WORKSHEET 5

THE CATEGORY OF SIMPLICIAL SETS

Exercise 1 (Sets vs simplicial sets).
We consider the functor $\mathrm{cst} : \mathsf{Set} \to \mathsf{sSet}$ which assigns the constant simplicial set

$$X_n := X\ , \quad d_i = \mathrm{id}\ , \quad \text{and} \quad s_i = \mathrm{id}$$

to any set X.

(1) Show that this functor is full, faithful, and representable.
We denote by $\mathrm{C}^\bullet$ the cosimplicial set which represents this constant functor.
(2) Show that the constant functor cst admits a left adjoint functor L and describe it.
(3) Show that the constant functor cst admits a right adjoint functor R and describe it.
(4) Show that the category Set of sets is equivalent to the category sSet_0 of simplicial sets of dimension 0.

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Exercise 2 (Toward the Dold–Kan correspondence).
The *Moore complex* $\mathrm{C}_*\mathfrak{X}$ of a simplicial set $\mathfrak{X}$ admits for chains in degree n the free $\mathbb{Z}$-module on the set of n-simplicies, i.e. $\mathrm{C}_n\mathfrak{X} := \mathbb{Z}\, X_n$, and for differential the alternated sum of the faces:

$$\mathrm{d} := \sum_{i=0}^{n} (-1)^i \mathrm{d}_i \ : \ \mathbb{Z}\, X_n \to \mathbb{Z}\, X_{n-1}\ .$$

(1) Show that $\mathrm{d}^2 = 0$.
(2) Show that the graded module $\mathrm{D}_*\mathfrak{X}$ whose chains in degree n are made up of the free $\mathbb{Z}$-module on the set of degenerate n-simplicies $\mathrm{D}_n\mathfrak{X} := \mathbb{Z}\left(X_n \backslash \mathrm{N}X_n\right)$ is a chain sub-complex of the Moore complex, i.e. $\mathrm{D}_*\mathfrak{X} \subset \mathrm{C}_*\mathfrak{X}$.

The *normalised Moore complex* $\mathrm{N}_*\mathfrak{X}$ of a simplicial set $\mathfrak{X}$ is the quotient chain complex of $\mathrm{C}_*\mathfrak{X}$ by $\mathrm{D}_*\mathfrak{X}$, i.e.

$$\mathrm{N}_*\mathfrak{X} := (\mathrm{C}_*\mathfrak{X}/\mathrm{D}_*\mathfrak{X}, \mathrm{d})\ .$$

(3) Show that both the Moore chain complex C_* and the normalised Moore chain complex N_* form a functor from simplicial sets to chain complexes over $\mathbb{Z}$.
(4) Show that the normalised Moore chain complex N_* admit a right adjoint functor and describe it.

REMARK. One can prove that the canonical projection $\mathrm{C}_*\mathfrak{X} \twoheadrightarrow \mathrm{N}_*\mathfrak{X}$ is a homotopy equivalence. The normalised Moore functor N_* is part of an equivalence of categories between the category of simplicial abelian groups and non-negatively graded chain complexes over $\mathbb{Z}$.

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Exercise 3 (Categories and simplicial sets). We denote by Cat the category of small categories. For any $n \in \mathbb{N}$, consider the category $\mathsf{Cat}[n] := \{0 \to 1 \to \cdots \to n\}$ associated to the poset $0 < 1 < \cdots < n$.

(1) Endow these categories $\mathsf{Cat}[n]$, for $n \in \mathbb{N}$, with a cosimplicial structure in order to get a cosimplicial (small) category denoted by $\mathsf{Cat}[\bullet]$.

(2) Describe the simplicial representation functor

$$\begin{array}{rccl} \mathfrak{N} \coloneqq \mathrm{R}_{\mathsf{Cat}[\bullet]} : & \mathsf{Cat} & \longrightarrow & \mathsf{sSet} \\ & \mathsf{C} & \longmapsto & \mathrm{Hom}_{\mathsf{Cat}}(\mathsf{Cat}[\bullet], \mathsf{C}) \end{array}$$

associated to this cosimplicial category. Its upshot $\mathfrak{N}\mathsf{C}$ is usually called the *nerve of a category* C.

(3) Show that the nerve functor admits a left adjoint functor and describe it.

(4) Show that the nerve functor is full and faithful. How do you interpret this result?

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Exercise 4 (Classifying space).

Let G be a group. It gives rise to a category made up of one object $*$ with set of automorphisms given by G. The *classifying space* $\mathrm{B}G$ of the group G is the simplicial set defined by the nerve of this category.

(1) Describe the classifying space $\mathrm{B}G$ of a group G.

We consider the collection $(\mathrm{E}G)_n \coloneqq G^{n+1}$ of sets, for $n \geqslant 0$, endowed with the following face and degeneracy maps

$$\mathrm{d}_i(g_0, \ldots, g_n) \coloneqq (g_0, \ldots, \widehat{g_i}, \ldots, g_n) \quad \text{et} \quad \mathrm{s}_i(g_0, \ldots, g_n) \coloneqq (g_0, \ldots, g_i, g_i, \ldots, g_n) \ .$$

We denote this data simply by $\mathrm{E}G$.

(1) Show that the assignment $\mathrm{E}G$ defines a functor from the category of groups to the category of simplicial sets.

(2) Show that the functor $\mathrm{E} : \mathsf{Grp} \to \mathsf{sSet}$ admits a left adjoint and describe it.

(3) Any n-simplex $(\mathrm{E}G)_n = G^{n+1}$ admits a left action from the group G under the formula

$$g.(g_0, \ldots, g_n) \coloneqq (gg_0, \ldots, gg_n) \ .$$

Show this endows $\mathrm{E}G$ with a structure of a simplicial G-module.

(4) We consider the orbits under this action with the induced face and degeneracy maps:

$$\mathrm{E}G/G \coloneqq \left((\mathrm{E}G)_n/G = G^{n+1}/G, \bar{\mathrm{d}}_i, \bar{\mathrm{s}}_i\right) \ .$$

Show that $\mathrm{E}G/G$ is a simplicial set isomorphic to the classifying space $\mathrm{B}G$.

(5) Show that $\mathrm{E}G \twoheadrightarrow \mathrm{E}G/G \cong \mathrm{B}G$ is a surjective morphism of simplicial sets and compute its "fiber".

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