Worksheet 3

CW complexes

Exercise 1 (Examples of CW complex).

(1) For $n \in \mathbb{N}$, is $\mathbb{R}^n$ a CW complex? It is a finite CW complex?
(2) Show that the product $X \times Y$ of two finite CW complexes is again a CW complex.
(3) Let (X, A) be a relative CW complex. Show that X/A is also a CW complex.
(4) Show that the n-times iterated (unpointed) suspension $\Sigma^n X$ of a CW complex is again a CW complex, where
$$\Sigma X \coloneqq \frac{X \times I}{X \times \partial I} .$$
(5) Let $f : X \to Y$ be a cellular map between CW complexes. Show that the factorisation of f
$$X \rightarrowtail \mathrm{Cyl}(f) \xrightarrow{\sim} Y$$
is the composite of two cellular maps.

Exercise 2 (C in CW complex).

(1) Show that every CW complex X is in set-theoretical bijection with
$$X \cong \bigsqcup_{n\in\mathbb{N}, j\in J_n} \mathring{D}^n ,$$
where $\mathring{D}^n = D^n \backslash \partial D^n$ is the interior of D^n, with $\mathring{D}^0 = D^0 = \{*\}$ by convention.
(2) Show that any compact subspace of a CW complex intersects only a finite number of cells.

Exercise 3 (Infinite CW complexes).

(1) Show that the n-dimension sphere S^n, for any $n \in \mathbb{N}$, admits a CW complex structure with two k-dimensional cells for any $0 \leqslant k \leqslant n$.
By passing to the colimit
$$S^0 \subset S^1 \subset S^2 \subset \cdots \subset S^\infty = \bigcup_{n\geqslant 0} S^n ,$$
this defines a new CW complex called the ***infinite dimensional sphere*** and denoted by S^∞.
(2) Let X be a CW complex which is the union (colimit) of an increasing sequence of CW subcomplexes
$$X_1 \subset X_2 \subset X_3 \subset \cdots \subset X = \bigcup_{n\geqslant 1} X_n$$
such that each inclusion $X_n \rightarrowtail X_{n+1}$ is nullhomotopic. Show that X is contractible.
(3) Show that S^∞ is contractible.

Exercise 4 (Extension Lemma).

(1) Let (X, A) be a relative CW complex and a map $f : A \to Y$ with Y path connected. Show that f extends to a map $F : X \to Y$

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ \big\downarrow & \nearrow_{\exists F} & \\ X & & \end{array}$$

if $\pi_{n-1}(Y) = 0$ for any n such that $X \backslash A$ has cells of dimension n.

(2) Let (X, A) be a CW pair such that A is contractible. Show that X retracts to A.

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Exercise 5 (Cellular approximation).

(1) Show that every map $f : X \to Y$ between CW complexes is homotopic to a cellular map.
HINT: We will assume the following result, see [Hatcher, Lemma 4.10]. Let $f : (X, A) \to (Z, Y)$ be a map of CW-complexes such that the dimension of A is less than n, $X \backslash A$ is made up of only one n-dimensional cell, the dimension of Y is less than m, $Z \backslash Y$ is made up of only one cell of dimension $N \geqslant m$. The map f is homotopic relative to A to a map $g : (X, A) \to (Z, Y)$ such that $g(X \backslash A)$ misses a point of $Z \backslash Y$.

(2) Prove that $\pi_k(S^n) = 0$ for $k < n$.

(3) Show that every map $f : (X, A) \to (Y, B)$ between CW pairs is homotopic to a cellular map.

(4) Let (X, A) be a CW pair such that $X \backslash A$ has only cells in dimension greater than n. Show that (X, A) is n-connected, that is $\pi_k(X, A) = 0$, for $k \leqslant n$.

(5) Let X be a CW complex. Show that the CW pair $\big(X, X^{(n-1)}\big)$ is n-connected and that the inclusion $X^{(n-1)} \rightarrowtail X$ induces isomorphisms

$$\pi_k\big(X^{(n-1)}\big) \xrightarrow{\cong} \pi_k(X)\ ,$$

for $k < n$, and a surjection $\pi_n\big(X^{(n-1)}\big) \twoheadrightarrow \pi_n(X)$.

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Exercise 6 (Postnikov tower).

(1) Let X be a CW complex. For $n \geqslant 1$, show that there exists a CW complex X_n for which $X \subset X_n$ is a CW subcomplex and which satisfies

$$\begin{cases} \pi_k(X_n) \cong \pi_k(X)\ , & \text{for } k \leqslant n\ , \\ \pi_k(X_n) \cong 0\ , & \text{for } k > n\ . \end{cases}$$

(2) Show that there exists maps $X_{n+1} \to X_n$ which make the following diagram commutative

$$\begin{array}{ccc} & & \vdots \\ & & \big\downarrow \\ & \nearrow & X_3 \\ & & \big\downarrow \\ & \nearrow & X_2 \\ & & \big\downarrow \\ X & \longrightarrow & X_1 \end{array}$$

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