



**WORKSHEET 3**

**CW COMPLEXES**

**Exercise 1** (Examples of CW complex).

- (1) For  $n \in \mathbb{N}$ , is  $\mathbb{R}^n$  a CW complex? It is a finite CW complex?
- (2) Show that the product  $X \times Y$  of two finite CW complexes is again a CW complex.
- (3) Let  $(X, A)$  be a relative CW complex. Show that  $X/A$  is also a CW complex.
- (4) Show that the  $n$ -times iterated (unpointed) suspension  $\Sigma^n X$  of a CW complex is again a CW complex, where

$$\Sigma X := \frac{X \times I}{X \times \partial I}.$$

- (5) Let  $f : X \rightarrow Y$  be a cellular map between CW complexes. Show that the factorisation of  $f$

$$X \longrightarrow \text{Cyl}(f) \xrightarrow{\sim} Y$$

is the composite of two cellular maps.



**Exercise 2** (C in CW complex).

- (1) Show that every CW complex  $X$  is in set-theoretical bijection with

$$X \cong \bigsqcup_{n \in \mathbb{N}, j \in J_n} \mathring{D}^n,$$

where  $\mathring{D}^n = D^n \setminus \partial D^n$  is the interior of  $D^n$ , with  $\mathring{D}^0 = D^0 = \{*\}$  by convention.

- (2) Show that any compact subspace of a CW complex intersects only a finite number of cells.



**Exercise 3** (Infinite CW complexes).

- (1) Show that the  $n$ -dimension sphere  $S^n$ , for any  $n \in \mathbb{N}$ , admits a CW complex structure with two  $k$ -dimensional cells for any  $0 \leq k \leq n$ .

By passing to the colimit

$$S^0 \subset S^1 \subset S^2 \subset \dots \subset S^\infty = \bigcup_{n \geq 0} S^n,$$

this defines a new CW complex called the *infinite dimensional sphere* and denoted by  $S^\infty$ .

- (2) Let  $X$  be a CW complex which is the union (colimit) of an increasing sequence of CW subcomplexes

$$X_1 \subset X_2 \subset X_3 \subset \dots \subset X = \bigcup_{n \geq 1} X_n$$

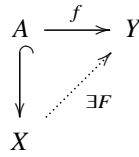
such that each inclusion  $X_n \hookrightarrow X_{n+1}$  is nullhomotopic. Show that  $X$  is contractible.

- (3) Show that  $S^\infty$  is contractible.



**Exercise 4** (Extension Lemma).

- (1) Let  $(X, A)$  be a relative CW complex and a map  $f : A \rightarrow Y$  with  $Y$  path connected. Show that  $f$  extends to a map  $F : X \rightarrow Y$



if  $\pi_{n-1}(Y) = 0$  for any  $n$  such that  $X \setminus A$  has cells of dimension  $n$ .

- (2) Let  $(X, A)$  be a CW pair such that  $A$  is contractible. Show that  $X$  retracts to  $A$ .



**Exercise 5** (Cellular approximation).

- (1) Show that every map  $f : X \rightarrow Y$  between CW complexes is homotopic to a cellular map.   
 HINT: We will assume the following result, see [Hatcher, Lemma 4.10]. Let  $f : (X, A) \rightarrow (Z, Y)$  be a map of CW-complexes such that the dimension of  $A$  is less than  $n$ ,  $X \setminus A$  is made up of only one  $n$ -dimensional cell, the dimension of  $Y$  is less than  $m$ ,  $Z \setminus Y$  is made up of only one cell of dimension  $N \geq m$ . The map  $f$  is homotopic relative to  $A$  to a map  $g : (X, A) \rightarrow (Z, Y)$  such that  $g(X \setminus A)$  misses a point of  $Z \setminus Y$ .
- (2) Prove that  $\pi_k(S^n) = 0$  for  $k < n$ .
- (3) Show that every map  $f : (X, A) \rightarrow (Y, B)$  between CW pairs is homotopic to a cellular map.
- (4) Let  $(X, A)$  be a CW pair such that  $X \setminus A$  has only cells in dimension greater than  $n$ . Show that  $(X, A)$  is  $n$ -connected, that is  $\pi_k(X, A) = 0$ , for  $k \leq n$ .
- (5) Let  $X$  be a CW complex. Show that the CW pair  $(X, X^{(n-1)})$  is  $n$ -connected and that the inclusion  $X^{(n-1)} \hookrightarrow X$  induces isomorphisms

$$\pi_k(X^{(n-1)}) \xrightarrow{\cong} \pi_k(X),$$

for  $k < n$ , and a surjection  $\pi_n(X^{(n-1)}) \twoheadrightarrow \pi_n(X)$ .

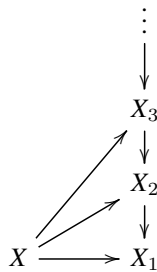


**Exercise 6** (Postnikov tower).

- (1) Let  $X$  be a CW complex. For  $n \geq 1$ , show that there exists a CW complex  $X_n$  for which  $X \subset X_n$  is a CW subcomplex and which satisfies

$$\begin{cases} \pi_k(X_n) \cong \pi_k(X), & \text{for } k \leq n, \\ \pi_k(X_n) \cong 0, & \text{for } k > n. \end{cases}$$

- (2) Show that there exists maps  $X_{n+1} \rightarrow X_n$  which make the following diagram commutative



✉ Bruno Vallette: vallette@math.univ-paris13.fr .

🌐 Web page: www.math.univ-paris13.fr/~vallette/ .