FREE MONOID IN MONOIDAL ABELIAN CATEGORIES

BRUNO VALLETTE

Abstract. We give an explicit construction of the free monoid in monoidal abelian categories when the monoidal product does not necessarily preserve coproducts. Then we apply it to several new monoidal categories that appeared recently in the theory of Koszul duality for operads and props. This gives a conceptual explanation of the form of the free operad, free dioperad and free properad.

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Introduction

The construction of the free monoid in monoidal categories is a general problem that appears in many fields of mathematics. In a monoidal category with denumerable coproducts, when the monoidal product preserves coproducts, the free monoid on an object V is well understood and is given by the words with letters in V (see [MacL1] Chapter VII Section 3 Theorem 2). In general, the existence of the free monoid has been established, under some hypotheses, by M. Barr in [B]. When the monoidal product preserves colimits over the simplicial category, E. Dubuc described in [D] a construction for the free monoid. A general categorical answer was given by G.M. Kelly in [K] when the monoidal product preserves colimit on one side. Once again, its construction requires the tensor product to preserve colimits. The problem is that the monoidal products that appeared recently in various domains do not share this general property.

In order to study the deformation theory of algebraic structures like algebras (e.g. associative, commutative, Lie algebras) and bialgebras (e.g. associative bialgebras, Frobenius bialgebras, Lie bialgebras, involutive Lie bialgebras), one models the operations acting on them with operads, properads or props. Like algebras for operads, it turns out that many types of (bi)algebras can be defined as particular modules over a monoid in a monoidal category. Moreover, in simple cases, one does not need the full machinery of props. For instance, Frobenius bialgebras and Lie bialgebras can be modelled by *dioperads* (see [G]) whereas associative bialgebras and involutive Lie bialgebras require the notion of *properads* (see [V]). These two notions are monoids which generate bigger props but such that the associated categories of models are the same. The (co)homology theories and the lax notion "up to homotopy" of a particular type of (bi)algebras are given by the Koszul duality of operads [GK], dioperads [G] or properads [V, MeVa]. To generalize Koszul duality theory from associative algebras [P] to operads, dioperads and properads, the first step is to extend to notions of bar and cobar constructions. These constructions are chain complexes whose underlying space is based on the (co)free (co)operad (respectively (co)dioperad and (co)properad).

This paper was motivated by these new examples of monoidal structures and the need to make explicit the associated free monoids for Koszul duality theories. When the monoidal product preserves coproducts on one side, G.M. Kelly gave a construction of the free monoid by means of a particular colimit in Equation (23.2) page 69 of [K] (see also H.J. Baues, M. Jibladze, A.Tonks in [BJT] Appendix B and C. Rezk [R] Appendix A). This construction applies to operads. Since the other monoidal products considered here do not preserve coproducts neither general colimits, we need to refine the arguments. For monoidal abelian categories verifying mild conditions, we produce a particular colimit which gives a general construction for the free monoid. Then we apply this result to make explicit the free monoid in various contexts. The construction of the free properad is new. For the other examples, the present construction gives a conceptual explanation for their particular form based on categories of graphs.

This paper is organized as follows. In the two first sections, we fix some conventions and recall the crucial notion of reflexive coequalizers. The general construction of the free monoid is given in Section 3. In Section 4, we define the notion of *split analytic functor*, which provides a sufficient condition to apply the results of the previous section. The last section is devoted to the description of the free properad, free $\frac{1}{2}$-prop, free dioperad, free special prop and free colored operad.

1. Conventions

We recall briefly the main definitions used throughout the text.

Let $(\mathcal{A}, \boxtimes, I)$ be a monoidal abelian category. One important goal here is to understand the behavior of the monoidal product with the coproduct of $\mathcal{A}$. In the sequel, we will denote the coproduct in an abelian category by

$$\begin{tikzcd} A \arrow[r, tail] \arrow[dr, "f"'] & A \oplus B \arrow[d, "f+g"] & B \arrow[l, tail] \arrow[dl, "g"] \\ & C. & \end{tikzcd}$$

When the maps f and g are evident the image of $f+g$ will be denote by $A+B$.

Definition (Multiplication functors)**.** For every object A of $\mathcal{A}$, we call *left multiplication functor* by A (respectively *right multiplication functor*), the functor defined by $L_A : X \mapsto A \boxtimes X$ (respectively $R_A : X \mapsto X \boxtimes A$).

Definition (Biadditive monoidal category)**.** When the left and right multiplication functors are additive for every object A of $\mathcal{A}$, the monoidal abelian category $(\mathcal{A}, \boxtimes, I)$ is said to be *biadditive*.

In a biadditive monoidal category, one knows how to construct classical objects such as free monoids. When the category is not biadditive, one can consider the following object to understand the default of the monoidal product to be additive.

Definition (Multilinear part)**.** Let A, B, X and Y be objects of $\mathcal{A}$. We call *multilinear part* in X the cokernel of the morphism

$$A \boxtimes Y \boxtimes B \xrightarrow{A\boxtimes i_Y \boxtimes B} A \boxtimes (X \oplus Y) \boxtimes B,$$

which is denoted by $A \boxtimes (\underline{X} \oplus Y) \boxtimes B$.

The multilinear part in X is naturally isomorphic to the kernel of the application

$$A \boxtimes (X \oplus Y) \boxtimes B \xrightarrow{A\boxtimes \pi_Y \boxtimes B} A \boxtimes Y \boxtimes B,$$

where π_Y is the projection $X \oplus Y \to Y$. The short exact sequence

$$\begin{tikzcd} A \boxtimes (\underline{X} \oplus Y) \boxtimes B \arrow[r, tail] & A \boxtimes (X \oplus Y) \boxtimes B \arrow[r, two heads, "A\boxtimes \pi_Y \boxtimes B"'] & A \boxtimes Y \boxtimes B \arrow[l, bend right, "A\boxtimes i_Y \boxtimes B"'] \end{tikzcd}$$

splits and we have naturally

$$A \boxtimes (X \oplus Y) \boxtimes B \cong A \boxtimes (\underline{X} \oplus Y) \boxtimes B \oplus A \boxtimes Y \boxtimes B.$$

A category is biadditive monoidal if and only if one has $A\boxtimes(\underline{X}\oplus Y)\boxtimes B = A\boxtimes X\boxtimes B$ for every objects A, B, X and Y.

Definition (The simplicial categories Δ and Δ_{face})**.** The class of objects of the *simplicial category* Δ is the set of finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$, for $n \in \mathbb{N}$. And the set of morphisms $Hom_\Delta([n], [m])$ is the set of order-preserving morphisms from $[n]$ to $[m]$.
For $i = 0, \ldots, n$, one defines the *face map* $\varepsilon_i \in Hom_\Delta([n], [n+1])$ by the following formula

$$\varepsilon_i(j) = \begin{cases} j & \text{if} \quad j < i, \\ j+1 & \text{if} \quad j \geq i. \end{cases}$$

The category Δ_{face} is the subcategory of Δ such that the sets of morphisms are reduced to the compositions of face maps (and the identities $id_{[n]}$).

Remark. The category Δ_{face} is also denoted by Δ^+ in the literature.

2. Reflexive coequalizers

In this section, we recall the properties of reflexive coequalizers which will play a crucial role in the sequel.

The multiplication functors L_A and R_A do not necessarily preserve cokernels, even when they are additive. Nevertheless, for some monoidal products (for instance the ones treated in the sequel), the multiplication functors preserve *reflexive coequalizers*, which is a weaker version of the notion of cokernel. For more details about reflexive coequalizers, we refer the reader to the book of P.T. Johnstone [J].

Definition (Reflexive coequalizer)**.** A pair of morphisms $X_1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} X_0$ is said to be *reflexive* is there exists a morphism $s_0 : X_0 \to X_1$ such that $d_0 \circ s_0 = d_1 \circ s_0 = id_{X_0}$. A coequalizer of a reflexive pair is called a *reflexive coequalizer.*

Proposition 1. *Let $\Gamma \; : \mathcal{A} \to \mathcal{A}$ be a functor in an abelian category $\mathcal{A}$. If Γ preserves reflexive coequalizers then it preserves epimorphisms.*

Proof. Let $B \xrightarrow{\pi} C$ be an epimorphism. Since $\mathcal{A}$ is an abelian category, π is the cokernel of its kernel $A \xhookrightarrow{i} B \xrightarrow{\pi} C$.
The cokernel π can be written as the reflexive coequalizer of the following pair

$$\begin{tikzcd} A \oplus B \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & B \arrow[l, bend right=40, "s_0"'] \arrow[r, two heads, "\pi"] & C, \end{tikzcd}$$

where $d_0 = i + id_B$, $d_1 = id_B$ and $s_0 = i_B$.
Since Γ preserves reflexive coequalizers, we get that $\Gamma(\pi)$ is the coequalizer of $(\Gamma(d_0), \Gamma(d_1))$. Therefore, $\Gamma(\pi)$ is an epimorphism. $\square$

A monoidal product is said to *preserve reflexive coequalizer* if multiplication functors L_A and R_A preserve reflexive coequalizers for every object A of $\mathcal{A}$.

Proposition 2 (Lemma 0.17 of [J])**.** *Let $(\mathcal{A}, \boxtimes, I)$ be a monoidal abelian category such that the monoidal product preserves reflexive coequalizers. Let*

$$\begin{tikzcd} M_1 \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & M_0 \arrow[l, bend right=40, "s_0"'] \arrow[r, two heads, "\pi"] & M \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} M'_1 \arrow[r, shift left, "d'_1"] \arrow[r, shift right, "d'_0"'] & M'_0 \arrow[l, bend right=40, "s'_0"'] \arrow[r, two heads, "\pi'"] & M' \end{tikzcd}$$

be two reflexive coequalizers. Then $M \boxtimes M'$ is the reflexive coequalizer of

$$\begin{tikzcd} M_1 \boxtimes M'_1 \arrow[r, shift left, "d_1 \boxtimes d'_1"] \arrow[r, shift right, "d_0 \boxtimes d'_0"'] & M_0 \boxtimes M'_0 \arrow[l, bend right=40, "s_0 \boxtimes s'_0"'] \arrow[r, two heads, "\pi \boxtimes \pi'"] & M \boxtimes M' . \end{tikzcd}$$

In a monoidal category, when the monoidal product preserves reflexive coequalizers, it shares the following crucial property with the multilinear part.

Proposition 3. *Let $(\mathcal{A}, \boxtimes, I)$ be a monoidal abelian category such that the monoidal product preserves reflexive coequalizers. Let V and W be two objects of $\mathcal{A}$ and let*

$\iota_A : A \hookrightarrow W$, $\iota_B : B \hookrightarrow W$ be two sub-objects of W. The following two objects are equal in $\mathcal{A}$

$$\mathrm{Im}\big(V \boxtimes (\underline{(A+B)} \oplus W)\big) = \mathrm{Im}(V \boxtimes (\underline{A} \oplus W)) + \mathrm{Im}(V \boxtimes (\underline{B} \oplus W)),$$

where Im *is understood to be the image of*

$$V \boxtimes (\underline{A} \oplus W) \hookrightarrow V \boxtimes (A \oplus W) \xrightarrow{V \boxtimes (\iota_A + id_W)} V \boxtimes W.$$

Proof. We prove first that the image $\mathrm{Im}(V \boxtimes (\underline{A} \oplus W))$ is equal to the kernel of $V \boxtimes W \xrightarrow{V \boxtimes \pi} V \boxtimes W/A$, where π is the cokernel of $A \hookrightarrow W$. This cokernel is the following reflexive coequalizer

$$\begin{tikzcd} A \oplus W \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & W \arrow[l, bend right, "s_0"'] \arrow[r, two heads, "\pi"] & W/A, \end{tikzcd}$$

where $d_0 = \iota_A + id_W$, $d_1 = \pi_W$ and $s_0 = i_W$. By the assumption, it is preserved by left tensoring with V

$$\begin{tikzcd} V \boxtimes (A \oplus W) \arrow[r, shift left, "V \boxtimes \pi_W"] \arrow[r, shift right, "V \boxtimes (\iota_A + id_W)"'] & V \boxtimes W \arrow[l, bend right, "V \boxtimes i_W"'] \arrow[r, two heads, "V \boxtimes \pi"] & V \boxtimes W/A\ . \end{tikzcd}$$

Hence the kernel of $V \boxtimes \pi$ is the image of $V \boxtimes (\iota_A + id_W) - V \boxtimes \pi_W$. Since $V \boxtimes (A \oplus W) = V \boxtimes W \oplus V \boxtimes (\underline{A} \oplus W)$ and since $V \boxtimes (\iota_A + id_W) - V \boxtimes \pi_W$ vanishes on the first component $V \boxtimes W$, the image of $V \boxtimes (\iota_A + id_W) - V \boxtimes \pi_W$ is given by the image on the second component $V \boxtimes (\underline{A} \oplus W)$. The multilinear part $V \boxtimes (\underline{A} \oplus W)$ is defined as the kernel of $V \boxtimes \pi_W$, so the image of $V \boxtimes (\iota_A + id_W) - V \boxtimes \pi_W$ on it, is equal to the image of $V \boxtimes (\iota_A + id_W)$.
Therefore, the left hand side of the equation is equal to the kernel of the following reflexive coequalizer

$$\begin{tikzcd} V \boxtimes (A \oplus B \oplus W) \arrow[r, shift left, "V \boxtimes \pi_W"] \arrow[r, shift right, "V \boxtimes (\iota_A + \iota_B + id_W)"'] & V \boxtimes W \arrow[l, bend right, "V \boxtimes i_W"'] \arrow[r, two heads, "V \boxtimes \pi"] & V \boxtimes W/(A+B)\ , \end{tikzcd}$$

that is the image of

$$\begin{aligned} V \boxtimes (\iota_A + \iota_B + id_W) - V \boxtimes \pi_W \quad &= \quad V \boxtimes (\iota_A + \iota_B + id_W) - V \boxtimes (\iota_B + id_W) \\ &\quad\quad + V \boxtimes (\iota_B + id_W) - V \boxtimes \pi_W. \end{aligned}$$

If we decompose $V \boxtimes (A \oplus B \oplus W)$ into

$$V \boxtimes (A \oplus B \oplus W) = V \boxtimes (\underline{A} \oplus B \oplus W) \oplus V \boxtimes (\underline{B} \oplus W) \oplus V \boxtimes W,$$

we can see that the image of $V \boxtimes (\iota_A + \iota_B + id_W) - V \boxtimes \pi_W$ on the first component is equal to $\mathrm{Im}(V \boxtimes (\underline{A} \oplus W))$. On the second component, it is equal to $\mathrm{Im}(V \boxtimes (\underline{B} \oplus W))$. And it vanished on the last component, which concludes the proof. $\square$

As a direct corollary, we get the same formula for a finite number of sub-objects $A_1, \ldots, A_n$ of W, that is

$$\mathrm{Im}\big(V \boxtimes (\underline{(\Sigma_i A_i)} \oplus W)\big) = \Sigma_i\ \mathrm{Im}(V \boxtimes (\underline{A_i} \oplus W)).$$

3. Construction of the free monoid

In this section, we give the construction of the free monoid. We work in monoidal abelian category $(\mathcal{A},\, \boxtimes,\, I)$ such that the monoidal product preserves reflexive coequalizers and sequential colimits.

Associated to every object V of $\mathcal{A}$, we consider the augmented object $V_+ := I \oplus V$. The injection of I in V_+ is denoted by $\eta\,:\, I \hookrightarrow V_+$ and the projection of V_+ in I is denoted by $\varepsilon\,:\, V_+ \twoheadrightarrow I$. We define $V_n := (V_+)^{\boxtimes n}$. By convention, we have $V_0 = (V_+)^0 = I$. Let $\mathcal{FS}(V)$ denote the coproduct $\bigoplus_{n\geqslant 0} V_n$.

This object is naturally endowed with degeneracy maps

$$\eta_i\,:\, V_n \cong (V_+)^{\boxtimes i} \boxtimes I \boxtimes (V_+)^{\boxtimes (n-i)} \xrightarrow{V_i \boxtimes \eta \boxtimes V_{n-i}} (V_+)^{\boxtimes i} \boxtimes V_+ \boxtimes (V_+)^{\boxtimes (n-i)} = V_{n+1}\ .$$

Remark. When V is an augmented monoid, consider the kernel $\bar{V}$ of the augmentation $V \to I$, called the *augmentation ideal*. There are face maps on $\mathcal{FS}(\overline{V})$ which define the categorical (or simplicial) bar construction on V (see [MacL1]).

In a biadditive monoidal category, the colimit of $\{V_n\}_n$ on the small category Δ_{face} is isomorphic to $\bigoplus_{n\in\mathbb{N}} V^{\boxtimes n}$, which corresponds to “words” in V. In this case, it gives the construction of the free monoid. In general, the colimit $\mathrm{Colim}_{\Delta_{\text{face}}} V_n$ is not preserved by the monoidal product. Therefore, one has to consider some quotient of V_n before taking the colimit on Δ_{face}.

Denote by $I \boxtimes A \xrightarrow{\lambda_A} A$ and $A \boxtimes I \xrightarrow{\rho_A} A$ the natural isomorphisms of the monoidal category $(\mathcal{A},\, \boxtimes,\, I)$. We define the morphism $\tau\,:\, V \to V_2$ by the following composition

$$V \xrightarrow{\lambda_V^{-1} \oplus \rho_V^{-1}} I \boxtimes V \oplus V \boxtimes I \xrightarrow{\eta \boxtimes i_V - i_V \boxtimes \eta} (I \oplus V) \boxtimes (I \oplus V) = V_2\ ,$$

where i_V is the inclusion $V \hookrightarrow I \oplus V$.
For every A and B two objects of $\mathcal{A}$, we consider the “relation” object

$$R_{A,B} := \mathrm{Im}\left(A \boxtimes (\underline{V} \oplus V_2) \boxtimes B \hookrightarrow A \boxtimes (V \oplus V_2) \boxtimes B \xrightarrow{A \boxtimes (\tau + id_{V_2}) \boxtimes B} A \boxtimes V_2 \boxtimes B \right).$$

We denote by $R_{i,\, n-i-2}$ the subobject $R_{V_i,\, V_{n-i-2}}$ of V_n for $0 \leq i \leq n-2$.

Definition $(\widetilde{V}_n)$**.** We define the object $\widetilde{V}_n$ by the formula

$$\widetilde{V}_n := \mathrm{Coker}\left(\bigoplus_{i=0}^{n-2} R_{i,n-i-2} \to V_n \right).$$

We denote by $R_n = \sum_{i=1}^{n-2} R_{i,n-i-2}$ the image of $\bigoplus_{i=0}^{n-2} R_{i,n-i-2}$ in V_n. Hence, we also denote $\widetilde{V}_n$ by $V_n/(\sum_{i=0}^{n-2} R_{i,n-i-2}) = Vn/R_n$ and the following short exact sequence by

$$0 \longrightarrow R_n \xhookrightarrow{i_n} V_n \xrightarrow{\pi_n} \widetilde{V}_n \longrightarrow 0.$$

Lemma 4.

(1) *The morphisms η_i between V_n and V_{n+1} induce morphisms $\widetilde{\eta}_i$ between the quotients $\widetilde{V}_n$ and $\widetilde{V}_{n+1}$.*

(2) *For every couple i, j, the morphisms $\widetilde{\eta}_i$ and $\widetilde{\eta}_j$ are equal.*

Proof.

(1) It is enough to see that

$$\begin{cases} \eta_i(R_{j,\,n-j-2}) \subset R_{j,\,n-j-1} & \text{if} \quad j \leq i-2, \\ \eta_i(R_{j,\,n-j-2}) \subset R_{j+1,\,n-j-2} & \text{if} \quad j \geq i, \\ \eta_i(R_{i-1,\,n-i-1}) \subset R_{i,\,n-i-1} + R_{i-1,\,n-i} & \text{for} \quad i = 1, \dots, n-1. \end{cases}$$

(2) Since $(\eta_i - \eta_{i+1})(V_n) \subset R_{i,\,n-i-1}$, one has $\widetilde{\eta}_i = \widetilde{\eta}_{i+1}$.

□

We denote by $\widetilde{\eta}$ any map $\widetilde{\eta}_i$.

Definition ($\mathcal{F}(V)$)**.** The object $\mathcal{F}(V)$ is defined by the following sequential colimit

$$\begin{tikzcd} I \arrow[r, "\widetilde{\eta}"] \arrow[dr, "j_0"'] & \widetilde{V}_1 = V_1 = V_+ \arrow[r, "\widetilde{\eta}"] \arrow[d, "j_1"] & \widetilde{V}_2 \arrow[r, "\widetilde{\eta}"] \arrow[dl, "j_2"'] & \widetilde{V}_3 \arrow[r, "\widetilde{\eta}"] \arrow[dll, "j_3"'] & \widetilde{V}_4 \cdots \arrow[dlll, "j_4"] \\ & \mathcal{F}(V) := \mathrm{Colim}_{\mathbb{N}} \widetilde{V}_n. & & & \end{tikzcd}$$

The colimit $\mathrm{Colim}_{\Delta_{\mathrm{face}}} V_n$ has been transformed to the sequential colimit $\mathrm{Colim}_{\mathbb{N}} \widetilde{V}_n$ by considering the quotients $\widetilde{V_n}$. The hypothesis that the monoidal product $\boxtimes$ preserves such colimits gives the following property.

Lemma 5. *For every object A of $\mathcal{A}$, the multiplication functors L_A and R_A preserve the previous colimit $\mathcal{F}(V)$. One has*

$$A \boxtimes \mathop{\mathrm{Colim}}_{\mathbb{N}} \widetilde{V}_n \cong \mathop{\mathrm{Colim}}_{\mathbb{N}} (A \boxtimes \widetilde{V}_n) \quad \textit{and} \quad \mathop{\mathrm{Colim}}_{\mathbb{N}} \widetilde{V}_n \boxtimes A \cong \mathop{\mathrm{Colim}}_{\mathbb{N}} (\widetilde{V}_n \boxtimes A).$$

We will now endow the object $\mathcal{F}(V)$ with a structure of monoid.

The unit $\widetilde{\eta}$ is given by the morphism $j_0 : I \to \mathcal{F}(V)$. The product is defined from the concatenation morphisms $V_n \boxtimes V_m \to V_{n+m}$. We consider

$$\mu_{n,\,m} \;:\; V_n \boxtimes V_m \xrightarrow{\sim} V_{n+m} \twoheadrightarrow \widetilde{V}_{n+m} \xrightarrow{j_{n+m}} \mathcal{F}(V)\ .$$

Proposition 6. *There exists a unique map $\widetilde{\mu}_{n,\,m} : \widetilde{V}_n \boxtimes \widetilde{V}_m \to \mathcal{F}(V)$ such that*

$$\begin{tikzcd} V_n \boxtimes V_m \arrow[r, two heads, "\pi_n \boxtimes \pi_m"] \arrow[dr, "\mu_{n,\,m}"'] & \widetilde{V}_n \boxtimes \widetilde{V}_m \arrow[d, "\widetilde{\mu}_{n,\,m}"] \\ & \mathcal{F}(V). \end{tikzcd}$$

Proof. The cokernels $\widetilde{V}_n$ are reflexive coequalizers of the pairs

$$\begin{tikzcd} R_n \oplus V_n \arrow[r, shift left, "d_1^n"] \arrow[r, shift right, "d_0^n"'] & V_n \arrow[l, bend right, "s_0^n"'] \arrow[r, two heads, "\pi_n"] & \widetilde{V}_n, \end{tikzcd}$$

where $d_0^n = i_n + id_{V_n}$, $d_1^n = \pi_{V_n}$ and $s_0^n = i_{V_n}$. Since the monoidal product $\boxtimes$ preserves reflexive coequalizers, we have, by Proposition 2, that $\pi_n \boxtimes \pi_m$ is the (reflexive) coequalizer of the pair $(d_0^n \boxtimes d_0^m,\, d_1^n \boxtimes d_1^m)$. Hence, the proof is given by the universal property of coequalizers. One just has to show that $\mu_{n,\,m}(d_0^n \boxtimes d_0^m) = \mu_{n,\,m}(d_1^n \boxtimes d_1^m)$. This relation comes from the diagram

$$\begin{tikzcd}
(R_n \oplus V_n) \boxtimes (R_m \oplus V_m) \arrow[r, "(i_n+id_{V_n})\boxtimes(i_m+id_{V_m})"] \arrow[d, "\pi_{V_n}\boxtimes\pi_{V_m}"] & V_n \boxtimes V_m \arrow[r, "\sim"] & V_{n+m} \arrow[d, "\pi_{n+m}"] \\
V_n \boxtimes V_m \arrow[r, "\sim"] & V_{n+m} \arrow[r, "\pi_{n+m}"] & \widetilde{V}_{n+m},
\end{tikzcd}$$

which is commutative by the following arguments. Since $(R_n \oplus V_n) \boxtimes (R_m \oplus V_m)$ decomposes as

$$\begin{aligned}
(R_n \oplus V_n) \boxtimes (R_m \oplus V_m) &\cong (\underline{R_n} \oplus V_n) \boxtimes (R_m \oplus V_m) \oplus V_n \boxtimes (R_m \oplus V_m) \\
&\cong \underbrace{(\underline{R_n} \oplus V_n) \boxtimes (R_m \oplus V_m)}_{(iii)} \oplus \underbrace{V_n \boxtimes (\underline{R_m} \oplus V_m)}_{(ii)} \oplus \underbrace{V_n \boxtimes V_m}_{(i)}
\end{aligned}$$

(see Section 1), it is enough to prove the relation on each component.
On (i), we have

$$V_n \boxtimes V_m \hookrightarrow (R_n \oplus V_n) \boxtimes (R_m \oplus V_m) \xrightarrow{(i_n+id_{V_n})\boxtimes(i_m+id_{V_m})} V_n \boxtimes V_m =$$
$$V_n \boxtimes V_m \hookrightarrow (R_n \oplus V_n) \boxtimes (R_m \oplus V_m) \xrightarrow{\pi_{V_n}\boxtimes\pi_{V_m}} V_n \boxtimes V_m = id.$$

On (ii) and on (iii), both composite vanish. The composite

$$V_n \boxtimes (\underline{R_m} \oplus V_m) \hookrightarrow (R_n \oplus V_n) \boxtimes (R_m \oplus V_m) \xrightarrow{\pi_{V_n}\boxtimes\pi_{V_m}} V_n \boxtimes V_m$$

is equal to zero by the definition of $V_n \boxtimes (\underline{R_m} \oplus V_m)$, which is the kernel of $id_{V_n} \boxtimes \pi_{V_m}$. The other composite

$$V_n \boxtimes (\underline{R_m} \oplus V_m) \hookrightarrow (R_n \oplus V_n) \boxtimes (R_m \oplus V_m) \xrightarrow{id\boxtimes(i_m+id)} V_n \boxtimes V_m \cong V_{n+m} \xrightarrow{\pi_{n+m}} \widetilde{V}_{n+m}$$

is also equal to zero because the image of $V_n \boxtimes (\underline{R_m} \oplus V_m) \hookrightarrow (R_n \oplus V_n) \boxtimes (R_m \oplus V_m) \xrightarrow{id\boxtimes(i_m+id)} V_n \boxtimes V_m \cong V_{n+m}$ is a sub-object of $R_{n+m} = \sum_{j=0}^{m+n-2} R_{j,m+n-j-2}$ by the following argument. Proposition 3 shows that this image is equal to the sum $\sum_{i=0}^{m-2} \mathrm{Im}\big(V_n \boxtimes (\underline{R_{i,m-i-2}} \oplus V_m)\big)$. Each $\mathrm{Im}\big(V_n \boxtimes (\underline{R_{i,m-i-2}} \oplus V_m)\big)$ is a sub-object of the image of the composite

$$V_n \boxtimes \big(\underline{V_i \boxtimes (\underline{V} \oplus V_2) \boxtimes V_{m-i-2}} \oplus V_m\big) \hookrightarrow V_n \boxtimes \big(V_i \boxtimes (V \oplus V_2) \boxtimes V_{m-i-2} \oplus V_m\big)\big)$$
$$\xrightarrow{V_n\boxtimes(V_i\boxtimes(\tau+id_{V_2})\boxtimes V_{m-i-2}+id_{V_m})} V_n \boxtimes V_m,$$

which is equal to the image of

$$V_{n+i} \boxtimes (\underline{V} \oplus V_2) \boxtimes V_{m-i-2} \hookrightarrow V_{n+i} \boxtimes (V \oplus V_2) \boxtimes V_{m-i-2}$$
$$\xrightarrow{V_{n+i}\boxtimes(\tau+id_{V_2})\boxtimes V_{m-i-2}} V_n \boxtimes V_m,$$

that is $R_{n+i,m-i-2}$.
We apply the same arguments to (iii). Since the image of $(\underline{R_n} \oplus V_n) \boxtimes (R_m \oplus V_m) \hookrightarrow (R_n \oplus V_n) \boxtimes (R_m \oplus V_m) \xrightarrow{(i_n+id)\boxtimes(i_m+id)} V_n \boxtimes V_m \cong V_{n+m}$ is a sub-object of the

sum $\sum_{i=0}^{n-2} R_{i,m+n-i-2}$, which is a sub-object of R_{n+m}, the same statement holds for (iii). $\square$

Lemma 7. *There exists a unique morphism $\widetilde{\mu}_{n,\,*}$ such that the following diagram is commutative*

$$\begin{tikzcd}
\widetilde{V}_n \boxtimes I \arrow[r, "\widetilde{V}_n \boxtimes \widetilde{\eta}"] \arrow[d, "\widetilde{\mu}_{n,\,0}"'] \arrow[drr] & \widetilde{V}_n \boxtimes \widetilde{V}_1 \arrow[r, "\widetilde{V}_n \boxtimes \widetilde{\eta}"] \arrow[dl, "\widetilde{\mu}_{n,\,1}"] \arrow[dr] & \widetilde{V}_n \boxtimes \widetilde{V}_2 \ \cdots \arrow[d] \arrow[dll, "\widetilde{\mu}_{n,\,2}"] \\
\mathcal{F}(V) & & \widetilde{V}_n \boxtimes \mathcal{F}(V) = \mathrm{Colim}_{\mathbb{N}}(\widetilde{V}_n \boxtimes \widetilde{V}_*). \arrow[ll, "\exists!\, \widetilde{\mu}_{n,\,*}"]
\end{tikzcd}$$

Proof. Since the morphisms $\widetilde{\mu}_{n,\,m}$ commute with the morphisms $\widetilde{V}_n \boxtimes \widetilde{\eta}$

$$\begin{tikzcd}
\widetilde{V}_n \boxtimes \widetilde{V}_m \arrow[r, "\widetilde{V}_n \boxtimes \widetilde{\eta}"] \arrow[d, "\widetilde{\mu}_{n,\,m}"'] & \widetilde{V}_n \boxtimes \widetilde{V}_{m+1} \arrow[dl, "\widetilde{\mu}_{n,\,m+1}"] \\
\mathcal{F}(V), &
\end{tikzcd}$$

we have by the universal property of colimits that there exists a unique map

$$\widetilde{\mu}_{n,\,*} \,:\, \operatorname*{Colim}_{\mathbb{N}}(\widetilde{V}_n \boxtimes \widetilde{V}_*) \to \mathcal{F}(V)$$

such that the diagram commutes. We conclude the proof with Lemma 5 which asserts that $\mathrm{Colim}_{\mathbb{N}}(\widetilde{V}_n \boxtimes \widetilde{V}_*) = \widetilde{V}_n \boxtimes \mathcal{F}(V)$. $\square$

Lemma 8. *There exists a unique morphism $\bar{\mu}$ such that the following diagram is commutative*

$$\begin{tikzcd}
I \boxtimes \mathcal{F}(V) \arrow[r, "\widetilde{\eta} \boxtimes \mathcal{F}(V)"] \arrow[d, "\widetilde{\mu}_{0,\,*}"'] \arrow[drr] & \widetilde{V}_1 \boxtimes \mathcal{F}(V) \arrow[r, "\widetilde{\eta} \boxtimes \mathcal{F}(V)"] \arrow[dl, "\widetilde{\mu}_{1,\,*}"] \arrow[dr] & \widetilde{V}_2 \boxtimes \mathcal{F}(V) \ \cdots \arrow[d] \arrow[dll, "\widetilde{\mu}_{2,\,*}"] \\
\mathcal{F}(V) & & \mathcal{F}(V) \boxtimes \mathcal{F}(V) = \mathrm{Colim}_{\mathbb{N}}(\widetilde{V}_n \boxtimes \mathcal{F}(V)). \arrow[ll, "\exists!\, \bar{\mu}"]
\end{tikzcd}$$

Proof. The arguments are the same. $\square$

Remark. The construction of $\bar{\mu}$ with the first colimit on the left and the second on the right gives the same morphism.

Proposition 9. *The object $\mathcal{F}(V)$ with the multiplication $\bar{\mu}$ and the unit $\bar{\eta}$ forms a monoid in the monoidal category $(\mathcal{A}, \boxtimes, I)$.*
Moreover, this monoid is augmented. We denote by $\bar{\mathcal{F}}(V)$ its ideal of augmentation.

Proof. The relation satisfied by the unit is obvious. The associativity of $\bar{\mu}$ comes from the associativity of the maps $\mu_{n,\,m}$.
The counit map ε is defined by taking the colimit of the maps

$$\begin{tikzcd}
R_n = \sum_{i=0}^{n-2} R_{i,\,n-2-i} \arrow[r] & V_n = (V \oplus I)^{\boxtimes n} \arrow[r, two heads, "\pi_n"] \arrow[d, "\varepsilon^{\boxtimes n}"'] & \widetilde{V}_n \arrow[dl, dashed, "\exists!\, \widetilde{\varepsilon^{\boxtimes n}}"] \\
 & I^{\boxtimes n} = I. &
\end{tikzcd}$$

$\square$

Theorem 10 (Free monoid)**.** *In a monoidal abelian category $(\mathcal{A}, \boxtimes, I)$ which admits sequential colimits and such that the monoidal product preserves sequential colimits and reflexive coequalizers, the monoid $(\mathcal{F}(V), \bar{\mu}, \bar{\eta})$ is free on V.*

Proof. The unit of adjunction is defined by

$$u_V \ : \ V \hookrightarrow V \oplus I \xrightarrow{j_1} \mathcal{F}(V).$$

For a monoid (M, ν, ζ), the counit $c_M \ : \mathcal{F}(M) \to M$ is given by the colimit of the following maps $\widetilde{\nu^n}$

$$\begin{tikzcd} R_n = \sum_{i=0}^{n-2} R_{i,\, n-2-i} \arrow[r] & M_n = (M \oplus I)^{\boxtimes n} \arrow[r, two heads, "\pi_n"] \arrow[d, "\nu^n \circ (M+\zeta)^{\boxtimes n}"'] & \widetilde{M}_n \arrow[dl, dashed, "\exists!\, \widetilde{\nu^n}"] \\ & M, & \end{tikzcd}$$

where the morphisms $\nu^n \ : \ M^{\boxtimes n} \to M$ represents $n-1$ compositions of the map ν with itself. The maps $\widetilde{\nu^n}$ are well defined since $\nu^n \circ (M+\zeta)^{\Box n}(R_{i,\, n-2-i}) = 0$, for every i, by associativity of ν.
One has immediately the two relations of adjunction

$$\mathcal{F}(V) \xrightarrow{\mathcal{F}(u_V)} \mathcal{F}(\mathcal{F}(V)) \xrightarrow{c_{\mathcal{F}(V)}} \mathcal{F}(V) \quad = \quad id_{\mathcal{F}(V)} \quad \text{and}$$

$$M \xrightarrow{u_M} \mathcal{F}(M) \xrightarrow{c_{\mathcal{F}(M)}} M \quad = \quad id_M.$$

□

4. Split analytic functors

In this section, we define the notion of *split analytic functor*. We show that a split analytic functor preserves reflexive coequalizers. We will use this proposition in the next section to show that the monoidal products, considered in the sequel, preserve reflexive coequalizers.

Let $\mathcal{A}$ be an abelian category. And denote by Δ_n the diagonal functor $\mathcal{A} \to \mathcal{A}^{\times n}$.

Definition (Homogenous polynomial functors)**.** We call a *homogenous polynomial functor of degree* n any functor $f \ : \ \mathcal{A} \to \mathcal{A}$ that can be written $f = f_n \circ \Delta_n$ with f_n a functor $\mathcal{A}^{\times n} \to \mathcal{A}$ additive in each input.

Definition (Split polynomial functor)**.** A functor $f \ : \ \mathcal{A} \to \mathcal{A}$ is called *split polynomial* if it is the direct sum of homogenous polynomial functors $f = \bigoplus_{n=0}^{N} f_{(n)}$.

The functors induced by monoidal products can not always be written with a finite sum of polynomial functors.

Definition (Split analytic functors)**.** We call *split analytic functor* any functor $f \ : \ \mathcal{A} \to \mathcal{A}$ equal to $f = \bigoplus_{n=0}^{\infty} f_{(n)}$ where $f_{(n)}$ is an homogenous polynomial functor of degree n.

Example. The Schur functor $\mathcal{S}_{\mathcal{P}} \ : \ \text{Vect} \to \text{Vect}$ associated to an $\mathbb{S}$-module $\mathcal{P}$ (a collection $\{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ of modules over the symmetric groups $\mathbb{S}_n$) defined by the following formula

$$\mathcal{S}_{\mathcal{P}}(V) := \bigoplus_{n=0}^{\infty} \mathcal{P}(n) \otimes_{\mathbb{S}_n} V^{\otimes n}$$

is a split analytic functor.

Proposition 11. *Let $f = \bigoplus_{n=0}^{N} f_{(n)}$ be a split analytic functor such that for every $n \in \mathbb{N}$, every $i \in [n]$ and every $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n \in \mathcal{A}$ the functor $X \mapsto f_n(X_1, \ldots, X_{i-1}, X, X_{i+1}, \ldots, X_n)$ preserves reflexive coequalizers. Then f preserves reflexive equalizers.*

Proof. Let $X_1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} X_0 \xrightarrow{\pi} X$ be a reflexive coequalizer. The result comes from the the formula

$$\sum_{i=1}^{n} f_n(X_0, \ldots, \underbrace{(d_0 - d_1)(X_1)}_{i^{\text{th}} \text{ place}}, \ldots, X_0) = (f_n(d_0, \ldots, d_0) - f_n(d_1, \ldots, d_1)) \circ \Delta_n(X_1).$$

The inclusion $\supset$ is always true since

$$f_n(d_0, \ldots, d_0) - f_n(d_1, \ldots, d_1) = \sum_{i=1}^{n} f_n(d_0, \ldots, d_0, \underbrace{d_0 - d_1}_{i^{\text{th}} \text{ place}}, d_1, \ldots, d_1).$$

The reverse inclusion $\subset$ lies on s_0 and comes from

$$\begin{aligned} & f_n(X_0, \ldots, X_0, (d_0 - d_1)(X_1), X_0, \ldots, X_0) \\ = \; & f_n(X_0, \ldots, d_0(X_1), \ldots, X_0) - f_n(X_0, \ldots, d_1(X_1), \ldots, X_0) \\ = \; & f_n(d_0 s_0(X_0), \ldots, d_0(X_1), \ldots, d_0 s_0(X_0)) - \\ & f_n(d_1 s_0(X_0), \ldots, d_1(X_1), \ldots, d_1 s_0(X_0)). \end{aligned}$$

□

5. Applications

The aim of this section is to apply the previous construction of the free monoid of new families of monoidal categories that appeared recently in the theory of Koszul duality. In order to understand the deformations of algebraic structures, one models them with an algebraic object (e.g. operads, colored operads, properads). This algebraic object turns out to be a monoid in an appropriate monoidal category. The best example is the notion of operad which is a monoid in the monoidal category of $\mathbb{S}$-modules with the composition product $\circ$ (see J.-L. Loday [L] or J.P. May [M])). The example of the free properad is new. The other free monoids given here were already known but the construction of Section 3 gives a conceptual explanation for their particular form.

5.1. **Free properad.** We recall the definition of the monoidal category of $\mathbb{S}$-bimodules with the connected composition product $\boxtimes_c$. For a full treatment of $\mathbb{S}$-bimodules and the related monoidal categories, we refer the reader to [V].

Definition ($\mathbb{S}$-bimodules)**.** An $\mathbb{S}$-*bimodule* $\mathcal{P}$ is a collection $(\mathcal{P}(m, n))_{m,\, n \in \mathbb{N}}$ of modules over the symmetric groups $\mathbb{S}_m$ on the left and $\mathbb{S}_n$ on the right, such that the two actions are compatible. We denote the category of $\mathbb{S}$-bimodules by $\mathbb{S}$-biMod.

An $\mathbb{S}$-bimodule models the operations with n inputs and m outputs acting on a type of algebraic structures (like algebras, bialgebras for instance).
In order to represent the possible compositions of these operations, we introduced in [V] a monoidal product $\boxtimes_c$ in the category of $\mathbb{S}$-bimodules. The product $\mathcal{Q} \boxtimes_c \mathcal{P}$ of two $\mathbb{S}$-bimodules is given by the sum on connected directed graphs with 2 levels $\mathcal{G}_c^2$ where the vertices of the first level are indexed by elements of $\mathcal{P}$ and the vertices of the second level are indexed by elements of $\mathcal{Q}$ (see Figure 1).

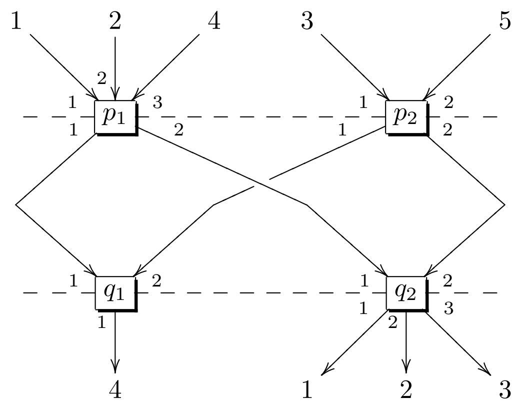


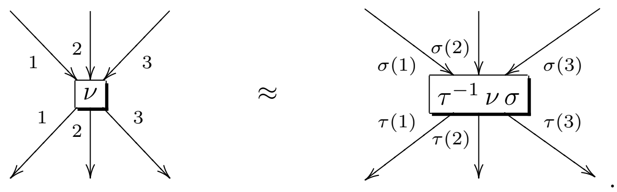
FIGURE 1. Example of an element of $\mathcal{Q} \boxtimes_c \mathcal{P}$.

We denote by $In(\nu)$ and $Out(\nu)$ the sets of inputs and outputs of a vertex ν of a graph. Let $\mathcal{N}_i$ be the set of vertices on the i^{th} level.

Definition (Connected composition product $\boxtimes_c$)**.** Given two $\mathbb{S}$-bimodules $\mathcal{P}$ and $\mathcal{Q}$, we define their product by the following formula

$$\mathcal{Q} \boxtimes_c \mathcal{P} := \left(\bigoplus_{g \in \mathcal{G}_c^2} \bigotimes_{\nu \in \mathcal{N}_2} \mathcal{Q}(|Out(\nu)|,\, |In(\nu)|) \otimes_k \bigotimes_{\nu \in \mathcal{N}_1} \mathcal{P}(|Out(\nu)|,\, |In(\nu)|) \right) \Big/ \approx\ ,$$

where the equivalence relation $\approx$ is generated by



The $\mathbb{S}$-bimodule I defined by the following formula

$$I := \begin{cases} I(1,\, 1) = k, & \\ I(m,\, n) = 0 & \text{otherwise.} \end{cases}$$

plays the role of the unit in the monoidal category ($\mathbb{S}$-bimod, $\boxtimes_c$). It corresponds to the identity operation.

Definition (Properads)**.** We call a *properad* a monoid $(\mathcal{P},\, \mu,\, \eta)$ in the monoidal category $(\mathbb{S}\text{-bimod},\, \boxtimes_c,\, I)$.

We are going to describe the free properad. To do that, we first show the following lemma.

Lemma 12. *For every pair* $(A,\, B)$ *of* $\mathbb{S}$*-bimodules, the functor*

$$\Phi_{A,\, B}\ :\ X \mapsto A \boxtimes_c X \boxtimes_c B$$

is a split analytic functor.

Proof. The $\mathbb{S}$-bimodule $A\boxtimes_c X\boxtimes_c B$ is given by the direct sum on 3-level connected graphs $\mathcal{G}_c^3$ such that the vertices of the first level are indexed by elements of B, the vertices of the second level are indexed by elements of X and the vertices of the third level are indexed by elements of A. Denote by $\mathcal{G}_{c,\, n}^3$ the set of 3-level graphs with n vertices on the second level. Therefore, the functor $\Phi_{A,\, B}$ can be written

$$\begin{aligned}
\Phi_{A,\, B}(X) &= A \boxtimes_c X \boxtimes_c B \\
&= \bigoplus_{n\in\mathbb{N}} \Big(\bigoplus_{g\in\mathcal{G}^3_{c,\, n}} \bigotimes_{\nu\in\mathcal{N}_1} A(|Out(\nu)|,\, |In(\nu)|) \otimes \bigotimes_{i=1}^{n} X(|Out(\nu_i)|,\, |In(\nu_i)|) \otimes \\
&\qquad \bigotimes_{\nu\in\mathcal{N}_3} B(|Out(\nu)|,\, |In(\nu)|) \Big) \Big/ \approx \\
&= \bigoplus_{n\in\mathbb{N}} \Phi_n(X, \ldots, X),
\end{aligned}$$

where Φ_n is an homogenous polynomial functor of degree n. $\square$

Proposition 13. *The category* $(\mathbb{S}\text{-}biMod,\, \boxtimes_c,\, I)$ *is a monoidal abelian category that preserves reflexive coequalizers and sequential colimits.*

Proof. For every $\mathbb{S}$-bimodule A, the left and right multiplicative functors $L_A := A \boxtimes_c \bullet$ and $R_A := \bullet \boxtimes_c A$ by A are split analytic functors by the previous lemma. Since the functors Φ_n preserve reflexive equalizers in each variable, they preserve reflexive coequalizers by Proposition 11. $\square$

This proposition allows us to apply Theorem 10. Let us interpret this construction in the framework of $\mathbb{S}$-bimodules.

Theorem 14. *The free properad on an* $\mathbb{S}$*-bimodule* V *is given by the sum on connected graphs (without level)* $\mathcal{G}$ *with the vertices indexed by elements of* V

$$\mathcal{F}(V) = \left(\bigoplus_{g\in\mathcal{G}_c} \bigotimes_{\nu\in\mathcal{N}} V(|Out(\nu)|,\, |In(\nu)|) \right) \Bigg/ \approx\ .$$

The composition μ *comes from the composition of directed graphs.*

Proof. The multilinear part in Y, denoted $A \boxtimes_c (X \oplus \underline{Y}) \boxtimes_c B$ is isomorphic to the sub-$\mathbb{S}$-bimodule of $A \boxtimes_c (X \oplus Y) \boxtimes_c B$ composed by 3-level connected graphs with the vertices of the second level indexed by elements of X and at least one element of Y. Let V be an $\mathbb{S}$-bimodule. Denote by $V_+ = I \oplus V$ the augmented $\mathbb{S}$-bimodule. Consider the $\mathbb{S}$-bimodule $V_n := (V_+)^{\boxtimes_c n}$ given by n-level connected graphs where the vertices are indexed by elements of V and I. The $\mathbb{S}$-bimodule $\widetilde{V_n} := \text{Coker}\left(\bigoplus_i R_{V_i,\, V_{n-i-2}} \to V_n\right)$ corresponds to the quotient of the $\mathbb{S}$-bimodule of n-level connected graphs by the relation $V \boxtimes_c I \simeq I \boxtimes_c V$, which is equivalent to forget the levels. $\square$

The notion of properad is a "connected" version of the notion of prop (see F.W. Lawvere [La], S. Mac Lane [MacL2] and J.F. Adams [A]). For more details about the link between these two notions we refer the reader to [V]. From the previous theorem, one can get the description of the free prop on an $\mathbb{S}$-bimodule V. We find the same construction of the free prop as B. Enriquez and P. Etingof in [EE] in terms of forests of graphs without levels.

Recall that we have the following inclusions of monoidal abelian categories (see [V] Section 1)

$$(Vect,\, \otimes_k,\, k) \hookrightarrow (\mathbb{S}\text{-Mod}, \circ,\, I) \hookrightarrow (\mathbb{S}\text{-biMod}, \boxtimes_c,\, I),$$

where the product $\circ$ of $\mathbb{S}$-modules corresponds to the composition of the related Schur functors. It can be represented by trees with 2 levels (see J.-L. Loday [L] and J.P. May [M]). A monoid for the product $\circ$ is called an *operad*. A direct corollary of the preceding theorem gives the free associative algebra and the free operad as the direct sum on trees without levels. Since the monoidal product $\circ$ of $\mathbb{S}$-modules preserves coproducts on the left, the free operad can be given by more simple colimit (see Kelly [K] Equation (23.2) page 69, Baues-Jibladze-Tonks [BJT] Appendix B and Rezk [R] Appendix A).

5.2. **Free $\frac{1}{2}$-prop.** On the category of $\mathbb{S}$-bimodules, one can define three other monoidal products. When one wants to model the operations acting on types of (bi)algebras defined by relations written with simple graphs (without loops for instance), there is no need to use the whole machinery of properads. It is enough to restrict to simpler types of compositions. That is we consider monoidal products based on these compositions. The main property is that the category of (bi)algebras over this more simple object is equal to the category of (bi)algebras of the associated properad. Therefore, in order to study the deformation theory of these (bi)algebras, it is enough to prove Koszul duality theory for the simpler monoid. (For more details on these notions, we refer the reader to the survey of M. Markl [Ma2]).

Denote by $\mathcal{G}_2^{\frac{1}{2}}$ the set of 2-level connected graphs such that every vertices of the first level has only one output or such that every vertices of the second level has only one input (see Figure 2).

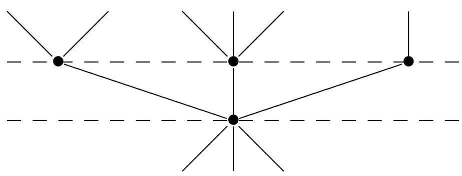


FIGURE 2. Example of a graph in $\mathcal{G}_2^{\frac{1}{2}}$.

Definition (Product $\square_{\frac{1}{2}}$)**.** Let $\mathcal{P}$, $\mathcal{Q}$ be two $\mathbb{S}$-bimodules. Their product $\mathcal{Q} \square_{\frac{1}{2}} \mathcal{P}$ is the restriction of the connected composition product $\mathcal{Q} \boxtimes_c \mathcal{P}$ on graphs of $\mathcal{G}_2^{\frac{1}{2}}$.

This product is associative and has I for unit. Therefore, ($\mathbb{S}$-biMod, $\Box_{\frac{1}{2}}$, I) is a monoidal abelian category. A monoid in this category is a $\frac{1}{2}$-prop, notion defined by M. Markl and A.A. Voronov in [MV] and introduced by M. Kontsevich [Ko, Ma]. Once again, we can apply Theorem 10. The free $\frac{1}{2}$-prop on an $\mathbb{S}$-bimodule V is given the sum on graphs with one vertex in the middle, grafted above by trees (without levels) and grafted below by reversed trees without levels (see Figure 3).

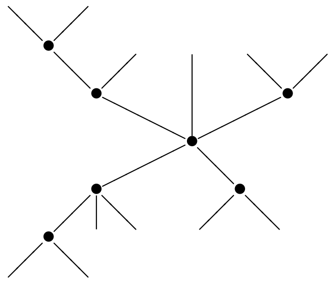


FIGURE 3. Example of the underlying graph in a free $\frac{1}{2}$- PROP.

5.3. **Free dioperad.** Wee Liang Gan in [G] considered the case when the permitted compositions are based on graphs of genus 0 (see Figure 4).

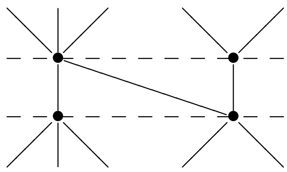


FIGURE 4. Example of a connected graph with 2 levels.

Definition (Product $\Box$)**.** The product $\mathcal{Q}\Box\mathcal{P}$ of two $\mathbb{S}$-bimodules is given by the restriction on 2-level connected graphs of genus 0 of the monoidal product $\boxtimes_c$.

Once again, this defines a new monoidal category structure on $\mathbb{S}$-bimodules. A monoid for this product corresponds to the notion of *dioperad* introduced in [G]. By the same arguments, Theorem 10 shows that the free dioperad on an $\mathbb{S}$-bimodule V is given by the direct sum of graphs of genus 0, without levels, whose vertices are indexed by elements of V.

For example, Lie bialgebras, Frobenuis algebras, infinitesimal bialgebras can be modelled by a dioperad (see [G]).

5.4. **Free special prop.** In order to give the resolution of the prop of bialgebras, M. Markl in [Ma] defined the notion of *special props*. It is corresponds to monoids in the monoidal category of $\mathbb{S}$-bimodules where the monoidal product is based only on composition called *fractions* ([Ma] definition 19). We can apply Theorem 10 in this case which gives the free special prop.
Notice that this notion of special props corresponds to the notion of *matrons* defined by S. Saneblidze R. Umble in [SU] and is related to the notion of $\frac{2}{3}$-prop of B. Shoikhet [Sh].

5.5. **Free colored operad.** Roughly speaking, a colored operad is an operad where the operations have colors indexing the leaves and the root. The composition of such operations is null if the colors of the roots of the inputs operations do not fit with the colors of the operation below. C. Berger and I. Moerdijk defined a monoidal product of the category of colored collections such that the related monoids are exactly colored operad (see Appendix of [BM]). Once again, Theorem 10 applies in this case and we get the description of the free colored operad by means of trees without levels.

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Laboratoire J.A. Dieudonné, Université de Nice Sophia-Antipolis, Parc Valrose, 06108 Nice Cedex 02, France
E-mail address : `brunov@math.unice.fr`
URL : `http://math.unice.fr/~brunov`