

Worksheet 6

HOMOTOPY THEORY OF SIMPLICIAL SETS

Exercise 1 (Kan complexes).

- (1) Prove that a simplicial set \mathfrak{X} is a Kan complex if and only if, for any $n \ge 2$, $0 \le k \le n$, and any collection $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \in X_{n-1}$ such that $d_{j-1}(x_i) = d_i(x_j)$, for $i < j, i, j \ne k$, there exists a $x \in X_n$ such that $d_i(x) = x_i$, for any $i \ne k$.
- (2) Show that the nerve of a group, viewed as a one-point category, is a Kan complex.
- (3) Do Kan complexes form the essential image of the singular functor?
- (4) Show that the underlying simplicial set of a simplicial group, i.e. after forgetting the group structures, forms a Kan complex.
- (5) Show that the standard *n*-simplicies Δ^n are not Kan complexes for $n \ge 1$.



Exercise 2 (Kan fibration). A *Kan fibration* is a morphism of simplicial sets $p: \mathfrak{X} \twoheadrightarrow \mathfrak{Y}$ which satisfies the following extension property: for every diagram as below, there exists a diagonal map $\Delta^n \to \mathfrak{X}$ which factors it

- (1) Give a characterisation of a Kan complex in terms of Kan fibrations.
- (2) Show that a continuous map $f: X \to Y$ is a Serre fibration if and only if the induced morphism of simplicial set $\operatorname{Sing}(f): \operatorname{Sing}(X) \to \operatorname{Sing}(Y)$ is a Kan fibration.
- (3) Give a combinatorial characterisation of a Kan fibration similar to that of a Kan complex given in Question (1) of Exercise 1.
- (4) Let $p: \mathfrak{X} \to \mathfrak{Y}$ be a Kan fibration and let $y \in Y_0$ be a vertex of \mathfrak{Y} . We denote by * the simplicial subset of \mathfrak{Y} generated by y_0 , that is its smallest simplicial subset containing y_0 . Show that (the *fiber*) $p^{-1}(*)$ is simplicial subset of \mathfrak{X} and a Kan complex.



Exercise 3 (Mapping space).

Let $i: \mathfrak{K} \hookrightarrow \mathfrak{L}$ be a monomorphism of simplicial sets and let $p: \mathfrak{X} \twoheadrightarrow \mathfrak{Y}$ be a Kan fibration. They induce the following commutative diagram

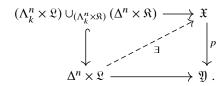
$$\begin{array}{ccc} \mathfrak{Hom}(\mathfrak{L},\mathfrak{X}) & \xrightarrow{p_*} \mathfrak{Hom}(\mathfrak{L},\mathfrak{Y}) \\ & & \downarrow^{i^*} & & \downarrow^{i^*} \\ \mathfrak{Hom}(\mathfrak{K},\mathfrak{X}) & \xrightarrow{p_*} \mathfrak{Hom}(\mathfrak{K},\mathfrak{Y}) \ . \end{array}$$

(4) Show that the induced map

$$\mathfrak{Hom}(\mathfrak{L},\mathfrak{X})\xrightarrow{(i^*,p_*)}\mathfrak{Hom}(\mathfrak{K},\mathfrak{X})\times_{\mathfrak{Hom}(\mathfrak{K},\mathfrak{Y})}\mathfrak{Hom}(\mathfrak{L},\mathfrak{Y})$$

is a Kan fibration.

HINT. Use the fact that p satisfies the following right lifting property:



- (5) Show that the pushforward $p_*: \mathfrak{Hom}(\mathfrak{R}, \mathfrak{X}) \to \mathfrak{Hom}(\mathfrak{R}, \mathfrak{Y})$ is a Kan fibration.
- (6) Show that the pullback $i^*: \mathfrak{Hom}(\mathfrak{L},\mathfrak{X}) \twoheadrightarrow \mathfrak{Hom}(\mathfrak{K},\mathfrak{X})$ is a Kan fibration when \mathfrak{X} is a Kan complex.



Exercise 4 (Topological homotopy vs simplicial homotopy).

- (1) Show that the geometric realisation functor $|\ |\ : \Delta \mathsf{Ens} \to \mathsf{Top}$ sends simplicial homotopies to a topological homotopies.
- (2) Show that the singular chain functor Sing : Top $\to \Delta$ Ens sends topological homotopies to simplicial homotopies.

REMARK. These two functors induce an equivalence of categories between the homotopy category of CW-complexes and the homotopy category of Kan complexes.



Exercise 5 (Classifying space again). We use the same objects and notations as in Exercise 4 of Worksheet 5.

- (1) Compute the simplicial homotopy groups of the nerve BG of a group.
- (2) Show that EG is contractible: there exist two morphisms of simplicial sets $f: EG \to * = \Delta^0$ and $g: * \to EG$ such that $fg \sim id_*$ and $gf \sim id_{EG}$.
- (3) Show that the morphism of simplicial sets $EG \twoheadrightarrow EG/G \cong BG$ is a Kan fibration.



Exercise • 6 (Homotopy groups).

- (1) Can one find a Kan complex \mathfrak{X} , such that each *n*-simplex X_n is finite, which models the circle?
- (2) Compute all the homotopy groups of spheres.

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