 \mathcal{P}}} \xrightarrow{\partial} \pi_1 A \to \pi_1 X \to \underset{\pi_1(X,A)}{\underbrace{\pi_0 \mathcal{P}}} \xrightarrow{\partial} \pi_0 A \xrightarrow{\iota_*} \overset{\parallel}{\pi_0 X}\right)$$

long exact sequence ⊛

Examples: $\mathbb{Z} \to \mathbb{R} \to S^1$

$\pi_n(S^1) \cong 0 \quad n \geq 2$

$\pi_1(S^1) \cong \mathbb{Z}$

$\mathbb{Z}/_2\mathbb{Z} \to S^d \to \mathbb{P}\mathbb{R}^d \quad d \geq 2$

↳ $\pi_n(S^d) \cong \pi_n(\mathbb{P}\mathbb{R}^d) \quad n \geq 2$

$\pi_1(\mathbb{P}\mathbb{R}^d) \cong \mathbb{Z}/_2\mathbb{Z}$

Prop:
- $\pi_n(S^d) = 0$ for $n < d$
- $\pi_n(S^n) \cong \mathbb{Z}$

Proof:

- admis!

Def: [Relative homotopy groups]

$$\pi_n(X,A) := \pi_{n-1}(\mathrm{Path}(X;*;A)) \cong \pi_0(\Omega^{n-1}\mathrm{Path}(X;*;A))$$

$\overset{\shortparallel}{\pi_n(X,A,*)}$

↳ Grp $n \geq 2$

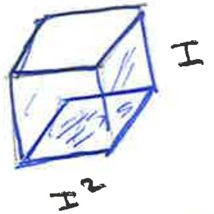
↳ Ab ≥ 3

$$\pi_n(X) = [\Sigma^{n-1}S^0;\underset{\underset{\mathrm{Path}(X;*;*)}{\shortparallel}}{\Omega X}]_* \longrightarrow [\Sigma^{n-1}S^0, \mathrm{Path}(X,*;A)]$$

$\boxed{\text{Thm}}$ ⊛ [long exact sequence of relative homotopy groups]

Define $J^n := \partial I^{n-1} \times I \cup I^{n-1} \times \{0\} \subset I^n$ et $J^1 = \{0\} \subset I$

Ex: **n=3**



n=2

↳ $\pi_n(X,A,*) \cong [(I^n;\partial I^n;J^n),(X,A,*)]$

with this $\partial: \pi_n(X,A) \longrightarrow \pi_{n-1}(A)$

$$(I^n,\partial I^n,J^n) \xrightarrow{\varphi} (X,A,*) \longmapsto ((I^{n-1}\times\{1\}, \partial I^{n-1}\times\{1\}) \xrightarrow[\varphi_{|I^{n-1}\times\{1\}}]{} A,*)$$

and $\iota_*: \pi_n(A) \longrightarrow \pi_n(X)$ & $\pi_n(X) \longrightarrow \pi_n(X,A)$

$(X,*,*) \longrightarrow (X,A,*)$

$\boxed{\text{Thm}}$ [Long exact sequence of a $\overset{\text{Serre}}{\text{fibration}}$]

$p: E \twoheadrightarrow B$ (Serre) fibration with B path connected

long exact sequence:

$$\to \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \cdots \pi_1(B) \xrightarrow{\partial} \pi_0(F) \xrightarrow{i_*} \pi_0(E) \to \{*\}$$

where $b_0 \in B$ base point

$e_0 \in F := p^{-1}(b_0)$ base point

Proof: idea: $\pi_n(E,F) \cong \pi_n(B)$ and apply ⊛ to $F \subset E$ ◻

other cases

↳ Hopf fibrations **Ex**

→ A few (very) computations

Props X contractible $\Rightarrow \pi_n(X) = 0 \quad \forall n \geq 0$

X discrete $\Rightarrow \pi_n(X) = 0 \quad \forall n \geq 1$

Proof:

$$\to \underset{\underset{0}{\shortparallel}}{\pi_n(F)} \to \underbrace{\pi_n(E) \xrightarrow{\sim} \pi_n(B)}_{n \geq 2} \to 0 \ \cdots$$

$$\pi_1(F) \to \pi_1(E) \to \pi_1(B) \to \pi_0(F)$$

$0 \quad \overset{g}{\cong} \quad \cong$ ◻

Prop: $p: E \to B$ (^universal) covering $\Rightarrow \pi_n(E) \cong \pi_n(B) \quad n \geq 2$

$\Rightarrow \pi_1(B) \cong \pi_0(F)$

(ie E simply connected

Proof ① $X \subset Cyl(f)$ cofibration où $Cyl(f) = \dfrac{X\times I \cup Y}{(x,1)\sim f(x)}$

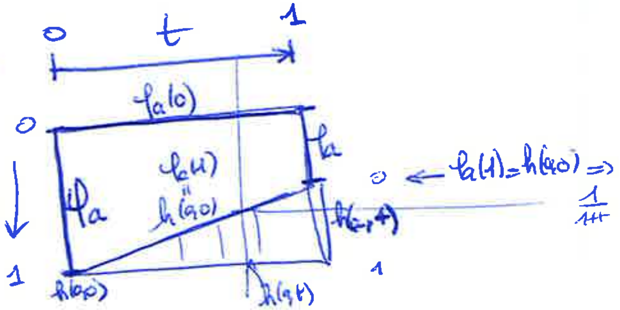
$$Cyl(f)\times 0 \cup X \xleftarrow{r} Cyl(f)\times I$$

$$(y,0) \longleftarrow\!\shortmid (y,s)$$

$$\longleftarrow\!\shortmid (x,t,s)$$

$$\begin{cases} \left(x, \dfrac{2t-s}{2-s}, 0\right) & \text{if } 2t-s \geq 0 \\ (x, 0, s-2t) & \text{if } 2t-s \leq 0 \end{cases}$$

idée:



$\leftarrow \varphi_a(1) = h(a,0) \Rightarrow \frac{1}{1+t}$

on pose (par exemple):

$$\varphi_{a,t}(s) := \begin{cases} \varphi_a((1+t)s) & \text{pour } 0 \leq s \leq \frac{1}{1+t} \\ h(a,(1+t)s-1) & \text{pour } \frac{1}{1+t} \leq s \leq 1 \end{cases}$$

□

Proof ③ Direct checking of HLP wrt A:

$$\begin{tikzcd} A \arrow[r, "g", "\partial_1"'] \arrow[d, "i_0"'] & Path(f) \arrow[d, "e'"] \\ A\times I \arrow[r, "h"'] \arrow[ur, dashed, "\exists H"] & Y \arrow[ul, phantom, "\textcircled{2}_2"] \end{tikzcd}$$

given $g: A \to Path(f)$

$a \mapsto (\underset{X}{\overset{\in}{x_a}}, \underset{I\to Y}{\overset{\in}{\varphi_a}})$ $\varphi_a(0) = f(x_a)$

$h: A\times I \to Y$

$(a,t) \mapsto h(a,t)$

$\square \Leftrightarrow h(a,0) = \varphi_a(1)$

$H: A\times I \to Path(f)$

$(a,t) \longmapsto (x_a, \underset{I\to Y}{\varphi_{a,t}})$ tq $\boxed{\varphi_{a,t}(0) = f(x_a) = \varphi_a(0)}$

tq $\textcircled{2}_1$ $H(a,0) = g(a)$

$\quad\quad \parallel$

$(x_a, \varphi_{a,0}) = (x_a, \varphi_a)$ ie $\boxed{\varphi_{a,0} = \varphi_a}$

$\textcircled{2}_2$ $e' H(a,t) = \boxed{\varphi_{a,t}(1) = h(a,t)}$

Proofs

⑤ CW complexes

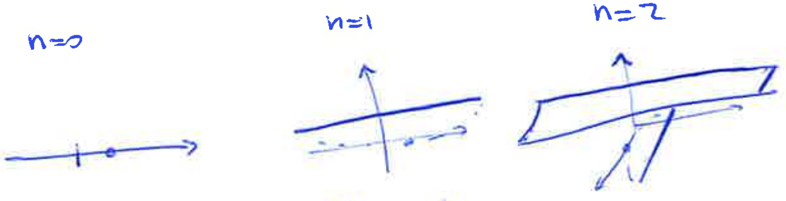
21/11/18

motivation: spaces given by "cells"

Ex 1: polyedral subdivision of S^2 (Euler ~ 1752) — 1st top. invariant

$$\chi(\downarrow) = \underset{\uparrow \atop \#\text{vertices}}{n(0)} - \underset{\uparrow \atop \#\text{edges}}{n(1)} + \underset{\uparrow \atop \#\text{faces}}{n(2)}$$ always the same

Ex 2: (paradigm) $\mathbb{P}^n\mathbb{R} = \mathbb{P}\mathbb{R}^{n+1}$



$\mathbb{P}^0\mathbb{R} = \mathbb{P}\mathbb{R} = \{*\}$

$\mathbb{P}^1\mathbb{R} \cong S^1$

$* \cup_{f_1} D^1$

$* \cup_{f_1} D^1 \cup_{f_2} D^2$ . . . . .

Unfolding:

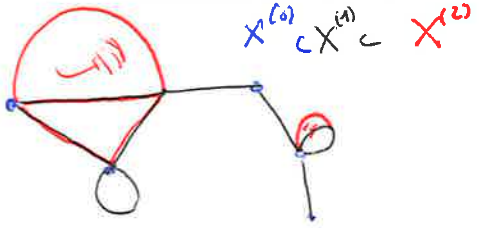
$$\begin{tikzcd} J_0 \times \overset{=\emptyset}{S^{-1}} \arrow[r, "\varphi_0"] \arrow[d, hook] & \overset{=\emptyset}{X^{(-1)}} \arrow[d] \\ J_0 \times D^0 \arrow[r, "\Phi_0"] & \ulcorner J_0 \times D^0 \end{tikzcd}$$

set of points

$$\begin{tikzcd} J_1 \times \overset{= \{0,1\}}{S^0} \arrow[r, "\varphi_1"] \arrow[d, hook] & X^{(0)} \arrow[d, hook] \\ \underset{\overset{\parallel}{I}}{J_1 \times D^1} \arrow[r, "\Phi_1"] & \ulcorner X^{(1)} \end{tikzcd}$$

$$X^{(1)} = \frac{X^{(0)} \sqcup J_1 \times D^1}{\varphi(j,0) \sim (j,0) \atop \varphi(j,1) \sim (j,1)}$$

$$\begin{tikzcd} J_2 \times S^1 \arrow[r] \arrow[d, hook] & X^{(1)} \arrow[d] \\ J_2 \times D^2 \arrow[r] & \ulcorner X^{(2)} \end{tikzcd}$$



Examples.

① $S^n \cong \{*\} \cup D^n$ with $\varphi_n : S^{n-1} \twoheadrightarrow \{*\}$

(already seen: $D^n/_{\partial D^n} \cong I^n/_{\partial I^n} \cong S^n$)

② Graph = dim 1 CW cx (def in fact)

Tree = ——————— simply connected ($\pi_1(X) = 0$)

Torus: 1, 2, 1 .

③ $\mathbb{P}^n\mathbb{R} \cong D^0 \cup_{f_1} D^1 \cup_{f_2} D^2 \cup_{f_3} D^3 \cup \ldots \cup D^n$ $\quad n \leq \infty$.

($\mathbb{P}^\infty\mathbb{R}$)

④ From the Hopf fibration. $S^1 \longrightarrow S^{2n+1} \xrightarrow{f_{2n+2}} \mathbb{P}^n\mathbb{C}$, we get

$$\mathbb{P}^n\mathbb{C} \cong D^0 \cup_{f_0} D^2 \cup \ldots \cup_{f_{2n+2}} D^{2n+2}$$

~~Heavy~~ **Def.** [CW complex] X topo $\underset{\overset{\nwarrow}{\text{or} \cong}}{=}$ colim$_n$ $X^{(n)}$

where

$$\emptyset = X^{(-1)} \subset X^{(0)} \subset X^{(1)} \subset \ldots \subset X^{(n)} \subset \ldots \subset \bigcup_{n=0}^{\infty} X^{(n)} = \operatorname{colim}_n X^{(n)}$$

with

$$\begin{tikzcd} \coprod_{j \in J_{n+1}} S^n = \overset{\swarrow \text{discrete}}{J_n} \times S^{n+1} \arrow[r, "\varphi_{n+1}"] \arrow[d, hook] & X^{(n)} \arrow[d] \\ \coprod_{j \in J_n} D^{n+1} = J_n \times D^{n+1} \arrow[r, "\Phi_{n+1}"] & \ulcorner X^{(n+1)} \end{tikzcd}$$

attaching map

characteristic map

$$X^{(n+1)} \cong \frac{X^{(n)} \sqcup J_n \times D^{n+1}}{\varphi_{n+1}(j,x) \sim (j,x)}$$

• "finite" if $X = X^{(n)}$ for some n → dim $X := n$ if $X^{(n-1)} \subsetneq X^{(n)} = X^{(n+1)} = \ldots$

• $X^{(n)}$: "n^{th}-skeleton"

Case $n = \infty$: endowed with the "weak topology" (or "colimit topology") defined by

$U \subset X$ is open if $U \cap X^{(n)}$ is open $\forall n$. "∞ approximated by n large enough"

→ $\underset{\text{compact}}{f : K} \longrightarrow \underset{\text{continuous}}{X} \Longleftrightarrow f : K \to X^{(n)}$ for some n.

→ $f : X \longrightarrow Y$ continuous iff $f|_{X^{(n)}} : X^{(n)} \longrightarrow Y$ continuous.

Ex: $\mathbb{P}^\infty\mathbb{R}$, $\mathbb{P}^\infty\mathbb{C}$,

Rk: Terminology:

C : Closure-finiteness: the closure of any cell intersects a finite # of other cells

W : Weak topology

Def: [Relative CW cx] (X,A)

Same def with $X^{(-1)}$ replaced by A

Rk: CW cx $= (X,\emptyset)$

A, $A \sqcup J_0 \times D^0$, ……

$\uparrow$ add points

Ex: A a sub CW cx of X

↳ (X,A) CW pair

21/11/18

Proposition: (X,A) relative CW cx $\to$ $A \hookrightarrow X$ cofibration

Propositions:

$$\begin{tikzcd} S^{n-1} \arrow[d, hook] \\ D^n \end{tikzcd} \text{ cof} \overset{Ex...}{\Rightarrow} \begin{tikzcd} J_n\times S^{n-1} \arrow[d] \\ J_n\times D^{n-1} \end{tikzcd} \text{ cof} \overset{Ex.}{\Rightarrow} \begin{tikzcd} X^{(n-1)} \arrow[d, hook] \\ X^{(n)} \end{tikzcd} \text{ cof}$$

et stable par compo (Ex...) $\square$

Rk: $\chi(X) := \sum_n (-1)^n |J_n|$ Euler characteristic

Thm : $\perp$ homotopy

Category?

Def: [Cellular map] $f : X \longrightarrow Y$ cellular if $f(X^{(n)}) \subset Y^{(n)}$

Exercise:

① $X \times Y$: CW cx with $(X\times Y)^{(n)} = \bigcup_{i+j=n} X^{(i)} \times Y^{(j)}$

$\uparrow \uparrow$ CW cxs

② (X,A) CW pair $\Rightarrow X/A$ CW cx.

③ X,Y CW cxs $f : X \to Y$ cellular

factorisation: $X \hookrightarrow Cyl(f) \overset{?}{\overset{\sim}{\twoheadrightarrow}} Y$ cellular

22/11/18

3 main (first) results: Whitehead, CW approximation, Hurewicz

$f : X \xrightarrow{\sim} Y \Rightarrow \pi_n(f)$ iso $\forall n$

$\Leftarrow$ True for X,Y CW cxs!

$\boxed{\text{Thm}}$ [Whitehead]

X,Y CW cxs.

x_0 y_0 base points in $X^{(0)}$ and $Y^{(0)}$ resp

$f : (X,x_0) \to (Y,y_0)$

if $\pi_n(X,x_0) \xrightarrow[\simeq]{\pi_n(f)} \pi_n(Y,y_0)$ $\forall n$ ⊛

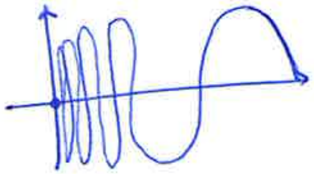
$\overset{\text{thm}}{\Longrightarrow}$ f ho eq

Def: ⊛ weak (homotopy) equiv

↳ Proof

Rk: The hypothesis CW cx is necessary:

Counter Ex: $X = \{x=0\} \cup \{y = \sin\frac{1}{x}\}$



↳ connected

2 path-connected components.

$Y = \{a,b\} \xrightarrow{f} X$ ~~[illegible]~~ weak ho eq

$a \longmapsto 0$

$b \longmapsto \neq 0$

But not a ho eq

since $\nexists f^{-1}$.

Rk: if X,Y CW cx st $\pi_n(X) \cong \pi_n(Y)$

$\not\Rightarrow$ ho eq since one needs a map.

Counterex: $X = S^3 \times \mathbb{P}^2\mathbb{R}$ & $Y = S^2 \times \mathbb{P}^3\mathbb{R}$

Same $\pi_1 \cong \mathbb{Z}/2\mathbb{Z}$ $(\pi_1(X\times Y) \cong \pi_1(X) \times \pi_1(Y))$

same universal cover $S^3 \times S^2$

$\Rightarrow$ same π_n $\forall n \geq 0$.

but smooth 5-manifolds with $Y = S^2 \times \mathbb{P}^3\mathbb{R}$ orientable

$X = S^3 \times \mathbb{P}^2\mathbb{R}$ non orientable

$\Rightarrow \neq H_5 \Rightarrow$ not ho. eq

Rk: algebraic test for ho eq.

13

~~Proof: [Whitehead Thm]~~

Lemma [Compression]

Let (X,A) relative CWcx st for each n st $X^{(n)} \neq X^{(n-1)}$
(Y,B) pour $B \neq \emptyset$ $\Rightarrow \pi_n(Y,B,y_0) = 0$

Then every map $(X,A) \to (Y,B)$ is homotopic $\overset{\wedge}{\text{rel } A}$ to a map $X \to B$

Proof: by induction on the skeleton $X^{(n)}$ of X.

initiation: let n_0 be the first n s.t. X has an n_0-cell

$$\begin{tikzcd}
J_{n_0}\times S^{n_0-1} \arrow[r, "f_{n_0}"] \arrow[d, tail] & A \arrow[r, "f"] \arrow[d, tail] & B \arrow[d, hook] \\
J_{n_0}\times D^{n_0} \arrow[r, "\Phi_{n_0}"] & X^{(n_0)} \arrow[r, "f"'] & Y
\end{tikzcd}$$

Thus. $f\circ\Phi_{n_0} : (J_{n_0}\times D^{n_0}; J_{n_0}\times S^{n_0-1}) \to (Y,B)$
$\Leftrightarrow \{(D^{n_0}, S^{n_0-1}) \to (Y,B)\}_{J_{n_0}}$

But $\pi_{n_0}(Y,B,y_0) = 0 \Rightarrow f\Phi_{n_0} \underset{J_{n_0}\times S^{n_0-1}}{\overset{H}{\sim}} g_{n_0} : J_{n_0}\times D^{n_0} \to B \subset Y$

for any t

$$\begin{tikzcd}
J_{n_0}\times S^{n_0-1} \arrow[r, "f_{n_0}"] \arrow[d, tail] & A \arrow[r, "f"] \arrow[d, tail] \arrow[dr, phantom, "2"] & B \arrow[d] \\
J_{n_0}\times D^{n_0} \arrow[r] \arrow[rr, bend right, "H(-,t)"'] \arrow[ur, phantom, "\lrcorner"] & X^{(n)} \arrow[r, dashed, "\exists h(-,t)"] & Y
\end{tikzcd}$$

and $h(-,0) = f : X^{(n_0)} \to Y$
$h(-,1) \subset B \subset Y$
$X^{(n_0)} \to B \subset Y$

$\underbrace{h(a,t) = f(a)}$

$f \xrightarrow[\text{rel } a]{h} h(-,1)$

Since $X^{(n)} \rightarrowtail X$ cofibration the homotopy $h(-,t) : X^{(n_0)} \to Y$ extends to a homotopy $H : X \to Y$

$$\begin{tikzcd}
X^{(n_0)} \arrow[r, bend left, "f"] \arrow[r, bend right, "h(-,1)"'] & Y
\end{tikzcd} \quad 0 \leq t \leq 1$$

$$\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[r, bend right, "H(-,1)"'] & Y
\end{tikzcd}$$

$\hookrightarrow H(X^{(n_0)}; I) = h(X^{(n_0)}; I)$
$\Downarrow$
$\begin{cases} H \text{ relative à } A. \\ H(\underset{X^{(n_0)}}{-},1) \subset B \end{cases}$

$$\begin{tikzcd}
X^{(n_0)} \arrow[r, "i_0"] \arrow[d, tail] & X^{(n_0)}\times I \arrow[d] \arrow[ddr, bend left, "h"] & \\
X \arrow[r, "i_0"] \arrow[drr, bend right, "f"'] & X\times I \arrow[dr, dashed, "H"] & \\
& & Y
\end{tikzcd}$$

$n \Rightarrow n+1$ use the very same argument

$A \leftrightarrow X^{(n)}$ $\Rightarrow$ a homotopy $H^{(n+1)} : X \to Y$
$X^{(n_0)} \leftrightarrow X^{(n+1)}$

$$\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[r, bend right, "H(-,1)"'] & Y
\end{tikzcd}$$

s.t. $H^{(n+1)}(X^{(n)}; I) = h^{(n+1)}(X^{(n)}, I)$
$\Downarrow$
$\begin{cases} H^{(n+1)} \text{ relative to } X^{(n)} \Rightarrow \text{rel to } A \\ H^{(n+1)}(-,1) \subset B \end{cases}$

if $n = \infty$: $H^{(0)} : 0 \to \frac{1}{2}$
$H^{(1)} : \frac{1}{2} \to \frac{3}{4}$
$H^{(2)} : \frac{3}{4} \to \frac{7}{8}$
$H^{(n)} : \frac{2^n-1}{2^n} \to \frac{2^{n+1}-1}{2^{n+1}}$
$\qquad\quad \overset{\shortparallel}{1-\frac{1}{2^n}} \qquad ; \qquad \overset{\shortparallel}{1-\frac{1}{2^{n+1}}}$

Cor. $h : B \to Y$ weak equivalence / n-connected
$\pi_k(B) \underset{\cong}{\xrightarrow{h_*}} \pi_k(Y)$ $k<n$
$\left(\pi_n(B) \xrightarrow{h_*} \pi_n(Y)\right)$

then

$h_* : [X;B] \xrightarrow{\cong} [X;Y]$ if $\left\{X \text{ CW } (\dim < n)\right.$

Proof: using the factorisation
$\underbrace{B \rightarrowtail \mathrm{Cyl}(h)} \xrightarrow{\sim} Y$
can thus assume $\Rightarrow$ h inclusion

The long exact sequence of relative homotopy groups:

$$\pi_n(B) \xrightarrow{\cong} \pi_n(Y) \to \overset{0}{\pi_n(Y,B)} \to \pi_{n-1}(B) \xrightarrow{\cong} \pi_{n-1}(Y) \to \overset{0}{\pi_{n-1}(Y,B)} \to \cdots$$

So compression lemma ⇒ every map $(X,\emptyset) \xrightarrow{f} (Y,B)$ is homotopic to a map $g: X \to B \subset Y$. i.e. surj holds
relative CW cx

→ inj: let $f,g: X \to B$ s.t. $f \sim g: X \to B \subset Y$

i.e. $\exists H$:
$$\begin{tikzcd} X\times I \arrow[r] & Y \\ X\times\partial I \arrow[u, hook] \arrow[r, "f\cdot g"'] & B \arrow[u, hook] \end{tikzcd}$$
by the compression lemma applied to the relative CW cx $(X\times I, X\times\partial I) \to (Y,B)$

H is homotopic to
$$\begin{tikzcd} h: X\times I \arrow[r] & B \\ X\times\partial I \arrow[u, hook] \arrow[ur, "(f\cdot g)"'] & \end{tikzcd}$$
relative to $X\times\partial I$

so $f \overset{h}{\sim} g$ ⇒ injectivity □

Rk: stronger statement:

$f: X \to Y$ an n-(weak) equiv $\pi_k \cong$ $k<n$, $\pi_k \twoheadrightarrow n$

$\dim X, Y \leq n$ ⇒ ho eq

↳ X finite CW cx $\dim X > 1$, X not contractible

has infinitely many non 0 π_n

Whitehead ⇒ low π_k ⇒ higher ones

Proof [thm Whitehead] $f: X \to Y$ iso $\pi_n(X) \xrightarrow[\cong]{f_*} \pi_n(Y)$ $\forall n \geq 0$

Cor ⇒ $[Y,X] \xrightarrow[\cong]{f_*} [X,X]$, $[g] \longmapsto [id_X]$ i.e. $fg \sim id_Y$ ⇒ g w.eq $\overset{\text{Cor}}{\Rightarrow}$ $[X,Y] \xrightarrow{g_*} [Y,Y]$, $[\varphi] \longmapsto [id_Y]$ i.e.

i.e. $g\varphi \sim id_Y$ ⇒ $fg\varphi \sim f$ ⇒ $\varphi \sim f$ and $gf \sim id_Y$ □

~~Def~~ 2 spaces are said to be weakly homotopy equivalent if $\exists f: X \to Y$ weak h.eq ⚠ not an equivalence relation (except for CW cx by the Whitehead thm

Cor: X CW cx contractible iff $\pi_n(X)=0$ $\forall n \geq 0$

Proof Consider $X \to *$ and apply Whitehead Thm □

Exercise

~~Proposition: A CW cx, $f: X \to Y$ weak h.eq $\Rightarrow f_*: [A,X] \xrightarrow{\cong} [A,Y]$ bijection~~

Exercise?

Thm [CW approximation]

For any space X, $\exists$ CW X_{cw} and a weak equivalence

$$X_{cw} \xrightarrow[\sim_w]{\ell_X} X$$

$\forall f: X \to Y$ $\exists$ map $X_{cw} \xrightarrow{F} Y_{cw}$ st

$$\begin{tikzcd} X_{cw} \arrow[r, "\ell_X"] \arrow[d, "F"'] \arrow[dr, phantom, "2"] & X \arrow[d, "f"] \\ Y_{cw} \arrow[r, "\ell_Y"'] & Y \end{tikzcd}$$

hc

unique up to ho

Rk: if X is n-connected ⇒ X_{cw} can be chosen to have one vertex and no k-cells for $k \leq n$.

↑ Def

idea: reconstru Thm 6.2

Proof: $\exists$ construction $X_{cw} \xrightarrow{\ell_X} X$ ....

Cor: $[X_{cw}, Y_{cw}] \xrightarrow[\cong]{\ell_{Y*}} [X_{cw}, Y]$

$F \longleftarrow f\,\ell_X$

$\exists!$, up to ho F

$\ell_Y \circ F \sim f\ell_X$ □

Hurewicz morphism

$f: S^n \to X \Rightarrow H_n(S^n) \xrightarrow{f_*} H_n(X)$

$H_n(S^n) = \mathbb{Z}$

$1 \longmapsto f_*(1)$

So this defines the Hurewicz morphism

$[S^n, X] \xrightarrow{\mathcal{H}} H_n(X)$

$[S^n, X] \cong \pi_n(X)$

$[f] \longmapsto f_*(1)$

Rk: group morph.

Example: $X = S^n$: $\tilde{\mathcal{H}}: \pi_n(S^n) \to \mathbb{Z}$

Def: $\tilde{\mathcal{H}}(f) \in \mathbb{Z}$: degree of f

Thm [Brouwer] $\tilde{\mathcal{H}}$ iso

(easy ⇒)

↳ particular case of .

Thm [Hurewicz]

st $\pi_k(X)=0$ $\forall k<n$ for $n \geq 1$

(i) $\tilde{H}_k(X)=0$ for $k<n$

(ii) $\mathcal{H}: H_n(X) \xrightarrow{\cong} \pi_n(X)$

Rk: relative form: (X,A) $(n-1)$-connected pair ($\pi_k(X,A)=0$ $\forall k<n$) of path connected space, $n \geq 2$, $A \neq \emptyset$

$\pi_n(X,A,x_0) \cong H_n(X,A)$ &

$(H_i(X,A)=0 \ \forall i<n)$

Cor: X, Y ~~(CW cxs)~~ CW cxs simply connected

$f: X \to Y$ homology iso ⇒ f h.eq

Proof: $f: X \hookrightarrow \underbrace{Cyl(f) \xrightarrow{\simeq} Y}_{\text{def retract}}$

$\pi_i(\cdot) \cong \pi_i(Y)$

$H_i(\cdot) \cong H_i(Y)$

$X \hookrightarrow Cyl(f) \Rightarrow$

$\cdots \to H_i(X) \xrightarrow{\cong} H_i(Y) \to H_i(Cyl(f), X) \xrightarrow{\cong} \cdots$

$H_i(Cyl(f),X) = 0$ by ⇒ and $\pi_i(Cyl, X) = 0$

$\pi_i(X) \xrightarrow{\cong} \pi_i(Y) \to \pi_i(Cyl \ldots$

$\Rightarrow H_n(Cyl(f), X) = 0 \Rightarrow X$ def retract of $Cyl(f)$ □