A combinatorial model for certain Taylor towers

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The Cast

The goal of this talk is to explain the relationship between these characters:

- The forgetful functor $U : s.Comm_{\mathbb{Q}} \to s.Mod_{\mathbb{Q}}$;
- The de Rham complex for $\mathbb{Q} \to B$:

$$\cdots \leftarrow \Omega^3_{B/\mathbb{Q}} \leftarrow \Omega^2_{B/\mathbb{Q}} \leftarrow \Omega_{B/\mathbb{Q}} \leftarrow B$$

- $U(B \otimes_{\mathbb{Q}} \mathrm{sk}_1 \Delta^{\bullet}_{*})$ where sk_1 is the simplicial 1-skeleton;
- Functor calculus Taylor towers.

The Cast

- Rezk: Connected $U(B \otimes_{H\mathbb{Q}} \mathrm{sk}_1 \Delta^\bullet_*)$ to the de Rham complex via the homotopy spectral sequence of this cosimplicial object.
- Goodwillie-Waldhausen: Connected the functor calculus tower of the forgetful functor from rational commutative ring spectra to $\mathcal{S}$-modules to the de Rham complex.

Prequel: calculus of functions

Recall that if $f : \mathbb{R} \to \mathbb{R}$ is C^∞, then

- The nth Taylor polynomial of f about b at x is

$$T_n^b f(x) = f(b) + f'(b)(x-b) + \cdots + \frac{f^{(n)}(b)(x-b)^n}{n!}$$

- The Taylor series of f about b at x is

$$T_\infty^b f(x) = \sum_{n \geq 0} \frac{f^{(n)}(b)(x-b)^n}{n!}$$

and if f is nice (analytic) then $T_\infty^b f(x) = f(x)$ for x near b (radius of convergence).

The setting

Let

- $\mathcal{C}_g$ be a simplicial model category; with initial object A and terminal object B;
- g is the unique map $g : A \to B$ in $\mathcal{C}_g$
- $\mathcal{S}$ a category of spectra (stable model category);
- $F : \mathcal{C}_g \to \mathcal{S}$ a functor which preserves weak equivalences.

Act 1: calculus of functors

A *Taylor tower* for F is any tower of functors and natural transformations

$$F \to \cdot \to P_nF(X) \to P_{n-1}F(X) \to \cdots \to P_1F(X) \to P_0F(X)$$

such that

- Each functor P_nF is polynomial degree n for some notion of degree n;
- The functors P_nF are universal amongst degree n functors with natural transformations from F.

Goodwillie's n-excisive tower $P_n F$:

- Polynomial degree n means n-excisive, i.e. $P_n F$ takes strongly homotopy co-cartesian n-cubes to homotopy cartesian n-cubes.
- Let $D_n F = \mathrm{hofib}\,(P_n F \to P_{n-1} F)$. Then

$$D_n F(X) \simeq \partial_n F \wedge_{h\Sigma_n} X^{\wedge n}.$$

- There are good notions of *analyticity* and *convergence*.
- If G is n-excisive and there is a natural transformation $F \to G$ with constants κ and c such that

  for any $A \to X \to B$ in $\mathcal{C}_g$ where $X \to B$ is k-connected for some $k \geq \kappa$, then $F(X) \to G(X)$ is at least $((n+1)k - c)$-connected

  then $P_n F \simeq G$.

See Goodwillie 92, Goodwillie 03, Kuhn 07, BJM15.

Johnson-McCarthy discrete calculus

The discrete calculus tower has a different notion of degree n:

- If $\mathcal{C}$ is pointed, F is linear iff it is reduced and

$$F(X \coprod Y) \simeq F(X) \times F(Y).$$

- For general functors from $\mathcal{C}_g$, the failure of F to be linear is measuerd by the 2nd cross effect, the iterated homotopy fiber of

$$\begin{tikzcd} F(X \coprod_A Y) \arrow[r] \arrow[d] & F(X \coprod_A B) \arrow[d] \\ F(B \coprod_A Y) \arrow[r] & F(B \coprod_A B) \end{tikzcd}$$

- The nth cross effect is defined by the iterated homotopy fiber of an n-cube which is $F(X_1 \coprod_A \cdots \coprod_A X_n)$ in its initial corner.

A functor with $cr_n F \simeq \star$ is degree n.

Johnson-McCarthy discrete calculus

If $\mathcal{C}$ is pointed, there is an adjuntion

$$Fun(\mathcal{C},\mathcal{S}) \underset{cr_n}{\overset{\Delta}{\leftrightarrows}} Fun(\mathcal{C}^n,\mathcal{S})$$

This produces a cotriple $\perp_n = \Delta \circ cr_n$. If $\mathcal{C}$ is not pointed, $\perp_n$ is still a cotriple. [Johnson-McCarthy 03, BJM15]

Definition (Johnson-McCarthy 03, BJM15)

The universal degree n approximation to $F : \mathcal{C}_g \to \mathcal{S}$ is

$$\Gamma_n^g F(X) = \mathrm{hocof}\left(|\perp_n^* F(X)| \to F(X)\right).$$

Agreement: end of Act I

Theorem (BJM15)

1. *When F commutes with realizations,*

$$\Gamma_n^g F \simeq P_n F.$$

2. *At the initial object A,*

$$\Gamma_n^g F(A) \simeq P_n F(A).$$

Note: 'commutes with realizations' is a mild condition; for example if F satisfies the limit axiom and is n-excisive, then F commutes with realizations by Mauer-Oats 01.

Act II: de Rham complex

Let $\mathcal{C}$ be the category of simiplicial commutative $A_\bullet$-algebras.

- The Kähler differentials Ω_{B_i/A_i} are the free B_i-module generate by symbols db subject to $d(b+c) = db + dc$ and $d(bc) = bdc + cdb$.
- The de Rham complex is $DR_A(B)$:

$$\cdots \leftarrow \Omega^3_{B_i/A_i} \leftarrow \Omega^2_{B_i/A_i} \leftarrow \Omega^1_{B_i/A_i} \leftarrow B_i$$

  where $\Omega^n_{B_i/A_i}$ is the nth exterior power of $\Omega^1_{B_i/A_i}$.
- Note that this is a cochain complex of A-modules. We will specialize to $A_\bullet = \mathbb{Q}_\bullet$, a constant simplicial object.

Here $A_\bullet$ is inital, but $B_\bullet$ isn't terminal.

An analogy

The terminal object B is like the center of expansion for the Taylor series of functions, b.

- $T_\infty^b f(x) = f(x)$ for x **near** b.
- $P_\infty F(X) \simeq F(X)$ for $X \to B$ highly connected.

If we want to let B vary, we should see what happens when we let b vary.

The Taylor series $T_n^b f(0)$ as a function of b

1. It is still a series:

$$T_n^b f(0) = f(b) - f'(b)b + \cdots + (-1)^n \frac{f^{(n)}(b)b^n}{n!}.$$

2. $T_n^b f(0)$ needn't be polynomial degree n, even if f was polynomial degree n.
3. $T_\infty^b f(0)$ wants to recover the **number** $f(0)$, not the function f.

$T^b_\infty f(0)$ wants to recover the **number** $f(0)$

If f is polynomial degree n, then

$$\begin{aligned}
\frac{d}{db} T^b_\infty f(0) &= \frac{d}{db} \sum_{k=0}^{n} \frac{(-1)^n f^{(n)}(b) b^n}{n!} \\
&= \sum_{k=0}^{n} \frac{(-1)^n f^{(n+1)}(b) b^n}{n!} + \sum_{k=0}^{n} \frac{n(-1)^n f^{(n)}(b) b^{n-1}}{n!} \\
&= 0
\end{aligned}$$

because the latter is a telescoping sum.

Hence, $T^b_\infty f$ is constant, and for $b = 0$ it is $f(0)$.

The varying center tower

Definition

Let $F : {}_A\backslash\mathcal{C} \to \mathcal{S}$. The varying center tower V_nF is defined by

$$V_nF(g : A \to X) = \Gamma_n^g F(A).$$

Properties of this tower:

1. V_nF is a functor.
2. There are natural transformations $V_nF \to V_{n-1}F$ making $\{V_nF\}_{n\geq 0}$ into a tower.
3. It is NOT a Taylor tower (not polynomial).
4. When it converges, it converges to the constant functor $F(A)$.

Back to de Rham

Theorem (Goodwillie-Waldhausen, BEJM)

Let $g : \mathbb{Q} \to B_\bullet$.

$$V_n U(g : \mathbb{Q} \to B_\bullet) \simeq DR^n_{\mathbb{Q}}(B_\bullet)$$

where $DR^n_{\mathbb{Q}}(B_\bullet)$ denotes the truncation of the de Rham complex at the n-th stage.

To explain this, recall $V_n U(g : \mathbb{Q} \to B_\bullet) = \Gamma^g_n U(\mathbb{Q})$, and in this case $\Gamma^g_n = P_n$. We will show:

1. $DR^n_{X_\bullet}(B_\bullet)$ is n-excisive as a functor of $\mathcal{C}_f$.
2. If $X_\bullet \to B_\bullet$ is $k \geq 1$ connected, then $U(X) \to DR^n_{\mathbb{Q}}(X_\bullet)$ is at least $(n+1)k - (n+1)$-connected.

Then at $X_\bullet = \mathbb{Q}$, $DR^n_{\mathbb{Q}}(B_\bullet) \simeq \Gamma^g_n U(\mathbb{Q}) =: V_n U(g : \mathbb{Q} \to B)$.

$DR^n_{X_\bullet}(B_\bullet)$ is n-excisive

For this, we work one Ω^n at a time.

- $\Omega^1_{B_\bullet/X_\bullet}$ is degree 1 because $\Omega^1_{B_\bullet/X_\bullet} = I/I^2$ where $I = \ker(B \otimes_X B \to B)$.
- $\Omega^n_{B_\bullet/X_\bullet}$ is degree n because $\Omega^n_{B_\bullet/X_\bullet} := (\Omega^1_{B_\bullet/X_\bullet})^{\otimes n}_{\Sigma_n}$ and rationally, this is $(\Omega^1_{B_\bullet/X_\bullet})^{\otimes n}_{h\Sigma_n}$.
- Functors of this form are homogeneous degree n by fundamental results of Goodwillie.

Connectivity

If $X_\bullet \to B_\bullet$ is $k > 1$ connected, then $U(X) \to DR^n_{X_\bullet}(B_\bullet)$ is $(n+1)k - (n+1)$-connected.

- Assume that $X_\bullet \to B_\bullet$ is an isomorphism in dimensions $\leq k$, injective in dimensions $> k$.
- Then $\Omega^1_{B_i/X_i} = 0$ for $0 \leq i \leq k$.
- Then $\Omega^m_{B_i/X_i} = 0$ for $0 \leq i \leq mk$.
- As a result, the map

$$DR_{X_\bullet}(B_\bullet) \to DR^n_{X_\bullet}(B_\bullet)$$

  is at least $(n+1)k - (n+1)$-connected.
- By the Poincare Lemma, each row satisfies $DR_{X_i}(B_i) \simeq X_i$

Evaluating at $X_\bullet = \mathbb{Q}$, we get

$$V_n U(\mathbb{Q} \to B_\bullet) = DR^n_{\mathbb{Q}}(B_\bullet).$$

Act III: Cosimplicial models

Rezk-Goodwillie-Waldhausen now says:

$$V_\infty U(\mathbb{Q} \to B) \simeq DR_{\mathbb{Q}}(B) \simeq Tot|B \otimes_{\mathbb{Q}} \mathrm{sk}_1 \Delta_*^\bullet|.$$

Theorem (BEJM)

Let $g : A \to B$ and $F : \mathcal{C}_g \to \mathcal{S}$. For any $n \geq 0$,

$$V_\infty F(g) \to Tot|\Gamma_\infty^g F(X \otimes_A \mathrm{sk}_n \Delta_*^\bullet)|$$

and the tower of spectra

$$\{V_k F(g)\}_{k\geq 1} = \{\Gamma_k^g F(A)\}_{k\geq 1}$$

is pro-equivalent to the Tot-tower of spectra

$$\{Tot^{m(k+1)}|\Gamma_k^g F(X \otimes_A \mathrm{sk}_n \Delta_*^\bullet)|\}_{k\geq 1}.$$

An outline of the proof

Work inductively, first proving it for linear functors:

Proposition (BEJM)

If F is degree 1 relative to g, then

$$F(A) \simeq \mathrm{Tot}^m \left|F(X \otimes_A \mathrm{sk}_n \Delta^\bullet_*)\right|$$

for all $m \geq n+1$. This gives a pro-equivalence of the constant tower $F(A)$ with the Tot-tower on the right.

then polynomial functors:

Proposition

For all $k \geq 1$ and $n \geq 0$, the map

$$V_k F(g) = \Gamma^g_k F(A) \to Tot^m |\Gamma^g_k F(X \otimes_A \mathrm{sk}_n \Delta^\bullet_*)|$$

is an equivalence for each $m \geq (n+1)k$. This gives a pro-equivalence.

The case $n = 1$

When F is degree 1,

$$F(X \otimes_A U) \simeq F(X) \otimes_{F(A)} U$$

for any finite non-empty set U because $F(X \otimes_A --)$ comes from a cocartesian cube.

Now,

$$F(A) \simeq F(X) \otimes_{F(A)} \emptyset = F(X) \otimes_{F(A)} \mathrm{Tot}^m \, \mathrm{sk}_n \, \Delta_*^\bullet$$

whenever $m \geq n + 1$.

In $\mathcal{S}$, the coproduct is weakly equivalent to a product, and Tot commutes with products:

$$F(A) \simeq \mathrm{Tot}^m(F(X) \otimes_{F(A)} \mathrm{sk}_n \, \Delta_*^\bullet) \simeq \mathrm{Tot}^m \, F(X \otimes_A \mathrm{sk}_n \, \Delta_*^\bullet).$$

Apply the geometric realization to finish.

Convergence

Let $\mathcal{C} = \mathit{Top}$ or $\mathcal{S}$. We say F is *weakly ρ-analytic relative to g* provided that for any object $A \to X \to B$ in $\mathcal{C}_g$ with $X \to B$ ρ-connected, $F(X) \simeq \Gamma^g_\infty F(X)$.

Theorem (BEJM)

If g is c-connected, F is weakly ρ-analytic relative to g, and F commutes with realizations, then

$$V_\infty F(g) \simeq \operatorname{Tot} |F(B \otimes_A \operatorname{sk}_n \Delta^\bullet_*)|$$

whenever $n \geq \rho - c - 1$ is non-negative.

As a special case,

$$V_\infty U(\mathbb{Q} \to B) \simeq \operatorname{Tot} |B \otimes_{\mathbb{Q}} \operatorname{sk}_1 \Delta^\bullet_*|.$$