Master 2: Homotopy Theories 2021-2022

**Worksheet 2**

**HOMOTOPY THEORY OF TOPOLOGICAL SPACES II**

**Exercise 1** (Homotopy invariance).

(1) Is the property of being a cofibration a homotopy invariant notion: if $i : A \rightarrowtail X$ is a cofibration and if $j : A \to X$ a map homotopy equivalent to $i$, is $j$ a cofibration?
(2) Is the property of being a fibration a homotopy invariant notion: if $p : E \twoheadrightarrow B$ is a fibration and if $q : E \to B$ a map homotopy equivalent to $p$, is $q$ a fibration?

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**Exercise 2** (Fibrations and cofibrations).

(1) Show that cofibrations are stable under composition: if $i : A \rightarrowtail B$ and $j : B \rightarrowtail C$ are two cofibrations then $ji : A \rightarrowtail C$ is a cofibration.
(2) Show that cofibrations are stable under coproduct: if $i : A \rightarrowtail X$ and $j : B \rightarrowtail Y$ are two cofibrations then $i \sqcup j : A \sqcup B \rightarrowtail X \sqcup Y$ is a cofibration.
(3) Show that cofibrations are stable under pushout: if $i : A \rightarrowtail X$ is a cofibration then

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "i"] & B \arrow[d, tail, "j"] \\ X \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & X \underset{f}{\sqcup} B \end{tikzcd}$$

the map $j : B \rightarrowtail X \underset{f}{\sqcup} B$ is a cofibration.
(4) Show the map $E \twoheadrightarrow \{*\}$ is a fibration.
(5) Show that fibrations are stable under composition: if $p : C \twoheadrightarrow D$ and $q : D \twoheadrightarrow E$ are two fibrations then $qp : C \twoheadrightarrow E$ is a fibration.
(6) Show that fibrations are stable under product: if $p : D \twoheadrightarrow A$ and $q : E \twoheadrightarrow B$ are two fibrations then $p \times q : D \times E \twoheadrightarrow A \times B$ is a fibration.
(7) Show that fibrations are stable under pullback: if $p : E \twoheadrightarrow B$ is a fibration then

$$\begin{tikzcd} X \underset{f}{\times} E \arrow[r] \arrow[d, two heads, "q"] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\ X \arrow[r, "f"] & X \underset{f}{\sqcup} B \end{tikzcd}$$

the map $q : X \underset{f}{\times} E \twoheadrightarrow X$ is a fibration.

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**Exercise 3** (Uniqueness of factorisation). We work in the category $\mathsf{Top}^A$ of maps under $A$: its objects are maps $i : A \to X$ with domain $A$ and its morphisms between two maps $i : A \to X$ and $j : A \to Y$ are maps $f : X \to Y$, such that $fi = j$:

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "j"] & \\ X \arrow[rr, "f"] & & Y . \end{tikzcd}$$

A *homotopy* $H : X \times I \to Y$ *under* $A$ between two such maps $f \sim g$ is a homotopy such that every map $H(-,t)$ lives in $\mathsf{Top}^A(i,j)$, for $t \in I$, i.e. $H(-,t)i = j$. This induces an equivalence relation called *homotopy equivalence under* $A$. We will admit the following (seminal) theorem.

Theorem. *Let $i : A \to X$ and $j : A \to Y$ be two cofibrations and let $f : X \to Y$ such that $fi = j$. If $f$ is a homotopy equivalence, then $f$ is a homotopy equivalence under $A$.*

(1) Show that if a map is a cofibration and a homotopy equivalence, then it is a deformation retract.
(2) Show the following uniqueness statement for the factorisation of a map into the composite of a cofibration with a homotopy equivalence: let

$$X \overset{}{\underset{j}{\rightarrowtail}} Z \xrightarrow[q]{\sim} Y \quad \text{and} \quad X \underset{j'}{\rightarrowtail} Z' \xrightarrow[q']{\sim} Y$$

be two such factorisations, then there exists a homotopy equivalence $k : Z \xrightarrow{\sim} Z'$ such that the following is commutative on the left-hand side and homotopy commutative on the right-hand side

$$\begin{tikzcd} & Z \arrow[dd, "h"] \arrow[dr, "q"] & \\ X \arrow[ur, tail, "j"] \arrow[dr, tail, "j'"'] & & Y \\ & Z' . \arrow[ur, "q'"'] & \end{tikzcd}$$

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**Exercise 4** (Fibrations). Let $p : E \to B$ be a fibration with $B$ path connected.

(1) Show that $p$ is surjective.
(2) Show that two fibers $p^{-1}(b)$ and $p^{-1}(b')$, for $b, b' \in B$, are homotopy equivalent.

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**Exercise 5** (Hopf fibration).

(1) Show that there exists a fiber bundle of the form $S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{P}^n\mathbb{C}$ .
(2) What can you say about the homotopy groups $\pi_n(\mathbb{P}^n\mathbb{C})$, for $n \geqslant 0$, of the complex projective spaces?
(3) Compute $\pi_2(S^2)$ and prove that $\pi_n(S^3) \cong \pi_n(S^2)$, for $n \geqslant 3$.

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**Exercise 6** (Real projective spaces).

(1) What can you say about $\pi_n(\mathbb{P}^1\mathbb{R})$, for $n \geqslant 0$?
(2) Show that the embedding $\mathbb{P}^k\mathbb{R} \hookrightarrow \mathbb{P}^n\mathbb{R}$ does not admit a retraction when $0 < k < n$.
(3) Let $d \geqslant 2$. Compute the homotopy groups $\pi_n(\mathbb{P}^d\mathbb{R})$ for $0 \leqslant n \leqslant d$.
(4) What can you say about $\pi_n(\mathbb{P}^d\mathbb{R})$, for $n \geqslant d + 1$ ?

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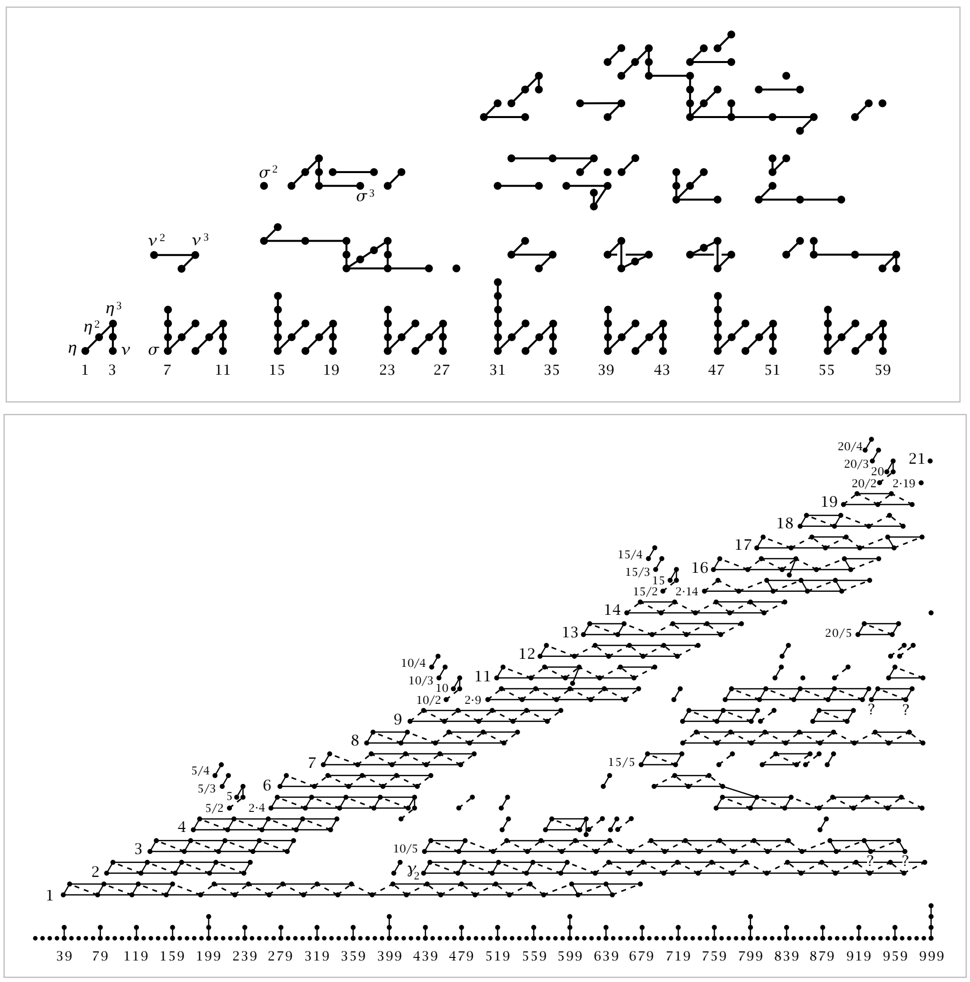
FIGURE 1. The first homotopy groups of spheres

$\pi_i(S^n)$

| $n \downarrow$ / $i \longrightarrow$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $\mathbb{Z}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{12}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_3$ | $\mathbb{Z}_{15}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2\times\mathbb{Z}_2$ |
| 3 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{12}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_3$ | $\mathbb{Z}_{15}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2\times\mathbb{Z}_2$ |
| 4 | 0 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}\times\mathbb{Z}_{12}$ | $\mathbb{Z}_2\times\mathbb{Z}_2$ | $\mathbb{Z}_2\times\mathbb{Z}_2$ | $\mathbb{Z}_{24}\times\mathbb{Z}_3$ | $\mathbb{Z}_{15}$ | $\mathbb{Z}_2$ |
| 5 | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{24}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{30}$ |
| 6 | 0 | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{24}$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{24}$ | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{24}$ | 0 |

FIGURE 2. Examples of tables in stable homotopy theory





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