Worksheet 4

SIMPLICIAL SETS

Exercise 1 (Simplicial approximation).

(1) Show that every CW complex X is homotopy equivalent to the geometric realization simplicial complex $\mathfrak{X}$.

(2) Show that every topological space X admits a weakly homotopy equivalent simplicial complex $\mathfrak{X}$.

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Exercise 2 (From Δ-complexes to simplicial sets).
Let $\Phi : \bar{\Delta} \rightarrowtail \Delta$ be the embedding of the reduced simplicial category in the simplicial category.

(1) Describe the pullback functor $(\Phi^{\mathrm{op}})^* : \mathsf{sSet} \to \Delta\mathsf{Cx}$ from the category of simplicial sets to the category of Δ-complexes, defined by $(\Phi^{\mathrm{op}})^*(\mathfrak{X}) \coloneqq \mathfrak{X} \circ \Phi^{\mathrm{op}}$, for any $\mathfrak{X} : \Delta^{\mathrm{op}} \to \mathsf{Set}$.

(2) Is the geometric realisation $|\mathfrak{X}|$ of a simplicial set $\mathfrak{X}$ in continuous bijection with the geometric realisation $|(\Phi^{\mathrm{op}})^*(\mathfrak{X})|_\Delta$ of its associated Δ-complex.

(3) Show that the functor $(\Phi^{\mathrm{op}})^*$ admits a left adjoint functor L given by

$$(\mathrm{L}\mathfrak{X})_n = \left\{(\varphi, x) \mid \varphi : [n] \twoheadrightarrow [m] \text{ non-decreasing}, x \in X_m\right\},$$

where the faces are given by $d_i(\varphi, x) \coloneqq (\varphi\delta_i, x)$, if $\varphi\delta_i$ is surjective, otherwise $\varphi\delta_i$ can be written in a unique way $\delta_j\psi$ with ψ surjective and then $d_i(\varphi, x) = (\psi, d_j(x))$. The degeneracies are given by $s_i(\varphi, x) \coloneqq (\varphi\sigma_i, x)$.

(4) Describe the image under L of the following Δ-complex $\mathfrak{X}$, which is a model for the circle with two cells:

$$X_0 = \{0\}, \quad X_1 = \{01\}, \quad X_2 = X_3 = \cdots = \emptyset \qquad \text{with} \qquad d_0(01) = d_1(01) = 0 .$$

(5) Let $\mathfrak{X}$ be a Δ-complex. Show that its geometrical realisation is in continuous bijection with the geometrical realisation of the associated simplicial set, i.e.

$$|\mathfrak{X}|_\Delta \cong |\mathrm{L}\mathfrak{X}| .$$

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Exercise 3 (n-skeleton). We consider the full sub-category Δ_n of the simplex category Δ made up of only the objects $[0], \ldots, [n]$; we denote this embedding of categories by $\Upsilon : \Delta_n \rightarrowtail \Delta$. Presheaves $\Delta_n^{\mathrm{op}} \to \mathsf{Set}$ over the category Δ_n are called *n-truncated simplicial sets*; we denote their category by $\Delta_n\mathsf{Set}$.

(1) Describe the pullback functor $(\Upsilon^{\mathrm{op}})^* : \mathsf{sSet} \to \Delta_n\mathsf{Set}$ from the category of simplicial sets to the category of n-truncated simplicial sets.

(2) Show that the functor $(\Upsilon^{\mathrm{op}})^*$ admits a left adjoint functor L given by $(\mathrm{L}\mathfrak{X})_m = X_m$, for $m \leqslant n$ and

$$(\mathrm{L}\mathfrak{X})_m = \left\{(\varphi, x) \mid \varphi : [m] \twoheadrightarrow [n] \text{ non-decreasing}, x \in X_n\right\}, \quad \text{for} \quad m > n .$$

(3) Show that $\mathrm{L}(\Upsilon^{\mathrm{op}})^*(\mathfrak{X}) \cong \mathrm{sk}_n\mathfrak{X}$ for every simplicial set $\mathfrak{X}$.

(4) Describe the unity of this adjunction.

(5) Using the identification of Question (3), show that the counit of adjunction amounts to the following inclusion of simplicial sets

$$\mathrm{L}\Upsilon^*(\mathfrak{X}) \cong \mathrm{sk}_n\mathfrak{X} \rightarrowtail \mathfrak{X}\ .$$

(6) Conclude that the restriction of this adjunction to the full sub-category of simplicial sets of dimension less than n is an equivalence of categories.

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Exercise 4 ((Co)limits).

(1) Show that the category sSet of simplicial sets admit all limits and all colimits.
HINT: Consider first the point-wise set-theoretical limits and colimits.

(2) Describe the initial simplicial set, the terminal simplicial set, the coproduct of simplicial sets, and the product of simplicial sets.

(3) Show that the n-skeleton of a simplicial set can be written as a pushout of the following type:

$$\begin{array}{ccc} \coprod_{x\in \mathrm{N}X_n} \partial\Delta^n & \longrightarrow & \mathrm{sk}_{n-1}\mathfrak{X} \\ \downarrow & \ulcorner & \downarrow \\ \coprod_{x\in \mathrm{N}X_n} \Delta^n & \longrightarrow & \mathrm{sk}_n\mathfrak{X}\ . \end{array}$$

(4) Show that the boundary $\partial\Delta^n$ of the standard n-simplex can be written as a coequalizer of the following type:

$$\coprod_{0\leqslant i<j\leqslant n} \Delta^{n-2} \rightrightarrows \coprod_{0\leqslant l\leqslant n} \Delta^{n-1} \longrightarrow \partial\Delta^n\ .$$

(5) Show that the k-th horn Λ^n_k can be written as a coequalizer of the following type:

$$\coprod_{0\leqslant i<j\leqslant n} \Delta^{n-2} \rightrightarrows \coprod_{\substack{0\leqslant l\leqslant n\\ l\neq k}} \Delta^{n-1} \longrightarrow \Lambda^n_k\ .$$

(6) Why the above question shows that the geometrical k-th horn $|\Lambda^n_k|$ can be written as a coequalizer of the following type:

$$\coprod_{0\leqslant i<j\leqslant n} |\Delta^{n-2}| \rightrightarrows \coprod_{\substack{0\leqslant l\leqslant n\\ l\neq k}} |\Delta^{n-1}| \longrightarrow |\Lambda^n_k|\ ?$$

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Exercise 5 (Prismatic decomposition).
The product $\mathfrak{X}\times\mathfrak{Y}$ of two simplicial sets $\mathfrak{X}$ et $\mathfrak{Y}$ is defined by

$$(\mathfrak{X}\times\mathfrak{Y})_n := X_n \times Y_n$$

endowed by the faces $d_i^{\mathfrak{X}} \times d_i^{\mathfrak{Y}}$ and degeneracies $s_i^{\mathfrak{X}} \times s_i^{\mathfrak{Y}}$.

(1) Describe the simplicial set $\Delta^1\times\Delta^1$, notably its non-degenerate simplicies, and its geometric realisation $|\Delta^1\times\Delta^1|$.

(2) Describe the non-degenerate simplicies of the simplicial set $\Delta^p\times\Delta^q$ and insist on the case of the non-degenerate $p+q$ simplicies. Show that $|\Delta^p\times\Delta^q| \cong |\Delta^p|\times|\Delta^q|$.

(3) Describe the simplicial set $\Delta^p\times\Delta^1$ and draw the associate triangulation of $|\Delta^2\times\Delta^1|$.

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✉ Bruno Vallette: vallette@math.univ-paris13.fr .
☞ Web page: www.math.univ-paris13.fr/~vallette/ .